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Natural Models, Second-order Logic & Categoricity in Set Theory

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Introduction

Among philosophically relevant logical results Zermelo's semi-categoricity theorem has received little to no attention, perhaps owing to its relative inaccessibility. The purpose of the present study is to offer a reasonably self-contained presentation of Zermelo's theorem that is accessible also to a philosopher with some knowledge of elementary set theory, say from van Dalen et al. (1975), Krivine and Miller (1973), Enderton (1977) or Hrbacek and Jech (1999), and of some basic notions from model theory, for which Rautenberg (2010) or Hodges (1997) is more than enough.[1]

With developments in model theory it has become clear that the present-day canonical foundation of mathematics, that is first-order Zermelo-Fraenkel set theory, fails horribly at unambiguous denotation. It does this to such an extent that in a modern framework semi-categoricity cannot be interpreted as a result on first-order models. If we however use full second-order models we salvage external, or 'true', semi-categoricity, although we then lose a sound and complete deductive calculus. With Henkin semantics we do have completeness, but retain only internal semi-categoricity.

Our approach will be as follows. After some preliminary remarks below concerning notation, we review some aspects of first-order set theory ZFC in the first section. The main result here is the existence of natural models of ZFC, assuming the existence of large cardinals. Section two deals with second-order logic and Henkin semantics in general and with well-founded structures and Mostowski's Collapsing Theorem in particular. The latter is used in section three to prove external semi-categoricity of second-order set theory ZFC^2 with respect to full models. The section closes with a proof of internal semi-categoricity of ZFC^2 with respect to all Henkin models.

Remark. We use classical logic throughout our investigations. In most of our proofs it seems there is no easy way around this, especially with respect to the excluded middle, although proving this use is essential is another matter altogether. Furthermore, intuitionistic and constructive set theory differ in some important respects from classical set theory. Adding for example forms of foundation to constructive set theory leads to the excluded middle with respect to bounded formulae or even full classical ZF (depending on the formulation of the foundation axiom to be added), as does assuming the class of ordinals is linearly ordered (see (Aczel and Rathjen, 2001)).

There is of course a rich and interesting discussion concerning the proper foundations of mathematics, going back to the difficulties Cantor had with his contemporaries followed by the foundational crisis surrounding Frege and Russell and by Brouwer's polemics, resulting in non-classical logics and set theories.[2] Pursuing this any further would however lead us too far afield.

Before continuing I would like to acknowledge the academic staff of both the Institute of Philosophy and the Mathematical Institute of Leiden University for their inspiring expertise. Specifically, I sincerely thank my supervisor Göran Sundholm for his enduring and knowledgeable guidance during the long process that has been the writing of this thesis. I am furthermore grateful to my family and friends for their endless love, help and support.

Preliminaries. Let **Set** be the class of all sets. We assume standard set-theoretic notions in our metatheory on **Set**. Let σ be the signature consisting of the symbol $\in$, which is sufficient to write the first- as well as the second-order formalization of our set theory. Fix the standard, countable set of first- and second-order variables, and collect them in the set Var. We agree that all first-order variables are lower case of the form $a, b, \ldots$ and that all second-order relational and functional variables are upper case of the form $\mathfrak{R}, \mathfrak{S}, \ldots$ and lower case of the form $\mathfrak{f}, \mathfrak{g}, \ldots$ respectively. Together with our standard logical connectives and quantifiers this induces the formal language $\mathcal{L}$, where the difference between first- and second-order should be clear from context.

If $\mathcal{M}$ is a model in a signature τ and ξ a symbol in τ we write $\xi^{\mathcal{M}}$ for the interpretation of ξ in $\mathcal{M}$ or even just ξ if the meaning is clear. Notice there is a distinction between well-formed formulae in $\mathcal{L}$ and formulae in our metatheory. Thus, $x \in y$ can be an informal statement about given sets x, y or a well-formed formula, to be interpreted in a model in the signature σ with free variables x, y.

As is customary, we do not distinguish between models and their underlying domains. Thus we write $x \in \mathcal{M}$ to mean x is an element of the domain associated to $\mathcal{M}$. Note Rautenberg (2010) and Hodges (1997) differ on the question what exactly a model is: Rautenberg distinguishes between models and structures but Hodges does not.[3] The difference however is only important in the case of open formulae, and need not bother us for it disappears when working informally, as we do. Thus if $\mathcal{M}$ is a model and $\varphi(x)$ a formula with free variable x, we say $\mathcal{M} \models \exists x \varphi(x)$ iff there is some $x \in \mathcal{M}$ such that $\mathcal{M} \models \varphi(x)$.[4]

Finally, we denote the class of ordinals by **Ord** and write them as α, β etc..[5] The cumulative hierarchy (of **Set**) is written as $V_0, V_1, \ldots, V_\omega, \ldots$ and their union as $\mathbf{V}$. It is a known result that $\mathbf{Set} = \mathbf{V}$ holds.[6] Recall that every element of **Set** has a well-defined rank, i.e. for all $x \in \mathbf{Set}$ there is a minimal $\alpha \in \mathbf{Ord}$ such that $x \in V_{\alpha+1}$, written as $\operatorname{rank} x$, for which it holds that $x \in y$ implies $\operatorname{rank} x < \operatorname{rank} y$. For a set of ordinals A we write $\bigcup A$ as $\sup A$.

1. First-order Set Theory

Let Replacement and Separation be the axiom schemata of (first-order) replacement and separation of ZFC respectively, and write Extensionality for the axiom of extensionality, likewise for the other axioms. We us the common definitional extension of our language consisting of the symbols $\bigcup, \bigcap, \mathcal{P}, \subset$, etc.. Thus, we are justified in using these symbols when they are convenient, but can prove our metatheoretic results by reducing everything to formulae written in σ.

1.1. Inaccessible Cardinals & Natural Models.

In what follows we review natural models of ZFC, assuming the existence of sufficiently large cardinals.

Definition 1.1.1. A cardinal number π is called *inaccessible* if it is uncountable and if it is:

(1) A *regular cardinal*, i.e. for all ordinals $\alpha < \pi$ and $f : \alpha \to \pi$ arbitrary we have $\sup f[\alpha] < \pi$;
(2) A *strong limit cardinal*, i.e. for any cardinal $\kappa < \pi$ we have $|\mathcal{P}\kappa| < \pi$.

Remark. Note that an infinite cardinal π is regular iff it equals its own cofinality, i.e. the smallest possible cardinality of any set A of ordinals such that $A \subset \pi = \sup A$ is π itself (Enderton, 1977, Cor. 9R).

Definition 1.1.2. For any ordinal number α, write $\mathcal{M}_\alpha$ for the σ-structure that has V_α as domain and interprets $\in$ as $\in$ on V_α. We call $\mathcal{M}_\alpha$ a *natural model* with *characteristic* α.

The assumption that an inaccessible cardinal exists leads to a natural model of ZFC. Because this is a well-known result, we state the theorem and only sketch the proof.[7]

Theorem 1.1.3 (Natural First-Order Models). *If π is an inaccessible cardinal, then $\mathcal{M}_\pi \models \mathsf{ZFC}$.*

Proof. The most interesting argument concerns the axiom schema of Replacement, which we save for last. First, for all $\alpha > 0$ it holds that $\mathcal{M}_\alpha \models \mathsf{Extensionality} + \mathsf{Regularity} + \mathsf{Union}$, since we have Extensionality and Regularity in our metatheory and since natural models are transitive. If λ is any limit ordinal, then we also have $\mathcal{M}_\lambda \models \mathsf{Pairing} + \mathsf{Power} + \mathsf{Choice}$. One again uses that Pairing, Power Set and Choice hold in the metatheory and observes these 'operations' only finitely increase the ranks of the sets involved. If $\beta > \omega$, then also $\mathcal{M}_\beta \models \mathsf{Infinity}$, for V_β then contains ω, which indeed is infinite.

We claim that $\mathcal{M}_\lambda \models \mathsf{Separation}$ for any limit ordinal λ, meaning any instantiation of the schema Separation holds in $\mathcal{M}_\lambda$. To see this, let $\varphi(x, \vec{y}, a)$ and $a, y_1, y_2, \ldots, y_n \in V_\lambda$ be given. Define b as $\{x \in a \mid \varphi(x, \vec{y}, a)\}$, which is a set by Separation on a in our metatheory. From $b \subset a$ and the fact that λ is a limit ordinal we deduce $b \in V_\lambda$. Because φ was arbitrary, this implies $\mathcal{M}_\lambda \models \mathsf{Separation}$.

Now suppose π is inaccessible. We will need the following two facts (the first is shown after the current proof, the latter is obvious):

(1) For any $x \in V_\pi$ we have $|x| < \pi$ because π is inaccessible;
(2) For all ordinals α and sets x we have $\operatorname{rank} x < \alpha$ iff $x \in V_\alpha$.

Let $\psi(x, \vec{y}, a, z)$ be a first-order formula, and moreover suppose $y_1, y_2, \ldots, y_n, a \in V_\pi$ are such that for all $x \in a$ and all $z_1, z_2 \in V_\pi$, if $\psi(x, \vec{y}, a, z_1)$ and $\psi(x, \vec{y}, a, z_2)$ both hold in $\mathcal{M}_\pi$, then $\mathcal{M}_\pi \models z_1 = z_2$. Thus, informally, the formula ψ with parameters $y_1, \ldots y_n, a$ induces a functional relation $x \mapsto z$ in $\mathcal{M}_\pi$ of which the domain is a subset of a. We need to show that inside $\mathcal{M}_\pi$ the image of this relation forms a set. Therefore, in order to prove the instantiation of Replacement on ψ in $\mathcal{M}_\pi$, we need an element $b \in V_\pi$ such that

$$\forall z \in V_\pi (z \in b \Leftrightarrow \exists x \in a(\psi(x, \vec{y}, a, z))).$$

Intuitively, this is a strong claim, because we have no control over the image under this functional relation of the elements in its domain, and thus the rank may increase dramatically, which is why we need π to be sufficiently big.

Define f as $\{\langle x, z\rangle \in V_\pi \times V_\pi \mid x \in a \wedge \psi(x, \vec{y}, a, z)\}$, which is a set by Separation in our metatheory. From the assumption on ψ it follows that f is a surjective function $\operatorname{dom} f \to \operatorname{ran} f$. Observe it suffices to show $\operatorname{ran} f \in V_\pi$, for then we can take $b = \operatorname{ran} f$ in the above claim.

To see the former indeed holds, i.e. that $\operatorname{ran} f \in V_\pi$, let first S be the set $\{\operatorname{rank} f(x) \mid x \in \operatorname{dom} f\} \subset \pi$ by Separation and observe that $|S| \leq |\operatorname{dom} f|$ holds. Moreover, from $a \in V_\pi$ it follows that $|a| < \pi$ by the first fact above. Combining this with $\operatorname{dom} f \subset a$ yields

$$|S| \leq |\operatorname{dom} f| \leq |a| < \pi.$$

Now with $S \subset \pi$ this implies $\sup S < \pi$, because π is inaccessible, so in particular regular. From $\operatorname{rank} f(x) \leq \sup S$ it follows that $f(x) \in V_{\sup S+1}$ for all $x \in \operatorname{dom} f$, using the second fact above. So $\operatorname{ran} f \subset V_{\sup S+1}$ and, because π is a limit ordinal, $\operatorname{rank} \operatorname{ran} f \leq \sup S + 1 < \pi$ holds, showing that indeed $\operatorname{ran} f \in V_\pi$.

To conclude, if π is an inaccessible cardinal, then in particular it is an uncountable limit ordinal (for it is an uncountable cardinal number), so $\mathcal{M}_\pi \models \mathsf{ZFC}$, which was to be shown. □

Note that in the proof above the fact that an inaccessible cardinal is a strong limit ordinal has not been used explicitly. It is however used in the proof below, and hence implicitly also in the proof above.

Lemma 1.1.4. *If π is an inaccessible cardinal and $x \in V_\pi$, then $|x| < \pi$.*

Proof. Note there is a $\xi < \pi$ such that $x \in V_\xi$. We show that $|V_\xi| < \pi$ by induction on ξ. Note $\xi = 0$ is impossible and $\xi = 1$ is trivial. If $\xi = \beta + 1$ for some ordinal β, then $|V_\beta| < \pi$ holds by the induction hypothesis, and $|V_\xi| = |\mathcal{P}V_\beta| < \pi$ holds because π is a strong limit ordinal. If on the other hand ξ is a limit ordinal, then consider the function $f : \xi \to \pi$, mapping $\gamma \in \xi$ to $|V_\gamma|$ (which is well-defined by the induction hypothesis). Because π is a regular cardinal, this gives us $\sup f[\xi] < \pi$, i.e. $|V_\xi| \leq \bigcup_{\gamma<\xi} |V_\gamma| < \pi$. Thus, from $x \in V_\xi \subsetneq V_\pi$ follows $|x| \leq |V_\xi| < \pi$ as desired (using transitivity of V_ξ). □

1.2. **First-order Models of Set Theory.** Let $\mathcal{M}$ be the class of all models of ZFC. An obvious question to ask is whether every $\mathscr{M} \in \mathcal{M}$ is a natural model of characteristic κ with κ an inaccessible cardinal. More generally, one can wonder about the possibilities of $|\mathscr{M}|$. Observe that Zermelo himself addressed these problems in 1930, claiming that any model of his set theory interpreting $\in$ as $\in$ must be of the form $\mathscr{M}_\alpha$ with α an inaccessible cardinal.[8]

On the other hand, we have the well known Skolem's paradox, asserting that if ZFC is consistent, it has a countable model. This is a result of the Löwenheim-Skolem theorems, implying that a theory in $\mathcal{L}$ which has an infinite model has a model of every infinite cardinality, which we will not prove here.[9] However, it must be noted that despite its name, no contradiction follows from Skolem's paradox. Although it is true one can show the existence of uncountable sets from ZFC using natural deduction, what one in effect does by such a proof is showing that for every model $\mathscr{M}$ of ZFC one has sets $x, y \in \mathscr{M}$ such that both are infinite, but no bijection *inside* $\mathscr{M}$ exists between x and y.[10] From Skolem's paradox it follows that there can very well be a bijection between x and y *outside* $\mathscr{M}$, i.e. in our metatheory.

Perhaps the most paradoxical feat is then that Zermelo claimed every model must be of inaccessible characteristic. How could this be true in the light of Skolem's paradox? Of course, Zermelo was well aware of Skolem's remarks concerning the apparent contradiction between the Löwenheim-Skolem theorems and the deduction of the existence of uncountable sets from ZFC (Ebbinghaus, 2006). In fact, Zermelo sharply rejects any possible finitistic conclusions one could try and draw from Skolem's paradox:

> *The 'ultrafinite antinomies of set theory', to which scientific reactionaries and anti-mathematicians appeal in their fight against set theory with such eager passion, are only apparent 'contradictions', due only to a confusion between set theory itself, which is non-categorically determined by its axioms, and the individual models representing it.* [...] *Thus, instead of leading to constriction and mutilation, the set-theoretic 'antinomies' lead, when understood correctly, to an as yet unforeseeable development and enrichment of the mathematical science.* (Zermelo, 1930)

A proposed solution to the problem of indeterminacy of our set theory lies in the use of second-order logic.[11] Indeed, as Kanamori (2010) remarks, the final formulation of Zermelo's work on axiomatic set theory (Zermelo, 1930) is to be regarded in a second-order fashion and as a reaction to the apparent anomalies found by Skolem in his first-order casting of Zermelo's earlier works on set theory, such as (Zermelo, 1904). As we will see in the next section, with the aid of quantification over classes of sets we can indeed show that every full model of second-order set theory is isomorphic to $\mathcal{M}_\pi$ for some inaccessible cardinal π. Thus, although set theory may be non-categorically determined by its axioms, it is nevertheless 'semi-categorical' if we understand it in a suitable second-order fashion.

Before we continue, let us mention some results that are ordinarily proven using natural deduction from ZFC, but which, after the necessary relativizations, hold for all models of ZFC. These results will be used in the proof of semi-categoricity.[12]

Notation. For any σ-structure $\mathcal{M}$ and $x \in \mathcal{M}$, let $\mathcal{E}x$ be the set $\{y \in \mathcal{M} \mid y \in^{\mathcal{M}} x\}$ and call it the *extension* of x.

Definition 1.2.1. Let $\mathcal{M} \models \mathsf{ZFC}$. An element $\alpha \in \mathcal{M}$ is called an *ordinal in* $\mathcal{M}$ if it is transitive and totally ordered with respect to $\in^{\mathcal{M}}$. Define $\mathcal{O}^{\mathcal{M}}$ as the set of all ordinals in $\mathcal{M}$. Call $x \in \mathcal{M}$ *well-ordered in* $\mathcal{M}$ if $\mathcal{E}x$ is totally ordered by $\in^{\mathcal{M}}$ and for any $s \subset^{\mathcal{M}} x$ such that $s \neq \varnothing^{\mathcal{M}}$ there is some minimal element in $\mathcal{E}s$ with respect to $\in^{\mathcal{M}}$. For $y \in \mathcal{M}$ call $y \cup^{\mathcal{M}} \{y\}^{\mathcal{M}}$ the *successor* of y and denote it by $\mathcal{S}y$.

Let $\mathcal{M}$ be a model of ZFC, interpreting $\in$ as say ϵ. Then we have a copy of the natural numbers in $\mathcal{M}$, meaning there is an $\omega \in \mathcal{M}$ such that $z \mathrel{\epsilon} \omega$ iff $z = \mathcal{S}^n \varnothing^{\mathcal{M}}$ for some $n \in \mathbb{N}$. Now let α be in $\mathcal{O}^{\mathcal{M}}$. Then α is well-ordered in $\mathcal{M}$ while $\mathcal{E}\alpha$ and $\mathcal{O}^{\mathcal{M}}$ are both well-ordered by ϵ. Furthermore, we have an *internal cumulative hierarchy* of $\mathcal{M}$, meaning we can define $V_0^{\mathcal{M}} := \varnothing^{\mathcal{M}}$, for ordinals β in $\mathcal{M}$ the set $V_{\mathcal{S}\beta}^{\mathcal{M}} := \mathcal{P}^{\mathcal{M}} V_\beta^{\mathcal{M}}$, and for limit ordinals λ in $\mathcal{M}$ the set $V_\lambda^{\mathcal{M}} := \bigcup_{\alpha\epsilon\lambda}^{\mathcal{M}} V_\alpha^{\mathcal{M}}$. Then for each $x \in \mathcal{M}$ there is some ordinal ζ in $\mathcal{M}$ such that $x \epsilon V_\zeta^{\mathcal{M}}$. We will abusively refer to this fact by writing $\mathcal{M} = \bigcup_{\xi\in O^{\mathcal{M}}}^{\mathcal{M}} V_\xi^{\mathcal{M}}$.

2. Second-order Logic & Well-founded Structures

There are many flavors of second-order logic available, not all of which are equivalent in the theorems they prove nor the models they allow. Naturally, also the metatheoretic results that hold for these flavors differ from one another. We will be considering Henkin semantics for the following reasons. On the one hand, in this setting full second-order logic can easily be captured, which we will use for the proof of external semi-categoricity. On the other hand, Henkin semantics has some nice metatheoretic properties that also allow for an internal semi-categoricity of second-order set theory. In our second-order logic we follow Shapiro (1991); Väänänen (2001, 2012); Väänänen and Wang (2015), and indicate some specific references where necessary.

Notation. In the following, fix a signature τ consisting of a binary relation symbol $<$.

2.1. Comprehension & Henkin Semantics.

Definition 2.1.1. The second-order axiom of *Comprehension* reads, for any second-order formula φ not containing $\mathfrak{R}$ free:

Comprehension. $\exists\mathfrak{R}\forall\vec{x}(\mathfrak{R}\vec{x} \leftrightarrow \varphi(\vec{x}))$.

We also have *Second-order Choice*, reading:

Choice^2. $\forall\mathfrak{R}(\forall\vec{x}\exists y\mathfrak{R}(\vec{x}, y) \rightarrow \exists\mathfrak{f}\forall\vec{x}\mathfrak{R}(\vec{x}, \mathfrak{f}(\vec{x})))$.

A *Henkin model* is a pair $\langle\mathcal{M}, \mathcal{G}\rangle$ with $\mathcal{M}$ a first-order model and $\mathcal{G}$ a collection of relations and functions satisfying Choice^2 + Comprehension. The latter determines the range of our second-order variables and contains, by comprehension, all second-order definable functions and relations of $\mathcal{M}$. If we take all relations and functions on $\mathcal{M}$ we get a *full* second-order model, which will be denoted simply by $\mathcal{M}$.

Our second-order deductive system will consist of Choice^2 + Comprehension and some straightforward rules for manipulating quantifiers and logical connectives, as given in (Shapiro, 1991, p. 66).

Remark. Observe with Choice^2 we can also derive Comprehension for functions:

fComprehension. $\forall\mathfrak{R}(\forall\vec{x}\exists! y\mathfrak{R}(\vec{x}, y) \rightarrow \exists\mathfrak{f}\forall\vec{x}\mathfrak{R}(\vec{x}, \mathfrak{f}(\vec{x})))$,

where the converse statement, that every function has a graph, can be derived from Comprehension (Shapiro, 1991, p. 67). Thus, giving second-order functional relations or second-order functions amounts to the same thing. Also observe that if $\mathfrak{R}$ is functional on $\mathfrak{S}$ we can derive the existence of an $\mathfrak{f}$ which is $\mathfrak{R}$ on $\mathfrak{S}$. Because of this, we will be abusing some notation and consider second-order definable partial functional relations as second-order definable partial functions.

The benefit of Henkin semantics, in contrast to full second-order logic, is that it allows for a complete proof system and is even compact, as shown by Henkin (1950). This is due to the fact that Henkin semantics can be reduced to many-sorted logic, which in turn can be reduced to first-order logic (see Väänänen (2001, 2014, §4), Shapiro (1991, Thm. 4.16, Thm. 4.17) and van Dalen (1994, § 4) for a modern presentation of these facts). However, we shall see that the downside of compactness is that it prohibits external semi-categoricity with respect to all Henkin models.

2.2. **Well-founded Structures: Internal or External?** Models of set theory are intended to be well-founded. This means every nonempty subset of such a model should have an $\in$-minimal element or, what amounts to the same thing, they should not allow for infinite descending $\in$-chains. However, we need to be careful with respect to the viewpoint from which we consider well-foundedness. For we have that ZFC proves that every nonempty x contains an $\in$-minimal element and disproves the existence of infinite descending $\in$-chains that are *first-order definable*. Akin to Skolem's paradox, with a standard compactness argument we can however prove the existence of models for ZFC that do have infinite descending $\in$-chains seen from the outside.

We want our models of second-order set theory to be truly well-founded in the following sense.

Definition 2.2.1 (External Well-foundedness)**.** A Henkin model $\langle \mathcal{M}, \mathcal{G} \rangle$ in τ is said to be *well-founded* if every nonempty $X \subset \mathcal{M}$ has a $<$-minimal element.

The reason we want this to hold is that we can perform well-founded induction on such models, which we will be needing in the proof of external semi-categoricity. Or, more precisely, well-founded induction allows for a certain representation of full models of second-order set theory by means of Mostowski's result below. This will drastically ease the proof of external semi-categoricity.

We also have a notion of internal well-foundedness with respect to Henkin models. This is used in the proof of internal semi-categoricity but defined below to contrast it with the external variant.

Notation. Extend the signature σ to $\hat{\sigma}$ by adding two unitary predicates V and V', and two binary predicates ϵ and ϵ'. Furthermore, we extend our language $\mathcal{L}$ with the logical (second-order) symbols $\subset, \bigcap, \times$, etc., with obvious interpretations (where the difference with the extralogical versions should be clear from context). We agree that for a Henkin model $\langle \mathcal{M}, \mathcal{G} \rangle$ in $\hat{\sigma}$ it always holds that (the interpretations of) $V, V', \epsilon, \epsilon'$ are in $\mathcal{G}$ and that $\langle \mathcal{M}, \mathcal{G} \rangle \models \epsilon \subset V \times V \wedge \epsilon' \subset V' \times V'$. We will also be committing the sin of writing $\langle \mathcal{M}, \mathcal{G} \rangle \models \vec{x} \in R$ while meaning $\langle \mathcal{M}, \mathcal{G} \rangle \models R(\vec{x})$, where the difference with the extralogical $\in$ should again be clear from context (we only do this when considering internal models of ZFC^2, where the extralogical $\in$ does not play any role).

Definition 2.2.2 (Internal Well-foundedness)**.** Let $\langle \mathcal{M}, \mathcal{G} \rangle$ be a Henkin model in $\hat{\sigma}$. We say $\langle V, \epsilon \rangle$ is *well-founded in* $\langle \mathcal{M}, \mathcal{G} \rangle$ if

$$\langle \mathcal{M}, \mathcal{G} \rangle \models \forall \mathfrak{R}((\mathfrak{R} \subset V \wedge \exists x \mathfrak{R}x) \rightarrow \exists m(\mathfrak{R}m \wedge \neg \exists y(\mathfrak{R}y \wedge y \,\epsilon\, m))),$$

and likewise for $\langle V', \epsilon' \rangle$.

Remark. Let $\langle \mathcal{M}, \mathcal{G} \rangle$ be a Henkin model in $\hat{\sigma}$. Then $\langle V, \epsilon \rangle$ is intended to be an internal model of second-order set theory, which roughly means that inside $\langle \mathcal{M}, \mathcal{G} \rangle$ it behaves like a model of ZFC^2 with domain V and interpretation ϵ of $\in$.

Note if $\langle \mathcal{M}, \mathcal{G} \rangle$ is externally well-founded with respect to ϵ then $\langle V, \epsilon \rangle$ must obviously be well-founded in $\langle \mathcal{M}, \mathcal{G} \rangle$. The converse fails however, for Henkin models can simply interpret V vacuously, while there are such models that are not externally well-founded, as we shall soon see.

2.3. **Mostowski's Collapsing Theorem.** The following is taken from (Jech, 2003, §6), with some necessary adjustments to our goals. In this part, we fix a Henkin model $\langle \mathcal{M}, \mathcal{G} \rangle$ in τ. For $x \in \mathcal{M}$ let again $\mathcal{E}x$ be the *extension* $\{y \in \mathcal{M} \mid y < x\}$ of x.

Definition 2.3.1. Introduce the following terminology with respect to $\langle \mathcal{M}, \mathcal{G} \rangle$:

(1) Call $\langle \mathcal{M}, \mathcal{G} \rangle$ *extensional* if for all $x, y \in \mathcal{M}$ distinct $\mathcal{E}x \neq \mathcal{E}y$ holds;
(2) The model $\langle \mathcal{M}, \mathcal{G} \rangle$ is said to be *transitive* if it interprets $<$ as $\in$ and if $x \in \mathcal{M}$ implies $x \subset \mathcal{M}$.

Theorem 2.3.2 (Well-Founded Induction)**.** *Suppose $\langle \mathcal{M}, \mathcal{G} \rangle$ is transitive and $\Phi \subset \mathcal{M}$ is such that for all $x \in \mathcal{M}$ it holds that $\mathcal{E}x \subset \Phi$ implies $x \in \Phi$. Then $\Phi = \mathcal{M}$.*

Proof. First observe $\langle \mathcal{M}, \mathcal{G} \rangle$ is well-founded. For if $X \subset \mathcal{M}$ is nonempty, by regularity in our metatheory we may take $y \in X$ such that $y \cap X = \emptyset$, which is $\in$- and whence $<$-minimal in X.

Now let Y be $\mathcal{M} - \Phi$. Suppose Y is nonempty. Because $\langle \mathcal{M}, \mathcal{G} \rangle$ is well-founded, we can pick $m \in Y$ minimal with respect to $<$ in Y. By minimality we must have $\mathcal{E}m \subset \Phi$, implying $m \in \Phi$, which is impossible. Whence Y is empty and the claim follows. $\square$

Remark. If $\langle \mathcal{M}, \mathcal{G} \rangle$ is well-founded, we can define the *rank* of an element $x \in \mathcal{M}$, written as $\operatorname{rank} x$, by induction by means of the function $\rho : \mathcal{M} \rightarrow \mathbf{Ord}$, sending $x \in \mathcal{M}$ to $\sup\{\rho(y) + 1 \mid y \in \mathcal{E}x\}$ (Jech, 2003, Thm. 2.27).

Lemma 2.3.3. *If two transitive Henkin models in τ are isomorphic, they are identical. Moreover, transitive Henkin models only allow trivial automorphisms.*

Proof. This is a simple well-founded induction, see (Jech, 2003, Thm. 6.7). $\square$

Theorem 2.3.4 (Mostowski's Collapsing Theorem)**.** *Suppose $\langle \mathcal{M}, \mathcal{G} \rangle$ is well-founded and extensional. Then there is a unique transitive Henkin model $\langle \mathcal{N}, \mathcal{F} \rangle$ in τ and a unique isomorphism π between the two.*

Proof. Without loss of generality assume $\langle \mathcal{M}, \mathcal{G} \rangle$ is nonempty. Notice $\langle \mathcal{M}, \mathcal{G} \rangle$ has a unique $<$-minimal element, say $m \in \mathcal{M}$: it has at least one by well-foundedness and it is unique by extensionality. Now for $x \in \mathcal{M}$ define

$$\pi(x) \coloneqq \{\pi(z) \mid z < x\}.$$

We claim π is a well-defined function on all of $\mathcal{M}$, using well-founded induction. For this, let Φ be the set of all elements in $\mathcal{M}$ on which π is well-defined. Obviously $\pi(m) = \varnothing$ is well-defined. And if $x \in \mathcal{M}$ is such that $\pi(y)$ is well-defined for all $y \in \mathcal{E}x$, then so is $\pi(x)$ by definition of π. Whence Φ equals $\mathcal{M}$.

Now let $\mathcal{N}$ be the image of $\mathcal{M}$ under π, which is a set by Replacement in our metatheory. Likewise, let $\mathcal{F}$ be the relations and functions on $\mathcal{N}$ induced by $\mathcal{G}$ under π. We let $\langle \mathcal{N}, \mathcal{F} \rangle$ interpret $<$ as $\in$. First observe $\langle \mathcal{N}, \mathcal{F} \rangle$ is well-founded. For if $Y \subset \mathcal{N}$ is nonempty, we can take $X \subset \mathcal{M}$ nonempty as $\pi^{-1}[Y]$ and take a $<$-minimal element $q \in X$. Then $\pi(q) \in Y$ is $\in$-minimal, for if $y \in Y$ is such that $y \in \pi(q)$, then we have $y = \pi(z)$ for some $z < q$ by definition of π. Hence $z \in \pi^{-1}[Y] = X$, which contradicts the minimality of q.

We claim that the extensionality of $\langle \mathcal{M}, \mathcal{G} \rangle$ implies π is injective. To derive a contradiction, suppose this is not the case. Then because $\langle \mathcal{N}, \mathcal{F} \rangle$ is well-founded, we can take $z \in \mathcal{N}$ of minimal rank such that there are $x, y \in \mathcal{M}$ distinct in the fiber of π above z. By extensionality, it follows $\mathcal{E}x \neq \mathcal{E}y$, and we may assume without loss of generality there is a $u \in \mathcal{M}$ in $\mathcal{E}x$ but not in $\mathcal{E}y$. Observe $\pi(u) \in \pi(x) = \pi(y)$ follows from $u < x$. By definition of π we then also have a $v \in \mathcal{E}y$ such that $\pi(u) = \pi(v)$, but for which $u \neq v$ must hold because $u \in \mathcal{E}x - \mathcal{E}y$ while $v \in \mathcal{E}y$. But this contradicts the assumption that z was of minimal rank, because $\pi(u) \in z$ implies $\operatorname{rank} \pi(u) < \operatorname{rank} z$.

It is clear π is also surjective, and hence a bijection. To see it is a homomorphism, i.e. that $x < y$ iff $\pi(x) \in \pi(y)$, first observe the implication from left to right is trivial. For the converse, assume $\pi(x) \in \pi(y)$. Then by definition of π we have that $\pi(x) = \pi(z)$ for some $z < y$. Because π is bijective, $x = z < y$ holds as desired.

We conclude π is indeed an isomorphism. Note from this also follows $\langle \mathcal{N}, \mathcal{F} \rangle$ is a Henkin model, and it is clear this model is transitive by construction. Finally, uniqueness follows from Lemma 2.3.3: if $\pi' : \langle \mathcal{M}, \mathcal{G} \rangle \to \langle \mathcal{N}', \mathcal{F}' \rangle$ is an isomorphism then so is $\pi'\pi^{-1} : \langle \mathcal{N}, \mathcal{F} \rangle \to \langle \mathcal{N}', \mathcal{F}' \rangle$. Whence $\langle \mathcal{N}', \mathcal{F}' \rangle$ is equal to $\langle \mathcal{N}, \mathcal{F} \rangle$ and $\pi'\pi^{-1}$ is the identity. $\square$

3. Second-order Set Theory

In this section we show our main result: second-order set theory ZFC^2 is externally semi-categorical with respect to full models and internally semi-categorical with respect to all Henkin models. The former means that for all full models $\mathcal{M}, \mathcal{N}$ of ZFC^2, we have either an isomorphism from $\mathcal{M}$ to an initial segment of $\mathcal{N}$ or the other way around, which is called a *quasi-isomorphism*. The later means, roughly, that in any Henkin model $\langle \mathcal{M}, \mathcal{G} \rangle$ containing structures that behave like models of ZFC^2, such structures are proven to be quasi-isomorphic inside $\langle \mathcal{M}, \mathcal{G} \rangle$.

3.1. **Axioms & Natural Models.** All but the Separation and Replacement schemata remain first-order in ZFC^2. With the power of second-order logic, these schemata are replaced by only two axioms, namely the following:

$\mathsf{Separation}^2$. $\forall \mathfrak{R} \forall \vec{y} a \exists b (b = \{x \in a \mid \mathfrak{R}(x, \vec{y}, a)\})$;
$\mathsf{Replacement}^2$. $\forall \mathfrak{f} \forall a \exists b \forall z (z \in b \leftrightarrow \exists x \in a (\mathfrak{f}(x) = z))$.

> *Remark.* Observe, with $\mathsf{fComprehension}$ our second-order formalization of Replacement is equivalent to:
>
> $\forall \mathfrak{R} \forall a (\forall x z z' (x \in a \wedge \mathfrak{R}(x, z) \wedge \mathfrak{R}(x, z') \to z = z') \to \exists b \forall z (z \in b \leftrightarrow \exists x \in a \mathfrak{R}(x, z)))$.

Proposition 3.1.1. *For all Henkin models $\langle \mathcal{M}, \mathcal{G} \rangle$ in σ satisfying ZFC^2 we have $\mathcal{M} \models \mathsf{ZFC}$.*

> *Remark.* This implies the first-order results mentioned in §1.2 hold in the second-order case as well.

Proof. Using $\mathsf{Comprehension}$, this is straightforward. □

The following is taken from Kanamori (2008, Thm. 1.3), and goes back to Zermelo himself. For any ordinal α we let $\mathcal{M}_\alpha$ be the full second-order σ-structure with underlying domain V_α, interpreting $\in$ as $\in$, and again call it *natural*.

Theorem 3.1.2. *A cardinal κ is inaccessble iff $\mathcal{M}_\kappa \models \mathsf{ZFC}^2$.*

Proof. For the forward direction the proof is similar to the first-order case.

Conversely, suppose κ is such that $\mathcal{M}_\kappa \models \mathsf{ZFC}^2$. We first claim κ must be regular. Suppose it is not. Then we have an $\alpha < \kappa$ and an $f : \alpha \to \kappa$ such that $\sup f[\alpha] \geq \kappa$ and, because $f[\alpha] \subset \kappa$, in fact $\sup f[\alpha] = \kappa$. Now because $f \subset \alpha \times \kappa \subset V_\kappa^2$, with second-order Replacement we have $f[\alpha] \in V_\kappa$ and thus $\kappa = \sup f[\alpha] \in V_\kappa$, which is a contradiction.

Next we show κ must be a strong limit cardinal. Suppose this is not the case. Then take $\lambda < \kappa$ such that $\kappa \leq |\mathcal{P}\lambda|$. As will be shown below $\mathcal{P}^{\mathcal{M}_\kappa}\lambda = \mathcal{P}\lambda$ holds. But $\kappa \leq |\mathcal{P}\lambda|$ gives us a surjection $f : \mathcal{P}\lambda \to \kappa$ and thus $f[\mathcal{P}\lambda] = \kappa \in V_\kappa$ by $\mathsf{Replacement}^2$, which again gives us a contradiction. □

The next result is needed to complete the prove above, but is also convenient for the proof of external semi-categoricity and nicely shows exactly where second-order Separation is used.

Lemma 3.1.3. *Let $\mathcal{M}$ be a transitive, full model in σ satisfying $\mathsf{Separation}^2 + \mathsf{Power}$. Then for all $x \in \mathcal{M}$ we have $\mathcal{P}x = \mathcal{P}^{\mathcal{M}}x$.*

Proof. Let $x \in \mathcal{M}$. First note that for all $y \in \mathcal{M}$ it holds that

$$y \in \mathcal{P}^{\mathcal{M}}x \Leftrightarrow y \subset^{\mathcal{M}} x \Leftrightarrow \forall z \in \mathcal{M}(z \in y \Rightarrow z \in x).$$

Now let y be in $\mathcal{P}^{\mathcal{M}}x$ and $z \in y$. Then because $y \in \mathcal{M}$ and $\mathcal{M}$ is transitive, we also get that $z \in \mathcal{M}$, and hence $z \in x$ by definition of $\mathcal{P}^{\mathcal{M}}x$, showing $\mathcal{P}x \supset \mathcal{P}^{\mathcal{M}}x$.

Conversely, let $y \subset x$ be arbitrary. Then again by transitivity $y \subset \mathcal{M}$ holds. Thus $x \cap y$ equals y, and it is an element of $\mathcal{M}$ by $\mathsf{Separation}^2$ from x. Because $\mathcal{M}$ interprets $\in$ as $\in$, it is now clear that $y \in \mathcal{P}^{\mathcal{M}}x$, showing $\mathcal{P}x \subset \mathcal{P}^{\mathcal{M}}x$. □

3.2. **External Semi-categoricity.** In contrast to the first-order case, full models of ZFC^2 are always well-founded, as the following result shows.

Proposition 3.2.1. *Let $\mathcal{M}$ be a full model in σ of ZFC^2. Then $\mathcal{M}$ is well-founded.*

Remark. Observe $\mathcal{M} \models \mathsf{Replacement}^2$ is necessary for this result to hold, as shown in (Uzquiano, 1999, Thm. 4), i.e. there are full models $\mathcal{N} \models \mathsf{ZFC}^2 - \mathsf{Replacement}^2$ that are not well-founded.

Proof. Let $X \subset \mathcal{M}$ be nonempty and $\omega \in \mathcal{M}$ the copy of $\mathbb{N}$ in $\mathcal{M}$. Write $<$ for $\in^{\mathcal{M}}$. Suppose X has no $<$-minimal element. Then we have a set $y := \{x_i \mid x_i \in \mathcal{M}, i < \omega\}$ such that $x_0 > x_1 > x_2 \ldots$. Now with second-order Replacement from ω it follows there is an $\tilde{y} \in \mathcal{M}$ with $p < \tilde{y}$ iff $p \in y$ for all $p \in \mathcal{M}$, and thus there must be a $z \in \mathcal{M}$ such that $z < \tilde{y}$ and $\tilde{y} \cap^{\mathcal{M}} z = \varnothing^{\mathcal{M}}$ by $\mathsf{Regularity}$, which is impossible. □

All is in place for the proof of external semi-categoricity.

Theorem 3.2.2. *Let $\mathcal{M}$ be a full model of ZFC^2 in σ. Then $\mathcal{M}$ is uniquely isomorphic to the natural model $\mathcal{M}_\kappa$ with $\kappa \cong \mathcal{O}^{\mathcal{M}}$.*

Proof. Observe by Proposition 3.2.1 $\mathcal{M}$ is well-founded, and it is obviously extensional. Thus, by Mostowski's Collapsing Theorem, it is uniquely isomorphic to a unique transitive Henkin model: itself a full model of ZFC^2. So we may assume without loss of generality that $\mathcal{M}$ is transitive. Therefore, by Lemma 3.1.3, for any $x \in \mathcal{M}$ it holds that $\mathcal{P}^{\mathcal{M}}x = \mathcal{P}x$.

Now let κ be the unique ordinal such that $\mathcal{O}^{\mathcal{M}} \cong \kappa$. Because $\mathcal{M}$ is transitive it interprets $\in$ as $\in$ and thus $\mathcal{O}^{\mathcal{M}}$ is in fact equal to κ.

We show by induction that $V_\xi^{\mathcal{M}} = V_\xi$ for any $\xi < \kappa$. Obviously, because $\mathcal{M}$ is transitive $V_0^{\mathcal{M}} = \varnothing^{\mathcal{M}} = \varnothing = V_0$ holds. Now suppose $\lambda < \kappa$ is such that $V_\alpha^{\mathcal{M}} = V_\alpha$ for any $\alpha < \lambda$. If $\lambda = \beta + 1$ for some ordinal β, then also

$$V_\lambda^{\mathcal{M}} = \mathcal{P}^{\mathcal{M}} V_\beta^{\mathcal{M}} = \mathcal{P} V_\beta^{\mathcal{M}} = \mathcal{P} V_\beta = V_\lambda.$$

Next suppose λ is a limit ordinal. Then we have

$$V_\lambda^{\mathcal{M}} = \{x \in \mathcal{M} \mid \exists \alpha \in \mathcal{O}_{<\lambda}^{\mathcal{M}}(x \in V_\alpha^{\mathcal{M}})\} = \{x \in \mathcal{M} \mid \exists \alpha < \lambda(x \in V_\alpha)\} = \mathcal{M} \cap V_\lambda.$$

Therefore, if $V_\lambda \subset \mathcal{M}$, then we indeed have $V_\lambda^{\mathcal{M}} = V_\lambda$. To see the former holds, let $x \in V_\lambda$ be given. Then there is some $\alpha < \lambda$ such that $x \in V_\alpha$, which is equal to $V_\alpha^{\mathcal{M}}$ by the induction hypothesis, and thus $x \in V_\alpha^{\mathcal{M}} \subset \mathcal{M}$ as desired.

Note $\mathcal{M}$ is identical to $\bigcup_{\alpha<\kappa}^{\mathcal{M}} V_\alpha^{\mathcal{M}}$, which we have shown to be identical to $\bigcup_{\alpha<\kappa}^{\mathcal{M}} V_\alpha$. Now by the same token as above the latter equals $\mathcal{M} \cap V_\kappa$, and again we have $V_\kappa \subset \mathcal{M}$, indeed showing $\mathcal{M} = V_\kappa$ holds. □

As an immediate consequence of the above we acquire the following.

Corollary 3.2.3 (External Semi-categoricity). *The theory* ZFC^2 *is semi-categorical with respect to full models. That is, for any two such models* $\mathcal{M}, \mathcal{N} \models \mathsf{ZFC}^2$ *in* σ *we can uniquely embed* $\mathcal{M}$ *as an initial segment into* $\mathcal{N}$*, or the other way around.*

Furthermore, we see that the existence of full models for ZFC^2 is independent of ZFC. For by Theorems 3.2.2 and 3.1.2 ZFC^2 has full models iff there are inaccessible cardinals. Because the latter leads to a natural model for ZFC, it is indeed consistent with ZFC to either assume or negate the existence of such cardinals and hence the consistency of full second-order set theory.

3.3. **Internal Well-foundedness.** The above strategy doesn't work for *all* Henkin models, for we can show there are models $\langle \mathcal{M}, \mathcal{G} \rangle$ of ZFC^2 that are not externally well-founded. Simply extend $\mathcal{L}$ by adding fresh constants $c_0, c_1, c_2, \ldots$, consider the sentences $c_0 \ni c_1 \ni c_2 \ni \ldots \ni c_k$ for $k \in \mathbb{N}_{>0}$ and let Σ be the collection of all of these sentences. Then $\mathsf{ZFC}^2 + \Sigma$ is finitely satisfiable by any second-order model of ZFC^2, and thus has a model by compactness. It is readily seen such a model cannot be externally well-founded.

Notwithstanding the above, we can use the notion of internal well-foundedness and the fact that second-order set theory has a finite axiomatization to prove internal semi-categoricity. To see what this means, define in the extended signature $\hat{\sigma}$ the second-order sentence $\mathsf{ZFC}^2(V, \epsilon)$, expressing that $\langle V, \epsilon \rangle$ satisfies all axioms of ZFC^2 relative to $\langle V, \epsilon \rangle$. Thus, for example, the relativized version of second-order Seperation becomes:

$\mathsf{Separation}(V, \epsilon)$. $\forall \mathfrak{R} \subset V \forall \vec{y} a \in V \exists b \in V (b = \{x \,\epsilon\, a \mid \mathfrak{R}(x, \vec{y}, a)\})$.

Now $\mathsf{ZFC}^2(V, \epsilon)$ is the conjunction of all relativized versions of the sentences in ZFC^2 (which are indeed finite in number, written as $\varphi(V, \epsilon)$ for $\varphi \in \mathsf{ZFC}^2$). Ditto for $\langle V', \epsilon' \rangle$.

Before continuing we agree on some more terminology. Let $\langle \mathcal{M}, \mathcal{G} \rangle$ be a Henkin model in $\hat{\sigma}$ satisfying the sentence $\mathsf{ZFC}^2(V, \epsilon)$. Thus $\langle \mathcal{M}, \mathcal{G} \rangle$ contains an internal model of second-order set theory, namely the interpretations of $\langle V, \epsilon \rangle$, from hereon simply written as V. Because the set of ordinals of any model of set-theory is second-order definable, we have a subset $\mathrm{Ord} \in \mathcal{G}$ of $\mathcal{M}$ consisting of the ordinals in V. Likewise, we have an internal copy $\omega \in V$ of $\mathbb{N}$ and an internal empty set (i.e. an ϵ-minimal element in all of V), written as $\emptyset$. Furthermore, for x in V we define the *extension* $\mathcal{E}x \in \mathcal{G}$ of x as $\{y \in V \mid \langle \mathcal{M}, \mathcal{G} \rangle \models y \,\epsilon\, x\}$, and write the internal power-set simply as $\mathcal{P}x$. If $\langle \mathcal{M}, \mathcal{G} \rangle \models \mathsf{ZFC}^2(V', \epsilon')$, we do the same as above with respect to $\langle V', \epsilon' \rangle$, but add a prime as in $\mathrm{Ord}' \subset V'$ etc.. In all these notations the difference with the 'real world' versions should be clear from the context.

Now for any subsets $X, Y \in \mathcal{G}$ of $\mathcal{M}$ such that $\langle \mathcal{M}, \mathcal{G} \rangle$ satisfies $\epsilon \subset X \times X \wedge \epsilon' \subset Y \times Y$, being an isomorphism from $\langle X, \epsilon \rangle$ to $\langle Y, \epsilon' \rangle$ inside $\mathcal{M}$ is second-order definable, and thus we have a second-order sentence $\mathsf{Iso}(\pi, \langle X, \epsilon \rangle, \langle Y, \epsilon' \rangle)$ expressing a given $\pi \in \mathcal{G}$ is an isomorphism from $\langle X, \epsilon \rangle$ to $\langle Y, \epsilon' \rangle$. Likewise, we can express an embedding $\mathsf{Embed}(\iota, \langle X, \epsilon \rangle, \langle Y, \epsilon' \rangle)$ as an isomorphism $\iota : X \to \iota[X]$ with $\iota[X] \subset Y$ *downwards-closed*, i.e. such that $x \in \iota[X] \wedge y \,\epsilon'\, x$ implies $y \in \iota[X]$. If $\tau \subset X \times Y$ is a relation such that $\mathsf{Embed}(\tau, \langle X, \epsilon \rangle, \langle Y, \epsilon' \rangle) \vee \mathsf{Embed}(\tau^{-1}, \langle Y, \epsilon' \rangle, \langle X, \epsilon \rangle)$ holds, we call it a *quasi-isomorphism*, which we write as $\mathsf{Qiso}(\tau, \langle X, \epsilon \rangle, \langle Y, \epsilon' \rangle)$.

To prove internal semi-categoricity, we need the following lemma.

Lemma 3.3.1. *Let $\langle \mathcal{M}, \mathcal{G} \rangle$ be a Henkin model in $\hat{\sigma}$ satisfying $\mathsf{ZFC}^2(V, \epsilon)$. Then $\langle V, \epsilon \rangle$ is well-founded in $\langle \mathcal{M}, \mathcal{G} \rangle$.*

Proof. Let $U \in \mathcal{G}$ be a nonempty subset of V. Take x in U and define $\bar{x}$ as the transitive closure $\bigcup_{n \in \mathbb{N}} \cup^n x$ of x, which is an element of V by $\mathsf{Replacement}^2(V, \epsilon)$. With $\mathsf{Separation}^2(V, \epsilon)$, take $w \in V$ such that $\mathcal{E}w$ is $\{y \in V \mid y \in U \cap \mathcal{E}\bar{x}\}$. Finally, by $\mathsf{Regularity}(V, \epsilon)$ let $z \,\epsilon\, w$ be such that $z \cap w = \emptyset$.

Observe $z \epsilon w$ implies $z \in U$ and $z \epsilon \bar{x}$, where the latter implies $z \subset \bar{x}$ by transitivity of $\bar{x}$. We claim z is ϵ-minimal in U. To the contrary, suppose $t \in U$ is such that $t \,\epsilon\, z$. Then $t \in \mathcal{E}\bar{x}$ and thus $t \,\epsilon\, w$ by construction of w, contradicting $w \cap z = \emptyset$. So no such $t \in U$ exists and z is as desired. ☐

3.4. **Internal Semi-categoricity.** Recall for $\langle \mathcal{M}, \mathcal{G} \rangle$ in $\hat{\sigma}$ satisfying $\mathsf{ZFC}^2(V, \epsilon)$ we have an internal cumulative hierarchy, with stages written as $V_\alpha \in V$ for $\alpha \in \mathrm{Ord}$. As a corollary of the above, we note it is provable in $\langle \mathcal{M}, \mathcal{G} \rangle$ that this is indeed the cumulative hierarchy of V in that $\forall x \in V \exists \alpha \in \mathrm{Ord}(x \,\epsilon\, V_\alpha)$ holds in $\langle \mathcal{M}, \mathcal{G} \rangle$. Likewise with respect to $\langle V', \epsilon' \rangle$.

Using internal well-foundedness we first acquire internal semi-categoricity with respect to the ordinals. The proof is taken from (Shapiro, n.d.).

Proposition 3.4.1. *Let $\langle \mathcal{M}, \mathcal{G} \rangle$ be a Henkin model in $\hat{\sigma}$ satisfying the sentences $\mathsf{ZFC}^2(V, \epsilon)$ and $\mathsf{ZFC}^2(V', \epsilon')$. Then $\langle \mathcal{M}, \mathcal{G} \rangle \models \exists \pi \mathsf{Qiso}(\pi, \langle \mathrm{Ord}, \epsilon \rangle, \langle \mathrm{Ord}', \epsilon' \rangle)$.*

Proof. For first-order variables x, y and a binary relation $\mathfrak{R}$ define the formula

$$\psi(x, y, \mathfrak{R}) \coloneqq \forall z \,\epsilon\, x \exists w \,\epsilon'\, y(\mathfrak{R}(z, w)) \wedge \forall w \,\epsilon'\, y \exists z \,\epsilon\, x(\mathfrak{R}(z, w)).$$

Call a set $R \subset \mathcal{M} \times \mathcal{M}$ *ordinal-closed* if $\langle \emptyset, \emptyset' \rangle \in R$, if $R \subset \mathrm{Ord} \times \mathrm{Ord}'$ and if furthermore for all $\langle x, y \rangle \in \mathrm{Ord} \times \mathrm{Ord}'$ it holds that $\psi(x, y, R)$ implies $R(x, y)$. Observe $\mathrm{Ord} \times \mathrm{Ord}' \in \mathcal{G}$ is ordinal-closed. Now define

$$T \coloneqq \bigcap \{R \in \mathcal{G} \mid R \text{ is ordinal-closed}\}.$$

Note we have $T \in \mathcal{G}$ because being ordinal-closed is second-order definable. It is clear that $\langle \emptyset, \emptyset' \rangle \in T$ and $T \subset \mathrm{Ord} \times \mathrm{Ord}'$ hold. Now let $\langle x, y \rangle$ be in $\mathrm{Ord} \times \mathrm{Ord}'$ such that $\psi(x, y, T)$ holds. Then $\langle x, y \rangle$ is in each ordinal-closed R, whence $T(x, y)$ holds, showing T is itself ordinal-closed.

Let $\langle x, y \rangle$ be in T. We claim that $\psi(x, y, T)$, i.e.

($\star$) For all $z \,\epsilon\, x$ there is $w \,\epsilon'\, y$ such that $T(z, w)$ and, conversely, for all $w \,\epsilon'\, y$ there is $z \,\epsilon\, x$ such that $T(z, w)$.

To the contrary, suppose $\neg\psi(x, y, T)$. Then trivially $\langle x, y\rangle \neq \langle\emptyset, \emptyset'\rangle$, so if we define T' as $T - \{\langle x, y\rangle\}$ then we have $\langle\emptyset, \emptyset'\rangle \in T'$ and $T' \subset \mathrm{Ord} \times \mathrm{Ord}'$. Now suppose $\langle p, q\rangle$ in $\mathrm{Ord} \times \mathrm{Ord}'$ is such that $\psi(p, q, T')$ holds. Then this implies $T(p, q)$ and we clearly have $\langle p, q\rangle \neq \langle x, y\rangle$, so in fact $T'(p, q)$ holds as well. Thus T' is ordinal-closed, giving us $T \subset T'$, which is impossible.

To see T is a bijective relation, suppose this is not the case. Let then Φ be $\{x \in \mathrm{Ord} \mid \exists yy'(T(x, y) \wedge T(x, y') \wedge y \neq y')\}$, which is second-order definable. Because $\langle V, \epsilon\rangle$ is well-founded in $\langle \mathcal{M}, \mathcal{G}\rangle$, we have an ϵ-minimal $a \in \Phi$, say with distinct $b_1, b_2 \in \mathrm{Ord}'$ such that $\langle a, b_1\rangle, \langle a, b_2\rangle \in T$ and $b_1 \epsilon' b_2$. Then define

$$S \coloneqq T - \{\langle a, b_2\rangle\}.$$

We show S is ordinal-closed. Obviously $S \subset \mathrm{Ord} \times \mathrm{Ord}'$. Furthermore, $b_1 \epsilon' b_2$ so $b_2 \neq \emptyset'$, implying $\langle\emptyset, \emptyset'\rangle \in S$. Now suppose $\langle p, q\rangle \in \mathrm{Ord} \times \mathrm{Ord}'$ is such that $\psi(p, q, S)$ holds. Observe we have $T(p, q)$, and assume $p = a \wedge q = b_2$. Then, by claim $(\star)$, the minimality of a and the fact that $T(a, b_1)$, for all $x \epsilon a$ there is a unique $y_x \epsilon' b_1$ with $T(x, y_x)$. Suppose $x \epsilon a$ is such that $T(x, b_1)$. Then by uniqueness of y_x we must have $b_1 = y_x \epsilon' b_1$, which is impossible. Therefore, because $b_1 \epsilon' b_2$, not for all $y \epsilon' b_2$ there is an $x \epsilon a$ such that $T(x, y)$, contradicting the claim $(\star)$ and $T(a, b_2)$. This implies $p \neq a \vee q \neq b_2$ and thus $S(p, q)$. As desired, this means S is ordinal closed, implying $T \subset S$, which is impossible.

We have seen Φ is empty. We can do the same with respect to Φ' defined as $\{y \in \mathrm{Ord}' \mid \exists xx'(T(x, y) \wedge T(x', y) \wedge x \neq x'\}$, this time using that $\langle V', \epsilon'\rangle$ is well-founded in $\langle \mathcal{M}, \mathcal{G}\rangle$, to see Φ' is empty. Therefore T was bijective after all.

Let Ψ resp. Ψ' be the sets $\{x \in \mathrm{Ord} \mid \exists y \in \mathrm{Ord}'\, T(x, y)\}$ resp. $\{y \in \mathrm{Ord}' \mid \exists x \in \mathrm{Ord}\, T(x, y)\}$, which are second-order definable. To see the domain of T is Ord or the range of T is Ord′, suppose $\Psi \neq \mathrm{Ord}$. Then take m minimal in $\mathrm{Ord} - \Psi$ and observe for all $x \in \mathrm{Ord}$ it holds $m \epsilon x \wedge x \in \Psi$ implies $m \in \Psi$ by claim $(\star)$, which is impossible by the assumption on m. Now suppose $\Psi' \neq \mathrm{Ord}'$. Again let n be minimal in $\mathrm{Ord}' - \Psi'$ and observe for all $y \in \mathrm{Ord}'$ that $n \epsilon' y \wedge y \in \Psi'$ is impossible. Now for each $x \epsilon m$ there must be some $y \in \mathrm{Ord}'$ such that $T(x, y)$ by minimality of m, and we have just seen for such y it must hold $y \epsilon' n$. Likewise, for all $y \epsilon' n$ there is $x \epsilon m$ with $T(x, y)$. Or, in other words, $\psi(m, n, T)$ and therefore $T(m, n)$ holds, using T is ordinal-closed. This, however, contradicts the assumption on m.

The above implies $\Psi = \mathrm{Ord}$ or $\Psi' = \mathrm{Ord}'$. Without loss of generality assume $\Psi = \mathrm{Ord}$, so that T induces a function $\pi : \mathrm{Ord} \to \mathrm{Ord}'$ by $\mathsf{fComprehension}$. What remains to show for the claim $\mathsf{Qiso}(\pi, \langle \mathrm{Ord}, \epsilon\rangle, \langle \mathrm{Ord}', \epsilon'\rangle)$ is that π is structure-preserving and that $\pi[\mathrm{Ord}]$ is downwards-closed. Both however are immediate consequences of $(\star)$, which finishes our proof. □

All is in place for our final result. The strategy will be similar as the previous proof and is inspired by (Väänänen and Wang, 2015).

Theorem 3.4.2 (Internal Semi-categoricity). *If $\langle \mathcal{M}, \mathcal{G}\rangle$ is a Henkin model in $\hat{\sigma}$ such that $\langle \mathcal{M}, \mathcal{G}\rangle \models \mathsf{ZFC}^2(V, \epsilon) \wedge \mathsf{ZFC}^2(V', \epsilon')$, then $\langle \mathcal{M}, \mathcal{G}\rangle \models \exists\tau \mathsf{Qiso}(\tau, \langle V, \epsilon\rangle, \langle V', \epsilon'\rangle)$.*

Proof. By Proposition 3.4.1, take $\pi \in \mathcal{G}$ such that $\langle \mathcal{M}, \mathcal{G}\rangle$ satisfies the sentence $\mathsf{Qiso}(\pi, \langle \mathrm{Ord}, \epsilon\rangle, \langle \mathrm{Ord}', \epsilon'\rangle)$ and let $\Pi \subset \mathrm{Ord} \times \mathrm{Ord}'$ be π considered as binary relation. Without loss of generality assume the domain of π is all of Ord. For a binary relation $\mathfrak{R}$ define the formula

$$\psi(x, y, \mathfrak{R}) := \forall z \,\epsilon\, x \exists w \,\epsilon'\, y(\mathfrak{R}(z, w)) \wedge \forall w \,\epsilon'\, y \exists z \,\epsilon\, x(\mathfrak{R}(z, w)).$$

Now call a binary $R \in \mathcal{G}$ *set-closed* if $\Pi \subset R$ and if for all $\langle x, y\rangle \in V \times V'$ it holds $\psi(x, y, R)$ implies $R(x, y)$. Define

$$T := \bigcap\{R \in \mathcal{G} \mid R \text{ is set-closed}\}.$$

Observe $T \in \mathcal{G}$. Because Π is contained in every set-closed set and $V \times V'$ itself is set-closed we have $\Pi \subset T \subset V \times V'$. We claim

($\star$) $\forall \langle x, y\rangle \in V \times V'(\psi(x, y, T) \Leftrightarrow T(x, y))$.

The forward direction is trivial. For the converse, suppose $T(x, y)$ and $\neg\psi(x, y, T)$ hold and let T' be $T - \{\langle x, y\rangle\}$. Observe, if $\Pi(x, y)$ holds then $\psi(x, y, \Pi)$ by the proof of Proposition 3.4.1 and hence $\psi(x, y, T)$ because $\Pi \subset T$, which is impossible. Thus we see that $\neg\Pi(x, y)$, i.e. that $\Pi \subset T'$. By the same argument as given in the proof of Proposition 3.4.1 we derive $T' \subset T$, which is impossible. Whence $T(x, y)$ implies $\psi(x, y, T)$ for all $\langle x, y\rangle \in V \times V'$.

In the following, for $\alpha \in \mathrm{Ord}$ write the restriction $T|_{\mathcal{E}V_\alpha} \subset \mathcal{E}V_\alpha \times \mathcal{E}'V'_{\pi(\alpha)}$ of T simply as $T|_\alpha$ (which is well-defined, as shown below). For the remainder of the proof, we employ the following strategy. We first show that T is:

i. Functional;
ii. An injective relation;
iii. Structure-preserving.

Then we show **iv.** that for all $\alpha \in \mathrm{Ord}$ it holds $T|_\alpha$ is an isomorphism to conclude that T is a quasi-isomorphism.

i. Let Φ be $\{x \in V \mid \exists y y' \in V'(y \neq y' \wedge T(x, y) \wedge T(x, y')\}$. Then Φ is second-order definable. Thus, by Lemma 3.3.1, if Φ is nonempty it has a minimal element, say $m \in V$. Take $y, y' \in V'$ distinct such that $T(m, y)$ and $T(m, y')$. Because $\mathcal{E}'y \neq \mathcal{E}'y'$, we may assume without loss of generality there is a $u \in V'$ such that $u \,\epsilon'\, y$ holds but not $u \,\epsilon'\, y'$.

Let $z \,\epsilon\, m$ be such that $T(z, u)$ (using $T(m, y)$, claim ($\star$) and $u \,\epsilon'\, y$). Likewise, using $z \,\epsilon\, m$ and $T(m, y')$, let $w \,\epsilon'\, y'$ be such that $T(z, w)$. Observe $\neg(u \,\epsilon'\, y') \wedge w \,\epsilon'\, y'$ implies $u \neq w$, and thus we have found a $z \in \Phi$ but with $z \,\epsilon\, m$, which is impossible by minimality of m. So Φ is empty and T is functional.

Because T is functional we may consider it as a function $\tau : \mathrm{dom}\, T \to \mathrm{ran}\, T$, which is just notational convenience in the remainder of the proof.

ii. To show T is injective, do the same as above but this time with respect to $\{y \in V' \mid \exists x x' \in V(x \neq x' \wedge T(x, y) \wedge T(x', y))\}$.

iii. Suppose $\langle x, y\rangle, \langle p, q\rangle \in V \times V'$ are such that $T(x, y)$ and $T(p, q)$ hold. Suppose $x \epsilon p$. Then with ($\star$) we have a $w \epsilon' q$ such that $T(x, w)$, and $y = w \epsilon' q$ follows from functionality of T. The converse if similar, using injectivity of T. Therefore, T is structure-preserving.

iv. We proceed by transfinite induction, which is justified by Lemma 3.3.1. For the base case observe $\mathcal{E}V_0 = \varnothing = \mathcal{E}'V'_{\pi(0)}$, so we have nothing to prove. Now suppose $T|_\alpha$ is an isomorphism. To show $T|_{\alpha+1}$ is one as well, by (i) - (iii) it suffices to show the restriction $T|_{\alpha+1}$ is well-defined and surjective.

For the first claim let $x \in V_{\alpha+1}$ be given. Observe $\langle \mathcal{M}, \mathcal{G}\rangle \models \forall z \epsilon x(z \epsilon V_\alpha)$ so $\mathcal{E}x \subset \mathcal{E}V_\alpha$. By induction hypothesis we have $\tau[\mathcal{E}x] \subset \mathcal{E}'V'_{\pi(\alpha)}$. Moreover, $\tau[\mathcal{E}x]$ is second-order definable. Thus, using $\langle \mathcal{M}, \mathcal{G}\rangle \models \mathsf{Separation}(V', \epsilon')$, we have an $y \in V'$ such that

$$\langle \mathcal{M}, \mathcal{G}\rangle \models \forall w \in V'(w \epsilon' y \leftrightarrow (w \in \mathcal{E}'V'_{\pi(\alpha)} \cap \tau[\mathcal{E}x])).$$

We show that $T(x, y)$ holds. By ($\star$), it suffices to prove $\psi(x, y, T)$. So first suppose we are given a $z \epsilon x$. Then by induction hypothesis there is a unique $w \in \mathcal{E}'V'_{\pi(\alpha)}$ such that $T(z, w)$. Now because $w \in \tau[\mathcal{E}x]$, by construction of y indeed $w \epsilon' y$. Conversely, suppose $q \epsilon' y$ is given. Then $q \in \tau[\mathcal{E}x]$ by construction of y and hence there is a $p \epsilon x$ such that $T(p, q)$. Therefore $T(x, y)$ indeed holds. Furthermore $y \subset' V'_{\pi(\alpha)}$ and thus $y \epsilon' V'_{\pi(\alpha+1)}$, showing $T|_{\alpha+1}$ is well-defined.

For the second claim, by symmetry, for given $y \in V'_{\pi(\alpha+1)}$ we derive the existence of an $x \in V_{\alpha+1}$ such that $T(x, y)$, so $T|_{\alpha+1}$ is surjective as well.

The case where $\lambda \in \mathrm{Ord}$ is a limit ordinal such that $T|_\alpha$ is an isomorphism for all $\alpha \epsilon \lambda$ is straightforward. As noted, with (i) - (iii) this implies (iv).

With $V = \bigcup_{\alpha \in \mathrm{Ord}} \mathcal{E}V_\alpha$ and (i) - (iv) we see $\tau : V \to V'$ is an isomorphism onto its image. What remains to show for the claim τ is a quasi-isomorphism is that $\tau[V]$ is downwards-closed. But this is easy by ($\star$), which concludes our proof. ☐

3.5. **Discussion.** We have seen full second-order logic allows for a semi-categorical set theory. However, as mentioned, we lack a sound and complete decision calculus for second-order logic. One can show this via categoricity of full second-order arithmetic in combination with Gödel's first incompleteness theorem. On the other hand, completeness with respect to Henkin semantics is due to the fact that it can be reduced to first-order logic. Let us first pause to reflect on these matters.

There are two questions I want to touch on shortly. The first is why any one should care about categoricity at all. The second is what semi-categoricity in the context of set theory shows. For the first, note that in many theories (such as the theory of groups) one of course is not interested in categoricity. But with foundational theories like arithmetic or set theory, the fear is that if our models are allowed to behave wildly, we cannot rule out pathological behaviors trickling down to the rest of our mathematics. Thus, if we are, say, doing arithmetic on $\mathbb{N}$ and someone asks us the ludicrous question what $\mathbb{N}$ precisely is, we of course say it is the set $\{0, 1, 2, \ldots\}$, taken with ordinary addition and multiplication as $+$ resp. $\cdot$.

But then our hypothetical friend demands us to be formal. We are doing mathematics after all. So we give him a set of axioms, such as first-order Peano arithmetic, and say:

> *These are the rules of arithmetic. When we say we are doing arithmetic, what we actually mean is we are following these rules using only strict logical principles.*

After some thought, our surprisingly witty partner remarks that he has found a set M with operations $+', \cdot'$ and elements $0', 1'$ that exactly satisfies our axioms, but which is itself uncountable. Now, if we really want to do arithmetic on the natural numbers, we are left with no choice but to cast our theory in second-order logic. We show him a proof of categoricity and contend we have reached the utmost rigor.

The next time we see our friend he tells us he has given it some thought. For we have shown all models of second-order arithmetic to be isomorphic, but what exactly is a model? More precisely, what is a set? We play the same game again and end up with full second-order set theory, which is sufficiently categorical for $\mathbb{N}$ to be uniquely determined in all of its models. He replies:

> *You showed me semi-categoricity of the second-order set theory you use. But while doing so, you quantified over sets, taking unions and intersections of sets and even power sets. How are you justified in these operations if set theory is the theory you want to build up?*

We reply with Henkin semantics, and tell him in this approach his concerns have been met. For all we need to be comfortable about to use this logic is $\mathsf{Comprehension} + \mathsf{Choice}^2$. Of course, he replies we then only have internal semi-categoricity, so it seems we are back to the start. For cannot the Henkin models (and hence their internal models of ZFC^2) themselves still behave in wild and unexpected ways?

This also touches on the second question, namely what semi-categoricity in the context of set theory shows. The discussion above has been obscured in our proofs by the fact that we took **Set** in our metatheory as if this were unambiguous. As our friend showed, it is not.[13] But such is life: in building our theories we have to start from somewhere. Moreover, we do not need logic (nor our friend) to tell us ambiguity is impossible to root out. When we are doing mathematics, or any other meaningful endeavor, we are always operating under a set of 'sanity conditions'. These need not be explicitly or even implicitly present to our minds. In fact, most of these 'conditions' only come to mind when someone or something is breaking them, which is hence a misfortune we cannot avoid in advance.

If we do not aim for eternal truth and unambiguous semantics, and accept there will always be hidden rules of the games we play that can be interpreted in unforeseeable ways, then the problem we had with our friend can be resolved. For we can simply say that our theories work, and that their meaning is derived from our use. They work to unify various parts of mathematics, to ground our logical practices and to aid understanding of our theories.[14]

Thus one should care about semi-categoricity in the context of foundational theories if one wants to know what is needed to be agreed upon in order to do unambiguous mathematics, although not all future pathological interpretations can be ruled out. Hence mathematical foundations point to our current knowledge of the 'sanity conditions' under which we do mathematics. And this, to my mind, is also what semi-categoricity is about: not eternally existing mathematical entities 'out there', but the dynamic structures of our mathematical understanding here and now.[15]

Notes

1. With respect to the exact formalism of our set theory, note in all of the mentioned references the notion of urelements is dropped (as we do). Some authors include the axiom of existence. We leave this axiom out (as Zermelo did) and take it for granted in our background logic, because it is irrelevant to our discussion.

2. See for example (Brouwer, 1913, 1949; Frege, 1884; Russell, 1919) for some of these discussions.

3. The question comes down to whether one demands a model to come with an interpretation of all variables (as Rautenberg does), or whether a model only interprets closed terms (as Hodges does). The difference is resolved when one realizes Hodges adds 'fresh constants' when he needs to interpret an open formula.

4. Thus, the latter may here be either understood as adding a fresh new constant x to our language, to be interpreted as $x \in \mathcal{M}$, or as picking out an image $x \in \mathcal{M}$ of $x \in \mathrm{Var}$ under the valuation $\mathrm{Var} \to \mathcal{M}$ associated to $\mathcal{M}$. This of course amounts to the same thing.

5. With ordinals we always mean von Neumann ordinals. Likewise, we always consider cardinals in the sense of the von Neumann cardinal assignment.

6. For this reason, Zermelo–Fraenkel set theory is also called the iterative conception of sets, where the latter is used to informally justify the former. See e.g. (Boolos, 1971; Parson, 1975).

7. See (Enderton, 1977, Thm. 9L) for a full proof.

8. Note of course this follows after translation into modern term, i.e. when urelements are discarded and the axiom of infinity is taken into consideration. See the introductory note by Kanamori (2010).

9. See (Rautenberg, 2010, §3.4) for a proof of Skolem's paradox. Note this result uses the fact that our $\mathcal{L}$ is finite, and thus countable, but generalizes to languages of any give cardinality, increasing the lower bound of the cardinality of the non-natural models with the cardinality of the given language.

10. There is thus no $f \in \mathcal{M}$ such that in $\mathcal{M}$ the first-order sentence expressing f is a bijection $x \to y$ holds.

11. Observe the indeterminacy of set theory is not 'fixable' in any effective way due to the results by Gödel.

12. A relativization means we consider interpreted versions of formulae. Thus if φ is of the form $\forall x \ldots$ and we relativize φ to a model $\mathcal{M}$ this becomes $\forall x \in \mathcal{M} \ldots$.

13. And, as Putnam (1980) points out, this ambiguity cannot be fixed for we can "Skolemize absolutely everything", including our metatheory (or our meta-metatheory, or ...).

14. It should be no surprise I draw from Wittgenstein in these reflections, although I do not want to go into the discussion of interpreting him. In particular, see (Wittgenstein, 1971) for his thoughts on 'hinge propositions', which accord to my 'sanity conditions', and (Wittgenstein, 1967, 1983) for his thoughts on rules and language games. See also Putnam (1980) who argues taking anything but use (namely something called 'interpretation') as a constituent of meaning is actually at the heart of Skolem's paradox, which "can only have crazy solutions". For "To speak as if this were my problem, 'I know how to use my language, but, now how shall I single out an interpretation?' is to speak nonsense. Either the use already fixes the 'interpretation' or *nothing* can.".

15. As an example of the dynamics in foundations, consider the fact the Zermelo sought arguments for the justification of his axioms from a cumulative conception of sets, on the basis of intuition. Thus, he took the existence of a large enough chain of ordinals and a faculty to take 'full power sets' as being intelligible, and deduced a cumulative hierarchy of sets. The latter is then of course a model for his set theory and is therewith intended as a justification for these axioms (Ebbinghaus, 2006). Zermelo even devised a (we would say, informal) definition of sets ("a class defined up to isomorphism"). Compare this to the situation in modern text books, where Zermelo's axioms are usually given prior to the notions of ordinals, cumulative hierarchies and models.

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