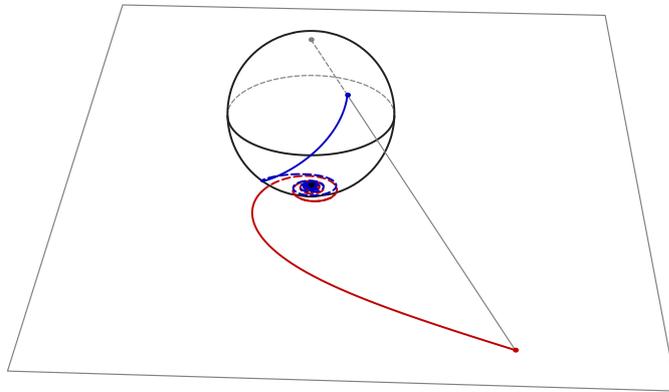# The supersymmetric non-linear sigma model on $SU(2N)$

THESIS

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The supersymmetric non-linear sigma model on $SU(2N)$

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Abstract

In this thesis, the construction of the supersymmetric non-linear sigma model is presented. This model is applied to the symmetry group $SU(2N)$. Several subgroups of this symmetry group are gauged, whereupon the particle spectrum is determined. The thesis concludes with an outlook on how to proceed.

We have to remember that what
we observe is not nature herself,
but nature exposed to our method
of questioning.

Werner Heisenberg,
Physics and Philosophy (1958)

Contents

Chapter **1**

Introduction

All phenomena currently observed can be explained as a result of gravitational, electromagnetic, weak nuclear, or strong nuclear interactions. Of these, gravity is the most familiar for everyday life, as it keeps the planets in orbit around the sun. The force of electromagnetism binds molecules to form gases, liquids, solids, and from these, life. The nuclear forces, as their name implies, act only over small, subatomic distances. The strong force binds protons and neutrons into atomic nuclei. The weak force acts on the resultant nuclei, and causes many to decay. Hence, if fermions and bosons are the building blocks of nature, these four interactions, called the fundamental interactions, form the mortar that binds the blocks into a single structure.

The collection of particles and their interactions form the basis of the Standard Model of elementary particles. Although it provides an extremely accurate description of nature[1], it is considered to be incomplete. For example, it does not explain the phenomena of dark matter and dark energy. It is also expected that new physics is needed at the Planck scale (10^{19} GeV), since at this scale gravitational effects become relevant. Although present experiments yield no conclusive signs of additional structure at at TeV scale, it would be surprising if no new discoveries would be made between the 16 orders of magnitude between the electroweak scale and the Planck scale. This by itself is already a strong suggestion of physics beyond the Standard Model, due to the Hierarchy Problem.[2–6] This implies that the Higgs potential is sensitive to any additions to the Standard Model.

Enter supersymmetry. Supersymmetry, which is the set of transformations relating bosons to fermions and vice versa, was discovered independently by Gervais and Sakita, Golfand and Likhtman, and Volkov and Akulov in the early 1970s.*[8–10] As will be explained further in this thesis, supersymmetry neatly resolves the hierarchy problem.[11] Furthermore, extrapolation of the β-functions and running coupling constants suggest that an approximately supersymmetric particle spectrum greatly facilitates the unification of the electro-weak and color gauge couplings at an energy scale near $10^{15} - 10^{16}$ GeV.[12]

As the Standard Model does not exhibit manifest supersymmetry, any realistic supersymmetric theory must necessarily be broken. The Minimal Supersymmetric Standard Model is an example of broken supersymmetry, where all the elementary particles have complementary partners. However, the mass splittings are largely achieved by hand, rather than a result of the theory itself.

It was discovered by Zumino that the scalar fields of supersymmetry must live in a Kähler manifold, with an explicit example being the Grassmannian manifold $U(N+M)/U(N) \times U(M)$.[13] This has since been extended to general

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*An ealy form of supersymmetry was introduced in the mid 1960s by Miyazawa.[7] Unlike modern supersymmtry, this early supersymmetry did not involve spacetime.

groups G/H.[14–18] A manifold belongs to the branch of mathematics called differential geometry, which has been given its own section further in this thesis. Even so, it is worthwhile to have at least an intuitive picture in mind before proceeding. A (smooth) manifold is a surface of arbitrary dimension. Examples include the surface of a sphere or a torus. A Kähler manifold is a complex manifold which satisfies additional requirements.

The breaking of supersymmetry and the requirement of Kähler manifolds motivated the development of supersymmetric coset models, in particular the coset G/H, where the global symmetry group G is broken down to H.[†] Usually, these symmetries are non-linear. The manifold parametrising these symmetries is described by the non-linear sigma model.[20] Research on these constructions have been meticulously studied, and with the completion of consistent supersymmetric models with non-linear realisations of $SU(5)$, $SO(10)$, E_6 or E_8 new possibilities for grand unification are now available.[21–23] However, there are two problems that arise when one considers the non-linear σ-models used for these coset models: Firstly, the models are not renormalisable. This by itself is not a problem, as the non-linear structure of the model is assumed to hold near the Planck scale. At this energy scale, supergravity must be taken in consideration. As supergravity theories themselves are not renormalisable, it is expected that non-renormalisable couplings might arise in the matter sector. At low energies, the theory should reduce to a renormalisable one.[24] Secondly, the pure non-linear σ-models suffer from anomalies.[25–27] An anomaly arises when a symmetry of the classical theory is not a theory of the quantum theory, implying that the theory is inconsistent. These anomalies can be cancelled by additional supermultiplets carrying representations of the original coset space.[21] Since this thesis is done at the classical level, this problem is mentioned only for completeness.

This thesis considers a construction based on the $U(N+M)/U(N) \times U(M)$ model.[28] The thesis is outlined as follows: in chapter 2, a short review of the mathematical formalisms needed is presented. This includes a short review of Kähler geometry, the Standard Model, as well as supersymmetry from the component formalism. In chapter 3 a construction of the non-linear σ-model and its coupling to matter fields is presented. Once this is done, the full gauge invariant supersymmetric σ-model is derived. In chapter 4 two subgroups of the full symmetry group are gauged, and the resulting particle spectrum is determined. In chapter 5 a procedure for the cancellation of anomalies is presented. In chapter 6 the results are presented, and an outlook on how to proceed is sketched.

Additionally, there are 5 appendices. Appendix A provides a short reference

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[†] For an early review, see [19].

to various Fierz identities introduced in section 2.3.4. Appendix B shows the transformation property of gauge fields necessary in section 3.2. Appendix C gives a detailed derivation of the non-linear transformation of the scalar fields in section 3.2.1. Appendix D then shows how supersymmetry can be made compatible with these non-linear transformations. Finally, appendix E shows how to restore supersymmetry on the non-linear σ-model upon gauging the non-linear symmetries. Of course, the appendices will be refered to if neccessary in the text.

Definitions and concepts

2.1 Differential geometry

In this text we make heavy use of Lie groups. As will be explained later, Lie groups are mathematical groups with the structure of a differentiable manifold. Hence, before delving into group theory, the formalism of differentiable geometry is first explained. A short extension to complex manifolds, the Kähler manifolds, is provided at the end. This section is meant to be a short review. Detailed treatments of differential geometry can be found in many textbooks and syllabi, such as [29–31]. Kähler geometry is covered extensively in [32].

2.1.1 A definition of differentiable manifolds

We can put a vector space structure on any n-dimensional euclidean space E_n isomorphic to $\mathbb{R}^n$. In particular, we can regard any point $P \in E_n$ as the origin of a unique n-dimensional vectorspace, called the tangent space $T_P E_n$. By a continuous choice of orthonormal bases $\{\mathbf{e}_1(P), \ldots, \mathbf{e}_n(P)\}$ of the tangent spaces we can construct the vector bundle $\cup_{P \in E_n} T_P E_n$. The tangent space $T_P E_n$ is then called the fibre over P. Within an open subset U of E_n the fibre bundle looks like $U \times \mathbb{R}^n$. Note that since E_n is isomorphic to $\mathbb{R}^n$, we can speak of the latter in favor of the former. Coordinates on $\mathbb{R}^n$ are given with respect to the standard cartesian basis.

Central in the study of differential geometry lies the notion of regular transformations, defined as follows:

Definition. *Let U be an open subset of $\mathbb{R}^n$. Let there be n differentiable functions $y^1 = y^1(x^1, \ldots, x^n)$, ..., $y^n(x^1, \ldots, x^n)$ of the cartesian coordinates $x^1, \ldots, x^n$ on U such that the jacobian is invertible everywhere on U. Then the y^i are regular coordinates and the coordinate transformation $x^1, \ldots, x^n \to y^1, \ldots, y^n$ is called a regular coordinate transformation.*

Definition. *A subset M of $\mathbb{R}$ is called a k-dimensional differentiable manifold if, given a $P \in M$, there exists a smooth coordinate system $(x^1, \ldots, x^n)$ defined in a neighbourhood U of P, such that*

$$M \cap U = \left\{ P \in U | x^{k+1}(P) = c_1, \ldots, x^n(P) = c_{n-k} \right\}. \tag{2.1}$$

In other words, there exists a regular coordinate transformation such that M looks like the k-dimensional hyperplane in $\mathbb{R}^n$, and looks locally like $\mathbb{R}^k$.

The above definition can be roughly stated as follows: a set M is a differentiable manifold if M can be covered by open collections U_α which look like open

subsets of (are diffeomorphic to) $\mathbb{R}^n$. Thus, for every U_α there exists a homeomorphism (a continuous bijective map with continuous inverse) $\phi_\alpha : U_\alpha \to V_\alpha$ for an open subset $V_\alpha \in \mathbb{R}^n$, such that if $U_\alpha \cap U_\beta \neq \varnothing$, the composition $\phi_\beta^{-1} \circ \phi_\alpha$ is smooth (that is, infinitely differentiable). Note that, for $P \in U_\alpha$, $\phi_\alpha(P) \in \mathbb{R}^n$ yields the coordinates of P. The doublet (U_α, ϕ_α) is called a chart on M. The collection of all charts is called an atlas.

The functions ϕ_α can be used to define differentiability for functions on manifolds:

Definition. *Let M and N be differentiable manifolds. A function $f : M \to N$ is differentiable if for $P \in M$ the function $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is differentiable in $\phi_\alpha(P)$. Here, $P \in U_\alpha$, $f(P) \in V_\beta$. (U_α, ϕ_α) and (V_β, ψ_β) are charts on M and N, respectively.*

For each point $P \in M$ we can again construct the tangent space:

Definition. *Let M be a differentiable manifold of dimension n, $P \in U_\alpha \subset M$ for an open neighbourhood U_α of P in M. A tangent vector to M in P is a map $X : C^\infty(U_\alpha) \to C^\infty(U_\alpha)$ such that*

1. *$X(af + bg) = aX(f) + bX(g)$, for $a, b \in \mathbb{R}$, $f, g \in C^\infty(U_\alpha)$.*

2. *$X(fg)(P) = f(P)X(g) + g(P)X(f)$.*

The collection of all tangent vectors to M in P form the tangent space $T_P M$.

Vectors in the tangent space are spanned by the partial derivative: $\partial_i = \partial/\partial x^i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the ith component is nonzero. The fibre bundle is again defined as the union of all tangent spaces $\cup_P T_P M$ and a vector field is a map $X : M \to T_P M$ such that $X(P) \in T_P M$. In addition, X is differentiable: $X(P) = X^i \partial_i$, where the coefficients X^i are C^∞ functions of the coordinates of P.

The dual to the tangent space is the cotangent space, denoted by $T_P(\mathbb{R}^n)^*$: $T_P(\mathbb{R}^n) \to \mathbb{R}$. If $x^1, \ldots, x^n$ are local coordinates then a basis on the cotangent space is $\{dx^1, \ldots, dx^n\}$, defined by $dx^i(\partial_j) = \delta^i_j$. An element of the cotangent space is called a covector or a 1-form. If $f : M \to \mathbb{R}$ is a differentiable function on M then the differential-1-form is defined such that for a tangent vector on $P \in M$: $df(X) = X(f)$. In terms of the coordinates x^i of a local neighbourhood U_α: $df = \partial_i f dx^i$.

2.1.2 The metric tensor

To make sense of concepts like angles and distance on $\mathbb{R}^n$ one defines the inner product $(,)$. For cartesian coordinates this is by definition $(\partial_i, \partial_j) = \delta_{ij}$. In

terms of regular coordinates $y^1, \dots, y^n$ we can define the metric tensor g. Its components transform according to a covariant tensor of rank two:

$$g_{ij} = \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \right). \tag{2.2}$$

The metric tensor is denoted in various ways:

$$g = \mathrm{d}s^2 = g_{ij}\mathrm{d}y^i \otimes \mathrm{d}y^j = g_{ij}\mathrm{d}y^i\mathrm{d}y^j. \tag{2.3}$$

The inverse metric tensor is defined by the rank 2 contravariant components g^{ij}:

$$g^{ij}g_{jk} = \delta^i_k. \tag{2.4}$$

Using the (inverse) metric we can lower (raise) the indices of vectors and 1-forms, e.g. $v^i = g^{ij}v_j$.

A useful property of differentiable manifolds is that they are "locally flat". Be this we mean that the metric of any differentiable manifold can be written in the canonical form

$$g_{ij} = \eta_{ij} = \mathrm{diag}(-1, \dots, -1, +1, \dots, +1, 0, \dots, 0). \tag{2.5}$$

More on this in the next section. The signature of the metric is determined by the positive and negative eigenvalues of the canonical form. Angles can now be defined as follows: the length of a vector $X \in T_P M$ is defined as $\sqrt{g_P(X, X)}$. The angle θ between two vectors $X, Y \in T_P M$ is defined by

$$\cos\theta = \frac{g_P(X, Y)}{\sqrt{g_P(X, X)}\sqrt{g_P(Y, Y)}}. \tag{2.6}$$

Distance is defined as follows: let $\gamma : [a, b] \to M$ be a smooth curve, and let $x^i, \dots, x^n$ be arbitrary regular coordinates. The length of γ is can now be defined to be

$$L_\gamma = \int_a^b \sqrt{g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}t}\frac{\mathrm{d}x^j}{\mathrm{d}t}}\,\mathrm{d}t. \tag{2.7}$$

The metric tensor also allows us to compare vectors in different tangent spaces: this is done using the covariant derivative ∇_i. The definition depends on whether its argument is a vector or a 1-form (or indeed a general rank (r,s) tensor):

$$\begin{aligned}
\nabla_i v^j &= \partial_i v^j + \Gamma^i_{jk}v^k, \\
\nabla_i w_j &= \partial_i w_j - \Gamma^k_{ij}w_k, \\
\nabla_i T^{jk}_{lm} &= \partial_i T^{jk}_{lm} + \Gamma^j_{ia}T^{ak}_{lm} + \Gamma^k_{ia}T^{ja}_{lm} - \Gamma^a_{il}T^{jk}_{am} - \Gamma^a_{im}T^{jk}_{la}.
\end{aligned} \tag{2.8}$$

The Γ are the Christoffel symbols, constructed from the metric tensor:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \right).$$ (2.9)

Geometrically, whenever a contravariant vector field X has a vanishing covariant derivative, it is said to be parallelly transported from one tangent space to another.

Since this is dependent of the path taken we can define this in terms of curves on M: suppose for a curve γ with parameter t and local coordinates $(x^i(t), \ldots, x^n(t))$, then the covariant derivative of a vector field X along γ is

$$\frac{DX^i}{dt} = \frac{dX^j}{dt} \nabla_j X^i.$$ (2.10)

If $\frac{DX^i}{dt} = 0$ then X is said to be parallel to γ. In addition, for $v \in T_P M$, $v' \in T_Q M$ is said to be the parallel transport of v to Q if there exists a parallel vector field X along γ such that $X(P) = v$ and $X(Q) = v'$.

2.1.3 Vielbeins

A vielbein is a set of vectors $\hat{e}_{(a)}$ in T_p of the manifold M satisfying

$$g(\hat{e}_{(a)}, \hat{e}_{(b)}) = \eta_{ab},$$ (2.11)

where η_{ab} is the canonical form of the metric. Similarly, one-forms $\hat{\theta}^{(a)}$ in T_p satisfy

$$\hat{\theta}^{(a)}(\hat{e}_{(b)}) = \delta^a_b.$$ (2.12)

We can express the coodinate basis $\hat{e}_{(i)} = \partial_i$ in this basis:

$$\hat{e}_{(i)} = e_i{}^a \hat{e}_{(a)}.$$ (2.13)

Inverses of the matrices $e_i{}^a$ satisfy

$$e^i{}_a e_j{}^a = \delta^i_j,$$
$$e_a{}^i e^i{}_b = \delta^a_b.$$

The vielbeins imply

$$g_{ij} = e_i{}^a e_j{}^b \eta_{ab},$$ (2.14)

which quantifies the statement that in this basis, the metric is "locally flat". Basis transformations can be realised using the (1,1) tensor

$$e = e_i{}^a dx^i \otimes \hat{e}_{(a)}.$$ (2.15)

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The advantage of this prescription is that in noncoordinate bases coordinates and bases (by local Lorentz transformations) can be transformed independently:

$$T^{a'i'}{}_{b'j'} = \Lambda^{a'}{}_a \frac{\partial x^{i'}}{\partial x^i} \Lambda^b{}_{b'} \frac{\partial x^j}{\partial x^{j'}} T^{ai}{}_{bj}. \tag{2.16}$$

2.1.4 Curvature

Another important object that can be constructed from the metric tensor is the Riemann tensor. This tensor quantifies the intrinsic curvature of the manifold. It is defined as

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} - \partial_l \Gamma^i_{jk} + \Gamma^i_{ka}\Gamma^a_{jl} - \Gamma^i_{ja}\Gamma^a_{jk}. \tag{2.17}$$

This definition follows immediately from the following identity:

$$\left[\nabla_i, \nabla_j\right] Z_k = R^l{}_{k\,ij} Z_l - T^l_{ij} \nabla_l Z_k, \tag{2.18}$$

where we've introduced the torsion tensor

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}. \tag{2.19}$$

If the connection coefficients turn out to be symmetric (that is, if the metric is free of torsion), equation (2.18) further simplifies to

$$\left[\nabla_i, \nabla_j\right] Z_k = R^l{}_{k\,ij} Z_l. \tag{2.20}$$

This form is known as the Ricci identity for the vector field Z.

2.1.5 p-forms and the exterior product

The notion of 1-forms can be generalised to any $(0, s)$-covariant tensor. We can proceed as follows. Let S be a tensor of rank (r, s), T a tensor of rank $(t, u$, then the tensor product $S \otimes T$ results in a tensor of rank $(r + t, s + u)$ according to

$$(S \otimes T)(v^1, \dots, v^r, w^1, \dots, w^t, x_1, \dots, x_s, y_1 \dots, y_u) =$$
$$S(v^1, \dots, v^r, x_1, \dots, x_s) T(w^1, \dots, w^t, y_1 \dots, y_u).$$

For a given covariant tensor T of rank s, we can define the antisymmatrisator A by

$$A(T)(v^1, \dots, v^s) = \frac{1}{s!} \sum_{\mathbb{P}} \epsilon^{\mathbb{P}(1)\dots\mathbb{P}(s)} T(v^{\mathbb{P}(1)}, \dots v^{\mathbb{P}(1)}), \tag{2.21}$$

where ϵ is the Levi-Civita symbol. We sum over all permutations $\mathbb{P}$ of $1, \dots, r$.

Denote the vector space of antisymmetric covariant rank k tensors on the manifold n-dimensional M by $\bigwedge^k M$ and the direct sum of these spaces by $\oplus_{r=0}^n \bigwedge M = \bigwedge M$. Then the exterior product $S \wedge T$ for covariant matrices of rank s and r, respectively, is defined as

$$S \wedge T = \frac{(r+s)!}{r!s!} A(S \otimes T) \tag{2.22}$$

Thus, $\bigwedge M$ is realised as a Grassmann algebra. This is important in the context of supersymmetry. Elements of $\bigwedge^p M$ are called p-forms.

Besides the external product, we can also define the external derivative:

Definition. *Let α, β respectively be a p- and q-form on a manifold M. The exterior derivative of a p-form on M is the operation $\mathrm{d} : \bigwedge^p(M) \to \bigwedge^{p+1}(M)$ for $p \in \mathbb{N}$ with the following properties:*

1. *$\mathrm{d}(\alpha + \beta) = \mathrm{d}\alpha + \mathrm{d}\beta$ if $p = q$.*

2. *$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p \alpha \wedge \mathrm{d}\beta$.*

3. *$\mathrm{d}^2\alpha = \mathrm{dd}\alpha = 0$.*

A p-form α is called closed if its exterior derivative vanishes ($\mathrm{d}\alpha = 0$) and exact if α is itself the exterior derivative of a $p-1$-form ($\alpha = \mathrm{d}\beta$). Poincaré's lemma guarantees that all closed forms on contractible manifolds are exact. This will be important when considering complex manifolds.

Definition. *Let M, N be differentiable manifolds and let $f : N \to \mathbb{R}$ be a differentiable function. Let $P \in M$. The pullback is the function $f_* : T_P M \to T_{f(P)} N$ such that for any $g : N \to \mathbb{R}$ and $X \in T_P M$*

$$f_*(X)(g) \circ f = X(g \circ f). \tag{2.23}$$

Definition. *Let M, N and f be given as above. Let ω be a covariant in $\left(T_{f(P)} N^*\right)^{\otimes k}$. The pullback $f^*\omega$ is the tensor*

$$f^*\omega(X_1, \ldots, X_n) = \omega(f_* X_1, \ldots, f_* X_n). \tag{2.24}$$

2.1.6   The Lie derivative and Killing vectors

A vector field X defines a flow through any point on the manifold M. The change of a tensor field along the flow of X is determined by the Lie derivative. Before we define the Lie derivative, we give the definition of the flow of X:

Definition. *Let X be a vector field on a smooth manifold M. The flow of X through $P \in M$ is a mapping $f_P : (-\alpha, \alpha) \times M \to M$ such that*

1. $f(0, P) = P$.

2. $\frac{\mathrm{d}}{\mathrm{d}t} f(t, P) = X(f(t, P))$.

For a vector field Y, the Lie derivative is defined as

$$L_X Y = [X, Y], \tag{2.25}$$

where $[X, Y] = X(Y) - Y(X)$ is the commutator or Lie bracket. The action of the Lie bracket on a differentiable function f is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)). \tag{2.26}$$

Thus, if X and Y are tangent vectors in $T_P M$, $[X, Y] \in T_P M$. The Lie derivative is important when considering isometries.

Definition. *Let M and N be smooth manifolds with metric tensors g_M and g_N, respectively. A differentiable function $f : M \to N$ is called an isometry if $f^* g_N = g_M$.*

Isometries are generated by Killing vector fields. Equivalently, translations along these vector fields leave the metric invariant.

Definition. *A vector field X on a smooth manifold M with metric tensor g is a Killing vector field if $L_X g = 0$. Equivalently, X is a Killing vector field if $\nabla_i X_j + \nabla_j X_i = 0$, where ∇_j is the covariant derivative on M.*

2.1.7 Kähler geometry

The concept of differential geometry can be extended to include complex coordinates $(z^\alpha, \overline{z}^{\underline{\alpha}})$, where $\overline{z}^{\underline{\alpha}}$ is the conjugate of z^α. Locally, the metric is

$$g_{\alpha, \underline{\beta}} = g \left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial \overline{z}^{\underline{\beta}}} \right). \tag{2.27}$$

Given any complex metric g with $g_{\alpha \underline{\beta}} = (g_{\beta \underline{\alpha}})^*$ (the metric is therefore hermitian), the fundamental two-form Ω in local holomorphic coordinates can be expressed as

$$\Omega = i g_{\alpha \underline{\beta}} \mathrm{d}z^\alpha \wedge \mathrm{d}\overline{z}^{\underline{\beta}}. \tag{2.28}$$

A Kähler manifold is given by a metric which satisfies[33]

$$g_{\alpha \underline{\beta}, \gamma} = g_{\gamma \underline{\beta}, \alpha}, \quad g_{\alpha \underline{\beta}, \underline{\gamma}} = g_{\alpha \underline{\gamma}, \underline{\beta}}. \tag{2.29}$$

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This condition is equivalent to the closure of Ω: $d\Omega = 0$. By Poincaré's lemma, Ω is exact. In fact this is equivalent to the existence of a scalar function K from which the metric can locally be determined by

$$g_{\alpha\underline{\beta}} = \frac{\partial^2 K}{\partial z^\alpha \partial \overline{z}^{\underline{\beta}}}. \tag{2.30}$$

This function is the Kähler potential. It is defined up to holomorphic transformations of the form

$$K(z^\alpha, \overline{z}^{\underline{\alpha}}) \rightarrow K'(z^\alpha, \overline{z}^{\underline{\alpha}}) = K(z^\alpha, \overline{z}^{\underline{\alpha}}) + F(z^\alpha) + \overline{F}(\overline{z}^{\underline{\alpha}}). \tag{2.31}$$

This implies that the Kähler potentials from different coordinate systems are related: if two local coordinate charts $\{z_i\}$ and $\{z_j\}$ have a non-empty intersection, the corresponding potentials satisfy

$$K_i(z_i, \overline{z}_i) = K_j(z_j, \overline{z}_j) + F_{(ij)}(z_j) + \overline{F}_{(ij)}(z_j). \tag{2.32}$$

In the absence of torsion, the Levi-Civita connection is nonzero only in the case of unmixed indices:

$$\Gamma_{\alpha\beta}{}^\gamma = g^{\underline{\lambda}\gamma} g_{\alpha\underline{\lambda},\beta}, \quad \overline{\Gamma}_{\underline{\alpha}\underline{\beta}}{}^{\underline{\gamma}} = g^{\gamma\underline{\lambda}} g_{\lambda\underline{\beta},\underline{\alpha}}, \tag{2.33}$$

where $g^{\gamma\underline{\lambda}} = (g^{-1})_{\underline{\lambda}\gamma}$ and the comma denotes differentiation with respect to the complex coordinates. The non-vanishing components of the Riemann tensor can be shown to be

$$
\begin{aligned}
R_{\alpha\underline{\beta}\gamma\underline{\delta}} &= g_{\delta\underline{\delta}} \Gamma_{\alpha\gamma}{}^\delta{}_{,\underline{\beta}} = g_{\alpha\underline{\delta},\gamma\underline{\beta}} - g_{\kappa\underline{\lambda}} \Gamma_{\alpha\gamma}{}^\kappa \overline{\Gamma}_{\underline{\beta}\underline{\delta}}^{\underline{\lambda}} \\
&= g^{\underline{\zeta}\zeta} g_{\alpha\underline{\zeta},\gamma} g_{\zeta\underline{\beta},\underline{\delta}}.
\end{aligned}
\tag{2.34}
$$

Coordinate transformations which leave the metric invariant are again the Killing vectors of the manifold, but since these vectors are generally complex, the Killing condition is slightly modified: let ζ^α be a Killing vector, then ζ^α satisfies

$$\zeta_{\underline{\beta},\alpha} + \overline{\zeta}_{\alpha,\underline{\beta}} = 0, \tag{2.35}$$

where $\zeta_{\underline{\beta}} = g_{\alpha\underline{\beta}} \zeta^\alpha$, and the comma denotes differentiation with respect to the manifold coordinates. Applying a second covariant derivative to (2.35) and using the complex equivalent of the Ricci identity (2.20)

$$\left[\nabla_{\underline{\beta}}, \nabla_\alpha \right] \zeta_{\underline{\alpha}} = R_{\alpha\underline{\beta}\underline{\beta}\underline{\alpha}} \zeta^\beta, \tag{2.36}$$

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we find that, after using the facts that the Killing vectors are holomorphic and the metric is covariantly constant, that the Killing vectors satisfy the following relation:

$$\nabla_{\underline{\alpha}}\nabla_{\underline{\beta}}\overline{\zeta}_\alpha = -R_{\alpha\underline{\beta}\beta\underline{\alpha}}\zeta^\beta. \tag{2.37}$$

As the Killing vectors represent invariances of the Kähler manifold, the Killing vectors must obey a Lie-algebra structure:

$$\zeta_A^\beta \zeta_{B,\beta}^\alpha - \zeta_B^\beta \zeta_{A,\beta}^\alpha = f_{AB}{}^C \zeta_C^\alpha, \tag{2.38}$$

where, as it turns out, the $f_{AB}{}^C$ are the Lie algebra's totally antisymmetric structure constants.

The complex structure of (2.35) allows the Killing vectors to be derived locally from a single real scalar function M:

$$\zeta_{\underline{\beta}} = -i\frac{\delta M}{\delta \overline{z}^{\underline{\beta}}}, , \tag{2.39}$$

$$\overline{\zeta}_\alpha = i\frac{\delta M}{\delta \overline{z}^\alpha}, \tag{2.40}$$

with

$$\zeta_{\underline{\beta}} = g_{\alpha\underline{\beta}}\zeta^\alpha. \tag{2.41}$$

These equations define M up to a constant of integration. However, it turns out to be convenient to choose these constants such that the potentials transform according to the adjoint representation of the Lie algebra (the details of which will be explained in the next section) of the Killing vectors:

$$\delta_i M_j = R_i^A \frac{\delta M_j}{\delta \overline{z}^B} + \overline{R}_i^{\underline{A}} \frac{\delta M_j}{\delta \overline{z}^{\underline{A}}} = f_{ij}{}^k M_k. \tag{2.42}$$

Using the Killing and Kähler potentials, it can be shown that the under Killing transformations the transfer functions in (2.31) take the following form:

$$\begin{aligned} F_i &= \frac{\delta K}{\delta z^\alpha}R_i^\alpha + iM_i, \\ \overline{F}_i &= \frac{\delta K}{\delta \overline{z}^{\underline{\beta}}}\overline{R}_i^{\underline{\beta}} - iM_i. \end{aligned} \tag{2.43}$$

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Under Killing transformations, these functions satisfy

$$
\begin{aligned}
\delta_i F_j - \delta_j F_i &= \frac{\delta^2 K}{\delta\phi^\alpha \delta\phi^\beta}\left(R_j^\alpha R_i^\beta - R_i^\alpha R_j^\beta\right) + \frac{\delta K}{\delta\phi^\alpha}\left(\frac{\delta R_j^\alpha}{\delta\phi^\beta}R_i^\beta - \frac{\delta R_i^\alpha}{\delta\phi^\beta}R_j^\beta\right) \\
&\quad + i\left(\frac{\delta M_j}{\delta\phi^\alpha}R_i^\alpha - \frac{\delta M_i}{\delta\phi^\alpha}R_j^\alpha\right) \\
&= f_{ij}{}^k\left(\frac{\delta K}{\delta\phi^\alpha}R_k^\alpha + i M_k\right) \\
&= f_{ij}{}^k F_k.
\end{aligned}
\tag{2.44}
$$

This follows immediately from the definition of the transfer functions (3.30) in the first step, the Lie algebra spanned by the Killing vectors (2.38) in the second step, and the use of equation (2.39) and the adjoint transformation property of the Killing potential (2.42) in the last step.

2.2 Lie algebras

Group theory is often useful in the description of physical phenomena; examples of areas of physics where group theory is relevant include crystallography, special relativity, quantum mechanics and particle physics. Problems in these branches of physics can often be greatly simplified by exploiting the symmetries of the models under study. As a symmetry often comprises a set of operations which leave a certain quantity invariant, these operations have certain properties in common which is described by mathematical groups. In this section, a short introduction to group theory is presented. The major concepts of group theory as they are used in this thesis are defined and elaborated upon. Extensive treatments of group theory, Lie algebra and their applications to physics can be found in [31, 34, 35].

2.2.1 Definitions of group theory

A group is a mathematical object satisfying the following definition:

Definition. *A group is a (non-empty) set $G = \{g_i\}$ which satisfies the following properties:*

- *It has a associative multiplication $\circ$ under which it is closed: $g_i \circ g_j = g_k \in G$, for $g_i, g_j \in G$.*

- *For $g_i, g_j, g_k \in G$ the multiplication is distributive. In other words, it satisfies $g_i \circ (g_j \circ g_k) = (g_i \circ g_j) \circ g_k$.*

- *$\exists\, e \in G$ such that $e \circ g_i = g_i \circ e = g_i\ \forall g_i \in G$. This is the (unique) identity element.*

- *$\forall g_i\ \exists\, g_i^{-1} : g_i \circ g_i^{-1} = g_i^{-1} \circ g_i = e$. From this definition we can see that, in group theory, left and right inverses are identical, and unique.*

In general the operation $\circ$ is not commutative: $g_i \circ g_j \neq g_j \circ g_i$. If the commutativity equation $g_i \circ g_j = g_j \circ g_i$ holds for all $g_i, g_j \in G$ then G is called abelian. Furthermore, a group can contain a finite or infinite number of elements. If it contains a finite number of elements it is said to be finite, if it contains an infinite number of elements it is said to be infinite.

example The set of invertible complex $n \times n$ matrices forms a group. This group is called the general linear group, and denoted $GL(n, \mathbf{C})$.

Definition. *A subset H of a group G is a subgroup if H itself is a group under the multiplication ∘:*

- *$e \in H$.*

- *If $h_1, h_2 \in H$, then $h_1 \cdot h_2 \in H$.*

- *If $h \in H$, then $h^{-1} \in H$.*

example The set of $n \times n$ matrices U satisfying $UU^\dagger = U^\dagger U = 1$ is a subgroup of $GL(n, \mathbf{C})$. This group is called the unitary group $U(N)$. If $V \in U(N)$ in addition satisfies $\det V = 1$ then these matrices form the subgroup $SU(N)$.

In order to compare different groups one considers structure preserving mappings between these groups. These mappings are homomorphisms, and are defined as follows:

Definition. *Let G, G' be groups with group multiplications $\circ$, $\circ'$, respectively. A map $f : G \to G'$ is called a homomorphism if it preserves the group structure, that is, if it obeys $f(g_1 \circ g_2) = f(g_1) \circ' f(g_2)$ for $g_1, g_2 \in G$. If in addition f is a bijection, f is called an isomorphism, and G and G' are said to be isomorphic, which is denoted as $G \simeq G'$.*

Definition. *Let G be a group and let $g_1, g_2, g_3 \in G$. An equivalence relation is a binary relation $\sim$ satisfying the following properties*

- *It is reflexive: $g_1 \sim g_1$.*

- *It is symmetric: if $g_1 \sim g_2$, then $g_2 \sim g_1$.*

- *It is trasitive: if $g_1 \sim g_2$ and $g_2 \sim g_3$, then $g_1 \sim g_3$.*

if $g_1 \sim g_2$ then the two elements are said to be equivalent. Equivalent elements form a set called an equivalence class.

Definition. *Let G be a group and H be a subgroup. Define the equivalence relation $\sim$ for $g_1, g_2 \in G$ as follows: $g_1 \sim g_2$ if and only if $g_1 = hg_2$, for $h \in H$. Equivalence classes obtained in this way are called left-cosets, and are denoted by gH. Similarly one can define right-cosets Hg. Sets of left-cosets gH form the coset group G/H.*

If the left-coset of a subgroup H of G is equal to the right-coset, H is called an invariant subset of G.

Definition. *Let G be a group and H_1, H_2 be subgroups of G. G is said to be the direct product of H_1, H_2 (denoted by $G = H_1 \times H_2$, if*

- $h_1 h_2 = h_2 h_1 \forall h_1 \in H_1, h_2 \in H_2$.

- $\forall g \in G$ *the following equation holds:* $g = h_1 h_2$, *for* $h_1 \in H_1, h_2 \in H_2$.

- *The decomposition of (ii) is unique.*

Definition. *Let G be a group and V be a Hilbert space. A representation is a homomorphism $T : G \rightarrow GL(V)$, where $GL(V)$ is the set of invertible linear operators on V, the latter is also called the representation space. The dimension of T is equal to the dimension of V. If T is injective the representation is said to be faithful.*

2 different classes of representations can be distinguised. First we need the following definitions.

Definition. *Let W be a linear subspace of a representation space V. W is called an invariant if for all $w \in W$ the orbit $\left\{ T_g(w) : g \in G \right\}$ is a subset of W: $T_g(W) \subset W$.*

The classes of representation of interest can then be distinguised as follows:

Definition. *A representation T is called reducible if there are invariant linear subspaces U and V of W such that $W = U \oplus V$. If a representation is not reducible it is irreducible.*

A finite dimensional representation T is a direct sum of irreducible representations:

$$T = \bigoplus_i m_i T^i, \tag{2.45}$$

with T^i the irreducible representations and m_i their degeneracies.

Lie groups and algebras

As noted in the beginning of the previous section, group theory in physics was introduced as a mathematical tool useful for the description of symmetries. A special kind of symmetry is a symmetry parametrised by a set of numbers, called a continuous symmetry. For example, the matrix

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.46}$$

is an element of $SU(2)^*$, and parametrised by a single parameter θ in a continuous and differentiable manner. Operations such as the group multiplication and the inverse map are therefore differentiable maps. A group exhibiting such a continuous symmetry is called a Lie group. The parameters of a Lie group can locally be used as coordinates in euclidean space. Thus, Lie groups are differentiable manifolds, equipped with a group structure:

---

*It is also an element of the group $SO(2)$, but as the latter is a subgroup of the former this does not pose any complications.

Definition. *A Lie group G is a differentiable manifold equipped with a group structure such that the group product $G \times G \to G$ and the inverse map $g \to g^{-1}$ are both differentiable. A n-dimensional manifold corresponds with an n-parameter Lie group.*

Consider the $SU(2)$ matrix in (2.46). This matrix represents a rotation in the plane: a vector $\mathbf{x}$ is transformed as

$$\mathbf{x} \to \mathbf{x}' = U\mathbf{x}.$$

Suppose we expand U about the identity. This means that we consider infinitesimal rotations, such that the infinitesimal transformation of a vector $\mathbf{x} \in \mathbb{R}^2$ is given by

$$\delta\mathbf{x} = (U - I)\mathbf{x} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R(\theta)\mathbf{x}.$$

From the group structure of rotations, we then find that

$$U(\theta) = (R(\theta/n))^n \implies U(\theta) = \lim_{n \to \infty} (R(\theta/n))^n = \lim_{n \to \infty} \left(I + \frac{\theta}{n}J \right)^n = e^{\theta J},$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The matrix exponential is defined by its formal power series. In fact, using the power series the group structure can be explicitly verified:

$$e^{\theta J} = \sum_{n=0}^{\infty} \frac{1}{n!}(\theta J)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n)!}(-1)^n \theta^{2n} I + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(-1)^n \theta^{2n+1} J$$

$$= \cos\theta I + \sin\theta J = U(\theta)$$

It is conventional to let J be hermitian. Thus U can be written as the exponential of the Pauli matrix σ_2. This approach works quite generally: any unitary matrix can be written as the exponential of a hermitian matrix. These hermitian matrices are said to generate the group of unitary matrices. More on $SU(N)$ in the next section.

Before proceeding we note that the above is a specific example of a general theorem: A Lie group is generated by a Lie algebra. The Lie algebra has additional properties, which we now define:

Definition. *Let G be an n-parameter Lie goup. A Lie algebra g is a vector space with an extra operation $[,] : g \times g \to g$, called the Lie-bracket, which has the following properties for $T_i \in g$:*

1. *It is linear:* $\left[\lambda T_i + \mu T_j, T_k\right] = \lambda[T_i, T_k] + \mu\left[T_j, T_k\right].$

2. *It is antisymmetric:* $\left[T_i, T_j\right] = -\left[T_j, T_i\right].$

3. *The Lie bracket satisfies the Jacobi identity:*

$$\left[T_i, \left[T_j, T_k\right]\right] + \left[T_j, \left[T_k, T_i\right]\right] + \left[T_k, \left[T_i, T_j\right]\right] = 0. \tag{2.47}$$

This defines a vector space isomorphic to $T_e G$, or the tangent space of G at the identity. Hence, the commutator of any two elements of g can be expressed as a linear combination of elements:

$$\left[T_i, T_j\right] = f_{ij}{}^k T_k. \tag{2.48}$$

The $f_{ij}{}^k$ are the (completely antisymmetric) structure constants of the algebra.

Plugging equation (2.48) into the Jacobi identity yields the requirement

$$f_{jk}{}^l f_{il}{}^m + f_{ki}{}^l f_{jl}{}^m + f_{ij}{}^l f_{kl}{}^m = 0. \tag{2.49}$$

2.2.2 $SU(N)$

In this thesis we will work extensively with the group $SU(2N)$, which is the set of $2N$-dimension matrices U with the property $UU^\dagger = 1$ and $\det U = 1$. In this part of the thesis, we will give the properties of the general $SU(N)$ group. The extension is trivial.

Any element of $SU(N)$ can be parametrised by $N^2 - 1$ traceless hermitian matrices:

$$U = e^{i\alpha_i T_i}, \quad T^\dagger = T. \tag{2.50}$$

In light of the example above, an important theorem is the Baker-Campbell-Hausdorff theorem, which states that form two square matrices X and Y:

$$e^X e^Y = e^Z, \tag{2.51}$$

with

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \dots. \tag{2.52}$$

The dots indicate higher order commutators of X and Y. The Baker-Campbell-Hausdorff theorem together with the parametrisation in equation (2.50) shows that the hermitian matrices satisfy equation (2.48). Hence, they form a basis of $SU(N)$.

The $N^2 - 1$ generators can be labelled as T_{ij}, with $i, j = 1, 2, \ldots, N$. In this case, the Lie bracket becomes

$$[T_{ij}, T_{kl}] = \delta_{jk} T_{il} - \delta_{il} T_{kj},$$

from which the structure constants can be easily found by inspection. Alternatively, a specific basis for the generators can be found by the method of generalised Pauli and Gell-Mann matrices.[†] Generators in this basis are denoted τ^i, and are normalised via $\mathrm{Tr}\left[\tau^i \tau^j\right] = \delta^{ij}/2$. The structure constants can be computed via

$$\begin{aligned} f^{ijk} &= -2i\mathrm{Tr}\left(\left[T^i, T^j\right]T^k\right) \\ &= \frac{1}{4i}\mathrm{Tr}\left(\left[\tau^i, \tau^j\right]\tau^k\right). \end{aligned} \tag{2.53}$$

A second set of completely symmetric structure constants can be computed via

$$\begin{aligned} d^{ijk} &= -2i\mathrm{Tr}\left(\left\{T^i, T^j\right\}, T^k\right) \\ &= \frac{1}{4i}\mathrm{Tr}\left(\left\{\tau^i, \tau^j\right\}\tau^k\right). \end{aligned} \tag{2.54}$$

2.2.3 Representations of $SU(N)$

In general, one can distinguish 2 representations of $SU(N)$ that are important for our purposes. These are listed below.

The defining representation The defining representation is the representation that defines the group (or algebra). Let α and β be N dimensional complex vectors. Equivalent to the definition given above, $U(N)$ can be defined as the set of matrices which leave the bilinear form

$$\bar{\alpha}\beta \tag{2.55}$$

invariant.[‡] Hence, an element in the fundamental representation of $SU(N)$ is the N-dimensional complex vector. The group elements realise linear transformations of the vector space spanned by these complex vectors. As the vector is complex, there exists a second representation, called the conjugate representation $\overline{N}$.

[†] A straightforward construction of these generalisations can be found in [36].

[‡]Here, pure phase transformations of the form $\alpha \to e^{i\theta}\alpha$ are omitted. This yields the requirement that the generators be traceless.

The adjoint representation   The adjoint representation $\mathrm{ad} : g \to gl(g)$ maps $X \in g$ to $\mathrm{ad}_X : g \to g$, with

$$\mathrm{ad}_X(Y) = [X, Y],$$

for $Y \in g$. From the Jacobi identity it follows that the adjoint representation is a derivation on g. The generators of $SU(N)$ belong to the adjoint representation; in fact, the representation matrices are constructed from the structure constants by

$$\left(T^b\right)^c_a = i f^{abc}.$$

The adjoint representation therefore presents a way for the elements of the Lie group to act on the elements on the algebra. For $SU(2N)$, the Lie algebra is the vector space of $N \times N$ traceless hermitian matrices, and the elements U of the Lie group acts on the adjoint representation as

$$V \to UVU^\dagger.$$

particle	colour multiplicity	isospin multiplicity	isospin I_3	hypercharge Y	electric charge $Q = Y + I_3$
ν_L	1	2	$+1/2$	$-1/2$	0
e_L	1	2	$-1/2$	$-1/2$	-1
ν_L^c	1	1	0	0	0
e_L^c	1	1	0	$+1$	$+1$
u_L	3	2	$+1/2$	$+1/6$	$2/3$
d_L	3	2	$-1/2$	$+1/6$	$-1/3$
u_L^c	3	1	0	$-2/3$	$-2/3$
d_L^c	3	1	0	$+1/3$	$+1/3$
g	8	1	0	0	0
W^+	1	3	$+1$	0	$+1$
W^0	1	3	0	0	0
W^-	1	3	-1	0	-1
B	1	1	0	0	0
H^+	1	2	$+1/2$	$+1/2$	$+1$
H^-	1	2	$-1/2$	$+1/2$	0

Table 2.1: *Particle content of the Standard Model, together with the charges with respect to the gauge group.*

2.3 The Standard Model

All known matter is composed of elementary particles, which fall in one of three catagories: leptons, quarks, and mediators. These parcticles and all their interactions, with the exception of gravity, are described by the theory known as the Standard Model. This section provides a short introduction of the concepts used to construct the mathematical tools used to derive the field equations for matter, and its interactions. For a detailed review, we refer to [37–40].

2.3.1 Dirac algebra

The Dirac matrices are taken to be normalised by the Clifford algebra

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}, \tag{2.56}$$

where the unit matrix on the right-hand side is implied. In Minkowski space (with metric signature mostly plus, so $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$) we choose representations such that γ_0 is anti-hermitian and

$$\gamma_a^\dagger = \gamma_0 \gamma_a \gamma_0. \tag{2.57}$$

Furthermore, we define the chirality matrix

$$\gamma_5 = \frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = i\gamma_0\gamma_1\gamma_2\gamma_3. \tag{2.58}$$

The Levi-Civita symbol ϵ_{abcd} is defined such that

$$\epsilon_{0123} = -\epsilon^{0123} = 1.$$

γ_5 has the property that it anti-commutes with any of the Dirac matrices and squares to unity. We define the spinor matrices by

$$\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b] = \frac{1}{2}(\gamma_a\gamma_b - \eta_{ab}) \tag{2.59}$$

It can be shown that the spinor matrices span a Lie algebra:

$$\left[\sigma_{\mu\nu}, \sigma_{\kappa\lambda}\right] = \eta_{\nu\kappa}\sigma_{\mu\lambda} - \eta_{\nu\lambda}\sigma_{\mu\kappa} - \eta_{\mu\kappa}\sigma_{\nu\lambda} + \eta_{\mu\lambda}\sigma_{\nu\kappa}. \tag{2.60}$$

Thus, the spinor matrices form a representation of the Lorentz group. A spinor is then defined as a four-component object ψ which transform as

$$\psi' = e^{\frac{1}{2}\omega^{\mu\nu}\sigma_{\mu\nu}}\psi \tag{2.61}$$

under Lorentz transformations.

Contractions with gamma matrices

Linear operators can be used to construct Lorentz invariant operators which act on spinors. Frequently, these operators are contracted with the Dirac γ matrices. A notation that is used throughout this thesis is the Feynman slash: for any operator O_μ the Feynman slash is defined as

$$\slashed{O} = \gamma^\mu O_\mu = \gamma \cdot O.$$

2.3.2 Charge conjugation

Given a spinor satisfying the free Dirac equation, we can define the charge-conjugate spinor by

$$\psi^c = C\overline{\psi}^T. \tag{2.62}$$

In the special case that a spinor is equal to its charge conjugate it is called a Majorana spinor. The anti-symmetric unitary matrix C is the charge conjugate matrix, satisfying the following properties:

- $C = -C^T$

- $C^{-1}\gamma_a C = -\gamma_a^T$

These properties imply that the extended Dirac algebra can be split into 10 symmetric elements:

$$\gamma_\mu C = \left(\gamma_\mu C\right)^T, \quad \sigma_{\mu\nu} C = \left(\sigma_{\mu\nu} C\right)^T, \tag{2.63}$$

and 6 anti-symmetric elements:

$$C = -C^T, \quad \gamma_5 C = -(\gamma_5 C)^T, \quad \gamma_5 \gamma_\mu C = -\left(\gamma_5 \gamma_\mu C\right)^T. \tag{2.64}$$

Contractions of Majorana spinors with elements of (2.63) and (2.64) satisfy flip properties, in which the order of contractions is reversed. Let Γ denote an element from the Dirac algebra. Then a general contraction of two spinors η and ϵ can be written as

$$\overline{\eta}\Gamma\epsilon. \tag{2.65}$$

Using equation (2.62) this can be written as

$$\mp\overline{\epsilon}\Gamma^T\eta. \tag{2.66}$$

Comparison with (2.63) and (2.64) then yields the following identities:

$$\overline{\eta}\epsilon = \overline{\epsilon}\eta, \tag{2.67}$$
$$\overline{\eta}\gamma_\mu\epsilon = -\overline{\epsilon}\gamma_\mu\eta, \tag{2.68}$$
$$\overline{\eta}\sigma_{\mu\nu}\epsilon = -\overline{\epsilon}\sigma_{\mu\nu}\eta, \tag{2.69}$$
$$\overline{\eta}\gamma_5\epsilon = \overline{\epsilon}\gamma_5\eta, \tag{2.70}$$
$$\overline{\eta}\gamma_\mu\gamma_5\epsilon = \overline{\eta}\gamma_\mu\gamma_5\eta, \tag{2.71}$$
$$\overline{\eta}\slashed{\partial}\epsilon = \overline{\epsilon}\slashed{\partial}\eta. \tag{2.72}$$

Equation (2.72) is a corollary of (2.68) when applied to the spinor action. As such, it is valid only under integration by parts. Using (2.72), another useful identity is readily proved:

$$\int \overline{\psi}\slashed{\partial}\psi \, \mathrm{d}^4 x = \int \left[\overline{\psi}_L\slashed{\partial}\psi_L + \overline{\psi}_R\slashed{\partial}\psi_R\right] \mathrm{d}^4 x$$
$$= \int \left[\overline{\psi}_L\slashed{\partial}\psi_L + \overline{\psi}_R\overleftarrow{\slashed{\partial}}\psi_R\right] \mathrm{d}^4 x$$
$$= \int \overline{\psi}_L\overleftrightarrow{\slashed{\partial}}\psi_L \, \mathrm{d}^4 x,$$

where $\alpha\overleftrightarrow{\partial_\mu}\beta \equiv \alpha\partial_\mu\beta - \partial_\mu(\alpha)\beta$. An equivalent expression holds for ψ_R.

2.3.3 Chirality

From (2.58) it follows that γ_5 squares to unity. Hence, the operator

$$P_\pm = \frac{1 \pm \gamma_5}{2}$$

are projection operators. This can be exploited to introduce the notion of chirality for spinors: a chiral spinor is defined as an eigenspinor of γ_5. In this text, the eigenspinors with eigenvalue $+1$ are called right-handed, while the eigenspinors with eigenvalue -1 are called left-handed. Hence, from any given Majorana spinor ψ we can construct a chiral spinor by

$$\psi_R = P_+\psi, \quad \psi_L = P_-\psi,$$

where

$$\begin{aligned}
\gamma_5\psi_R &= \psi_R, \\
\gamma_5\psi_L &= -\psi_L.
\end{aligned} \tag{2.73}$$

From this follows the property that right-handed and left-handed spinors are each others charge conjugate:

$$\begin{aligned}
\psi_L &= \frac{1-\gamma_5}{2}\psi = \frac{1-\gamma_5}{2}C\overline{\psi}^T \\
&= C\frac{1+\gamma_5^T}{2}\overline{\psi}^T = C\overline{\frac{1+\gamma_5}{2}\psi}^T \\
&= (\psi_R)^C,
\end{aligned}$$

and similarly for ψ_R. From this follows that a chiral spinor can only be a solution of the free Dirac equation if it describes a massless particle. More on this later.

Charge conjugation identities for chiral spinors are similar to equations (2.67)-(2.72) with the difference being that the spinors on the right hand side are replaced by their charge conjugates, for example:

$$\overline{\eta}_R\gamma_\mu\epsilon_R = -\overline{\epsilon}_L\gamma_\mu\eta_L. \tag{2.74}$$

2.3.4 Fierz decomposition

The set $\Gamma = (1, \gamma_a, \sigma_{ab}, \gamma_5\gamma_a, \gamma_5)$ forms a basis on the vector space of 4×4 matrices. Hence, any 4-dimensional matrix M can be decomposed into a linear combination of the elements of Γ as[41]

$$M = \alpha + \alpha^a\gamma_a + \frac{1}{2}\alpha^{ab}\sigma_{ab} + \alpha_5^a\gamma_5\gamma_a + \alpha_5\gamma_5, \tag{2.75}$$

where

$$\alpha = \frac{1}{4}\operatorname{Tr}[M], \quad \alpha^a = \frac{1}{4}\operatorname{Tr}[M\gamma^a], \quad \alpha^{ab} = -\operatorname{Tr}\left[M\sigma^{ab}\right]$$

$$\alpha_5^a = -\frac{1}{4}\operatorname{Tr}[M\gamma_5\gamma^a], \quad \alpha_5 = \frac{1}{4}\operatorname{Tr}[M\gamma_5] \tag{2.76}$$

This decomposition is called Fierz decomposition.

For convenience, some important Fierz identities for general Majorana spinors are listed below

$$\psi_L\overline{\psi}_R = -\frac{1}{2}\overline{\psi}_R\psi_L\frac{1-\gamma_5}{2}, \tag{2.77}$$

$$\psi_R\overline{\psi}_L = -\frac{1}{2}\overline{\psi}_L\psi_R\frac{1+\gamma_5}{2}, \tag{2.78}$$

$$\epsilon_L\overline{\eta}_L = -\frac{1}{2}\overline{\eta}_L\gamma_\mu\epsilon_L\gamma^\mu\frac{1+\gamma_5}{2}, \tag{2.79}$$

$$\epsilon_R\overline{\eta}_R = -\frac{1}{2}\overline{\eta}_R\gamma_\mu\epsilon_R\gamma^\mu\frac{1-\gamma_5}{2}, \tag{2.80}$$

$$\overline{\psi}_R\psi_L\overline{\psi}_L\psi_R = \frac{1}{2}\left(\overline{\psi}_L\gamma_\mu\psi_L\right)\left(\overline{\psi}_L\gamma^\mu\psi_L\right). \tag{2.81}$$

Short proofs are provided in appendix A.

2.3.5 Symmetries and field equations

The concepts of symmetries and actions quantifying the statements in the previous sections.

First, recall that the action is defined as

$$S = \int \mathcal{L}(\phi, \partial_\mu\phi)\, \mathrm{d}^4x, \tag{2.82}$$

with $\mathcal{L}$ the lagrangian for one or more fields ϕ_k and their derivatives. Let G be an n dimensional Lie group. We formally define a symmetry as a transformation of G, defined below, which leave the action invariant:[§]

$$\phi_k \to \phi_k + \delta_i\phi_k \implies \delta_i S = 0. \tag{2.83}$$

Of special interest are the set of transformations which depend on the variational principle of the action. Assuming that boundary terms vanish, equation (2.83)

[§] In general, a symmetry can be decomposed as a linear combination of symmetries of the form in (2.83): $\delta\phi_k = \epsilon^i\delta_i\phi_k$, with $i = 1,\dots,n$, and ϵ^i the infinitesimal parameters of the symmetries i.

implies for a general transformation the lagrangian satisfies

$$\delta S = \int \left[\frac{\delta \mathcal{L}}{\delta \phi_k} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_k)} \right) \right] \delta \phi_k \, \mathrm{d}^4 x = 0. \tag{2.84}$$

Summation over the ks is implied. Since this must vanish for arbitrary field transformations, the term in curly brackets must be zero for each k. Hence, we arrive at the Euler-Lagrange equations of motion:

$$\frac{\delta \mathcal{L}}{\delta \phi_k} = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_k)} \right). \tag{2.85}$$

Note that in the presence of continuous symmetries the boundary term gives rise to a conserved Noether currents: under a infinitesimal transformation $\phi_k \to \phi_k + \epsilon^i \delta_i \phi_k$, the lagrangian remains invariant up to a total derivative:

$$\mathcal{L} \to \mathcal{L} + \epsilon^i \partial_\mu K_i^\mu. \tag{2.86}$$

Comparing this with the general transformation of the lagrangian

$$\epsilon^i \delta_i \mathcal{L} = \epsilon^i \left[\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_k)} \delta_i \phi_k \right) + \overbrace{\left\{ \frac{\delta \mathcal{L}}{\delta \phi_k} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_k)} \right) \right\}}^{=0,\ \text{by equation (2.85)}} \delta_i \phi_k \right] \tag{2.87}$$

the currents can be seen to be given by

$$J_i^\mu = \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi_k)} \delta_i \phi_k - K_i^\mu, \tag{2.88}$$
$$\partial_\mu J_i^\mu = 0,$$

where summation over k is again implied. The Noether currents defines a charge operators which is constant in time:

$$Q_i = \int J_i^0 \, \mathrm{d}^3 x. \tag{2.89}$$

This charge operator is a generator of G acting on the Hilbert space of quantum states. The fields are transformed according to

$$[iQ_i, \phi_k] = \delta_i \phi_k. \tag{2.90}$$

As was will be seen later, a symmetry of the action is not neccessarily a symmetry of the vacuum:

$$\text{either } Q_i \, |0\rangle = 0, \tag{2.91a}$$

$$\text{or } Q_i \, |0\rangle \neq 0. \tag{2.91b}$$

In the first case, Q_i is a symmetry of both the action and the vacuum. In the second state, the symmetry of the action is not a symmetry of the vacuum, and hence not a symmetry of the physical states. In this case the symmetry is said to be spontaneously broken. Through Goldstone's theorem, the broken symmetries imply the existence of massless bosons (Nambu-Goldstone bosons).

   We now turn to some important examples. This illustrates the procedures outlined above. In addition, it provides us with the field equations needed later. The rest of this section is concerned only with the actions of various fields and their field equations. Spontaneously symmetry breaking is further elaborated upon in section 2.5, in the context of the Higgs mechanism.

   Consider the Klein-Gordon action, which describes a real spin-0 scalar field Φ:

$$S_{KG} = -\frac{1}{2} \int \left[\partial_\mu \Phi \partial^\mu \Phi + m^2 \Phi^2 \right] \mathrm{d}^4 x. \tag{2.92}$$

Using the Euler-Lagrange equations (2.85), we find

$$\Box \Phi = m^2 \Phi, \tag{2.93}$$

with $\Box = \partial_\mu \partial^\mu$ the d'Alembert operator. This is the Klein-Gordon equation. By the prescription $i\partial_0 = E$, $i\boldsymbol{\nabla} = \mathbf{p}$, it guarantees the energy-momentum relation

$$-E^2 + p^2 = -m^2, \tag{2.94}$$

known from special relativity. The Klein-Gordon equation only fixes the energy-momentum condition. If a field has additional properties, such as colour, flavour or spin, additional constraints are required; fermions must satisfy the Dirac equation

$$(\slashed{\partial} + m)\Psi = 0. \tag{2.95}$$

It is readily checked that the Dirac equation follows from the Dirac action

$$S_D = \frac{i}{2} \int \overline{\Psi} [\slashed{\partial} + m]\Psi \, \mathrm{d}^4 x. \tag{2.96}$$

The Dirac equation implies the Klein-Gordon equation. To see this, act on (2.95) from the left by $(\slashed{\partial} - m)$. The Klein-Gordon equation then follows as a

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consequence of the Dirac algebra. Note that the charge conjugate Ψ^C of a spinor Ψ satisfies the Dirac equation as well:

$$
\begin{aligned}
(\partial\!\!\!/ + m)\Psi^C &= (\partial\!\!\!/ + m)C\overline{\Psi}^T \\
&= C\left(-(\gamma^\mu)^T \partial_\mu + m\right)\overline{\Psi}^T.
\end{aligned}
$$

The latter equation is the transpose of

$$
-\overline{\Psi}\left(-\overleftarrow{\partial\!\!\!/} + m\right)C,
$$

which by equation (2.57) can be written as

$$
-\Psi^\dagger\left(\overleftarrow{\partial\!\!\!/}^\dagger + m\right)\gamma_0 C.
$$

This is proportional to the hermitian conjugate of the Dirac equation for Ψ, which equals zero by assumption. Additionally, for a Majorana spinor ψ satisfying (2.95) we can write

$$
\partial\!\!\!/\psi_L + m\psi_R = -(\partial\!\!\!/\psi_R + m\psi_L) \tag{2.97}
$$

However, the left-hand side of (2.97) is right-chiral, while the right-hand side is left-chiral. Hence, both terms must be zero, and we find the chiral form of the Dirac equation:

$$
\begin{aligned}
\partial\!\!\!/\psi_L + m\psi_R &= 0, \\
\partial\!\!\!/\psi_R + m\psi_L &= 0.
\end{aligned} \tag{2.98}
$$

This shows that chiral spinors solve the Dirac equation only when they are massless.

Thirdly, consider for a massive spin-1 vector boson A_μ. It is described by the Proca equation

$$
\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = 0. \tag{2.99}
$$

The Proca is obtained from the Proca action

$$
S_P = -\int\left[\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \frac{1}{2}m^2 A^\mu A_\mu\right] d^4x. \tag{2.100}
$$

In the case of $m = 0$ (2.99) reduces to the Maxwell equations in the vacuum. Contrary to the massless case, the Proca equation implies a fixed gauge for A_μ: contracting with ∂_ν yields the condition

$$
m^2 \partial_\nu A^\nu = 0. \tag{2.101}
$$

This implies that, unless $m = 0$, the Proca equation is never gauge invariant.[¶]

¶The Proca equation encodes some freedom, however. See [42].

2.4 Gauge theory

Recall that the field equations for a matter field ϕ follow from the Euler-Lagrange equations (2.85)

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = 0, \tag{2.102}$$

obtained by the principle of least action. We therefore want to construct a lagrangian, which properly reproduces the behaviour of matter and its interactions with the mediators of the fundamental forces. The fundamental interactions can be derived from the principle of local gauge invariance. Since the fields in the standard model transform linearly in the fundamental or adjoint representation of the gauge group, we'll first cover linear gauge transformations. We'll finish this section with the generalisation to non-linear gauge transformations.

Consider a complex Dirac field ψ. We want the action, and hence the lagrangian of this field to be invariant if we transform ψ by an abelian phase factor

$$\psi \to \psi' = e^{i\theta} \psi, \tag{2.103}$$

where θ is a constant angle. Clearly, the mass term $\overline{\psi}\psi$ is invariant under this transformation, both globally (if θ is constant) and locally (if θ is a function of spacetime):

$$\overline{\psi}\psi \to \overline{\psi}'\psi' = \overline{\psi}e^{-i\theta}e^{i\theta}\psi = \overline{\psi}\psi.$$

If the transformation is global another gauge invariant term is the kinetic term $\overline{\psi}\partial\!\!\!/\psi$. If this gauge transformation is to hold locally though, this symmetry is broken:

$$\overline{\psi}'\partial\!\!\!/\psi' = \overline{\psi}\partial\!\!\!/\psi + i\overline{\psi}\psi\partial\!\!\!/\theta. \tag{2.104}$$

To fix this, we introduce a field A_μ which couples to ψ with a strength e and transforms as

$$A_\mu \to A'_\mu = A_\mu - \frac{i}{e}\partial_\mu \theta \tag{2.105}$$

and define the gauge covariant derivative as

$$\nabla_\mu \psi = (\partial_\mu - ieA_\mu)\psi. \tag{2.106}$$

This covariant derivative commutes with the gauge transformation

$$\begin{aligned}
\nabla_\mu \psi \to \nabla'_\mu \psi' &= (\partial_\mu - ieA_\mu - \partial_\mu \theta)e^{i\theta}\psi \\
&= e^{i\theta}(\partial_\mu - ieA_\mu)\psi,
\end{aligned} \tag{2.107}$$

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and as a result the covariant kinetic term is gauge invariant. Another quantity that commutes with gauge transformations of the form (2.103) is the commutator of two covariant derivatives:

$$\left[\nabla_\mu, \nabla_\nu\right] = -ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right),$$

to which we associate the field strength $F_{\mu\nu}$ of the gauge field A_μ. Since the covariant derivatives provide translations through spacetime, the field strength tensor can be thought of as the flux across closed spacetime loop. Thus, including only renormalisable terms compatible with CPT invariance, the complete lagrangian is

$$\mathcal{L} = \overline{\psi}\slashed{\nabla}\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m\overline{\psi}\psi. \tag{2.108}$$

It can be checked that this reproduces the correct equations of motion by substituting (2.108) in (2.102) for the appropriate fields.

2.4.1 Linear Yang-Mills theory

In general, a field can transform linearly under a continuous group of transformation, represented by unitary $n \times n$ matrices $U = \exp(i\theta)$ generated by hermitian generators T_a of the symmetry group:

$$\psi \to \psi' = U\psi, \qquad \theta = \theta^a T_a. \tag{2.109}$$

For each of these generators we can assign a gauge field A_μ^a, and we can then define a general gauge covariant derivative

$$\nabla_\mu = \partial_\mu - igA_\mu, \tag{2.110}$$

where $A_\mu = A_\mu^a T_a$ is the Lie algebra valued gauge field. We require the field to transform non-homogeneously under gauge transformations

$$A_\mu \to A_\mu' = A_\mu + \frac{i}{e}\nabla_\mu\theta = A_\mu + \frac{i}{e}\partial_\mu\theta - \left[A_\mu, \theta\right]. \tag{2.111}$$

The field strength tensor of the gauge field is again defined as the commutator of the covariant derivatives:

$$\left[\nabla_\mu, \nabla_\nu\right] = -igF_{\mu\nu}, \tag{2.112}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig\left[A_\mu, A_\nu\right] = F_{\mu\nu}^a T_a. \tag{2.113}$$

The field strength belongs to the adjoint representation of the gauge group:

$$F_{\mu\nu} \rightarrow F'_{\mu\nu} = F_{\mu\nu} - ig\left[F_{\mu\nu}, \theta\right] \tag{2.114}$$

The Yang-Mills lagrangian is then

$$\mathcal{L} = \overline{\psi}\slashed{\nabla}\psi - \frac{1}{2}\,\mathrm{Tr}\left[\left(F_{\mu\nu}F^{\mu\nu}\right)\right] + m\overline{\psi}\psi. \tag{2.115}$$

2.4.2 General Yang-Mills theory

The above discussion only concerns gauge transformations which are linear. This is a limitation, since the lagrangian of the σ-model is invariant under non-linear transformations. In this section, we generalise the notion of gauge symmetry to include symmetries of any kind, which may or may not be linear.

Consider a set of n fields ϕ^A, with $A = 1, \ldots n$, which for simplicity are taken to be classical commution fields. Suppose we have a set of m transformations

$$\delta_i\phi^A = R_i^A[\phi], \tag{2.116}$$

where the R_i^A are local functions of the fields ϕ^A and their derivatives. These transformations define infinitesimal (global) symmetries if to first order they leave the action invariant, that is if

$$\delta_i S = \delta_i\phi^A \frac{\delta S}{\delta\phi^A} = R_i^A \frac{\delta S}{\delta\phi^A} = 0 \tag{2.117}$$

irrespective of the field equations of the fields ϕ^A. In terms of the lagrangian $\mathcal{L}$, this implies

$$\delta_i S = \int\left[\delta_i\phi^A \frac{\delta\mathcal{L}}{\delta\phi^A} + \delta\left(\partial_\mu\phi^A\right)\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^A}\right]\mathrm{d}^4x, \tag{2.118}$$

where

$$\delta_i\left(\partial_\mu\phi^A\right) = \frac{\delta R_i^A}{\delta\phi^B}\partial_\mu\phi^B, \tag{2.119}$$

by the chain rule. Trivially, the composition of two infinitesimal symmetry transformations R_i^A and R_j^B is again a symmetry:

$$\delta_i\delta_j S = \delta_i 0 = 0.$$

In particular, the commutator of two transformations leaves the action invariant. If we assume that the set of transformations is complete, this implies that the transformation functions span a Lie algebra

$$[\delta_i, \delta_j] S = \delta_i \left(R_j^A \frac{\delta S}{\delta \phi^A} \right) - \delta_j \left(R_i^A \frac{\delta S}{\delta \phi^A} \right)$$

$$= R_i^B \frac{\delta R_j^A}{\delta \phi^B} \frac{\delta S}{\delta \phi^A} + R_i^A R_j^B \frac{\delta^2 S}{\delta \phi^B \delta \phi^A} - R_j^B \frac{\delta R_i^A}{\delta \phi^B} \frac{\delta S}{\delta \phi^A} - R_j^A R_i^B \frac{\delta^2 S}{\delta \phi^B \delta \phi^A}$$

$$= \left(R_i^B \frac{\delta R_j^A}{\delta \phi^B} - R_j^B \frac{\delta R_i^A}{\delta \phi^B} \right) \frac{\delta S}{\delta \phi^A} = 0.$$

Since our set is assumed to be complete, any transformation can be decomposed as a linear combination of transformations:

$$\left(R_i^B \frac{\delta R_j^A}{\delta \phi^B} - R_j^B \frac{\delta R_i^A}{\delta \phi^B} \right) = f_{ij}{}^k R_k^A, \tag{2.120}$$

where $f_{ij}{}^k$ are antisymmetric structure functions, as they may depend on the fields ϕ^A. Hence, the R_i^A span a Lie algebra.

We now define local a local symmetry to be a set of transformations dependent on parameters which are smooth functions over spacetime:

$$\delta_\xi \phi^A = \xi^i(x) \delta_i \phi^A = \xi^i(x) R_i^A[\phi]. \tag{2.121}$$

Since now the transformations are spacetime dependent, the derivative picks up an extra term proportional to the derivative of the gauge parameter. The action then becomes dependent on the choice of gauge. In order to restore gauge invariance, we define the covariant derivative to be

$$D_\mu \phi^A = \partial_\mu \phi^A - A_\mu^i \delta_i \phi^A = \partial_\mu \phi^A - A_\mu^i R_i^A[\phi], \tag{2.122}$$

where the vector fields A_μ^i provide the compensating transformation:

$$\delta_\alpha D_\mu \phi^A = \partial_\mu \alpha^i R_i^A + \alpha^i \partial_\mu R_i^A - \delta_\alpha A_\mu^i R_i^A - A_\mu^i \delta_\alpha R_i^A$$

$$= \alpha^i \frac{\delta R_i^A}{\delta \phi^B} \left(\partial_\mu \phi^B - A_\mu^j R_j^B \right) + \partial_\mu \alpha^i R_i^A$$

$$+ \alpha^i \frac{\delta R_i^A}{\delta \phi^B} R_j^B A_\mu^j - A_\mu^i \frac{\delta R_i^A}{\delta \phi^B} \alpha^j R_j^B - \delta_\alpha A_\mu^i R_i^A i$$

$$= \alpha^i \frac{\delta R_i^A}{\delta \phi^B} D_\mu \phi^B + \partial_\mu \alpha^i R_i^A + \left(R_i^B \frac{\delta R_j^A}{\delta \phi^B} - R_j^B \frac{\delta R_i^A}{\delta \phi^B} \right) A_\mu^i \alpha^j - \delta_\alpha A_\mu^i R_i^A$$

$$\tag{2.123}$$

Thus, if we impose that the R_i^A once again form a Lie algebra and subsequently define

$$\delta_\alpha A_\mu^i = \partial_\mu \alpha^i + f_{jk}{}^i A_\mu^j \alpha^k \qquad (2.124)$$

the gauge invariance is restored. With the covariant derivative defined as in equation (2.122), we can prove the generalised Ricci identity. First we define

$$D_\mu \partial_\nu \phi^A = \partial_\mu \left(\partial_\nu \phi^A \right) - A_\mu^i \frac{\delta R_i^A}{\delta \phi^B} \left(\partial_\nu \phi^B \right).$$

The Ricci identity then follows thus:

$$\begin{aligned}
\left[D_\mu, D_\nu \right] \phi^A ={} & \left[\partial_\mu, \partial_\nu \right] \phi^A - \left(D_\mu A_\nu^i - D_\nu A_\mu^i \right) R_i^A - \left(A_\nu^i \partial_\mu \phi^B - A_\mu^i \partial_\nu \phi^B \right) \frac{\delta R_i^A}{\delta \phi^B} \\
& + A_\mu^j A_\nu^i \left(R_j^B \frac{\delta R_i^A}{\delta \phi^B} - R_i^B \frac{\delta R_i^A}{\delta \phi^B} \right) - \left(A_\mu^i \partial_\nu \phi^B - A_\nu^i \partial_\mu \phi^B \right) \frac{\delta R_i^A}{\delta \phi^B} \\
={} & - F_{\mu\nu}^i R_i^A, \qquad (2.125)
\end{aligned}$$

where

$$F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + f_{jk}{}^i A_\mu^j A_\nu^k \qquad (2.126)$$

is the generalised Yang-Mills field strength tensor. It once again transforms adjointly under the gauge group:

$$\delta_\alpha F_{\mu\nu}^i = f_{jk}{}^i F_{\mu\nu}^j \alpha^k$$

and

$$\delta_\alpha \left(D_\mu F_{\nu\lambda}^i \right) = f_{jk}{}^i \left(D_\mu F_{\nu\lambda} \right)^j \alpha^k,$$

where $D_\mu F_{\nu\lambda}^i$ is defined to be

$$D_\mu F_{\mu\nu}^i = \partial_\mu F_{\nu\lambda}^i + f_{jk}{}^i A_\mu^j F_{\nu\lambda}^k. \qquad (2.127)$$

Finally, it satisfies the Bianchi identity:

$$D_\mu F_{\nu\lambda}^i + D_\nu F_{\lambda\mu}^i + D_\lambda F_{\mu\nu}^i = 0.$$

Note that this can be written compactly as

$$\epsilon^{\kappa\lambda\mu\nu} D_\lambda F_{\mu\nu}^i = 0. \qquad (2.128)$$

2.5 Higgs mechanism

The covariant derivative uniquely determines the coupling of fermions to the gauge bosons, once the charges under the symmetry groups are established. From table 2.1, it follows that left-handed and right-handed helicity states belong to different representations of the gauge groups. This poses no problem for the kinetic terms, as a spinor ψ can be neatly decomposed in its left-handed and right-handed parts:

$$\overline{\psi}\slashed{\nabla}\psi = \overline{\psi}_L\slashed{\nabla}\psi_L + \overline{\psi}_R\slashed{\nabla}\psi_R. \tag{2.129}$$

However, mass terms are forbidden by gauge invariance: mass terms for Dirac fermions can be written down as

$$m_\psi\overline{\psi}\psi = m_\psi\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right). \tag{2.130}$$

However, mass terms of the form (2.130) are no longer gauge invariant. To correct this a complex scalar field H is introduced which transforms in such a way under the gauge group to make the term

$$gH\overline{\psi}\psi, \tag{2.131}$$

with coupling parameter g, invariant. It is then possible to construct gauge invariant mass terms for spinors if H exhibits spontaneous symmetry breaking, or in other words, if the potential that describes H has a non-trivial minimum.

A lagrangian which exhibits spontaneous symmetry breaking is

$$\mathcal{L}_H = \left|\nabla_\mu H\right|^2 + \mu^2|H|^2 - \lambda|H|^4, \tag{2.132}$$

where $\mu^2, \lambda > 0$. The potential has a non-zero minimum at

$$\langle|H|\rangle = \sqrt{\frac{\mu^2}{2\lambda}}. \tag{2.133}$$

Hence, when one expands about this minimum, it is found that (2.131) reduces to a mass term for fermions:

$$gH\overline{\psi}\psi \to g\langle H\rangle\,\overline{\psi}\psi = g\sqrt{\frac{\mu^2}{2\lambda}}\overline{\psi}\psi. \tag{2.134}$$

Furthermore, this mechanism gives mass to the gauge bosons corresponding to the symmetries that are spontaneously broken. To see how this comes about, suppose H is described by (2.132) plus the kinetic term:

$$\mathcal{L} = -\frac{1}{2}|\partial_\mu H|^2 + \mu^2|H|^2 - \lambda|H|^4. \tag{2.135}$$

This lagrangian has a global $U(1)$ symmetry. If we promote this to a local symmetry, we introduce a covariant derivative

$$\nabla_\mu H = (\partial_\mu - ieA_\mu)H, \tag{2.136}$$

and therefore have to include the Yang-Mills lagrangian for the gauge field A_μ in the bosonic part of the action. Expanding the components of $H = H_1 + iH_2$ about the nontrivial minimum (2.132) by defining:

$$H_1 \equiv \sqrt{\frac{\mu^2}{2\lambda}} + \alpha, \tag{2.137}$$
$$H_2 \equiv \beta,$$

we find the modified form of (2.135):

$$\mathcal{L} = -\frac{1}{2}\left(\partial_\mu\alpha\partial^\mu\alpha + 4\mu^2\alpha^2\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\frac{\mu^2 e^2}{2\lambda}\left(A_\mu + \frac{\sqrt{2\lambda}}{e\mu}\partial_\mu\beta\right)^2 \tag{2.138}$$
$$+ \dots$$

where we have neglected a constant and the dots stand for higher order terms that are irrelevant for this example. It can now be seen that the field β can be eliminated upon making the redefinition

$$A_\mu + \frac{\sqrt{2\lambda}}{e\mu}\partial_\mu\beta \to \widetilde{A}_\mu. \tag{2.139}$$

Plugging equation (2.138) into (2.102) for each of the fields and comparison with (2.93) and (2.99) shows that the fields α and $\widetilde{A}_\mu$ have become massive, while the field β disappears from the physical spectrum.[||]

2.6   Anomalies

The analysis of this thesis is done mostly at the classical level. Nevertheless, it would be advantageous if the theory is also respected at the quantum level. This is possible only if the theory is free of anomalies. Simply put, anomalies are violations of the classical symmetries of the theory. Chiral theories in four-dimensional spacetime develop anomalies from triangle diagrams, where the fermion runs in a loop.[37] This is pictured in figure 2.1. If the fermion couples to gauge fields at the vertices the anomaly is called a gauge anomaly. Gauge

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[||] This is an example of the unitary gauge, where all Goldstone fields are "gauged away".
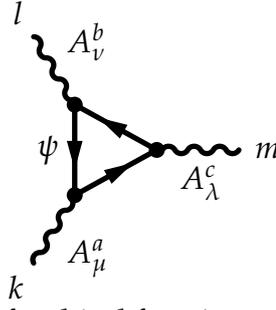
Figure 2.1: *Triangle diagram of a chiral fermion ψ coupling to the gauge bosons of the gauge groups a, b and c. The gauge bosons have momenta k, l and m, respectively.*

anomalies destroy the theory, as their presence implies time-like polarisation states for the gauge fields. This breaks down the unitarity of the theory, by the non-conservation of the $U(1)$ axial vector current

$$J_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi. \tag{2.140}$$

To see how this comes about, recall that for a continuous symmetry there exists a conserved Noether current (2.88) satisfying

$$\partial_\mu J^\mu = 0 \iff p_\mu J^\mu = 0. \tag{2.141}$$

Here p_μ is the incoming momentum of the gauge field coupling to the current. This consistency condition should hold for the triangle diagram in 2.1. Using the Feynman rules, the triangle diagram must be proportional to a function $T_{\mu\nu\lambda}$ of the external momenta and the quantity [37, 43–45]

$$\pm \operatorname{Tr}\left[T_a\{T_b, T_c\}\right], \tag{2.142}$$

where the T_i are the generators of the symmetry groups of the gauge bosons and the trace is take over all fermion species. The sign follows from the eigenvalues of the chirality operator γ_5 for right-handed fermions and left-handed fermions (equation (2.73)). By equation (2.141):

$$k_\mu J^\mu = l_\mu J^\mu = m_\mu J^\mu = 0, \tag{2.143}$$

which in the case of the diagram 2.1, if momentum is conserved, implies

$$k^\mu T_{\mu\nu\lambda} = l^\nu T_{\mu\nu\lambda} = (k+l)^\lambda T_{\mu\nu\lambda} = 0, \tag{2.144}$$

where $m = k + l$. However, to avoid divergencies in the diagram one needs a regularisation scheme, and no regularisation scheme is able to satisfy all three conditions simultaneously.[45] The only way for (2.141) to be satisfied is if all

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contributions to (2.142) vanish, that is if the contributions of the right-handed fermion species exactly cancel the contributions of the left-handed fermion species. Hence, the anomalies cancel automatically if the gauge bosons couple identically to the left-handed and right-handed fermions.** For a chiral theory, anomalies can be avoided if the chiral fermions in the theory carry exactly the right quantum numbers for all the contributions to triangle diagrams to cancel. The pure non-linear σ-model has nonvanishing triangle contributions, and hence the particle spectrum should be extended. Since this is to be done in a supersymmetric way, the extra fermions are part of chiral multiplets, or matter multiplets. From a phenomenological point of view, it is preferable to include these multiplets in such a way that it preserves the original non-linear symmetries. This procedure is presented for the non-linear σ-model in chapter 5.

** However, comparison with table 2.1 shows that the Standard Model is free of anomalies.

2.7 Supersymmetry

This section provides some historical background which motivated, or at least stimulated, the development of supersymmetry. After that, it describes the mathematical formalism needed, such as the construction of supermultiplets and the relation of supersymmetry with Kähler geometry. For more extensive treatments, see [46–48].

The Standard Model acccurately describes the three fundamental interactions (electromagnetism combined with the weak and strong interactions) into a single theoretical framework. To date, it provides the most accurate predictions: precision tests of QED determine the fine-structure constant α with an uncertainty of less than one part per billion. However, it is still far from complete.

Since the Standard Model provides no description of gravity at the quantum scale, it has to be extended at the reduced Planck scale $M_P = 1/\sqrt{8\pi G} \approx 2.4 \cdot 10^{18}$ GeV, when gravitational effects become relevant. This differs from the electroweak scale by 16 orders of magnitude in energy. The cause that M_P/M_W is so large can be traced to the question why the mass of the Higgs boson is much smaller than the Planck mass.

Experimentally, it is known that that vacuum expectation value (2.133) $\langle H \rangle \approx 174$ GeV[49] and $m_H^2 \approx (125 \, \text{GeV})^2$. However, one would expect that quantum corrections to m_H^2 would make the Higgs mass comparable to the Planck mass. Each massive fermion particle contributes to the Higgs mass by a Yukawa coupling $\mathcal{L} = -\lambda_f \bar{f} H f$. Figure 2.2a shows a one-loop correction from a massive Dirac fermion ψ with mass m_ψ. Using the Feynman rules yields the correction, to leading order,

$$\Delta m_H^2 = -\frac{|\lambda_\psi|^2}{8\pi^2} \Lambda_{UV}^2 + \dots, \tag{2.145}$$

where Λ_{UV} is the energy scale at which the current theory breaks down, and new physics is needed to describe the high-energy behavior. Similarly, a massive complex scalar ϕ with mass m_ϕ coupling to the Higgs field yields a Lagrangian $\mathcal{L} = -\lambda_S |H|^2 |\phi|^2$. A one-loop correction to the squared Higgs mass is shown in figure 2.2b. The Feynman rules then yield a contribution

$$\Delta m_H^2 = \frac{\lambda_\phi}{16\pi^2} \Lambda_{UV}^2 + \dots. \tag{2.146}$$

Furthermore, the coupling constants λ_ϕ and λ_ψ are proportional to the masses m_ϕ and m_ψ, respectively, and thus the quantum corrections are more sensitive to the heavy particles. The fact that despite these contributions the Higgs mass is so much smaller than the Planck mass indicates a miraculous fine-tuning between the quantum corrections and the bare Higgs mass. The simplest way of bringing
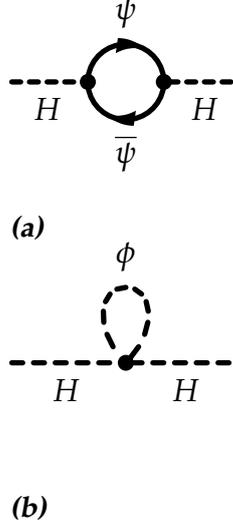
(a)

(b)

Figure 2.2: *One-loop quantum corrections to the squared Higgs mass due to (a) a fermion ψ and (b) a complex scalar ϕ.*

this about is if for each massive fermion there exist two complex scalar fields with $\lambda_\phi = |\lambda_\psi|$, then the contributions from (2.145) and (2.146) exactly cancel. This proposes a symmetry relating bosons and fermions called a supersymmetry.

To each particle in the Standard Model it associates a new particle, called the superpartner, which belongs to obeys different statistics than the original particle. This is realised by the introduction of an anticommuting chiral operator Q, which turns bosonic states into fermionic states, and vice versa:

$$Q\,|\text{boson}\rangle = |\text{fermion}\rangle\,, \quad Q\,|\text{fermion}\rangle = |\text{boson}\rangle\,. \tag{2.147}$$

As spinors are in general complex objects one can make a similar prescription for the complex conjugate $Q^\dagger$. Furthermore, since Q and $Q^\dagger$ are spin changing operators, supersymmetry must be a spacetime symmetry. This strongly restricts any quantum field theory that incorporates supersymmetry[50, 51]: the supersymmetry operators must satisfy the so-called supersymmetry algebra

$$\{Q, \overline{Q}\} = 2\slashed{P} \tag{2.148}$$

$$\{Q, Q\} = \{\overline{Q}, \overline{Q}\} = 0 \tag{2.149}$$

$$[P_\mu, Q] = [P_\mu, \overline{Q}] = 0, \tag{2.150}$$

where P_μ is the generator of spacetime translations. From these commutation relations one can immediately derive that the commutator of two supersymmetry

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transformations is in fact a translation. Using equations (2.90), (2.47), (2.67):

$$[\delta_\eta, \delta_\epsilon]\psi = -\left[[\overline{\eta}Q, \overline{Q}\epsilon], \psi\right] + \left[[\overline{\epsilon}Q, \overline{Q}\eta], \psi\right] \tag{2.151}$$

$$= -\overline{\eta}\left[\{Q, \overline{Q}\}, \psi\right]\epsilon \tag{2.152}$$

$$= -2\overline{\eta}\gamma^\mu\epsilon P_\mu\psi \tag{2.153}$$

$$= 2i\overline{\eta}\gamma^\mu\epsilon\partial_\mu\psi. \tag{2.154}$$

It can straightforwardly be shown that the transformation rules given in the rest of the text indeed satisfy this relation: see appendix D. These relations also imply that superpartners have the same mass, as the supersymmetry generators commute with the squared-mass operator P^2. Particle states with their superpartner constitute irreducible representations of the supersymmetry algebra as supermultiplets, and hence every supermultiplet contains both fermionic and bosonic states. The supersymmetry generators also commute with gauge transformations, and hence superpartners belong to the same representation of the gauge group.

The representations of supersymmetry contain equal numbers of bosonic and fermionic states. This can be proved by introducing the operator $(-1)^{2s}$, where s is the particle spin. Since this operator has eigenvalues $+1$ for bosons and -1 for fermions, it must anticommute with the operators Q and $\overline{Q}$. Thus, consider the trace of $(-1)^{2s}\slashed{P}$, taken over a representation of the supersymmetry algebra with momentum p_μ:

$$\mathrm{Tr}\left[(-1)^{2s}\slashed{P}\right] = \gamma^\mu \sum_i \left\langle i \left| (-1)^{2s} P_\mu \right| i \right\rangle$$

$$= \gamma^\mu p_\mu \mathrm{Tr}\left[(-1)^{2s}\right]$$

$$= \gamma^\mu p_\mu (n_b - n_f). \tag{2.155}$$

where n_b and n_f are the numbers of bosons and fermions in the multiplet, respectively. One can also rewrite the trace using equations (2.148)-(2.150):

$$2\sum_i \left\langle i \left| (-1)^{2s}\slashed{P} \right| i \right\rangle = \sum_i \left\langle i \left| (-1)^{2s}[Q\overline{Q} + \overline{Q}Q] \right| i \right\rangle. \tag{2.156}$$

Since momentum commutes with the supercharges, any combination of Q and $\overline{Q}$ acting on a state in the multiplet yield another state in the multiplet with the same momentum p_μ. One therefore has a completeness relation, and can insert

the identity in (2.156), with the result

$$
\begin{aligned}
2 \operatorname{Tr}\left[(-1)^{2s} \not{P}\right] &= \sum_i \langle i|(-1)^{2s} Q\overline{Q}|i\rangle + \sum_{i,j} \langle i|(-1)^{2s}\overline{Q}|j\rangle \langle j|Q|i\rangle \\
&= \sum_i \langle i|(-1)^{2s} Q\overline{Q}|i\rangle + \sum_{i,j} \langle j|Q|i\rangle \langle i|(-1)^{2s}\overline{Q}|j\rangle \\
&= \sum_i \langle i|(-1)^{2s} Q\overline{Q}|i\rangle + \sum_i \langle i|Q(-1)^{2s}\overline{Q}|i\rangle \\
&= \sum_i \langle i|(-1)^{2s} Q\overline{Q}|i\rangle - \sum_i \langle i|(-1)^{2s} Q\overline{Q}|i\rangle \\
&= 0.
\end{aligned}
\tag{2.157}
$$

Hence, the number of bosonic and fermionic states in the multiplet are equal, as claimed.

A distinction can be made between two supermultiplets: if we extend a chiral (Majorana) spinor supersymmetrically the resulting multiplet, it and its superpartner (the sfermion) is called a chiral multiplet. If a gauge boson is promoted to a supermultiplet it forms a vector multiplet with its superpartner, the gaugino.

2.7.1 The Wess-Zumino model: the chiral multiplet

Consider a complex scalar field Z and a single left-chiral spinor ψ_L. The action for these fields is

$$
S = \int \left[-\partial\overline{Z} \cdot \partial Z - m^2 \overline{Z}Z + i\overline{\psi}_L \not{\partial}\psi_L + \frac{im}{2}\left(\overline{\psi}_R \psi_L + \overline{\psi}_L \psi_R\right) \right] \mathrm{d}^4 x
\tag{2.158}
$$

This action is invariant under the supersymmetry transformations

$$
\begin{aligned}
\delta Z &= -i\sqrt{2}\,\overline{\epsilon}_R \psi_L, \\
\delta \psi_L &= \sqrt{2}\left(\not{\partial}Z\epsilon_R - m\overline{Z}\epsilon_L\right),
\end{aligned}
\tag{2.159}
$$

where the left-chiral spinor ϵ_L is the infinitesimal parameter. Because the supersymmetry transformations depend on the parameter m, the closure of the supersymmetry algebra is dependent on the field equations for Z and ψ_L, that is, the action is only invariant under supersymmetry on-shell. In order for the supersymmetry algebra to close homogeneously off-shell, one typically introduces complex fields H and $F(Z)$ to the action:

$$
\begin{aligned}
S = \int \Big[&-\partial\overline{Z} \cdot \partial Z + i\overline{\psi}_L \not{\partial}\psi_L + \overline{H}H \\
&+ HF(Z) + \overline{H}\,\overline{F}(\overline{Z}) + \frac{i}{2}F'(Z)\overline{\psi}_R\psi_L + \frac{i}{2}\overline{F}'(\overline{Z})\overline{\psi}_L\psi_R \Big] \mathrm{d}^4 x.
\end{aligned}
\tag{2.160}
$$

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Equation (2.160) is invariant under the supersymmetry transformations

$$\delta Z = -i\sqrt{2}\bar{\epsilon}_R \psi_L, \tag{2.161}$$

$$\delta\psi_L = \sqrt{2}(\slashed{\partial} Z\epsilon_R + H\epsilon_L), \tag{2.162}$$

$$\delta H = -i\sqrt{2}\bar{\epsilon}_L \slashed{\partial}\psi_L, \tag{2.163}$$

for any holomorphic function F and reproduces (2.158) upon elimination of the auxiliary field H by its field equation and setting $F(Z) = mZ$. Using the procedure of section 2.3.5, the Noether current (called the supercurrent) is found:

$$J_\mu = (\slashed{\partial} Z + \overline{F})\gamma_\mu\psi_R + (\slashed{\partial}\overline{Z} + F)\gamma_\mu\psi_L. \tag{2.164}$$

The vanishing divergence of the supercurrent is implied by the field equations.

Now, the commutator of supersymmetry transformations is again a symmetry of the theory, independent of the field equations:

$$\left[\delta_\eta, \delta_\epsilon\right] X = a^\mu \partial_\mu X, \tag{2.165}$$

where $a^\mu = 2i\bar{\eta}\gamma^\mu\epsilon$ and $X = Z, \psi_L, H$. The triplet of fields (Z, ψ_L, H) is called the chiral multiplet.

2.7.2 Vector multiplets

It is also possible to promote a massless vector field V_μ to a supermultiplet. This effectively reverses the procedure mentioned in the above section, and therefore the supermultiplet of a vector boson (called a vector multiplet) comprises the boson itself, a Majorana spinor and an auxiliary field to close the supersymmetry algebra off-shell. The supersymmetric action is

$$\mathcal{S}_V = \int \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{i}{2}\bar{\lambda}\slashed{\partial}\lambda + \frac{1}{2}D^2 \right] \mathrm{d}^4 x, \tag{2.166}$$

where

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - \left[V_\mu, V_\nu\right] \tag{2.167}$$

is the field-strength tensor associated with the vector field. It is straightforward to check that the supersymmetry transformations are

$$\begin{aligned}
\delta V_\mu &= -i\bar{\epsilon}\gamma_\mu\lambda, \\
\delta\lambda &= \left(-F_{\mu\nu}\sigma^{\mu\nu} + iD\gamma_5\right)\epsilon, \\
\delta D &= \bar{\epsilon}\gamma_5\slashed{\partial}\lambda.
\end{aligned} \tag{2.168}$$

The supercurrent for the vector multiplet is

$$J_\mu = F^{\nu\lambda}\sigma_{\nu\lambda}\gamma_\mu\lambda. \tag{2.169}$$

It can readily be checked that the divergence of J_μ vanishes upon use of the field equations.

2.7.3   Construction of a supersymmmetric lagrangian

Note that since the auxiliary field D for a vector multiplet is transformed into a total derivative, a term linear in D is automatically supersymmetric:

$$S_D = \xi \int D\,\mathrm{d}^4 x \implies \delta S_D = \xi \int \partial_\mu(\alpha^\mu)\,\mathrm{d}^4 x \simeq 0$$

This provides the motivation for the following: We can decompose the transformation rules (2.168) in terms of chiral spinors:

$$\begin{aligned}
\delta V_\mu &= -i\bar{\epsilon}_R\gamma_\mu\lambda_R - i\bar{\epsilon}_L\gamma_\mu\lambda_L, \\
\delta D &= \bar{\epsilon}_L\partial\!\!\!/\lambda_L - \bar{\epsilon}_R\partial\!\!\!/\lambda_R.
\end{aligned} \tag{2.170}$$

Equation (2.170) follows from the fact that for two Majorana spinors ψ, χ:

$$\bar{\psi}\gamma_\mu\chi = \left(\bar{\psi}_L + \bar{\psi}_R\right)\gamma_\mu(\chi_L + \chi_R).$$

However, focussing on the term $\bar{\psi}_L\gamma_\mu\chi_R$:

$$\begin{aligned}
\bar{\psi}_L\gamma_\mu\chi_R &= \psi_L^\dagger\gamma_0\gamma_5^2\gamma_\mu\chi_R \\
&= (\gamma_5\psi_L)^\dagger\gamma_0\gamma_\mu\gamma_5\chi_R \\
&= -\bar{\psi}_L\gamma_\mu\chi_R,
\end{aligned}$$

And similarly for $\bar{\psi}_R\gamma_\mu\chi_L$. Thus, the terms of mixed chirality vanish. Similarly,

$$\begin{aligned}
\delta\lambda_L &= \frac{1-\gamma_5}{2}\delta\lambda = -\left(F_{\mu\nu}\sigma^{\mu\nu} + iD\right)\epsilon_L, \\
\delta\lambda_R &= \frac{1+\gamma_5}{2}\delta\lambda = \left(-F_{\mu\nu}\sigma^{\mu\nu} + iD\right)\epsilon_R.
\end{aligned} \tag{2.171}$$

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Using these transformation rules, it is possible to construct a real vector multiplet (W_μ, Λ, D) out of a chiral multiplet (Z, ψ_L, H):

$$
\begin{aligned}
W_\mu =& \frac{i}{2}\left(K_{,z}\partial Z - K_{,\bar{z}}\partial_\mu \overline{Z}\right) - \frac{1}{2}K_{,z\bar{z}}\overline{\psi}_L \gamma_\mu \psi_L, \\
\Lambda_L =& \frac{i}{\sqrt{2}}K_{,z\bar{z}}\left(\slashed{\partial}Z\psi_R - \left[\overline{H} + \frac{i}{2}\overline{\Gamma}_{\bar{z}\bar{z}}^{\;\;\bar{z}}\overline{\psi}_L\psi_R\right]\psi_L\right), \\
\Lambda_R =& -\frac{i}{\sqrt{2}}K_{,z\bar{z}}\left(\slashed{\partial}\overline{Z}\psi_R - \left[H + \frac{i}{2}\Gamma_{zz}^{\;\;z}\overline{\psi}_R\psi_L\right]\psi_R\right), \\
D =& -K_{,z\bar{z}}\partial\overline{Z}\cdot\partial Z + \frac{i}{2}K_{,z\bar{z}}\overline{\psi}_L\overset{\leftrightarrow}{\slashed{D}}\psi_L \\
& + K_{,z\bar{z}}\left(\overline{H} + \frac{i}{2}\overline{\Gamma}_{\bar{z}\bar{z}}^{\;\;\bar{z}}\overline{\psi}_L\psi_R\right)\left(H + \frac{i}{2}\Gamma_{zz}^{\;\;z}\overline{\psi}_R\psi_L\right) - \frac{1}{4}R_{\bar{z}z\bar{z}z}\overline{\psi}_R\psi_L\overline{\psi}_L\psi_R,
\end{aligned}
$$
(2.172)

with the shorthand notation:

$$
\begin{aligned}
K_{,z\bar{z}} &= \frac{\partial^2 K}{\partial\overline{Z}\partial Z}, \\
\Gamma_{zz}^{\;\;z} &= K_{,z\bar{z}}^{-1}K_{,z\bar{z}z}, \\
\overline{\Gamma}_{\bar{z}\bar{z}}^{\;\;\bar{z}} &= K_{,z\bar{z}}^{-1}K_{,z\bar{z}\bar{z}}, \\
R_{\bar{z}z\bar{z}z} &= K_{,z\bar{z}z\bar{z}} - K_{,z\bar{z}}\Gamma_{zz}^{\;\;z}\overline{\Gamma}_{\bar{z}\bar{z}}^{\;\;\bar{z}}, \\
D_\mu\psi_L &= \partial_\mu\psi_L + \partial_\mu Z\Gamma_{zz}^{\;\;z}\psi_L,
\end{aligned}
$$
(2.173)

transform as the components of a scalar multiplet for any well-behaved function K. We can then recover equation (2.160) by constructing an action which is purely linear in D, as discussed above. Written out in full, the lagrangian is

$$
\begin{aligned}
\mathcal{L} = -\int\Big[& g_{z\bar{z}}\left(\partial\overline{Z}\cdot\partial Z - \frac{i}{2}\overline{\psi}_L\overset{\leftrightarrow}{\slashed{\partial}}\psi_L - \overline{H}H\right) - \frac{i}{2}\left(g_{z\bar{z},z}\partial_\mu Z - g_{z\bar{z},\bar{z}}\partial_\mu\overline{Z}\right)\overline{\psi}_L\gamma^\mu\psi_L \\
& -\frac{i}{2}\left(g_{z\bar{z},\bar{z}}H\overline{\psi}_L\psi_R + g_{z\bar{z},z}\overline{H}\overline{\psi}_R\psi_L\right) + \frac{1}{4}g_{z\bar{z},z\bar{z}}\overline{\psi}_R\psi_L\overline{\psi}_L\psi_R\Big]\;\mathrm{d}^4x.
\end{aligned}
$$
(2.174)

2.7.4 The superpotential

Similarly, the auxiliary field H for a chiral multiplet transforms in to a total derivative as well. Starting from the basic chiral multiplet (ϕ, ψ_L, H), it is possible to construct a field which transforms according to (2.163). To wit:

$$
A = W'(Z)H + \frac{i}{2}W''(Z)\overline{\psi}_R\psi_L.
$$
(2.175)

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The holomorphic field W is called the superpotential. Since A is complex, a real contribution to the supersymmetric lagrangian can obtained by adding A with its complex conjugate to (2.174)

$$\Delta S = \int \left[W'(Z)H + \overline{W}'(\overline{Z})\overline{H} + \frac{i}{2}W''(Z)\overline{\psi}_R\psi_L + +\frac{i}{2}\overline{W}''(\overline{Z})\overline{\psi}_L\psi_R \right] \mathrm{d}^4x. \tag{2.176}$$

This modifies the equations of motion for the auxiliary field H. The superpotential is only mentioned here for completeness, and will not be used in the analysis in this thesis.

2.7.5 Kähler geometry in supersymmetry

Note that the function K in equation (2.172) is only determined up to transformations of the form

$$K(Z,\overline{Z}) \to K'(Z,\overline{Z}) = K(Z,\overline{Z}) + F(Z) + \overline{F}(\overline{Z}). \tag{2.177}$$

Such transformations leave the chiral and auxiliary components unchanged, while the bosonic transforms as

$$\delta W_\mu = \frac{i}{2}\left(F'(Z)\partial_\mu Z - \overline{F}'(\overline{Z})\partial_\mu \overline{Z} \right) = \partial_\mu \alpha, \tag{2.178}$$

with $\alpha = \frac{i}{2}\left(F - \overline{F} \right)$. Thus, (2.177) corresponds with an abelian gauge transformation of W_μ.

Comparison of the above construction with the properties of the Kähler potential in section 2.1.7 shows that a chiral supermultiplet is embedded in a Kähler manifold. The process of generalising this to a family of multiplets is then straightforward. Generalising (2.174) to r complex scalar multiplets, we find the general action:

$$S = -\int \left[g_{\alpha\underline{\alpha}}\left(\partial\overline{Z}^{\underline{\alpha}} \cdot \partial Z^\alpha - \frac{i}{2}\overline{\psi}_L^{\underline{\alpha}}\overleftrightarrow{\partial}\psi_L^\alpha - \overline{H}^{\underline{\alpha}}H^\alpha \right) \right.$$
$$- \frac{i}{2}\left(g_{\alpha\overline{\alpha},\beta}\partial_\mu Z^\beta - g_{\alpha\underline{\alpha},\underline{\beta}}\partial_\mu\overline{Z}^{\underline{\beta}} \right)\overline{\psi}_L^{\underline{\alpha}}\gamma^\mu\psi_L^\alpha \tag{2.179}$$
$$\left. - \frac{i}{2}\left(g_{\alpha\underline{\alpha},\underline{\beta}}H^\alpha\overline{\psi}_L^{\underline{\alpha}}\psi_R^{\underline{\beta}} + g_{\alpha\underline{\alpha},\beta}\overline{H}^{\underline{\alpha}}\overline{\psi}_R^\alpha\psi_L^\beta \right) + \frac{1}{4}g_{\alpha\underline{\alpha},\beta\underline{\beta}}\overline{\psi}_R^\alpha\psi_L^\beta\overline{\psi}_L^{\underline{\alpha}}\psi_R^{\underline{\beta}} \right] \mathrm{d}^4x.$$

Here $g_{\alpha\overline{\beta}}$ is the Kähler metric (2.30). Summation over $\alpha, \underline{\alpha}, \beta, \underline{\beta}$ is implied, and take the values $1, \ldots, r$. Eliminating the auxiliary fields H^α by their field equations (compare (2.172))

$$H^\alpha = -\frac{i}{2}\Gamma_{\beta\gamma}^{\ \ \alpha}\overline{\psi}_R^\alpha\psi_L^\beta, \tag{2.180}$$

The action can be written purely in the multiplet components:

$$S = \int \left[-g_{\alpha\underline{\alpha}} \partial \overline{Z}^{\underline{\alpha}} \cdot \partial Z^{\alpha} + \frac{i}{2} g_{\alpha\underline{\alpha}} \overline{\psi}_L^{\underline{\alpha}} \overleftrightarrow{\slashed{D}} \psi_L^{\alpha} - \frac{1}{4} R_{\alpha\underline{\alpha}\beta\underline{\beta}} \overline{\psi}_L^{\underline{\alpha}} \psi_R^{\beta} \overline{\psi}_R^{\underline{\alpha}} \psi_L^{\beta} \right] \mathrm{d}^4 x, \quad (2.181)$$

with $R_{\alpha\underline{\alpha}\beta\underline{\beta}}$ the Riemann tensor (2.34) and

$$D_\mu \psi_L^\alpha = \partial_\mu \psi_L^\alpha + \partial_\mu Z^\gamma \Gamma_{\gamma\beta}^{\ \ \alpha} \psi_L^\beta. \quad (2.182)$$

2.7.6 The mass formula

In the beginning of this section, it was proved that all particles in the same supermultiplet must share the same mass. This creates a major dilemma: since the photon is massless, the photino should also be massless. Since it belongs to the same representation of the electromagnetic gauge group, it couples to the charged matter in the same way as the photon, and therefore should not be difficult to be observed. The absence of superpartners places the requirement that supersymmetry be broken for any realistic field theory.

There is a rule which relates the masses in a given multiplet. For unbroken supersymmetry this rule is simple: all particles have the same mass. For broken supersymmetry, the masses of particles are related by the supertrace. The supertrace is the sum of squared masses, weighed by the particle spins:

$$\mathrm{STr}\, m^2 = \mathrm{Tr} \left[\sum_J (-1)^{2J} (2J+1) m_J^2 \right]. \quad (2.183)$$

Since a supermultiplet comprises a spin-0 scalar field, a spin-1 vector field, and a spin-½ spinor field, the supertrace becomes

$$\mathrm{STr}\, m^2 = \mathrm{Tr} \left[m_0^2 - 2m_{1/2}^2 + 3m_1^2 \right]. \quad (2.184)$$

The value of the supertrace is defined to vanish:

$$\mathrm{STr}\, m^2 = m^2 (n_b - n_f) = 0, \quad (2.185)$$

with one again n_b and n_f the number of bosonic and fermionic states in the multiplet, respectively.[24]

Chapter 3

The non-linear σ-model

Having introduced all the neccessary notation and mathematical tools, we now turn to the construction of the non-linear σ-model. We shall proceed as follows: first, we shall show how the bosonic model naturally emerges by restricting fields to take values in the target manifold. Second, the results will be cast in the form of Kähler geometry and, in light of the requirements for supersymmetry, consider the coupling of (fermionic) matter fields to the to scalar model, and end this section with the supersymmetric extension of the σ-model.

3.1 Parametrisation of the coset $SU(2N)/SU(N)^2 \times U(1)$

An element of $Z \in SU(2N)$ can be partitioned as

$$\mathbf{Z} = \begin{pmatrix} U & V \\ X & Y \end{pmatrix}, \tag{3.1}$$

where U, V, X, and Y are $N \times N$ complex matrices. The requirement that $\mathbf{Z}$ be unitary yields a set of constraints to these partitions:

$$\begin{aligned} UU^\dagger + VV^\dagger &= 1, \\ XX^\dagger + YY^\dagger &= 1, \\ UX^\dagger + VY^\dagger &= 0. \end{aligned} \tag{3.2}$$

From these constraints we can surmise that the partition of $\mathbf{Z}$ can be parametrised as

$$\begin{aligned} U &= e^{iL} \left(1 + \phi\phi^\dagger \right)^{-1/2}, \\ V &= e^{iL} \left(1 + \phi\phi^\dagger \right)^{-1/2} \phi, \\ X &= -e^{iK} \left(1 + \phi^\dagger\phi \right)^{-1/2} \phi^\dagger, \\ Y &= e^{iK} \left(1 + \phi^\dagger\phi \right)^{-1/2}, \end{aligned} \tag{3.3}$$

as can be readily checked by substitution. In this parametrisation, ϕ is an arbitrary complex $N \times N$ matrix, while L and K are $N \times N$ hermitian matrices. The square roots in equation (3.3) are defined by their power series expansion. As pure elements of the symmetry group, the fields $\mathbf{Z}$ and ϕ are dimensionless. The physical fields can then be obtained by introducing a parameter f of dimension inverse mass such that

$$\mathbf{Z} \to f\mathbf{Z}, \quad \phi \to f\phi. \tag{3.4}$$

For simplicity, f will be set to unity for the remainder of this thesis.

The parametrisation (3.3) can be written as

$$\mathbf{Z} = \mathbf{\Omega Z}_0,$$

where

$$\mathbf{Z}_0 = \begin{pmatrix} \left(1 + \phi\phi^\dagger\right)^{-1/2} & \left(1 + \phi\phi^\dagger\right)^{-1/2}\phi \\ -\left(1 + \phi^\dagger\phi\right)^{-1/2}\phi^\dagger & \left(1 + \phi^\dagger\phi\right)^{-1/2} \end{pmatrix}, \qquad \mathbf{\Omega} = \begin{pmatrix} e^{iL} & 0 \\ 0 & e^{iK} \end{pmatrix}. \qquad (3.5)$$

Succesive $S[U(N) \times U(N)]$ gauge transformations $\mathbf{\Omega}' = \mathrm{diag}(e^{i\Lambda}, e^{i\varkappa})$ from the left therefore result in a transformed phase factor via

$$e^{iL} \to e^{iL'} = e^{i\Lambda}e^{iL},$$

with an analagous transformation rule holding for e^{iK}. Since we can group all $\mathbf{Z}$ which differ by a transformation of the form

$$\mathbf{Z}' = \mathbf{\Omega Z} = \begin{pmatrix} e^{iL} & 0 \\ 0 & e^{iK} \end{pmatrix}\mathbf{Z}, \qquad (3.6)$$

we can define the left coset space of $SU(2N)$ by setting $L = K = 0$. The requirements for $\mathbf{\Omega}$ are that its generators be traceless and $\det \mathbf{\Omega} = 1$, since $\mathbf{\Omega} \in S[U(N) \times U(N)]$. The resulting coset space is denoted $SU(2N)/S[U(N)^2]$. The coset space $SU(2N)/SU(N)^2 \times U(1)$ is then constructed by omitting a diagional $U(1)$ charge.[28]

3.2   Construction of the lagrangian

The parametrisation (3.3) demonstrates that the coset space $SU(2N)/S[U(N) \times U(N)]$ can be considered as a smooth manifold, with the matrices ϕ and $\phi^\dagger$ as coordinates. Next, a lagrangian with the proper symmetries is to be written down. A suitable form exploits the $SU(N)^2 \times U(1)$ symmetry in a gauge invariant way:[52]

$$\mathcal{L} = -\mathrm{Tr}\left[\mathcal{D}_\mu Z^\dagger \mathcal{D}^\mu Z\right], \qquad (3.7)$$

where

$$\mathcal{D}_\mu Z = \left[\partial_\mu - ig\begin{pmatrix} A_\mu & 0 \\ 0 & B_\mu \end{pmatrix}\right]Z. \qquad (3.8)$$

The covariant derivative in equation (3.7) is defined such that the condition $ZZ^\dagger = 1$ is respected: by using the Euler-Lagrange equations for the fields

A^μ, B^μ, A_μ and B_μ can be expressed in terms of the fields ϕ and $\phi^\dagger$. To do so, one can expand equation (3.7), neglect the terms not proportional to the gauge fields, take derivatives, take partial traces and set the resulting expressions equal to zero. From this, we find

$$
\begin{aligned}
A_\mu &= \frac{i}{2g}\left(U \overleftrightarrow{\partial_\mu} U^\dagger + V \overleftrightarrow{\partial_\mu} V^\dagger\right), \\
B_\mu &= \frac{i}{2g}\left(X \overleftrightarrow{\partial_\mu} X^\dagger + Y \overleftrightarrow{\partial_\mu} Y^\dagger\right).
\end{aligned}
\tag{3.9}
$$

Substitution of the parametrisation in (3.3) in (3.9) yields

$$
\begin{aligned}
A_\mu &= \frac{i}{2g}\left\{\left(1+\phi\phi^\dagger\right)^{-\frac{1}{2}}\left(\phi \overleftrightarrow{\partial_\mu}\phi^\dagger\right)\left(1+\phi\phi^\dagger\right)^{-\frac{1}{2}}\right. \\
&\qquad\left. +\left[\left(1+\phi\phi^\dagger\right)^{-\frac{1}{2}},\partial_\mu\left(1+\phi\phi^\dagger\right)^{-\frac{1}{2}}\right]\right\}, \\
B_\mu &= \frac{i}{2g}\left\{\left(1+\phi^\dagger\phi\right)^{-\frac{1}{2}}\left(\phi^\dagger \overleftrightarrow{\partial_\mu}\phi\right)\left(1+\phi^\dagger\phi\right)^{-\frac{1}{2}}\right. \\
&\qquad\left. +\left[\left(1+\phi^\dagger\phi\right)^{-\frac{1}{2}},\partial_\mu\left(1+\phi^\dagger\phi\right)^{-\frac{1}{2}}\right]\right\}.
\end{aligned}
\tag{3.10}
$$

It can straightforwardly though tedeuously be checked that under a gauge transformation of the form (3.6) the field A_μ and B_μ transform according to

$$
\begin{aligned}
A_\mu \to A'_\mu &= e^{iL} A_\mu e^{-iL} + \frac{i}{g}e^{iL}\partial_\mu e^{-iL}, \\
B_\mu \to B'_\mu &= e^{iK} B_\mu e^{-iK} + \frac{i}{g}e^{iK}\partial_\mu e^{-iK}.
\end{aligned}
\tag{3.11}
$$

Thus, the fields transform as proper gauge fields, and as a result, the derivative in (3.8) and the gauge transformation (3.6) commute. For reference, the transformation properties of gauge fields is elaborated upon in B. Upon substitution of the gauge fields in equation (3.7) we find, up to a factor of 2:[28, 53, 54]

$$
\mathcal{L} = -\mathrm{Tr}\left(\left(1+\phi\phi^\dagger\right)^{-1}\partial_\mu\phi\left(1+\phi^\dagger\phi\right)^{-1}\partial^\mu\phi^\dagger\right).
\tag{3.12}
$$

In addition to the gauge transformations on the left, the lagrangian also has proper $SU(2N)$ gauge symmetry from the right. This is covered in the next section.

3.2.1 Global gauge transformations of $SU(2N)/SU(N)^2 \times U(1)$

The lagrangian also has a global symmetry of the form

$$Z' = ZZ^*, \tag{3.13}$$

where Z^* is a proper $SU(2N)$ matrix. This transformation forces ϕ to transform non-linearly.[52] Because of this, the symmetry does not respect the coset in general; to restore the gauge $L = K = 0$ one has to multiply Z' with a restoring $SU(N)^2$ transformation Ξ from the left: $\Xi ZZ^* \in SU(2N)/SU(N)^2 \times U(1)$. Infinitesimal global transformations of the form (compare (3.3))

$$\mathbf{Z}^* = \begin{pmatrix} 1 + i\Lambda & \epsilon \\ -\epsilon^\dagger & 1 + i\underline{\Lambda} \end{pmatrix} \tag{3.14}$$

yield the following transformations of the fields:

$$\begin{aligned}
\phi \to \phi' &= \phi + \epsilon + \phi \epsilon^\dagger \phi - i\Lambda\phi + i\phi\underline{\Lambda}, \\
\phi^\dagger \to \phi'^\dagger &= \phi^\dagger + \epsilon^\dagger + \phi^\dagger \epsilon \phi^\dagger + i\phi^\dagger \Lambda - i\underline{\Lambda}\phi^\dagger, \\
e^{iL} \to e^{iL'} &= e^{iL}\left(1 + i\Lambda + i\Omega(\phi, \phi^\dagger)\right), \\
e^{iK} \to e^{iK'} &= e^{iK}\left(1 + i\underline{\Lambda} + i\underline{\Omega}(\phi, \phi^\dagger)\right).
\end{aligned} \tag{3.15}$$

The details can be found in appendix C. The $U(1)$ charge can be isolated from the hermitian matrices by defining

$$\Lambda \to \widetilde{\Lambda} + \alpha, \tag{3.16}$$

$$\underline{\Lambda} \to \underline{\widetilde{\Lambda}} + \beta, \tag{3.17}$$

$$\xi = \alpha - \beta. \tag{3.18}$$

$\widetilde{\Lambda}, \underline{\widetilde{\Lambda}}$ are traceless hermitian matrices, while ξ is the $U(1)$ central charge. The non-linear transformation for the field ϕ then becomes

$$\delta\phi = \epsilon + \phi\epsilon^\dagger\phi - i\Lambda\phi + i\phi\underline{\Lambda} - i\xi\phi. \tag{3.19}$$

In the rest of the text, the tildes on the traceless matrices will be dropped. The hermitian matrices $\Omega, \underline{\Omega}$ arise as the compensating transformation to restore the coset. Substitution of (3.15) into (3.13), we find that these matrices satisfy

$$\begin{aligned}
\left\{\Omega, \left(1 + \phi\phi^\dagger\right)^{-1/2}\right\} &= i\left[\left(1 + \phi\phi^\dagger\right)^{-1/2}\phi\epsilon^\dagger - \epsilon\phi^\dagger\left(1 + \phi\phi^\dagger\right)^{-1/2}\right], \\
\left\{\underline{\Omega}, \left(1 + \phi^\dagger\phi\right)^{-1/2}\right\} &= i\left[\left(1 + \phi^\dagger\phi\right)^{-1/2}\phi^\dagger\epsilon - \epsilon^\dagger\phi\left(1 + \phi^\dagger\phi\right)^{-1/2}\right].
\end{aligned} \tag{3.20}$$

3.2.2 Gauging the global symmetry

The symmetries of the model can be specified purely in terms of $SU(N)$ transformations on the field ϕ. Since these transformations are non-linear, it is instructive to first realise the gauging of the whole symmetry group as a linear $SU(2N)$ matrix. This allows for a convenient check when gauging the symmetries in a non-linear fashion in section 4.

The gauging of the global symmetry allows vector bosons to be coupled to the model. To do this, the covariant derivative in (3.8) is supplemented by a coupling to the gauge group acting from the right:

$$\partial_\mu \mathbf{Z} \to \nabla_\mu \mathbf{Z},$$

$$\nabla_\mu \mathbf{Z} = \partial_\mu \mathbf{Z} - ie\mathbf{Z} \begin{pmatrix} V_\mu & Q_\mu \\ -Q_\mu^\dagger & W_\mu \end{pmatrix}. \tag{3.21}$$

V_μ, W_μ belong to the linear subgroups $SU(N)$ of $SU(2N)$, while $Q_\mu, Q_\mu^\dagger$ belong to the complementary group of $SU(2N)/SU(N)^2 \times U(1)^*$. Thus:

$$\mathcal{D}_\mu \mathbf{Z} = \left(\nabla_\mu - ig \begin{pmatrix} A_\mu & 0 \\ 0 & B_\mu \end{pmatrix} \right) \mathbf{Z}. \tag{3.22}$$

Eliminating the auxiliary gauge fields A_μ and B_μ in terms of ϕ we find that (3.9) still holds, provided we introduce covariant derivatives:

$$A_\mu = \frac{i}{2g} \left(U \overleftrightarrow{\nabla}_\mu U^\dagger + V \overleftrightarrow{\nabla}_\mu V^\dagger \right),$$

$$B_\mu = \frac{i}{2g} \left(X \overleftrightarrow{\nabla}_\mu X^\dagger + Y \overleftrightarrow{\nabla}_\mu Y^\dagger \right), \tag{3.23}$$

where

$$\nabla_\mu U = \partial_\mu U - ieUV_\mu + ieVQ_\mu^\dagger,$$
$$\nabla_\mu V = \partial_\mu V - ieVW_\mu - ieUQ_\mu,$$
$$\nabla_\mu X = \partial_\mu X - ieXV_\mu + ieYQ_\mu^\dagger, \tag{3.24}$$
$$\nabla_\mu Y = \partial_\mu Y - ieYW_\mu + ieXQ_\mu.$$

If we substitute the above back into the lagrangian we find

$$\mathcal{L} = -\text{Tr}\bigg[\nabla_\mu V^\dagger \nabla^\mu V + \nabla_\mu U^\dagger \nabla^\mu U + \nabla_\mu X^\dagger \nabla^\mu X + \nabla_\mu Y^\dagger \nabla^\mu Y$$
$$+ \frac{1}{4} \left(V \overleftrightarrow{\nabla}_\mu V^\dagger + U \overleftrightarrow{\nabla}_\mu U^\dagger \right)^2 + \frac{1}{4} \left(X \overleftrightarrow{\nabla}_\mu X^\dagger + Y \overleftrightarrow{\nabla}_\mu Y^\dagger \right)^2 \bigg]. \tag{3.25}$$

---

*Note that the gauge transformations generated by $Q_\mu, Q_\mu^\dagger$ introduce the non-linearity in the transformation of ϕ. See (3.14) and (3.15).

In terms of the parametrisation of equation (3.3) (up to a factor of 2):[28]

$$\mathcal{L} = -\operatorname{Tr}\left[\left(1 + \phi\phi^\dagger\right)^{-1}\nabla_\mu\phi\left(1 + \phi^\dagger\phi\right)^{-1}\nabla^\mu\phi^\dagger + e^2 Q_\mu^\dagger Q^\mu\right], \qquad (3.26)$$

where

$$\nabla_\mu\phi = \partial_\mu\phi - ie\left(\phi W_\mu - V_\mu\phi\right). \qquad (3.27)$$

The symmetry is thus spontaneously broken and the fields of the non-linear gauge transformations become massive. Since the gauge fields are dynamic, one still needs to add the Yang-Mills Lagrangians (2.115) of $V_\mu, W_\mu, Q_\mu, Q_\mu^\dagger$ to (3.26).

3.2.3 Kähler geometry on the coset

In this section the general concepts of section 2.1.7 will be applied to the grassmanian manifolds, in particular the coset space $SU(2N)$. The components of the metric, connection coefficients and the curvature will be computed.

The Kähler potential for the grassmanian manifolds is[13]

$$K\left(\phi, \phi^\dagger\right) = \log \det(1 + \phi\phi^\dagger). \tag{3.28}$$

It can straightforwardly be checked that the Kähler potential is invariant under the transformations (3.15), up to the ambiguity mentioned in equation (2.31). The variation of the Kähler potential can be written as

$$\delta K = \delta\phi \cdot \frac{\delta K}{\delta\phi} + \frac{\delta K}{\delta\phi^\dagger}\delta\phi^\dagger$$
$$= F + \overline{F}, \tag{3.29}$$

where the functions F and $\overline{F}$ are defined by

$$F = \zeta \cdot \frac{\delta K}{\delta\phi} + iM, \quad \overline{F} = \overline{\zeta} \cdot \frac{\delta K}{\delta\phi^\dagger} - iM, \tag{3.30}$$

and

$$\zeta = \delta\phi = \phi' - \phi$$
$$= \epsilon + \phi\epsilon^\dagger\phi - i\Lambda\phi + i\phi\underline{\Lambda} - i\xi\phi. \tag{3.31}$$

The functions $F, \overline{F}$ are (anti)holomorphic, as can be proved by using equations (2.39), (2.40), (2.41), as well as the property that the Killing vectors are holomorphic. Explicitly:

$$F[\phi] = \text{Tr}\left[\epsilon^\dagger\phi\right]. \tag{3.32}$$

Hence, K satisfies the required transformation properties for the Kähler potential. The functions ζ are therefore the Killing vectors of the metric.

From the Kähler potential one can compute the metric using equation (2.30). It is actually convenient to do so in terms of the metric components. One then proceeds as follows: from the identity $\log \det X = \text{Tr}\left[\log X\right]$ one obtains a formal power series for the Kähler potential. Differentiating this series with respect to the components of $\phi^\dagger$ one obtains the formal power series

$$\frac{\delta K}{\delta\phi^{\dagger\, j}_{\ \ \beta}} = \left[\left(1 + \phi\phi^\dagger\right)^{-1}\phi\right]^\beta_{\ j} = \sum_{n=0}^{\infty} \frac{1}{n}\left(-\phi\phi^\dagger\right)^n\phi. \tag{3.33}$$

The metric then obtained by differentiating this expression with respect to the components of ϕ, with the result

$$g_{\alpha\,j}^{\,i\,\beta} = \left(1 + \phi\phi^\dagger\right)^{-1}{}_{i}^{\,j}\left(1 + \phi^\dagger\phi\right)^{-1}{}_{\alpha}^{\,\beta}. \tag{3.34}$$

Using equation (3.34), the lagrangian of the coset model can be simplified to

$$\mathcal{L} = \text{Tr}\left[g\partial_\mu\phi^\dagger\partial^\mu\phi\right]. \tag{3.35}$$

The inverse metric immediately follows from the form of the metric:

$$g^{-1}{}_{\beta\,i}^{\,j\,\alpha} = \left(1 + \phi\phi^\dagger\right)_i^{\,j}\left(1 + \phi^\dagger\phi\right)_\beta^{\,\alpha} \tag{3.36}$$

For completeness, the connection and curvarture components can then be immediately computed from the metric and its inverse, using equations (2.33) and (2.34):

$$\Gamma_{\alpha\,\gamma\,j}^{\,i\,k\,\beta} = (g^{-1})_{l\,j}^{\,\delta\,\beta}\frac{\delta g_{\alpha\,l}^{\,i\,\delta}}{\delta\phi_k^{\,\gamma}} \tag{3.37}$$

$$= -\delta_j^i\delta_\gamma^\beta\left(\left(1 + \phi^\dagger\phi\right)^{-1}\phi^\dagger\right)_\alpha^{\,k} - \delta_j^k\delta_\alpha^\beta\left(\left(1 + \phi^\dagger\phi\right)^{-1}\phi^\dagger\right)_\gamma^{\,i}, \tag{3.38}$$

$$\mathcal{R}_{\alpha\,j\,\gamma\,m}^{\,i\,\beta\,k\,\delta} = \frac{\delta^2 g_{\alpha\,j}^{\,i\,\beta}}{\delta\phi_k^{\,\gamma}\delta\phi^{\dagger\,m}_{\,\lambda}} - g_{\kappa\,a}^{\,b\,\lambda}\Gamma_{\alpha\,\gamma\,i}^{\,i\,k\,\kappa}\overline{\Gamma}_{j\,m\,\lambda}^{\,\beta\,\delta\,a} \tag{3.39}$$

$$= -\left(g_{\gamma\,j}^{\,i\,\beta}g_{\alpha\,m}^{\,k\lambda} + g_{\alpha\,j}^{\,k\,\beta}g_{\gamma\,m}^{\,i\,\lambda}\right). \tag{3.40}$$

Returning to the Killing vectors, from (3.30) it is straightforward to compute the Killing potential:

$$M = i\left(\zeta\cdot\frac{\delta M}{\delta\phi} - F\right) = -i\left(\overline{\zeta}\cdot\frac{\delta M}{\delta\phi^\dagger} - \overline{F}\right)$$

$$= -i\,\text{Tr}\left[\left(1 + \phi^\dagger\phi\right)^{-1}\left(\epsilon^\dagger\phi - \phi^\dagger\epsilon + i\phi^\dagger\Lambda\phi - i\underline{\Delta}\phi^\dagger\phi + i\xi\phi^\dagger\phi\right)\right]. \tag{3.41}$$

Of course, a field independent constant can freely be added to (3.41), as the Killing vectors are obtained from the potential by equations (2.39) and (2.40). It can be shown that the Killing vectors (3.31) satisfy equation (2.38), that is, they span a Lie algebra:

$$\zeta_A\cdot\frac{\delta\zeta_B}{\delta\phi} - \zeta_B\cdot\frac{\delta\zeta_A}{\delta\phi} = \zeta_C, \tag{3.42}$$

where the ζ_i are given by equation (3.31) and[†]

$$\epsilon_C = i\Big(\epsilon_{[A}\underline{\Lambda}_{B]} + \Lambda_{[A}\epsilon_{B]}\Big), \tag{3.43}$$

$$\epsilon_C^\dagger = i\Big(\underline{\Lambda}_{[A}\epsilon_{B]}^\dagger + \epsilon_{[A}^\dagger \Lambda_{B]}\Big), \tag{3.44}$$

$$\Lambda_C = i\Big(\epsilon_{[A}\epsilon_{B]}^\dagger + [\Lambda_A, \Lambda_B]\Big), \tag{3.45}$$

$$\underline{\Lambda}_C = i\Big(\epsilon_{[A}^\dagger \epsilon_{B]} + [\underline{\Lambda}_A, \underline{\Lambda}_B]\Big). \tag{3.46}$$

The Killing potentials M_i then transform adjointly:

$$\delta_1 M_2 = M_3, \tag{3.47}$$

provided the constants in (3.41) are chosen to be

$$C = -\frac{1}{2}\operatorname{Tr}\left[\underline{\xi}\right]. \tag{3.48}$$

[†]For equations (3.43)-(3.46) we adopt the convention that Λ, $\underline{\Lambda}$ are again general hermitian matrices.

3.3 Matter coupling in the σ-model

In this section the coupling of matter fields to the sigma model is addressed. Matter fields ψ are introduced as linear representations T of the symmetry group. To wit, it transforms as follows under the subgroups U of the symmetry transformations (3.13)[19]:

$$\psi \rightarrow \psi' = \psi T(U). \tag{3.49}$$

Nonlinear transformations under elements Z of the full gauge group can then be obtained by

$$\psi \rightarrow \psi' = \psi T(U(Z)). \tag{3.50}$$

This is illustrated explicitly below.

Consider a field ψ^α in the fundamental representation of the proper subgroup $SU(N)$. Using the above procedure, this field can be extended to a representation of the full symmetry group by defining its complementary $SU(2N)$ components according to[28]

$$\psi^{\underline{\alpha}} U^{\dagger i}_{\underline{\alpha}} = -\psi^{\underline{i}} U^{\dagger i}_{\underline{i}} \tag{3.51}$$

or

$$\psi^{\underline{i}} X^{\dagger \alpha}_{\underline{i}} = -\psi^{\underline{\alpha}} Y^{\dagger \alpha}_{\underline{\alpha}}. \tag{3.52}$$

In this sense, the partitions U, V, X, Y and their hermitian conjugates can be regarded as generalised N-bein fields, since they enable a extension of vectors from the local linear subgroups to the global symmetry group, or vice versa. We can use the parametrisation to solve these equations in terms of the manifold coordinates:

$$\psi^{\underline{i}} = -\psi^{\underline{\alpha}} \left(V^\dagger U^{\dagger -1} \right)^{\underline{i}}_{\underline{\alpha}} = -\psi^{\underline{\alpha}} \phi^{\dagger i}_{\underline{\alpha}}, \tag{3.53}$$

$$\chi^{\underline{i}} = -\psi^{\underline{i}} \left(X^\dagger Y^{\dagger -1} \right)^{\underline{\alpha}}_{\underline{i}} = -\psi^{\underline{i}} \phi^{\underline{\alpha}}_{\underline{i}}. \tag{3.54}$$

With this prescription, the components of ψ transform under Z as

$$\delta \psi^{\underline{i}} = \psi^{\underline{j}} \left(i\Lambda - \phi \epsilon^\dagger \right)^{\underline{i}}_{\underline{j}} \tag{3.55}$$

$$\delta \chi^{\underline{i}} = \psi^{\underline{j}} \left(i\underline{\Lambda} + \phi \epsilon^\dagger \right)^{\underline{i}}_{\underline{j}} \tag{3.56}$$

It will be seen later that this is exactly the type of gauge transformation necessary for gauge invariance to be compatible with supersymmetry.

It is now possible to write down a Lagrangian for spinor fields coupled to the σ-model. Consider $\psi^{\underline{\alpha}}$ in the fundamental representation of $SU(N)$. We can then introduce an auxiliary spinor χ according to equation (3.51):

$$\chi U^\dagger + \psi V^\dagger = 0. \tag{3.57}$$

Then the vector (ψ, χ) spans a linear representation of $SU(2N)$, and thus a invariant Lagrangian is

$$\mathcal{L} = -i\overline{\psi}\overleftrightarrow{\partial}\psi - i\overline{\chi}\overleftrightarrow{\partial}\chi. \tag{3.58}$$

Upon substitution of equation (3.53) we find the non-linear realisation:

$$\mathcal{L} = -\operatorname{Tr}\left[i\left(1 + \phi^\dagger\phi\right)\left(\overline{\psi}\overleftrightarrow{\partial}\psi\right) - i\left(\overline{\psi}\gamma^\mu\psi\right)\left(\phi^\dagger\overleftrightarrow{\partial_\mu}\phi\right)\right]. \tag{3.59}$$

If we then make the redefinition

$$\chi = \left(1 + \phi\phi^\dagger\right)^{1/2}\psi\left(1 + \phi^\dagger\phi\right)^{1/2}, \tag{3.60}$$

we find, after a Fierz rearrangement of the four-fermion term,[28]

$$\mathcal{L} = -\operatorname{Tr}\left[g\left(\partial_\mu\phi^\dagger\partial^\mu\phi - i\overline{\chi}\overleftrightarrow{D}\chi + \mathcal{R}\overline{\chi}\chi\overline{\chi}\chi\right)\right], \tag{3.61}$$

with

$$\mathcal{R} = \frac{\delta^2 g}{\delta\phi\delta\phi^\dagger} - g^{-1}\frac{\delta g}{\delta\phi}\frac{\delta g}{\delta\phi^\dagger}, \tag{3.62}$$

$$D_\mu\chi = \partial_\mu\chi + \Gamma\partial_\mu\phi\chi. \tag{3.63}$$

These are the analogues of (2.181) and (2.182) for the grassmannian σ-manifolds.

3.4 Supersymmetric extension of the σ-model

From the previous section it is seen that the $SU(2N)/SU(N)^2 \times U(1)$ enables supersymmetry once matter fields have been coupled to the model. An off-shell supersymmetric lagrangian can now be presented. First, introduce a chiral spinor ψ_L and an auxiliary field H transforming under the adjoint representation of the gauge group. From the analysis at 2.7 we are now in a position to write down a Lagrangian invariant under supersymmetry transformations:[13, 55, 56]

$$\mathcal{L} = -\,\mathrm{Tr}\left[g\left(\partial_\mu \phi^\dagger \partial^\mu \phi - i\overline{\psi}_L \overleftrightarrow{\slashed{\partial}} \psi_L - \overline{H}H \right) - i\left(\frac{\delta g}{\delta \phi^\dagger}\overline{\psi}_L \psi_R H + \frac{\delta g}{\delta \phi}\overline{H}\overline{\psi}_R \psi_L \right) \right.$$
$$\left. -i\left(\partial_\mu \phi \frac{\delta g}{\delta \phi} - \frac{\delta g}{\delta \phi^\dagger}\partial_\mu \phi^\dagger \right)\overline{\psi}_L \gamma^\mu \psi_L + \frac{\delta^2 g}{\delta \phi \delta \phi^\dagger}\overline{\psi}_R \psi_L \overline{\psi}_L \psi_R \right]. \tag{3.64}$$

This Lagrangian is invariant under the supersymmetry transformations[‡]

$$\delta \phi = -i\overline{\epsilon}_R \psi_L,$$
$$\delta \psi_L = \frac{1}{2}(\slashed{\partial}\phi \epsilon_R + H\epsilon_L), \tag{3.65}$$
$$\delta H = -i\overline{\epsilon}_L \slashed{\partial}\psi_L.$$

In section 3.2.1 it was shown that the coordinates ϕ posses the posses the non-linear symmetry (3.15), which are of the form

$$\delta \phi = \xi R[\phi].$$

This gauge invariance is to be compatible with supersymmetry. Thus, we require the gauge transformations to commute with supersymmetry:

$$[\delta_G, \delta_S]X = 0, \tag{3.66}$$

where δ_G denotes a gauge transformation, δ_S a supersymmetry transformation, and $X = \phi, \psi_L, H$. From this, we can surmise the gauge transformations for ψ_L

[‡]Note that the sfermionic fields are rescaled compared to the D term lagrangian of (2.172). As a result, this lagrangian realises a different representation of the supersymmetry algebra, with $[\delta_\eta, \delta_\epsilon] = \frac{i}{2}\overline{\eta}\gamma^\mu \epsilon \partial_\mu$.

and H:

$$
\begin{aligned}
[\delta_G, \delta_S]\phi &= \bar{\epsilon}_R \delta_G \psi_L - \xi \delta_S R[\phi] \\
&= \bar{\epsilon}_R \delta_G \psi_L - \xi \frac{\delta R}{\delta \phi} \delta_S \phi \\
&= \bar{\epsilon}_R \left(\delta \psi_L - \xi \frac{\delta R}{\delta \phi} \psi_L \right) \\
&= 0.
\end{aligned}
$$

So

$$
\delta_G \psi = \xi \frac{\delta R}{\delta \psi_L} \psi_L. \tag{3.67}
$$

Similarly,

$$
\begin{aligned}
2[\delta_G, \delta_S]\psi_L &= \slashed{\partial} \delta_G \phi \epsilon_R + \delta_G H \epsilon_L - \xi \delta_S \left(\frac{\delta R}{\delta \phi} \psi_L \right) \\
&= \delta_G H \epsilon_L + \xi \frac{\delta R}{\delta \phi} \slashed{\partial} \phi \epsilon_R - \xi \frac{\delta R}{\delta \phi} \slashed{\partial} \phi \epsilon_R - \xi \left[\frac{\delta R}{\delta \phi} H - 2i \frac{\delta^2 R}{\delta \phi \delta \phi} \psi_L \bar{\psi}_R \right] \epsilon_L \\
&= 0.
\end{aligned}
$$

Equation (2.77) allows us to write the double-spinor term as

$$
-\frac{1}{2} \bar{\psi}_R \psi_L \frac{1 - \gamma_5}{2},
$$

and hence the gauge transformations of H take the form

$$
\delta_G H = \xi \left[\frac{\delta R}{\delta \phi} H + i \frac{\delta^2 R}{\delta \phi \delta \phi} \bar{\psi}_R \psi_L \right]. \tag{3.68}
$$

The global symmetries can now be promoted to local symmetries. Doing this requires the usual description of introducing covariant derivatives

$$
\begin{aligned}
\nabla_\mu \phi &= \partial_\mu \phi - e A_\mu R[\phi], \\
\nabla_\mu \psi_L &= \partial_\mu \psi_L - e A_\mu \frac{\delta R}{\delta \phi} \psi_L.
\end{aligned} \tag{3.69}
$$

The gauge bosons A_μ require the introduction of a vector multiplet $V = (A_\mu, \lambda, D)$, in the adjoint representation of the local Killing vectors, for each gauge boson.

The supersymmetry transformations for this multiplet take the form

$$\delta A_\mu = -\frac{i}{2}\bar{\epsilon}\gamma_\mu\lambda, \tag{3.70}$$

$$\delta\lambda = \frac{1}{2}\left(-\sigma^{\mu\nu}F_{\mu\nu} + iD\gamma_5\right)\epsilon, \tag{3.71}$$

$$\delta D = \frac{1}{2}\bar{\epsilon}\gamma_5\slashed{\nabla}\lambda, \tag{3.72}$$

Where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - e\left[A_\mu, A_\nu\right], \tag{3.73}$$
$$\nabla_\mu\lambda = \partial_\mu\lambda - e\left[A_\mu, \lambda\right].$$

To make sure that the supersymmetry algebra still closes for all fields according to (2.165), the supersymmetry transformations have to be modified slightly[§]:

$$\delta\phi = -i\bar{\epsilon}_R\psi_L, \tag{3.74}$$

$$\delta\psi_L = \frac{1}{2}(\slashed{\nabla}\phi\epsilon_R + H\epsilon_L), \tag{3.75}$$

$$\delta H = -i\bar{\epsilon}_L(\slashed{\nabla}\psi_L - e\lambda_R R[\phi]). \tag{3.76}$$

Gauge invariance introduces couplings between the chiral and the vector multiplets. As a result, we have to add additional couplings to the lagrangian to make the model as a whole supersymmetric. The complete gauged and supersymmetric lagrangian is then

$$\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{chiral} + \mathcal{L}_c. \tag{3.77}$$

where

$$\mathcal{L}_{YM} = -\text{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\bar{\lambda}\slashed{\nabla}\lambda - \frac{1}{2}D^2\right], \tag{3.78}$$

$\mathcal{L}_{chiral}$ is (3.64) with the derivatives replaced by covariant derivatives, and

$$\mathcal{L}_c = -\text{Tr}\left[2ieg\left(\bar{\psi}_L\lambda_R R + \overline{R}\bar{\lambda}_R\psi_L\right) + eDM\right], \tag{3.79}$$

with M the Killing potential, compensates the added contributions under a supersymmetry transformation. If the gauged symmetry group has a $U(1)$ factor we can additionally include a Fayet-Iliopoulos term:

$$\mathcal{L}_{FI} = \text{Tr}\left[\alpha e D_{U(1)}\right], \tag{3.80}$$

[§]This modification is dictated by the supersymmetry algebra. A derivation can be found in D.

as discussed in section 2.7.3. Plugging in the equations of motion

$$H = -ig^{-1}\frac{\delta g}{\delta \phi}\overline{\psi}_R \psi_L,$$ (3.81)

$$D = e\left(M + \alpha \delta_{U(1)}\right),$$ (3.82)

the final lagrangian becomes

$$\begin{aligned}
\mathcal{L} = &- \text{Tr}\left[g\left(\nabla_\mu \phi^\dagger \nabla^\mu \phi - i\overline{\psi}_L \overleftrightarrow{\nabla} \psi_L\right) + 2ie\left(g\overline{R\lambda}_R \psi_L + Rg\overline{\psi}_L \lambda_R\right)\right] \\
&- \text{Tr}\left[-i\left(\nabla_\mu \phi \frac{\delta g}{\delta \phi} - \frac{\delta g}{\delta \phi^\dagger}\nabla_\mu \phi^\dagger\right)\overline{\psi}_L \gamma^\mu \psi_L + \mathcal{R}\overline{\psi}_R \psi_L \overline{\psi}_L \psi_R\right] \\
&- \text{Tr}\left[\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\overline{\lambda}\overleftrightarrow{\nabla}\lambda\right] - V
\end{aligned}$$ (3.83)

The $\delta_{U(1)}$ is a shorthand; it is nonzero only if the gauged symmetry group includes the $U(1)$ factor. $\mathcal{R}$ is the curvature tensor (2.34) evaluated in our coordinate system:

$$\mathcal{R} = \frac{\delta^2 g}{\delta\phi\delta\phi^\dagger} - g^{-1}\frac{\delta g}{\delta\phi}\frac{\delta g}{\delta\phi^\dagger}.$$ (3.84)

The evolution of the system is now determined by the scalar potential

$$V = \frac{e^2}{2}\text{Tr}\left[\left(M + \alpha\delta_{U(1)}\right)^2\right].$$ (3.85)

Its exact form depends on which symmetries are gauged. Since the potential comprises a sum of squares, it vanishes only if each symmetry has a vanishing Killing potential. This is generally not the case, as M generally has non-zero constants demanded by (2.42). Thus, supersymmetry is generally not preserved.

Gauging the non-linear symmetries

There are various ways to proceed, now that the scalar potential has been constructed, depending on which symmetries are gauged. In this section, we consider the gauging of the whole symmetry group, as well as the full stability group $SU(N)^2 \times U(1)$. In this section we make extensive use of the results from section 2.1.7. We now apply this to the grassmannian coset space:

Recall that the isometries of the coset were obtained by multiplication from the right by a hermitian matrix:

$$Z' = ZU, \tag{4.1}$$

where Z is given by (3.1)-(3.3) and U is given by (3.14). In the language of Yang-Mills theory, the non-linear transformations can then be written as

$$\delta\phi = \epsilon + \phi\epsilon^\dagger\phi - i\Lambda\phi + i\phi\Lambda - i\xi\phi$$
$$= \Xi \cdot R. \tag{4.2}$$

The parameter independent Killing vectors can then be obtained from the Killing potentials To do so, it is convenient to write down the Killing potential in terms of $SU(2N)$ representations. As an element of the full symmetry group, we find:

$$M = \begin{pmatrix} \frac{\phi\phi^\dagger}{1+\phi\phi^\dagger} - \frac{1}{2} & -\frac{1}{1+\phi\phi^\dagger}\phi \\ -\frac{1}{1+\phi^\dagger\phi}\phi^\dagger & -\frac{\phi^\dagger\phi}{1+\phi^\dagger\phi} + \frac{1}{2} \end{pmatrix}, \tag{4.3}$$

$$R = \begin{pmatrix} -i\phi & i \\ -i\phi^2 & i\phi \end{pmatrix}. \tag{4.4}$$

Two final remarks on the notation that we've used so far: First, in this derivation we have chosen linear representations of the full gauge group. The non-linear representations can be obtained by explicitly evaluating the various matrix products. For example, if the whole $SU(2N)$ group is gauged:

$$\nabla_\mu\phi = \partial_\mu\phi - \left(Q_\mu + \phi Q_\mu^\dagger\phi - iV_\mu\phi + i\phi W_\mu\right), \tag{4.5}$$

as dictated by equation (3.15). Similarly, for the fermionic fields

$$\nabla_\mu\psi_L = \partial_\mu\psi_L - \left(\psi_L Q_\mu^\dagger\phi + \phi Q_\mu^\dagger\psi_L - iV_\mu\psi_L + i\psi_L W_\mu\right), \tag{4.6}$$

in accordance to equation (3.69). Furthermore, this choice of representation means that equation (3.83) carries an implicit sum of the Yang-Mills lagrangians over the subgroups of $SU(2N)$ as partitioned in section 3.2.1. Here, we denote V_μ and W_μ as the gauge bosons belonging to the $SU(N)$ group acting from the left and right, respectively, as in section 3.2. Similarly, the gauge bosons denoted by

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$Q_\mu, Q_\mu^\dagger$ belong to the remaining subspace of the $SU(2N)/SU(N)^2 \times U(1)$ coset. Should we gauge the global $U(1)$ subgroup, we call the gauge boson A_μ. Finally, the gauginos corresponding to these gauge bosons are denoted by $\lambda^v, \lambda^w, \lambda^q, \lambda^a$, respectively.

4.1 A note on the stability of the system

At this point a few remarks are in order: generally, some Kähler metrics contain zero modes. This means that in the ground state one or more of the kinetic terms vanish. If this is the case, the model is said to be singular, as mass or curvature terms in the lagrangian diverge.[23] Another possibility is the appearance of ghosts; in this case some of the fields carry negative kinetic energy, and the model becomes unstable.[37] Examples of such singular manifolds are the anomaly free manifolds $SO(2N)/U(N)$ and $SU(2)/U(1)$.[22, 23] In the pure σ-model, however, the metric is non-singular.

Recall that the Kähler metric of the $SU(2N)$ model is

$$g = \left(1 + \phi\phi^\dagger\right)^{-1} \otimes \left(1 + \phi^\dagger\phi\right)^{-1} \tag{4.7}$$

If the manifold is to be free of ghosts and zero modes, this metric must be positive definite:

$$\det g = \frac{1}{1 + \phi\phi^\dagger}\frac{1}{1 + \phi^\dagger\phi} > 0. \tag{4.8}$$

The graph of (4.8) is plotted in figure 4.1. As it is nonnegative in any neighbourhood, we conclude that our model is nonsingular as is.

4.2 Gauging the full group

Gauging the full $SU(2N)$ group, we find that the scalar potential is constant:

$$V_{sc} = \frac{e^2}{2}\,\mathrm{Tr}\left[\begin{matrix} \frac{1}{1+\phi\phi^\dagger}\phi\phi^\dagger - \frac{1}{2} & -\frac{1}{1+\phi\phi^\dagger}\phi \\ -\frac{1}{1+\phi^\dagger\phi}\phi^\dagger & -\frac{1}{1+\phi^\dagger\phi} + \frac{1}{2} \end{matrix}\right]^2 = \frac{e^2 N}{4} > 0. \tag{4.9}$$

Thus, supersymmetry is broken. The scalar fields ϕ are therefore the Goldstone fields of the theory. The scalar degrees of freedom are absorbed by the gauge bosons of the non-linear symmetries. As a result these gauge bosons become massive, as can be seen in the unitary gauge $\phi = 0$. The covariant derivatives
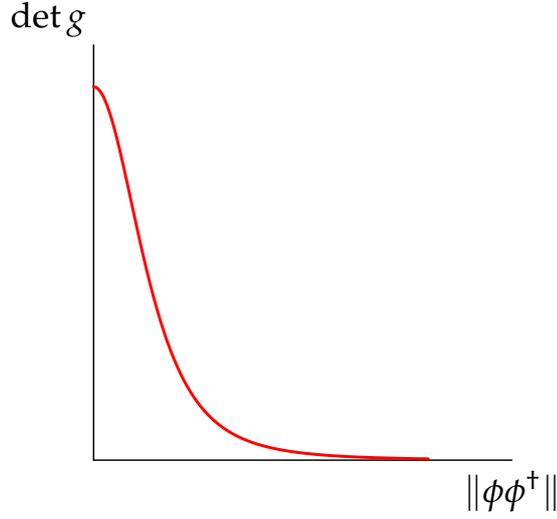
Figure 4.1: *determinant of the metric as a function of $\|\phi\phi^\dagger\|$.*

are:

$$\nabla_\mu\phi = \partial_\mu\phi - e\left(Q_\mu - \phi Q_\mu^\dagger\phi - iV_\mu\phi + i\phi W_\mu\right) \to -eQ_\mu, \qquad (4.10)$$

$$\nabla_\mu\psi_L = \partial_\mu\psi_L - e\left(\psi_L Q_\mu^\dagger\phi + \phi Q_\mu^\dagger\psi_L - iV_\mu\psi_L + i\psi_L W_\mu\right)$$
$$\to \partial_\mu\psi_L + ie(V_\mu\psi_L - \psi_L W_\mu). \qquad (4.11)$$

Where the expressions to the right of the arrows in equations (4.10) and (4.11) are evaluated in the unitary gauge. From this we can make several remarks: first, the constant value of (4.10) indicates that the non-linear symmetry is spontaneously broken. Hence, the gauge field becomes massive. Furthermore, the sfermion field ψ_L decouples from gauginos of the linear subgroup, instead combining with the gauginos of the nonlinear subgroup, forming a massive Dirac fermion. The spectrum therefore comprises two massless scalar multiplets, as well as a massive doublet comprising one Dirac fermion and one gauge boson. As supersymmetry is broken, this doublet no longer comprises a supermultiplet.

The preceding paragraph can be shown quantitatively: in the unitary gauge, the metric becomes constant:

$$g \to 1. \qquad (4.12)$$

Expanding the kinetic term of the coordinates in the unitary gauge, we find that the non-linear symmetry is broken:

$$\nabla_\mu\phi^\dagger\nabla^\mu\phi = e^2 Q_\mu^\dagger Q^\mu. \qquad (4.13)$$

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Particle	Ψ	Q_μ	λ^w	W_μ	λ^v	V_μ
Squared mass	$2e^2$	$2e^2$	0	0	0	0

Table 4.1: *Mass spectrum of the physical fields after gauging the full symmetry group.*

Considering the bosonic part of the lagrangian up to terms of quadratic order in the fields[*] then yields

$$\mathcal{L}_b = \mathrm{Tr}\left[\left(-\frac{1}{4}\left(\partial_\mu Q_\nu - \partial_\nu Q_\mu\right)\left(\partial_\mu Q_\nu^\dagger - \partial_\nu Q_\mu^\dagger\right) - \frac{1}{2}\left(2e^2 Q^\mu Q_\mu^\dagger\right)\right)\right.$$
$$-\frac{1}{4}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right) \tag{4.14}$$
$$\left.-\frac{1}{4}\left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\left(\partial_\mu W_\nu - \partial_\nu W_\mu\right)\right] + \dots$$

Computing the field equations and comparing with we see that the bosons of the unbroken symmetries remain massless, whereas the boson belonging to the non-linear transformations gains a mass $m_q^2 = 2e^2$.

Next, we consider the fermionic part of the lagrangian, again up to terms of quadratic order. Using equations (3.83) and (4.4), we see that the only non-zero Yukawa couplings are those of the sfermion ψ_L with Q_μ, producing the fermionic lagrangian

$$\mathcal{L}_f = i\mathrm{Tr}\left[\overline{\psi}_L \overset{\leftrightarrow}{\partial} \psi_L + \frac{1}{2}\overline{\lambda^q}_L \overset{\leftrightarrow}{\partial} \lambda^q{}_L + \frac{1}{2}\overline{\lambda^v}_L \overset{\leftrightarrow}{\partial} \lambda^v{}_L + \frac{1}{2}\overline{\lambda^w}_L \overset{\leftrightarrow}{\partial} \lambda^w{}_L\right.$$
$$\left.+2e\left(\overline{\lambda^q}_R\psi_L + \overline{\psi}_L\lambda^q{}_R\right)\right] + \dots \tag{4.15}$$

Defining

$$\Psi = \sqrt{2}\psi_L - \lambda^q_R, \tag{4.16}$$

the lagrangian is readily diagonalised. Up to a global factor of 2 (4.15) becomes

$$\mathcal{L}_q = i\,\mathrm{Tr}\left[\overline{\lambda^v}\overset{\leftrightarrow}{\partial}\lambda^v + \overline{\lambda^w}\overset{\leftrightarrow}{\partial}\lambda^w + \overline{\Psi}\overset{\leftrightarrow}{\partial}\Psi + \sqrt{2}e\overline{\Psi}\Psi\right] + \dots \tag{4.17}$$

Again turning to the equations of motion and comparing with, we surmise that the gauginos remain massless, whereas the Dirac spinor Ψ gains a mass $m_\Psi^2 = 2e^2$. For easy reference, the mass spectrum is given in table 4.1. We see that in the multiplets (W_μ, λ^w), (V_μ, λ^v) the number of helicity states are equal,

---

[*]There are higher order terms, of course, but these are interaction terms. The lagrangian up to second order yields the free model, from which the field equations of section 2.3.5 can be derived.

therefore forming full scalar multiplets. The doublet (Q_μ, Ψ) misses one scalar state, thereby no longer forming a supermultiplet.[†]

4.3 Gauging the full stability group

In this section it is convenient to perform the following decomposition for the scalar field ϕ:[‡]

$$\phi = e^{i\xi} P, \tag{4.18}$$

where both ξ and P are hermitian[§]. Since this decomposition mirrors a linear gauge transformation

$$P \to P' = e^{i\xi} P, \tag{4.19}$$

the field ξ disappears from the physical spectrum, serving as a Goldstone field. If the potential admits any minima, say at $P = \phi_0$, fluctuations about these minima can be parametrised by

$$\phi \to \phi_0 + \rho, \tag{4.20}$$

where now, ρ is the Higgs boson of the theory.

Next to the full symmetry group, a subgroup with interesting features is the maximal subgroup $SU(N)^2 \times U(1)$. In this case, the abelian component allows us to include a Fayet-Iliopoulos term in (3.83). The potential (3.85) then takes the form

$$V = \frac{e_1^2}{2} \operatorname{Tr} \left[\frac{\phi \phi^\dagger}{1 + \phi \phi^\dagger} \right]^2 + \frac{e_2^2}{2} \operatorname{Tr} \left[\frac{\phi^\dagger \phi}{1 + \phi^\dagger \phi} \right]^2 + e_3^2 \operatorname{Tr} \left[\frac{\phi \phi^\dagger}{1 + \phi \phi^\dagger} - \left(\frac{1}{2} - \alpha \right) \right]^2. \tag{4.21}$$

Depending on the value of α, the potential may or may not have nontrivial minima. This will be expanded upon in the next sections.

Expanding the covariant derivative for the coordinate field, using (4.20):

$$\begin{aligned} \nabla_\mu \phi = \partial_\mu \rho &- i\left(-e_1 V_\mu \rho + e_2 \rho W_\mu \right) - ie_3 A_\mu \rho \\ &- i\left(-e_1 V_\mu \phi_0 + e_2 \phi_0 W_\mu \right) - ie_3 A_\mu \phi_0. \end{aligned} \tag{4.22}$$

[†]In fact, this scalar state is related to the theory of anomlies. The anomaly cancellation requires the embedding of the N^2 dimensional Kähler manifold into a larger $2N^2$ dimensional manifold. As a result, extra scalar degrees of freedom are introduced. See [23].

[‡]Any square complex matrix can be decomposed in this manner. For a short proof, see for example [57].

[§]Actually, we only require P to be positive definite, in the sense that any quadratic form of P is real and nonnegative. For a complex matrix this implies that P is hermitian.

In preparation of what is to come, we define

$$q^2 = e_1^2 + e_2^2, \tag{4.23}$$

$$\sin\theta = \frac{e_1}{q}, \tag{4.24}$$

$$\cos\theta = \frac{e_2}{q}, \tag{4.25}$$

$$X_\mu = \cos\theta W_\mu - \sin\theta V_\mu, \tag{4.26}$$

$$Y_\mu = \sin\theta W_\mu + \cos\theta W_\mu. \tag{4.27}$$

A similar construction can be made for the gauginos λ_R^x, λ_R^y. In terms of these new gauge fields, the covariant derivative becomes:

$$\nabla_\mu\phi \to \nabla_\mu\rho - iq\phi_0 X_\mu - ie_3\phi_0 A_\mu + C_\mu, \tag{4.28}$$

where

$$\nabla_\mu\rho = \partial_\mu\rho - i\frac{q}{2}\left(\{\rho, X_\mu\} + \cos 2\theta\left[\rho, X_\mu\right] + \sin 2\theta\left[\rho, Y_\mu\right]\right) - ie_3 A_\mu\rho, \tag{4.29}$$

$$C_\mu = i\frac{q}{2}\left(\cos 2\theta\left[\phi_0, X_\mu\right] + \sin 2\theta\left[\phi_0, Y_\mu\right]\right). \tag{4.30}$$

It will later turn out that C vanishes. From equation (4.28) it can be surmised that, if the potential has nontrivial minima, the gauge fields X_μ and A_μ become massive. If that turns out to be the case, the global symmetry is spontaneously broken to $SU(N)$. Two cases will be considered; the case $\alpha = 0$, where both supersymmetry and the internal symmetry are broken, and the case $\alpha = 1/2$, where both supersymmetry and the internal symmetry are preserved. For both cases, the potential is sketched in figure 4.2.

4.3.1 Including the Fayet-Iliopoulos term

If $\alpha = 1/2$, the potential can be seen to be

$$V = \frac{e_1^2 + e_2^2 + 2e_3^2}{2} \operatorname{Tr}\left[\frac{\phi\phi^\dagger}{1 + \phi\phi^\dagger}\right]^2. \tag{4.31}$$

Thus, $\phi_0 = 0$, and subsequently both supersymmetry and the internal symmetry are preserved. Using equation (4.20) we proceed analogously to the case of the full symmetry group. Using the parametrisation (4.20), the metric and potential can be expanded up to quadratic order:

$$g = 1 + \mathcal{O}\left(\rho^2\right), \tag{4.32}$$

$$V = \mathcal{O}\left(\rho^4\right). \tag{4.33}$$

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Collecting terms up to quadratic order the bosonic part of (3.83) is

$$\mathcal{L}_b = - \operatorname{Tr} \left[\partial_\mu \rho \partial^\mu \rho \right] - \frac{1}{4} \operatorname{Tr} \left[(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \right] \\ - \frac{1}{4} \operatorname{Tr} \left[(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 \right] - \frac{1}{4} \operatorname{Tr} \left[(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 \right] \tag{4.34}$$

The Yukawa couplings in equation (3.83) vanish up to second order. Thus the fermionic part of the lagrangian can, after appropriately renormalising the fields, be written as

$$\mathcal{L}_f = \frac{i}{2} \operatorname{Tr} \left[\overline{\psi_L} \overleftrightarrow{\partial} \psi_L + \overline{\lambda^v}_R \overleftrightarrow{\partial} \lambda^v_R + \overline{\lambda^w}_R \overleftrightarrow{\partial} \lambda^w_R + \overline{\lambda^a}_R \overleftrightarrow{\partial} \lambda^a_R \right] \tag{4.35}$$

The physical spectrum can therefore be seen to comprise the massless multiplets (ρ, ψ_L), (A_μ, λ^a), (V_μ, λ^w), (W_μ, λ^w).

4.3.2 Gauging without the Fayet-Iliopoulos term

On the other hand, if we set $\alpha = 0$, the potential becomes

$$V = \frac{q^2}{2} \operatorname{Tr} \left[\frac{\rho^2}{1 + \rho^2} \right]^2 + \frac{e_3^2}{2} \operatorname{Tr} \left[\frac{1}{4} - \frac{\rho^2}{(1 + \rho^2)^2} \right], \tag{4.36}$$

and hence has nontrivial extrema. Demanding that the variation of the potential with respect to ρ vanish, we find

$$\frac{\delta V}{\delta \rho} = \left(1 + \rho^2 \right)^{-3} \rho \left(\left[2q^2 + e_3^2 \right] \rho^2 - e_3^2 \right) = 0, \tag{4.37}$$

which occurs if $\rho = 0$ or

$$\rho_0^2 = \frac{e_3^2}{e_3^2 + 2q^2}. \tag{4.38}$$

It can straightforwardly be checked that the trivial solution corresponds to a local maximum, which is therefore unstable. The ground state of the system therefore is ρ_0. From figure 4.2 it can be seen that both the internal symmetry and supersymmetry are broken. As a result, the particle spectrum is split into two vector multiplets, one that remains massless, whereas the other becomes massive, in addition to the Higgs boson. As noted previously, the internal symmetry group breaks down to $SU(N)$.

Since the ground state is degenerate, the field ϕ can freely be expanded about the real ρ_0:

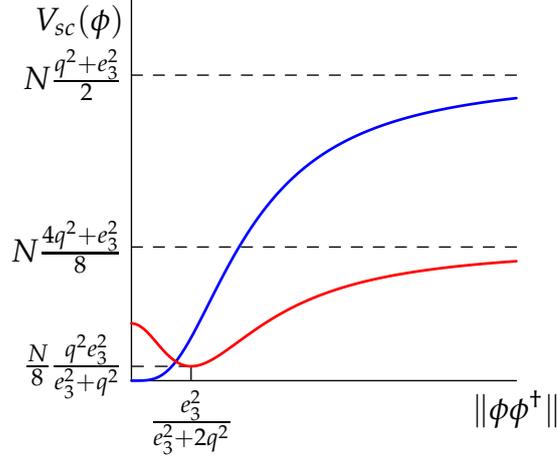$$\phi = \rho_0 + \sqrt{\mathcal{A}} \rho. \tag{4.39}$$

Figure 4.2: *Graphs of the potential with $\alpha = 1/2$ (blue), and $\alpha = 0$ (red).*

$\mathcal{A}$ is a normalisation factor, chosen so that the field ρ has a properly normalised kinetic term. Again expanding the metric and the potential up to second order:

$$g = \frac{1}{(1+\rho_0^2)^2} + \mathcal{O}\left(\rho^2\right), \tag{4.40}$$

$$V = \frac{N}{8}\frac{q^2 e_3^2}{q^2 + e_3^2} + \frac{e_3^2}{2}\frac{\mathcal{A}^2}{(1+\rho_0^2)^2}\frac{e_3^2 + 2q^2}{e_3^2 + q^2}\,\mathrm{Tr}\left[\rho^2\right] + \mathcal{O}\left(\rho^4\right), \tag{4.41}$$

$$\frac{1}{(1+\rho_0^2)^2} = \frac{1}{4}\left(\frac{e_3^2 + 2q^2}{e_3^2 + q^2}\right)^2. \tag{4.42}$$

The bosonic part of the lagrangian, up to second order in the fields, is

$$
\begin{aligned}
\mathcal{L}_b =\ & -\frac{\mathcal{A}^2}{(1+\rho_0^2)^2}\,\mathrm{Tr}\left[\left(\partial_\mu\rho\partial^\mu\rho + \frac{e_3^2}{2}\frac{e_3^2 + 2q^2}{e_3^2 + q^2}\rho^2\right)\right] \\
& - \mathrm{Tr}\left[\left(\frac{1}{4}\left(\partial_\mu X_\nu - \partial_\nu X_\mu\right)^2 + \frac{1}{2}\left(\frac{q^2 e_3^2}{2}\frac{2q^2 + e_3^2}{\left(q^2 + e_3^2\right)^2}\right)X_\mu X^\mu\right)\right] \\
& - \mathrm{Tr}\left[\frac{1}{4}\left(\partial_\mu Y_\nu - \partial_\mu Y_\nu\right)^2\right] \\
& - \mathrm{Tr}\left[\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\mu A_\nu\right)^2 + \frac{1}{2}\left(\frac{e_3^4}{2}\frac{2q^2 + e_3^2}{\left(e_3^2 + q^2\right)^2}A_\mu A^\mu\right)\right] + \dots,
\end{aligned}
\tag{4.43}
$$

whence it follows that for the scalar field ρ to have a properly normalised kinetic term the normalisation satisfies $\mathcal{A} = 1 + \rho_0^2$. Using the Euler-Lagrange equations

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(2.93) and (2.99), the following squared masses are obtained:

$$m_\rho^2 = \frac{1}{2} \frac{e_3^4 + 2q^2 e_3^2}{e_3^2 + q^2}, \tag{4.44}$$

$$m_X^2 = \frac{q^2 e_3^2}{2} \frac{2q^2 + e_3^2}{(q^2 + e_3^2)^2}, \tag{4.45}$$

$$m_Y^2 = 0, \tag{4.46}$$

$$m_A^2 = \frac{e_3^4}{2} \frac{2q^2 + e_3^2}{(e_3^2 + q^2)^2}. \tag{4.47}$$

Having determined the scalar field masses, we move on to consider the fermionic sector in the action up to quadratic terms:

$$\mathcal{L}_f = i \left[g\overline{\psi}_L \overleftrightarrow{\not{\partial}} \psi_L + \frac{1}{2}\overline{\lambda^x}\not{\partial}\lambda^x + \frac{1}{2}\overline{\lambda^y}\not{\partial}\lambda^y + \frac{1}{2}\overline{\lambda^a}\not{\partial}\lambda^a \right.$$
$$\left. +2igq\left(\overline{\psi}_L \lambda_R^x R^x - \overline{R^x \lambda^x}_R \psi_L\right) + 2ige_3\left(\overline{\psi}_L \lambda_R^a R^a - \overline{R^a \lambda^a}_R \psi_L\right)\right] + \dots . \tag{4.48}$$

Plugging in the explicit form of the Killing vectors $R^x \approx i\rho_0$, $R^a \approx -i\rho_0$ and the metric from equation (4.40), equation (4.48) can be written as

$$\mathcal{L}_f = i \left[\overline{\chi}_L \overleftrightarrow{\not{\partial}} \chi_L + \frac{1}{2}\overline{\lambda^x}_R \overleftrightarrow{\not{\partial}} \lambda_R^x + \frac{1}{2}\overline{\lambda^a}_R \overleftrightarrow{\not{\partial}} \lambda_R^a + \frac{1}{2}\overline{\lambda^y}_R \overleftrightarrow{\not{\partial}} \lambda_R^y \right.$$
$$\left. +iqe_3 \frac{\sqrt{2q^2 + e_3^2}}{e_3^2 + q^2}\left(\overline{\chi}_L \lambda_R^x - \overline{\lambda^x}_R \chi_L\right) - ie_3^2 \frac{\sqrt{2q^2 + e_3^2}}{e_3^2 + q^2}\left(\overline{\chi}_L \lambda_R^a - \overline{\lambda^a}_R \chi_L\right)\right], \tag{4.49}$$

where

$$\chi_L = \frac{1}{1+\rho_0^2}\psi_L.$$

The lagrangian can then be diagonalised by defining

$$\Psi = \chi_L - i\lambda_R^x, \tag{4.50}$$

$$\Phi = -i\chi_L + \lambda_R^a. \tag{4.51}$$

In terms of the spinors Ψ and Φ, equation (4.48) simplifies to

$$\mathcal{L}_f = i \operatorname{Tr}\left[\overline{\Psi}\not{\partial}\Psi + qe_3 \frac{\sqrt{2q^2 + e_3^2}}{q^2 + e_3^2}\overline{\Psi}\Psi + \overline{\Phi}\not{\partial}\Phi + e_3^2 \frac{\sqrt{2q^2 + e_3^2}}{q^2 + e_3^2}\overline{\Phi}\Phi + \overline{\lambda^y}\not{\partial}\lambda^y \right]. \tag{4.52}$$

Comparison with (2.95) yields the fermion masses:

$$m_\Psi = e_3 q \frac{\sqrt{2q^2 + e_3^2}}{e_3^2 + q^2}, \quad m_\Phi = e_3^2 \frac{\sqrt{2q^2 + e_3^2}}{e_3^2 + q^2}, \quad m_Y = 0. \tag{4.53}$$

Furthermore, combining (4.44)-(4.47) and (4.53), a straightforward calculation shows that the supertrace formula is satisfied:

$$\begin{aligned}
\mathrm{STr}\, m^2 &= \sum m_0^2 - 2 \sum m_{1/2}^2 + 3 \sum m_1^2 \\
&= m_\rho^2 - 2\left(m_\Psi^2 + m_\Phi^2\right) + 3\left(m_X^2 + m_A^2\right) \\
&= \frac{1}{2} \frac{e_3^4 + 2q^2 e_3^2}{e_3^2 + q^2} - 2\left(e_3^2 q^2 + e_3^4\right) \frac{e_3^2 + 2q^2}{\left(e_3^2 + q^2\right)^2} + \frac{3}{2}\left(e_3^4 + q^2 e_3^2\right) \frac{e_3^2 + 2q^2}{\left(e_3^2 + q^2\right)} \\
&= 0. \tag{4.54}
\end{aligned}$$

Chapter **5**

Anomaly cancellation

It has been mentioned in the introduction that the pure σ-model is anomalous. These arise from the coupling of the bosonic coordinate fields to chiral fermions. As anomalies make a theory inconsistent, one has to find a way to remove them. One way of cancelling anomalies is by introducing Wess-Zumino counter terms to the action.[58, 59] Another method is to introduce chiral multiplets in such a way as to make the coefficients of the anomalies vanish. In light of Grand Unification, the latter option is preferable, as allows for the inclusion of additional families of fermions (quarks and leptons).

5.1 Matter coupling

In this section, an outline will be provided on how to add these matter multiplets to the model in such a way that the original symmetries are respected. The simplest way to do this is by introducing additional supermultiplets which transform as a tangent vector of the manifold. More general matter representations can then be constructed by considering tensor products of these supermultiplets.

First we show how to couple additional matter fields to the non-linear sigma-model. Starting from the scalar field A of this multiplet, which transforms as a tangent vector. In component notation:

$$\delta A^\alpha = \xi_i \frac{\delta R^{i\alpha}}{\delta \phi^\beta} A^\beta, \tag{5.1}$$

in analogy with equation (3.67). Proceeding analogously as in section 3.4, this yields the following transformation rules under the isometries of the manifold:[60, 61]

$$\delta A^\alpha = \xi_i \frac{\delta R^{i\alpha}}{\delta \phi^\beta} A^\beta, \tag{5.2}$$

$$\delta \chi_L^\alpha = \xi_i \left(\frac{\delta R^{i\alpha}}{\delta \phi^\beta} \chi_L + \frac{\delta^2 R^{i\alpha}}{\delta \phi^\beta \delta \phi^\gamma} A^\beta \psi_L^\gamma \right), \tag{5.3}$$

$$\delta N^\alpha = \xi_i \left(\frac{\delta R^{i\alpha}}{\delta \phi^\beta} N^\alpha + \frac{\delta^2 R^{i\alpha}}{\delta \phi^\beta \delta \phi^\gamma} \left[2i \overline{\psi}_R^\beta \chi_L^\gamma + A^\beta N^\gamma \right] + i \frac{\delta^3 R^{i\alpha}}{\delta \phi^\beta \delta \phi^\gamma \delta \phi^\delta} A^\beta \overline{\psi}_R^\gamma \psi_L^\delta \right). \tag{5.4}$$

Hence, the scalar multiplet does not transform homogeneously as a vector. The auxiliary component does not pose any problems, as it is eliminated by its field equation. For the spinor component, a covariant spinor can be defined

$$\hat{\chi}_L^\alpha = \chi_L^\alpha + X_L^\alpha. \tag{5.5}$$

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The requirement that $\hat{\chi}_L^\alpha$ transform as a vector then requires that

$$\delta^i X_L^\alpha = \frac{\delta R^{i\alpha}}{\delta\phi^\beta} X_L^\beta - \frac{\delta^2 R^{i\alpha}}{\delta\phi^\beta \delta\phi^\gamma} A^\beta \psi_L^\gamma. \tag{5.6}$$

Comparison with (3.68), (3.81) and (5.2) shows that the proper prescription is

$$\hat{\chi}_L^\alpha = \chi_L^\alpha - \Gamma_{\beta\gamma}^\alpha A^\beta \psi_L^\gamma. \tag{5.7}$$

This prescription ensures that kinetic terms of the form

$$\mathcal{L}_k = -g_{\alpha\underline{\beta}} \nabla_\mu \overline{A}^{\underline{\beta}} \nabla^\mu A^\alpha - g_{\alpha\underline{\beta}} \overline{\hat{\chi}}_L^{\underline{\beta}} \overset{\leftrightarrow}{\nabla} \hat{\chi}_L^\alpha \tag{5.8}$$

are invariant under the isometries of the manifold, provided the metric (3.34) is used. We are now in a position to couple additional matter to the sigma model. To begin, we interpret the fields (ϕ^α, A^α) as the coordinates of a manifold of dimension $2N$. The original Kähler manifold is a submanifold of this larger object. A suitable manifold is given by the new potential

$$\mathcal{K} = K(\phi, \overline{\phi}) + g_{\alpha\underline{\beta}} \overline{A}^{\underline{\beta}} A^\alpha, \tag{5.9}$$

with K as in (3.28). It can readily be checked that equation (5.9) carries the same Killing vectors of the original manifold. Using equation (2.30) and (5.9), the complete metric can be seen to be

$$G_{i\underline{j}} = \begin{pmatrix} \frac{\delta^2 \mathcal{K}}{\delta\overline{\phi}^{\underline{\beta}} \delta\phi^\alpha} & \frac{\delta^2 \mathcal{K}}{\delta\overline{\phi}^{\underline{\beta}} \delta A^\alpha} \\ \frac{\delta^2 \mathcal{K}}{\delta\overline{A}^{\underline{\beta}} \delta\phi^\alpha} & \frac{\delta^2 \mathcal{K}}{\delta\overline{A}^{\underline{\beta}} \delta A^\alpha} \end{pmatrix} = \begin{pmatrix} g_{\alpha\underline{\beta}} + g_{\alpha\underline{\beta},\gamma\underline{\delta}} \overline{A}^{\underline{\delta}} A^\gamma & g_{\alpha\underline{\beta},\gamma} \overline{A}^{\underline{\gamma}} \\ g_{\alpha\underline{\beta},\gamma} A^\gamma & g_{\alpha\underline{\beta}} \end{pmatrix}. \tag{5.10}$$

The indices $i, \underline{j} = 1, \ldots, 2N$ label the new manifold. Using (5.10), the complete lagrangian can be found by using equation (2.179). Explicit forms can be found in [23, 60, 61]. Finally, the Killing vectors of the new manifold are derived from the new Killing potential:

$$\mathcal{M}_k = M_k + i \overline{A}^{\underline{\alpha}} R_{k\underline{\alpha},\beta} A_\beta. \tag{5.11}$$

The Killing vectors

$$\delta_k \phi^\alpha = R_k^\alpha, \quad \delta_k A^\alpha = \frac{\delta R_k^\alpha}{\delta\phi^\beta} A^\beta, \tag{5.12}$$

then follow from equations (2.39), (2.41) and (2.37), with the understanding that the complete metric (5.10) is used.

5.2 Anomaly cancellation

The transformation rules for the multiplet introduced above are completely fixed in terms of the ones for $(\phi^\alpha, \psi_L^\alpha)$. This is insufficient for the cancellation of anomalies[21]: consider superfield transforming as a tensor of rank p:

$$\delta^i A^{\alpha_1 \alpha_2 \dots \alpha_p} = \frac{\delta R^{i\alpha_1}}{\delta \phi^\beta} A^{\beta \alpha_2 \dots \alpha_p} + \dots + \frac{\delta R^{i\alpha_p}}{\delta \phi^\beta} A^{\alpha_1 \alpha_2 \dots \beta}, \tag{5.13}$$

$$\delta^i \hat{\chi}_L^{\alpha_1 \dots \alpha_p} = \frac{\delta R^{i\alpha_1}}{\delta \phi^\beta} \hat{\chi}_L^{\beta \dots \alpha_p} + \dots + \frac{\delta R^{i\alpha_p}}{\delta \phi^\beta} \hat{\chi}_L^{\alpha_1 \dots \beta}, \tag{5.14}$$

The chiral $U(1)$ charge of this field is fixed be a relative weight of p. However, for phenomenologically interesting models more freedom in charge assignment is required.[21, 61] This can be acomplished by the introduction of complex line bundles. To begin, a scalar matter multiplet S can be introduced whose components $(A, \hat{\chi}_L)$ transform as a complex line bundle of weight k:

$$\delta_i w = k F_i A, \tag{5.15}$$

$$\delta_i \eta_L = k \frac{\delta F_i}{\delta \phi^\beta} A \psi_l^\beta. \tag{5.16}$$

This defines a new representation of the Killing algebra because of equation (2.44). This equation guarantees that the commutator of two Killing transformations satisfies (2.38). From the line bundle, the transformation rules for tensors (5.13), (5.14) can be modified by defining

$$\mathcal{A}^{\alpha_1 \dots \alpha_p} = w \mathcal{A}^{\alpha_1 \dots \alpha_p}, \tag{5.17}$$

and similarly for the spinor component. The new field $\mathcal{A}$ transforms according to

$$\delta_i \mathcal{A}^{\alpha_1 \dots \alpha_p} = \lambda F_i \mathcal{A}^{\alpha_1 \dots \alpha_p} + \frac{\delta R_i^{\alpha_1}}{\delta \phi^\beta} \mathcal{A}^{\beta \alpha_2 \dots \alpha_p} + \dots + \frac{\delta R_i^{\alpha_p}}{\delta \phi^\beta} \mathcal{A}^{\alpha_1 \dots \alpha_{p-1} \beta}. \tag{5.18}$$

With the prescription the $U(1)$ charges can be adjusted.[21] The Kähler potential in (5.9) needs to be adjusted to correct for the additional transformations in (5.18). From (5.18) it follows that the quantity

$$e^{-\lambda K} \overline{w} w, \tag{5.19}$$

where K is the Kähler potential of the σ-model (3.28), is invariant under the full set of isometries of the Killing vectors. Hence, to cancel the anomalies we define the new Kähler potential

$$\mathcal{K} = K + e^{-\lambda K} g_{\alpha \underline{\beta}} \overline{A}^{\underline{\beta}} A^\alpha. \tag{5.20}$$

Note that the construction for the line bundle uses only local quantities. To guarantee that this construction works over the entire manifold, certain charge quantisations must be met. In particular, the holomorphic transfer functions defined in (2.32) must satisfy the so-called cocycle condition[62]

$$\frac{1}{2\pi}\left(F_{(ij)}\left(\phi^k\right) + F_{(jk)}\left(\phi^i\right) + F_{(ki)}\left(\phi^j\right)\right) \in \mathbb{Z}. \tag{5.21}$$

In terms of the Kähler form introduced in equation (2.28), it is required that

$$\frac{1}{2\pi}\int_{C_2} \Omega(K) \in \mathbb{Z}, \tag{5.22}$$

for any closed 2-cycle C_2.[62] This condition implies that the manifold is Kähler-Hodge[32]. Equation (5.21) guarantees that the above construction works over the entire manifold. The anomalies can thus be cancelled, and the internal symmetries gauged consistently as in section 3.4.

Discussion and Outlook

In this thesis the construction of the supersymmetric non-linear σ-model was presented. Using the formalisms of both real differential and Kähler geometry, Lie group theory, and classical field theory reviewed in the first chapter, the bosonic σ-model was constructed, starting from a parametrisation of the symmetry group $SU(2N)$ which was realised non-linearly. The model was then extended by the coupling to matter fields, which provided a natural supersymmetric extension. The demand that the action be simultaneously invariant under gauge and supersymmetry transformations yielded a unique way to determine the coupling of gauge fields to the spinor and auxiliary components of the supermultiplet: by the vanishing of the commutator of gauge and supersymmetry transformations.

After the construction of the supersymmetric σ-model was completed, two subgroups of $SU(2N)$ were gauged: the full symmetry group $SU(2N)$ itself, and the linear subgroup $SU(N)^2 \times U(1)$. It was found that the promotion of these subgroups from global to local gauge invariance introduced terms to the action which broke supersymmetry. This could be fixed by adjusting the suppersymmetry transformation rules, and using the properties of these rules to introduce scalar and Yukawa couplings to the action. By construction, these terms provided exactly the right contributions to restore the supersymmetry of the model. However, it was found that the Killing potential of the theory required non-zero constants.

Once the model was supersymmetrically gauged, the auxiliary fields of the chiral and vector multiplets were eliminated in terms of their field equations. For the vector field auxiliaries, this gave rise to a scalar potential for the scalar fields of the chiral multiplet. The exact shape of the potential depended on which subgroup of $SU(2N)$ was gauged, and in the case of the linear subgroup if the Fayet-Iliopoulos term was included.

For the full symmetry group $SU(2N)$ it was found that the potential was a non-zero constant. The particle spectrum of the theory could be found by implementing the unitary gauge, in which the goldstone bosons vanished. It was then found that the non-linear symmetries of the theory were broken, yielding an incomplete massive multiplet. In addition, two massless $SU(N)$ multiplets were found.

For the linear subgroup $SU(N)^2 \times U(1)$, it was found that the presence of the Fayet-Iliopoulos term was crucial for the preservation of both supersymmetry and the internal symmetry: If the term was included, the potential obtained a global minimum at a vanishing vacuum. Since this minimum vanished, the spectrum comprised one chiral multiplet and three vector multiplets, all massless. However, if the Fayet-Illiopoulis term was neglected the constants in the Killing potential forced the potential to have a non-zero minimum. Since this minimum

was unequal to zero, both the internal symmetry and supersymmetry were broken. The spinors of the chiral multiplet combined with some of the gauginos of the vector multiplet to form Dirac spinors, producing a spectrum of massive and massless particles in which supersymmetry was no longer manifest.

It has been noted several times in this thesis that the non-linear σ-model is anomalous. This does not pose a problem for the analysis done in this thesis, as it was done at the classical level. However, it would be preferable if the structure of the theory would be respected at the quantum level. An extension of the model which ensures that quantum corrections do not destroy the theory was considered in the final part of the thesis. By utilising a line bundle construction, additional matter multiplets can be introduced. The chiral charge of these multiplets can be adjusted in such a way that the anomaly coefficients can be made to vanish.

One might wonder where to go from here. Once the anomalies are cancelled, the symmetries of the model can be gauged consistently. Additionaly, as the model should at low energies reduce to the Standard Model, superpotential terms can be introduced. This facilitates the breaking of the symmetry group to the Standard Model group $SU(3) \times SU(2) \times U(1)$. In section 2.7.4 it was found that the only requirement on the superpotential is that it be holomorphic in the fields ϕ^α. From this, a general superpotential can be seen to be

$$W(\phi) = L_\alpha \phi^\alpha + \frac{1}{2} M_{\alpha\beta} \phi^\alpha \phi^\beta + \frac{1}{3!} y_{\alpha\beta\gamma} \phi^\alpha \phi^\beta \phi^\gamma + \dots, \qquad (6.1)$$

where L, M and y are quantities symmetric in their indices. Using this prescription the anomaly-free construction of a realistic model which exhibits hidden supersymmetry on $SU(2N)$ can truly begin.

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Fierz identities

In this appendix short proofs for the Fierz identities in section 2.3.4 will be provided. All spinors will be Majorana anti-commuting Grassmann numbers, unless otherwise specified.

Proof of equations (2.77) **and** (2.78): Note that for any two chiral spinors ψ_R and χ_L the coefficients in (2.75) for $\chi_L\overline{\psi}_R$ take the form

$$\alpha = -\frac{1}{4}\overline{\psi}_R\chi_L, \tag{A.1}$$

$$\alpha_\mu = -\frac{1}{4}\overline{\psi}_R\gamma_\mu\chi_L = 0, \tag{A.2}$$

$$\alpha_{\mu\nu} = -\overline{\psi}_R\sigma_{\mu\nu}\chi_L = \overline{\chi}_R\sigma_{\mu\nu}\psi_L, \tag{A.3}$$

$$\alpha_{5\mu} = \frac{1}{4}\overline{\psi}_R\gamma_5\gamma_\mu\chi_L = \frac{1}{4}\overline{\psi}_R\gamma_\mu\chi_L = -\alpha_\mu = 0, \tag{A.4}$$

$$\alpha_5 = -\frac{1}{4}\overline{\psi}_R\gamma_5\chi_L = \frac{1}{4}\overline{\psi}_R\chi_L = -\alpha. \tag{A.5}$$

Hence, if $\chi = \psi$, $\alpha_{\mu\nu} = 0$, and we find equation (2.77). The proof for equation (2.78) is identical except with the replacement $L \leftrightarrow R$, and the different eigenvalue for γ_5.

Proof of equations (2.79) **and** (2.80): Again, for chiral spinors ψ_L and χ_L, the matrix coefficients of $\chi_L \overline{\psi}_L$ are

$$\alpha = -\frac{1}{4}\overline{\psi}_L \chi_L = 0 \tag{A.6}$$

$$\alpha_\mu = -\frac{1}{4}\overline{\psi}_L \gamma_\mu \chi_L \tag{A.7}$$

$$\alpha_{\mu\nu} = \overline{\psi}_L \sigma_{\mu\nu} \chi_L = 0 \tag{A.8}$$

$$\alpha_{5\mu} = \frac{1}{4}\overline{\psi}_L \gamma_5 \gamma_\mu \chi_L = -\alpha_\mu \tag{A.9}$$

$$\alpha_5 = \frac{1}{4}\overline{\psi}_L \gamma_5 \chi_L = -\alpha = 0 \tag{A.10}$$

Thus,

$$
\begin{aligned}
\chi_L \overline{\psi}_L &= \frac{1}{2}\overline{\psi}_L \gamma_\mu \chi_L \frac{1 - \gamma_5}{2} \gamma^\mu \\
&= \frac{1}{2}\overline{\psi}_L \gamma_\mu \chi_L \gamma^\mu \frac{1 + \gamma_5}{2}.
\end{aligned}
\tag{A.11}
$$

Again, the proof for (2.80) is identical except with the replacement $L \leftrightarrow R$, and the different eigenvalue for γ_5.

Proof of equation (2.81): This the Fierz identity (2.79) applied to the two left-chiral spinors in the middle, with the subsequent use of the dual identity (2.68) for right-chiral spinors.

B

Appendix

Transformation rules for local gauge fields

The defining feature of the covariant derivative is that it commutes with gauge transformations. In this particular case, this implies that under local $S[U(3) \times U(3)]$ gauge transformation $\mathbf{Z} \to \mathbf{Z}' = \mathbf{\Omega}\mathbf{Z}$,

$$\mathcal{D}_\mu \mathbf{Z} \to \mathbf{\Omega}\mathcal{D}_\mu \mathbf{Z}.$$

We define

$$\mathbf{\Omega} = \begin{pmatrix} e^{i\Lambda} & 0 \\ 0 & e^{i\underline{\Lambda}} \end{pmatrix},$$

with $\Lambda, \underline{\Lambda}$ hermitian. Recall that the covariant derivative has the form

$$\mathcal{D}_\mu \mathbf{Z} = \left(\partial_\mu - igS_\mu\right)\mathbf{Z}.$$

Thus, under a $S[U(3) \times U(3)]$ transformation:

$$
\begin{aligned}
\mathcal{D}_\mu \mathbf{Z} \to \mathcal{D}'_\mu \mathbf{Z}' &\left(\partial_\mu - igS'_\mu\right)\mathbf{\Omega}\mathbf{Z} \\
&= \mathbf{\Omega}\partial_\mu \mathbf{Z} + \mathbf{\Omega}\overleftarrow{\partial_\mu}\mathbf{Z} - igS'_\mu\mathbf{\Omega}\mathbf{Z} \\
&= \mathbf{\Omega}\left(\partial_\mu - igS_\mu\right)\mathbf{Z}.
\end{aligned}
\tag{B.1}
$$

The last step is required by the gauge covariance. From (B.1) we see that

$$
\begin{aligned}
S'_\mu &= \mathbf{\Omega}S_\mu\mathbf{\Omega}^\dagger - \frac{i}{g}\mathbf{\Omega}\overleftarrow{\partial_\mu}\mathbf{\Omega}^\dagger \\
&= \mathbf{\Omega}S_\mu\mathbf{\Omega}^\dagger + \frac{i}{g}\mathbf{\Omega}\partial_\mu\mathbf{\Omega}^\dagger.
\end{aligned}
$$

In our case S_μ depends on the gauge fields

$$S_\mu = \begin{pmatrix} A_\mu & 0 \\ 0 & B_\mu \end{pmatrix}.$$

Therefore, the transformation rule for the gauge fields is

$$A_\mu[\Lambda] = \frac{i}{g} e^{i\Lambda} \partial_\mu e^{-i\Lambda} + e^{i\Lambda} A_\mu[0] e^{-i\Lambda},$$

$$B_\mu[\underline{\Delta}] = \frac{i}{g} e^{i\underline{\Delta}} \partial_\mu e^{-i\underline{\Delta}} + e^{i\underline{\Delta}} B_\mu[0] e^{-i\underline{\Delta}},$$

as claimed in equation (3.11).

Non-linear transformations of manifold coördinates

We can parametrize any global $SU(2N)$ transformation by

$$\Xi = \begin{pmatrix} e^{i\Lambda}\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2} & e^{i\Lambda}\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2}\epsilon \\ -e^{i\Delta}\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2}\epsilon^\dagger & e^{i\Delta}\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2} \end{pmatrix}.$$

It is readily checked that in the infinitesimal case this reproduces (3.14). Multiplication of $\mathbf{Z}$ with Ξ from the right yields

$$\mathbf{Z}\Xi = \begin{pmatrix} e^{iL}\left(1 + \phi\phi^\dagger\right)^{-1/2} & e^{iL}\left(1 + \phi\phi^\dagger\right)^{-1/2}\phi \\ -e^{iK}\left(1 + \phi^\dagger\phi\right)^{-1/2}\phi^\dagger & e^{iK}\left(1 + \phi^\dagger\phi\right)^{-1/2} \end{pmatrix}\begin{pmatrix} e^{i\Lambda}\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2} & e^{i\Lambda}\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2}\epsilon \\ -e^{i\Delta}\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2}\epsilon^\dagger & e^{i\Delta}\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2} \end{pmatrix}$$
$$= \begin{pmatrix} (\mathbf{Z}\Xi)_1^1 & (\mathbf{Z}\Xi)_1^2 \\ (\mathbf{Z}\Xi)_2^1 & (\mathbf{Z}\Xi)_2^2 \end{pmatrix},$$

where

$$(\mathbf{Z}\Xi)_1^1 = e^{iL}\left(1 + \phi\phi^\dagger\right)^{-1/2}e^{i\Lambda}\left[1 - e^{-i\Lambda}\phi e^{i\Delta}\epsilon^\dagger\right]\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2},$$

$$(\mathbf{Z}\Xi)_1^2 = e^{iL}\left(1 + \phi\phi^\dagger\right)^{-1/2}e^{i\Lambda}\left[\epsilon + e^{-i\Lambda}\phi e^{i\Delta}\right]\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2},$$

$$(\mathbf{Z}\Xi)_2^1 = -e^{iK}\left(1 + \phi^\dagger\phi\right)^{-1/2}e^{i\Delta}\left[e^{-\Delta}\phi^\dagger e^{i\Lambda} + \epsilon^\dagger\right]\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2},$$

$$(\mathbf{Z}\Xi)_2^2 = e^{iK}\left(1 + \phi^\dagger\phi\right)^{-1/2}e^{i\Delta}\left[1 - e^{-\Delta}\phi^\dagger e^{i\Lambda}\epsilon\right]\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2}.$$

Note that, for convenience, we can define

$$\xi = e^{-i\Lambda}\phi e^{i\Delta}, \quad \xi^\dagger = e^{-\Delta}\phi^\dagger e^{i\Lambda}.$$

This definition implies

$$\left(1 + \phi\phi^\dagger\right)^{-1/2} e^{i\Lambda} = e^{i\Lambda}\left(1 + \xi\xi^\dagger\right)^{-1/2}, \quad \left(1 + \phi^\dagger\phi\right)^{-1/2} e^{i\underline{\Lambda}} = e^{i\underline{\Lambda}}\left(1 + \xi^\dagger\xi\right)^{-1/2},$$

and hence

$$(Z\Xi)_1^1 = e^{iL}e^{i\Lambda}\left(1 - \xi\xi^\dagger\right)^{-1/2}\left[1 + \xi\epsilon^\dagger\right]\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2}, \tag{C.1}$$

$$(Z\Xi)_1^2 = e^{iL}e^{i\Lambda}\left(1 + \xi\xi^\dagger\right)^{-1/2}\left[\epsilon + \xi\right]\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2}, \tag{C.2}$$

$$(Z\Xi)_2^1 = -e^{iK}e^{i\underline{\Lambda}}\left(1 + \xi^\dagger\xi\right)^{-1/2}\left[\xi^\dagger + \epsilon^\dagger\right]\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2}, \tag{C.3}$$

$$(Z\Xi)_2^2 = e^{iK}e^{i\underline{\Lambda}}\left(1 + \xi^\dagger\xi\right)^{-1/2}\left[1 - \xi^\dagger\epsilon\right]\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2}. \tag{C.4}$$

From this, we see that the gauge is not respected. To restore the gauge, we multiply by

$$\begin{pmatrix} e^{iL}e^{i\Lambda}e^{i\Omega}e^{-i\Lambda}e^{-iL} & 0 \\ 0 & e^{iK}e^{i\underline{\Lambda}}e^{i\underline{\Omega}}e^{-i\underline{\Lambda}}e^{-iK} \end{pmatrix},$$

where $\Omega, \underline{\Omega}$ are chosen to recover a particular gauge choice. We then see that we can identify

$$e^{iL'} = e^{iL}e^{i\Lambda}e^{i\Omega}, \quad e^{iK'} = e^{iK}e^{i\underline{\Lambda}}e^{i\underline{\Omega}}.$$

Furthermore, from equations (C.1) and (C.2), we want

$$\left(1 + \xi\xi^\dagger\right)^{-1/2}\left[1 - \xi\epsilon^\dagger\right]\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2} = \left(1 + \phi'\phi'^\dagger\right)^{-1/2},$$

$$\left(1 + \xi\xi^\dagger\right)^{-1/2}\left[\epsilon + \xi\right]\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2} = \left(1 + \phi'\phi'^\dagger\right)^{-1/2}\phi'$$

$$= \left(1 + \xi\xi^\dagger\right)^{-1/2}\left[1 - \xi\epsilon^\dagger\right]\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2}\phi'.$$

Thus,

$$\phi' = \left(1 + \epsilon\epsilon^\dagger\right)^{1/2}\left[1 - \xi\epsilon^\dagger\right]^{-1}\left[\epsilon + \xi\right]\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2}.$$

Similarly, from (C.3) and (C.4):

$$\phi'^\dagger = \left(1 + \epsilon\epsilon^\dagger\right)^{1/2}\left[1 - \xi^\dagger\epsilon\right]^{-1}\left[\xi^\dagger + \epsilon^\dagger\right]\left(1 + \epsilon\epsilon^\dagger\right)^{-1/2}.$$

It is readily checked that in the infinitesimal case, to first order in $\Lambda, \underline{\Lambda}, \epsilon, \epsilon^\dagger$:

$$\phi' = \left(1 + \epsilon\epsilon^\dagger\right)^{1/2}\left[1 - e^{-i\Lambda}\phi e^{i\underline{\Lambda}}\epsilon^\dagger\right]^{-1}\left[\epsilon + e^{-i\Lambda}\phi e^{i\underline{\Lambda}}\right]\left(1 + \epsilon^\dagger\epsilon\right)^{-1/2}$$

$$= \left[1 + (1 - i\Lambda)\phi(1 + i\underline{\Lambda})\epsilon^\dagger\right]\left[\epsilon + (1 - i\Lambda)\phi(1 + i\underline{\Lambda})\right]$$

$$= \phi + \epsilon + \phi\epsilon^\dagger\phi - i\Lambda\phi + i\underline{\Lambda}\phi,$$

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Appendix D

Supersymmetric gauge invariance

Recall that the bare supersymmetry transformations are

$$\delta\phi = -i\bar{\epsilon}_R\psi_L, \tag{D.1}$$

$$\delta\psi_L = \frac{1}{2}(\slashed{\partial}\phi\epsilon_R + H\epsilon_L), \tag{D.2}$$

$$\delta H = -i\bar{\epsilon}_L\slashed{\partial}\psi_L \tag{D.3}$$

for the chiral multiplet and

$$\delta A_\mu = -\frac{i}{2}\bar{\epsilon}\gamma_\mu\lambda, \tag{D.4}$$

$$\delta\lambda = \frac{1}{2}\left(-\sigma^{\mu\nu}F_{\mu\nu} + iD\gamma_5\right)\epsilon, \tag{D.5}$$

$$\delta D = \frac{1}{2}\bar{\epsilon}\gamma_5\slashed{\partial}\lambda, \tag{D.6}$$

for the vector multiplet. Furthermore, the theory is invariant under the transformation

$$\delta_G\phi = \xi R[\phi],$$

for a gauge transformation δ_G. The commutator of two supersymmetry transformations gives a translation in spacetime:

$$\left[\delta_\eta, \delta_\epsilon\right]X = \frac{i}{2}\bar{\eta}\gamma^\mu\epsilon\partial_\mu X.$$

Demanding that the gauge invariance be local, we promote the derivatives to covariant derivatives. This forces us to modify the SUSY transformation rules,

as the couplings to the vector multiplet introduce new terms due to the variation of the vector field A_μ. For the scalar and gaugino fields, we propose the ansatz

$$\delta\phi = -i\bar\epsilon_R\psi_L, \tag{D.7}$$

$$\delta\psi_L = \frac{1}{2}(\slashed\nabla\phi\epsilon_R + H\epsilon_L). \tag{D.8}$$

The transformation rule for the auxiliary field is constrained by the supersymmetry algebra. We check that the rules for ϕ and ψ_L generate the same algebra for the scalar field:

$$\begin{aligned}
2[\delta_\eta, \delta_\epsilon]\phi &= -2i(\bar\epsilon_R\delta_\eta\psi_L - \bar\eta_R\delta_\epsilon\psi_L) \\
&= -i(\bar\epsilon_R\gamma^\mu\eta_R - \bar\eta_R\gamma^\mu\epsilon_R)\nabla_\mu\phi \\
&\quad - i(\bar\epsilon_R\eta_L - \bar\eta_R\epsilon_L)H,
\end{aligned}$$

but since by the chiral versions of (2.67) and (2.68), this reduces to

$$[\delta_\eta, \delta_\epsilon]\phi = \frac{i}{2}\bar\eta\gamma^\mu\epsilon\nabla_\mu\phi. \tag{D.9}$$

For the gaugino field:

$$\begin{aligned}
2[\delta_\eta, \delta_\epsilon]\psi_L &= -i\gamma^\mu(\epsilon_R\bar\eta_R - \eta_R\bar\epsilon_R)(\nabla_\mu\psi_L) + \frac{ie}{2}\gamma^\mu(\epsilon_R\bar\eta - \eta_R\bar\epsilon)\gamma_\mu\lambda R \\
&\quad + \epsilon_L\delta_\eta H - \eta_L\delta_\epsilon H.
\end{aligned}$$

Using the chiral version of (2.67) and equation (2.80), the commutator simplifies to

$$2[\delta_\eta, \delta_\epsilon]\psi_L = \frac{i}{2}(\bar\eta\gamma_\alpha\epsilon)\gamma^\mu\gamma^\alpha\left(\nabla_\mu\psi_L - \frac{e}{2}\gamma_\mu\lambda_R R\right) + \epsilon_L\delta_\eta H - \eta_L\delta_\epsilon H. \tag{D.10}$$

Inserting the Dirac algebra (2.56), we then find

$$2[\delta_\eta, \delta_\epsilon]\psi_L = i\bar\eta\gamma^\mu\epsilon\nabla_\mu\psi_L - \frac{i}{2}\bar\eta\gamma_\mu\epsilon\gamma^\mu(\slashed\nabla\psi_L - e\lambda_R R) + \epsilon_L\delta_\eta H - \eta_L\delta_\epsilon H.$$

Hence,

$$\epsilon_L\delta_\eta H - \eta_L\delta_\epsilon H = \frac{i}{2}\bar\eta\gamma_\alpha\epsilon\gamma^\alpha[\slashed\nabla\psi_L - e\lambda_R R].$$

Notice that we can insert the projector $(1 + \gamma_5)/2$ before the brackets on the right-hand side. This allows us (using equation (2.79)) to perform the Fierz transformation

$$\frac{1}{2}\bar\eta\gamma^\mu\epsilon\gamma_\mu\frac{1 + \gamma_5}{2} = -(\epsilon_L\bar\eta_L - \eta_L\bar\epsilon_L)$$

Thus, we find

$$\epsilon_L \delta_\eta H - \eta_L \delta_\epsilon H = -i(\epsilon_L \bar{\eta}_L - \eta_L \bar{\epsilon}_L)(\slashed{\nabla} \psi_L - e\lambda_R R).$$

Hence, we find the transformation rule for the auxiliary field:

$$\delta H = -i\bar{\epsilon}_L(\slashed{\nabla} \psi_L - e\lambda_R R). \tag{D.11}$$

As a final check, we compute the commutator of two susy transformations for H: first, note that

$$
\begin{aligned}
2\delta_\eta \delta_\epsilon H &= -2i\bar{\epsilon}_L \delta_\eta(\slashed{\nabla} \psi_L - e\lambda_R R) \\
&= -ie \left[\slashed{\nabla} \slashed{\nabla} \phi \eta_R + \slashed{\nabla} H \eta_L + ie\gamma_\mu \bar{\eta} \gamma_\mu \lambda \frac{\delta R}{\delta \phi} \psi_L \right. \\
&\quad \left. - e \left(\sigma^{\mu\nu} F_{\mu\nu} \eta_R + iD\eta_R \right) R + 2ie\lambda_R \frac{\delta R}{\delta \phi} \bar{\eta}_R \psi_L \right].
\end{aligned}
\tag{D.12}
$$

The first term can be rewritten as

$$
\begin{aligned}
\slashed{\nabla} \slashed{\nabla} \phi &= \gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu \phi \\
&= \frac{1}{2} \gamma^\mu \gamma^\nu \left(\nabla_\mu \nabla_\nu + \nabla_\nu \nabla_\mu + [\nabla_\mu, \nabla_\nu] \right) \phi.
\end{aligned}
$$

Using equations (2.56) and (2.125), this can be simplified to

$$\frac{1}{2} \nabla_\mu \nabla^\mu \phi - e\sigma^{\mu\nu} F_{\mu\nu} R.$$

Hence, (D.12) can be written as

$$
\begin{aligned}
2\delta_\eta \delta_\epsilon H &= -i\bar{\epsilon}_R \eta_R \nabla^2 \phi - ie\bar{\epsilon}_L \eta_R DR - i\bar{\epsilon}_L \gamma^\nu \eta_L \nabla_\nu H \\
&\quad + e\bar{\epsilon}_L \gamma^\mu \bar{\eta} \gamma_\mu \lambda \frac{\delta R}{\delta \phi} \psi_L + 2e\bar{\epsilon}_L \lambda_R \frac{\delta R}{\delta \phi} \bar{\eta}_R \psi_L.
\end{aligned}
\tag{D.13}
$$

The first two terms are invariant if we interchange the order of SUSY transformations. Hence, the commutator is

$$
\begin{aligned}
2[\delta_\eta, \delta_\epsilon] H &= i\bar{\eta} \gamma^\mu \epsilon \nabla_\mu H + 2e \left(\bar{\epsilon}_L \lambda_R \frac{\delta R}{\delta \phi} \eta_R \psi_L - \bar{\eta}_L \lambda_R \frac{\delta R}{\delta \phi} \epsilon_R \psi_L \right) \\
&\quad + e \left(\bar{\epsilon}_L \gamma^\mu \bar{\eta} \gamma_\mu \lambda \frac{\delta R}{\delta \phi} \psi_L - \bar{\eta}_L \gamma^\mu \bar{\epsilon} \gamma_\mu \lambda \frac{\delta R}{\delta \phi} \psi_L \right) \\
&= i\bar{\eta} \gamma^\mu \epsilon \nabla_\mu H + \left(\bar{\eta}_L \gamma_\mu \lambda_L \frac{\delta R}{\delta \phi} \bar{\epsilon}_L \gamma^\mu \psi_L - \bar{\epsilon}_L \gamma_\mu \lambda_L \frac{\delta R}{\delta \phi} \bar{\eta}_L \gamma^\mu \psi_L \right). \tag{D.14}
\end{aligned}
$$

However,

$$\bar{\epsilon}_L \gamma_\mu \lambda_L \frac{\delta R}{\delta \phi} \bar{\eta}_L \gamma^\mu \psi_L = \bar{\lambda}_R \gamma_\mu \epsilon_R \frac{\delta R}{\delta \phi} \bar{\psi}_R \gamma^\mu \eta_R$$

$$= -\frac{1}{2} \bar{\eta}_L \gamma_\mu \gamma_\nu \gamma^\mu \lambda_L \frac{\delta R}{\delta \phi} \bar{\epsilon}_L \gamma^\nu \psi_L$$

$$= \bar{\eta}_L \gamma_\nu \lambda_L \frac{\delta R}{\delta \phi} \bar{\epsilon}_L \gamma^\nu \psi_L.$$

In this derivation we used equations (2.80) and (2.68) in the first step and the identity

$$\bar{\eta}_L \gamma_\mu \gamma_\nu \gamma^\mu \lambda_L = -2\bar{\eta}_L \gamma_\nu \lambda_L$$

in the second step.

Therefore, the last two terms in (D.14) cancel, and we are left with

$$[\delta_\eta, \delta_\epsilon] H = \frac{i}{2} \bar{\eta} \gamma^\mu \epsilon \nabla_\mu H, \tag{D.15}$$

where

$$\nabla_\mu H = \partial_\mu H - eA_\mu \left(\frac{\delta R}{\delta \phi} H + i \frac{\delta^2 R}{\delta \phi^2} \bar{\psi}_R \psi_L \right). \tag{D.16}$$

Construction of the gauge invariant
σ-model

In this appendix it will be shown that the gauge invariant version of (3.64) together with (3.79) are invariant under the supersymmetry transformations (3.74)-(3.76) and (3.70)-(3.72). This will be done by construction, starting from (3.64). It will be assumed that the reader is familiar with the global gauge invariant case. For details, see [48]. Equation (3.64) is by construction invariant under supersymmetry as the D-term of a vector superfield.[*] The first step is the promotion of partial derivatives to covariant ones, as explained in section 3.4. These derivatives introduce extra terms, due to the variation of the vector potential A_μ, under supersymmetry transformations. These terms, along with the extra terms due to the extra variation of H, can then be cast in the form of (3.79).

We can split (3.64) into 3 parts:

$$\mathcal{L} = \mathrm{Tr}\left[L_{kin} + L_{int} + L_{coupling}\right], \tag{E.1}$$

with

$$L_{kin} = -g\left(\nabla_\mu \phi^\dagger \nabla^\mu \phi + i\overline{\psi}_L \overleftrightarrow{\nabla} \psi_L + \overline{H}H\right), \tag{E.2}$$

$$L_{int} = i\frac{\delta g}{\delta \phi^\dagger}\overline{\psi}_L \psi_R H + i\frac{\delta g}{\delta \phi}\overline{H}\overline{\psi}_R \psi_L + i\left(\nabla_\mu \phi \frac{\delta g}{\delta \phi} - \frac{\delta g}{\delta \phi^\dagger}\nabla_\mu \phi^\dagger\right)\overline{\psi}_L \gamma^\mu \psi_L$$
$$- \frac{\delta^2 g}{\delta \phi^2}\overline{\psi}_R \psi_L \overline{\psi}_L \psi_R. \tag{E.3}$$

---

[*] Indeed, since the chiral part of (3.77) is invariant under supersymmetry one might wonder one can create such a superfield using covariant derivatives. This is indeed the case, and (3.79) is then demanded by the closure of the algebra. See for example [48]

$L_{coupling}$ corresponds to (3.79). The variation of each part will be considered seperately.

Variation of L_{kin}

For the kinetic term of the coordinates, we pick up extra terms due to the SUSY variation of A_μ. To wit:

$$-\frac{ie}{2}g\left(\nabla_\mu\phi^\dagger\bar\epsilon_L\gamma^\mu\lambda_L R + \nabla_\mu\phi^\dagger\bar\epsilon_R\gamma^\mu\lambda_R R + \overline{R}\bar\epsilon_L\gamma^\mu\lambda_L\nabla_\mu\phi + \overline{R}\bar\epsilon_R\gamma^\mu\lambda_R\nabla_\mu\phi\right). \tag{E.4}$$

Similarly, the extended transformation rule for H introduces new terms due to the gaugino field:

$$ieg\left(\overline{H}\bar\epsilon_L\lambda_R R + \overline{R}\lambda_R\epsilon_L H\right)$$

Substituting the transformation rules for the sfermion field (3.75), we have the following equality:

$$\begin{aligned} ieg\left(\overline{H}\bar\epsilon_L\lambda_R R + \overline{R}\lambda_R\epsilon_L H\right) &= 2ieg(\delta\overline\psi_L\lambda_R R + \overline{R}\lambda_R\delta\psi_L) \\ &+ ieg\left(\bar\epsilon_R\gamma^\mu\lambda_R\nabla_\mu\phi^\dagger - \overline{R}\lambda_R\gamma^\mu\epsilon_R\nabla_\mu\phi\right). \end{aligned} \tag{E.5}$$

Adding (E.4) and (E.5), we find that gauge invariance adds the following terms to the lagrangian:

$$\begin{aligned} &2ieg\left(\delta\overline\psi_L\lambda_R R + \overline{R}\lambda_R\delta\psi_L\right) + \frac{e}{2}\left(iRg\nabla_\mu\phi^\dagger - \nabla_\mu\phi g\overline{R}\right)\left(\bar\epsilon_R\gamma^\mu\lambda_R - \bar\epsilon_L\gamma^\mu\lambda_L\right) \\ &= 2ieg\left(\delta\overline\psi_L\lambda_R R + \overline{R}\lambda_R\delta\psi_L\right) - \frac{e}{2}\nabla_\mu M\bar\epsilon\gamma_5\gamma^\mu\lambda \end{aligned} \tag{E.6}$$

where M is the Killing potential, with

$$\nabla_\mu M = \nabla_\mu\phi\frac{\delta M}{\delta\phi} + \frac{\delta M}{\delta\phi^\dagger}\nabla_\mu\phi^\dagger = \left(iRg\nabla_\mu\phi^\dagger - \nabla_\mu\phi g\overline{R}\right).$$

After integration by parts, we find a total contribution of

$$\Delta L = 2ieg\left(\delta\overline\psi_L\lambda_R R + \overline{R}\lambda_R\delta\psi_L\right) + eM\delta D. \tag{E.7}$$

The sfermionic kinetic term can be split up in two parts: first, the coupling to A_μ yields a correction compared to the globally gauge invariant case. Second, the covariant derivatives introduce the complication that, unlike partial derivatives, they do not commute. Since this is crucial for supersymmetry to hold, the coveriant derivatives need to be handled carefully.

First the variation of the sfermionic fields:

$$\frac{i}{2}g\big(\overline{\psi}_L\gamma^\mu\nabla_\mu\delta\psi_L - \nabla_\mu(\overline{\psi}_L)\gamma^\mu\delta\psi_L\big) + \text{h.c},$$

Substituting (3.75) we find, after integration by parts:

$$\frac{i}{2}\overline{\psi}_L\gamma^\mu\gamma^\nu\big(\nabla_\mu\nabla_\nu + \nabla_\mu\nabla_\nu\big)\phi\epsilon_R + \dots, \tag{E.8}$$

where the dots indicate terms that are identical to the case prior to imposing local gauge invariance, except with covariant derivatives instead of partial derivatives. Writing $\nabla_\mu\nabla_\nu = \nabla_\nu\nabla_\mu + [\nabla_\mu, \nabla_\nu]$, we find that compared to the partial derivatives the covariant derivatives add the term

$$\frac{i}{2}g\overline{\psi}_L\gamma^\mu\gamma^\nu[\nabla_\mu,\nabla_\nu]\phi\epsilon_R = -ieg\overline{\psi}_L\sigma^{\mu\nu}F_{\mu\nu}R\epsilon_R,$$

due to the Ricci identity (2.125) for ϕ. Using (3.71), we find that variation of the kinetic terms of the sfermions add

$$2ieg\overline{\psi}_L\delta\lambda_R R + eg\overline{\psi}_L D\epsilon_R + \text{ h.c} = 2ieg\big(\overline{\psi}_L\delta\lambda_R R + \overline{R}\delta\overline{\lambda}_R\psi_L\big) \\ + e(Rg\overline{\psi}_L\epsilon_R - \overline{\epsilon}_R\psi_L g\overline{R})D \tag{E.9}$$

to the lagrangian. The latter term can be written as $e\delta MD$. Moving on to the variation of A_μ: the coupling to A_μ in the covariant derivative picks up extra terms:

$$-ieg\overline{\psi}_L\gamma^\mu\delta A_\mu\frac{\delta R}{\delta\phi}\psi_L = -\frac{e}{2}g\overline{\psi}_L\gamma^\mu\big(\overline{\epsilon}_R\gamma_\mu\lambda_R + \overline{\epsilon}_L\gamma_\mu\lambda_L\big)\frac{\delta R}{\delta\phi}\psi_L. \tag{E.10}$$

The first term can by means of the Fierz rearrangement (2.80) be written as

$$eg\overline{\psi}_L\lambda_R\overline{\epsilon}_R\frac{\delta R}{\delta\phi}\psi_L = \frac{ie}{2}g\overline{\psi}_L\lambda_R\delta R,$$

And comparison with (E.7) motivates us to write

$$-ieg\overline{\psi}_L\gamma^\mu\delta A_\mu\frac{\delta R}{\delta\phi}\psi_L = 2ieg\overline{\psi}_L\lambda_R\delta R$$
$$+ \frac{e}{2}g\overline{\psi}_L\gamma^\mu\big(\overline{\epsilon}_L\gamma_\mu\lambda_R - \overline{\epsilon}_R\gamma_\mu\lambda_R\big)\frac{\delta R}{\delta\phi}\psi_L. \tag{E.11}$$

Adding (E.11) to its hermitian conjugate yields a total contribution of

$$2ieg\big(\delta\overline{R}\lambda_R\psi_L + \overline{\psi}_L\lambda_R\delta R\big) - \frac{e}{2}\overline{\psi}_L\left(\frac{\delta R}{\delta\phi}g + g\frac{\delta\overline{R}}{\delta\phi^\dagger}\right)\gamma^\mu\big(\overline{\epsilon}\gamma_5\gamma_\mu\lambda\big)\psi_L. \tag{E.12}$$

Note that the Killing condition reads

$$\frac{\delta R}{\delta \phi} g + g \frac{\delta \overline{R}}{\delta \phi^\dagger} = 0,$$

so the second term in (E.12) vanishes. Adding up all the variations, we see that these can be compensated by letting

$$L_{coupling} = -2ieg\left(\overline{\psi}_L \lambda_R R + \overline{R}\lambda_R \psi_L\right) - eMD. \tag{E.13}$$

This puts the requirement on the Killing potential that it transform adjointly under gauge transformations. We can then show that this is indeed the correct expression, as the remaining variations of $L_{coupling}$ cancel against the extra terms we get from the variation of L_{int}. This will be done in the next section.

Variation of L_{int}

The only thing left to check is if the variation of the metric in (E.13) cancels the extra contributions of L_{int}. Computing the variation of the metric in $-2ieg\overline{\psi}_L\lambda_R R$ yields

$$-2ie\left(\delta\phi\frac{\delta g}{\delta\phi} + \frac{\delta g}{\delta\phi^\dagger}\delta\phi^\dagger\right)\overline{\psi}_L\lambda_R R = -2e\left(\overline{\epsilon}_R\psi_L\frac{\delta g}{\delta\phi} + \frac{\delta g}{\delta\phi^\dagger}\overline{\psi}_L\epsilon_R\right)\overline{\psi}_L\lambda_R R. \tag{E.14}$$

Using the cyclic property of (E.1) and the Fierz rearrangements (2.78) and (2.80) we can write (E.14) as

$$e\left(\overline{\epsilon}_R\gamma_\mu\lambda_R\right)R\frac{\delta g}{\delta\phi}\left(\overline{\psi}_L\gamma^\mu\psi_L\right) + e\frac{\delta g}{\delta\phi^\dagger}\left(\overline{\psi}_L\psi_R\right)\left(\overline{\epsilon}_L\lambda_R\right)R.$$

Comparison with (3.76) and (E.3) then shows that the second term cancels against the correction due to H. Using (3.70) and (E.3), we can write the first term as

$$e\left(\overline{\epsilon}_R\gamma_\mu\lambda_R\right)R\frac{\delta g}{\delta\phi}\left(\overline{\psi}_L\gamma^\mu\psi_L\right) = \frac{e}{2}\left(\overline{\epsilon}_R\gamma_\mu\lambda_R - \overline{\epsilon}_L\gamma_\mu\lambda_L\right)R\frac{\delta g}{\delta\phi}\left(\overline{\psi}_L\gamma^\mu\psi_L\right)$$
$$+ ie\delta A_\mu R\frac{\delta g}{\delta\phi}\left(\overline{\psi}_L\gamma^\mu\psi_L\right). \tag{E.15}$$

Thus, the last term of (E.15) cancels against the correction of the covariant derivative in (E.3). Finally, taking the first term and adding its hermitian conjugate yields

$$-\frac{e}{2}\left(\overline{\epsilon}\gamma_5\gamma_\mu\lambda\right)\left(R\frac{\delta g}{\delta\phi} + \frac{\delta g}{\delta\phi^\dagger}\overline{R}\right)\left(\overline{\psi}_L\gamma^\mu\psi_L\right),$$

which is proportional to an infinitesimal translation of the metric along the Killing vectors. Hence, it vanishes identically. This completes the gauge invariant extension.

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