Distribution of Entanglement in Multipartite Quantum states

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Abstract

We have extended measures of bipartite entanglement to measures of multipartite entanglement for pure states. To better grasp the different ways in which a multipartite state may be entangled, we first give a more general definition of entanglement that is based on partitions of the particles. Then we present a measure corresponding to this definition and use both analytic and computational methods to gain insight in the manner in which the W_n and GHZ_n states are entangled. Although the overall entanglement of two general multipartite states is not straightforwardly comparable we conclude that the W_n states are less entangled than GHZ_n states for every partition of the particles.

Contents

Chapter 1

Introduction

Quantum entanglement is a resource for quantum computing. Quantum computing is a vast and highly active field of research in contemporary theoretical physics as well as in contemporary theoretical computer science. Since entanglement is one of the keystone principles in building a quantum computer, it is useful to know about properties of the distribution of entanglement in multipartite states. Because of the potentially vast processing power of quantum computers, quantum computers may in the future enable us to find cures for diseases by doing advanced simulations which are currently not yet feasible. Furthermore, research into entanglement generally contributes to our knowledge about the quantum theoretical description of nature. For these reasons it is useful and interesting to look at aspects of higher order entanglement in detail.

In this thesis we investigate the distribution of entanglement between subsystems in pure entangled states of multiple parts. First, we will gain a general insight into entanglement by looking at well established theoretical matter, which is part of most master programmes in Theoretical Physics. Concretely, definitions and useful as well as required properties of some quantitative measures of entanglement will be acquired, both for the bipartite case and the more general multipartite case. Then explicit calculations for concrete measures of α-entanglement are performed, specifically for the symmetric states GHZ_n and W_n. A comparison is made between the entanglement of the GHZ_n states and the W_n states, facilitated by computational methods as well as analytical results.

Chapter 2

The basics of quantum mechanics

In Quantum Mechanics, the state of a system is described by a *state vector*. A state vector is a normalized vector in a complex Hilbert space. A Hilbert space H is a vector space with an inner product defined on its elements, which is complete with respect to the norm induced by the inner product. This means that the limit of any Cauchy sequence in H converges in H.

We make use of the following conventions, which are widely used in physics: A vector in H is called a *ket* and is denoted by $|a\rangle \in H$, and its conjugate transpose is called *bra* and denoted by $\langle a|$. The inner product of $|a\rangle, |a'\rangle \in H$ is then defined as $\langle |a\rangle, |a'\rangle\rangle := \langle a| \cdot |a\rangle$, and removing redundant symbols, normally written as $\langle a|a'\rangle$. In addition, if a state $|c\rangle \in H$ can be denoted as $|c\rangle = \lambda_1 |a\rangle + \lambda_2 |b\rangle$ for some $\lambda_1, \lambda_2 \in \mathbb{C}, |a\rangle, |b\rangle \in H$, physicists tend to say that $|c\rangle$ is *in a superposition* of $|a\rangle$ and $|b\rangle$.

If H is a complex Hilbert space, we have that the following properties need to hold for all $|a\rangle, |b\rangle, |c\rangle \in H$ and $\lambda \in \mathbb{C}$:
$\langle a|b\rangle = \overline{\langle b|a\rangle}$ (where the overline denotes complex conjugation)
$\langle a|b + \lambda c\rangle = \langle a|b\rangle + \lambda \langle a|c\rangle$ and consequently $\langle \lambda a|b\rangle = \overline{\lambda} \langle a|b\rangle$.
We define the norm as $\|a\| := \sqrt{\langle a|a\rangle}$. Then also:
$\langle a|a\rangle = \|a\|^2 = 0 \iff |a\rangle = 0 \in H$ holds.

H additionally needs to be *complete* which means that given a Cauchy sequence $\{|a_i\rangle\}_{i \in \mathbb{N}} \subseteq H$, it needs to be the case that $\lim_{i \to \infty} |a_i\rangle = |a\rangle$ exists in H.

Finally, we note that if two Hilbert spaces H and H' are of equal dimension, then they are isomorphic.

Qubits and qudits An object of great interest in the field of quantum physics, and especially in quantum computing, is the qubit. A qubit is defined as a system that is described by the following state vector:

$$|\psi\rangle := c_1 |0\rangle + c_2 |1\rangle \in \mathbb{C}^2,$$

where $|c_1|^2 + |c_2|^2 = 1$ and $c_1, c_2 \in \mathbb{C}$[1]. Almost all explicit calculations that will be performed in this bachelor thesis, will be calculations on qubits.
Somewhat more generally, a qu*d*it is defined as:

$$|\psi_d\rangle := \sum_{i=1}^{d} c_i |i\rangle ,$$

Again with $\sum_{i=1}^{d} |c_i|^2 = 1$ and $c_i \in \mathbb{C}$ for all i. In case of $d = 3$ a qudit is often called a *qutrit*. So $|\psi_3\rangle = \frac{1}{\sqrt{3}} |0\rangle + \frac{1}{\sqrt{3}} |1\rangle + \frac{1}{\sqrt{3}} |2\rangle$ would be an example of a (state vector describing a) qutrit.
So far we have only seen Hilbert spaces suited for describing systems that consist of only one single part. In order to approach the concept of Quantum *entanglement*, we will define the *tensor product* $H_1 \otimes H_2$ of two Hilbert spaces.

Tensor products: a practical approach We are given two Hilbert spaces H_1, H_2 with (finite) dimensions $\dim(H_1) = N$ and $\dim(H_2) = M$. We may keep the dimensions finite, as we are mainly interested in qudits (or even only qu*b*its). For the same reason, we only have to consider Hilbert spaces of the form $\mathbb{C}^d$. We will define the tensor product of H_1 and H_2 in a minute, but first we give a short motivation as to why we want to use this product.

Suppose $|a\rangle \in H_1$ were to describe the state of system 1, and $|b\rangle \in H_2$ were to describe the state of system 2. If we would be asked to describe the state of the combined system, it seems reasonable to simply take the tuple (pair) $\{|a\rangle, |b\rangle\}$. We want the tuple $\{|a\rangle, |b\rangle\}$ to live in a Hilbert space again, because if we had not known anything about the two distinct parts of the combined system beforehand, we would have intended to describe it with a state vector from a single Hilbert space. Applying the principle of *success through simplicity*, perhaps $H_1 \times H_2 = \mathbb{C}^N \times \mathbb{C}^M = \mathbb{C}^{N+M}$ contains what we need to properly describe the joint system. Its addition and scalar multiplication are defined component-wise, and its zero vector is just the tuple $\{0_{H_1}, 0_{H_2}\}$[1]. Sadly, we run into a problem: The kets $|a\rangle$ and $\lambda |a\rangle$ (for

[1] This space is also called the *direct sum* of H_1 and H_2 and denoted $H_1 \oplus H_2$.

$\lambda \in \mathbb{C} \setminus \{0\}$) describe the same state. Our scalar multiplication cannot handle that: As per definition, $(\lambda |p\rangle, \lambda |q\rangle) = \lambda(|p\rangle, |q\rangle)$ holds, whereas we actually need the equality $(\lambda |a\rangle, |b\rangle) = \lambda(|a\rangle, |b\rangle)$ to be true, in order to be able to properly normalize our states. This definition of scalar multiplication also doesn't leave room for superpositions of states. Since experimental results do indicate the existence of superposition, the space $H_1 \times H_2$ cannot adequately describe quantum states.

We instead introduce the *tensor product* of H_1 and H_2, which *does* nicely meet our requirements. In order to construct it, choose an orthonormal basis $B_1 := \{|1\rangle_1, ..., |n\rangle_1\}$ for H_1 and $B_2 := \{|1\rangle_2, ..., |m\rangle_2\}$ for H_2. Then the space that has $B := B_1 \times B_2$ as its basis[2], is called the tensor product of H_1 and H_2. We denote it by $H_1 \otimes H_2$. We continue to use the bra-ket notation for elements of B, and write $|i\rangle \otimes |j\rangle$ for the element $\{|i\rangle_1, |j\rangle_2\} \in B$. The symbol $\otimes$ used here is just there for notation and has no further meaning. Note that at this stage we are not finished. First and foremost, it is not clear whether there is a corresponding element from the tensor product for two states $|a\rangle \in H_1$, $|b\rangle \in H_2$. Secondly, we have not yet defined an inner product on $H_1 \otimes H_2$. An arbitrary element $|c\rangle \in H_1 \otimes H_2$ looks like this:

$$|c\rangle = \sum_{|i\rangle \otimes |j\rangle \in B} c_{ij} |i\rangle \otimes |j\rangle = \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij} |i\rangle \otimes |j\rangle ,$$

for some scalars $c_{ij} \in \mathbb{C}$. We can now define a product of vectors $|a\rangle \in H_1$, $|b\rangle \in H_2$, productively also denoted with $\otimes$. Decompose $|a\rangle$ in the basis B_1 and $|b\rangle$ in the basis B_2. So $|a\rangle = \sum_{i=1}^{N} a_i |i\rangle_1$ and $|b\rangle = \sum_{j=1}^{M} b_i |j\rangle_2$. The *tensor product of vectors* is then simply defined as

$$|a\rangle \otimes |b\rangle := \sum_{|i\rangle \otimes |j\rangle \in B} a_i b_j |i\rangle \otimes |j\rangle = \sum_{i=1}^{N} \sum_{j=1}^{M} a_i b_i |i\rangle \otimes |j\rangle .$$

A relevant fact which we will not prove here, is that the tensor product of two Hilbert spaces is independent of the bases chosen in its construction. Lastly a remark about notation: We often abbreviate $|a\rangle \otimes |b\rangle$ as $|a \otimes b\rangle$ or sometimes even as $|ab\rangle$.

If A is a linear operator on H_1, and B is a linear operator on H_2, then we define $A \otimes B$ such that $A \otimes B |a \otimes b\rangle = |Aa \otimes Bb\rangle$.

[2] $B_1 \times B_2$ is the Cartesian product of B_2 and B_2; the set of *all* tuples $\{|i\rangle_1, |j\rangle_2\}$ with $|i\rangle_1 \in B_1$ and $|j\rangle_2 \in B_2$.

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Inner product on $H_1 \otimes H_2$ We would like $H_1 \otimes H_2$ to be a Hilbert space, and not merely a vector space. To achieve that, all we have to do[3] is define an inner product on it. Given orthonormal bases $B_1 := \{|1\rangle_1, ..., |n\rangle_1\}$ for H_1 and $B_2 := \{|1\rangle_2, ..., |m\rangle_2\}$ for H_2, we define the inner product of basis vectors $|i \otimes j\rangle, |k \otimes l\rangle \in H_1 \otimes H_2$ as:

$$\langle i \otimes j | k \otimes l \rangle := \langle i|k\rangle \langle j|l\rangle = \delta_{i,k}\delta_{j,l}.$$

Then the definition of the inner product of arbitrary vectors in $H_1 \otimes H_2$ follows from the sesquilinearity of the inner products on H_i. Note that although given $|\psi_i\rangle, |\chi_i\rangle \in H_i$, this reduces to $\langle \psi_1 \otimes \psi_2 | \chi_1 \otimes \chi_2 \rangle = \langle \psi_1 | \chi_1 \rangle \cdot \langle \psi_2 | \chi_2 \rangle$, not every element of $H_1 \otimes H_2$ can be written as such a product.

Entanglement It is high time to give a definition of entanglement. We say that $|\Phi\rangle \in H_1 \otimes H_2$ is an *entangled state*, if it can *not* be written as a product $|\psi_1 \otimes \psi_2\rangle$ for any $|\psi_1\rangle \in H_1$ and $|\psi_2\rangle \in H_2$.[1]

Bear in mind that a state vector can always be denoted as a sum of multiple terms, depending on the basis chosen. This means that a state being denoted with more than one term does not imply that it be entangled per se, for example:

$$|\psi_1\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|1\rangle + |0\rangle) \otimes (|1\rangle + |0\rangle).$$

Extending this definition to tensor products of more than two Hilbert spaces, we say that $|\Phi\rangle \in \bigotimes_{i=1}^{n} H_i$ is an *entangled state*, if it can *not* be written as a product $\bigotimes_{i=1}^{n} |\psi_i\rangle$ for any $|\psi_i\rangle \in H_i$.

Finally, a vector is called *separable* if it *can* be written as a tensor product of vectors $|\psi_i\rangle \in H_i$.

Density matrices Sometimes we don't know which state a system is in, and we only know a that is in one of a handful of states with associated probabilities. In this situation the concept of a density matrix is useful.

Given states $|\psi_i\rangle \in H$ and associated probabilities $\{p_1, ..., p_n\} \subseteq [0, 1]$ (so that $\sum_{i=1}^{n} p_i = 1$) we define the corresponding *density matrix* as:

$$\rho := \sum_i p_i |\psi_i\rangle \langle \psi_i|.$$

[3] If the obtained space turns out not to be complete with respect to the distance induced by this inner product, we take the completion. The completeness of the tensor product is immediate in case of finite-dimensional Hilbert spaces, which is always the case in this thesis.

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Mixed and pure states In case ρ is of the form $\rho = |\psi\rangle \langle\psi|$ for some state $|\psi\rangle \in H$, we call it a *pure state*. Otherwise, ρ is called a *mixed state*.[4] At this point we are calling both vectors and matrices a kind of *state*. If it's not specified explicitly which is meant (by writing $|a\rangle \in H$ or $\rho : H \to H$), the bra-ket notation will still indicate which object we are dealing with. Moreover, the use of the term "state" when considering matrices can be justified as well. If two systems are described by the same density matrix, then the probability distributions of the outcomes of measurements on the two systems will also be equal. Consequently it makes sense to say that the two systems are in the same state.

The partial trace and reduced density matrices The following two definitions will soon prove useful: Given a density matrix $\rho : H_1 \otimes H_2 \to H_1 \otimes H_2$ and orthonormal bases B_1 for H_1 and B_2 for H_2, we define the *partial trace over subsystem 2*:

$$\mathrm{Tr}_2(\rho) = \sum_{a,a' \in B_1} \sum_{b \in B_2} |a\rangle \langle ab| \rho |a'b\rangle \langle a'| .$$

We also define the *reduced density matrix of subsystem 1*: $\rho_1 := \mathrm{Tr}_2(\rho)$.
The partial trace over subsystem 1 and reduced density matrix of subsystem 2 are defined analogously.

The generalisation for density matrices on tensor products of more than two Hilbert spaces goes as follows: Given a density matrix $\rho : \bigotimes_{i=1}^n H_i \to \bigotimes_{i=1}^n H_i$ and a subsystem K - i.e. some of the H_i, let K be the set containing their indices - we define the *partial trace over subsystem K*:

$$\mathrm{Tr}_K(\rho) = \sum_{a,a' \in B_{\overline{K}}} \sum_{b \in B_K} |a\rangle \langle ab| \rho |a'b\rangle \langle a'| ,$$

where B_K is some orthonormal basis for subsystem K (explicitly: $\mathrm{span}(B_K) = \bigotimes_{i \in K} H_i$), and $\overline{K}$ is the subsystem complement to K.
Then again, the associated *reduced density matrix* is defined as $\rho_{\overline{K}} := \mathrm{Tr}_K(\rho)$.

[4] Perhaps it is more sensible to say "mixture of states", but we will stick to the conventional term.

The Schmidt decomposition theorem A theorem about pure bipartite states that will be essential later in this thesis, is the *Schmidt decomposition theorem*. It states: Given a state vector $|a\rangle \in H_1 \otimes H_2$ in a bipartite quantum system, there exist orthonormal states $|i_1\rangle$ for H_1 and $|i_2'\rangle$ for H_2 such that:

$$|a\rangle = \sum_{i=1}^{k} \sqrt{p_i}\, |i_1\rangle\, |i_2'\rangle\,,$$

where $p_i \in \mathbb{R}_{>0}$ are uniquely determined up to order, and $\sum_{i=1}^{k} p_i = 1$, and $k \leq \min\{\dim(H_1), \dim(H_2)\}$. We call p_i the *Schmidt coefficients* and k the Schmidt rank of $|a\rangle$[1, 2]. We will not prove this theorem here, but it is worth mentioning that the p_i turn out to be the eigenvalues of the reduced density matrix ρ_1 of the state. This also immediately shows that there is no basis dependency in the determination of the p_i.

Lemma Expressing a state in its Schmidt decomposition can greatly simplify the calculation of a partial trace. We can use the Schmidt decomposition of a given state as follows: Consider a state vector in a bipartite hilbert space $H = H_K \otimes H_{\overline{K}}$ and find its Schmidt decomposition. The basis vectors $|i_K\rangle$ for H_K and $|i'_{\overline{K}}\rangle$ for $H_{\overline{K}}$ obtained from the Schmidt decomposition theorem can then be extended to orthonormal bases B_K of H_K and $B_{\overline{K}}$ of $H_{\overline{K}}$. If we then take the partial trace over $\overline{K}$ using precisely those bases, we obtain that the reduced matrix ρ_K expressed in the basis B_K is the diagonal matrix with p_i on the i-th row (and zeros on any rows after the k-th). Equally, when we take the partial trace over K using B_K and $B_{\overline{K}}$, we obtain $\rho_{\overline{K}}$ expressed in the basis $B_{\overline{K}}$ is also a diagonal matrix with p_i on the i-th row (and zeros after the k-th row).

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Chapter 3

Measures of bipartite entanglement

The definition of entanglement we currently have, only enables us to make a distinction between states that are entangled, and states that are not entangled. Perhaps we should be a little more demanding than that.

To magnify the inadequacy of our current definition of entanglement, consider the following states for $\varepsilon \in [0,1]$:

$$|\Psi\rangle_\varepsilon := \sqrt{1-\varepsilon}\,|01\rangle + \sqrt{\varepsilon}\,|10\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2.$$

The vectors $|\Psi\rangle_\varepsilon$ are normalized for all $\varepsilon \in [0,1]$, as $(\sqrt{1-\varepsilon})^2 + (\sqrt{\varepsilon})^2 = 1$. Note that only for $\varepsilon = 0 \vee \varepsilon = 1$ we have a separable state, as we then obtain $|\Psi\rangle_0 = |01\rangle$ and $|\Psi\rangle_1 = |10\rangle$. The state is entangled for all $\varepsilon \in (0,1)$ though.

Now consider the analogous definition of "nonzeroness" of a number. Using this reasoning we would say that the number 2 is just as nonzero as the number 12; they would both just be called *nonzero*. But we *do* of course distinguish them. The absolute value serves as our measure of "nonzeroness", even for complex numbers, and in a similar manner it would be nice if we had some measure of "nonseparableness".

Needless to say, a measure of entanglement should not depend on the basis chosen to represent a state. We can fortunately make use of the Schmidt decomposition to avoid that.

Participation ratio Let $|a\rangle$ be a pure state in $H_1 \otimes H_2$. We then define the *participation ratio* to be:

$$\mathrm{pr}(|a\rangle) := \frac{1}{\sum_{i=1}^{k} p_i^2},$$

where p_i are the Schmidt coefficients of $|a\rangle$ and k is the Schmidt rank of $|a\rangle$[2].

Since the eigenvalues of the square of a matrix are equal to the squares of the eigenvalues, and the trace of a matrix is equal to the sum of its eigenvalues, we can rewrite this to get:

$$\mathrm{pr}(|a\rangle) = \frac{1}{\mathrm{Tr}(\rho_1^2)}.$$

Since the Schmidt coefficients add up to 1, or equivalently since $Tr(\rho_1) = 1$, it is easy to see that the participation ratio takes values in $[1, k]$. It is equal to k precisely when ρ_1 has $\frac{1}{k}$ as its only eigenvalues (with multiplicity k), and it is equal to 1 when ρ_1 has 1 as its only nonzero eigenvalue (with multiplicity 1).

Calculating the participation ratio for our exemplary state $|\Psi\rangle_\varepsilon$ we get $\mathrm{pr}(|\Psi\rangle_\varepsilon) = \frac{1}{(1-\varepsilon)^2+\varepsilon^2}$. In the following graph we see what this expression looks like:

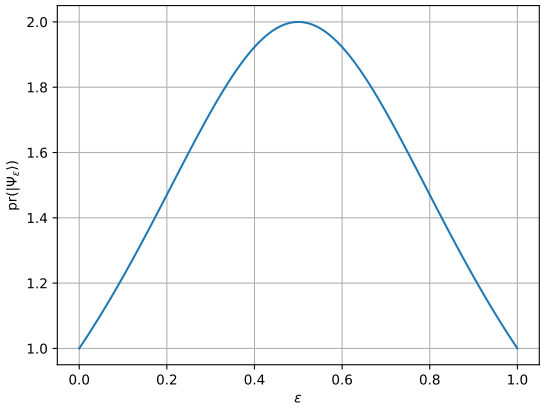

Figure 3.1: *The participation ratio of $|\Psi_\varepsilon\rangle$ as a function of ε.*

Figure 3.1 shows us that the participation ratio satisfies at least the minimal demand we would like a measure of entanglement to satisfy, namely that it is minimal for separable states ($\varepsilon = 0, 1$).

fact: mixedness of reduced states corresponds to entangledness of whole state.[3] Some insight into why this is the case can be provided by the property that when a reduced density matrix is a pure state, then it is associated with a single state vector. Then there is no entanglement present between the

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part associated with that reduced density matrix and the rest of the system. Because of this fact, we are able to define measures of mixedness of reduced states as measures of entanglement of the whole state.

Let in the following part $\rho : H_1 \otimes H_2 \to H_1 \otimes H_2$ be an arbitrary pure density matrix describing a bipartite system.

Von Neumann entropy Possibly the most well known measure of mixedness is the von Neumann entropy, given by:

$$S_{vN}(\rho) := -\mathrm{Tr}(\rho \ln(\rho)).$$

Here, $\ln(\rho)$ is the matrix logarithm of ρ. It is defined as follows:

Since ρ is Hermitian, it is diagonalisable. Writing P for the matrix of eigenvectors and ρ_D for the corresponding diagonal matrix, we define $\ln(\rho) := P \ln(\rho_D) P^{-1}$, where $\ln(\rho_D)$ is defined by $\ln(\rho_D)_{ii} := \ln(\rho_{ii})$. This logarithm is ill-defined in case there are zeros on the diagonal. Fortunately, in case of the von Neumann entropy the problem can be evaded by imposing a continuity condition. To achieve this we will rewrite the von Neumann entropy in terms of the eigenvalues of ρ.

Since the trace of a matrix is equal to the sum of its eigenvalues, we have that:

$$S_{vN}(\rho) = -\sum_i \lambda_i \ln(\lambda_i),$$

where λ_i are the eigenvalues of ρ. In order not be hindered by infinities and to preserve continuity we define $\lambda \ln(\lambda)$ to be zero, or equivalently we only take the sum over the nonzero eigenvalues of ρ. A significant physical argument in favour of defining $\lambda \ln(\lambda)$ equal to zero, is that adding states that occur with probability zero to a system should not affect the entropy of the system.

If ρ is a pure state, the corresponding measure of entanglement is thus given by:

$$E_{S_{vN}}(\rho) := S_{vN}(\mathrm{Tr}_2(\rho)) = S_{vN}(\rho_1).$$

Again rewriting this to a sum of the eigenvalues gives us:

$$E_{S_{vN}}(\rho) = -\sum_i p_i \ln(p_i),$$

where p_i are the eigenvalues of ρ_1, which are the Schmidt coefficients of ρ.

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Tsallis entropies For $q \in \mathbb{R}_{>0}$ we define the *Tsallis entropies*[3, 4]:

$$\mathrm{Ts}_q(\rho) := \frac{1}{1-q}(\mathrm{Tr}(\rho^q) - 1)$$

A fact we will not show here is that the limit case $q \to 1$ coincides with the von Neumann entropy. The case where $q = 2$ is called the *linear entropy*, which is worth noting since it is a built-in function in QuTiP, a Python library we will use later on in this thesis.

In fact, the Tsallis entropy can be expressed in terms of eigenvalues of ρ as well:

$$\mathrm{Ts}_q(\rho) = \frac{1}{1-q}(\sum_i(\lambda_i^q) - 1).$$

This is the case since if A is a matrix and λ is an eigenvalue of A, then A^n has eigenvalues λ^n.

Analogous to the definition of entanglement based on the von Neumann entropy we get for a pure state ρ:

$$E_{Ts_q}(\rho) := \mathrm{Ts}_q(\mathrm{Tr}_2(\rho)) = \mathrm{Ts}_q(\rho_1),$$

and this measure of entanglement can therefore also be expressed in terms of the Schmidt coefficients:

$$E_{Ts_q}(\rho) = \frac{1}{1-q}(\sum_i(p_i^q) - 1).$$

LOCC Any viable measure of the entanglement of a system shall not increase on average by means of local operations (i.e. operations on the distinct parts of the system) and classical communications (i.e. using information about the result of an earlier operation carried out on one part of the system in performing an operation on another part of the system)[3, 5, 6]. We abbreviate local operations and classical communications by *LOCC*. Unfortunately there is no simple way to mathematically capture the concept of LOCC, but fundamentally both the von Neumann entropy and the Tsallis entropies (for $q > 0$) satisfy the imposed condition. In chapter 5 we will elaborate on this.

Chapter 4

Partitions and α-entangledness

A better definition of multipartite entanglement We might have a reason to doubt the usefulness of our current definition of entanglement, when applied to systems consisting of more than two parts. Consider the following entangled state:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|010\rangle + |100\rangle).$$

We can rewrite it at such:

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \otimes |0\rangle .$$

So far we denoted the standard basis of $\mathbb{C}^2$ as $\{|0\rangle, |1\rangle\}$, and the standard basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ as $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. A particle that can classically be in four states, would be described by a state living in $\mathbb{C}^4$ and analogously we would denote the standard basis vectors as $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$. But nothing stops us from using the notation $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. With that in mind, we might just as well view $|\psi_2\rangle$ as a state living in $\mathbb{C}^4 \otimes \mathbb{C}^2$. But then it isn't entangled anymore!

Partitions. This idea of grouping certain particles can be formalized by looking at *partitions* of the particles. Given a set L, we call α a *partition* of L if it is a set of subsets of L and the following properties are satisfied:

1. $\varnothing \notin \alpha$
2. $\bigcup_{K \in \alpha} K = L$

3. $K, K' \in \alpha : K \neq K' \implies K \cap K' = \varnothing$

Given some set L, we call the partition $\alpha = \{L\}$ the *trivial partition*.

Next, we will apply this definition and create a more general definition of entanglement based upon it. Say we look at a system of n particles. Let $|\Psi\rangle \in \bigotimes_{i=1}^{n} H_i$ be the (a) state vector associated with these particles.

Definition: α-entangled. First, we label our n particles. We write $L := \mathbb{Z} \cap [1, n] = \{1, 2, \ldots, n\}$ for the set of labels. Then let α be a partition of L. We say that $|\Psi\rangle$ is *α-separable*, if for every $K \in \alpha$ there exists a state vector $|\psi_K\rangle \in \bigotimes_{i \in K} H_i$ such that $|\Psi\rangle = \bigotimes_{K \in \alpha} |\psi\rangle_K$ holds.

$|\Psi\rangle$ is called *α-entangled*, if it *cannot* be expressed as a tensor product $|\Psi\rangle = \bigotimes_{K \in \alpha} |\psi\rangle_K$ of states $|\psi_K\rangle \in \bigotimes_{i \in K} H_i$.

Examples Before we continue, we wish to get rid of the excess of brackets when writing out our partitions of particles. From now on we denote a partition $\alpha = \{K_1, \ldots, K_i, \ldots\}$ as $\alpha = K_1|\ldots|K_i|\ldots$. Any case where we will explicitly write out α will involve at most $n = 9$ particles, so we will then also leave out the commas and brackets which are part of the usual notation of $K \in \alpha$.

In the case of two particles, there exists only a single nontrivial partition of the labels. Then we have $n = 2$ and therefore $L = \{1, 2\}$, so the one nontrivial partition is $\alpha = 1|2$. This means that for $n = 2$, the new definition of α-entanglement brings nothing new compared to our original definition of entanglement.

Things get a bit more interesting if we look at $n = 3$. There are 5 different partitions, namely $\alpha = L, \alpha = 12|3, \alpha = 13|2, \alpha = 1|23$, and $\alpha = 1|2|3$. The state $|\psi_2\rangle$ is then indeed α-separable for $\alpha = 12|3$, since $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is a state in $\bigotimes_{i \in 12} H_i = H_1 \otimes H_2$, and $|0\rangle$ lives in $\bigotimes_{i \in 3} H_i = H_3$. It is also α-entangled for all other nontrivial α.

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Chapter 5

Quantifying multipartite entanglement

As we have seen, depending on the way in which the parts of a system get partitioned, it can change if we call a given state entangled or not. For this reason it is unreasonable to try to assign a single number to a given state, that is supposed to represent the "degree of entanglement" of said state. Nonetheless, we could attempt to quantify the degree of α-entanglement. That would mean that given some state, we obtain a number for every partition of its parts. We will denote the degree of α-entanglement of a given pure state ρ as $f_\alpha(\rho)$.

Defining f_α Given a pure state $\rho : \bigotimes_{i=1}^n H_i \to \bigotimes_{i=1}^n H_i$ and a partition α of its parts $L = \mathbb{Z} \cap [1, n]$, we define its f_α as follows:

$$f_\alpha(\rho) := \sum_{K \in \alpha} F(\mathrm{Tr}_{\overline{K}}(\rho)),$$

Where F is an entropy such as the von Neumann entropy or a Tsallis-q entropy. In this expression, each term $F(\mathrm{Tr}_{\overline{K}}(\rho))$ can be interpreted as the (bipartite) entanglement of ρ, where the system is perceived as the bipartition $H_K \otimes H_{\overline{K}}$. In that respect, f_α is a multipartite analogue to the previously defined measures of bipartite entanglement.

Apart from the analogy with the bipartite approach, there are a number other of reasons for defining a measure of multipartite entanglement the way we do here. We stress that this thesis only covers the description of the entanglement of *pure* states. In that case the most important require-

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ment for f_α to be a good measurement of entanglement, is that f should be an *entanglement monotone*[3].

Entanglement monotone Given an arbitrary pure state ρ which becomes $\rho' = \sum_i p'_i \rho'_i$ through applying some pure[1] LOCC operations, a function $f : (\bigotimes_{i=1}^n H_i \to \bigotimes_{i=1}^n H_i) \to \mathbb{R}$ that assigns a real number to a density matrix[2] is called an *entanglement monotone*, if $f(\rho) \geq \sum_i p'_i f(\rho'_i)$ holds.

From theorem 3 in Szalay it follows that for some $K \in \alpha$ the function $F \circ \mathrm{Tr}_K$ is an entanglement monotone if F is the von Neumann entropy, and if F is a Tsallis-q entropy for $q > 0$ as well. He also shows that sums of entanglement monotones again are entanglement monotones. As a consequence, f_α in the way we have defined above is also an entanglement monotone.

Some further properties a good measure of α-entanglemend should satisfy, is that it is zero for α-separable states, and larger than zero for α-entangled states. Finally, we want that f_α is larger than f_β *if and only if* the partition α is *finer* than the partition β. To illustrate what *finer* means: $1|2|34|567$ is finer than $12|34|567$ which in turn is finer than $1234|567$, and $1|23|4567$ is neither finer nor *coarser* (i.e. less fine) than any of the former three partitions. This also illustrates the fact that we cannot orden partitions linearly by their coarseness, since some partitions can simply not be compared. It turns out that f_α indeed has this property when F is taken to be the von Neumann entropy or the Tsallis-q entropy for $q > 1$ (not $q > 0$).

f_α can also be interpreted as a measure of statistical distinguishability[3]. For this thesis the concept of statistical distinguishability has not been examined, but we think it should still be referenced here as it is another argument for defining f_α in this manner.

Symmetric states The number of partitions of n particles grows extremely fast. For only 5 particles, there are already 52 different partitions. For state vectors that are symmetric under permutations of the particles, it is not necessary to consider *all* partitions, since many of them are then equivalent and will give the same result for f_α. The number of partitions of a set of n elements *up to permutations* is equal to the number of ways n can be written as a sum of positive integers, which still grows very rapidly. But

[1] This means that ρ'_i have to be pure states.

[2] We actually don't need to define f for *every* density matrix - only for pure states. We might abuse this fact to write $f(|a\rangle)$ where it should be $f(|a\rangle\langle a|)$. Wherever we write $f(|a\rangle)$, read $f(ket2dm(|a\rangle))$ where $ket2dm(|a\rangle) := |a\rangle\langle a|$.

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for 5 particles, there are only 7 inequivalent partitions. The following table shows that the number of partitions modulo permutations is still manageable for the first few n, whereas the total number of partitions is not:

$\|L\| = n$	number of partitions[7]	up to permutations[8]
0	1	1
1	1	1
2	2	2
3	5	3
4	15	5
5	52	7
6	203	11
7	877	15
8	4140	22
9	21147	30
10	115975	42

Now if we look at the actual partitions up to permutations, the relation with the integer partitions becomes more clear:

$n = 4$:	$n = 5$:
1234	12345
	1234|5
123|4	123|45
	123|4|5
12|34	12|34|5
12|3|4	12|3|4|5
1|2|3|4	1|2|3|4|5

As we can see, the integer partition of n is found in the number of elements in each part of a partition. Explicitly, $5 = 5, 5 = 4 + 1, 5 = 3 + 2, 5 = 3 + 1 + 1$ and so on. The map the other way is given as follows: Let a be an integer partition of n and write it as a sequence with elements ordened by their size (so e.g. $a = \{1, 1, 3\}$ when $n = 5$). Then the partition of L corresponding to this integer partition is given by

$$\alpha = \bigcup_{j=1}^{|\alpha_{int}|} \left\{ \mathbb{Z} \cap (\sum_{i=1}^{j-1} a_i, \sum_{i=1}^{j} a_i] \right\},$$

where we follow the common convention that the empty sum is taken equal to zero ($\sum_{i=1}^{0} a_i = 0$). Writing this out for $a = \{1, 1, 3\}$ yields $\alpha = \{\mathbb{Z} \cap (0, 1], \mathbb{Z} \cap (1, 2], \mathbb{Z} \cap (2, 5]\} = \{\{1\}, \{2\}, \{3, 4, 5\}\} = 1|2|345$, for example.

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Chapter 6

Analytical results for the α-entanglement of W_n and GHZ_n

As the von Neumann entropy and the Tsallis-q entropy for $q > 1$ satisfy all the requirements for a proper measure of multipartite entanglement of pure states, we will now investigate the associated functions f_α and attempt to conclude whether they are able to adequately distinguish the entanglement of the GHZ and W states and their n-dimensional generalisations.

The GHZ and W states are defined as follows:

$$|GHZ\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle),$$

and

$$|W\rangle := \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle).$$

Both live in $(\mathbb{C}^2)^{\otimes 3}$. These states are symmetric under permutation of their parts. Also, both states are α-entangled for all nontrivial partitions α. This means that in order to get a good picture of the manner in which these states are entangled, it is not necessary to consider *all* partitions of the particles, and considering for instance only the following three suffices: $1|2|3, 1|23$ and 123. Since the trivial partition is really uninteresting, we will leave it out from now on.

It is quite straightforward to generalize the GHZ and W states from 3 to general dimension $n \geq 2$. for readability we will write $|0_n\rangle := |0\rangle^{\otimes n}$ and $|1_n\rangle := |1\rangle^{\otimes n}$ in the following parts. We have:

$$|GHZ_n\rangle := \frac{1}{\sqrt{2}}(\underbrace{|000\ldots0\rangle}_{n} + \underbrace{|111\ldots1\rangle}_{n}) = \frac{1}{\sqrt{2}}(|0_n\rangle + |1_n\rangle),$$

and

$$|W_n\rangle := \frac{1}{\sqrt{n}}(\underbrace{|100\ldots0\rangle}_{n} + |010\ldots0\rangle + ... + |000\ldots1\rangle) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \bigotimes_{j=1}^{n} |\delta_{ij}\rangle .$$

These states live in $H = (\mathbb{C}^2)^{\otimes n}$.

If we want to know values for the α-entanglement for these states, we will have to find the eigenvalues of the reduced density matrices. It's not needed to take into account which particles exactly we trace out, because of the permutational symmetry of the states. Explicitly:

Like before, we label the particles with $L := \mathbb{Z} \cap [1, n]$. Let $K \subsetneq L$ be arbitrarily given, and $K \neq \emptyset$, and write $|K| = p$, $|\overline{K}| = n - p = q$. We will first calculate the eigenvalues of the partial trace tracing out $\overline{K}$ for the GHZ_n state, and then do the same for the W_n state.

If we view H as the bipartite Hilbert space $H_K \otimes H_{\overline{K}}$, then the GHZ_n state can be expressed as follows:

$$|GHZ_n\rangle = \frac{1}{\sqrt{2}} |0_p\rangle \otimes |0_q\rangle + \frac{1}{\sqrt{2}} |1_p\rangle \otimes |1_q\rangle .$$

Now we note that the state is written in its Schmidt decomposition, since the states $|1_p\rangle$ and $|0_p\rangle$ are orthonormal and the states $|1_q\rangle$ and $|0_q\rangle$ as well. Therefore our lemma in chapter 2 implies that the nonzero eigenvalues of the reduced matrix $\mathrm{Tr}_{\overline{K}}(|GHZ_n\rangle \langle GHZ_n|)$ are simply $p_1 = p_2 = \frac{1}{2}$.

We can rewrite the W_n states to obtain a Schmidt decomposition for it

as well:

$$\begin{aligned}
|W_n\rangle &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \bigotimes_{j=1}^{n} |\delta_{ij}\rangle \\
&= \frac{1}{\sqrt{n}} \left(\sum_{i=1}^{p} \bigotimes_{j=1}^{n} |\delta_{ij}\rangle + \sum_{i=p+1}^{n} \bigotimes_{j=1}^{n} |\delta_{ij}\rangle \right) \\
&= \frac{1}{\sqrt{n}} \sum_{i=1}^{p} \bigotimes_{j=1}^{p} |\delta_{ij}\rangle \otimes |0_q\rangle + \frac{1}{\sqrt{n}} \sum_{i=1}^{q} |0_p\rangle \otimes \bigotimes_{j=1}^{q} |\delta_{ij}\rangle \\
&= \sqrt{\frac{p}{n}} \, |W_p\rangle \otimes |0_q\rangle + \sqrt{\frac{q}{n}} \, |0_p\rangle \otimes |W_q\rangle \, .
\end{aligned}$$

Because $|W_p\rangle$ and $|W_q\rangle$ are normalized sums of only terms of the form $|00\ldots1\ldots0\rangle$, it is the case that $|W_p\rangle$ is orthonormal to $|0_p\rangle$ and $|W_q\rangle$ is orthonormal to $|0_p\rangle$. Thus we have indeed written $|W_n\rangle$ in its Schmidt decomposition. With this fact we can use our lemma again and conclude that the nonzero eigenvalues of $\mathrm{Tr}_{\overline{K}}(|W_n\rangle \langle W_n|)$ are given by $p_1 = \frac{p}{n}$ and $p_2 = \frac{q}{n}$.

Now we can calculate the measures of bipartite entanglement for any bipartition H as $H_K \otimes H_{\overline{K}}$, and then use them to find expressions for the measures of α-entanglement. Remember how the von Neumann entropy and Tsallis-q entropies were formulated in terms of eigenvalues:

$$S_{vN}(\rho) = -\sum_{i=1}^{k} \lambda_i \ln(\lambda_i),$$

and

$$\mathrm{Ts}_q(\rho) = \frac{1}{1-q}\left(\sum_{i=1}^{k} (\lambda_i^q) - 1\right).$$

Using $F = S_{vN}$ we then obtain the following expression for f_α in terms of the Schmidt coefficients. We have:

$$f_\alpha(\rho) = \sum_{K \in \alpha} F(\mathrm{Tr}_{\overline{K}}(\rho)) = -\sum_{K \in \alpha} \sum_{i=1}^{k_K} p_{K,i} \ln(p_{K,i}),$$

where k_K is the Schmidt rank and $p_{K,i}$ are the Schmidt coefficients of ρ following from the Schmidt decomposition where the bipartition of H is given by $H_K \otimes H_{\overline{K}}$.

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Likewise, we can obtain an expression for f_α in terms of the Schmidt coefficients using $F = \mathrm{Ts}_q$:

$$f_\alpha(\rho) = \sum_{K\in\alpha} F(\mathrm{Tr}_{\overline{K}}(\rho)) = \sum_{K\in\alpha} \frac{1}{1-q}(\sum_{i=1}^{k_K}(p_{K,i}^q) - 1).$$

We have now come to the point where we are able to give explicit expressions for both $f_\alpha(GHZ_n)$ and $f_\alpha(W_n)$ for both entropies. Using the von Neumann entropy we obtain for $\rho = |GHZ_n\rangle\langle GHZ_n|$:

$$f_\alpha(GHZ_n) = -\sum_{K\in\alpha}\left(\frac{1}{2}\ln(\frac{1}{2}) + \frac{1}{2}\ln(\frac{1}{2})\right) = -|\alpha|\ln(\frac{1}{2}) = |\alpha|\ln(2),$$

and for $\rho = |W_n\rangle\langle W_n|$ the expression becomes:

$$f_\alpha(W_n) = -\sum_{K\in\alpha}\left(\frac{p_K}{n}\ln(\frac{p_K}{n}) + \frac{q_K}{n}\ln(\frac{q_K}{n})\right),$$

where p_K and q_K are defined by $p_{K,1} = \frac{p_K}{n}$ and $p_{K,2} = \frac{q_K}{n}$.

Likewise, using the Tsallis-a entropy (where we write a to avoid using q twice) the GHZ_n state gives us:

$$f_\alpha(GHZ_n) = \sum_{K\in\alpha}\frac{1}{1-a}\left((\frac{1}{2})^a + (\frac{1}{2})^a - 1\right) = \frac{1}{1-a}\sum_{K\in\alpha}(2^{1-a}-1) = |\alpha|\frac{2^{1-a}-1}{1-a},$$

and finally for the W_n state we get:

$$f_\alpha(W_n) = \frac{1}{1-a}\sum_{K\in\alpha}\left((\frac{p_K}{n})^a + (\frac{q_K}{n})^a - 1\right).$$

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Chapter 7

Computational results for the α-entanglement of W_n and GHZ_n

In the previous chapter we have derived concrete expressions for the α-entanglement of a given pure state, when we choose F to be either the von Neumann entropy or a Tsallis-$q_{>1}$ entropy. However, other functions F need not be expressable in terms of eigenvalues per se. But even if that were possible, for an arbitrary (pure) state it is impractical to calculate f_α for a given α by hand, as determining the Schmidt coefficients for even a single $K \in \alpha$ may already be a great challenge - not to mention calculating f_α for *all* partitions, even when only permutationally symmetric states would be considered. For this reason, we have written a program in Python 3 thankfully making use of the QuTiP 4 library (Quantum Toolbox in Python)[9, 10]. Our program is in principle capable of calculating $f_\alpha(\rho)$ for any F[1] and any partition α, and any pure state ρ. In case of permutationally symmetric states, it can generate all partitions α and thus give a complete picture of how the entanglement is distributed over all α.

[1] F needs to be expressable in python code, such that e.g. infinite series may only be approximated

As the time it takes to generate the data increases substantially with the number of partitions, we have limited ourselves to $n = 14$. Beyond $n = 6$ it is not possible to give a (readable) explicit display of all partitions α on the horizontal axis[2]. These have been the determining factors for which n we generate plots for. Hence we only consider $n = 3$ for historical reasons, and $n = 6$ and $n = 14$ for the reasons pointed out here. We have decided against displaying graphs for other $n < 14$, because all observations can already be made from the cases $n = 6$ and $n = 14$. For the same reason we omit graphs where $F = Ts_q$ for $q \neq 2, 3$ is chosen. The corresponding data has been generated though, and this data (as well as the code generating it) can be requested from the author. For readability, the von Neumann entropy is calculated (in essence just scaled) using the binary logarithm instead of the natural logarithm. Finally, as this is a lot more convenient in Python, the particles are labeled from 0 through $n - 1$ instead of from 1 through n.

In figures 7.1, 7.2 and 7.3 we show the values f_α takes for all permutationally different partitions α for several different n, using the von Neumann entropy as our function F. In figure 7.1 we show the results for the original GHZ and W states ($n = 3$).

Then in figures 7.4 and 7.5, to better observe effects of the size of elements of the partitions, we have generated plots displaying $\frac{f_\alpha}{|\alpha|}$. In this way we eliminate the effect that the number of elements of a partition may have on the α-entanglement.

Figures 7.6 and 7.7 were generated to display for all permutationally different partitions α the α-entanglement of $|GHZ_6\rangle$ and $|W_6\rangle$ as figure 7.2, but with the Tsallis-2 entropy as F in figure 7.6, and with the Tsallis-3 entropy in figure 7.7.
Similarly 7.8 and 7.9 were generated show the same data as 7.2, but with the Tsallis-2 entropy as F in figure 7.6, and with the Tsallis-3 entropy in figure 7.7.

[2]Instead, we have ordened the partitions lexicographically and labeled them accordingly.

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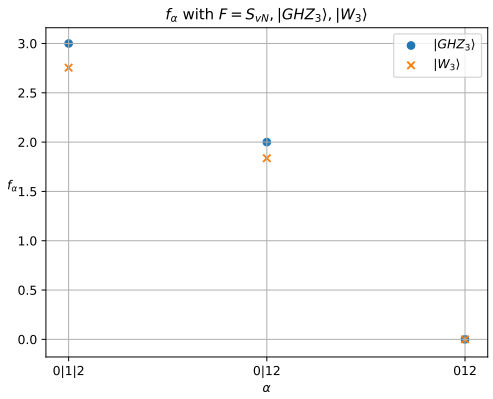

Figure 7.1: *The original states GHZ and W ($n = 3$).*

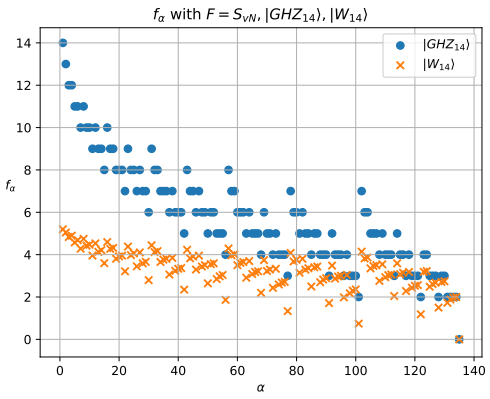

Figure 7.2: *$n = 14$, the largest n where we can still do the calculation.*

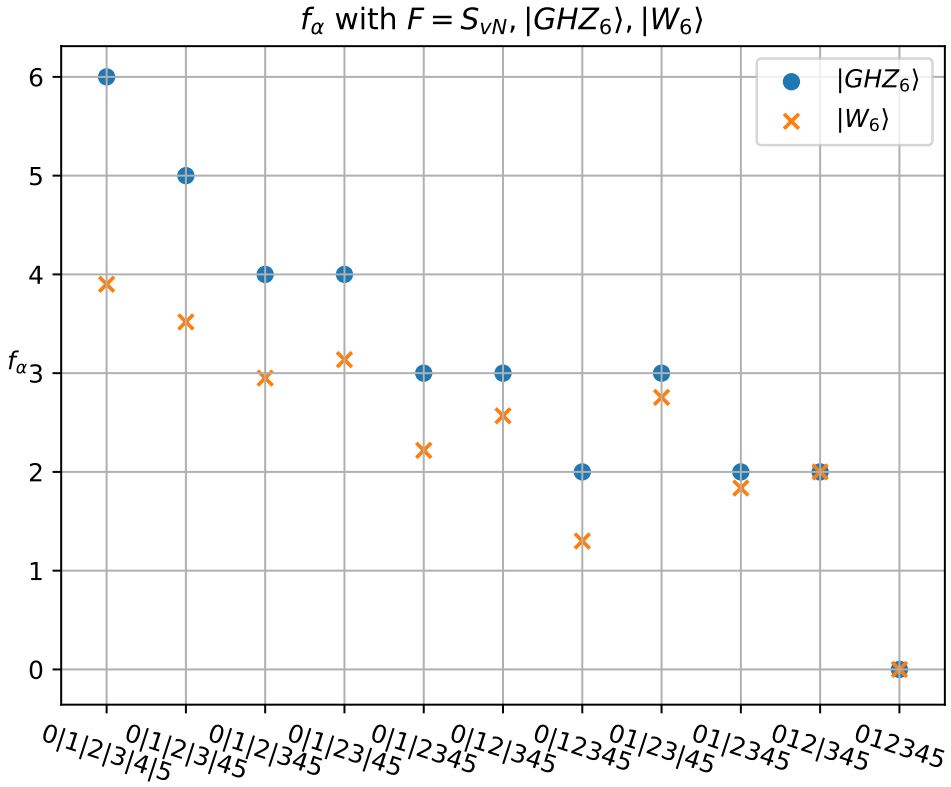

Figure 7.3: *$n = 6$, the largest n where displaying all α is still possible.*

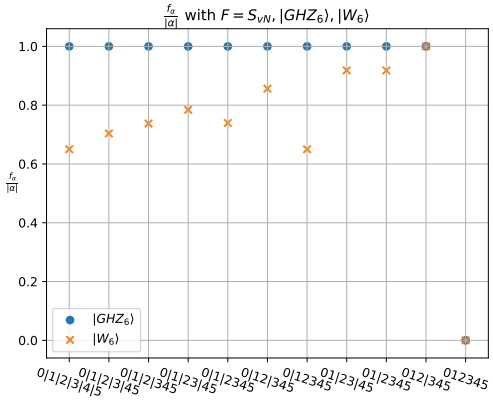

Figure 7.4: *$f_\alpha / |\alpha|$ for the von Neumann entropy and $n = 6$.*

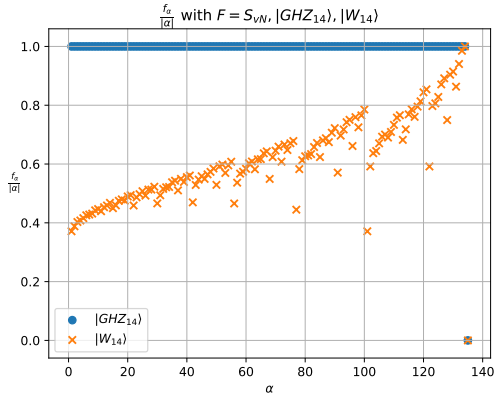

Figure 7.5: *$f_\alpha / |\alpha|$, $n = 14$.*

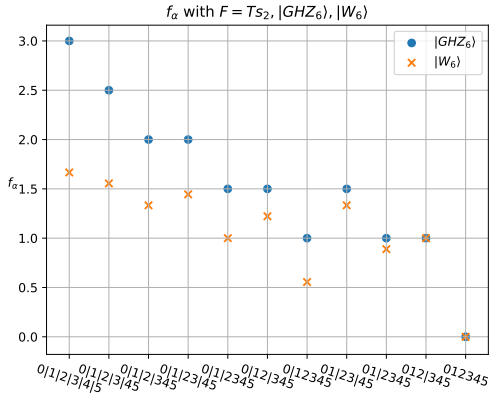

Figure 7.6: *f_α for Tsallis-2 and $n = 6$.*

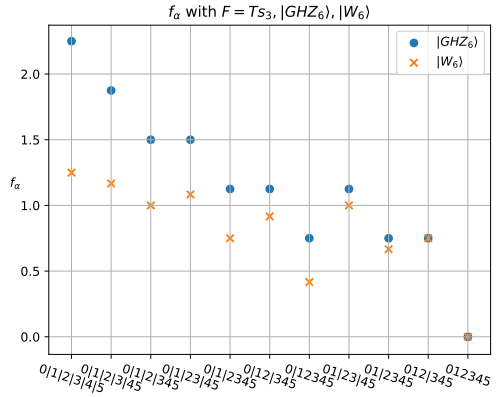

Figure 7.7: *f_α for Tsallis-3 and $n = 6$.*

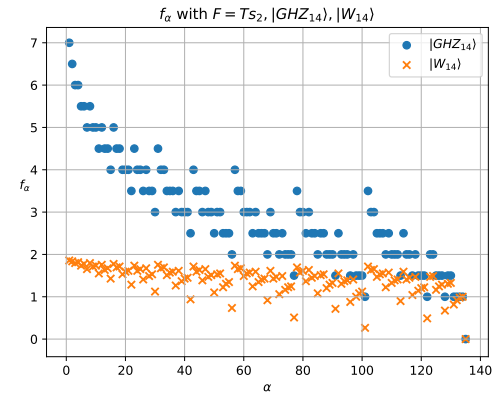

Figure 7.8: *f_α for Tsallis-2 and $n = 14$.*

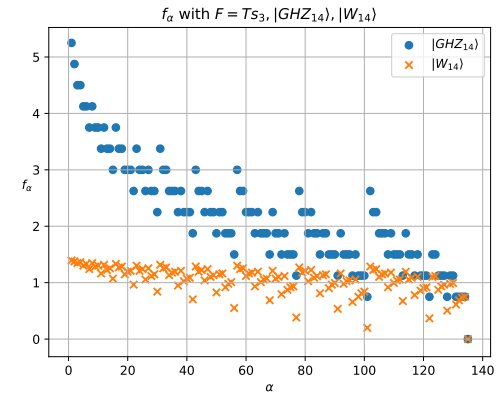

Figure 7.9: *f_α for Tsallis-3 and $n = 14$.*

Chapter 8

Conclusion and discussion

In figure 7.1 we can readily observe that lower entanglement corresponds with coarser partitions. As it should be, f_α is strictly positive for both states for all α except $\alpha = \{L\}$, where it is zero. This behaviour continues to be observable in figures 7.2 and 7.3. There it also becomes clear that $f_\alpha(W_n) \leq f_\alpha(GHZ_n)$ holds for all α, with equality when $\alpha = \{L\}$ and when α is a partition of two equally sized parts. This is in accordance with the analytic expressions we found in the previous chapter. A partition of two parts of equal size namely gives us Schmidt coefficients for the W_n state of $p_1 = p_2 = \frac{1}{2}$, equal to those of the GHZ_n state.
The fact that $f_\alpha(W_n) = f_\alpha(GHZ_n)$ holds when α is a partition of two equally sized parts also means that for odd n, the W_n states are *always* (for all nontrivial partitions α) less α-entangled than the GHZ_n states. Furthermore, we observe in figure 7.3 that any chain of partitions $\alpha_1, \alpha_2, \ldots$ ordened from crude to fine corresponds with a chain of inequalities $f_{\alpha_1}(|\psi\rangle) \leq f_{\alpha_2}(|\psi\rangle) \leq \ldots$, for both $|\psi\rangle = |GHZ_n\rangle$ and $|\psi\rangle = |W_n\rangle$.

In the previous chapter we derived that the dependency of $f_\alpha(GHZ_n)$ on α is completely given by the partition size $|\alpha|$, which is clearly reflected in the figures. Interestingly, figure 7.4 shows us that for the W_n state a lower value for f_α seems to correspond to a large difference in size of the elements of the partition, or just to the presence of small and large elements compared to n. This may be explained by the fact that when a Schmidt coefficient of the W_n state is very close to either 0 or 1, then the term $F(\mathrm{Tr}_{\overline{K}})$ for the corresponding K becomes very small.
Figure 7.5 on the other hand shows the general trend that the degree of "α-entanglement per part" appears to be higher for partitions with less parts. Note that the partitions $0|1|234$ and $0|12345$ break this pattern. We

conjecture that partitions with very large and/or small elements generally do this.

Finally, by means of figures 7.6, 7.7, 7.8 and 7.9 we can somewhat examine the effect of the particular function we choose for F on the α-entanglement as calculated. This might tell us whether we are actually observing properties of the states or if we are merely seeing properties of our F. We note that everything we have observed so far using figures 7.1, 7.2 and 7.3 (where F was the von Neumann entropy), continues to be true using either the Tsallis-2 or the Tsallis-3 entropy. We do note that statements about α-entanglement for differing partitions α and different states may change depending on which F we choose. For example, comparing figures 7.3 and either 7.6 or 7.7, we can see that

$$f_{0|1|2345}(GHZ_n) < f_{0|1|23|45}(W_n)$$

holds true in case we use the von Neumann entropy, and in contrast

$$f_{0|1|2345}(GHZ_n) > f_{0|1|23|45}(W_n)$$

is the case when we choose either the Tsallis-2 or the Tsallis-3 entropy.

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