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Gravity and Connections on Vector Bundles

Bachelor's Thesis

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Abstract

The main subject of this thesis is a reformulation of Einstein's equation. In this reformulation, the variable is not a metric, but a connection on a vector bundle. Nevertheless, we can associate a Riemannian metric to a connection. This allows us to relate the new formulation to the usual formulation, i.e. this allows us to argue that the new formulation is in fact a reformulation of Einstein's equation. Since the physically significant metrics are of Lorentzian signature, we also consider modifying the new formulation in an attempt to make it suitable for Lorentzian metrics.

Contents

1 Introduction

Between 1907 and 1915, Albert Einstein developed his theory of general relativity. The central equation in this theory is called *Einstein's equation*[1]. The variable in Einstein's equation is a metric on a 4-dimensional smooth manifold called *spacetime*. The nonlinear character of Einstein's equation makes it very difficult to find exact solutions. Therefore, one might wonder whether it is possible to formulate Einstein's equation in a more clever way. The main subject of this thesis is the formulation presented in [6]. In this new formulation (or reformulation), the main variable is not a metric, but a connection on a real vector bundle. To this connection, we associate a Riemannian metric. If the connection satisfies a particular equation, the associated metric will solve Einstein's equation.

There are a number of reasons for studying this new formulation. One of the reasons is that the new formulation puts Einstein's equation in the framework of a special type of theory, a *Yang-Mills theory*. A nice property of a (classical) Yang-Mills theory is that we understand how to quantize it, i.e. putting general relativity in the framework of a Yang-Mills theory gives a possible route to quantum gravity. Another reason for studying this new formulation is the following. As mentioned, we associate a Riemannian metric to a connection. However, metrics of physical significance are of Lorentzian signature. To make sure that the metric associated to the connection is Lorentzian we are forced to replace the real vector bundle by a complex vector bundle. Also, we need to impose extra conditions on the connection, called *reality conditions*. In the end, we get a real (Lorentzian) metric even though we are working with complex numbers. This property may prove useful in answering the following question: given a complex solution of Einstein's equation, can we find a corresponding real solution? This question arises when one tries to find solutions to Einstein's equation with the Hopf field[2] as a source term. Namely, complex solutions of a similar nature have been found, but it is unknown whether these complex solutions give rise to real solutions.

[1] Unless stated otherwise, with Einstein's equation we mean Einstein's equation in vacuum with cosmological constant.

[2] The Hopf field is an electromagnetic field with knotted field lines. See for instance [3, p. 11].

The structure of this thesis is as follows. The mathematical concepts needed to discuss the reformulation will be explained in chapter 2. In chapter 3, we will first explain the usual formulation of Einstein's equation. After this, we derive some technical results that will allow us to construct a metric from a connection. We will then give a detailed explanation of the reformulation presented in [6]. Finally, we consider modifying the new formulation in an attempt to make it suitable for Lorentzian metrics.

2 Preliminaries

In this thesis we wish to discuss different formulations of Einstein's equation. To do this properly we need to consider a number of concepts from differential geometry. This chapter is meant to introduce the reader to some of these concepts and the conventions that will be used. The first two sections are about smooth manifolds, differential forms and tensors. We assume that the reader is familiar with these topics, i.e. we will mostly introduce notations and conventions in these sections. After that matrix Lie groups and Lie algebras will be discussed, which will allow us to consider vector bundles. Then, we will arrive at the most important concepts: connections on vector bundles and curvature. Finally, the Hodge star operator will be defined. This operator will give rise to a notion of self- and anti-self-duality.

2.1 Smooth manifolds and differential forms

Many physical theories can be formulated using calculus on $\mathbb{R}^n$. However, Einstein's equation requires a more general notion of calculus. In this generalisation we replace $\mathbb{R}^n$ by a smooth manifold. We assume that the reader is familiar with some basic concepts from differential geometry, namely smooth manifolds and differential forms. This section is meant to introduce the notation that will be used regarding these topics.

In this thesis, the word *smooth* will mean of class C^∞, i.e. infinitely differentiable. Let X be a topological space and assume that X is locally Euclidean of dimension n, second countable and Hausdorff. Strictly speaking, a *(real) n-dimensional smooth manifold* is a pair $(X, \mathscr{A})$, where X is as above and $\mathscr{A}$ is a *smooth structure*[1] on X. An element of $\mathscr{A}$ is called a *chart on X*. Often, we will not explicitly mention the smooth structure and just call X an n-dimensional smooth manifold. Throughout this document $\mathbb{K}$ will denote an element of $\{\mathbb{R}, \mathbb{C}\}$. Since $\mathbb{R}$ and $\mathbb{R}^2$ are smooth manifolds, the canonical identification of $\mathbb{C}$ with $\mathbb{R}^2$ makes sure that $\mathbb{K}$ is always equipped with the structure of a (real) smooth manifold. The ring of smooth

[1] The definition of a smooth structure can be found in chapter 1 of [11].

functions from X to $\mathbb{K}$ will be denoted by $C^\infty(X,\mathbb{K})$. To every point $x \in X$, we associate an n-dimensional real vector space. Namely, the *tangent space* T_xX. This is the vector space of *derivations at* x, i.e. the vector space of linear[2] maps $X : C^\infty(X,\mathbb{R}) \to \mathbb{R}$ satisfying[3]

$$X(f \cdot g) = X(f) \cdot g(x) + f(x) \cdot X(g)$$

for all $f, g \in C^\infty(X,\mathbb{R})$. Consider the n-dimensional smooth manifold $\mathbb{R}^n$ and let $x \in \mathbb{R}^n$ be a point. The derivations $\partial_1|_x, \dots, \partial_n|_x \in T_x\mathbb{R}^n$, defined by

$$\partial_i|_x(f) = \frac{\partial f}{\partial x^i}(x),$$

form a basis of $T_x\mathbb{R}^n$. Let $dx^1|_x, \dots, dx^n|_x \in T_x^*\mathbb{R}^n := (T_x\mathbb{R}^n)^*$ denote the dual basis of $\partial_1|_x, \dots, \partial_n|_x$. Also, let $dx^i : \mathbb{R}^n \to \coprod_{x\in\mathbb{R}^n} T_x^*\mathbb{R}^n$ be defined by $dx^i(x) = dx^i|_x$ for all $x \in X$.
Let V be a $\mathbb{K}$-vector space and let J_k be the subspace of

$$V^{\otimes k} = \underbrace{V \otimes \dots \otimes V}_{k \text{ times}}$$

generated by elements of the form $v_1 \otimes \dots \otimes v_k$ where $v_i = v_j$ for some $i \neq j$. The quotient $\Lambda^k(V) = V^{\otimes k}/J_k$ is called[4] the *k-th exterior power of* V. Let $v_1 \wedge \dots \wedge v_k$ denote the image of $v_1 \otimes \dots \otimes v_k \in V^{\otimes k}$ under the quotient map. The exterior power satisfies the following universal property.

Proposition 2.1.1. Let W be a $\mathbb{K}$-vector space. For every alternating multilinear map $f : V^k \to W$, there exists a unique linear map $\tilde{f} : \Lambda^k(V) \to W$ such that $\tilde{f}(v_1 \wedge \dots \wedge v_k) = f(v_1, \dots, v_k)$ for all $v_1, \dots, v_k \in V$.

Proof. See [14, p. 59]. □

Let $f : V \to W$ be a linear map and define $\Lambda^k(f) : \Lambda^k(V) \to \Lambda^k(W)$ as the unique linear map satisfying $\Lambda^k(f)(v_1 \wedge \dots \wedge v_k) = f(v_1) \wedge \dots \wedge f(v_k)$. The *wedge product* $\wedge : \Lambda^k(V) \times \Lambda^l(V) \to \Lambda^{k+l}(V)$ is the unique bilinear map satisfying

$$(v_1 \wedge \dots \wedge v_k) \wedge (v_1' \wedge \dots \wedge v_l') = v_1 \wedge \dots \wedge v_k \wedge v_1' \wedge \dots \wedge v_l'.$$

Suppose that V is a real vector space. We will write

$$\Lambda^k(V,\mathbb{K}) = \begin{cases} \Lambda^k(V) & \text{if } \mathbb{K} = \mathbb{R} \\ \Lambda^k(V)_{\mathbb{C}} & \text{if } \mathbb{K} = \mathbb{C}, \end{cases}$$

[2] Of course, $C^\infty(X,\mathbb{K})$ is also a $\mathbb{K}$-vector space.
[3] More details about this definition of the tangent space can be found in chapter 3 of [11].
[4] This definition only makes sense for $k \geq 2$. We define $\Lambda^0(V) = \mathbb{K}$ and $\Lambda^1(V) = V$.

where $\Lambda^k(V)_{\mathbb{C}}$ denotes the complexification[5] of $\Lambda^k(V)$. An element of $\Lambda^k(V,\mathbb{K})$ will be called a $\mathbb{K}$-*valued k-form* on V. A *(smooth)* $\mathbb{K}$-*valued differential k-form* is a smooth[6] map

$$\omega : X \to \coprod_{x\in X} \Lambda^k(T_x^*X,\mathbb{K}) =: \Lambda^k(T^*X,\mathbb{K})$$

with $\omega(x) \in \Lambda^k(T_x^*X,\mathbb{K})$ for all $x \in X$. We will simplify notation by writing $\Lambda^k(T^*X)$ instead of $\Lambda^k(T^*X,\mathbb{R})$. Let $\Omega^k(X,\mathbb{K})$ denote the $C^\infty(X,\mathbb{K})$-module of $\mathbb{K}$-valued differential k-forms. Note that we can identify $\Omega^0(X,\mathbb{K})$ with $C^\infty(X,\mathbb{K})$. Also, let $d : \Omega^k(X,\mathbb{R}) \to \Omega^{k+1}(X,\mathbb{R})$ be the *exterior derivative* (see for instance [11, p. 306]). Since we can identify $\Omega^k(X,\mathbb{C})$ with the complexification of $\Omega^k(X,\mathbb{R})$, we also get a linear[7] map $d : \Omega^k(X,\mathbb{C}) \to \Omega^{k+1}(X,\mathbb{C})$ defined by $d(z\otimes\omega) = z\otimes d\omega$ for all $z \in \mathbb{C}$ and $\omega \in \Omega^k(X,\mathbb{R})$.

2.2 Tensors

In this section, we explain the conventions that will be used regarding tensors. In particular, we give a definition of a metric on a vector space and we introduce our signature conventions. Also, the identification of tensors with multilinear maps will be discussed. We conclude with a brief explanation of orientations on vector spaces and orientation preserving maps.
Throughout this section $V, V_1, \ldots, V_m$ and W will all denote finite dimensional $\mathbb{K}$-vector spaces. In particular, let n be the dimension of V.

Definition 2.2.1. An element of

$$T_s^r(V) := V^{\otimes r} \otimes (V^*)^{\otimes s}$$

is called a *tensor of type* (r,s) *on* V.

We will often want to define a linear map from $V_1 \otimes \ldots \otimes V_m$ to W by specifying the images of *pure tensors*, i.e. elements of the form $v_1 \otimes \ldots \otimes v_m$. We will use the universal property of tensor products, i.e. the following proposition, to make sure that such a definition gives rise to a unique well-defined linear map.

Proposition 2.2.2. For every multilinear map $f : V_1 \times \ldots \times V_m \to W$, there exists a unique linear map $\tilde{f} : V_1 \otimes \ldots \otimes V_m \to W$ such that $\tilde{f}(v_1 \otimes \ldots \otimes v_m) = f(v_1,\ldots,v_m)$ for all $v_i \in V_i$.

Proof. See [11, p.265]. $\square$

[5] We define the complexification of a real vector space as in [14, p. 53].

[6] For now, we define smoothness of ω using charts on X. See for instance [12, p. 206]. In section 2.4, we will see that $\Lambda^k(T^*X,\mathbb{K})$ is the total space of a $\mathbb{K}$-vector bundle. Therefore, we can define $\mathbb{K}$-valued differential k-forms as sections of $\Lambda^k(T^*X,\mathbb{K})$. Also, $\coprod$ denotes the disjoint union.

[7] Of course, $\Omega^k(X,\mathbb{K})$ is also a $\mathbb{K}$-vector space.

Remark 2.2.3. Let $\mathrm{Mult}_s^r(V)$ denote the vector space of multilinear maps from $(V^*)^r \times V^s$ to $\mathbb{K}$. It is not uncommon to call elements of $\mathrm{Mult}_s^r(V)$ a tensor of type (r,s) on V as well. The reason for this is that $T_s^r(V)$ and $\mathrm{Mult}_s^r(V)$ are canonically isomorphic: let $\varphi : V^r \times (V^*)^s \to \mathrm{Mult}_s^r(V)$ be the map that sends $(v_1, \dots, v_r, \alpha^1, \dots, \alpha^s)$ to

$$(\beta^1, \dots, \beta^r, w_1, \dots, w_s) \mapsto \beta^1(v_1) \cdot \ldots \cdot \beta^r(v_r) \cdot \alpha^1(w_1) \cdot \ldots \cdot \alpha^s(w_s).$$

It is easily shown that φ is multilinear. Hence, we get a linear map $\tilde{\varphi} : T_s^r(V) \to \mathrm{Mult}_s^r(V)$ as in Proposition 2.2.2. One can show that $\tilde{\varphi}$ is an isomorphism.

Another useful identification is the following.

Proposition 2.2.4. The linear map $\varphi : W \otimes V^* \to \mathrm{Hom}(V, W)$ that sends $w \otimes v^*$ to $v \mapsto v^*(v) \cdot w$ is an isomorphism.

Proof. See [14, p. 51]. $\square$

For the rest of this section, we will assume $\mathbb{K} = \mathbb{R}$. Let us look at an important example of a tensor.

Definition 2.2.5. A symmetric bilinear map $g : V \times V \to \mathbb{R}$ is called a *metric on V* if it is *non-degenerate*, i.e.

$$\flat_g : V \to V^*, \ v \mapsto g(v, \cdot) := (w \mapsto g(v, w))$$

is an isomorphism.

Note that Remark 2.2.3 allows us to identify a metric on V with a tensor of type (0,2) on V. Let g be a metric on V. A basis $e_1, \dots, e_n$ of V is called *g-orthonormal* or just *orthonormal* if $|g(e_i, e_j)| = \delta_{ij}$ for all $i, j \in \{1, \dots, n\}$. Define $s \in \mathbb{N}$ by

$$s = \#\{i : g(e_i, e_i) = -1\}.$$

We will call g a metric of *signature* $(n-s, s)$. Let $\mathscr{M}^{n-s,s}(V)$ denote the set of metrics of signature $(n-s, s)$ on V. We call an element $g \in \mathscr{M}^{n,0}(V)$ a *Riemannian* metric and an element $g \in \mathscr{M}^{1,n-1}(V)$ a *Lorentzian* metric. Also, let $\mathscr{C}^{n-s,s}(V)$ denote the quotient of $\mathscr{M}^{n-s,s}(V)$ and the following equivalence relation:

$$g' \sim g \iff g' = c \cdot g \text{ for some } c > 0.$$

An element $[g] \in \mathscr{C}^{n-s,s}(V)$ is called a *conformal class*.

Remark 2.2.6. In differential geometry one often identifies $\Lambda^k(V^*)$ with $\mathrm{Alt}^k(V)$, the vector space of alternating multilinear maps from V^k to $\mathbb{R}$. This identification is constructed as follows. First, let $\sigma \in S_k$ be a permutation and consider the linear map $s_\sigma : (V^*)^{\otimes k} \to (V^*)^{\otimes k}$ defined by

$$s_\sigma(\alpha^1 \otimes \ldots \otimes \alpha^k) = \alpha^{\sigma(1)} \otimes \ldots \otimes \alpha^{\sigma(k)}.$$

Now define

$$T^k_{\mathrm{Alt}}(V^*) = \{T \in (V^*)^{\otimes k} : s_\sigma(T) = \mathrm{sgn}(\sigma) \cdot T \text{ for all } \sigma \in S_k\}.$$

Using the universal property of exterior powers, it can be shown that there exists a unique isomorphism $\varphi : \Lambda^k(V^*) \to T^k_{\mathrm{Alt}}(V^*)$ satisfying

$$\varphi(\alpha^1 \wedge \ldots \wedge \alpha^k) = \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \cdot \alpha^{\sigma(1)} \otimes \ldots \otimes \alpha^{\sigma(k)},$$

for all $\alpha^1, \ldots, \alpha^k \in V^*$. Finally, one can show that we can identify $T^k_{\mathrm{Alt}}(V^*)$ with $\mathrm{Alt}^k(V)$ via the restriction of the isomorphism constructed in Remark 2.2.3. It follows that $\Lambda^k(V^*)$ is canonically isomorphic to $\mathrm{Alt}^k(V)$. More details about the previous identifications can be found in [14, p. 55].

Next, we will briefly discuss orientations on vector spaces. Since $\dim \Lambda^n(V) = 1$, we can define the following equivalence relation on $\Lambda^n(V) \setminus \{0\}$:

$$\omega' \sim \omega \iff \omega' = \lambda\omega \text{ for some } \lambda > 0.$$

Let $\mathfrak{O}(V) = (\Lambda^n(V) \setminus \{0\})/\sim$ denote the quotient set. Clearly, $\mathfrak{O}(V)$ consists of precisely two elements. An element of $\mathfrak{O}(V)$ is called an *orientation on V*. Note that a nonzero element $\omega \in \Lambda^n(V)$ uniquely determines an orientation on V, namely $[\omega]$. Therefore, we will sometimes call a nonzero element of $\Lambda^n(V)$ an orientation on V as well.

Definition 2.2.7. A pair $(V, \mathfrak{o})$, where V is a finite dimensional $\mathbb{R}$-vector space and $\mathfrak{o}$ is an orientation on V, is called an *oriented vector space*. A basis $e_1, \ldots, e_n \in V$ of V with $e_1 \wedge \ldots \wedge e_n \in \mathfrak{o}$ is called a *(positively) oriented basis of* $(V, \mathfrak{o})$.

Let $\mathfrak{o}_V$ and $\mathfrak{o}_W$ be orientations on V and W, respectively. Also, let $f : V \to W$ be an isomorphism (assuming one exists). Then $\Lambda^n(f) : \Lambda^n(V) \to \Lambda^n(W)$ is also an isomorphism. Therefore, the map

$$\mathfrak{O}(f) : \mathfrak{O}(V) \to \mathfrak{O}(W),\ [\omega] \mapsto [\Lambda^n(f)(\omega)]$$

is well-defined. We call f *orientation-preserving* if $\mathfrak{O}(f)(\mathfrak{o}_V) = \mathfrak{o}_W$. Note that if n is odd, f or $-f$ is always[8] orientation-preserving. If $V = W$, one can show that $\Lambda^n(f)$ corresponds to multiplying by $\det(f)$ (see [14, p. 61]). So, in this situation f is orientation-preserving if and only if $\det(f) > 0$.
Finally, we note that there is a canonical bijection between $\mathfrak{O}(V)$ and $\mathfrak{O}(V^*)$. Namely, let $e_1, \ldots, e_n$ be a basis of V and let $e^1, \ldots, e^n$ be its dual basis. Also, define $f : V \to V^*$ as the unique linear map that sends e_i to e^i. One can check that $\mathfrak{O}(f) : \mathfrak{O}(V) \to \mathfrak{O}(V^*)$ is a bijection and independent of the choice of basis.

[8]This follows from the fact that $\Lambda^n(\pm f) = \pm\Lambda^n(f)$ if n is odd.

2.3 Matrix Lie groups and Lie algebras

Let $\mathrm{GL}(k,\mathbb{K})$ denote the group of invertible $k \times k$ matrices with coefficients in $\mathbb{K}$. Before discussing vector bundles, we have to consider a special type of subgroup of $\mathrm{GL}(k,\mathbb{K})$. This type of subgroup is called a matrix Lie group. A matrix Lie group can be used to characterise extra structure on a vector bundle. This will be discussed in section 2.4. To a matrix Lie group G, we associate a set of matrices $\mathfrak{g}$ called the Lie algebra of G. It turns out that $\mathfrak{g}$ has the structure of a so called real Lie algebra. Analogously, the Lie algebra of a matrix Lie group can be used to characterise extra structure of a connection on a vector bundle. This is explained in section 2.5.
Note that we can identify $\mathrm{GL}(k,\mathbb{C})$ with a subset of $\mathbb{C}^{k^2}$. Therefore, the subspace topology gives rise to a topology on $\mathrm{GL}(k,\mathbb{C})$.

Definition 2.3.1. A subgroup $G \subseteq \mathrm{GL}(k,\mathbb{C})$ is called a *matrix Lie group* if G is a closed subset of $\mathrm{GL}(k,\mathbb{C})$.

Example 2.3.2. Let $\mathrm{SO}(k) \subseteq \mathrm{GL}(k,\mathbb{C})$ denote the subgroup consisting of real matrices A with $A^\top A = I$ and $\det A = 1$. In [7, p. 6], it is shown that $\mathrm{SO}(k)$ is a matrix Lie group.

Let $\mathrm{Mat}(k,\mathbb{K})$ denote the vector space of $k \times k$ matrices with coefficients in $\mathbb{K}$. Also, let $\exp : \mathrm{Mat}(k,\mathbb{C}) \to \mathrm{GL}(k,\mathbb{C})$ denote the *matrix exponential*.

Definition 2.3.3. Let $G \subseteq \mathrm{GL}(k,\mathbb{C})$ be a matrix Lie group. The set

$$\mathfrak{g} = \{A \in \mathrm{Mat}(k,\mathbb{C}) : \exp(t \cdot A) \in G \text{ for all } t \in \mathbb{R}\}$$

is called the *Lie algebra of G*.

The Lie algebra of a matrix Lie group has more structure than a set. Namely, it has the structure of a real Lie algebra.

Definition 2.3.4. A pair $(\mathscr{L}, [\cdot,\cdot])$ is called a *real Lie algebra* if the following holds:

- $\mathscr{L}$ is a real vector space.
- $[\cdot,\cdot] : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ is a bilinear map with the following properties:
  - $[x,y] = -[y,x]$ for all $x, y \in \mathscr{L}$.
  - $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$ for all $x, y, z \in \mathscr{L}$.

Theorem 2.3.5. The subset $\mathfrak{g} \subseteq \mathrm{Mat}(k,\mathbb{C})$ is closed under addition and scalar multiplication by real numbers. Also, $AB - BA \in \mathfrak{g}$ for all $A, B \in \mathfrak{g}$.

Proof. See [7, p. 43]. □

So, equipping $\mathfrak{g}$ with the usual matrix addition and real scalar multiplication, establishes $\mathfrak{g}$ as a real vector space. Also, define

$$[\cdot,\cdot] : \mathrm{Mat}(k,\mathbb{C}) \times \mathrm{Mat}(k,\mathbb{C}) \to \mathrm{Mat}(k,\mathbb{C}),\ (A,B) \mapsto AB - BA.$$

A straightforward verification shows that $[\cdot,\cdot]$ satisfies

$$[A,[B,C]] + [C,[A,B]] + [B,[C,A]] = 0$$

for all $A,B,C \in \mathrm{Mat}(k,\mathbb{C})$. Using Theorem 2.3.5, it follows that $(\mathfrak{g},[\cdot,\cdot]|_{\mathfrak{g}\times\mathfrak{g}})$ is a real Lie algebra.

Example 2.3.6. Let $\mathfrak{so}(k)$ denote the Lie algebra of $\mathrm{SO}(k)$. In [7, p. 40], it is shown that the Lie algebra of $\mathrm{SO}(k)$ is equal to the real antisymmetric $k \times k$ matrices:

$$\mathfrak{so}(k) = \{A \in \mathrm{Mat}(k,\mathbb{R}) : A^\top = -A\}.$$

2.4 Vector bundles

A vector bundle makes precise the idea of attaching a vector space to each point of a smooth manifold. Therefore, it allows one to generalise the notion of vector fields. These generalised vector fields are called sections. Many objects in differential geometry can be interpreted as sections of vector bundles. For instance, all the objects appearing in Einstein's equation. First, we define what vector bundles and sections are. After this, we discuss how new vector bundles can be constructed from old ones. These new vector bundles allow us to define objects like metrics and orientations in the context of vector bundles. Finally, we will define vector bundles with extra structure using matrix Lie groups.

Definition 2.4.1. Let X be an n-dimensional smooth manifold. A $\mathbb{K}$-*vector bundle of rank k over X* is a 3-tuple $(E,\pi,\mathscr{C})$ with the following properties:

- E (called the *total space*) is a smooth manifold, $\pi : E \to X$ is a smooth map and $E_x := \pi^{-1}(x)$ (called the *fibre of E over x*) is endowed with the structure of a k-dimensional $\mathbb{K}$-vector space for all $x \in X$.
- $\mathscr{C}$ is a *trivialising cover*, i.e. a set $\{(U_i,\psi_i) : i \in I\}$ with the following properties:
    - $X = \bigcup_{i\in I} U_i$ and $U_i \subseteq X$ is an open subset for all $i \in I$.
    - $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^k$ is a diffeomorphism such that

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\psi_i"] \arrow[dr, "\pi"'] & U_i \times \mathbb{K}^k \arrow[d, "\pi_1"] & (x,v) \arrow[d, mapsto] \\ & U_i & x \end{tikzcd}$$

commutes for all $i \in I$.

- For all $i \in I$ and $x \in U_i$, the map $\psi_{i,x} : E_x \to \mathbb{K}^k$ defined by $\psi_i(e) = (x, \psi_{i,x}(e))$ is linear, and hence an isomorphism.

Let $(E, \pi, \mathscr{C})$ be as in the previous definition. For all $i, j \in I$, the map $g_{ij} : U_i \cap U_j \to \mathrm{GL}(k, \mathbb{K})$, defined by

$$(\psi_i \circ \psi_j^{-1})(x, v) = (x, g_{ij}(x)v),$$

is called a *transition function*. It can be shown that the transition functions are smooth[9] maps (see [11, p. 107]). An element $(U, \psi) \in \mathscr{C}$ is called a *local trivialisation* and U is called a *trivialising neighbourhood*. Also, a $\mathbb{K}$-vector bundle will be called *real* if $\mathbb{K} = \mathbb{R}$ and *complex* if $\mathbb{K} = \mathbb{C}$.

Definition 2.4.2. Let $(F, \rho, \mathscr{D})$ be a $\mathbb{K}$-vector bundle of rank l over X. A smooth map $f : E \to F$ is called a *bundle map* if $\pi = \rho \circ f$ and $f|_{E_x} : E_x \to F_x$ is a linear map for all $x \in X$.

Let P be a property of a linear map. We say that a bundle map $f : E \to F$ has property P if $f|_{E_x} : E_x \to F_x$ has property P for all $x \in X$. For instance, we call a bundle map $f : E \to F$ an isomorphism (or a *bundle isomorphism*) if $f|_{E_x} : E_x \to F_x$ is an isomorphism for all $x \in X$.

Definition 2.4.3. A smooth map $\sigma : X \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_X$ is called a *(smooth global) section of* E. Let $U \subseteq X$ be an open subset. A smooth map $\sigma : U \to E$ satisfying $\pi \circ \sigma = \mathrm{id}_U$ is called a *(smooth) local section of* E. Let $\mathscr{A}^0(E)$ denote the set of global sections of E.

Equipping $\mathscr{A}^0(E)$ with pointwise addition and scalar multiplication makes it into a $\mathbb{K}$-vector space. We can also multiply sections pointwise by elements of $C^\infty(X, \mathbb{K})$, i.e. $\mathscr{A}^0(E)$ also has a $C^\infty(X, \mathbb{K})$-module structure. Let $\sigma \in \mathscr{A}^0(E)$ be a section and recall the bundle map $f : E \to F$. Instead of $\sigma(x)$, we will sometimes write σ_x. Also, we will occasionally write $f(\sigma)$ instead of $f \circ \sigma \in \mathscr{A}^0(F)$.

Definition 2.4.4. Let $e_1, \ldots, e_k : U \to E$ be local sections. We call $e_1, \ldots, e_k$ a *local frame of* E if $e_1(x), \ldots, e_k(x)$ is a basis of E_x for all $x \in U$.

[9] Note that we can identify $\mathrm{GL}(k, \mathbb{K})$ with an open subset of $\mathbb{K}^{k^2}$. Therefore, $\mathrm{GL}(k, \mathbb{K})$ can be equipped with the structure of an open submanifold. This justifies that we can call the transition functions smooth.

Given a local trivialisation $(U,\psi) \in \mathscr{C}$, we can always construct a local frame: define $e_1,\dots,e_k : U \to E$ by $e_i(x) = \psi^{-1}(x,\tilde{e}_i)$, where $\tilde{e}_i$ is the i-th member of the standard basis of $\mathbb{K}^k$. We call $e_1,\dots,e_k$ the *local frame induced by* (U,ψ).

Example 2.4.5. A simple example of a $\mathbb{K}$-vector bundle is the following. Define $E = X \times \mathbb{K}^k$ and let $\pi : E \to X$ be the projection onto the first factor. Also, define $\mathscr{C} = \{(X,\mathrm{id}_E)\}$. It is easily verified[10] that $(E,\pi,\mathscr{C})$ is a $\mathbb{K}$-vector bundle of rank k over X. It is called the $\mathbb{K}$-*trivial bundle*.

We now wish to construct new vector bundles from old ones. To do this efficiently, we will need the following lemma.

Lemma 2.4.6. Let $\{E_x : x \in X\}$ be a family of k-dimensional $\mathbb{K}$-vector spaces. Define $E = \coprod_{x\in X} E_x$ and let $\pi : E \to X$ be the map that sends an element of E_x to x. Also, let $\mathscr{C} = \{(U_i,\psi_i) : i \in I\}$ be a set with the following properties:

- $X = \bigcup_{i\in I} U_i$ and $U_i \subseteq X$ is an open subset for all $i \in I$.
- $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{K}^k$ is a bijection such that the diagram in Definition 2.4.1 commutes for all $i \in I$.
- The map $\psi_{i,x} : E_x \to \mathbb{K}^k$, defined by $\psi_i(e) = (x,\psi_{i,x}(e))$, is linear for all $i \in I$ and $x \in U_i$.
- The map $g_{ij} : U_i \cap U_j \to \mathrm{GL}(k,\mathbb{K})$, defined by

  $$(\psi_i \circ \psi_j^{-1})(x,v) = (x, g_{ij}(x)v),$$

  i.e. $g_{ij}(x) = \psi_{i,x} \circ \psi_{j,x}^{-1}$, is smooth for all $i,j \in I$.

Then there exists a unique topology and smooth structure on E such that $(E,\pi,\mathscr{C})$ is a $\mathbb{K}$-vector bundle of rank k over X.

Proof. See [11, p. 108]. □

Example 2.4.7. Let $(E,\pi,\mathscr{C})$ be a $\mathbb{K}$-vector bundle of rank k over the smooth manifold X and write $\mathscr{C} = \{(U_i,\psi_i) : i \in I\}$. Define $E^* = \coprod_{x\in X} E_x^*$ and let $\pi^* : E^* \to X$ be the map that sends an element of E_x^* to x. Let $(U_i,\psi_i) \in \mathscr{C}$ be a local trivialisation and $x \in U_i$ a point. Write

$$(\psi_{i,x}^{-1})^\top : E_x^* \to (\mathbb{K}^k)^*,\ \alpha \mapsto \alpha \circ \psi_{i,x}^{-1}.$$

Also, let $\varphi : \mathbb{K}^k \to (\mathbb{K}^k)^*$ be the linear map that sends the standard basis to the corresponding dual basis. We get an isomorphism $\psi_{i,x}^* : E_x^* \to \mathbb{K}^k$ defined by

[10] We should also specify the vector space structure on the fibres. Since $E_x = \{x\} \times \mathbb{K}^k$, we can just add and multiply by scalars in the second factor.

$$\begin{tikzcd} E_x^* \arrow[r, "(\psi_{i,x}^{-1})^\top"] \arrow[rr, bend right, "\psi_{i,x}^* := \varphi^{-1} \circ (\psi_{i,x}^{-1})^\top"'] & (\mathbb{K}^k)^* \arrow[r, "\varphi^{-1}"] & \mathbb{K}^k \end{tikzcd}$$

Now define

$$\psi_i^* : \pi^{*-1}(U_i) \to U_i \times \mathbb{K}^k,\ e \mapsto (\pi^*(e), \psi_{i,\pi^*(e)}^*(e)).$$

One can check that $\mathscr{C}^* := \{(U_i, \psi_i^*) : i \in I\}$ satisfies all the requirements of Lemma 2.4.6. So, according to Lemma 2.4.6 there exists a unique topology and smooth structure on E^* such that $(E^*, \pi^*, \mathscr{C}^*)$ is a vector bundle. It is called the *dual bundle of* $(E, \pi, \mathscr{C})$.

Example 2.4.8. Let $(F, \rho, \mathscr{D})$ be a $\mathbb{K}$-vector bundle of rank l over the smooth manifold X and write $\mathscr{D} = \{(V_j, \phi_j) : j \in J\}$. Define $E \otimes F = \coprod_{x \in X} E_x \otimes F_x$ and let $\tilde{\pi} : E \otimes F \to X$ be the map that sends an element of $E_x \otimes F_x$ to x. Let $(U_i, \psi_i) \in \mathscr{C}$ and $(V_j, \phi_j) \in \mathscr{D}$ be local trivialisations and $x \in U_i \cap V_j$ a point. Also, let $\varphi : \mathbb{K}^{kl} \to \mathbb{K}^k \otimes \mathbb{K}^l$ be the linear map that sends the standard basis $e_1, e_2, \ldots, e_{kl}$ to $e_1 \otimes e_1, e_1 \otimes e_2, \ldots, e_k \otimes e_l$. We get an isomorphism $(\psi_i \otimes \phi_j)_x : E_x \otimes F_x \to \mathbb{K}^{kl}$ defined by

$$\begin{tikzcd} E_x \otimes F_x \arrow[r, "\psi_{i,x} \otimes \phi_{j,x}"] \arrow[rr, bend right, "(\psi_i \otimes \phi_j)_x := \varphi^{-1} \circ (\psi_{i,x} \otimes \phi_{j,x})"'] & \mathbb{K}^k \otimes \mathbb{K}^l \arrow[r, "\varphi^{-1}"] & \mathbb{K}^{kl} \end{tikzcd}$$

Now define

$$\psi_i \otimes \phi_j : \tilde{\pi}^{-1}(U_i \cap V_j) \to (U_i \cap V_j) \times \mathbb{K}^{kl}, e \mapsto \big(\tilde{\pi}(e), (\psi_i \otimes \phi_j)_{\tilde{\pi}(e)}(e)\big).$$

One can check that $\mathscr{C} \otimes \mathscr{D} := \{(U_i \cap V_j, \psi_i \otimes \phi_j) : i \in I, j \in J\}$ satisfies all the requirements of Lemma 2.4.6. So, according to Lemma 2.4.6 there exists a unique topology and smooth structure on $E \otimes F$ such that $(E \otimes F, \tilde{\pi}, \mathscr{C} \otimes \mathscr{D})$ is a vector bundle. It is called the *tensor product bundle*.

Similarly, we can define a vector bundle with total space $\Lambda^r(E) := \coprod_{x \in X} \Lambda^r(E_x)$. Also, define $\mathrm{End}(E) = E \otimes E^*$ and note that Proposition 2.2.4 tells us that we can identify $E_x \otimes E_x^*$ with $\mathrm{End}(E_x) := \{f : E_x \to E_x : f \text{ is linear}\}$.
For the moment, assume $\mathbb{K} = \mathbb{R}$. The previously constructed bundles allow us to define some new objects. First note, Remark 2.2.3 says that we can identify an element of $E_x^* \otimes E_x^*$ with a bilinear map from $E_x \times E_x$ to $\mathbb{R}$. This identification is used in the following definition.

Definition 2.4.9. A section $g \in \mathscr{A}^0(E^* \otimes E^*)$ is called a *metric on* E if $g(x)$ is a metric on E_x for all $x \in X$.

We call a metric g on E of *signature* $(k-s,s)$ if $g(x)$ is of signature $(k-s,s)$ for all $x \in X$. Let $\mathscr{M}^{k-s,s}(E)$ denote the set of metrics on E of signature $(k-s,s)$. An element $g \in \mathscr{M}^{k,0}(E)$ is called a *Riemannian* metric and an element $g \in \mathscr{M}^{1,k-1}(E)$ is called a *Lorentzian* metric. Again, we can introduce an equivalence relation on the set of metrics $\mathscr{M}^{k-s,s}(E)$:

$$g' \sim g \iff g' = f \cdot g \text{ for some } f : X \to \mathbb{R}_{>0}.$$

An element $[g] \in \mathscr{C}^{k-s,s}(E) := \mathscr{M}^{k-s,s}(E)/\sim$ is called a *conformal class*.

Example 2.4.10. Suppose that E is equipped with a Riemannian metric g. Let $\mathfrak{so}(E_x)$ be defined by

$$\mathfrak{so}(E_x) = \{f \in \mathrm{End}(E_x) : g_x(f(v),w) + g_x(v,f(w)) = 0 \text{ for all } v,w \in E_x\}.$$

Note that $\mathfrak{so}(E_x)$ is a subspace of $\mathrm{End}(E_x)$. Now define $\mathfrak{so}(E) = \coprod_{x\in X} \mathfrak{so}(E_x)$ and let $\pi_{\mathfrak{g}} : \mathfrak{so}(E) \to X$ be the map that sends an element of $\mathfrak{so}(E_x)$ to x. As before, the trivialising cover $\mathscr{C}$ can be used to construct a set $\mathscr{C}_{\mathfrak{g}}$ that satisfies all the conditions of Lemma 2.4.6. So, the conclusion of Lemma 2.4.6 gives us a topology and smooth structure on $\mathfrak{so}(E)$ such that $(\mathfrak{so}(E), \pi_{\mathfrak{g}}, \mathscr{C}_{\mathfrak{g}})$ is a vector bundle.

Note that an element $f \in \mathrm{End}(E_x)$ is an element of $\mathfrak{so}(E_x)$ if and only if the matrix representation of f is antisymmetric in a g_x-orthonormal basis.

Definition 2.4.11. A section $\omega \in \mathscr{A}^0(\Lambda^k(E))$, where k denotes the rank of E, is called an *orientation on* E if $\omega(x) \neq 0$ for all $x \in X$. Two orientations ω and ω' on E are called *equivalent* if $\omega' = f \cdot \omega$ for some $f : X \to \mathbb{R}_{>0}$.

Example 2.4.12. An important example of a real vector bundle is the following. Define $TX = \coprod_{x\in X} T_xX$ and let $\pi_T : TX \to X$ be the map that sends an element of T_xX to x. In [11, p. 106], it is shown how to define a topology, smooth structure and trivialising cover $\mathscr{C}_T$ such that $(TX, \pi_T, \mathscr{C}_T)$ is a vector bundle over X. It is called the *tangent bundle*. A metric on the tangent bundle is called a *metric on* X and an orientation on $T^*X := (TX)^*$ is called an *orientation on* X. If an orientation on X exists we call X *orientable*. A nowhere vanishing section of $\Lambda^n(T^*X)$ is also sometimes called a *volume form*.

Let $\mathbb{K}$ be arbitrary again. The previous example (together with Lemma 2.4.6) allows us to define a $\mathbb{K}$-vector bundle over X with total space $\Lambda^r(T^*X,\mathbb{K}) = \coprod_{x\in X} \Lambda^r(T_x^*X,\mathbb{K})$.

Definition 2.4.13. A section $\omega \in \mathscr{A}^0(\Lambda^r(T^*X,\mathbb{K}) \otimes E)$ is called an *E-valued r-form*. The set of all E-valued r-forms will be denoted by $\mathscr{A}^r(E)$.

Let $\omega \in \mathscr{A}^r(E)$ be an E-valued r-form. Note that elements of $\Lambda^r(T_x^*X,\mathbb{K})$ can be identified with alternating multilinear maps from $(T_xX)^r$ to $\mathbb{K}$ (see Remark 2.2.6). So, $\omega(x) \in \Lambda^r(T_x^*X,\mathbb{K}) \otimes E_x$ can be identified with an alternating multilinear map from $(T_xX)^r$ to E_x by inserting tangent vectors in the first factor. Therefore, it makes sense to write $\omega(V_1,\dots,V_r) \in \mathscr{A}^0(E)$ for all $V_1,\dots,V_r \in \mathscr{A}^0(TX)$. Consider the following linear map

$$\Omega^r(X,\mathbb{K}) \otimes \mathscr{A}^0(E) \to \mathscr{A}^r(E),\ \omega \otimes \sigma \mapsto (x \mapsto \omega(x) \otimes \sigma(x)).$$

It turns out that this map is an isomorphism (see [15, p. 180]), i.e. we can identify $\mathscr{A}^r(E)$ with $\Omega^r(X,\mathbb{K}) \otimes \mathscr{A}^0(E)$.

Definition 2.4.14. Let $G \subseteq \mathrm{GL}(k,\mathbb{K})$ be a matrix Lie group. The vector bundle $(E,\pi,\mathscr{C})$ is called a *G-bundle* if all the transition functions map into the matrix Lie group G.

Example 2.4.15. Suppose that $(E,\pi,\mathscr{C})$ is a real $\mathrm{SO}(k)$-bundle. This just means that the fibres have extra structure: let $x \in X$ be a point and let $(U,\psi) \in \mathscr{C}$ be a local trivialisation with $x \in U$. Also, let $e_1,\dots,e_k : U \to E$ be the local frame induced by (U,ψ). Define a Riemannian metric g_x on E_x by declaring that $e_1(x),\dots,e_k(x)$ is orthonormal and define an orientation $\omega_x \in \Lambda^k(E_x)$ by $\omega_x = e_1(x) \wedge \ldots \wedge e_k(x)$. The definitions of g_x and ω_x do not depend on the choice of local trivialisation precisely because the transition functions are $\mathrm{SO}(k)$-valued. So, $g \in \mathscr{A}^0(E^* \otimes E^*)$ defined by $g(x) = g_x$ is a Riemannian metric on E and $\omega \in \mathscr{A}^0(\Lambda^k(E))$ defined by $\omega(x) = \omega_x$ is an orientation on E. By definition, the local frames induced by local trivialisations are oriented and orthonormal.
Conversely, suppose that $(E,\pi,\mathscr{C})$ is a real vector bundle equipped with a Riemannian metric g and an orientation ω. Write $\mathscr{C} = \{(U_i,\psi_i) : i \in I\}$ and let $e_1,\dots,e_k : U_i \to E$ be the local frame induced by $(U_i,\psi_i) \in \mathscr{C}$. Using the Gram-Schmidt process, we find an orthonormal local frame $e'_1,\dots,e'_k : U_i \to E$. Without loss of generality, we can assume that U_i is connected. Reordering the orthonormal frame will then result in an oriented orthonormal frame $e''_1,\dots,e''_k : U_i \to E$. Define $\psi''_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ by[11] $\psi''^{-1}_i(x,v) = v^\alpha e''_\alpha(x)$. Now let $\mathscr{C}'' = \{(U_i,\psi''_i) : i \in I\}$. One can check that $(E,\pi,\mathscr{C}'')$ is an $\mathrm{SO}(k)$-bundle precisely because the local frames induced by local trivialisations in $\mathscr{C}''$ are oriented and orthonormal.

2.5 Connections and curvature

Our previous discussion of vector bundles allows us to consider the most important mathematical concepts of this thesis: connections on vector bundles and curvature. While our intuitive understanding of curvature does not directly relate to the definitions below, this is the notion needed to describe Einstein's equation. First, we will define what a connection is and show that we can locally describe a connection

[11] Throughout this thesis, we will be using the *Einstein summation convention*: we sum over an index if it appears as a subscript and a superscript (unless stated otherwise).

using differential 1-forms. After this, the curvature of a connection will be defined. Again, we can give a local description of curvature using differential forms. Finally, we consider connections on G-bundles, i.e. vector bundles that are equipped with extra structure. We will define what it means for a connection to be compatible with this extra structure.

Let $(E, \pi, \mathscr{C})$ be a $\mathbb{K}$-vector bundle of rank k over a smooth manifold X.

Definition 2.5.1. A *connection on E* is a linear map $D : \mathscr{A}^0(E) \to \mathscr{A}^1(E)$ with

$$D(f \cdot \sigma) = df \otimes \sigma + f \cdot D\sigma \tag{2.1}$$

for all $f \in C^\infty(X, \mathbb{K})$ and $\sigma \in \mathscr{A}^0(E)$. Also, we will write $D_V\sigma = (D\sigma)(V) \in \mathscr{A}^0(E)$ for all $\sigma \in \mathscr{A}^0(E)$ and $V \in \mathscr{A}^0(TX)$.

Let D be a connection on E. We can use D to construct connections on other vector bundles. For instance, define $D^* : \mathscr{A}^0(E^*) \to \mathscr{A}^1(E^*)$ by[12]

$$(D^*\sigma^*)(\sigma) = d(\sigma^*(\sigma)) - \sigma^*(D\sigma)$$

for all $\sigma \in \mathscr{A}^0(E)$ and $\sigma^* \in \mathscr{A}^0(E^*)$. It is easily verified that D^* does indeed define a connection. Also, define $\mathrm{End}(D) : \mathscr{A}^0(\mathrm{End}(E)) \to \mathscr{A}^1(\mathrm{End}(E))$ by[13]

$$\mathrm{End}(D)(\sigma \otimes \sigma^*) = D\sigma \otimes \sigma^* + \sigma \otimes D^*\sigma^*.$$

Again, one can check that $\mathrm{End}(D)$ defines a connection. Let $e_1, \ldots, e_k : U \to E$ be a local frame induced by a local trivialisation $(U, \psi) \in \mathscr{C}$.

Definition 2.5.2. The 1-forms $A_i^j \in \Omega^1(U, \mathbb{K})$, defined by $De_i = A_i^j \otimes e_j$, are called the *local connections forms induced by (U, ψ)*.

Let $\sigma : U \to E$ be a local section and define $\sigma^i : U \to \mathbb{K}$ by $\sigma = \sigma^i e_i$. Equation (2.1) shows

$$D\sigma = d\sigma^i \otimes e_i + \sigma^i \cdot A_i^j \otimes e_j = (d\sigma^i + \sigma^j \cdot A_j^i) \otimes e_i,$$

i.e. D is completely determined by its local connection forms. Next, the curvature of a connection will be defined. For this, we need an extension of the connection to E-valued forms. Let $D : \mathscr{A}^p(E) \to \mathscr{A}^{p+1}(E)$ be the linear map defined by[14]

$$D(\omega \otimes \sigma) = d\omega \otimes \sigma + (-1)^p \omega \wedge D\sigma.$$

[12] Write $\sigma^*(\sigma) : X \to \mathbb{K}$, $x \mapsto \sigma_x^*(\sigma(x))$. Also, consider bilinear maps from $\mathscr{A}^1(E^*) \times \mathscr{A}^0(E)$ and $\mathscr{A}^0(E^*) \times \mathscr{A}^1(E)$ to $\Omega^1(X, \mathbb{K})$ defined by $(\omega \otimes \sigma^*, \sigma) \mapsto \sigma^*(\sigma) \cdot \omega$ and $(\sigma^*, \omega \otimes \sigma) \mapsto \sigma^*(\sigma) \cdot \omega$. Let $(D^*\sigma^*)(\sigma)$ and $\sigma^*(D\sigma)$ denote the images of $(D^*\sigma^*, \sigma)$ and $(\sigma^*, D\sigma)$ under the previous maps.

[13] Consider bilinear maps from $\mathscr{A}^1(E) \times \mathscr{A}^0(E^*)$ and $\mathscr{A}^0(E) \times \mathscr{A}^1(E^*)$ to $\mathscr{A}^1(\mathrm{End}(E))$ defined by $(\omega \otimes \sigma, \sigma^*) \mapsto \omega \otimes (\sigma \otimes \sigma^*)$ and $(\sigma, \omega \otimes \sigma^*) \mapsto \omega \otimes (\sigma \otimes \sigma^*)$. Let $D\sigma \otimes \sigma^*$ and $\sigma \otimes D^*\sigma^*$ denote the images of $(D\sigma, \sigma^*)$ and $(\sigma, D^*\sigma^*)$ under the previous maps.

[14] Let $\wedge : \Omega^p(X, \mathbb{K}) \times \mathscr{A}^q(E) \to \mathscr{A}^{p+q}(E)$ be the bilinear map defined by $(\omega, \eta \otimes \sigma) \mapsto (\omega \wedge \eta) \otimes \sigma$.

Definition 2.5.3. The linear map $F_D := D \circ D : \mathscr{A}^0(E) \to \mathscr{A}^2(E)$ is called the *curvature of* D.

The curvature also associates locally defined differential forms to (U, ψ).

Definition 2.5.4. The 2-forms $F_j^i \in \Omega^2(U, \mathbb{K})$, defined by $F_D e_i = F_i^j \otimes e_j$, are called the *local curvature forms induced by* (U, ψ).

The local curvature forms can be expressed in terms of the local connection forms. Namely, we have

$$\begin{aligned} F_D e_i &= D(A_i^j \otimes e_j) = dA_i^j \otimes e_j - A_i^j \wedge De_j \\ &= dA_i^j \otimes e_j - (A_i^j \wedge A_j^k) \otimes e_k = (dA_i^j + A_k^j \wedge A_i^k) \otimes e_j, \end{aligned} \tag{2.2}$$

i.e. $F_i^j = dA_i^j + A_k^j \wedge A_i^k$. Next, we will show that F_D can be identified with an $\mathrm{End}(E)$-valued 2-form. A straightforward verification shows $D(f \cdot \omega) = df \wedge \omega + f \cdot D\omega$ for all $f \in C^\infty(X, \mathbb{K})$ and $\omega \in \mathscr{A}^q(E)$. It follows that F_D satisfies

$$\begin{aligned} F_D(f \cdot \sigma) &= D(df \otimes \sigma + f \cdot D\sigma) \\ &= d^2 f \otimes \sigma - df \wedge D\sigma + df \wedge D\sigma + f \cdot (D \circ D)\sigma \\ &= f \cdot F_D(\sigma), \end{aligned}$$

for all $f \in C^\infty(X, \mathbb{K})$ and $\sigma \in \mathscr{A}^0(E)$. Consider $F_D(\sigma)_x$ for some $\sigma \in \mathscr{A}^0(E)$ and $x \in U$. Define $\sigma^i : U \to \mathbb{K}$ by $\sigma|_U = \sigma^i e_i$. The identity above shows

$$F_D(\sigma)_x = \sigma^i(x) \cdot F_D(e_i)_x,$$

i.e. $F_D(\sigma)_x$ only depends on $\sigma(x)$. Therefore, we get a well-defined linear map $(F_D)_x : E_x \to \Lambda^2(T_x^*X) \otimes E_x$ defined by $(F_D)_x(e) = F_D(\sigma)_x$, where $\sigma \in \mathscr{A}^0(E)$ is any section with $\sigma(x) = e$. Proposition 2.2.4 shows that we can identify $(F_D)_x$ with an element of $\Lambda^2(T_x^*X) \otimes E_x \otimes E_x^* = (\Lambda^2(T^*X) \otimes \mathrm{End}(E))_x$. So, F_D can be identified with a section of $\Lambda^2(T^*X) \otimes \mathrm{End}(E)$, namely $x \mapsto (F_D)_x$. Thus, we may write $F_D \in \mathscr{A}^2(\mathrm{End}(E))$. The fact that we can consider F_D to be an element of $\mathscr{A}^2(\mathrm{End}(E))$ allows us to formulate the following theorem.

Theorem 2.5.5. The curvature $F_D \in \mathscr{A}^2(\mathrm{End}(E))$ satisfies $\mathrm{End}(D)F_D = 0$. This property is called the *Bianchi identity*.

Proof. See [10, p. 542]. □

Now assume that $(E, \pi, \mathscr{C})$ is a G-bundle for some matrix Lie group $G \subseteq \mathrm{GL}(k, \mathbb{K})$ and let $\mathfrak{g}$ denote the corresponding Lie algebra.

Definition 2.5.6. A connection D on E is called a *G-connection* if the following holds for all local trivialisations $(U, \psi) \in \mathscr{C}$: let A_i^j be the local connection forms induced by (U, ψ). The matrix $A_{x,v} \in \mathrm{Mat}(k, \mathbb{K})$, defined by $(A_{x,v})_i^j = (A_i^j)_x(v)$, is an element of $\mathfrak{g}$ for all $x \in U$ and $v \in T_xX$.

Keep in mind: even though we do not explicitly mention it, whether or not a connection is a G-connection depends on the trivialising cover.

Example 2.5.7. Suppose that $(E, \pi, \mathscr{C})$ is a real $\mathrm{SO}(k)$-bundle and let D be an $\mathrm{SO}(k)$-connection on E. Recall that the Lie algebra of $\mathrm{SO}(k)$ is equal to the subspace of antisymmetric matrices. So, the local connection forms A_i^j induced by $(U, \psi) \in \mathscr{C}$ satisfy $A_i^j = -A_j^i$ for all $(U, \psi) \in \mathscr{C}$. Equation (2.2) proves that the local curvature forms also satisfy $F_i^j = -F_j^i$. Let $u, v \in T_xX$ be tangent vectors at $x \in X$. Unwinding some identifications shows that $(F_i^j)_x(u, v)$ is just the (i, j)-th entry of the matrix representation of $(F_D)_x(u, v) \in \mathrm{End}(E_x)$ in[15] $e_1(x), \ldots, e_k(x)$. So, the matrix representation of $(F_D)_x(u, v)$ is antisymmetric in an orthonormal basis[16]. Therefore, we have $(F_D)_x(u, v) \in \mathfrak{so}(E_x)$, i.e. F_D is a section of $\Lambda^2(T^*X) \otimes \mathfrak{so}(E)$. By definition, F_D is an element of $\mathscr{A}^2(\mathfrak{so}(E))$.

2.6 Hodge star operator

The Hodge star operator is a linear map $*$ from $\Lambda^k(V)$ to $\Lambda^{n-k}(V)$, where V is a real n-dimensional vector space. So, if $n = 4$ and $k = 2$, we see that $*$ is a linear map from $\Lambda^2(V)$ to itself. It turns out that $*^2 = 1$ or $*^2 = -1$, from which we can deduce that $*$ can only have two possible eigenvalues. This gives rise to a notion of self- and anti-self-duality. In this section, we will explain how the Hodge star operator is defined and consider some of its properties. We will also consider the Hodge star operator in the context of smooth manifolds.

Let V be an n-dimensional $\mathbb{R}$-vector space and g a metric on V. Also, let k be an integer with $1 \leq k \leq n-1$. Using the universal property of exterior powers twice, one can show that there exists a unique bilinear map $\langle \cdot, \cdot \rangle_g : \Lambda^k(V) \times \Lambda^k(V) \to \mathbb{R}$ satisfying

$$\langle v_1 \wedge \ldots \wedge v_k, v_1' \wedge \ldots \wedge v_k' \rangle_g = \det\big(g(v_i, v_j')\big). \tag{2.3}$$

Using an orthonormal basis of V, it is straightforward to check that $\langle \cdot, \cdot \rangle_g$ is symmetric and non-degenerate, i.e. $\langle \cdot, \cdot \rangle_g$ defines a metric on $\Lambda^k(V)$. Let ω be an orientation on V.

Proposition 2.6.1. There exists a unique element $\mathrm{vol}(g, \omega) \in \Lambda^n(V)$ with

$$\big|\langle \mathrm{vol}(g, \omega), \mathrm{vol}(g, \omega) \rangle_g\big| = 1 \qquad \text{and} \qquad \mathrm{vol}(g, \omega) \in [\omega].$$

We call $\mathrm{vol}(g, \omega) \in \Lambda^n(V)$ the *volume form of g and ω*.

[15] $e_1, \ldots, e_k : U \to E$ denotes the local frame induced by $(U, \psi) \in \mathscr{C}$.

[16] Recall that an $\mathrm{SO}(k)$-bundle is naturally equipped with a Riemannian metric. Also, the local frames induced by local trivialisations are orthonormal with respect to this metric.

Proof. Let $\nu \in [\omega]$ be a nonzero n-form. Write $c = 1/\sqrt{|\langle \nu, \nu \rangle_g|}$. Clearly, $c \cdot \nu$ satisfies both conditions, which proves existence. Let $\nu' \in \Lambda^n(V)$ be another element satisfying the conditions above and write $\nu' = \lambda \cdot (c \cdot \nu)$. The first condition shows $\lambda^2 = 1$, i.e. $\lambda = \pm 1$. The second condition shows $\lambda = 1$. Therefore, we have proved uniqueness. □

Note that $\mathrm{vol}(g, \omega)$ only depends on ω through $[\omega]$. Therefore, we will occasionally write $\mathrm{vol}(g, [\omega])$ instead of $\mathrm{vol}(g, \omega)$. The definition of $\mathrm{vol}(g, \omega)$ shows that we have $\mathrm{vol}(g, \omega) = e_1 \wedge \ldots \wedge e_n$ for all oriented orthonormal bases $e_1, \ldots, e_n$ of V.

Theorem 2.6.2. There exists a unique linear map $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$ satisfying

$$\eta \wedge (*\eta') = \langle \eta, \eta' \rangle_g \cdot \mathrm{vol}(g, \omega)$$

for all $\eta, \eta' \in \Lambda^k(V)$. We call $*$ the *Hodge star operator induced by g and ω.*

Proof. See [10, p. 408]. □

Sometimes, we will write $*_{g,\omega}$ or $*_{g,[\omega]}$ to stress the dependence of $*$ on g and ω. The previous definition of the Hodge star operator is very non-constructive. However, in an oriented orthonormal basis the Hodge star operator is easily computed.

Proposition 2.6.3. Let $e_1, \ldots, e_n$ be an oriented orthonormal basis of V and write $\epsilon_i = g(e_i, e_i)$. Also, let $i_1, \ldots, i_k \in \{1, \ldots, n\}$ be distinct integers and write $\{i_{k+1}, \ldots, i_n\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. We have

$$*(e_{i_1} \wedge \ldots \wedge e_{i_k}) = \pm(\epsilon_{i_1} \cdot \ldots \cdot \epsilon_{i_k}) e_{i_{k+1}} \wedge \ldots \wedge e_{i_n},$$

where the sign is chosen such that $\pm e_{i_1} \wedge \ldots \wedge e_{i_n} = e_1 \wedge \ldots \wedge e_n$.

Proof. See [10, p. 409]. □

Assume for the moment that $n = 4$ and $k = 2$. In this situation, $*$ is a linear map from $\Lambda^2(V)$ to itself. Let $e_1, \ldots, e_4$ be an oriented orthonormal basis of V.

Remark 2.6.4. Consider $\lambda \cdot g$ for some $\lambda > 0$ and note that the $\tilde{e}_i = e_i/\sqrt{\lambda}$ form an oriented $(\lambda \cdot g)$-orthonormal basis. Write $\{i_1, \ldots, i_4\} = \{1, \ldots, 4\}$. Proposition 2.6.3 shows

$$*_{\lambda \cdot g, \omega}(\tilde{e}_{i_1} \wedge \tilde{e}_{i_2}) = \pm(\epsilon_{i_1} \cdot \epsilon_{i_2}) \tilde{e}_{i_3} \wedge \tilde{e}_{i_4},$$

where the sign is such that $\pm \tilde{e}_{i_1} \wedge \ldots \wedge \tilde{e}_{i_4} = \tilde{e}_1 \wedge \ldots \wedge \tilde{e}_4$. Cancelling factors of $\sqrt{\lambda}$ on both sides of the previous equations shows that $*_{g,\omega}$ and $*_{\lambda \cdot g, \omega}$ must be equal[17]. So, all the metrics in the conformal class $[g]$ determine the same Hodge star operator. We also see

$$\mathrm{vol}(\lambda \cdot g, \omega) = \tilde{e}_1 \wedge \ldots \wedge \tilde{e}_4 = e_1 \wedge \ldots \wedge e_4/\lambda^2 = \mathrm{vol}(g, \omega)/\lambda^2. \tag{2.4}$$

[17] This reasoning also shows $*_{\lambda \cdot g, \omega} = *_{g,\omega}$ for $\lambda < 0$. We only have to replace $\sqrt{\lambda}$ by $\sqrt{|\lambda|}$.

Now assume that g is Riemannian. It turns out[18] that $*^2 = 1$. So, the possible eigenvalues of $*$ are ± 1. An element $\eta \in \Lambda^2(V)$ is called *self-dual* if $*\eta = \eta$ and *anti-self-dual* if $*\eta = -\eta$. Let $\Lambda^2_+(V)$ denote[19] the subspace of self-dual 2-forms and $\Lambda^2_-(V)$ the subspace of anti-self-dual 2-forms. Note that every element $\eta \in \Lambda^2(V)$ can be written as $\eta = \eta_+ + \eta_-$, where $\eta_\pm = (\eta \pm *\eta)/2$. Using $*^2 = 1$, we see that η_+ is self-dual and η_- is anti-self-dual. Clearly, we also have $\Lambda^2_+(V) \cap \Lambda^2_-(V) = \{0\}$. Therefore

$$\Lambda^2(V) = \Lambda^2_+(V) \oplus \Lambda^2_-(V). \tag{2.5}$$

Example 2.6.5. Consider the following 2-forms

$$\Sigma_1^\pm = e_1 \wedge e_2 \pm e_3 \wedge e_4, \qquad \Sigma_2^\pm = e_1 \wedge e_3 \mp e_2 \wedge e_4, \qquad \Sigma_3^\pm = e_1 \wedge e_4 \pm e_2 \wedge e_3.$$

Using Proposition 2.6.3, one can check that the Σ_i^+ are self-dual and the Σ_i^- are anti-self-dual. Also, note that the $\Sigma_i^\pm$ are independent. Since $\dim \Lambda^2(V) = \binom{4}{2} = 6$, (2.5) shows $\Lambda^2_\pm(V) = \operatorname{span}\{\Sigma_1^\pm, \Sigma_2^\pm, \Sigma_3^\pm\}$, i.e. $\dim \Lambda^2_\pm(V) = 3$.

Suppose now that g is Lorentzian. It turns out[18] that $*^2 = -1$. Therefore, $*$ has no real eigenvalues. However, we still want a notion of self- and anti-self-duality. This is achieved by considering the complexification $\Lambda^2(V, \mathbb{C})$ of $\Lambda^2(V)$. Note that we can extend $* : \Lambda^2(V) \to \Lambda^2(V)$ to a $\mathbb{C}$-linear map

$$* : \Lambda^2(V, \mathbb{C}) \to \Lambda^2(V, \mathbb{C}), \; z \otimes \eta \mapsto z \otimes (*\eta).$$

Since $*^2 = -1$, the possible eigenvalues of $*$ are $\pm \mathrm{i}$. Again, we call $\eta \in \Lambda^2(V, \mathbb{C})$ self-dual if $*\eta = \mathrm{i}\eta$ and anti-self-dual if $*\eta = -\mathrm{i}\eta$ and we let $\Lambda^2_+(V, \mathbb{C})$ denote the subspace of self-dual 2-forms and $\Lambda^2_-(V, \mathbb{C})$ the subspace of anti-self-dual 2-forms. As before, we have $\Lambda^2(V, \mathbb{C}) = \Lambda^2_+(V, \mathbb{C}) \oplus \Lambda^2_-(V, \mathbb{C})$ and $\dim \Lambda^2_\pm(V, \mathbb{C}) = 3$.
Let n and $1 \le k \le n-1$ be arbitrary again. In differential geometry, we want to consider the Hodge star operator on $\Lambda^k(V^*)$, where V is the tangent space. However, usually we will be given a metric on V, not on V^*. So, we need to define a metric on V^* in terms of a metric on V.

Definition 2.6.6. Write $\sharp_g = \flat_g^{-1} : V^* \to V$. The metric $g^{-1} : V^* \times V^* \to \mathbb{R}$, defined by

$$g^{-1}(\alpha, \beta) = g(\sharp_g \alpha, \sharp_g \beta),$$

is called the *inverse metric of g*.

Let $\omega \in \Lambda^n(V^*)$ be an orientation on V^*. To ease the notation, we will write $\mathrm{vol}(g, \omega)$ and $*_{g,\omega}$ instead of $\mathrm{vol}(g^{-1}, \omega)$ and $*_{g^{-1},\omega}$. Next, we will consider the Hodge star operator in the context of smooth manifolds.

[18] Proposition 2.6.3 can be used to show this. Also, it can be found in [10, p. 410].
[19] Occasionally, we will write $\Lambda^2_\pm(V, \mathbb{R})$ instead of $\Lambda^2_\pm(V)$.

Let X be an n-dimensional smooth manifold equipped with a metric g and an orientation ω. The linear map $*_{g,\omega} : \Omega^k(X,\mathbb{K}) \to \Omega^{n-k}(X,\mathbb{K})$, defined by

$$(*_{g,\omega}\eta)(x) = *_{g(x),\omega(x)}\eta(x),$$

is called the *Hodge star operator induced by g and ω*. As before, instead of $*_{g,\omega}$ we will often drop the subscripts and simply write $*$. Also, we let $\mathrm{vol}(g,\omega) \in \Omega^n(X,\mathbb{R})$ be defined by $\mathrm{vol}(g,\omega)_x = \mathrm{vol}(g(x),\omega(x))$. Note that the definition of the Hodge star operator can be extended to vector bundle-valued forms. Let $(E,\pi,\mathscr{C})$ be a $\mathbb{K}$-vector bundle over X and define

$$* : \mathscr{A}^k(E) \to \mathscr{A}^{n-k}(E),\ \eta \otimes \sigma \mapsto (*\eta) \otimes \sigma.$$

Now assume $n = 4$ and $k = 2$. As before, the previous definitions give rise to a notion of self- and anti-self-duality. Let[20] $\Lambda^2_+(T^*_xX,\mathbb{K})$ denote the subspace of self-dual 2-forms of the Hodge star operator induced by $g(x)$ and $\omega(x)$. The vector bundle[21]

$$\Lambda^2_+(T^*X,\mathbb{K}) = \coprod_{x\in X} \Lambda^2_+(T^*_xX,\mathbb{K})$$

is called the *bundle of $\mathbb{K}$-valued self-dual 2-forms induced by g and ω*.

[20] This only makes sense if $\mathbb{K} = \mathbb{R}$ when $*^2 = 1$ and $\mathbb{K} = \mathbb{C}$ when $*^2 = -1$.

[21] Strictly speaking, we have only defined $\Lambda^2_+(T^*X,\mathbb{K})$ as a set. The projection is defined as the unique map $\pi : \Lambda^2_+(T^*X,\mathbb{K}) \to X$ with $\pi^{-1}(x) = \Lambda^2_+(T^*_xX,\mathbb{K})$ and Lemma 2.4.6 can be used to define a topology and smooth structure on $\Lambda^2_+(T^*X,\mathbb{K})$.

3 General relativity

The main subject of this chapter is the reformulation of Einstein's equation presented in [6]. Before discussing this new formulation, we introduce the usual formulation of Einstein's equation. After this, we derive some technical results needed for the new formulation. These results will allow us to construct a metric from a so called definite connection. This can be used to relate the formalism presented in [6] to the usual formulation of Einstein's equation, i.e. this allows us to argue that the new formulation is indeed a reformulation of Einstein's equation. Finally, we will give a detailed explanation of the reformulation.
A solution of Einstein's equation in the usual formulation is a metric on a 4-dimensional smooth manifold. From a mathematical point of view, such a metric is allowed to have any signature. In section 3.3, we will discuss a reformulation of Einstein's equation for Riemannian metrics, i.e. metrics of signature $(4,0)$. The metrics of physical significance however, are of Lorentzian signature. Therefore, the technical results needed for the reformulation will not only be considered for Riemannian metrics but also for Lorentzian metrics. In section 3.4, we discuss how these technical results for Lorentzian metrics can be used to modify the formalism of section 3.3 in an attempt to make it suitable for Lorentzian metrics.

3.1 Einstein 4-manifolds

In this section, the usual formulation of Einstein's equation is introduced. In this formulation, the main variable is a metric on a 4-dimensional smooth manifold X called *spacetime*. To this metric, we associate the so called Levi-Civita connection. Using the curvature of the Levi-Civita connection, we can derive two objects: the Ricci tensor and the scalar curvature. Einstein's equation is then easily formulated in terms of the Ricci tensor and the metric. Finally, we will compute the scalar curvature of a metric that solves Einstein's equation.
Let X be a 4-dimensional smooth manifold, $\omega \in \Omega^4(X,\mathbb{R})$ an orientation and g a

Riemannian metric on X. As discussed[1] in the previous chapter, the orientation and metric allow us to construct a trivialising cover $\mathscr{C}$ such that $(TX, \pi_T, \mathscr{C})$ is an SO(4)-bundle. To formulate Einstein's equation we need to consider a special type of connection on TX. This connection is defined using the following definition.

Definition 3.1.1. Let D be a connection on TX. The bilinear map

$$T_D : \mathscr{A}^0(TX) \times \mathscr{A}^0(TX) \to \mathscr{A}^0(TX),$$

defined[2] by $T_D(V,W) = D_V W - D_W V - [V,W]$, is called the *torsion* of D. A connection on TX with vanishing torsion is called *torsion free*.

The previous definition allows us to define the desired connection.

Definition 3.1.2. A torsion free SO(4)-connection D on TX is called a[3] *g-Levi-Civita connection*.

It turns out that the following holds: among all the SO(4)-connections on TX there is a unique connection that is also torsion free, i.e. there exists a unique g-Levi-Civita connection ∇ on TX. The proof of this can be found in [10, p. 550].
Let F_∇ denote the curvature of ∇ and note that it is a section of $\Lambda^2(T^*X) \otimes \mathrm{End}(TX)$. Since $\Lambda^2(T_x^*X)$ is canonically isomorphic[4] to a subspace of $T_x^*X \otimes T_x^*X$, it follows that we can identify a section of $\Lambda^2(T^*X)$ with a section of $T^*X \otimes T^*X$. Also, by definition $\mathrm{End}(TX) = TX \otimes T^*X$. After switching the order of $\Lambda^2(T^*X)$ and $\mathrm{End}(TX)$, we see that F_∇ can be identified with a section of[5] $TX \otimes T^*X^{\otimes 3}$. Now consider the following bundle map

$$C_2^1 : TX \otimes T^*X^{\otimes 3} \to T^*X \otimes T^*X,\ v_1 \otimes \alpha^1 \otimes \alpha^2 \otimes \alpha^3 \mapsto \alpha^2(v_1) \cdot \alpha^1 \otimes \alpha^3.$$

Definition 3.1.3. The section $\mathrm{Ric}(g) : X \to T^*X \otimes T^*X$, defined by $\mathrm{Ric}(g) = C_2^1(F_\nabla)$, is called the *Ricci tensor of g*.

It is now straightforward to write down Einstein's equation.

Definition 3.1.4. A Riemannian metric g on X with

$$\mathrm{Ric}(g) = \Lambda \cdot g, \tag{3.1}$$

for some $\Lambda \in \mathbb{R}$, is called an *Einstein metric*.

[1] See Example 2.4.15. Keep in mind that an orientation on X is an orientation on T^*X, not on TX. However, there exists a canonical bijection between orientations on a vector space and orientations on its dual (see section 2.2). Therefore, the reasoning in Example 2.4.15 is still valid.

[2] $[V,W]$ denotes the *Lie bracket of V and W*. A definition can be found in [11, p. 90].

[3] Note that $\mathscr{C}$ depends on g. So, whether or not a connection on TX is an SO(4)-connection depends on g. In this definition, we made the dependence on g explicit.

[4] This was shown in Remark 2.2.6.

[5] We write $T^*X^{\otimes 3}$ instead of $T^*X \otimes T^*X \otimes T^*X$.

Equation (3.1) is called *Einstein's equation in vacuum* or just *Einstein's equation*. Suppose that g satisfies $\mathrm{Ric}(g) = \Lambda \cdot g$. We call (X, g) an *Einstein 4-manifold* and Λ the *cosmological constant of* g. Let $C_1^1 \circ \sharp_g^1 : T^*X \otimes T^*X \to \mathbb{R}$ be the map defined by

$$(C_1^1 \circ \sharp_g^1)(\alpha^1 \otimes \alpha^2) = \alpha^2(\sharp_{g(x)}\alpha^1),$$

where $\alpha^1 \otimes \alpha^2 \in T_x^*X \otimes T_x^*X$.

Definition 3.1.5. The map $s_g : X \to \mathbb{R}$, defined by $s_g = (C_1^1 \circ \sharp_g^1) \circ \mathrm{Ric}(g)$, is called the *scalar curvature of* g.

Suppose that g is an Einstein metric with cosmological constant Λ and let $e_1, \dots, e_4$ be a $g(x)$-orthonormal basis of T_xX. Also, let $e^1, \dots, e^4$ denote the corresponding dual basis. We have[6] $g(x) = \sum_{i=1}^4 e^i \otimes e^i$ and $\sharp_{g(x)} e^i = e_i$. So

$$(C_1^1 \circ \sharp_g^1)(g(x)) = \sum_{i=1}^{4} e^i(e_i) = 4.$$

Since $\mathrm{Ric}(g) = \Lambda \cdot g$, it follows that

$$s_g = (C_1^1 \circ \sharp_g^1) \circ \mathrm{Ric}(g) = \Lambda \cdot (C_1^1 \circ \sharp_g^1) \circ g = 4\Lambda, \tag{3.2}$$

where 4Λ denotes the constant map $x \mapsto 4\Lambda$. Of course, we would also like to formulate Einstein's equation for metrics of Lorentzian signature. This can be done in the same way as above: we only have to replace SO(4) by SO(1, 3).

3.2 Urbantke metric

A metric g on an oriented 4-dimensional $\mathbb{R}$-vector space V determines a 3-dimensional subspace of $\Lambda^2(V^*, \mathbb{K})$, namely the subspace of self-dual 2-forms $\Lambda^2_+(V^*, \mathbb{K})$. As discussed in the previous chapter, scaling the metric with a nonzero real number does not change the corresponding Hodge star operator on the 2-forms, i.e. all the metrics in the conformal class $[g]$ determine the same subspace of self-dual 2-forms. Conversely, one might ask whether a 3-dimensional subspace $S \subseteq \Lambda^2(V^*, \mathbb{K})$ uniquely determines a conformal class and orientation. In this section, we will show that this is indeed the case if S satisfies some conditions. The so called Urbantke metric will give us a way to explicitly construct a metric in the desired conformal class. In the next section, we wish to construct a Riemannian metric on a smooth manifold such that the curvature of a given connection is self-dual. Therefore, we will also discuss the previous topics in the context of smooth manifolds.

[6] We are writing an explicit summation here because the summed over indices are both superscripts.

Definition 3.2.1. Let $v \in V$ be a vector. Also, let $k \geq 2$ be an integer and define $\iota_v : \Lambda^k(V^*, \mathbb{K}) \to \Lambda^{k-1}(V^*, \mathbb{K})$ by[7]

$$(\iota_v \omega)(v_1, \ldots, v_{k-1}) = \omega(v, v_1, \ldots, v_{k-1})$$

for all $v_1, \ldots, v_{k-1} \in V$. We call ι_v the *interior product*.

Definition 3.2.2. Let $T = (\Sigma^1, \Sigma^2, \Sigma^3)$ be a triple of $\mathbb{K}$-valued 2-forms on V^* and let $\omega \in \Lambda^4(V^*, \mathbb{K})$ be a nonzero 4-form. Define an $\mathbb{R}$-bilinear map $\mathrm{Urb}(T, \omega) : V \times V \to \mathbb{K}$ by

$$\mathrm{Urb}(T, \omega)(v, w) \cdot \omega = \epsilon_{ijk} \cdot \iota_v \Sigma^i \wedge \iota_w \Sigma^j \wedge \Sigma^k,$$

where ϵ_{ijk} denotes the *Levi-Civita symbol*. We call $\mathrm{Urb}(T, \omega)$ the *Urbantke metric of T and ω*. The Urbantke metric is defined analogously in the context of smooth manifolds, i.e. when $\Sigma^1, \Sigma^2, \Sigma^3$ and ω are differential forms on some 4-dimensional smooth manifold.

Define $\tilde{\Sigma}^i = A^i_j \Sigma^j$ for some matrix $A \in \mathrm{Mat}(3, \mathbb{K})$. Write $\tilde{T} = (\tilde{\Sigma}^1, \tilde{\Sigma}^2, \tilde{\Sigma}^3)$. One can check that $\mathrm{Urb}(\tilde{T}, \omega) = \det A \cdot \mathrm{Urb}(T, \omega)$. Note that the Urbantke metric need not define a metric at all. However, imposing a number of conditions on T will ensure that $\mathrm{Urb}(T, \omega)$ does define a metric.
Let $\omega \in \Lambda^4(V^*, \mathbb{K})$ be a nonzero 4-form and consider the $\mathbb{K}$-bilinear map $\langle \cdot, \cdot \rangle_\omega$: $\Lambda^2(V^*, \mathbb{K}) \times \Lambda^2(V^*, \mathbb{K}) \to \mathbb{K}$ defined by

$$\langle F, G \rangle_\omega \cdot \omega = F \wedge G.$$

Now assume $\mathbb{K} = \mathbb{R}$ and let $S \subseteq \Lambda^2(V^*)$ be a subspace. We call S a *definite subspace* if $\langle \cdot, \cdot \rangle_\omega|_{S \times S}$ is definite, i.e. positive- or negative-definite, for all nonzero 4-forms $\omega \in \Lambda^4(V^*)$.

Theorem 3.2.3. Let $S \subseteq \Lambda^2(V^*)$ be a definite 3-dimensional subspace. Then there exists a unique element $(C, \mathfrak{o}) \in \mathscr{C}^{4,0}(V) \times \mathfrak{O}(V^*)$ with

(1) The subspace of self-dual 2-forms of the Hodge star operator induced by C and $\mathfrak{o}$ is equal to S.

Let $\nu \in \Lambda^4(V^*)$ be a nonzero 4-form. The previous shows that there exists a unique element $(g, \mathfrak{o}) \in \mathscr{M}^{4,0}(V) \times \mathfrak{O}(V^*)$ with

(2) The subspace of self-dual 2-forms of $*_{g,\mathfrak{o}}$ is equal to S.

(3) $\mathrm{vol}(g, \mathfrak{o}) = \pm \nu$.

Proof. Note that we can pick $\omega \in \Lambda^4(V^*) \setminus \{0\}$ to be such that $\langle \cdot, \cdot \rangle_\omega|_{S \times S}$ is positive-definite. In [4], it is shown that a conformal class $C \in \mathscr{C}^{4,0}(V)$ exists such that $(C, [\omega])$ satisfies (1).

[7] Note that we can identify a k-form with an alternating $\mathbb{R}$-multilinear map from V^k to $\mathbb{K}$.

Suppose that $(C', \mathfrak{o}')$ also satisfies (1). In [8], it is shown that mapping an element of $\mathscr{C}^{4,0}(V) \times \mathfrak{O}(V^*)$ to the corresponding Hodge star operator is one-to-one. So, we have proven the first statement of this theorem if we show that $(C, [\omega])$ induces the same Hodge star operator as $(C', \mathfrak{o}')$. Let $*$ be the Hodge star operator induced by $(C, [\omega])$ and $*'$ the Hodge star operator induced by $(C', \mathfrak{o}')$. We already know that

$$*|_S = \mathrm{id}_S = *'|_S.$$

Using the defining property of the Hodge star operator, one can show that the subspace of anti-self-dual 2-forms is equal to

$$\{\eta \in \Lambda^2(V^*) : \eta \wedge \eta' = 0 \text{ for all self-dual 2-forms } \eta'\},$$

i.e. it is uniquely determined by the subspace of self-dual 2-forms. Therefore, $*$ and $*'$ determine the same subspace of anti-self-dual 2-forms $S^\perp$. This shows

$$*|_{S^\perp} = -\mathrm{id}_{S^\perp} = *'|_{S^\perp}.$$

Since $\Lambda^2(V^*) = S \oplus S^\perp$, it follows that $* = *'$. The second statement of this theorem follows from the fact that $\Lambda^4(V^*)$ is 1-dimensional and[8] $\mathrm{vol}(c \cdot g, \omega) = c^2 \cdot \mathrm{vol}(g, \omega)$ for all $c > 0$ and $g \in \mathscr{M}^{4,0}(V)$. □

The previous theorem can be used to show that the Urbantke metric allows us to explicitly construct a metric in the desired conformal class.

Corollary 3.2.4. Let $S \subseteq \Lambda^2(V^*)$ be a definite 3-dimensional subspace and let $\omega \in \Lambda^4(V^*)$ be a nonzero 4-form such that $\langle \cdot, \cdot \rangle_\omega|_{S \times S}$ is positive-definite. Also, let $\tilde{\Sigma}^1, \tilde{\Sigma}^2, \tilde{\Sigma}^3$ be a basis of S and write $\tilde{T} = (\tilde{\Sigma}^1, \tilde{\Sigma}^2, \tilde{\Sigma}^3)$. Then $\mathrm{Urb}(\tilde{T}, \omega)$ is a positive- or negative-definite metric and $(\mathrm{Urb}(\tilde{T}, \omega), [\omega])$ satisfies (2) from Theorem 3.2.3.

Proof. As was shown in the proof of Theorem 3.2.3, there exists a $g \in \mathscr{M}^{4,0}(V)$ such that $(g, [\omega])$ satisfies (2) from Theorem 3.2.3. Let $e^1, \ldots, e^4$ be an oriented g^{-1}-orthonormal basis of $(V^*, [\omega])$. Consider the following forms:

$$\Sigma^1 = e^1 \wedge e^2 + e^3 \wedge e^4, \qquad \Sigma^2 = e^1 \wedge e^3 - e^2 \wedge e^4, \qquad \Sigma^3 = e^1 \wedge e^4 + e^2 \wedge e^3.$$

Example 2.6.5 shows that the Σ^i are self-dual with respect to $*_{g,\omega}$, i.e. the Σ^i form a basis for S. Write $T = (\Sigma^1, \Sigma^2, \Sigma^3)$. A long, yet straightforward, calculation shows that

$$\mathrm{Urb}(T, \mathrm{vol}(g, \omega)) = 6 \cdot g.$$

Now, let $A \in \mathrm{GL}(3, \mathbb{R})$ be such that $\tilde{\Sigma}^i = A^i_j \Sigma^j$. Also, write $\mathrm{vol}(g, \omega) = c \cdot \omega$ for some $c > 0$. We have

$$\begin{aligned} \mathrm{Urb}(\tilde{T}, \omega) &= \det A \cdot \mathrm{Urb}(T, \omega) = \det A \cdot (c \cdot \mathrm{Urb}(T, c \cdot \omega)) \\ &= c \cdot \det A \cdot \mathrm{Urb}(T, \mathrm{vol}(g, \omega)) = (6c \cdot \det A) \cdot g. \end{aligned}$$

[8] This identity follows from $(c \cdot g)^{-1} = c^{-1} \cdot g^{-1}$ and Remark 2.6.4.

Recall[9] that multiplying the metric by a nonzero real number does not change the corresponding Hodge star operator on the 2-forms. Therefore, we are done. □

Next, it will be shown how the derived results can be used in the context of smooth manifolds. The following theorem and corollary will turn out to be crucial ingredients for the reformulation explained in the next section. Let X be a connected 4-dimensional smooth manifold.

Theorem 3.2.5. Let $\mathscr{C} = \{(U_i, T_i) : i \in I\}$ be a set with the following properties:

- $X = \bigcup_{i\in I} U_i$ and $U_i \subseteq X$ is a connected open for all $i \in I$.
- $T_i = (\Sigma_i^1, \Sigma_i^2, \Sigma_i^3)$ and the Σ_i^α are elements of $\Omega^2(U_i, \mathbb{R})$.
- Let $i \in I$ be an index and $x \in U_i$ a point. The subspace

$$S_{x,i} = \operatorname{span}\left\{\Sigma_i^1(x), \Sigma_i^2(x), \Sigma_i^3(x)\right\}$$

  is definite and 3-dimensional. Let $j \in I$ be an index with $x \in U_j$. We have $S_{x,j} = S_{x,i}$. So, define $S_x = S_{x,i}$.

Also, let $\nu \in \Omega^4(X, \mathbb{R})$ be a volume form on X. There exists a unique sign $\mathfrak{s} \in \{\pm 1\}$ and a unique Riemannian metric g on X with:

- The Σ_i^α are self-dual with respect to $*_{g,\mathfrak{s}\cdot\nu}$.
- $\operatorname{vol}(g, \mathfrak{s}\cdot\nu) = \mathfrak{s}\cdot\nu$.

Proof. We can apply the second part of theorem 3.2.3 to S_x and $\nu(x)$ to get an element $(g_x, \mathfrak{o}_x) \in \mathscr{M}^{4,0}(T_xX) \times \mathfrak{O}(T_x^*X)$ for all $x \in X$. Also, let $\mathfrak{s}_x$ be the unique sign defined by $\mathfrak{s}_x \cdot \nu(x) \in \mathfrak{o}_x$. These definitions ensure that

- The $\Sigma_i^\alpha(x)$ are self-dual with respect to $*$, where $*$ is the Hodge star operator induced by g_x and $\mathfrak{s}_x \cdot \nu(x)$, for all $x \in U_i$.
- $\operatorname{vol}(g_x, \mathfrak{s}_x \cdot \nu(x)) = \mathfrak{s}_x \cdot \nu(x)$ for all $x \in X$.

Therefore, it is sufficient to show that $x \mapsto \mathfrak{s}_x$ is a constant map and $x \mapsto g_x$ is a smooth map. Write $g : X \to T^*X \otimes T^*X,\ x \mapsto g_x$.
Let (U_i, T_i) be an element of $\mathscr{C}$. Define $X_\nu : U_i \to \operatorname{Mat}(3, \mathbb{R})$ by $\Sigma_i^\alpha \wedge \Sigma_i^\beta = X_\nu^{\alpha\beta} \cdot \nu|_{U_i}$. First note that $X_\nu(x)$ is just a matrix representation of $\langle\cdot,\cdot\rangle_{\nu(x)}|_{S_x \times S_x}$. So, since S_x is a definite subspace, it follows that $X_\nu(x)$ is a positive- or negative-definite matrix for all $x \in U_i$. Also, note that $\det(X_\nu) : U_i \to \mathbb{R}_{\neq 0}$ is smooth and therefore certainly continuous. Since U_i is connected, it follows that[10] $\operatorname{im}(\det(X_\nu))$ is also connected, i.e. $\det(X_\nu(x)) > 0$ for all $x \in U_i$ or $\det(X_\nu(x)) < 0$ for all $x \in U_i$. Therefore, there exists a sign $\mathfrak{s}$ such that $\det(X_{\mathfrak{s}\cdot\nu}(x)) = \mathfrak{s} \cdot \det(X_\nu(x)) > 0$ for all $x \in U_i$. Note that a

[9] This was shown in Remark 2.6.4.
[10] The image of $\det(X_\nu)$ is denoted by $\operatorname{im}(\det(X_\nu))$.

negative-definite 3×3 matrix has negative determinant, i.e. $X_{\mathfrak{s}\cdot\nu}$ must be positive-definite on the whole of U_i.
We can now apply Corollary 3.2.4. First note that $\mathrm{Urb}(T_i, \mathfrak{s}\cdot\nu|_{U_i})$ defines a smooth section $U_i \to T^*X \otimes T^*X$ because it is defined in terms of smooth sections $\Sigma_i^1, \Sigma_i^2, \Sigma_i^3$ and $\mathfrak{s}\cdot\nu|_{U_i}$. According to Corollary 3.2.4, $\tilde{g} = \mathrm{Urb}(T_i, \mathfrak{s}\cdot\nu|_{U_i})$ is a metric on U_i such that the Σ_i^α are self-dual with respect to the Hodge star operator induced by $\tilde{g}$ and $\mathfrak{s}\cdot\nu|_{U_i}$. It can be shown that the signature of a smooth metric is constant on a connected open[11], i.e. Corollary 3.2.4 also shows that $\tilde{g}(x)$ is positive-definite for all $x \in U_i$ or $\tilde{g}(x)$ is negative-definite for all $x \in U_i$. Thus, there exists a sign $\tilde{\mathfrak{s}}$ such that $\tilde{g}' = \tilde{\mathfrak{s}}\cdot\tilde{g}$ is a Riemannian metric on U_i.
Note that $\mathrm{vol}(\tilde{g}', \mathfrak{s}\cdot\nu|_{U_i}) = f\cdot(\mathfrak{s}\cdot\nu|_{U_i})$ for some smooth function $f : U_i \to \mathbb{R}_{>0}$. As argued before, the volume form of $g' = \tilde{g}'/\sqrt{f}$ is equal to

$$\mathrm{vol}(g', \mathfrak{s}\cdot\nu|_{U_i}) = \mathfrak{s}\cdot\nu|_{U_i}.$$

Because[12] of uniqueness, Theorem 3.2.3 shows that we must have $g' = g|_{U_i}$ and $\mathfrak{s} = \mathfrak{s}_x$ for all $x \in U_i$. Smoothness of g' and the fact that X is covered by the U_i now shows that g is smooth. We have also shown that $x \mapsto \mathfrak{s}_x$ is locally constant. Since X is connected, we can conclude that $x \mapsto \mathfrak{s}_x$ is a constant map. □

Corollary 3.2.6. Assume that X is orientable and let $\mathscr{C}$ be a set as in Theorem 3.2.5. There exists an orientation $\omega_{\mathscr{C}}$ on X (which is unique up to equivalence) and a unique conformal class $C_{\mathscr{C}} \in \mathscr{C}^{4,0}(TX)$ such that the Σ_i^α are self-dual with respect to the Hodge star operator induced by $C_{\mathscr{C}}$ and $\omega_{\mathscr{C}}$.

Proof. Let $\nu \in \Omega^4(X, \mathbb{R})$ be a volume form and let $\mathfrak{s}$ and g be as in Theorem 3.2.5. Define $\omega_{\mathscr{C}} = \mathfrak{s}\cdot\nu$ and $C_{\mathscr{C}} = [g]$. By construction, $\omega_{\mathscr{C}}$ and $C_{\mathscr{C}}$ are such that the Σ_i^α are self-dual with respect to the Hodge star operator induced by $C_{\mathscr{C}}$ and $\omega_{\mathscr{C}}$. Theorem 3.2.3 proves the uniqueness. □

The previous results are all about Riemannian metrics. Because the physically relevant metrics are of Lorentzian signature, we will now discuss similar results for Lorentzian metrics. As explained before, when working with metrics of Lorentzian signature we switch to $\mathbb{K} = \mathbb{C}$.
Let $S \subseteq \Lambda^2(V^*, \mathbb{C})$ be a subspace. We call S a *non-degenerate subspace* if it satisfies

- $\langle\cdot,\cdot\rangle_\omega|_{S\times S}$ is non-degenerate for all nonzero 4-forms $\omega \in \Lambda^4(V^*, \mathbb{C})$.
- $\Sigma \wedge \overline{\Sigma'} = 0$ for[13] all $\Sigma, \Sigma' \in S$.

[11] This can be found in [1, p. 62].

[12] Recall that multiplying a metric with a nowhere vanishing scalar function does not change the corresponding Hodge star operator on the 2-forms. So, the Σ_i^α are still self-dual with respect to the Hodge star operator defined by g' and $\mathfrak{s}\cdot\nu|_{U_i}$.

[13] The complexification $W_{\mathbb{C}}$ of a real vector space W always comes equipped with an $\mathbb{R}$-linear map $\overline{\cdot} : W_{\mathbb{C}} \to W_{\mathbb{C}}, z \otimes w \mapsto \bar{z} \otimes w$, where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Theorem 3.2.7. Let $S \subseteq \Lambda^2(V^*, \mathbb{C})$ be a non-degenerate 3-dimensional subspace. Then there exists a unique element $(C, \mathfrak{o}) \in \mathscr{C}^{1,3}(V) \times \mathfrak{O}(V^*)$ with

(1) The subspace of self-dual 2-forms of the Hodge star operator induced by C and $\mathfrak{o}$ is equal to S.

Let $\nu \in \Lambda^4(V^*)$ be a nonzero 4-form. The previous shows that there exists a unique element $(g, \mathfrak{o}) \in \mathscr{M}^{1,3}(V) \times \mathfrak{O}(V^*)$ with

(2) The subspace of self-dual 2-forms of $*_{g,\mathfrak{o}}$ is equal to S.

(3) $\mathrm{vol}(g, \mathfrak{o}) = \pm\nu$.

Proof. Let $\omega \in \Lambda^4(V^*)$ be an $\mathbb{R}$-valued nonzero 4-form. Note that $\langle \cdot, \cdot \rangle_\omega|_{S \times S}$ is symmetric and non-degenerate. So, because we are working over $\mathbb{C}$, there exists a basis $\tilde{\Sigma}^1, \tilde{\Sigma}^2, \tilde{\Sigma}^3$ of S with $\langle \tilde{\Sigma}^i, \tilde{\Sigma}^j \rangle_\omega = \delta^{ij}$. Let $s \in \mathbb{C}$ be a square root of 2i and define $\Sigma^i = s \cdot \tilde{\Sigma}^i$. We have

$$\Sigma^i \wedge \Sigma^j = \langle \Sigma^i, \Sigma^j \rangle_\omega \cdot \omega = s^2 \langle \tilde{\Sigma}^i, \tilde{\Sigma}^j \rangle_\omega \cdot \omega = 2\mathrm{i}\delta^{ij}\omega \tag{3.3}$$

and

$$\Sigma^i \wedge \overline{\Sigma}^j = 0. \tag{3.4}$$

In [13, p. 8], it is shown that if a triple of $\mathbb{C}$-valued 2-forms satisfies (3.3) and (3.4), then there exists a Lorentzian metric $g \in \mathscr{M}^{1,3}(V)$ such that

$$S = \mathrm{span}\{\Sigma^1, \Sigma^2, \Sigma^3\}$$

is the subspace of self- or anti-self-dual 2-forms corresponding to $*_{g,\omega}$. Since $\mathrm{vol}(g, -\omega) = -\mathrm{vol}(g, \omega)$, it follows that $*_{g,-\omega} = -*_{g,\omega}$. Therefore, there exists a sign $\mathfrak{s} \in \{\pm 1\}$ such that $(g, [\mathfrak{s} \cdot \omega])$ satisfies (2). So, $([g], [\mathfrak{s} \cdot \omega])$ satisfies (1). Uniqueness is proved in the same way as in Theorem 3.2.3, i.e. by using a result from [8]. The second statement of this theorem is again a consequence of the fact that $\Lambda^4(V^*)$ is 1-dimensional and $\mathrm{vol}(c \cdot g, \mathfrak{s} \cdot \omega) = c^2 \cdot \mathrm{vol}(g, \mathfrak{s} \cdot \omega)$ for all $c > 0$. □

Again, the Urbantke metric gives us an explicit way to compute a metric in the desired conformal class.

Theorem 3.2.8. Let $S \subseteq \Lambda^2(V^*, \mathbb{C})$ be a non-degenerate 3-dimensional subspace and let $\Sigma^1, \Sigma^2, \Sigma^3$ be a basis of S satisfying (3.3) for some nonzero 4-form $\omega \in \Lambda^4(V^*)$. Also, write $T = (\Sigma^1, \Sigma^2, \Sigma^3)$. Then there exists a unique $c \in \{\pm 1, \pm i\}$ such that $g = c \cdot \mathrm{Urb}(T, \omega)$ is a Lorentzian metric and S is the subspace of self- or anti-self-dual 2-forms corresponding to $*_{g,\omega}$.

Proof. See [13, p. 8]. □

Theorem 3.2.7 and Corollary 3.2.8 can be used to derive results similar to Theorem 3.2.5 and Corollary 3.2.6 for Lorentzian metrics. Of course, the set $\mathscr{C}$ in Theorem

3.2.5 will need to have slightly different properties: we demand S_x to be a non-degenerate subspace instead of a definite subspace and allow the Σ_i^α to be $\mathbb{C}$-valued, i.e. $\Sigma_i^\alpha \in \Omega^2(U_i, \mathbb{C})$. The idea of the proof is the same as before: define a global (not necessarily smooth) Lorentzian metric using Theorem 3.2.7 and use the Urbantke metric, i.e. Theorem 3.2.8, to prove local smoothness. Unlike before, we can not use Theorem 3.2.8 directly. This is because the Σ_i^α do not necessarily satisfy (3.3). A variant of the Gram-Schmidt process shows that appropriate linear combinations[14] of the Σ_i^α will satisfy (3.3) on a possibly smaller open set $U_i' \subseteq U_i$.

3.3 Riemannian reformulation

We discussed the usual formulation of Einstein's equation in section 3.1. The subject of this section is a different formulation, namely the one presented in [6]. In this formulation, the variable is not a metric, but a connection on a vector bundle. Given a connection D, we can construct a vector bundle-valued 2-form Φ_D. This vector bundle-valued 2-form may or may not satisfy $D\Phi_D = 0$. The previous equation will turn out to be the equivalent of Einstein's equation in the following sense: we associate a Riemannian metric g_D to the connection and this metric is Einstein if Φ_D satisfies $D\Phi_D = 0$. First, we will show how the metric associated to the connection is constructed. This construction will depend on a choice of a real number Λ. This real number will turn out to be the cosmological constant. After this, the construction of Φ_D will be explained. Finally, we will give a sketch of the proof of Theorem 3.3.2 and illustrate the formalism by applying it to an example.
Let X be a 4-dimensional smooth manifold and assume that X is orientable and connected. Also, let $(E, \pi, \mathscr{C})$ be a real SO(3)-bundle of rank 3 over X and D an SO(3)-connection on E. The curvature of D will be denoted by[15] $F_D \in \mathscr{A}^2(\mathfrak{so}(E))$. We now wish to construct a Riemannian metric g_D on X such that the so called *Yang-Mills equations* hold. The Yang-Mills equations are given by[16]

$$DF_D = 0 \qquad \text{and} \qquad D * F_D = 0.$$

Note that the first Yang-Mills equation always holds because of the Bianchi identity. If F_D happens to be self-dual, then $D * F_D = DF_D = 0$, i.e. both the Yang-Mills equations are satisfied automatically. Therefore, we will construct a Riemannian metric g_D and an orientation $\omega_X \in \Omega^4(X, \mathbb{R})$ such that F_D is self-dual with respect to the Hodge star operator induced by g_D and ω_X. In this situation, we call D a *self-dual instanton*.

[14] With this, we mean linear combinations with coefficients in $C^\infty(U_i', \mathbb{C})$.
[15] Recall that the curvature of an SO(3)-connection is $\mathfrak{so}(E)$-valued. See Example 2.5.7.
[16] For simplicity, we write D instead of $\mathrm{End}(D)$.

The first step in constructing the previously mentioned metric is constructing a conformal class of Riemannian metrics using Corollary 3.2.6. Let

$$\mathscr{C}_{\mathfrak{g}} = \{(U_i, \psi_i) : i \in I\}$$

be a trivialising cover of $\mathfrak{so}(E)$ with[17] connected trivialising neighbourhoods. Also, let $e_1^i, e_2^i, e_3^i : U_i \to \mathfrak{so}(E)$ be the local frame induced by (U_i, ψ_i) and write[18]

$$F_D|_{U_i} = F_i^1 \otimes e_1^i + F_i^2 \otimes e_2^i + F_i^3 \otimes e_3^i,$$

for some 2-forms $F_i^1, F_i^2, F_i^3 \in \Omega^2(U_i, \mathbb{R})$. Write $T_i = (F_i^1, F_i^2, F_i^3)$. Suppose that there exists a $j \in I$ with $U_i \cap U_j \neq \varnothing$ and let $x \in U_i \cap U_j$ be a point. If $A \in \mathrm{GL}(3, \mathbb{R})$ is the matrix defined by $e_\beta^j(x) = A_\beta^\alpha \cdot e_\alpha^i(x)$, then we have

$$\begin{aligned} F_i^\alpha(x) \otimes e_\alpha^i(x) = F_D(x) &= F_j^\beta(x) \otimes e_\beta^j(x) \\ &= F_j^\beta(x) \otimes A_\beta^\alpha \cdot e_\alpha^i(x) = A_\beta^\alpha \cdot F_j^\beta(x) \otimes e_\alpha^i(x), \end{aligned}$$

i.e. $F_i^\alpha(x) = A_\beta^\alpha \cdot F_j^\beta(x)$. This shows that

$$\mathrm{span}\{F_i^1(x), F_i^2(x), F_i^3(x)\} = \mathrm{span}\{F_j^1(x), F_j^2(x), F_j^3(x)\}.$$

So, define $S_x = \mathrm{span}\{F_i^1(x), F_i^2(x), F_i^3(x)\}$. Note that we can only apply Corollary 3.2.6 if S_x is a definite 3-dimensional subspace of $\Lambda^2(T_x^*X)$. This motivates the following definition.

Definition 3.3.1. We call D a *definite connection* if S_x is a definite 3-dimensional subspace of $\Lambda^2(T_x^*X)$ for all $x \in X$.

We will assume that D is a definite connection, i.e. $\{(U_i, T_i) : i \in I\}$ satisfies all the requirements of Corollary 3.2.6. Therefore, there exists an orientation $\omega_X \in \Omega^4(X, \mathbb{R})$ (which is unique up to equivalence) and a unique conformal class $C_D \in \mathscr{C}^{4,0}(TX)$ such that the F_i^α are self-dual with respect to the Hodge star operator induced by C_D and ω_X. Let $\Lambda_D^+(T^*X)$ denote the bundle of $\mathbb{R}$-valued self-dual 2-forms induced by C_D and ω_X and note that the fibre over $x \in X$ is equal to S_x. We see that F_D is a section of $\Lambda_D^+(T^*X) \otimes \mathfrak{so}(E)$, i.e. F_D is self-dual with respect to the Hodge star operator induced by C_D and ω_X.
The next step in constructing g_D is determining a volume form ν_D. This volume form allows us to define g_D, namely we define g_D as the unique metric in C_D with $\mathrm{vol}(g_D, \omega_X) = \nu_D$. Before discussing this, we need to define what the sign of a definite connection is.

[17]Given an arbitrary trivialising cover, we can always construct a trivialising cover with connected trivialising neighbourhoods. So, there is no harm in assuming this.
[18]Note that we are not summing over i.

To define the sign of D we need an orientation on $\Lambda^+_D(T^*X)$. It turns out that the conformal class C_D naturally induces an orientation ω^+ on $\Lambda^+_D(T^*X)$. An explanation of this can be found in [6, p. 3]. Note that E is also equipped with an orientation: since E is an SO(3)-bundle, it is naturally equipped[19] with a Riemannian metric $g_E : X \to E^* \otimes E^*$ and an orientation $\omega_E : X \to \Lambda^3(E)$. Later, it will be shown that F_D can be identified with a bundle isomorphism $E \to \Lambda^+_D(T^*X)$. The connection D is called *positive-definite* if $F_D : E \to \Lambda^+_D(T^*X)$ is orientation-preserving and *negative-definite* otherwise. Write $\operatorname{sgn}(D) = 1$ if D is positive-definite and $\operatorname{sgn}(D) = -1$ if D is negative-definite. Since the fibres of E and $\Lambda^+_D(T^*X)$ are 3-dimensional, it follows[20] that $\operatorname{sgn}(D) \cdot F_D$ is always orientation-preserving.

A number of identifications will be needed to construct ν_D. These identifications will be used throughout this section. Let V be a 3-dimensional $\mathbb{R}$-vector space equipped with a Riemannian metric g_V and an orientation $\mathfrak{o}_V$. Using an oriented orthonormal basis of V, it is not hard to show that the following maps[21] are isomorphisms:

$$V \to V^*,\ v \mapsto g_V(v, \cdot) \qquad \text{and} \qquad V \to \mathfrak{so}(V),\ v \mapsto (w \mapsto v \times w := *(v \wedge w)).$$

Let E_x be the fibre of E over $x \in X$. Note that E_x is equipped with a metric $g_E(x)$ and an orientation $\omega_E(x)$. So, the previous identifications give rise to bundle isomorphisms $E \to E^*$ and $E \to \mathfrak{so}(E)$. Let us return to the construction of ν_D.

We first pick an arbitrary positively oriented volume form $\nu \in \Omega^4(X, \mathbb{R})$, i.e. a volume form with $\nu = f \cdot \omega_X$ for some smooth map $f : X \to \mathbb{R}_{>0}$. Using the theorems from the previous section, we see that there exists a unique metric $g_\nu \in C_D$ with $\operatorname{vol}(g_\nu, \omega_X) = \nu$. The metric in the definition of the Hodge star operator also gives us a Riemannian metric $\langle \cdot, \cdot \rangle_{g_\nu}$ on $\Lambda^+_D(T^*X)$. Since F_D is a section of $\Lambda^+_D(T^*X) \otimes \mathfrak{so}(E)$, the previously discussed identifications show that we can identify F_D with a section of $\Lambda^+_D(T^*X) \otimes E^*$. Therefore, Proposition 2.2.4 allows us to view $F_D(x)$ as a linear map from E_x to $\Lambda^+_D(T^*X)_x$. So, F_D defines a bundle map

$$F_D : E \to \Lambda^+_D(T^*X), e \mapsto F_D(\pi(e))(e).$$

It is easily seen that the image of $F_D(x)$ equals S_x, which is 3-dimensional by assumption. Since E_x is 3-dimensional as well, it follows that $F_D(x)$ is an isomorphism, i.e. F_D is a bundle isomorphism. Therefore, we can use the bundle map interpretation of F_D to define a new metric on E. Namely, we can define $g_{E,\nu} \in \mathscr{M}^{4,0}(E)$ by

$$g_{E,\nu}(s, s') = \left\langle F_D(s), F_D(s') \right\rangle_{g_\nu}$$

for all sections $s, s' \in \mathscr{A}^0(E)$. Applying the construction of M in Lemma A.1 fibrewise to E, g_E and $g_{E,\nu}$ gives rise to bundle isomorphisms $M_\nu : E \to E$ and

[19] This was shown in Example 2.4.15.

[20] This was discussed at the end of section 2.2.

[21] The definition of $\mathfrak{so}(V)$ is the same as in Example 2.4.10.

$\sqrt{M_\nu} : E \to E$. Use $\sqrt{M_\nu}$ to define the following scalar function:

$$\mathrm{Tr}\sqrt{M_\nu} : X \to \mathbb{R}, x \mapsto \mathrm{Tr}\ \sqrt{M_\nu}\Big|_{E_x},$$

where Tr denotes the trace of endomorphisms. Also, let $\Lambda \in \mathbb{R}$ be a nonzero real number whose sign agrees with the sign of D. Now define[22]

$$\nu_D = \frac{1}{\Lambda^2}\left(\mathrm{Tr}\sqrt{M_\nu}\right)^2 \cdot \nu$$

and, as mentioned before, let g_D be the unique Riemannian metric in C_D with $\mathrm{vol}(g_D, \omega_X) = \nu_D$, i.e. $g_D = g_{\nu_D}$. Also, write $M_D = M_{\nu_D}$.
We should check that the definition of ν_D is independent of the choice of ν. Let $e_1, e_2, e_3 : U \to E$ be a g_E-orthonormal local frame induced by a local trivialisation $(U, \psi) \in \mathscr{C}$ and let $M^{ij} : U \to \mathbb{R}$ be the matrix representation of M_ν in e_1, e_2, e_3. The previously discussed identifications allow us to identify F_D with a section of $\Lambda^+_D(T^*X)\otimes E$, i.e. we can write $F_D|_U = F^i \otimes e_i$ for some 2-forms $F^1, F^2, F^3 \in \Omega^2(U, \mathbb{R})$. Suppose that the following identity holds

$$F^i \wedge F^j = M^{ij} \cdot \nu. \tag{3.5}$$

Since the left hand side of (3.5) does not depend on ν, it follows that scaling ν by $\lambda : X \to \mathbb{R}_{>0}$ corresponds to scaling M_ν by $1/\lambda$. This shows that the definition of ν_D does not depend on ν. This scaling property also shows that

$$\left(\mathrm{Tr}\sqrt{M_D}\right)^2 = \Lambda^2.$$

But M_D is positive-definite, so its trace is positive. Since the sign of Λ agrees with the sign of D, it follows that $\mathrm{Tr}\sqrt{M_D} = \mathrm{sgn}(D) \cdot \Lambda$. We will now prove that (3.5) holds.
Let $e^1, e^2, e^3 : U \to E^*$ be the dual frame of e_1, e_2, e_3. Since e_1, e_2, e_3 is an orthonormal frame, it follows that e_i corresponds to e^i under the identification $E \cong E^*$. So, we may also write $F_D|_U = \sum_{i=1}^3 F^i \otimes e^i$, which shows that F_D, viewed as a bundle map, maps e_i to F^i. Using that the F^i are self-dual and that $\nu = \mathrm{vol}(g_\nu, \omega_X)$, it follows that

$$\begin{aligned} F^i \wedge F^j &= F^i \wedge *F^j = \langle F^i, F^j \rangle_{g_\nu} \cdot \nu = \langle F_D(e_i), F_D(e_j) \rangle_{g_\nu} \cdot \nu \\ &= g_{E,\nu}(e_i, e_j) \cdot \nu = M^{ij} \cdot \nu. \end{aligned}$$

The last equality holds because of Lemma A.1 (2).
Now that we have established what the metric associated to the connection is, we can formulate an equation in terms of objects defined by D and argue that this equation is the equivalent of Einstein's equation. More precisely, we will construct

[22] Lemma A.1 tells us that $\sqrt{M_\nu}$ is positive-definite. This shows that $\mathrm{Tr}\sqrt{M_\nu}$ is nowhere vanishing.

an E-valued 2-form Φ_D and argue that $D\Phi_D = 0$ is the equivalent of Einstein's equation.

Recall that E and $\Lambda_D^+(T^*X)$ are equipped with Riemannian metrics g_E and $\langle\cdot,\cdot\rangle_{g_D}$, respectively. The E-valued 2-form Φ_D will be constructed using an isometric bundle map $E \to \Lambda_D^+(T^*X)$. Such a bundle map can be used to compare SO(3)-connections on E with SO(3)-connections on $\Lambda_D^+(T^*X)$. This will allow us to use a powerful theorem about connections on $\Lambda_D^+(T^*X)$ to our advantage. As mentioned before, the curvature can be identified with a bundle isomorphism $F_D : E \to \Lambda_D^+(T^*X)$. However, F_D will not always be an isometric bundle map. This problem can be tackled by using Lemma A.1. Let $M_D^{-1/2}$ denote the inverse of $\sqrt{M_D}$. Now define

$$\Phi_D = \operatorname{sgn}(D) \cdot F_D \circ M_D^{-1/2} : E \to \Lambda_D^+(T^*X). \tag{3.6}$$

Lemma A.1 (3) says that Φ_D is an isometric bundle map. Also, note that $M_D^{-1/2}$ is positive-definite, i.e. it has positive determinant. Therefore, $M_D^{-1/2}$ is orientation-preserving[23]. As argued before, $\operatorname{sgn}(D)\cdot F_D$ is also orientation-preserving. It follows that Φ_D is orientation-preserving.

We mentioned before that we would construct an E-valued 2-form Φ_D. So, we should clarify how the bundle map $\Phi_D : E \to \Lambda_D^+(T^*X)$ can be identified with an E-valued 2-form. As with the curvature F_D, we can identify a bundle map from E to $\Lambda_D^+(T^*X)$ with a section of $\Lambda_D^+(T^*X) \otimes E^*$. Since $E \cong E^*$, it follows that we can identify Φ_D with a section of $\Lambda_D^+(T^*X) \otimes E$, i.e. an E-valued 2-form. The equation $D\Phi_D = 0$ is the equivalent of Einstein's equation in the following sense.

Theorem 3.3.2. Let X be a 4-dimensional smooth manifold and assume that X is orientable and connected. Also, let $(E, \pi, \mathscr{C})$ be an SO(3)-bundle of rank 3 over X, D a definite SO(3)-connection on E and $\Lambda \in \mathbb{R}$ a nonzero real number whose sign agrees with the sign of D. Finally, let g_D and Φ_D be as explained above. If $D\Phi_D = 0$, then g_D satisfies $\operatorname{Ric}(g_D) = \Lambda \cdot g_D$, i.e. g_D is Einstein.

An important ingredient in proving the previous is a theorem about connections on $\Lambda_D^+(T^*X)$. To formulate this theorem we need to introduce some definitions. In section 3.1 we have defined what the Levi-Civita connection on the tangent bundle is. A similar notion on the bundle of $\mathbb{R}$-valued self-dual 2-forms will now be introduced. For this, we need to define what the torsion of a connection on $\Lambda_D^+(T^*X)$ is. Let

$$\nabla : \mathscr{A}^0(\Lambda_D^+(T^*X)) \to \mathscr{A}^1(\Lambda_D^+(T^*X)) = \mathscr{A}^0(T^*X \otimes \Lambda_D^+(T^*X))$$

be a connection on $\Lambda_D^+(T^*X)$. Also, let $x \in X$ be a point. Since $\Lambda_D^+(T^*X)_x$ is canonically isomorphic to a subspace of $T_x^*X \otimes T_x^*X$, it follows that we can identify sections of $T^*X \otimes \Lambda_D^+(T^*X)$ with sections of $T^*X \otimes T^*X \otimes T^*X$. Using this

[23] This was shown at the end of section 2.2.

identification, we can[24] define a linear map

$$\sigma : \mathscr{A}^0(T^*X \otimes \Lambda_D^+(T^*X)) \to \mathscr{A}^0(\Lambda^3(T^*X)),$$

called *skew-symmetrisation*, by[25]

$$\sigma(\omega^1 \otimes \omega^2 \otimes \omega^3) = \frac{1}{1! \cdot 2!} \sum_{\tau \in S_3} \operatorname{sgn}(\tau) \cdot \omega^{\tau(1)} \otimes \omega^{\tau(2)} \otimes \omega^{\tau(3)}.$$

So $\sigma \circ \nabla$ is a linear map from the self-dual 2-forms $\Omega_D^+(X,\mathbb{R})$ to $\Omega^3(X,\mathbb{R})$. It is a natural question to ask whether $\sigma \circ \nabla$ coincides with the exterior derivative d. This motivates the following definition.

Definition 3.3.3. Let ∇ be a connection on $\Lambda_D^+(T^*X)$. The linear map

$$\tau(\nabla) = d - \sigma \circ \nabla : \Omega_D^+(X,\mathbb{R}) \to \Omega^3(X,\mathbb{R})$$

is called the *torsion of* ∇. A connection on $\Lambda_D^+(T^*X)$ with vanishing torsion is called *torsion free*.

Recall that $\Lambda_D^+(T^*X)$ is equipped with a Riemannian metric $\langle \cdot,\cdot \rangle_{g_D}$ and an orientation ω^+. So, as discussed in the previous chapter, there exists a trivialising cover that makes $\Lambda_D^+(T^*X)$ into an SO(3)-bundle. We can now mimic the definition of the Levi-Civita connection on the tangent bundle.

Definition 3.3.4. A torsion free SO(3)-connection ∇ on $\Lambda_D^+(T^*X)$ is called a g_D-*Levi-Civita connection*.

As with the tangent bundle, one can show that there exists a unique Levi-Civita connection on $\Lambda_D^+(T^*X)$. The proof can be found in [5]. We can now formulate the theorem needed for the proof of Theorem 3.3.2.

Theorem 3.3.5. Let X be a 4-dimensional smooth manifold, $\omega_X \in \Omega^4(X,\mathbb{R})$ an orientation, g a Riemannian metric on X and let $\Lambda_+^2(T^*X)$ denote the bundle of $\mathbb{R}$-valued self-dual 2-forms induced by g and ω_X. The metric g is Einstein if and only if the g-Levi-Civita connection ∇ on $\Lambda_+^2(T^*X)$ is a self-dual instanton, i.e. the curvature of ∇ is self-dual.

See [2] for details about the previous theorem. Let ∇ denote the g_D-Levi-Civita connection on $\Lambda_D^+(T^*X)$. Some properties of F_∇ will also be needed to prove Theorem 3.3.2. These properties will be discussed next.
Suppose that g_D is an Einstein metric. Theorem 3.3.5 says that F_∇ is self-dual, i.e. it is a section of $\Lambda_D^+(T^*X) \otimes \mathfrak{so}(\Lambda_D^+(T^*X))$. As before, we have bundle isomorphisms

$$\Lambda_D^+(T^*X) \to \left(\Lambda_D^+(T^*X)\right)^* \qquad \text{and} \qquad \Lambda_D^+(T^*X) \to \mathfrak{so}(\Lambda_D^+(T^*X)).$$

[24] Since $\Lambda^3(T_x^*X)$ is canonically isomorphic to a subspace of $T_x^*X \otimes T_x^*X \otimes T_x^*X$, we can also identify sections of $\Lambda^3(T^*X)$ with sections of $T^*X \otimes T^*X \otimes T^*X$.

[25] The coefficient $1/(1! \cdot 2!)$ in the definition of σ is chosen such that $\sigma(\alpha \otimes \beta) = \alpha \wedge \beta$ for all $\alpha \in \mathscr{A}^0(T^*X)$ and $\beta \in \mathscr{A}^0(\Lambda_D^+(T^*X))$. See [11, p. 299] for more details about this coefficient.

So, we can identify F_∇ with a section of $\Lambda^+_D(T^*X)\otimes(\Lambda^+_D(T^*X))^*$. Proposition 2.2.4 now shows that we can interpret F_∇ as a bundle map $F_\nabla : \Lambda^+_D(T^*X)\to\Lambda^+_D(T^*X)$. It turns out that $\mathrm{Tr}(F_\nabla)=s_{g_D}/4$, where $s_{g_D} : X\to\mathbb{R}$ denotes the scalar curvature of g_D. Since $\mathrm{Ric}(g_D)=\Lambda'\cdot g_D$ for some $\Lambda'\in\mathbb{R}$, (3.2) shows

$$\mathrm{Tr}(F_\nabla)=(4\Lambda')/4=\Lambda'.$$

Also, F_∇ is self-adjoint. More details about the previous facts can be found in [2]. A sketch of the proof of Theorem 3.3.2 will now be given.

Proof of Theorem 3.3.2 (sketch). We can use Φ_D and D to construct an SO(3)-connection $\Phi_{D*}(D)$ on $\Lambda^+_D(T^*X)$. First note that $\Phi_D : E\to\Lambda^+_D(T^*X)$ gives rise to a linear map

$$\mathscr{A}^0(\Phi_D):\mathscr{A}^0(E)\to\mathscr{A}^0(\Lambda^+_D(T^*X)),\ s\mapsto\Phi_D\circ s.$$

Also, define $\mathrm{id}_{\Lambda^1}\otimes\Phi_D : \Lambda^1(T^*X)\otimes E\to\Lambda^1(T^*X)\otimes\Lambda^+_D(T^*X)$ by

$$(\mathrm{id}_{\Lambda^1}\otimes\Phi_D)(\omega\otimes e)=\omega\otimes\Phi_D(e)$$

and let $\mathscr{A}^1(\Phi_D):=\mathscr{A}^0(\mathrm{id}_{\Lambda^1}\otimes\Phi_D):\mathscr{A}^1(E)\to\mathscr{A}^1(\Lambda^+_D(T^*X))$. We define $\Phi_{D*}(D)$ as the linear map that makes the following diagram commute

$$\begin{tikzcd}
\mathscr{A}^0(E) \arrow[r, "D"] \arrow[d, "\mathscr{A}^0(\Phi_D)"'] & \mathscr{A}^1(E) \arrow[d, "\mathscr{A}^1(\Phi_D)"] \\
\mathscr{A}^0(\Lambda^+_D(T^*X)) \arrow[r, "\Phi_{D*}(D)"] & \mathscr{A}^1(\Lambda^+_D(T^*X))
\end{tikzcd}$$

One can check that $\Phi_{D*}(D)$ does indeed define a connection. Also, you can show that $\Phi_{D*}(D)$ is an SO(3)-connection precisely because Φ_D is an isometry. We will now show that $\Phi_{D*}(D)$ is torsion free.

As argued before, Φ_D can be thought of as an E-valued 2-form. So, $D\Phi_D$ is a section of $\Lambda^3(T^*X)\otimes E$. Since $E\cong E^*$ and because of Proposition 2.2.4, we can identify $D\Phi_D$ with a bundle map $E\to\Lambda^3(T^*X)$. So, we get a bundle map

$$\tau_D:=(D\Phi_D)\circ\Phi_D^{-1}:\Lambda^+_D(T^*X)\to\Lambda^3(T^*X).$$

In [5], it is shown that $\mathscr{A}^0(\tau_D):\Omega^+_D(X,\mathbb{R})\to\Omega^3(X,\mathbb{R})$ coincides with $\tau(\Phi_{D*}(D))$. Therefore, the assumption $D\Phi_D=0$ shows us that $\Phi_{D*}(D)$ is torsion free.

Thus, we have shown that $\Phi_{D*}(D)$ is the g_D-Levi-Civita connection on $\Lambda^+_D(T^*X)$. One can also show that $\Phi_{D*}(D)$ is a self-dual instanton precisely because D is. Theorem 3.3.5 tells us that g_D must therefore be an Einstein metric, i.e. $\mathrm{Ric}(g_D)=\Lambda'\cdot g_D$ for some $\Lambda'\in\mathbb{R}$. It remains to be shown that $\Lambda'=\Lambda$.

The equality $\Phi_{D*}(D) = \nabla$ and the fact that Φ_D is an orientation-preserving isometry allows us to derive $F_D = F_\nabla \circ \Phi_D$. Also, (3.6) shows us that $\Phi_D^{-1} \circ F_D = \mathrm{sgn}(D) \cdot \sqrt{M_D}$. Therefore, we have the following

$$\Phi_D^{-1} \circ F_\nabla \circ \Phi_D = \Phi_D^{-1} \circ F_D = \mathrm{sgn}(D) \cdot \sqrt{M_D}. \tag{3.7}$$

Let $e_1, e_2, e_3 : U \to E$ be a local frame of E. Note that the matrix representation of $\Phi_D^{-1} \circ F_\nabla \circ \Phi_D$ in e_1, e_2, e_3 is the same as the matrix representation of F_∇ in $\Phi_D(e_1), \Phi_D(e_2), \Phi_D(e_3)$. So, it follows that $\mathrm{Tr}(\Phi_D^{-1} \circ F_\nabla \circ \Phi_D) = \mathrm{Tr}(F_\nabla)$. We conclude

$$\Lambda' = \mathrm{Tr}(F_\nabla) = \mathrm{Tr}(\Phi_D^{-1} \circ F_\nabla \circ \Phi_D) = \mathrm{sgn}(D) \cdot \mathrm{Tr}\sqrt{M_D} = (\mathrm{sgn}(D))^2 \cdot \Lambda = \Lambda.$$

□

Let us now consider an example to which we can apply the explained formalism.

Example 3.3.6. Write $X = \mathbb{R}^4$ and equip X with the orientation

$$\omega_X = dx^1 \wedge \ldots \wedge dx^4 \in \Omega^4(X, \mathbb{R}).$$

Also, let $(E, \pi, \mathscr{C})$ be the $\mathbb{R}$-trivial bundle of rank 3 over X. Since $\mathscr{C}$ consists of only one local trivialisation, $(E, \pi, \mathscr{C})$ is an SO(3)-bundle and we can define a connection D on E by specifying the local connection forms $A^i_j \in \Omega^1(X, \mathbb{R})$ in this local trivialisation. Let $f : X \to \mathbb{R}$ be defined by $f(x^1, x^2, x^3, x^4) = \tilde{f}(x^1)$, where $\tilde{f} : \mathbb{R} \to \mathbb{R}$ is some smooth map. Note that

$$df = \frac{\partial f}{\partial x^i} dx^i = f' dx^1,$$

where $f' : X \to \mathbb{R}$ denotes $(x^1, \ldots, x^4) \mapsto d\tilde{f}/dx|_{x=x^1}$. Let the A^i_j be defined by

$$(A^i_j)_{1\le i,j\le 3} = \begin{pmatrix} 0 & -A^3 & A^2 \\ A^3 & 0 & -A^1 \\ -A^2 & A^1 & 0 \end{pmatrix},$$

where $A^i = f \cdot dx^{i+1}$. By construction, D is an SO(3)-connection[26]. Using (2.2), we can compute the local curvature forms $F^i_j \in \Omega^2(X, \mathbb{R})$. The F^i_j are given by

$$(F^i_j)_{1\le i,j\le 3} = \begin{pmatrix} 0 & -F^3 & F^2 \\ F^3 & 0 & -F^1 \\ -F^2 & F^1 & 0 \end{pmatrix},$$

where

$$F^1 = f' dx^1 \wedge dx^2 + f^2 dx^3 \wedge dx^4, \qquad F^2 = f' dx^1 \wedge dx^3 - f^2 dx^2 \wedge dx^4,$$
$$F^3 = f' dx^1 \wedge dx^4 + f^2 dx^2 \wedge dx^3.$$

[26] Recall that the Lie algebra of SO(3) is equal to the set of real antisymmetric 3×3 matrices.

We would like D to be a definite connection. By definition, this means that[27] $S_x = \operatorname{span}\{F^1(x), F^2(x), F^3(x)\}$ needs to be a definite subspace. Define the matrix function $\tilde{X} : X \to \operatorname{Mat}(3, \mathbb{R})$ by

$$F^i \wedge F^j = \tilde{X}^{ij} \omega_X.$$

Note that $\tilde{X}(x)$ is just a matrix representation of $\langle \cdot, \cdot \rangle_{\omega_X(x)}|_{S_x \times S_x}$ for all $x \in X$. So, D is definite if and only if $\tilde{X}$ is definite on the whole of X. Computation shows

$$\tilde{X} = 2f'f^2 \cdot I,$$

where I denotes the identity matrix. So, $\tilde{X}(x)$ is definite for all $x \in X$ if and only if f and f' do not have zeros. Take for instance $\tilde{f}(x) = e^x$ for all $x \in \mathbb{R}$. This choice has the property $f' = f$. So, $\tilde{X} = 2f^3 \cdot I$ and

$$F^1 = dx^1 \wedge A^1 + A^2 \wedge A^3, \qquad F^2 = dx^1 \wedge A^2 - A^1 \wedge A^3,$$
$$F^3 = dx^1 \wedge A^3 + A^1 \wedge A^2.$$

We could now use Corollary 3.2.4 to compute a metric that makes the curvature self-dual. However, suppose that dx^1, A^1, A^2, A^3 is an orthonormal local frame. Example 2.6.5 shows[28] that the F^i are self-dual. Therefore, define

$$g = dx^1 \otimes dx^1 + \sum_{i=1}^{3} A^i \otimes A^i = dx^1 \otimes dx^1 + f^2 \sum_{i=2}^{4} dx^i \otimes dx^i.$$

Since dx^1, A^1, A^2, A^3 is g^{-1}-orthonormal, the previous reasoning shows $C_D = [g]$. Consider $\Lambda = \operatorname{sgn}(D) \cdot 3\sqrt{2}$ and note that the sign of Λ agrees with the sign of the connection. Define $\nu = \operatorname{vol}(g, \omega_X) = dx^1 \wedge A^1 \wedge A^2 \wedge A^3 = f^3 \omega_X$ and recall that the matrix representation of M_ν satisfies $F^i \wedge F^j = M^{ij} \nu$. Since $\tilde{X} = 2f^3 \cdot I$, it follows that

$$2f^3 \delta^{ij} \omega_X = F^i \wedge F^j = M^{ij} \nu = M^{ij} \cdot f^3 \omega_X,$$

i.e. $M_\nu = 2 \cdot \mathrm{id}_E$. So,

$$\nu_D = \frac{1}{\Lambda^2} \left(\operatorname{Tr} \sqrt{M_\nu} \right)^2 \cdot \nu = \nu.$$

Therefore, $g_D = g$. Also, $\Phi_D = \pm F_D \circ M_D^{-1/2} = \pm F_D / \sqrt{2}$. So, the Bianchi identity[29] shows that we must have $D\Phi_D = \pm DF_D / \sqrt{2} = 0$. According to theorem 3.3.2, it follows that $\operatorname{Ric}(g_D) = \Lambda \cdot g_D$, i.e. g_D is Einstein.

[27] Let $e_1, e_2, e_3 : X \to E$ denote the local frame induced by (X, id_E). The identification $E \cong \mathfrak{so}(E)$ gives rise to a corresponding local frame $\mathfrak{s}e_1, \mathfrak{s}e_2, \mathfrak{s}e_3 : X \to \mathfrak{so}(E)$. One can check that $F_D = F^i \otimes \mathfrak{s}e_i$, which justifies our definition of S_x.

[28] Note that dx^1, A^1, A^2, A^3 is oriented: $dx^1 \wedge A^1 \wedge A^2 \wedge A^3 = f^3 \omega_X$ and $f^3(x) > 0$ for all $x \in X$.

[29] This is actually a little more subtle. Recall the induced connection $\operatorname{End}(D)$ on $\operatorname{End}(E)$. The Bianchi identity says $\operatorname{End}(D) F_D = 0$. However, we have identified F_D with an E-valued 2-form via $E \cong \mathfrak{so}(E)$ and ask whether $DF_D = 0$ holds. Using a local trivialisation, one can show that D and the restriction of $\operatorname{End}(D)$ to $\mathfrak{so}(E)$-valued forms coincide under the identification $E \cong \mathfrak{so}(E)$. This justifies the conclusion.

3.4 Lorentzian reformulation

In general relativity, the metrics of interest are of Lorentzian signature. However, the reformulation discussed in the previous section associates a Riemannian metric g_D to a definite SO(3)-connection D. Therefore, the following question arises: can we modify the formalism in section 3.3 in such a way that the metric associated to the connection is a Lorentzian metric? The answer is: yes and no.
Yes, we can associate a conformal class of Lorentzian metrics to a connection. This is achieved as before: define the conformal class by demanding that the curvature of the connection is self-dual. Results like Theorem 3.2.7 and Theorem 3.2.8 make sure that this is indeed possible if the local curvature forms F^i satisfy

$$F^i \wedge F^j = \tilde{X}^{ij}\omega_X \qquad \text{and} \qquad F^i \wedge \overline{F}^j = 0 \tag{3.8}$$

for some matrix function $\tilde{X} : U \to \mathrm{GL}(3,\mathbb{C})$ and orientation $\omega_X \in \Omega^4(X,\mathbb{R})$. The resulting conformal class of Lorentzian metrics is considered to be a real object constructed from a complex object, namely the $\mathbb{C}$-valued local curvature forms. Therefore, the conditions (3.8) are often called *reality conditions*.
Notice the following: the self-dual 2-forms in Lorentzian signature are complex-valued. Therefore, we need to consider a complex vector bundle instead of a real one. More specifically, we need to consider a complex[30] $\mathrm{SO}(3,\mathbb{C})$-bundle of rank 3 and an $\mathrm{SO}(3,\mathbb{C})$-connection. The curvature of such a connection is valued in a 3-dimensional vector bundle as before. As with an SO(3)-bundle, we also get extra structure on the fibres. Namely, the fibres are equipped with a non-degenerate bilinear form and a nonzero $\mathbb{C}$-valued 3-form: the bilinear form is defined by declaring that its matrix representation is equal to the identity matrix in a basis induced by a local trivialisation and the 3-form is just the wedge product of this basis. The previous definitions are independent of the chosen local trivialisation precisely because the transition functions are $\mathrm{SO}(3,\mathbb{C})$-valued. The extra structure ensures that we still have bundle isomorphisms analogous to $E^* \cong E \cong \mathfrak{so}(E)$. So, most constructions from section 3.3 can be repeated when replacing a real SO(3)-bundle with a complex $\mathrm{SO}(3,\mathbb{C})$-bundle. However, a problem does arise.
Even though we can construct a conformal class of Lorentzian metrics, we do not know what metric in this conformal class we should choose. Recall the bundle map $M_\nu : E \to E$ from the previous section. We used the self-adjoint positive-definite square root of M_ν to construct the volume form ν_D. This volume form allowed us to specify a metric in the conformal class. Of course, there is an equivalent of a self-adjoint positive-definite map in the context of complex vector spaces: a Hermitian positive-definite map. However, the fibres of an $\mathrm{SO}(3,\mathbb{C})$-bundle are not (naturally) equipped with a Hermitian metric. So, we can not construct an equivalent of ν_D in

[30] $\mathrm{SO}(3,\mathbb{C})$ denotes the group of complex matrices $A \in \mathrm{GL}(3,\mathbb{C})$ with $A^\top A = I$ and $\det A = 1$.

the same way. To decide which square root we need, we should ask ourselves the following question: why did we use the self-adjoint positive-definite square root of M_ν to construct ν_D?
The reason is the following fact: F_∇ is definite if $D\Phi_D = 0$. This follows from (3.7). Let us elaborate on why this motivates the choice of square root. Squaring equation (3.7) shows

$$M_D = \Phi_D^{-1} \circ F_\nabla^2 \circ \Phi_D.$$

Since self-adjoint positive-definite square roots are unique, definiteness of F_∇ shows that the self-adjoint positive-definite square root of M_D must equal $\pm\Phi_D^{-1} \circ F_\nabla \circ \Phi_D$, i.e. this choice of square root ensures that we can identify $\sqrt{M_D}$ with $\pm F_\nabla$ via Φ_D. This is why[31] making sure that $\mathrm{Tr}\sqrt{M_D} = \pm\Lambda$ holds is equivalent to choosing the cosmological constant of g_D.
So, an answer to "what square root should we use in the modified formulation?" is: we should use the square root $\sqrt{M_D'}$ that can be identified with F_∇', where M_D' and F_∇' are the equivalents of M_D and F_∇ in the modified formulation. However, a priori we do not know F_∇'. So, since we do not have an equivalent of "F_∇ is definite", we still do not have a criterion that can be used to pick the right square root. It turns out that this problem is not easily solved (K. Krasnov, personal communication, May 24, 2016). However, there is an alternative.
Namely, in [9, p. 13] a different reformulation of Einstein's equation is presented. As with the previously discussed formulation, the connection is still considered a variable. Also, we still associate a metric to the connection in the same way. However, more variables are introduced. The idea is again the same though: we formulate equations in terms of the connection and the other variables. If these equations are satisfied, the associated metric will solve Einstein's equation. The main advantage of this formulation over the previous one is that using these other variables ensures that square roots of bundle maps are not needed. Looking further into this formulation might be an interesting topic for future research.

[31] As discussed before, the trace of F_∇ is equal to the cosmological constant of g_D.

4 Conclusion

In this thesis, we have looked at a reformulation of Einstein's equation, namely the one presented in [6]. First, we explained the mathematical concepts needed for this reformulation. The most crucial concepts were vector bundles and connections. Next, we derived some technical results that can be used to construct a metric from a definite connection. After this, we have given a detailed explanation of the reformulation presented in [6]. The new formulation relates to the usual formulation through the fact that we associate a Riemannian metric to a definite connection. Theorem 3.3.5 is the key result that allowed us to prove that the new formulation is in fact a reformulation of Einstein's equation. Finally, we considered modifying the new formulation to make it suitable for Lorentzian metrics. However, the fact that we have to take square roots of bundle maps turns out to be a significant problem. Nevertheless, in [9] a different reformulation is presented. This reformulation uses a lot of the same ideas but avoids having to take square roots of bundle maps.

Appendix

In chapter 3, we have used the results stated in the following lemma.

Lemma A.1. Let V and W be finite dimensional $\mathbb{R}$-vector spaces and let g_V and g_W be Riemannian metrics on V and W, respectively. Also, let $f : V \to W$ be an isomorphism and define $\tilde{g}_V : V \times V \to \mathbb{R}$ by $\tilde{g}_V(v_1, v_2) = g_W(f(v_1), f(v_2))$ for all $v_1, v_2 \in V$. Now define $M = \sharp_{g_V} \circ \flat_{\tilde{g}_V} : V \to V$. The following results hold:

(1) M is self-adjoint and positive-definite with respect to g_V, i.e. we have

$$g_V(M(v_1), v_2) = g_V(v_1, M(v_2)) \qquad \text{and} \qquad g_V(M(v), v) > 0$$

for all $v_1, v_2 \in V$ and $v \in V \setminus \{0\}$.

(2) The matrix representation M^{ij} of M in a g_V-orthonormal basis $e_1, \dots, e_n$ is given by $M^{ij} = \tilde{g}_V(e_i, e_j)$.

(3) There exists a unique linear map $\sqrt{M} : V \to V$ with the following: $\sqrt{M}$ is self-adjoint and positive-definite with respect to g_V and $\sqrt{M} \circ \sqrt{M} = M$. Moreover, $\sqrt{M}$ is invertible and its inverse $M^{-1/2} : V \to V$ is such that $\phi = f \circ M^{-1/2} : V \to W$ is an isometry, i.e. we have

$$g_V(v_1, v_2) = g_W(\phi(v_1), \phi(v_2))$$

for all $v_1, v_2 \in V$.

Proof. From $\sharp_{g_V} = \flat_{g_V}^{-1}$, we deduce $\flat_{g_V} \circ M = \flat_{\tilde{g}_V} : V \to V^*$. Using this, we see that

$$\tilde{g}_V(v_1, v_2) = \flat_{\tilde{g}_V}(v_1)(v_2) = \flat_{g_V}(M(v_1))(v_2) = g_V(M(v_1), v_2) \tag{A.1}$$

holds for all $v_1, v_2 \in V$. Since $\tilde{g}_V$ is symmetric and positive-definite, it follows that M is self-adjoint and positive-definite with respect to g_V. Let M^{ij} be the matrix representation of M in an orthonormal basis $e_1, \dots, e_n$. Using (A.1), it follows that

$$\tilde{g}_V(e_i, e_j) = \tilde{g}_V(e_j, e_i) = g_V(M(e_j), e_i) = g_V(M^{kj} e_k, e_i) = M^{kj} \delta_{ki} = M^{ij}.$$

A standard result from linear algebra says that a self-adjoint positive-definite map has a unique self-adjoint positive-definite square root. So, let $\sqrt{M} =: s$ be as in (3) and note that it is invertible because M is. Since $M \circ s^{-1} = s$, we have

$$\begin{aligned} g_W(\phi(v_1), \phi(v_2)) &= g_W\big(f(s^{-1}(v_1)), f(s^{-1}(v_2))\big) = \tilde{g}_V(s^{-1}(v_1), s^{-1}(v_2)) \\ &= g_V\big(M(s^{-1}(v_1)), s^{-1}(v_2)\big) = g_V(s(v_1), s^{-1}(v_2)) \\ &= g_V\big(v_1, s(s^{-1}(v_2))\big) = g_V(v_1, v_2) \end{aligned}$$

for all $v_1, v_2 \in V$. $\square$

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