A.J.J.M. DE KLERK

KNOTS AND ELECTROMAGNETISM

BACHELOR'S THESIS

SUPERVISED BY

dr. R.I. van der Veen, dr. J.W. Dalhuisen,
and prof. dr. D. Bouwmeester

25 JUNE 2016

Mathematical Institute
Leiden Institute of Physics

Leiden University

CONTENTS

1 INTRODUCTION

Over the past decades, the interaction between knot theory and physics has been of great interest. Both integral curves and zero sets that form knots, as well as invariants from knot theory, have arisen in physical theories. Particularly interesting is the occurrence of the Hopf fibration in many different areas of physics [22]. In electromagnetism, the Hopf fibration arises as the field line structure of an electromagnetic field, which we will refer to as the Hopf field. Its mathematical elegance, and its physical relevance in relation to plasma physics were the main motivation for our research.

One of our initial goals was to derive the Hopf field in a way not relying on the topological model of electromagnetism by Ranada [17, 18], or the ad hoc choice of Bateman variables in [12]. Employing a construction by Synge [21] allowed us to achieve this goal and gave rise to Bateman variables for the Hopf field. Then, during a study of generalisations of the Hopf field proposed by Kedia et al. in [12], we discovered a way of constructing electromagnetic fields such that the intersection of their zero set with an arbitrary spacelike slice in Minkowski space is a given algebraic link. These linked zero sets of electromagnetic fields in spacelike slices, also called optical vortices in physics, have already been studied both experimentally and theoretically in [5, 7, 13]. The main difference between this previous work and our result lies in the fact that our construction yields exact solutions to the Maxwell equations, while this prior work concerns paraxial fields. We should note that an exception is a paper by Bialynicki-Birula [6], upon which we build.

This thesis starts with a study of Minkowski space and operators induced by its pseudo-Riemannian metric in chapter 2. Then we go on to formulate electromagnetism in terms of differential forms in chapter 3. Chapter 2 and 3 show how many well known and some less well known results from physics arise naturally from this mathematical formalism. Furthermore, these chapters provide the necessary background for our treatment of the main results in chapter 4. In this chapter, we show how the Hopf field can be derived from a solution of the scalar wave equation. Finally, after a short digression on algebraic links, we show how self-dual electromagnetic fields can be derived such that its optical vortices are a given algebraic link.

2 MINKOWSKI SPACE

In the absence of gravitational effects, Minkowski space is the appropriate mathematical description of spacetime. It combines the spatial dimensions and time into a single four-dimensional whole with a non-Euclidian geometry. This geometry contains information about important physical concepts as we will see in section 2.1. Apart from the study of Minkowski space itself, we will also study operators on Minkowski space in section 2.2 and section 2.3. These operators will allow us to formulate electromagnetism in the formalism of differential forms in chapter 3.

2.1 THE GEOMETRY OF MINKOWSKI SPACE

An understanding of Minkowski space can help us understand electromagnetism or any other theory of physics compatible with the special theory of relativity. Therefore we devote this section to a discussion of the geometry of Minkowski space and its phyiscal interpretation.

DEFINITION 2.1.1: Let V be an n-dimensional real vector space and let g be a bilinear form on V. Then g is said to be

- *symmetric* if $g(v,w) = g(w,v)$ for all $v, w \in V$.
- *non-degenerate* if $g(v,w) = 0$ for all $w \in V$ implies that $v = 0$.

A symmetric non-degenerate bilinear form on a real vector space V is called a pseudo-Riemannian metric on V.

Note that a pseudo-Riemannian metric is very similar to a metric, only it need not be non-negative or satisfy the triangle inequality.

THEOREM 2.1.2: Let V be an n-dimensional real vector space and let g be a pseudo-Riemannian metric on V. Then there exists a basis $\{e_1, \dots, e_n\}$ for V such that $g(e_i, e_j) = \pm\delta_{ij}$; such a basis is called orthonormal. Furthermore, the number of elements e_j in different orthonormal bases that satisfy $g(e_j, e_j) = 1$ is the same.

Proof. See, for example, theorem 1.1.1 in [16]. □

The final property in theorem 2.1.2 allows us to unambiguously define the following property of pseudo-Riemannian metrics.

DEFINITION 2.1.3: Let V be an n-dimensional real vector space, let g be a pseudo-Riemannian metric on V, and let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for V. Then the signature of g is a doublet of numbers (p, q) where p and q are equal to the number of elements of $\{e_1, \ldots, e_n\}$ such that $g(e_i, e_i)$ is -1 and 1 respectively.

DEFINITION 2.1.4: Minkowski space $\mathcal{M}$ is a four-dimensional real vector space endowed with a pseudo-Riemannian metric η of signature $(1, 3)$. An element $v \in \mathcal{M}$ is said to be timelike if $\eta(v, v) < 0$, lightlike if $\eta(v, v) = 0$, and spacelike if $\eta(v, v) > 0$.

To see how the geometrical structure of Minkowski space is consistent with our every day experience of three-dimensional space and time as seperate entities, we investigate special subspaces of Minkowski space.

DEFINITION 2.1.5: A linear subspace $T \subset \mathcal{M}$ is timelike if it is spanned by a timelike vector $t \in \mathcal{M}$. Furthermore, a linear subspace $S \subset \mathcal{M}$ is spacelike if it is the orthogonal complement of a timelike subspace.

Since the pseudo-Riemannian metric on Minkowski space restricted to a timelike subspace T of $\mathcal{M}$ is non-degenerate, we can conclude from proposition 8.18 in [20] that $\mathcal{M} = T \oplus T^{\perp}$. Now, since η has signature $(1, 3)$, we can conclude that $T^{\perp}$ is spanned by three spacelike vectors, so that the restriction of η to this subspace is Euclidian. Therefore, we would like to identify a spacelike subspace with our spatial dimensions, but there are infinitely many spacelike subspaces. This multitude of choices for a spatial dimension will turn out to be the mathematical equivalent of the principle of relativity. However, despite suggestive nomenclature, it remains unclear how the evolution of time is incorporated in the geometry of Minkowski space. To this end we will consider more general subsets of $\mathcal{M}$.

DEFINITION 2.1.6: An affine subspace $\sum$ of $\mathcal{M}$ is said to be a spacelike slice if it can be written as $\sum = t + S$, where $t \in \mathcal{M}$ is timelike, and $S = \langle t \rangle^{\perp}$ is a spacelike subspace of $\mathcal{M}$.

Thus, given a fixed $t \in \mathcal{M}$ that is timelike, we get an orthogonal spacelike subspace $S = \langle t \rangle^{\perp}$ and we can write

$$\mathcal{M} = \bigsqcup_{\lambda \in \mathbb{R}} \lambda t + S$$

Such a way of writing $\mathcal{M}$ as a disjoint union of spacelike slices is called a splitting of Minkowski space. Given such a splitting of Minkowski space, we can interpret the spacelike slices as the spatial dimensions parametrised by the timelike direction which we identify with time. However, the issue that there are infinitely many different ways of writing Minkowski space as the disjoint union of spacelike slices of this form remains. We will resolve this issue after studying the geometry of Minkowski space with respect to charts.

DEFINITION 2.1.7: A frame of reference is a chart $(\mathcal{M}, h, \mathbb{R}^4)$, induced by a choice of basis $\{e_0, \ldots, e_3\}$ for $\mathcal{M}$, where h is given by

$$h : \mathcal{M} \to \mathbb{R}^4,\ x^\mu e_\mu \mapsto (x^0, \ldots, x^3)$$

Furthermore, we note that $x^\mu e_\mu$ is supposed to denote the summation of $x^\mu e_\mu$ over μ from zero to three. The omission of summation signs is common practice in physics and is called the Einstein summation convention. It states that if an index appears both in an upper and a lower position, it should be summed over.

Note that theorem 2.1.2 implies that there exists an orthonormal basis $\{e_0, \ldots, e_3\}$ for $\mathcal{M}$, which we order such that $\eta(e_0, e_0) = -1$. Then, any $v, w \in \mathcal{M}$ can be written as $v = v^\mu e_\mu$ and $w = w^\nu e_\nu$ and $\eta(v, w)$ is given by

$$\eta(v, w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

With respect to such a basis the matrix representation of η is diagonal with $\eta_{00} = -1$ and $\eta_{ii} = 1$ for $1 \leq i \leq 3$. Thus, using the Einstein summation convention, we may write $\eta(v, w) = v^\mu \eta_{\mu\nu} w^\nu$. As is common in physcis textbooks, we will use greek letters to denote summation over all four coordinates of Minkowski space, and latin letters to denote summation over x^1, x^2, x^3.

DEFINITION 2.1.8: An inertial frame of reference is a chart on $\mathcal{M}$ induced by an ordered orthonormal basis such that the first element of the basis e_0 satisfies $\eta(e_0, e_0) = -1$.

Note that choosing an ordered orthonormal basis $(e_0, \ldots, e_3)$ such that $\eta(e_0, e_0) = -1$, induces a splitting of Minkowski space by taking the spacelike subspace to be spanned by e_1, e_2, and e_3 and by taking e_0 as the timelike element in the splitting. Conversely, a splitting of spacetime gives an orthonormal basis. To see this, note that we can take the timelike element of $\mathcal{M}$ in the splitting of spacetime to be e_0, and obtain three orthonormal basis vectors from the corresponding spacelike subspace S using the Gram-Schmidt procedure.

DEFINITION 2.1.9: Let V be an n-dimensional real vector space endowed with a pseudo-Riemannian metric g. Then a diffeomorphism $f : V \to V$ is called an isometry if $g(f(v), f(w)) = g(v, w)$ holds for all $v, w \in V$.

The set of isometries together with composition forms a group, which in the case of Minkowski space is called the Poincaré group. The subgroup of linear isometries of the Poincaré group is called the Lorentz group.

PROPOSITION 2.1.10: Let V be an n-dimensional real vector space endowed with a pseudo-Riemannian metric g and let $f : V \to V$ be a linear map. Then f is an isometry if and only if f maps orthonormal bases to orthonormal bases.

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis, and suppose f maps orthonormal bases to orthonormal bases. Then it follows that

$$g(f(e_i), f(e_j)) = g(e'_i, e'_j) = \delta_{ij}$$

Since f is linear and g is bilinear we can conclude from this that f is an isometry. Conversely, suppose that f is an isometry. Then we have

$$g(f(e_i), f(e_j)) = g(e_i, e_j) = \delta_{ij}$$

This shows that $\{f(e_1), \ldots, f(e_n)\}$ is orthonormal. □

Thus different inertial frames and hence different splittings of Minkowski space, are related to each other by isometries. Choosing an inertial frame for Minkowski space is thought of in physics as putting an observer with a clock and a set of rulers somewhere in space. Different observers with consistently oriented directions of space and time are related to each other by Lorentz transformations that are positive definite on the first component of any inertial frame and preserve the orientation on Minkowski space. The group of such transformations is a subgroup of the Lorentz group called the restricted Lorentz group. In their classical form, the laws of physics are stated in terms of derivatives with respect to time and space as measured by such observers. However, we just discussed the ambiguity there is in this description. This problem is resolved by demanding that the laws of physics have the same 'form' in different inertial frames, i.e. that the laws of physics are symmetric under the restricted Lorentz group. In the formalism of differential forms which we will employ throughout this thesis, this comes down to the following: if F is a solution of a physical law, and $f : \mathcal{M} \to \mathcal{M}$ is an element of the restricted Lorentz group, then $f^*(F)$ also has to be a solution of the equation. Often the laws of physics have more symmetry than required by this discussion, but we will not go into this deeply. For example, we will not determine the most general groups under which an equation is symmetric, but if it is just as easy to prove that an equation is symmetric under the entire Lorentz group instead of just the restricted Lorentz group, we will show this instead.

2.2 THE HODGE STAR OPERATOR

The pseudo-Riemannian metric that Minkowski space is endowed with, induces an operator on differential forms called the Hodge star operator. This operator will be necessary to formulate the wave equation and Maxwell's equations in terms of differential forms.

PROPOSITION 2.2.1: Let V be an n-dimensional real vector space and let g be a pseudo-Riemannian metric on V. Then the map $T : V \to V^*,\ v \mapsto g(v, .)$ is an isomorphism.

Proof. Because V is finite dimensional we know that $\dim V = \dim V^*$ so that it is sufficient to show that T is linear and injective. Let $v, w \in V$ and $\lambda, \mu \in \mathbb{R}$, then by bilinearity of g we have that

$$T(\lambda v + \mu w) = g(\lambda v + \mu w, .) = \lambda g(v, .) + \mu g(w, .)$$

This shows that T is linear. Now let $v \in V$ and suppose that $T(v) = 0$, then it must hold for every $w \in V$ that $g(v, w) = 0$ which is the case if and only if $v = 0$ by the non-degeneracy of g. This shows that T is injective and concludes the proof. $\square$

This isomorphism between V and V^* induced by the pseudo-Riemannian metric g on V, induces a pseudo-Riemannian metric on $\Omega^1(V)$. To see this, note that

$$\langle\cdot|\cdot\rangle : \Omega^1(V) \times \Omega^1(V) \mapsto \mathbb{R},\ (\omega, \tau) \mapsto g(T^{-1}(\omega), T^{-1}(\tau))$$

satisfies the required properties. It can be shown that this induces a pseudo-Riemannian metric on $\Omega^k(V)$ by using the universal property of $\Lambda^k(V^*) = \Omega^k(V)$.

LEMMA 2.2.2: Let V be an n-dimensional real vector space and let g be a pseudo-Riemannian metric on V. Then there is a pseudo-Riemannian metric $\langle\cdot|\cdot\rangle^k$ on $\Omega^k(V)$, such that for one-forms $\omega^1, \ldots, \omega^k$, $\tau^1, \ldots, \tau^k \in \Omega^1(V)$ we have

$$\langle \omega^1 \wedge \cdots \wedge \omega^k | \tau^1 \wedge \cdots \wedge \tau^k \rangle^k = \det[\langle \omega^i, \tau^j \rangle]$$

Proof. See, for example, lemma 9.14 as well as the remarks preceeding and following it in [14]. $\square$

THEOREM 2.2.3: Let V be an n-dimensional real vector space, let g be a pseudo-Riemannian metric on V, and fix an orientation $\mathrm{Vol} \in \Omega^n(V)$. Then there is a unique linear map $\star : \Omega^k(V) \to \Omega^{n-k}(V)$, called the Hodge star operator, such that for any $\omega, \tau \in \Omega^k(V)$ it satisfies

$$\omega \wedge \star\tau = \langle \omega, \tau \rangle^k \cdot \mathrm{Vol}$$

Proof. See, for example, theorem 9.22 in [14]. $\square$

Even though this is a nice coordinate-independent definition of the Hodge star operator, in calculations it is convenient to have a more concrete expression for the Hodge star operator. The following proposition shows that for orthonormal bases such an explicit description is particularly simple. Since we will only consider orthonormal bases, we will not require a more general explicit description.

PROPOSITION 2.2.4: Let V be an n-dimensional real vector space endowed with a pseudo-Riemannian metric g on V with signature (p, q), and fix an orientation $\mathrm{Vol} \in \Omega^n(V)$. Furthermore, let $(e_1, \ldots, e_n)$ be an ordered orthonormal basis that is positively oriented, let $(e^1, \ldots, e^n)$ be a corresponding ordered dual basis, and let $\{i_1, \ldots, i_k\}$ and $\{i_{k+1}, \ldots, i_n\}$ be disjoint subsets of $\{1, \ldots, n\}$. Then we have

$$\star(e^{i_1} \wedge \cdots \wedge e^{i_k}) = \pm(-1)^p e^{i_{k+1}} \wedge \cdots \wedge e^{i_n}$$

where we take the plus sign if $e^{i_1} \wedge \cdots \wedge e^{i_k} \wedge e^{i_{k+1}} \wedge \cdots \wedge e^{i_n}$ is equal to the orientation on V, and the minus sign otherwise.

Proof. See, for example, proposition 9.23 in [14]. □

Proposition 2.2.4 allows us to easily see how the Hodge star operator acts on Minkowski space, and spacelike subspaces thereof.

EXAMPLE 2.2.5: First we choose an inertial frame and agree to denote the coordinate functions corresponding to our choice of ordered orthonormal basis (e_0, e_1, e_2, e_3) by (t, x, y, z). Then the natural orientation on V corresponding to this choice of basis, is $dt \wedge dx \wedge dy \wedge dz$. Now we can apply proposition 2.2.4 to determine explicitly how the Hodge star acts on the basis for the differential forms induced by the coordinate functions. For the bases of the one-forms and three-forms induced by the chosen coordinate functions we get

$$\begin{aligned} \star dt &= -dx \wedge dy \wedge dz & \star(dx \wedge dy \wedge dz) &= -dt \\ \star dx &= -dy \wedge dz \wedge dt & \star(dx \wedge dy \wedge dt) &= -dz \\ \star dy &= -dz \wedge dx \wedge dt & \star(dz \wedge dx \wedge dt) &= -dy \\ \star dz &= -dx \wedge dy \wedge dt & \star(dy \wedge dz \wedge dt) &= -dx \end{aligned}$$

Furthermore, the natural basis for the two-forms induced by our choice of coordinate functions satisfies

$$\begin{aligned} \star(dx \wedge dt) &= dy \wedge dz & \star(dx \wedge dy) &= -dz \wedge dt \\ \star(dy \wedge dt) &= dz \wedge dx & \star(dz \wedge dx) &= -dy \wedge dt \\ \star(dz \wedge dt) &= dx \wedge dy & \star(dy \wedge dz) &= -dx \wedge dt \end{aligned}$$

Finally, the natural bases for the zero-forms and the four-forms satisfy

$$\star 1 = -dt \wedge dx \wedge dy \wedge dz \qquad \text{and} \qquad \star\,(dt \wedge dx \wedge dy \wedge dz) = 1$$

The spacelike subspace S of $\mathcal{M}$ induced by this choice of orthonormal basis is spanned by e_1, e_2, and e_3. This subspace is naturally enowed a with pseudo-Riemannian metric given by $\eta|_S$, and we can take its orientation to be $dx \wedge dy \wedge dz$. These choices induce a Hodge star operator on S which we will denote by $\star_S$. This restricted Hodge star operator acts on the bases of one-forms and two-forms on S induced by the coordinate functions as

$$\begin{aligned} \star_S dx &= dy \wedge dz & \qquad \star_S(dx \wedge dy) &= dz \\ \star_S dy &= dz \wedge dx & \qquad \star_S(dz \wedge dx) &= dy \\ \star_S dz &= dx \wedge dy & \qquad \star_S(dy \wedge dz) &= dx \end{aligned}$$

Finally, we have that

$$\star_S 1 = dx \wedge dy \wedge dz \qquad \text{and} \qquad \star_S \left(dx \wedge dy \wedge dz\right) = 1$$

Furthermore, since S can be viewed as a three-dimensional Euclidian space in its own right, there is an exterior derivative operator for forms on S which we will denote by d_S. Since we will be working exclusively in Minkowski spacetime, this example allows us to compute the Hodge star of any differential form we will be interested in without having to refer to the abstract definitions.

In example 2.2.5, we see that applying the Hodge star operator twice either gives the identity on $\Omega^k(V)$ or the identity multiplied by a minus sign. This result turns out to be true in general, and whether we get a this extra minus sign or not, is determined by the signature of the metric as well as what type of form we start with.

PROPOSITION 2.2.6: Let V be an n-dimensional real vector space, and let g be a pseudo-Riemannian metric on V with signature (p, q). Then for any $\omega \in \Omega^k(V)$ it holds that

$$\star^2 \omega = (-1)^{p+k(n-k)} \omega$$

Proof. See, for example, proposition 9.25 in [14]. □

In particular, proposition 2.2.6 implies that the Hodge star operator is an isomorphism with inverse given by

$$\star^{-1} : \Omega^k(V) \to \Omega^{n-k}(V), \; \omega \mapsto (-1)^{p+k(n-k)} \star \omega$$

It turns out that isometries not only play a special role with respect to the pseudo-Riemannian metric g that V is endowed with, but also with respect to the Hodge star operator through its dependence on g.

PROPOSITION 2.2.7: Let V be an n-dimensional real vector space, let g be a pseudo-Riemannian metric, and let $f : V \to V$ be an isometry. Then for any $\omega \in \Omega^k(V)$ we have

$$f^*(\star\omega) = \star f^*(\omega) \qquad \text{or} \qquad f^*(\star\omega) = -\star f^*(\omega)$$

if f is orientation preserving or orientation reversing respectively.

Proof. Let $(e_1, \dots, e_n)$ be a positively oriented orthonormal basis for V with corresponding ordered dual basis $(e^1, \dots, e^n)$. Then a basis of the forms is given by $\{e^{i_1} \wedge \cdots \wedge e^{i_k}\}$ if we let i_1 through i_k take on all possible values in $\{1, \dots, n\}$. Therefore, it is sufficient to prove the proposition for forms of this type. Let $f : V \to V$ be an isometry, then it maps $(e_1, \dots, e_n)$ to another orthonormal basis $(f(e_1), \dots, f(e_n))$ as shown in proposition 2.1.10. This new basis $(f(e_1), \dots, f(e_n))$ is positively oriented if f is orientation preserving and negatively oriented otherwise. First we suppose that f is orientation preserving. Let $\{i_1, \dots, i_k\}$ and $\{i_{k+1}, \dots, i_n\}$ be disjoint subsets of $\{1, \dots, n\}$, then proposition 2.2.4 tells us that

$$\begin{aligned} f^*(\star(e^{i_1} \wedge \cdots \wedge e^{i_k})) &= f^*(\pm(-1)^p e^{i_{k+1}} \wedge \cdots \wedge e^{i_n}) \\ &= \pm(-1)^p f^*(e^{i_{k+1}}) \wedge \cdots \wedge f^*(e^{i_n}) \\ &= \star(f^*(e^{i_1}) \wedge \cdots \wedge f^*(e^{i_k})) \\ &= \star f^*(e^{i_1} \wedge \cdots \wedge e^{i_k}) \end{aligned}$$

Here p denotes the first component of the signature (p, q) of g. Noting that we get an extra minus sign if f is orientation reversing because of the definition of the ± 1 in proposition 2.2.4 concludes the proof. □

Having introduced the Hodge star operator, we are ready to formulate the theory of electromagnetism in chapter 3. However, as is often the case, things will simplify if we make the right definitions. Therefore we will first introduce two more operators, the codifferential operator in this section and the Laplace-Beltrami operator in section 2.3.

DEFINITION 2.2.8: Let V be an n-dimensional real vector space and let g be a pseudo-Riemannian metric on V. Then the codifferential operator is defined to be

$$\delta : \Omega^k(V) \to \Omega^{k-1}(V), \ \omega \mapsto (-1)^k \star^{-1} d \star \omega$$

Here the seemingly arbitrary factor of $(-1)^k$ is introduced so that δ is the adjoint of the induced pseudo-Riemannian metric we have on the space of k-forms by proposition 2.2.2.

PROPOSITION 2.2.9: Let V be an n-dimensional real vector space, let g be a pseudo-Riemannian metric, and let $f : V \to V$ be an isometry. Then for any $\omega \in \Omega^k(V)$ we have $f^*(\delta\omega) = \delta f^*(\omega)$.

Proof. First we note that the codifferential is just $\star d \star$ with possibly an extra factor of -1 depending on the metric and the type of form we let it act on. Let $f : V \to V$ be an isometry and note that if f is orientation preserving, the pullback with respect to f commutes with the Hodge star operator and hence the codifferential operator according to 2.2.7. Now suppose that f is orientation reversing, then the same proposition implies that we get an extra minus sign every time we interchange the pullback with respect to f and the Hodge star operator. Since the codifferential contains the Hodge star operator twice, these minus signs cancel. □

As mentioned, we will formulate electromagnetism in terms of differential forms in chapter 3. However, we will do so in terms of complex-valued differential forms on $\mathcal{M}$ instead of the real-valued differential forms discussed here. Therefore we note that in such cases we take the linear extension of the Hodge star operator to $\Omega^k(\mathcal{M}, \mathbb{C})$, i.e. the space of complex-valued differential k-forms. Furthermore, we note that all the results from this section are also valid for this linear extension.

2.3 THE LAPLACE-BELTRAMI OPERATOR

In the previous section, we introduced the Hodge star operator and showed how it gave rise to the codifferential operator. There is a certain combination of the codifferential operator and the exterior derivative that we will encounter in this thesis. Therefore, we devote this section the study of this new operator called the Laplace-Beltrami operator, which on Minkowski spacetime acts as a generalisation of the wave equation for differential forms.

DEFINITION 2.3.1: Let V be an n-dimensional real vector space and let g be a pseudo-Riemannian metric. Then the Laplace-Beltrami operator is a linear map $\Delta : \Omega^k(V, \mathbb{C}) \to \Omega^k(V, \mathbb{C})$ given by $\Delta = d\delta + \delta d$.

Before continuing, we will show that for zero-forms on Minkowski space the Laplace-Beltrami operator gives rise to the wave equation we are familiar with. Let $W \in C^\infty(\mathcal{M}, \mathbb{C})$, then we get

$$\Delta W = (d\delta + \delta d)W = \delta dW = (-1)^4 \star^{-1} d \star dW$$

because $\star W$ is a four-form, and taking the exterior derivative gives a five-form which is zero on Minkowski space showing that $\delta W = 0$. Also, since $\star^{-1}$ is an isomorphism, it follows that $\Delta W = 0$ if and only if $d \star dW = 0$. Calculating $d \star dW$ explicitly gives

$$\begin{aligned} d \star dW &= d \star (\partial_x W dx + \partial_y W dy + \partial_z W dz + \partial_t W dt) \\ &= d(\partial_x W dy \wedge dz \wedge dt + \partial_y W dz \wedge dx \wedge dt \\ &+ \partial_z W dx \wedge dy \wedge dt + \partial_t W dx \wedge dy \wedge dz) \\ &= (\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2) W dx \wedge dy \wedge dz \wedge dt \end{aligned}$$

So we see that $\Delta W = 0$ if and only if $(\partial_x^2 + \partial_y^2 + \partial_z^2 - \partial_t^2)W = 0$. We will refer to a smooth function $W \in C^\infty(\mathcal{M}, \mathbb{C})$ satisfying $\Delta W = 0$ as a solution of the wave equation. Solutions of the wave equation will turn out the be useful in section 3.5 which is why we consider an important example here.

EXAMPLE 2.3.2: Let $\tilde{W} : \mathcal{M} \to \mathbb{C}$ be given by

$$\tilde{W}(t,x,y,z) = (r^2 - t^2)^{-1}$$

where $r^2 = x^2 + y^2 + z^2$. Then we can check that W is a solution of the wave equation, but it will have singularities. Therefore we shift the time variable by a constant imaginary factor, which we choose to be $-i$, giving $W : \mathcal{M} \to \mathbb{C}$ given by

$$W(t,x,y,z) = (r^2 - (t-i)^2)^{-1}$$

Here the denominator is given by

$$r^2 - (t-i)^2 = r^2 - t^2 + 1 - 2it$$

Thus the imaginary part is zero if and only if $t = 0$, but in this case the real part reads $r^2 + 1$ which is never zero, showing that W has no singularities. Now let us check that W is a solution to the wave equation. The first derivatives of W are given by

$$\begin{aligned} \partial_t W &= 2(t-i)(r^2 - (t-i)^2)^{-2} \\ \partial_{x_i} W &= -2x_i(r^2 - (t-i)^2)^{-2} \end{aligned}$$

Here x_i denotes x,y or z. The second derivatives can be found with the chain rule giving

$$\begin{aligned} \partial_t^2 W &= 2(r^2 - (t-i)^2)^{-2} + 8(t-i)^2(r^2 - (t-i)^2)^{-3} \\ &= (2r^2 + 6(t-i)^2)(r^2 - (t-i)^2)^{-3} \\ \partial_{x_i}^2 W &= -2(r^2 - (t-i)^2)^{-2} + 8x_i^2(r^2 - (t-i)^2)^{-3} \\ &= (-2(r^2 - (t-i)^2) + 8x_i^2) \end{aligned}$$

So we see that

$$\begin{aligned} \partial_x^2 W + \partial_y^2 W + \partial_z^2 W &= (-6(r^2 - (t-i)^2) + 8r^2)(r^2 - (t-i)^2)^{-3} \\ &= (2r^2 + 6(t-i)^2)(r^2 - (t-i)^2)^{-3} = \partial_t^2 W \end{aligned}$$

This shows that W is indeed a solution of the wave equation.

Now let us return to our general discussion of the Laplace-Beltrami operator. Due to the symmetry in its definition, it behaves nicely with respect to the exterior derivative and the Hodge star operator as shown in the next proposition.

PROPOSITION 2.3.3: Let $\omega \in \Omega^k(V, \mathbb{C})$, then the Laplace-Beltrami operator satisfies $\Delta d\omega = d\Delta\omega$ and $\star\Delta\omega = \Delta \star \omega$.

Proof. Let $\omega \in \Omega^k(V)$, where the symmetric non-degenerate bilinear form on V has signature (p,q) and dimension n, Then we have

$$\begin{aligned} \Delta d\omega &= (d\delta + \delta d)d\omega = d\delta d\omega \\ &= d(d\delta + \delta d)\omega = d\Delta\omega \end{aligned}$$

because $d^2 = 0 = \delta^2$. Now we note that

$$\begin{aligned}
\star\delta d\omega &= \star(-1)^{k+1} \star^{-1} d \star d\omega \\
&= (-1)^{k+1+n-k+p+(k+1)(n-k-1)} d(-1)^{n-k} \star^{-1} d \star \star^{-1} \omega \\
&= (-1)^{k+1+n-k+p+(k+1)(n-k-1)} d\delta \star^{-1} \omega \\
&= (-1)^{k+1+n-k+p+(k+1)(n-k-1)+p+k(n-k)} d\delta \star \omega \\
&= d\delta \star \omega
\end{aligned}$$

Similarly we can prove that $\star d\delta\omega = \delta d \star \omega$. Combining these facts gives

$$\begin{aligned}
\star\Delta\omega &= \star(d\delta + \delta d)\omega \\
&= \delta d \star \omega + d\delta \star \omega \\
&= \Delta \star \omega
\end{aligned}$$

This concludes the proof. □

The final property of the Laplace-Beltrami operator we will need is the following.

PROPOSITION 2.3.4: Let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ then we have that $\Delta A = 0$ if and only if the components of A with respect to some basis satisfy the wave equation, i.e. if $A = A_\mu dx^\mu$ then $\Delta A = 0$ if and only if $\Delta A_\mu = 0$ for all $\mu \in \{0, 1, 2, 3\}$.

Proof. The proof of this proposition is a matter of applying example 2.2.5 a number of times. Despite being straightforward, the expressions become tedious. Therefore the proof is omitted. □

3 ELECTROMAGNETISM

In this chapter, we will formulate electromagnetism in the language of differential forms. Since we will be interested in electromagnetic fields in vacuum, we will not incorporate source terms in our treatment. Furthermore, we will not take into account any curvature of spacetime due to the energy density of the electromagnetic field as predicted by the general theory of relativity. Instead, we will assume that spacetime is modelled by Minkowski space as described in chapter 2. Finally, we note that parts of section 3.1, 3.2, and 3.3 are based on [2] and exercises therein.

3.1 MAXWELL'S EQUATIONS

In electromagnetism, the objects of study are electric and magnetic fields, which were classically viewed as distinct time-dependent vector fields on $\mathbb{R}^3$. However, it turned out that this description was less adequate in the context of special relativity because the electric and magnetic fields change according to the inertial frame of reference we choose. This observation indicated that the electric and magnetic fields are part of the same phenomenon called the electromagnetic field, which we will describe by smooth complex-valued two-forms on Minkowski space.

DEFINITION 3.1.1: A complex-valued two-form $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ is called an electromagnetic field if

$$dF = 0 \quad \text{and} \quad \delta F = 0$$

These equations are called the first and second Maxwell equation respectively. After choosing an inertial frame of reference for $\mathcal{M}$, we can write

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = E \wedge dt + B$$

provided that we define $E = (F_{i0} - F_{0i})dx^i$ and $B = \frac{1}{2}F_{ij}dx^i \wedge dx^j$. Furthermore, we define E_k and B_k to be the complex-valued functions on $\mathcal{M}$ such that $E = E_k dx^k$ and $\star_S B = B_k dx^k$ for any $k \in \{1, 2, 3\}$. From this we see that the real parts of E and B can be identified with vector fields on $\mathbb{R}^3$, which we will refer to as the electric and magnetic field respectively. We will denote the electric field by $\vec{E}$ and the magnetic field by $\vec{B}$.

With the choice for E and B as in definition 3.1.1, we can write down four equations for E and B equivalent to Maxwell's equations. To find the first two of these equations, we plug $F = E \wedge dt + B$ into the first Maxwell equations giving

$$0 = dF = d(E \wedge dt + B) = d_S E \wedge dt + \partial_t B \wedge dt + d_S B$$

Thus we see that the first Maxwell equation corresponds to

$$d_S E + \partial_t B = 0 \qquad \text{and} \qquad d_S B = 0$$

Noting that $\delta F = 0$ is equivalent to $d \star F = 0$ because $\star^{-1}$ is an isomorphism, we plug $F = E \wedge dt + B$ into $d \star F$. With example 2.2.5, this can be seen to give

$$\begin{aligned} 0 &= d(\star_S E - \star_S B \wedge dt) \\ &= d_S \star_S E + \partial_t \star_S E \wedge dt - d_S(\star_S B) \wedge dt \end{aligned}$$

Thus the second Maxwell equation corresponds to

$$\partial_t \star_S E - d_S \star_S B = 0 \qquad \text{and} \qquad d_S \star_S E = 0$$

These four equations for E and B to which Maxwell's equations as defined in 3.1.1 are equivalent, are closer to the form in which one usually first encounters Maxwell's equations. However, one of the several disadvantages of this formulation is that symmetry of Maxwell's equations under the restricted Lorentz group is not manifest.

PROPOSITION 3.1.2: Maxwell's equations are symmetric under the Lorentz group.

Proof. Let $f : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism and let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be an electromagnetic field. Then $f^*(F)$ also solves the first Maxwell equation because the pullback commutes with the exterior derivative giving

$$d(f^* F) = f^*(dF) = f^*(0) = 0$$

However, the second Maxwell equation is not invariant under general diffeomorphisms. Therefore we now assume f to be an isometry, in which case proposition 2.2.9 implies that

$$\delta f^*(F) = f^*(\delta F) = f^*(0) = 0$$

This concludes the proof. □

DEFINITION 3.1.3: Let F be an electromagnetic field and choose a frame of reference giving E and B as in definition 3.1.1. Then the field lines of the electromagnetic field are the integral curves of the vector fields corresponding to the real parts of E and B in $\mathbb{R}^3$ at a fixed time.

Since we know from definition 3.1.1 that E and B depend on the choice of basis, we know that the field lines will depend on our choice of basis.

REMARK 3.1.4: The first Maxwell equation states that an electromagnetic field is in particular a closed complex-valued two-form. Since Minkowski space is a vector space, it follows that $H^k(\mathcal{M}, \mathbb{C}) = 0$ for any $k \in \mathbb{Z}_{\geq 1}$. Therefore, there exists an $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ such that $F = dA$ for any electromagnetic field F.

DEFINITION 3.1.5: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be an electromagnetic field and let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ be such that $F = dA$. Then A is called a potential for F.

It turns out that all the information of an electromagnetic field F is contained in a potential for F, and that we can write down an equation for complex-valued one-forms that is equivalent to Maxwell's equations.

PROPOSITION 3.1.6: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$, and let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ such that $F = dA$. Then F is an electromagnetic field if and only if $\delta dA = 0$. If $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ satisfies $\delta dA = 0$ we call it a potential.

Proof. Suppose we have an electromagnetic field $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ and let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ be a potential for F. Plugging $F = dA$ in Maxwell's equations gives

$$ddA = 0 \qquad \text{and} \qquad \delta dA = 0$$

This shows the implication from left to right. Conversely, let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ such that $\delta dA = 0$ and take $F = dA$. Then we get

$$dF = ddA = 0 \qquad \text{and} \qquad \delta F = \delta dA = 0$$

This concludes the proof. □

3.2 GAUGE FREEDOM

It turns out that an electromagnetic field does not correspond to a unique potential. This freedom in the choice of potential is called gauge freedom and can be used to make a potential satisfy additional conditions. When sufficient conditions have been imposed to remove any freedom in the choice of the potential, we say that the gauge has been fixed.

PROPOSITION 3.2.1: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be an electromagnetic field, let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ be a potential for F, and let $C \in \Omega^1(\mathcal{M}, \mathbb{C})$. Then $A + C$ is a potential for F if and only if $C = dc$ for some $c \in C^\infty(\mathcal{M}, \mathbb{C})$.

Proof. Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be an electromagnetic field, let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ be a potential for F, let $C \in \Omega^1(\mathcal{M}, \mathbb{C})$, and suppose $A + C$ is also a potential for F. Then it must hold that

$$F = d(A + C) = dA + dC = F + dC$$

implying that $dC = 0$. Since $H^1(\mathcal{M}, \mathbb{C}) = 0$, there exists a $c \in C^\infty(\mathcal{M}, \mathbb{C})$ such that $C = dc$ proving the implication from left to right. Conversely, let $c \in C^\infty(\mathcal{M}, \mathbb{C})$ and let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ be a potential for an electromagnetic field $F \in \Omega^2(\mathcal{M}, \mathbb{C})$. Then $A + dc$ is a potential for F because we have

$$d(A + dc) = dA + ddc = dA = F$$

This concludes the proof. $\square$

Thus a potential for an electromagnetic field is only determined up to the addition of a closed complex-valued one-form. There are many additional conditions one can impose, but we will only make use of the Lorentz gauge which is defined as follows.

DEFINITION 3.2.2: Let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ then it is is said to be in the Lorentz gauge if $\delta A = 0$.

In general, when a gauge is chosen in one inertial frame, it need not necessarily be satisfied in another. This is the case if the gauge condition is not symmetric under the restricted Lorentz group. However, the Lorentz gauge does have this property.

PROPOSITION 3.2.3: The Lorentz gauge condition is symmetric under the Lorentz group.

Proof. Let $f : \mathcal{M} \to \mathcal{M}$ be an isometry, and let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ be a potential in the Lorentz gauge. Then $f^*(F)$ satisfies the Lorentz gauge because proposition 2.2.9 implies that

$$\delta f^*(A) = f^*(\delta A) = f^*(0) = 0$$

This concludes the proof. $\square$

Even though a potential in the Lorentz gauge satisfies an additional condition, it does not fix the gauge entirely, i.e. there is still some freedom in the choice of potential. To see this, let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ be a potential in the Lorentz gauge and let $c \in C^\infty(\mathcal{M}, \mathbb{C})$ be a function satisfying $\Delta c = 0$. Then $A + dc$ also satisfies the Lorentz gauge condition because of the remark following definition 2.3.1 we have

$$\delta(A + dc) = \delta A + \Delta c = 0 = \Delta c = 0$$

Thus $A + dc$ and A correspond to the same electromagnetic field.

REMARK 3.2.4: Let $A \in \Omega^1(\mathcal{M}, \mathbb{C})$ and suppose that it satisfies the Lorentz gauge condition. Then we know from proposition 3.1.6 that A is a potential if and only if

$$\delta dA = 0 = (\delta d + d\delta)A = \Delta A = 0$$

Thus by propostion 2.3.4 we see that in the Lorentz gauge the components of a potential have to satisfy the wave equation.

3.3 SELF-DUALITY

In dimension four the Hodge star operator has the special property that it maps two-forms to two-forms. Furthermore, we know by proposition 2.2.6 that for any $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ we have that $\star^2 F = -F$. Thus as a linear operator on $\Omega^2(\mathcal{M}, \mathbb{C})$ the Hodge star operator has $\pm i$ as its eigenvalues.

DEFINITION 3.3.1: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ then it is is called self-dual if $\star F = iF$ and anti-self-dual if $\star F = -iF$.

Since we are ultimately interested in electromagnetic fields, we will use the conventions established in section 3.1 even though the complex-valued two-forms we consider in this section need not be electromagnetic fields unless stated otherwise. Thus we write any $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ as $F = E \wedge dt + B$ where E and B are defined as in 3.1.1. Now the Hodge dual of F is given by $\star F = \star_S E - \star_S B \wedge dt$. Thus we see that F is self-dual if and only if $\star_S B = -iE$ and anti-self-dual if and only if $\star_S B = iE$. More explicitly, F is self-dual if and only if $B_k = -iE_k$ and anti-self-dual if and only if $B_k = iE_k$ for all $k \in \{1, 2, 3\}$. Furthermore, we note that if $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ is a self-dual or anti-self-dual complex-valued two-form that solves the first Maxwell equation, it also automatically satisfies the second Maxwell equation.

PROPOSITION 3.3.2: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$, then we can write F as a sum of self-dual and anti-self-dual two-forms F_+ and F_- respectively.

Proof. Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ and define

$$F_- = \frac{1}{2}(F + i \star F) \quad \text{and} \quad F_+ = \frac{1}{2}(F - i \star F)$$

Note that

$$\star F_+ = \frac{1}{2}(\star F + iF) = \frac{i}{2}(F - i \star F) = iF_+$$

as well as

$$\star F_- = \frac{1}{2}(\star F - iF) = -\frac{i}{2}(F + i \star F) = -iF_-$$

This shows that F_+ is self-dual and F_- is anti-self-dual. Noting that $F = F_+ + F_-$ concludes the proof. □

Proposition 3.3.2 implies that, in principle, it is sufficient to consider self-dual and anti-self-dual electromagnetic fields since any solution can be written as a sum of such fields.

PROPOSITION 3.3.3: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ and consider the parity inversion map $P : \mathcal{M} \to \mathcal{M}$, $(t, x, y, z) \mapsto (t, -x, -y, -z)$. Then $P^*(F)$ is anti-self-dual if and only if F is self-dual.

Proof. Note that pullback by a parity transformation leaves differential forms of the form $dx^i \wedge dx^j$ unchanged, and flips the sign of differential forms of the form $dx^i \wedge dx^0$. Therefore, the the components of E gain a minus sign with respect to the components of B after pullback by a parity transformation. This observation, combined with the proof below definition 3.3.1 that F is self-dual if and only if $B_k = iE_k$ and anti-self-dual if and only if $B_k = -iE_k$, proves the proposition. □

The dimension of $\Omega^2(\mathcal{M}, \mathbb{C})$ is six, and by the previous proposition we know that P^* maps from self-dual elements of $\Omega^2(\mathcal{M}, \mathbb{C})$ to anti-self-dual elements of $\Omega^2(\mathcal{M}, \mathbb{C})$. Furthermore, we know that

$$\mathrm{id}_{\Omega^2(\mathcal{M},\mathbb{C})} = (P \circ P)^* = P^* \circ P^*$$

showing that P^* is a bijection. This observation, combined with the result from proposition 3.3.2 that any element of $\Omega^2(\mathcal{M}, \mathbb{C})$ can be written as a sum of self-dual and anti-self-dual two-forms, shows that the dimension of the subspace of self-dual two-forms in Minkowski spacetime is three just like the dimension of the subspace of anti-self-dual two-forms.

For the remainder of this thesis we will only consider self-dual two-forms because we can obtain the corresponding anti-self-dual two-form by pullback with the parity inversion map. Furthermore, all the results we derive for self-dual electromagnetic fields in this section, also hold for anti-self-dual fields with similar proofs.

DEFINITION 3.3.4: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be an electromagnetic field, then the invariants of F are $\star(F \wedge F)$ and $\star(F \wedge \star F)$.

Taking an orthonormal basis for $\mathcal{M}$ and writing $F = E \wedge dt + B$ as introduced in definition 3.1.1 allows us to rewrite the fundamental invariants of F in terms of the components of E and B. Then $\star(F \wedge F)$ is given by

$$\star(F \wedge F) = -2(E_x B_x + E_y B_y + E_z B_z)$$

and $\star(F \wedge \star F)$ is given by

$$\star(F \wedge \star F) = -(E_x^2 + E_y^2 + E_z^2 - B_x^2 - B_y^2 - B_z^2)$$

Furthermore, it turns out that

$$\star(\star F \wedge \star F) = \star(F \wedge F)$$

The reason that these quantities are interesting is that they are Lorentz-invariant scalars. This means that they take the same value in inertial frames related by elements from the restricted Lorentz group. To see this, we first recall that different inertial frames are related by elements in the Lorentz group. We know from proposition 2.2.7 that the pullback with respect to such a map commutes with the Hodge star operator, so that we get

$$f^*(\star(F \wedge F)) = \star[f^*(F \wedge F)] = \star[(f^*F) \wedge (f^*F)]$$

as well as

$$f^*(\star[F \wedge \star F]) = \star[f^*(F \wedge \star F)] = \star[(f^*F) \wedge (f^* \star F)]$$

It turns out that the invariants of an electromagnetic field F vanish if F is self-dual.

PROPOSITION 3.3.5: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be a self-dual electromagnetic field then the fundamental invariants of F vanish.

Proof. Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be a self-dual electromagnetic field, then we have

$$\star(F \wedge F) = \star(\star F \wedge \star F) = \star(iF \wedge iF) = - \star (F \wedge F)$$

This shows that $\star(F \wedge F) = 0$, but we also have

$$\star(F \wedge \star F) = i \star (F \wedge F) = 0$$

Thus both the fundamental invariants of a self-dual electromagnetic field are zero. □

As a consequence, the electric and magnetic field corresponding to self-dual electromagnetic fields are orthogonal. To see this, note that

$$\star(F \wedge \star F) = \mathrm{Re}[E_x]^2 - \mathrm{Im}[E_x]^2 + 2i\mathrm{Re}[E_x]\mathrm{Im}[E_x] + \dots$$

Here we did not write down the similar terms we get for the y and z components of E and the components of B. Due to self-duality we know that for any $k \in \{x, y, z\}$ we have $B_k = -iE_k$ which implies that

$$\mathrm{Im}[E_k] = \mathrm{Re}[B_k]$$

Combining this with the fact that $\star(F \wedge \star F) = 0$ gives

$$\mathrm{Re}[E_x]\mathrm{Re}[B_x] + \mathrm{Re}[E_y]\mathrm{Re}[B_y] + \mathrm{Re}[E_z]\mathrm{Re}[B_z] = 0$$

i.e. $\vec{E} \cdot \vec{B} = 0$. We know that $\vec{E}$ and $\vec{B}$ depend on the inertial frame chosen. However, we only used the Lorentz-invariant expression $\star(F \wedge \star F) = 0$, and the self-duality condition to show that $\vec{E} \cdot \vec{B} = 0$. Therefore we can conclude that this holds regardless of the inertial frame we choose. As a consequence of this, [10] implies that a self-dual electromagnetic field without zeros satisfies something called the frozen field condition. This condition means that the field lines are not 'broken' under the time evolution, but deform smoothly. The evolution of the field is then described by a conformal deformation of space, i.e. a map that does preserves the metric up to scaling.

3.4 THE BATEMAN CONSTRUCTION

In this section we will discuss a method of constructing self-dual electromagnetic fields that is commonly referred to as the Bateman construction since it is originally due to Bateman [4]. This construction will be essential to our method of constructing self-dual electromagnetic fields with linked optical vortices.

Let $\alpha, \beta : \mathcal{M} \to \mathbb{C}$ be two smooth complex functions on Minkowski space, then $F := d\alpha \wedge d\beta$ is a closed two-form on $\mathcal{M}$. Hence it solves the first Maxwell equation. Note that F also solves the second Maxwell equation if F is self-dual. Writing out the self-duality condition $\star F = iF$ for $F = d\alpha \wedge d\beta$ gives that the self-duality condition is equivalent to the Bateman condition for α and β:

$$\nabla\alpha \times \nabla\beta = i(\partial_t \alpha \nabla\beta - \partial_t \beta \nabla\alpha)$$

Though this construction gives a different way of obtaining electromagnetic fields, we must admit that to use it we still have to solve a difficult equation. However, once we have found solutions to these equations, we have the freedom to construct a whole family of other self-dual solutions as explained in the next proposition.

PROPOSITION 3.4.1: Let $\alpha, \beta : \mathcal{M} \to \mathbb{C}$ be smooth functions satisfying the Bateman condition and let $f, g : \mathbb{C}^2 \to \mathbb{C}$ be arbitrary smooth functions. Then $F = df(\alpha, \beta) \wedge dg(\alpha, \beta)$ is a self-dual electromagnetic field.

Proof. Let $\alpha, \beta \in C^\infty(\mathcal{M}, \mathbb{C})$ such that $d\alpha \wedge d\beta$ is a self-dual electromagnetic field, and let $f, g : \mathbb{C}^2 \to \mathbb{C}$ be smooth. Then $df(\alpha, \beta) \wedge dg(\alpha, \beta)$ is a closed two-form on $\mathcal{M}$ that can be written as

$$\begin{aligned} df(\alpha,\beta) \wedge dg(\alpha,\beta) &= (\partial_\alpha f d\alpha + \partial_\beta f d\beta) \wedge (\partial_\alpha g d\alpha + \partial_\beta g d\beta) \\ &= \partial_\alpha f \partial_\beta g d\alpha \wedge d\beta + \partial_\beta f \partial_\alpha g d\beta \wedge d\alpha \\ &= (\partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g) d\alpha \wedge d\beta \end{aligned}$$

Since $d\alpha \wedge d\beta$ is self-dual we also know that

$$\begin{aligned} \star df(\alpha,\beta) \wedge dg(\alpha,\beta) &= \star(\partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g) d\alpha \wedge d\beta \\ &= (\partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g) \star (d\alpha \wedge d\beta) \\ &= (\partial_\alpha f \partial_\beta g - \partial_\beta f \partial_\alpha g) i d\alpha \wedge d\beta \\ &= i df(\alpha,\beta) \wedge dg(\alpha,\beta) \end{aligned}$$

Thus $df(\alpha, \beta) \wedge dg(\alpha, \beta)$ is also a self-dual electromagnetic field. □

It turns out that any self-dual electromagnetic field can be obtained by the Bateman construction.

THEOREM 3.4.2: Let $F \in \Omega^2(\mathcal{M}, \mathbb{C})$ be a self-dual electromagnetic field, then there are $\alpha, \beta : \mathcal{M} \to \mathbb{C}$ satisfying the Bateman condition such that $F = d\alpha \wedge d\beta$ holds.

Proof. This result is due to Hogan who showed it, albeit using a different formalism, in [9]. □

3.5 SUPERPOTENTIAL THEORY

It turns out that one can obtain solutions to the Maxwell equations by combining solutions to the wave equation with constant two-forms. Our treatment of this method in this section is based on the approach taken by Synge in section 13 of chapter 9 of [21], although he uses a different formalism.

PROPOSITION 3.5.1: Let $K \in \Omega^2(\mathcal{M}, \mathbb{C})$ be a constant two-form and let $W \in \mathbb{C}^\infty(\mathcal{M}, \mathbb{C})$ be a solution of the wave equation. Then $A = \star(dW \wedge K)$ is a potential, i.e. $F = dA$ is an electromagnetic field.

Proof. First note that A is in the Lorentz gauge because

$$\delta A = \delta \star (dW \wedge K) = \star^{-1} d \star \star (dW \wedge K) = \star^{-1} dd(WK) = 0$$

by proposition 2.2.6 and the fact that $d^2 = 0$. Now we know from section 3.2 that A is a potential if and only if $\Delta A = 0$.

$$\Delta A = \Delta \star d(WK) = \star d \Delta(WK)$$

Here we could swap Δ with d as well as $\star$ by proposition 2.3.3. Now note that $\Delta(WK) = 0$ if and only if the components of WK satisfy the wave equation by proposition 2.3.4. This condition is satisfied because W satisfies the wave equation and K has constant components concluding the proof. □

Because of the special properties of self-dual electromagnetic fields, we would like to know when the electromagnetic fields corresponding to potentials as constructed in 3.5.1 are self-dual. It turns out that if we write out $\star d \star (dW \wedge K) = id \star (dW \wedge K)$ respectively $\star d \star (dW \wedge K) = -id \star (dW \wedge K)$, we find that K has to satisfy

$$K_{xy} = -iK_{zt}, \quad K_{zx} = -iK_{yt}, \quad K_{yz} = iK_{xt}$$

in the self-dual case and

$$K_{xy} = iK_{zt}, \quad K_{zx} = iK_{yt}, \quad K_{yz} = -iK_{xt}$$

in the anti-self-dual case.

4

KNOTS IN ELECTROMAGNETISM

The purpose of this chapter is to derive the Hopf field, and to show how knot theory can be implemented in electromagnetism. To this end, we will first review the main result from the bachelor thesis of Ruud van Asseldonk [1] in section 4.1. We will then use the results from this section to show how the Hopf field can be obtained from the construction discussed in section 3.5. Finally, after a digression on algebraic links, we will arrive at the main result of this thesis in section 4.4. Here, we will give a constructive proof that self-dual electromagnetic fields exist with the special property that the intersection of their zero set with an arbitrary spacelike slice in Minkowski space is a given algebraic link.

4.1 SOLENOIDAL VECTOR FIELDS

In this section we will discuss a method of constructing solenoidal vector fields on $\mathbb{R}^3$, i.e. vector fields $\vec{B} : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfy $\nabla \cdot \vec{B} = 0$. Furthermore, we will show how this construction can be used to derive a vector field with the property that its integral curves are all linked circles. However, as we did throughout this thesis, we will work with differential forms instead of vector fields. Therefore we note that solenoidal vector fields correspond to two-forms $B \in \Omega^2(\mathbb{R}^3)$ that satisfy $dB = 0$, or one-forms $E \in \Omega^1(\mathbb{R}^3)$ that satisfy $d \star_S E = 0$. The treatment we present in this section is based on section 4.3 of [1], the bachelor thesis of Ruud van Asseldonk.

REMARK 4.1.1: From the discussion following definition 3.1.1, we know that a magnetic field is a solenoidal vector field, but to get a solution to Maxwell's equations we also need a solenoidal electric field that is coupled to the magnetic field in the correct way. Since the construction we will discuss in this section gives only one solenoidal field, it is not a method of constructing electromagnetic fields. The reason we decided to include this section is that it does give a good handle on the structure of the field lines, which will prove useful in section 4.2.

REMARK 4.1.2: Let N be a two-dimensional manifold, then any $\omega \in \Omega^2(N)$ is closed. Now let $f : \mathbb{R}^3 \to N$ be a smooth map, then $f^*(\omega)$ is a closed form on $\mathbb{R}^3$, because the pullback and the exterior derivative commute, i.e.

$$df^*(\omega) = f^*(d\omega) = f^*(0) = 0$$

Thus this gives a method of constructing closed two-forms on $\mathbb{R}^3$. This remark in itself is rather trivial, but it turns out to be interesting because the fibre structure of f is closely related to the structure of the field lines as shown by theorem 4.1.3.

THEOREM 4.1.3: Let N be a two-dimensional smooth manifold, let $\omega \in \Omega^2(N)$, and let $f : \mathbb{R}^3 \to N$ be a smooth map. Then the fibres of f coincide with the field lines of $f^*(\omega)$ where the latter is non-zero.

Proof. Let N be a two-dimensional manifold, let $f : \mathbb{R}^3 \to N$ be a smooth map, and let $\omega \in \Omega^2(N)$. Let $(U, h, \mathbb{R}^2)$ be some chart, where $h : U \to \mathbb{R}^2$ is given by $p \mapsto (q^1, q^2)$. Then we can write

$$\omega|_U = \omega_U \cdot dq^1 \wedge dq^2$$

for some $\omega_U \in C^\infty(U)$. Now, on $f^{-1}(U)$, which is open because f is smooth, so in particular continuous, $f^*(\omega)$ is given by

$$f^*(\omega|_U) = (\omega_U \circ f) \cdot df^1 \wedge df^2$$

The vector field on V corresponding to this form on V is given by $(\omega_U \circ f)\nabla f^1 \times \nabla f^2$. Unless this vector field is zero at some point, which corresponds to $f^*(\omega)$ being zero on this point, the vector field is orthogonal to the integral curves of f. To see this, note that the gradients of f^1 and f^2 are orthogonal to the level curves of f, so the outer product of these gradients at a point will be a tangent vector of the level curve of f through this point provided that this outer product is non-zero. Thus the integral curves of the vector field corresponding to the form $f^*(\omega)$ correspond to the level curves of f where the former is non-zero. $\square$

There are many possibilities for the two-manifold N and the map $f : \mathbb{R}^3 \to N$ in theorem 4.1.3. However, we will restrict our attention to the specific case where $N = \mathbb{S}^2$, and the map $f : \mathbb{R}^3 \to \mathbb{S}^2$ is derived from a map called the Hopf map. The Hopf map is the restriction of

$$\tilde{H} : \mathbb{C}^2 \to \mathbb{P}^1(\mathbb{C}), \ (z_1, z_2) \mapsto (z_1 : z_2)$$

to $\mathbb{S}^3$ viewed as the subset of $\mathbb{C}^2$ of unit norm. Since we can identify $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{S}^2$, we can view this as a map from $\mathbb{S}^3$ to $\mathbb{S}^2$. Composing the Hopf map with the inverse stereographic projection from $\mathbb{R}^3$ to $\mathbb{S}^2$, gives the map

$$\phi : \mathbb{R}^3 \to \mathbb{S}^2, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \frac{4\left(y\left(x^2+z^2-1\right)-2xz+y^3\right)}{\left(x^2+y^2+z^2+1\right)^2} \\ -\frac{4\left(z\left(x^2+y^2-1\right)+2xy+z^3\right)}{\left(x^2+y^2+z^2+1\right)^2} \\ \frac{8\left(x^2+1\right)}{\left(x^2+y^2+z^2+1\right)^2} - \frac{8}{x^2+y^2+z^2+1} + 1 \end{pmatrix}$$

The fibres of this map are circles which are all linked with each other. We will formally introduce linking in definition 4.3.4, the intuitive notion of linking should suffice for now. However, there is also one fibre that is not a circle, but a straight line. For more details on this map and its fibre structure see chapter 3 of [1]. Note that forms on $\mathbb{S}^2$ can be viewed as forms on $\mathbb{R}^3$. If we do this, one of the orientation forms on $\mathbb{S}^2$ is given by

$$\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$$

Taking the pullback of ω with ϕ gives

$$\begin{aligned}\phi^*(\omega) = &-\frac{32(xz+y)}{(x^2+y^2+z^2+1)^3}dx \wedge dy \\ &-\frac{32(xy-z)}{(x^2+y^2+z^2+1)^3}dz \wedge dx \\ &-\frac{16\left(x^2-y^2-z^2+1\right)}{(x^2+y^2+z^2+1)^3}dy \wedge dz\end{aligned}$$

Though we have taken the same approach as in [1], we end up with a form that differs by a minus sign and a transformation $x \mapsto -x$. This difference is due to the choice we made in identifying $\mathbb{R}^4$ with $\mathbb{C}^2$.

4.2 THE HOPF FIELD

The Hopf field is a self-dual electromagnetic field such that at $t = 0$ the field lines of both the electric and the magnetic field have the structure of the Hopf fibration, and are mutually orthogonal. In this section, we will show a new way of constructing the Hopf field using the method discussed in section 3.5.

To obtain an electromagnetic field from the construction in section 3.5, we need a solution to the wave equation. We take this to be the solution without singularities found in example 2.3.2, i.e.

$$W(t,x,y,z) = (x^2+y^2+z^2-(t-i)^2)^{-1}$$

The construction also requires a $K \in \Omega^2(\mathcal{M}, \mathbb{C})$ with constant coefficients, which we choose to be

$$K = -dz \wedge dx - i \cdot dy \wedge dz - dx \wedge dt + i \cdot dy \wedge dt$$

This choice of K guarantees that the resulting electromagnetic field will be self-dual by the discussion following proposition 3.5.1. Furthermore, proposition 3.5.1 guarantees that $A = \star(dW \wedge K)$ is a potential, which is given by

$$\begin{aligned} A = &\frac{-2(y+ix)}{(x^2+y^2+z^2-(t-i)^2)^2}dz + \frac{2(y+ix)}{(x^2+y^2+z^2-(t-i)^2)^2}dt \\ &+ \frac{-2it+2iz-2}{(x^2+y^2+z^2-(t-i)^2)^2}dx + \frac{-2t+2z+2i}{(x^2+y^2+z^2-(t-i)^2)^2}dy \end{aligned}$$

Now the electromagnetic field corresponding to A is

$$\begin{aligned} F = &\frac{4(t-x+iy-z-i)(t+x-i(y+1)-z)}{(x^2+y^2+z^2-(t-i)^2)^3}dy \wedge dz \\ &- \frac{4i(t-ix-y-z-i)(t+ix+y-z-i)}{(x^2+y^2+z^2-(t-i)^2)^3}dz \wedge dx \\ &- \frac{8i(t-z-i)(y+ix)}{(x^2+y^2+z^2-(t-i)^2)^3}dx \wedge dy + \dots \end{aligned}$$

Here we did not write down the terms determining the electric field, because they are fixed by self-duality when the terms determining the magnetic field are given. Note that at $t = 0$ we have $4\mathrm{Re}[B] = \phi^*(\omega)$, where $\phi^*(\omega)$ is the two-form we determined in section 4.1. This shows that the field lines of the magnetic field have the same structure as the field lines of the vector field corresponding to $\phi^*(\omega)$, which is that of the Hopf fibration. The same holds for the electric field by self-duality, which also implies that the field line structure is preserved in time due to the final result of section 3.3. We would now like to determine Bateman variables for this electromagnetic field, i.e. maps $\tilde{\alpha}, \tilde{\beta} : \mathcal{M} \to \mathbb{C}$ such that $F = d\tilde{\alpha} \wedge d\tilde{\beta}$. Note that for such a field, a potential is given by $A = \tilde{\alpha} d\tilde{\beta} = -\tilde{\beta} d\tilde{\alpha}$, so we might hope to obtain Bateman variables for the Hopf field from the potential. If we compare the components of the potential, we see that two of them differ by a minus sign, and the other two differ by a factor of i. Thus, this gives us two natural choices to take for $\tilde{\alpha}, \tilde{\beta}$, but these can never be Bateman variables for the field since it would give a field with a factor $(r^2-(t-i)^2)^{-4}$. Therefore, we take the components without the square in the denominator, i.e.

$$\tilde{\alpha}(t,x,y,z) = \frac{2i-2t+2z}{r^2-(t-i)^2} \quad \text{and} \quad \tilde{\beta}(t,x,y,z) = \frac{2(ix+y)}{r^2-(t-i)^2}$$

It turns out that these do, indeed, function as Bateman variables for the Hopf field, but we have some freedom here. We can take a factor of i from $\tilde{\alpha}$ to $\tilde{\beta}$ and add 1 to $i\tilde{\alpha}$ without changing the field. This gives new Bateman variables given by

$$\alpha(t,x,y,z) = \frac{r^2-t^2-1+2iz}{r^2-(t-i)^2} \quad \text{and} \quad \beta(t,x,y,z) = \frac{2(x-iy)}{r^2-(t-i)^2}$$

These are the form of Bateman variables used in [12], and will form the basis of our discussion in section 4.4.

4.3 ALGEBRAIC LINKS

In this section, we will give a short exposition on algebraic link theory. We will only cover parts of the theory that are relevant for our construction in section 4.4. More information can be found in the references used to write this section, which are [3, 8, 15, 19, 23].

DEFINITION 4.3.1: A link is a pair (X, L), where X is an oriented manifold diffeomorphic to $\mathbb{R}^3$, and L is an one-dimensional submanifold of X diffeomorphic to $\sqcup_{i=1}^{n} S^1$ for some $n \in \mathbb{Z}_{\geq 1}$ with the orientation induced by the standard orientation on $\mathbb{S}^1$. If $n = 1$, a link is said to be a knot.

We should note that whenever we refer to the main result of section 4.4, we identify links with one-dimensional manifolds diffeomorphic to $\sqcup_{i=1}^{n} S^1$ for some $n \in \mathbb{Z}_{\geq 1}$. This is done for brevity, but in all theorems, propositions, and proofs we will use the precise definition of a link stated above.

DEFINITION 4.3.2: Two links (X, L) and (X', L') are said to be equivalent if there exists an orientation preserving diffeomorphism $\varphi : X \to X$ such that $\varphi(L) = L'$, and such that $\varphi^*(\omega) = \omega'$, where ω and ω' are the orientations on L and L' respectively. Two equivalent links are denoted by $(X, L) \cong (X', L')$.

In section 4.2, we encountered the concept of linking in our brief description of the Hopf fibration. Even though, intuitively, it is clear what is meant by this, we will now formally define the notion of linking. To do so, we need the notion of a Seifert surface.

THEOREM 4.3.3: Let (X, K) be a knot, then there is a compact, two-dimensional orientable submanfold S of X, such that $\partial S = K$. We take S to have the orientation induced by the orientation on K. Such a surface S is called a Seifert surface for the knot (X, K).

Proof. See chapter 5 in [19] □

Even though there can be multiple Seifert surfaces for a given knot, it can be used to unambiguously define the linking number of two knots as we will now show.

DEFINITION 4.3.4: Let (X, K) and (X, K') be knots such that $K' \cap K = \emptyset$, and let S be a Seifert surface for K. We may assume that S intersects K' transversely in finitely many points, because if it does not we can deform its interior until it does. Let $\varphi : X \to \mathbb{R}^3$ be a diffeomorphism between X and $\mathbb{R}^3$, and let ω be the orientation on X induced by the standard orientation on $\mathbb{R}^3$. Now, let $p \in K' \cap S$, and take $v_1, v_2 \in T_pS$ such that $\varphi_*(v_1)$, and $\varphi_*(v_2)$ are right-handed with respect to the standard orientation on $\mathbb{R}^3$. Also take a $w \in T_pK$ such that $\omega'(w) > 0$ for the orientation ω on $\mathbb{S}^1$. Finally, we define $\sigma(p)$ to be equal to one if $\omega(v_1, v_2, w) > 0$, and to be equal to -1 otherwise. Then the linking number of the knots is defined to be

$$\mathcal{L}(K, K') = \sum_{p \in K' \cap S} \sigma(p)$$

Note that this definition is independent of the chosen Seifert surface for K. To see this, let S' be another Seifert surface for K, and reverse the orientation on S'. Then $S \cup S'$ is a closed surface in X, and hence for any $p \in K' \cap (S \cup S')$ such that $\sigma(p) = 1$, there is another $p' \in K' \cap (S \cup S')$ such that $\sigma(p') = -1$.

In section 4.4, we will show how we can implement a certain class of links in electromagnetism. Therefore we will restrict our attention to this subclass, called the algebraic links. However, before being able to say what makes a link algebraic, we need the notion of a plane curve.

DEFINITION 4.3.5: A complex plane curve is the zero set of a polynomial $h \in \mathbb{C}[v, w]$ viewed as a map $h : \mathbb{C}^2 \to \mathbb{C}$, $(v, w) \mapsto h(v, w)$ that satisfies $h(0, 0) = 0$ and has an isolated singularity or a simple point in the origin. We will always work in $\mathbb{C}^2$, so we will simply refer to such a zero set as a plane curve.

Note that $\mathbb{C}[v, w]$ is a unique factorisation domain, i.e. any $h \in \mathbb{C}[v, w]$ can be written as a product of irreducible elements and a unit in $\mathbb{C}[v, w]$. Furthermore, this factorisation is unique up to ordering of the irreducible factors and multiplication of the irreducible factors by unit elements.

DEFINITION 4.3.6: Let C be a plane curve associated to a $h \in \mathbb{C}[v, w]$, then a branch of C is the zero set of an irreducible factor of h.

Now we are ready to define algebraic links, but before we do so, we should say that we will denote the three-sphere of norm ϵ in $\mathbb{C}^2$ by $\mathbb{S}^3_\epsilon$. Furthermore, we note that for any $p \in \mathbb{S}^3$ the stereographic projection gives a diffeomorphism between $\mathbb{S}^3_\epsilon \backslash \{p\}$ and $\mathbb{R}^3$.

DEFINITION 4.3.7: A link (X, L) is said to be algebraic if there exists a plane curve C, an $\epsilon \in \mathbb{R}_{>0}$, and a $p \in \mathbb{S}^3_\epsilon \backslash C$ such that

$$(X, L) \cong (\mathbb{S}^3_\epsilon \backslash \{p\}, C \cap \mathbb{S}^3_\epsilon)$$

REMARK 4.3.8: Instead of describing an algebraic link as the intersection of a plane curve C with a sphere $\mathbb{S}^3_\epsilon$, it can also be described by the intersection of a plane curve with

$$\partial(D^2(\epsilon) \times D^2(\delta)) = \{(v, w) \in \mathbb{C}^2 || v| = \epsilon, |w| \leq \delta \text{ or } |v| \leq \epsilon, |w| = \delta\}$$

Here ϵ and δ can be chosen such that the square sphere intersects with C in a solid torus, where $|x| = \epsilon$. This alternative description of algebraic links was proposed by Kähler in [11] and will prove useful when describing the topology of algebraic links.

It turns out that every polynomial $h \in \mathbb{C}[v,w]$ that satisfies $h(0,0) = 0$ and has an isolated singularity or a simple point in the origin gives rise to a link.

PROPOSITION 4.3.9: Let $h \in \mathbb{C}[v,w]$ such that $h(0,0) = 0$ and such that h has an isolated singularity or a simple point in the origin and let C denote the plane curve corresponding to h. Then there is an $\epsilon \in \mathbb{R}_{>0}$ and a $p \in \mathbb{S}^3_\epsilon$ such that $(\mathbb{S}^3_\epsilon \backslash \{p\}, C \cap \mathbb{S}^3_\epsilon)$ is a link. Such a link is algebraic by construction.

Proof. See lemma 5.2.1 and the remarks following it in [23]. □

The ϵ in the definition of an algebraic link is necessary because there may be other singularities in the plane. The idea is to choose ϵ small enough such that $\mathbb{S}^3_\epsilon$ does not enclose any singularity outside of the origin. However, this still leaves a range of choices for ϵ, but we will now see that all values in this range give equivalent links.

LEMMA 4.3.10: Let C be a plane curve, then for $\epsilon, \epsilon' \in \mathbb{R}_{>0}$ small enough, there are $p \in \mathbb{S}^3_\epsilon$ and $p' \in \mathbb{S}^3_{\epsilon'}$ such that

$$(\mathbb{S}^3_\epsilon \backslash \{p\}, C \cap \mathbb{S}^3_\epsilon) \cong (\mathbb{S}^3_{\epsilon'} \backslash \{p'\}, C \cap \mathbb{S}^3_{\epsilon'})$$

Proof. See, for example, lemma 5.2.2 in [23]. □

Having discussed how algebraic links arise from zero sets of polynomials, we will now elaborate on how a polynomial determines the topology of the algebraic link it induces.

PROPOSITION 4.3.11: Let (X, L) be an algebraic link induced by a plane curve C corresponding to a polynomial $h \in \mathbb{C}[v,w]$. Then the number of connected components of L is equal to the number of irreducible factors into which h can be decomposed.

Proof. See the paragraph following lemma 5.2.1 in [23]. □

It turns out that every component of an algebraic link is completely determined by a corresponding irreducible factor of the polynomial that induces it; see section 2.3 in [23]. Therefore, we will restrict our treatment to knots corresponding to irreducible polynomials. To say more about the topology of the knot that an irreducible polynomial $h \in \mathbb{C}[v,w]$ induces, we will solve $h(v,w) = 0$ for w in terms of v. That such a solution can be obtained is a result due to Newton, and convergence of the solution for w was later shown by Puiseux'.

THEOREM 4.3.12: Let $h \in \mathbb{C}[v,w]$ such that $h(0,0) = 0$, then the equation $h(v,w) = 0$ has a convergent power series solution of the form $v = t^n$, $w = \sum_{k=1}^{\infty} a_k t^k$ for some $n \in \mathbb{N}$. Such a solution is called a Puiseux' expansion.

Proof. See, for example, theorem 2.2.1 and section 2.2 in [23]. □

The proof to theorem 4.3.12 gives successive approximations for w in terms of v of the form

$$w_0 = a_0 v^{\frac{p_0}{q_0}}$$
$$w_1 = v^{\frac{p_0}{q_0}}(a_0 + a_1 v^{\frac{p_1}{q_1}})$$
$$\vdots$$

Such an expression for w in terms of fractional powers of v will be reffered to as a Newton expansion. The corresponding Puiseux' expansion is then obtained by taking $v = t^n$ and substituting it into the expansion for w. Here n is chosen to be equal to $n = q_0 \cdot q_1 \cdots$ which is finite as shown in the proof. Thus, such a solution is characterised by the exponents, determined by (p_i, q_i) in the successive approximations. We can take these pairs to be coprime, and we will refer to them as the Newton pairs.

We will now give a description of the topology of a knot based on the Newton pairs and remark 4.3.8. Before doing so, we note that a knot (X, K) is equivalent to the embedding $\iota_K : K \to X$ of K in X. This description of a knot is better suited to describe the topology of a knot, so we will use it for now. First consider the simplest case, where there is only one Newton pair (p, q), where we take p and q to be coprime as noted before. Then the corresponding Newton expansion is $w = v^{p/q}$, and substituting $v = \epsilon e^{i\theta q}$ gives $w = \epsilon^{p/q} e^{i\theta p/q}$. With these choices for v and w, we have a parametrisation (v, w) of the knot that lies on a torus. As v goes around the circle of radius ϵ once, v goes p/q times around the circle of radius $\epsilon^{p/q}$. The curve $(\epsilon e^{i\theta q}, \epsilon^{p/q} e^{i\theta p/q})$ describing the knot, closes after v has gone around the toroidal direction of the torus p times, and w has gone around the toroidal direction of the torus q times. Such a knot is called a (p, q) torus knot. To deal with the more general situation with more than one Newton pair, we need the concept of a cable knot.

DEFINITION 4.3.13: Let (X, K) be a knot with corresponding embedding ι_K. Then a tubular neighbourhood of ι_K is an embedding of the solid torus $\tau : \mathbb{S}^1 \times D^2 \to X$ in X such that $\tau(t, 0) = \iota(t)$ for all $t \in \mathbb{S}^1$.

It turns out that such a tubular neighbourhood always exists, but it is not unique. To see this, note that given a tubular neighbourhood τ of a knot ι_K, we can obtain another as follows. Define

$$\varphi_t : \mathbb{S}^1 \times D^2 \to \mathbb{S}^1 \times D^2, \ (s, d) \mapsto (s, s^t d)$$

for any $t \in \mathbb{Z}$. Then $\iota_K \circ \varphi_t$ is also a tubular neighbourhood of ι_K, which is not equal to ι_K. The difference between both embeddings is a t-fold twist in the toroidal direction.

DEFINITION 4.3.14: Let (X, K) be a knot with corresponding embedding ι_K, and let τ be a tubular neighbourhood of ι_K such that the restriction of τ to $\mathbb{S}^1 \times \{(0,1)\}$, viewed as a knot, has linking number equal to zero with ι_K. Then, if $\iota' : \mathbb{S}^1 \to \mathbb{R}^3$ is a (p,q) torus knot on $\mathbb{S}^1 \times D^2$, $\tau \circ \iota'$ is a (p,q) cable knot on ι.

Suppose we have more general Newton pairs (p_i, q_i), then the first term in the expansion still describes a (p_0, q_0) torus knot. Now consider the next successive approximation

$$w_1 = v^{\frac{p_0}{q_0}}(a_0 + a_1 v^{\frac{p_1}{q_1}})$$

The additional term can be seen as perturbation to the (p_0, q_0) torus knot by substituting $v = \epsilon e^{i\theta}$, where ϵ is small. The resulting knot can then be seen as lying on a tubular neighbourhood of the (p_0, q_0) torus knot, going around its poloidal direction p_1 times, and its toroidal direction q_1 times. However, we should stress that (v_1, w_1) need not describe a (p_1, q_1) cable knot over a (p_0, q_0) torus knot. This is due to the fact that the toroidal direction of the embedded torus is in general not unknotted as required in the definition of a tubular neighbourhood. What kind of knot we do get is discussed in theorem 4.3.17. We should note that there is another problem with this description of the topology of a knot. Namely, this description seems to imply that every successive Newton pair changes the knot. It turns out that this is not the case, as we will now go on to show.

DEFINITION 4.3.15: Let $h \in \mathbb{C}[v, w]$ be irreducible such that $h(0, w) \neq 0$ and let $v = t^n$, $w = \sum_{k=1}^{\infty} a_k t^k$ be a solution of $h(v, w) = 0$. Now we define

$$\gamma_1 = \min\{k | a_k \neq 0 \text{ and } k \nmid n\} \qquad \text{and} \qquad e_1 = \gcd\{n, \beta_1\}$$

as well as

$$\gamma_{i+1} = \min\{k | a_k \neq 0 \text{ and } e_i \nmid n\} \qquad \text{and} \qquad e_{i+1} = \gcd\{e_i, \beta_{i+1}\}$$

until $e_g = 1$ which always happens; see for example section 2.3 in [23]. Then the Puiseux' characteristic of h is defined to be $(n; \gamma_1, \ldots, \gamma_g)$

It turns out that the Puiseux' characteristic determines the topology of the link completely. This fact is of great practical use when we would like to determine topology of a knot corresponding to the zero set of an irreducible polynomial $h \in \mathbb{C}[v, w]$, because it implies that we only need to determine finitely many terms of the Puiseux' expansion.

LEMMA 4.3.16: Let $h, h' \in \mathbb{C}[v, w]$ be irreducible with corresponding zero sets C and C' respectively. If h and h' have the same Puiseux' characteristic, then there is an $\epsilon \in \mathbb{R}_{>0}$ and a $p \in \mathbb{S}^3_\epsilon$ such that

$$(\mathbb{S}^3_\epsilon \backslash \{p\}, C \cap \mathbb{S}^3_\epsilon) \cong (\mathbb{S}^3_\epsilon \backslash \{p\}, C' \cap \mathbb{S}^3_\epsilon)$$

Proof. See proposition 5.3.1 in [23]. ☐

Let $h \in \mathbb{C}[v,w]$, and suppose we have a solution for $h(v,w) = 0$ for w in terms of v with Newton pairs (p_i, q_i). Note that this solution can equivalently be described by an expansion of the form

$$w = v^{\frac{m_1}{n_1}} + v^{\frac{m_2}{n_1 n_2}} + \dots$$

The pairs (m_i, n_i) are called the Puiseux' pairs. The Newton pairs are related to the Puiseux' pairs by $p_i = n_i$, $q_1 = m_1$, and $q_i = m_i - m_{i-1}n_i$.

THEOREM 4.3.17: Let $h \in \mathbb{C}[v,w]$ be irreducible, with Puiseux' pairs (m_1, n_1), ..., (m_g, n_g). Then the algebraic knot induced by h is equivalent to an iterated torus knot of type (r_i, n_i), where $r_1 = m_1$, and $r_i = m_i - m_{i-1}n_i + r_{i-1}n_{i-1}n_i$ for $i \geq 2$.

Proof. See proposition 2.3.9 in [3] ☐

Thus far we have shown that an irreducible polynomial induces a knot and we have discussed how this polynomial determines the topology of this knot. We will now see that given a Puiseux' expansion we can also construct an irreducible polynomial $h \in \mathbb{C}[v,w]$ such that the given expansion solves $h(v,w) = 0$.

THEOREM 4.3.18: Let $v = t^n$, $w = \sum_{k=1}^{\infty} a_k t^k$ be a Puiseux' expansion and define

$$u_s = - \sum_{k \equiv s \pmod{n}} a_k x^{(k-s)/n}$$

for any $s \in \{0, \dots, n-1\}$. Then there is an irreducible $h \in \mathbb{C}[x,y]$ such that $h(v,w) = 0$. Furthermore, this polynomial h is given by

$$h(x,y) = \det \begin{pmatrix} y+u_0 & u_1 & u_2 & \cdots & \cdots & u_{n-2} & u_{n-1} \\ xu_{n-1} & y+u_0 & u_1 & u_2 & \cdots & \cdots & u_{n-2} \\ xu_{n-2} & xu_{n-1} & \ddots & \ddots & \ddots & & \vdots \\ \vdots & xu_{n-2} & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & u_2 \\ xu_2 & \vdots & & \ddots & \ddots & \ddots & u_1 \\ xu_1 & xu_2 & \cdots & \cdots & xu_{n-2} & xu_{n-1} & y+u_0 \end{pmatrix}$$

Proof. See the corollary on page 57 of [8]. ☐

Having introduced all the elements from algebraic link theory that we will require for the rest of this thesis, we will now treat some examples.

EXAMPLE 4.3.19: Let $p, q \in \mathbb{Z}$ be coprime, then $h \in \mathbb{C}[v, w]$ given by $h(v, w) = v^p + w^q$ is irreducible. Note that $v^p + w^q = 0$ is easily solved to give $w = -v^{p/q}$ so that the Newton pair corresponding to h is (p, q). Therefore, we can conclude from the discussion preceding definition 4.3.13 that $(\mathbb{S}^3_\epsilon \backslash \{s\}, h^{-1}(0) \cap \mathbb{S}^3_\epsilon)$ describes a (p, q) torus knot for some $s \in \mathbb{S}^3_\epsilon$ and some $\epsilon \in \mathbb{R}_{>0}$ small enough.

EXAMPLE 4.3.20: Consider $h \in \mathbb{C}[v, w]$ given by $h(v, w) = v^2 + w^2 = (v + iw)(v - iw)$. Since h factors into two irreducible components, it describes a link. Each of the components is easily seen to describe a circle and it turns out that these circles are linked. This link is called the Hopf link.

The previous two examples illustrate how we can determine the topology of the link induced by its zero set. However, due to theorem 4.3.18, we can also start with the Newton pairs of a knot in mind and construct the corresponding polynomial. We will now show how this can be used to construct the polynomial corresponding to an iterated torus knot.

EXAMPLE 4.3.21: In this example we will show explicitly how an irreducible polynomial $h \in \mathbb{C}[x, y]$ can be constructed given the Newton pairs $(2, 3)$ and $(3, 2)$. The zero set of h will then correspond to an iterated torus knot of type $(3, 2)$,$(13, 3)$ by theorem 4.3.17. Given the Newton pairs, we have a Newton expansion of the form

$$y = x^{3/2}(1 + x^{2/3}) = x^{3/2} + x^{13/6}$$

This gives the following Puiseux' expansion:

$$\begin{cases} x &= t^6 \\ y &= t^9 + t^{13} \end{cases}$$

Then, according to theorem 4.3.18, the irreducible polynomial corresponding to this Puiseux' expansion is given by

$$\begin{aligned} h(x, y) &= \begin{pmatrix} y & -x^2 & 0 & -x & 0 & 0 \\ 0 & y & -x^2 & 0 & -x & 0 \\ 0 & 0 & y & -x^2 & 0 & -x \\ -x^2 & 0 & 0 & y & -x^2 & 0 \\ 0 & -x^2 & 0 & 0 & y & -x^2 \\ -x^3 & 0 & -x^2 & 0 & 0 & y \end{pmatrix} \\ &= y^6 - 3y^4x^3 + 3y^2x^6 - 6y^2x^8 - x^9 - 2x^{11} - x^{13} \end{aligned}$$

4.4 LINKED OPTICAL VORTICES

In this section we will show how the freedom in the Bateman construction can, in combination with the Bateman variables for the Hopf field from [12], be used to construct a family of self-dual electromagnetic fields with linked optical vortices. To be more precise, the intersection of the zero set of an electromagnetic field in this family with any spacelike slice in Minkowski space will have the same structure, and can be that of any algebraic link.

THEOREM 4.4.1: Let (X, L) be an algebraic link, and let $t \in \mathcal{M}$ which, as described in section 2.1, induces a splitting of Minkowski space

$$\mathcal{M} = \bigsqcup_{\lambda \in \mathbb{R}} \lambda t + \langle t \rangle^{\perp} := \bigsqcup_{\lambda \in \mathbb{R}} \Sigma_{\lambda}$$

Then there is a self-dual electromagnetic field F such that

$$(X, L) \cong (\Sigma_{\lambda}, \{p \in \mathcal{M} | F(p) = 0\} \cap \Sigma_{\lambda})$$

for any $\lambda \in \mathbb{R}$.

Proof. Let (X, L) be a link, with corresponding $h \in \mathbb{C}[v, w]$ and $\epsilon \in \mathbb{R}_{>0}$ such that

$$(X, L) \cong (\mathbb{S}^3_{\epsilon} \backslash \{p\}, h^{-1}(0) \cap \mathbb{S}^3_{\epsilon})$$

for some $p \in \mathbb{S}^3_{\epsilon}$. Also, let $t \in \mathcal{M}$ and denote the spacelike slices of the induced splitting of Minkowski space by Σ_{λ} for $\lambda \in \mathbb{R}$.

Now, let $\alpha, \beta : \mathcal{M} \to \mathbb{C}$ be the Bateman variables for the Hopf field from [12]. Then we can obtain Bateman variables for a scaled version of the Hopf field by taking $\alpha_{\epsilon}, \beta_{\epsilon} : \mathcal{M} \to \mathbb{C}$ to be α and β multiplied by $\sqrt{\epsilon/2}$. Then it can be checked that for any $t_* \in \mathbb{R}$ it holds that

$$|\alpha_{\epsilon}(t_*, x, y, z)|^2 + |\beta_{\epsilon}(t_*, x, y, z)|^2 = \epsilon$$

This shows that $(\alpha_{\epsilon}, \beta_{\epsilon})|_{\Sigma_{t_*}}$ maps into $\mathbb{S}^3_{\epsilon}$. Since $d\alpha_{\epsilon} \wedge d\beta_{\epsilon}$ is a scaled version of the Hopf field, it is in particular an electromagnetic field without zeros. Now we take $f, g : \mathbb{C}^2 \to \mathbb{C}$ to be given by

$$f(z_1, z_2) = \int h(z_1, z_2) dz_1 \qquad \text{and} \qquad g(z_1, z_2) = z_2$$

Then proposition 3.4.1 implies that $F = df(\alpha_{\epsilon}, \beta_{\epsilon}) \wedge dg(\alpha_{\epsilon}, \beta_{\epsilon})$ is a self-dual electromagnetic field given by

$$\begin{aligned} F &= df(\alpha_{\epsilon}, \beta_{\epsilon}) \wedge dg(\alpha_{\epsilon}, \beta_{\epsilon}) \\ &= (\partial_{\alpha_{\epsilon}} f \partial_{\beta_{\epsilon}} g - \partial_{\beta_{\epsilon}} f \partial_{\alpha_{\epsilon}} g) d\alpha_{\epsilon} \wedge d\beta_{\epsilon} \\ &= h(\alpha_{\epsilon}, \beta_{\epsilon}) d\alpha_{\epsilon} \wedge d\beta_{\epsilon} \end{aligned}$$

Restricted to Σ_0, the map $(\alpha_{\epsilon}, \beta_{\epsilon}) : \mathcal{M} \to \mathbb{S}^3_{\epsilon}$ is the inverse stereographic projection, which is a diffeomorphism from $\mathbb{R}^3$ to $\mathbb{S}^3 \backslash \{(0, i)\}$. Therefore we can conclude that

$$(\Sigma_0, \{p \in \Sigma_0 | h(\alpha_{\epsilon}(p), \beta_{\epsilon}(p)) = 0\} \cap \Sigma_0) \cong (X, L)$$

provided that two conditions are satisfied. The first condition is that $(0, i)$, the point not in the image of the inverse stereographic projection, is not mapped to zero under h. If this is the case, we choose another $h' \in \mathbb{C}[v, w]$ that corresponds to an equivalent link but does not map $(0, i)$ to zero and replace h by h' in the discussion above. The second condition is that Σ_0 and $\{p \in \Sigma_0 | h(\alpha_\epsilon(p), \beta_\epsilon(p)) = 0\} \cap \Sigma_0$ have the right orientations for the diffeomorphism to be orientation preserving. Since we have not specified their orientations yet, we have the freedom to choose them in such a way that this condition is satisfied.

Now we will show that the same result holds if we replace Σ_0 with a general spacelike slice Σ_λ. First we note that we calculated the rank of $(\alpha_\epsilon, \beta_\epsilon)|_{\Sigma_\lambda}$ to be equal to three for any $\lambda \in \mathbb{R}$. This shows that $(\alpha_\epsilon, \beta_\epsilon)$ has at least rank three, but since the three-sphere is a three-dimensional manifold, the rank of $(\alpha_\epsilon, \beta_\epsilon)$ is also at most three. Therefore we can conclude that the rank of $(\alpha_\epsilon, \beta_\epsilon)$ is precisely three. Furthermore, we note that $h|_{\mathbb{S}^3_\epsilon}$ has rank two, as shown in the proof of lemma 6.1 in [15]. Thus, $0 \in \mathbb{C}$ is a regular value of $h(\alpha, \beta)$, which together with the regular value theorem, implies that the zero set of $h(\alpha, \beta)$ in Minkowski space is a two-dimensional manifold. For notational convenience we define

$$L_\lambda = \{p \in \Sigma_\lambda | h(\alpha_\epsilon(p), \beta_\epsilon(p)) = 0\} \cap \Sigma_\lambda$$

From the observation that the rank of $(\alpha_\epsilon, \beta_\epsilon)|_{\Sigma_\lambda}$ is equal to three it also follows that this map is a local diffeomorphism. This implies that L_λ is empty or diffeomorphic to a disjoint union of circles. Suppose that L_λ is empty, then there must be a $T \in \mathbb{R}$ such that L_T is a discrete set because the zero set of F is a two-dimensional manifold and we know its intersection with L_0 to be non-empty. However, this gives a contradiction with the fact that $(\alpha_\epsilon, \beta_\epsilon)|_{\Sigma_T}$ is a local diffeomorphism and h has no isolated zeros in $\mathbb{S}^3_\epsilon$. Furthermore, the fact that the zero set of F in $\mathcal{M}$ is a two-dimensional submanifold implies that for any $\lambda' \in \mathbb{R}$ and any $p' \in L_{\lambda'}$ there is an open neighbourhood $U_{p'}$ of p' in the zero set of h. The intersection of the union of these opens for all $p' \in L_{\lambda'}$ with $\Sigma_{\lambda'+\epsilon'}$ is then equal to $L_{\lambda'+\epsilon'}$ for $\epsilon' \in \mathbb{R}_{>0}$ small enough. This implies that there is a diffeomorphism from $(\Sigma_{\lambda'}, L_{\lambda'})$ to $(\Sigma_{\lambda'+\epsilon'}, L_{\lambda'+\epsilon'})$. If we now choose the orientations of Σ_λ and L_λ such that this diffeomorphism is orientation preserving for all $\lambda \in \mathbb{R}$ and $\epsilon' \in \mathbb{R}_{>0}$, we can conclude that

$$(\Sigma_{\lambda'}, L_{\lambda'}) \cong (\Sigma_{\lambda'+\epsilon'}, L_{\lambda'+\epsilon'})$$

for some $\epsilon' \in \mathbb{R}_{>0}$. This concludes the proof. □

Due to the constructive nature of our proof to theorem 4.4.1, we can explicitly write down expressions for self-dual electromagnetic fields with linked optical vortices. We will now apply this to some specific examples, and visualise their zero sets numerically. This numerical visualisation was done using Mathematica, which calculated the surfaces in $\mathbb{R}^3$ at which the real and imaginary parts of $h(\alpha_\epsilon, \beta_\epsilon)$ are zero. These surfaces are the transparent orange surfaces in the figures we will encounter in the section. The zero sets of $F = h(\alpha_\epsilon, \beta_\epsilon) d\alpha_\epsilon \wedge d\beta_\epsilon$ are then determined by numerically computing the intersection of these surfaces, which are indicated in blue.

EXAMPLE 4.4.2: Here we will consider the case where the link (X, L) in theorem 4.4.1 is a torus knot. From example 4.3.19 we know that a polynomial $h \in \mathbb{C}[v, w]$ corresponding to a (p, q) torus knot is given by $h(v, w) = v^p + w^q$. Hence we choose

$$f(\alpha, \beta) = \frac{1}{p+1}\alpha^{p+1} + \beta^q \alpha \qquad \text{and} \qquad g(\alpha, \beta) = \beta$$

Then a self-dual electromagnetic field with a (p, q) torus knot as its zero set is given by

$$\begin{aligned} F &= df(\alpha, \beta) \wedge dg(\alpha, \beta) \\ &= (\alpha^p + \beta^q) d\alpha \wedge d\beta \end{aligned}$$

Numerical visualisations of the zero set of F for the case where $p = 3$ and $q = 2$ at $t = 0$ and $t = 3$ are shown in figure 1 and 2 respectively.

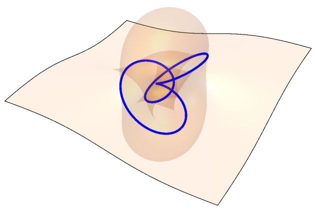

Figure 1: Numerical visualisation of the optical vortex at $t = 0$.

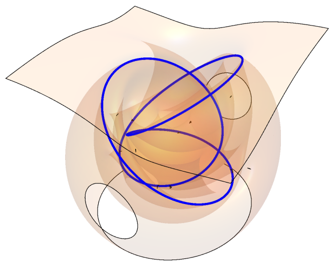

Figure 2: Numerical visualisation of the optical vortex at $t = 3$.

EXAMPLE 4.4.3: In this example we will consider the case where the link (X, L) in theorem 4.4.1 is the Hopf link. From example 4.3.20 we know that the polynomial $h \in \mathbb{C}[v, w]$ corresponding to the Hopf link is given by $h(v, w) = v^2 + w^2$. Hence we choose

$$f(\alpha, \beta) = \frac{1}{3}\alpha^3 + \beta^2 \alpha \qquad \text{and} \qquad g(\alpha, \beta) = \beta$$

Then a self-dual electromagnetic field with the Hopf link as its zero set is given by

$$\begin{aligned} F &= df(\alpha, \beta) \wedge dg(\alpha, \beta) \\ &= (\alpha^2 + \beta^2) d\alpha \wedge d\beta \end{aligned}$$

Numerical visualisations of the zero set of F at $t = 0$ and $t = 3$ are shown in figure 3 and 4 respectively.

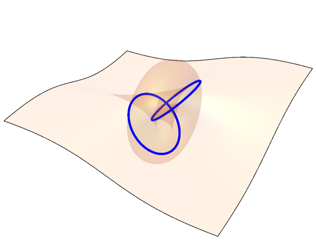

Figure 3: Numerical visualisation of the optical vortex at $t = 0$.

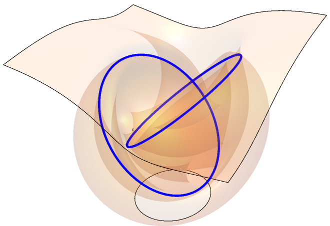

Figure 4: Numerical visualisation of the optical vortex at $t = 3$.

EXAMPLE 4.4.4: In example 4.3.21 we showed how an irreducible polynomial $h \in \mathbb{C}[x, y]$ could be constructed with Newton pairs $(2, 3)$ and $(3, 2)$. The zero set of h has been shown to correspond to an iterated torus knot of type $(3, 2)$, $(13, 3)$. By taking f and g as in the proof to theorem 4.4.1 we obtain a self-dual electromagnetic field $F = h(\alpha_\epsilon, \beta_\epsilon) d\alpha_\epsilon \wedge d\beta_\epsilon$. The zero set of this field at $t = 0$ as well as a parametrisation of the knot are visualised in figure 5 and 6 respectively.

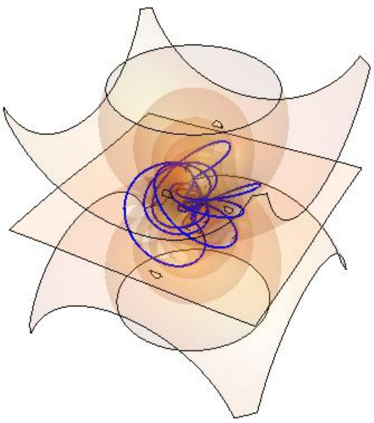

Figure 5: Numerical visualisation of the optical vortex at $t = 0$.

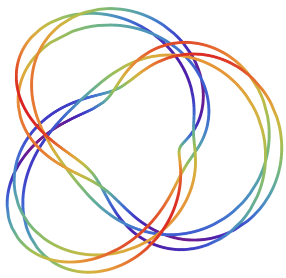

Figure 6: An iterated torus knot of type $(2, 3)$, $(3, 13)$.

5 CONCLUSION

In this thesis, we have studied Minkowski space, operators induced by its pseudo-Riemannian metric, and formulated electromagnetism in the language of differential forms. Using this formalism, we have shown how the Hopf field can be derived from a solution of the scalar wave equation, and we have shown how Bateman variables for the Hopf field arise from this derivation. Our final result is that we have have constructively proven that self-dual electromagnetic fields exist such that the intersection of the zero set of the field with any space-like affine subspace of Minkowski space describes an arbitrary algebraic link. In other words, we have discovered a new class of electromagnetic fields with knotted and linked optical vortices that preserve their structure under the time evolution.

Several directions for future research present themselves. First of all, it would be interesting to study the surfaces to which the field lines of the electromagnetic fields discussed in section 4.4 are constrained. We have obtained implicit expressions for these surfaces in some cases, but we have not been able to describe their structure in detail. It would also be interesting to see if electromagnetic fields exist for which the field lines form iterated torus knots or non-algebraic knots. Last, but certainly not least, it would be interesting to see if the electromagnetic fields with linked optical vortices discussed in section 4.4 could be created in the lab. Creating the exact electromagnetic fields we propose is not an experimental possibility, but it could be possible to create approximations to the exact solutions with the same vortex structure. We are looking into this question in collaboration with experimental physicists in the Quantum Optics Group.

BIBLIOGRAPHY

[1] R.R.L.J. van Asseldonk. "The Hopf Map in Magnetohydrodynamics". Bachelor's Thesis. Leiden University, 2015.

[2] John Baez and Javier P. Muniain. *Gauge fields, knots and gravity*. Singapore: World Scientific Publishing Co. Pte. Ltd., 1994.

[3] M. Barwick. "The Newton Polygon and the Puiseux Characteristic". Master's Thesis. The University of Sydney, 2011.

[4] H. Bateman. *The mathematical analysis of electric wave-motion*. London: Cambridge University Press, 1915.

[5] M. V. Berry and M. R. Dennis. "Topological events on wave dislocation lines: birth and death of loops, and reconnection". In: *J. Phys. A: Math. Theor.* 40.1 (2007), pp. 65–74.

[6] I. Bialynicki-Birula. "Electromagnetic vortex lines riding atop null solutions of the Maxwell equations". In: *J. Opt. A* 6 (2004), pp. 181–183.

[7] M.R. Dennis, R.P. King, B. Jack, K. O'Holleran and M.J. Padgett. "Isolated optical vortex knots". In: *Nat. Phys.* 6.2 (2010), pp. 118–121.

[8] D. Eisenbud and W. D. Neumann. *Three-dimensional link theory and invariants of plane curve singularities*. Princeton: Princeton University Press, 1985.

[9] P.A. Hogan. "Bateman electromagnetic waves". In: *Proc. R. Soc.A* 396 (1984), pp. 199–204.

[10] W. T. M. Irvine. "Linked and knotted beams of light, conservation of helicity and the flow of null electromagnetic fields". In: *J. Phys. A* 43.38 (2010).

[11] E. Kähler. "Über die Verzweigung einer algebraischen Funktion zweier Veränderlichen in der Umgebung einer singulären Stelle". In: *Math. Z.* 30 (1929), pp. 188–204.

[12] H. Kedia, I. Bialynicki-Birula, D. Peralta-Salas and W. T. M. Irvine. "Tying knots in light fields". In: *Phys. Rev. Lett.* 111.15 (2013), pp. 1–5.

[13] J. Leach, M.R. Dennis, J. Courtial and M.J. Padgett. "Knotted threads of darkness". In: *Nat.* 432.7014 (2004), p. 165.

[14] J. M. Lee. *Manifolds and Differential Geometry*. Providence: American Mathematical Society, 2009.

[15] J. Milnor. *Singular points of complex hypersurfaces*. Princeton: Princeton University Press, 1968.

[16] G. L. Naber. *The Geometry of Minkowski Spacetime*. New York: Springer, 2012.

[17] A. F. Rañada. "A topological theory of the electromagnetic field". In: *Lett. Math. Phys.* 18.2 (1989), pp. 97–106.

[18] A. F. Ranada. "Knotted solutions of the Maxwell equations in vacuum". In: *J.Phys. A* 23 (1990), pp. 815–820.

[19] D. Rolfsen. *Knots and Links*. Providence: AMS Chelsea, 2003.

[20] M. Stoll. *Linear Algebra II*. 2015. URL: http://pub.math.leidenuniv.nl/~luijkrmvan/linalg2/2015/LinAlg2-2015.pdf.

[21] J. L. Synge. *Relativity: The Special Theory*. Amsterdam: North-Holland Publishing Company, 1972.

[22] H.K. Urbantke. "The Hopf fibration—seven times in physics". In: *J. Geom. Phys.* 46 (2003), pp. 125–150.

[23] C.T.C. Wall. *Singular Points of Plane Curves*. Cambridge: Cambridge University Press, 2004.