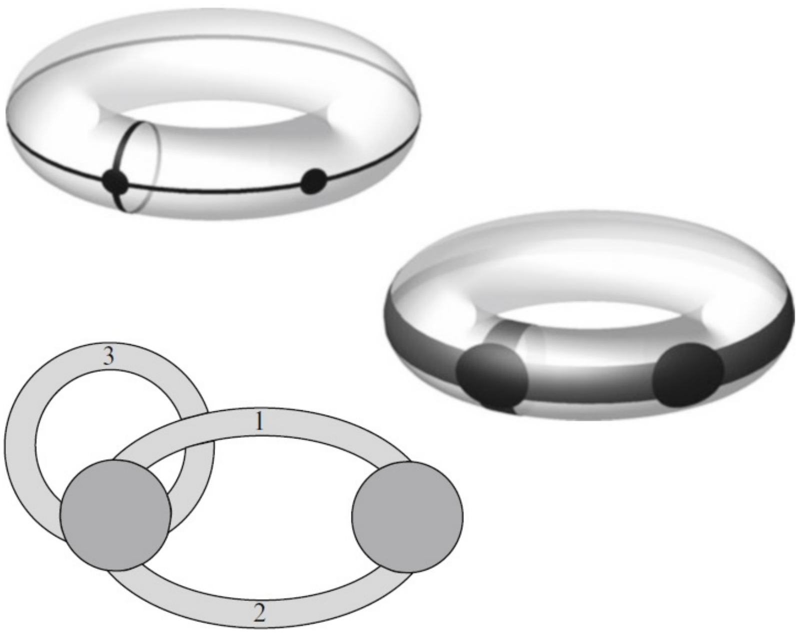
Examples of quantum moduli algebras via Hopf algebra gauge theory on ribbon graphs



THESIS

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Author :	Kevin van Helden
Student ID :	1328670
Supervisor :	Roland van der Veen
2^{nd} corrector :	Jan Willem Dalhuisen

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Kevin van Helden

Institituut-Lorentz, Leiden University
P.O. Box 9500, 2300 RA Leiden, The Netherlands

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Abstract

In this thesis we will try to find explicit examples and characterisations of the quantum moduli algebras on ribbon graphs with one vertex. First, we will study the classical case of group gauge theory, in which we identify the moduli algebra with the function algebra on the moduli space of flat connections. Secondly, the group gauge theory case will be extended to the group algebra case, for which we show that the quantum moduli algebra is isomorphic to the moduli algebra in the group gauge theory case. Thirdly, we will give a general construction of how to obtain quantum moduli algebras of semisimple finite-dimensional Hopf algebras, and we will identify this construction with the construction of the quantum moduli algebra in the group algebra case. Fourthly, we will be examining the situation in which our Hopf algebra is the Drinfel'd double of a group algebra. After giving some examples, we will show that the quantum moduli algebra in the case of the Drinfel'd double is isomorphic to the quantum moduli algebra in the group algebra case.

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Chapter 1

Physical motivations

In physics, the notion of gauge theory is vital in understanding the forces in the standard model. Gauge theories describe fields that form a solution to a set of physical differential equations, conserving the symmetry of a so-called gauge group. The concept of gauge theories is most clearly understood when illustrated by an example (the following example comes from [1]). In classical isospin gauge theory, one can view a nucleon as a field $\psi : \mathbb{M} \to \mathbb{C}^2$ and a pion as a field $\phi : \mathbb{M} \to \mathbb{C}^3$, where $\mathbb{M}$ is Minkowski space, since the nucleon is an isospin doublet (i.e. a superposition of a proton and a neutron) and the pion is an isospin triplet (i.e. a superposition of π^+, π and π^-). For this to become a gauge theory, it is necessary, as Yang and Mills discovered, to find equations such that if ϕ and ψ are solutions, then

$$\begin{aligned} \tilde{\psi} : \mathbb{M} &\to \mathbb{C}^2 & \tilde{\phi} : \mathbb{M} &\to \mathbb{C}^3 \\ x &\mapsto U_{1/2}(g(x))(\psi(x)) & x &\mapsto U_1(g(x))(\phi(x)) \end{aligned}$$

are also solutions of those same equations for all functions $g : \mathbb{M} \to SU(2)$. (In the above functions, $SU(2)$ is the special unitary group of 2×2-matrices, and U_j stands for the spin-j-representation of $SU(2)$. Those representations can also be found in [1].)

Changing a field by $\psi \to \tilde{\psi}$ or by $\phi \to \tilde{\phi}$ in the manner as displayed above is called a gauge transformation. If one wants to create a gauge theory with a different gauge group, one can follow the above approach, changing the fields and the representations accordingly. One of the most well-known examples is that of electromagnetism. (This example and a more elaborated derivation and explanation is to be found in [1].) In that case, the gauge group is $U(1) = \{e^{i\theta} \mid \theta \in [0, 2\pi]\}$, and the electromagnetic vector potential A has to satisfy Maxwell's equations (which in the case $dA = F = B + E \wedge dt$ reduce to)

$$\star\, d \star dA = J, \tag{1.1}$$

where $J = (\rho, \mathbf{j})$ is constructed from the charge density ρ and the current density $\mathbf{j}$. Gauge transformations for this gauge group are given by the formula $\tilde{A} = A + df$, where $f : \mathbb{R}^4 \to \mathbb{R}$ is a real-valued smooth function from spacetime. Using the facts that d is linear and that $d(df) = 0$, one can see that such a gauge transformation indeed does not change Maxwell's equations, and hence the equations are invariant under the action of the gauge group on the electromagnetic field.

In order to be able to sensibly use this theory in a quantum mechanical context, we need to quantize gauge theories. One approach to quantize such theories is called the Hamiltonian approach. For a system with an n-dimensional configuration space with generalized coordinates

$q_1, \ldots, q_n$, one can consider (classical) observables as smooth functions from the phase space, which is a $2n$-dimensional space with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$, where $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, to $\mathbb{R}$ ($\mathcal{L}$ is the Lagrangian of the system). Moreover, there exists a bracket operation for two observables: for f, g observables, one can define the Poisson bracket:

$$\{f, g\} := \frac{\partial f}{\partial \dot{p}_i} \frac{\partial g}{\partial \dot{q}^i} - \frac{\partial g}{\partial \dot{p}_i} \frac{\partial f}{\partial \dot{q}^i}, \tag{1.2}$$

in which we use the Einstein summation convention. The idea of quantizing a theory consists of changing classical observables f to quantum observables $\hat{f}$ which are self-adjoint operators on the Hilbert space of square integrable functions on our configuration space, such that the commutator of two quantum observables $\hat{g_1}, \hat{g_2}$ coincides with the Poisson bracket [1]. As example, two of the most elementary quantum observables in a one-dimensional quantum system are given by $\hat{x}$ and $\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$. The commutator of those two observables is quite well-known:

$$[\hat{x}, \hat{p}] = i\hbar. \tag{1.3}$$

These commutation relations between the different observables constitute a multiplication (in this case even an algebra) structure, and this algebra is conveniently called the algebra of observables. It is this object that we are attempting to explicitly determine in the next chapters, in the case that our quantum system is defined on $\Sigma \times \mathbb{R}$, where Σ is a compact 2-dimensional manifold (a smooth surface that locally looks like the plane, which we can approach in two dimensions in a more algebraic and eventually simpler setting than other dimensional manifolds), and that our gauge theory is Chern-Simons theory.
Chern-Simons theory is a special type of Lagrangian mechanics: here, the Lagrangian $\mathcal{L}_{CS}$ is given by

$$\mathcal{L}_{CS} = \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{1.4}$$

where A is a vector potential and $\wedge$ the wedge product, which is alternating and multilinear.* The corresponding action is then given by

$$S_{CS}(A) = \int_{\Sigma \times \mathbb{R}} \mathcal{L}_{CS}. \tag{1.5}$$

This action, however, yields a trivial result in the classical setting: if we minimize the action, the curvature $F = dA + A \wedge A$ vanishes, implying that the curvature of A must be flat for this Lagrangian system to have a minimal action. Another remarkable property of this theory is, if we consider another vector potential A' that differs from A only by a gauge transformation, that then

$$S_{CS}(A') - S_{CS}(A) = 8\pi^2 m$$

for some integer number $m \in \mathbb{Z}$. (This is due to the integrability of the second Chern class, and is explained in [1].) This implies that the Chern-Simons action is not gauge invariant. To solve this, one only uses the exponential

$$e^{\frac{ik}{4\pi} S_{CS}(A)}$$

with another integer $k \in \mathbb{Z}$ in calculations, since this quantity, considering the previous remark, is gauge invariant. Moreover, if one assumes the existence of a non-zero cosmological constant Λ, one can define the Chern-Simons state

$$\Psi_{CS} = e^{-\frac{6}{\Lambda} S_{CS}(A)}, \tag{1.6}$$

*For a formal construction of Equation 1.4, please see Appendix C.

which satisfies the properties for being in the physical state space for a quantum gauge theory for gravity [1]. Hence, understanding the quantization of Chern-Simons theory could contribute to a greater insight in a theory of quantum gravity. It has to be noted, however, that as of yet it is still unclear if this solution gives rise to a physically feasible system, or whether this can only be viewed as a toy model from which one can gain a more useful insight in theories of quantum gravity. The case we will be studying will be suited only for the purpose of understanding theories on quantum gravity, since we will only obtain examples for manifolds with two spatial dimensions and one temporal dimension, whereas spacetime is 4-dimensional.
We will investigate this by studying Hopf algebra gauge theory, since it has been claimed to be a mathematically axiomatic setting to calculate the algebra of observables in Chern-Simons theory on $\Sigma \times \mathbb{R}$, where Σ is a compact orientable 2-dimensional manifold [2].

Another motivation for studying gauge theory of Hopf algebras consists of the close relationship between this theory and the Kitaev model.
The Kitaev model is a spin 1/2-model on a honeycomb lattice, in which only nearest neighbour interactions are taken into account, but the interaction strength for a particle is different in all directions. In Figure 1.1, an example of a hexagon in the honeycomb lattice is given. The different

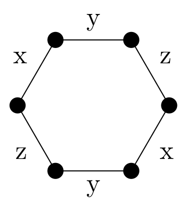

Figure 1.1: *A hexagon from the honeycomb lattice as described in the Kitaev model. The dots denote the positions of particles with spin 1/2, and the lines correspond to the interactions between those particles. Identical letters labelled to interactions imply identical interaction strengths.*

types of interaction (indicated by different letters) are called x-, y- and z-links, corresponding to the letters in Figure 1.1.
Then the Hamiltonian of this system can be written down:

$$H = -\sum_{v \in \{x,y,z\}} J_v \sum_{v\text{-link } i} \sigma_j^i \sigma_k^i, \tag{1.7}$$

where J_x, J_y, J_z are the different interaction strengths for the x-, y- and z-links respectively, and σ_j^i, σ_k^i the spin values of the two particles interacting by link i [3].
This model has a couple of interesting features worth noting. As a first, the Kitaev model has an exact solution arrived at by using Majorana operators, i.e. by describing the spin operators as Majorana fermions. Then this model can be seen as Majorana fermion hopping problem with a $\mathbb{Z}/2\mathbb{Z}$ gauge theory on the hopping matrix element. The fact that this model has an exact solution is remarkable, since it is not frequently seen in models in condensed matter physics [4].
Moreover, the excitations of the model can be considered to be anyons. Anyons are quasiparticles that are only to be found in a 2-dimensional system. In order to explain the concept of anyons, consider a 2-dimensional system in which two identical particles a, b with states ψ_a, ψ_b respectively live. If one then exchanges the two particles by rotating them around the other particle, their composite wavefunction $|\psi_a\psi_b\rangle$ changes by a phase $e^{i\theta}$: $|\psi_a\psi_b\rangle \to e^{i\theta}\,|\psi_a\psi_b\rangle$. If one exchanges the particles again, one obtains another factor $e^{i\theta}$. In any 3- or higher dimensional system, the

resulting trajectories can be continuously deformed to the identity or constant transformation, implying that $e^{2i\theta} = 1$, giving two solutions for θ: $\theta = 0, \pi$. In the first case, the discussed particles are bosons, in the second case, they are fermions. In a two dimensional system, it is not necessarily true that those trajectories can be continuously deformed into the constant transformation, which implies that the equation $e^{2i\theta} = 1$ is not a constraint in this system: θ can have **any** value, hence the name anyon [5–7]. Anyons are useful in understanding topological properties of the models in which they occur, and since they intrinsically carry information on the topological properties of a model, they are thus contributing to the construction of topological quantum field theories. Anyons are also considered a helpful tool in a more practical area of current research: the topological properties of certain types of anyons and the braiding of those seem to allow one to make universal quantum computations [3, 8]. (Note that the braiding of anyons is not per se uniquely determined: one needs to make a choice of Hopf algebra in order to be able to calculate those braidings.) This process is called topological quantum computation and it is thought of as a new approach to create fault-tolerant quantum computers [9].
Secondly, the Kitaev model can be derived axiomatically as a Hopf algebra gauge theory. In particular, it has been found that finding the algebra of operators on the protected space for a Kitaev model with a Hopf algebra H is equivalent to finding the quantum moduli algebra for the combinatorial quantization of Chern-Simons theory for $\mathcal{D}(H)$, the Drinfel'd double of H [10]. (For a definition of $\mathcal{D}(H)$ in the case that H is a group algebra, see Example 2.9). This implies that finding explicit examples of the quantum moduli algebra also immediately leads to analogous examples in the Kitaev model. By enhancing our understanding in the combinatorial quantization of Chern-Simons theory, we can thus contribute to the research on topological quantum field theory and topological quantum computing.
In this thesis, we will therefore investigate Hopf algebra gauge theory on ribbon graphs and attempt to find explicit examples of quantum moduli algebras. Firstly, in Chapter 2, we will introduce some concepts, such as that of a delta function and a Hopf algebra, which we will use frequently in the following chapters. In Chapter 3, we will study the classical case of group gauge theory, and try to motivate why this approach is justified for finding the algebra of observables in Chern-Simons theory. In Chapter 4, we will rewrite the theory from the previous chapter in the Hopf algebra gauge theory formalism. Subsequently, we will give a general construction for obtaining the quantum moduli algebra of a more general type of Hopf algebra in Chapter 5. Using this new construction, we will give some explicit examples of quantum moduli algebras in the case that our Hopf algebra is the Drinfel'd double of a group algebra in Chapter 6, followed by a proof which characterises these Drinfel'd double algebras. Lastly, we will conclude by summarising our main results and discussing possible further research in Chapter 7.

Chapter 2

Prerequisites

It is assumed that the reader has basic knowledge of group theory, graph theory and of linear algebra. The reader should be comfortable with concepts such as finite groups, group homomorphisms, linear maps, algebra morphisms, duality and tensor products.
First we shall discuss some basic facts and notation concerning function algebras.

Definition 2.1 *Let A be a set, and $\mathbb{F}$ a field. We define $Fun_{\mathbb{F}}(A) = Fun(A) := \{\varphi : A \to \mathbb{F}\}$.*

Lemma 2.2 *The set $Fun(A)$ is an algebra with pointwise addition, scaling and multiplication.*

Proof. This is clear, since $\mathbb{F}$ is an algebra. □
We will not explicitly mention the field $\mathbb{F}$ any further, but it is implicitly assumed in the following parts, that is, if it is not mentioned, all our vector spaces will be $\mathbb{F}$-vector spaces. Moreover, the theory in this thesis, if nothing else is mentioned, applies to all fields $\mathbb{F}$, but it is important to note that $\mathbb{F}$ is fixed throughout the remainder of the work.

Definition 2.3 *Let A be a set and $a \in A$. We define the **delta function** $\delta_a \in Fun(A)$ by*

$$\delta_a(b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{else.} \end{cases}$$

Next, we introduce some notation we will use extensively in the later chapters.
Let V be a vector space, and let $\alpha, \beta \in V$. For $j \leq n \in \mathbb{N}$, we define $(\alpha)_j \in V^{\otimes n}$ as the tensor product in which all components are 1, except for the j-th component, which is α. Analogously, for $i, j \leq n \in \mathbb{N}$ with $i \neq j$, we define $(\alpha \otimes \beta)_{ij} \in V^{\otimes n}$ as the tensor product in $V^{\otimes n}$ in which all components are 1, except for the i-th component, which is α, and the j-the component, which is β.
One structure that will be used extensively throughout this thesis will be that of a Hopf algebra. Therefore we will define this beforehand, as well as giving some examples we consider to be worthwhile looking into more extensively in the remainder of the thesis. The structure of the definition of a Hopf algebra in this chapter is largely inspired by [11].
In order to define a Hopf algebra, we need to know what an algebra and a coalgebra are.

Definition 2.4 *An **algebra** (also called an $\mathbb{F}$-algebra) K is a vector space K over a base field $\mathbb{F}$ with a bilinear multiplication map $\cdot : K \otimes K \to K, a \otimes b \mapsto a \cdot b$ such that*

$(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in K$; (associativity) (1)

there exists an element $1 \in K$ such that $1 \cdot a = a = a \cdot 1$ for all $a \in K$. (unit element) (2)

We will usually denote $ab = a \cdot b$ for the multiplication in K. It is insightful for the further construction to note that the above axioms of associativity and the existence of a unit element can also be expressed in saying that the diagrams in Figure 2.1 commute. In this figure, a new linear map $\eta : \mathbb{F} \to K$ is introduced.

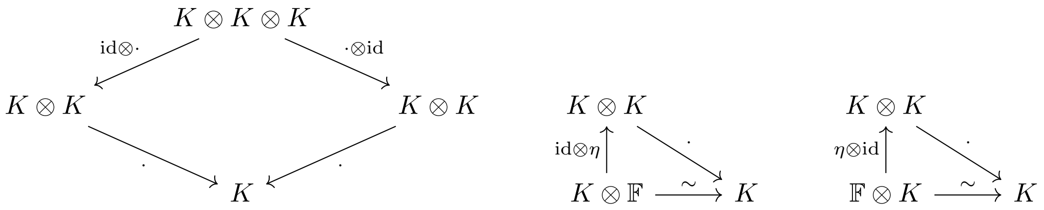

Figure 2.1: *Three commutative diagrams equivalent to the axioms of Definition 2.4. On the left, associativity is expressed. In the middle and on the right, the existence of a unit element is expressed.*

Definition 2.5 *A **coalgebra** (also called an $\mathbb{F}$-coalgebra) K is a vector space K over a base field $\mathbb{F}$ with a linear comultiplication map $\Delta : K \to K \otimes K$, and a linear counit map $\epsilon : K \to \mathbb{F}$ such that*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta; \quad \textit{(coassociativity)} \tag{1}$$
$$(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta, \tag{2}$$

in which we use the identification $K \otimes \mathbb{F} \simeq K \simeq \mathbb{F} \otimes K$.

In the following, we will frequently use Sweedler notation for the comultiplication in K: for $a \in K$, we write $\Delta(a) = a_{(1)} \otimes a_{(2)}$, implicitly assuming summations if necessary. Analogously to the case of the algebra, we find that the above axioms of associativity and the existence of a unit element can also be expressed in saying that the diagrams in Figure 2.2 commute. It is also worthwhile to note that the diagrams in Figure 2.2 are the same as in Figure 2.1, but with the arrows reversed, and Δ replaced by $\cdot$, and ϵ replaced by η.

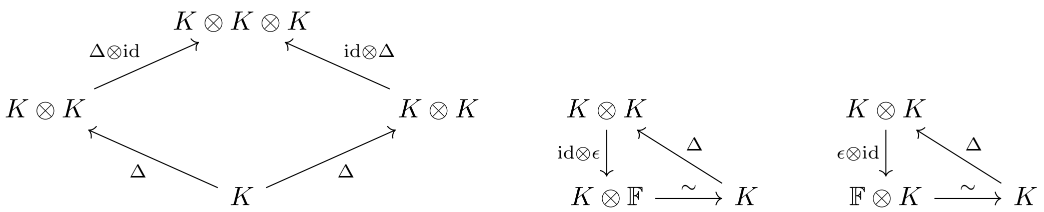

Figure 2.2: *Three commutative diagrams equivalent to the axioms of Definition 2.5. On the left, coassociativity is expressed. In the middle and on the right, the existence of a counit element is expressed.*

Finally, using the previous definitions, we can define a Hopf algebra.

Definition 2.6 *A **Hopf algebra** K is an algebra K that is also a coalgebra with a linear map*

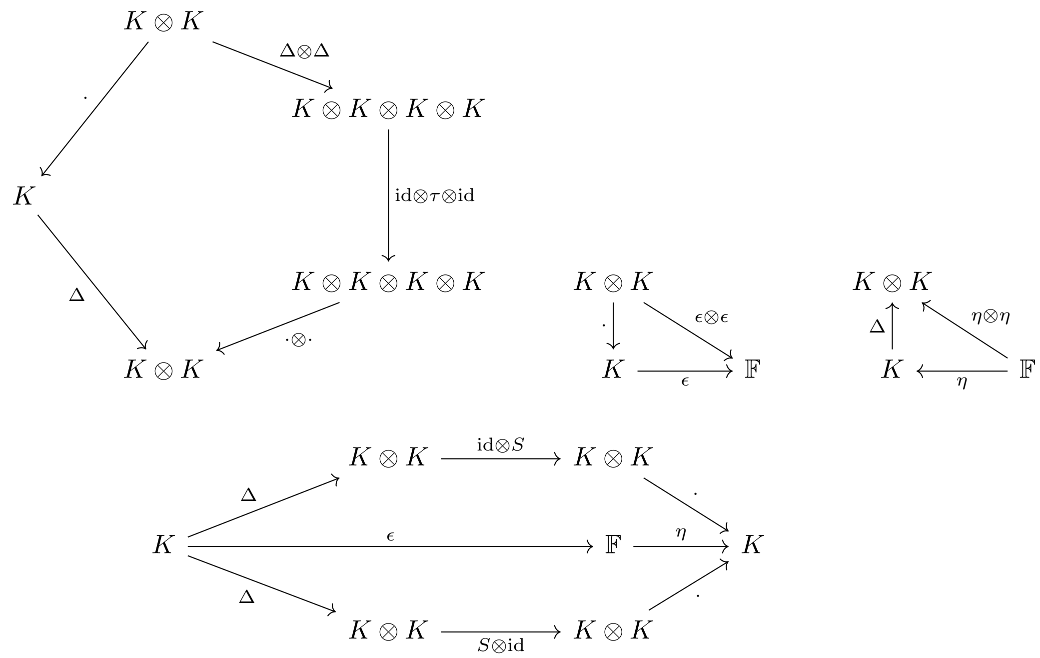

Figure 2.3: *Four commutative diagrams equivalent to the axioms of the Hopf algebra. On the top left, axiom (1) of Definition 2.6 is expressed. In this diagram, $\tau : K \otimes K \to K \otimes K$ is the linear map satisfying $\tau(g \otimes h) = h \otimes g$ for all $g, h \in K$. On the top middle, axiom (3) is expressed. On the top right, axiom (2) is expressed. On the bottom, axiom (5) is expressed.*

$S : K \to K$, *satisfying the following properties:*

$$\Delta(hg) = \Delta(h)\Delta(g) \text{ for all } g, h \in K; \quad (1)$$
$$\Delta(1) = 1 \otimes 1; \quad (2)$$
$$\epsilon(hg) = \epsilon(h)\epsilon(g) \text{ for all } g, h \in K; \quad (3)$$
$$\epsilon(1) = 1; \quad (4)$$
$$\cdot \circ (id \otimes S) \circ \Delta = \eta \cdot \epsilon = \cdot \circ (S \otimes id) \circ \Delta. \quad (5)$$

Note that we define the multiplication in $K \otimes K$ componentwise.

Note that axioms (1)-(4) are equivalent to saying that $\Delta : K \to K \otimes K$ and $\epsilon : K \to \mathbb{F}$ are algebra homomorphisms. Furthermore, in this case, it is also possible to express these axioms in a diagram (we will ignore the fourth axiom, which is an evident consequence of axiom (2) in Definition 2.5.) Those diagrams are to be found in Figure 2.3. Now we give some examples of Hopf algebras. The proof that they are in fact Hopf algebras can be found in [2]. We will also give definitions of R for Example 2.7 and 2.9, on which we will elaborate more in Chapter 5.

Example 2.7 *Let G be a finite group. The set $\mathbb{F}[G] = \{\sum_{g\in G} \lambda_g g \mid \lambda_g \in \mathbb{F}\}$ is a Hopf algebra when given the following operations for $g, h \in G$:*

$$g \cdot h = gh, \quad 1 = e, \quad \Delta(g) = g \otimes g, \quad \epsilon(g) = 1, \quad S(g) = g^{-1}, \quad R = e \otimes e.$$

Example 2.8 *Let G be a finite group. The set $\mathrm{Fun}(G) = \{f : G \to \mathbb{F}\}$ is a Hopf algebra when given the following operations for $g, h \in G$:*

$$\delta_g \cdot \delta_h = \delta_g(h)\delta_g, \quad 1 = \sum_{g\in G} \delta_g, \quad \Delta(g) = \sum_{u\in G} u \otimes u^{-1}g, \quad \epsilon(\delta_g) = \delta_g(e), \quad S(\delta_g) = \delta_{g^{-1}},$$

and this Hopf algebra is isomorphic to the dual Hopf algebra of $\mathbb{F}[G]$. *(See also Lemma 4.2.)*

Example 2.9 *Let* G *be a finite group. The vector space* $\mathcal{D}(\mathbb{F}[G]) = \mathrm{Fun}(G) \otimes \mathbb{F}[G]$ *is called the* ***Drinfel'd double*** *of* G *and is a Hopf algebra when given the following operations for* $g, h, g', h' \in G$*:*

$$\begin{aligned} (\delta_h \otimes g) \cdot (\delta_{h'} \otimes g) &= \delta_{g^{-1}hg}(h')\delta_h \otimes gg', & 1 &= 1 \otimes e, \\ \Delta(\delta_h \otimes g) &= \sum_{u,v\in G \,:\, uv=h} (\delta_v \otimes g) \otimes (\delta_u \otimes g), & \epsilon(\delta_h \otimes g) &= \delta_h(e), \\ S(\delta_h \otimes g) &= \delta_{g^{-1}h^{-1}g} \otimes g^{-1}, & R &= \sum_{g\in G} (1 \otimes g) \otimes (\delta_g \otimes e). \end{aligned}$$

Chapter 3

Group gauge theory

Let Γ be a connected directed graph, E the set of all edges on Γ, V the set of all vertices on Γ, and denote for an oriented edge e the edge with reverse orientation by e^{-1} and the starting vertex of e by $s(e)$ and the target vertex of e by $t(e)$. We write E for the cardinality of E (it will be clear in the notation when E is a set or a number) and we will assume that $1 \leq E < \infty$.

Definition 3.1 *A **cyclic ordering** on Γ is a family of bijective functions $\{c_v : \Gamma_v \to \mathbb{Z}/n_v\mathbb{Z} \mid v \in V\}$, where n_v is the valence of v and $\Gamma_v = \{e^\epsilon \mid e \in E,\ \epsilon \in \{\pm 1\}, s(e^\epsilon) = v\}$ the set of all edge ends at v.*

Definition 3.2 *A **ribbon graph** is a directed graph Γ with a cyclic ordering.*

As can be seen in Example 3.6, some ribbon graphs can be distinguished only by their cyclic ordering. From now on, we assume that Γ is a ribbon graph.

Definition 3.3 *A **path** p on Γ is a sequence $(e_i^{\epsilon_i})_{i=1}^n$, where $e_i \in E$ and $\epsilon_i \in \{\pm 1\}$, such that $v_i := t(e_i^{\epsilon_i}) = s(e_{i+1}^{\epsilon_{i+1}})$, $1 \leq i \leq n-1$.*

Definition 3.4 *A **face path** of Γ is a path $p = (e_i^{\epsilon_i})_{i=1}^n$ such that $v_n := t(e_n^{\epsilon_n}) = s(e_1^{\epsilon_1})$ and that $c_{v_i}(e_{i+1}^{\epsilon_{i+1}}) = c_{v_i}(e_i^{-\epsilon_i}) + 1$ for $1 \leq i \leq n-1$ and $c_{v_n}(e_1^{\epsilon_1}) = c_{v_n}(e_n^{-\epsilon_n}) + 1$, and that $e_k^{\epsilon_k} \neq e_l^{\epsilon_l}$ for all $1 \leq k \neq l \leq n$.*

Note that this relation is indeed well-defined, since we know for $1 \leq i \leq n$ that $t(e_i^{\epsilon_i}) = v_i$, and thus that $s(e_i^{-\epsilon_i}) = t(e_i^{\epsilon_i}) = v_i$. One can easily find the face paths of a ribbon graph by simply starting at one edge, following it from the starting vertex to the target vertex (or vice versa) and then continuing with the edge which is next in line with the respect to the ordering on that vertex, until one uses the same edge with the same orientation twice.

Definition 3.5 *A **face** $[p]$ is an equivalence class of face paths subject to relation $\sim$: $p = (e_i^{\epsilon_i})_{i=1}^n \sim p' \Leftrightarrow$ there exists a $m \in \mathbb{Z}/n\mathbb{Z}$ such that $p' = (e_{i+m}^{\epsilon_{i+m}})_{i=1}^n$.*
We denote the set of all faces of Γ by F.

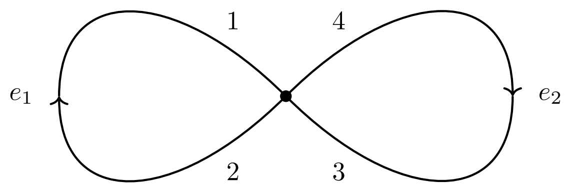

Figure 3.1: *An example of a ribbon graph, which will be called* Γ_1.

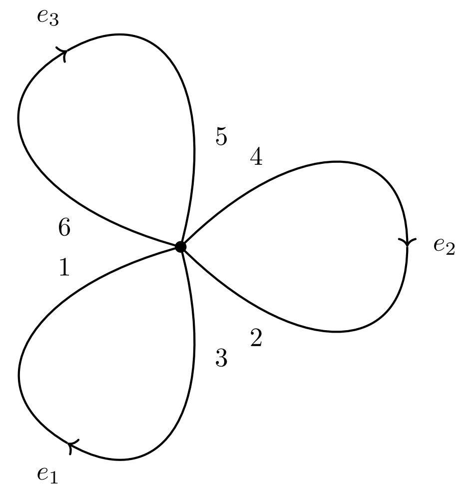

Figure 3.2: *An example of a ribbon graph, which will be called* Γ_2.

Example 3.6 In Figure 3.1 and 3.2, two examples of ribbon graphs are given, in which the cyclic ordering is already displayed in the graph. The set of all faces of Γ_1, as shown in Figure 3.1, is equal to $F_{\Gamma_1} = \{[(e_1)], [(e_2)], [(e_1^{-1}, e_2^{-1})]\}$. The set of all faces of Γ_2, as shown in Figure 3.2, is equal to $F_{\Gamma_2} = \{[(e_2^{-1}, e_3^{-1}, e_1^{-1}, e_2, e_1)], [(e_3)]\}$. Denote the face path $(e_2^{-1}, e_3^{-1}, e_1^{-1}, e_2, e_1)$ of Γ_2 by p_1. Also consider the ribbon graph Γ_2' be removing e_3 from Γ_2. Then Γ_1 and Γ_2' are identical as graphs, but the faces of Γ_2' are given by $F_{\Gamma_2'} = \{[(e_2^{-1}, e_1^{-1}, e_2, e_1)]\}$. Thus we can conclude that a different cyclic ordering of the same graph can yield a different ribbon graph, since it alters the (number of) faces of the graph.

In order to make a connection between the group gauge theory on ribbon graphs and that on 2-dimensional compact oriented manifolds, we need to note that for every ribbon graph, there is a unique 2-dimensional compact oriented manifold $\Sigma_\Gamma \subset \mathbb{R}^3$ up to homeomorphism, such that the geometric realisation $|\Gamma|$ of the ribbon graph can be embedded into Σ_Γ as a filling ribbon graph [12]. That is to say, there exists a map $\varphi : |\Gamma| \to \Sigma_\Gamma$ such that $\varphi : |\Gamma| \to \varphi(|\Gamma|)$ is homeomorphism and such that the connected components of $\Sigma_\Gamma \setminus \varphi(|\Gamma|)$ are diffeomorphic to disks. Here, the geometric realisation of Γ is given by the topological space $E \cup E^{-1} \times [0, 1]/\sim$, where $E^{-1} := \{e^{-1} \mid e \in E\}$ (note that $E \cup E^{-1}$ here has the discrete topology,) and where $\sim$ is the equivalence relation given by

- $(e, t) \sim (e^{-1}, 1 - t)$;

- $(e, 0) \sim (f, 0)$ if $s(e) = s(f)$;
- $(e, 0) \sim (f, 0)$ if $t(e) = t(f)$

for all $e, f \in E \cup E^{-1}$ and $t \in [0, 1]$ [12].
Conversely, it has also been shown that every 2-dimensional compact oriented manifold Σ is homeomorphic to Σ_Γ for a certain ribbon graph Γ [12]. This shows that there is a correspondence between ribbon graphs and 2-dimensional compact oriented manifolds by associating Γ with Σ_Γ. Hence, our theory can be transported via this correspondence to the Chern-Simons theory on 2-dimensional compact oriented manifolds.
Let G denote a finite group.

Definition 3.7 (Holonomy) *Given a path $p = (e_i^{\epsilon_i})_{i=1}^n$ on Γ, we define the group-theoretic* ***holonomy along*** p *as*

$$\begin{aligned} \mathrm{Hol}_{G,p} : G^{\times E} &\to G \\ (g_1, \dots, g_E) &\mapsto g_{e_n}^{\epsilon_n} \circ \dots \circ g_{e_1}^{\epsilon_1}. \end{aligned}$$

One can see holonomy as a discrete version of parallel transport. For motivation, in the case for a trivial fiber bundle, if we take Σ to be a 2-dimensional compact oriented manifold, F a manifold, $\pi : \Sigma \times F \to \Sigma$ the projection on the first factor, and $\gamma : [0, 1] \to \Sigma$ a (continuous) path, we can define parallel transport P_γ over γ to be the map

$$\begin{aligned} P_\gamma : \{\gamma(0)\} \times F \times [0, 1] &\to \Sigma \times F \\ (p, t) &\mapsto \tilde{\gamma}_p(t), \end{aligned}$$

where $\tilde{\gamma}$ is the unique lift of γ so that the diagram

$$\begin{array}{ccc} & & (\Sigma \times F, p) \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \pi \\ ([0,1], 0) & \xrightarrow{\gamma} & (\Sigma, \gamma(0)) \end{array}$$

of pointed topological spaces commutes [13, 14].
This can be modified to the discrete case by substituting the manifold Σ by the geometric realisation of the ribbon graph Γ and by substituting the manifold F by a (in our case finite) group G. Then, we can define the discrete version of parallel transport P'_γ over the (continuous path) $\gamma : [0, 1] \to |\Gamma|$ to be the map

$$\begin{aligned} P'_\gamma : \{\gamma(0)\} \times G \times [0, 1] &\to |\Gamma| \times G \\ (p, t) &\mapsto \tilde{\gamma}_p(t), \end{aligned}$$

where $\tilde{\gamma}$ is the unique lift of γ so that the diagram

$$\begin{array}{ccc} & & (|\Gamma| \times G, p) \\ & \overset{\tilde{\gamma}}{\nearrow} & \downarrow \pi' \\ ([0,1], 0) & \xrightarrow{\gamma} & (|\Gamma|, \gamma(0)) \end{array}$$

of pointed topological spaces (we take the discrete topology on G) commutes. Here $\pi' : |\Gamma| \times G \to |\Gamma|$ is the projection on the first factor.
If we now assume that $\gamma = \gamma_e : [0, 1] \to |\Gamma| = E \cup E^{-1} \times [0, 1]/ \sim, t \mapsto (e, t)$ for a certain

$e \in E \cup E^{-1}$, we find that γ exactly travels along the edge e. Note furthermore that $\tilde{\gamma}_p(0)$, where $p = (\gamma(0), g) \in |\Gamma| \times G$, uniquely defines $\tilde{\gamma}_p$: we have that $\tilde{\gamma}_p(t) = (\gamma(t), g)$. Therefore, it seems to make sense to call $g \in G$ the connection corresponding to the edge e and hence to call $(g_1, \ldots, g_E) \in G^{\times E}$ the connection in general. We can then define the holonomy for the path $p = (e)$ in terms of this connection by using Definition 3.7, and we can multiply this with the holonomy to obtain Definition 3.7 again.
Also note that via the embedding $\varphi : |\Gamma| \to \Sigma$, we can view the faces of Γ as loops in Σ, and hence, we can view the holonomy along a face path as the curvature of the connection inside the enclosed area by the face path [2]. Since we know that the minimal action of Chern-Simons theory is given by the condition that the curvature is flat [1], it seems natural to demand, in order to get physical solutions, that the holonomy along all faces yields the neutral element. Hence, it is common, in this set-up, to consider $g \in G$ as a connection and a flat connection if the holonomy along all the face paths (and thus along all the faces) is trivial.

Example 3.8 Choosing $G = S_3$ and $\Gamma = \Gamma_2$ from Figure 3.2, we find that

$$\mathrm{Hol}_{G,p_1}((1\ 2), (1\ 2\ 3), (1)) = (1\ 2\ 3)^{-1}(1)^{-1}(1\ 2)^{-1}(1\ 2\ 3)(1\ 2) = (1\ 2\ 3).$$

Definition 3.9 *Given a path p, we define* $\mathrm{Hol}^*_{G,p} : \mathrm{Fun}(G) \to \mathrm{Fun}(G^{\times E})$ *by* $\mathrm{Hol}^*_{G,p}(\varphi)(x) = \varphi(\mathrm{Hol}_{G,p}(x))$.

Although $\mathrm{Hol}_{G,p}$ is not a linear map, it is still sensible to use the notation for a dual map in the notation of the previous definition. Later on, we will extent $\mathrm{Hol}_{G,p}$ multilinearly and then we will be able to define an actual dual map in an analogous fashion.

Lemma 3.10 *Let p be a face path. The map* $R^*_p : \mathrm{Fun}(G^{\times E}) \to \mathrm{Fun}(G^{\times E}), \alpha \mapsto \mathrm{Hol}^*_{G,p}(\delta_e) \cdot \alpha$ *is identical to $R^*_{p'}$ if $p \sim p'$.*

Proof. Unwrapping definitions gives us that

$$R^*_p(\alpha)(x) = \begin{cases} \alpha(x) & \text{if } \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else.} \end{cases}$$

Since relations remain relations under cyclic permutations, we find that $\mathrm{Hol}_p(x) = e \Leftrightarrow \mathrm{Hol}_{p'}(x) = e$. □
Note that R^*_p respects multiplication.

Definition 3.11 *Let $\varphi \in \mathrm{Fun}(G^{\times E})$ and $x = (x_1, \ldots, x_E) \in G^{\times E}$. Then G acts on $\mathrm{Fun}(G^{\times E})$ as conjugation by*

$$g(\varphi)(x) := \varphi(gx_1g^{-1}, \ldots, gx_Eg^{-1}).$$

For the remainder of this chapter, assume that Γ has only one vertex. We will assume this in all the cases when defining invariant subalgebras, since this makes the definitions easier to work with.

Definition 3.12 *The* ***algebra of invariant functions*** *is defined as*

$$\mathrm{Fun}_{inv}(G^{\times E}) := \{\varphi : G^{\times E} \to \mathbb{F} \mid \textit{for all } g \in G : \ g(\varphi) = \varphi\} \subset \mathrm{Fun}(G^{\times E}).$$

We note that it is easy to see that $\mathrm{Fun}_{inv}(G^{\times E})$ is indeed a subalgebra of $\mathrm{Fun}(G^{\times E})$. For moduli algebra, we only consider a certain type of ribbon graph.

Definition 3.13 *Let Γ be a ribbon graph. It is called of* ***type*** $(3,2)$ *if every vertex is at least 3-valent and has at least two faces.*

Now let Γ be of type (3,2).

Definition 3.14 (Moduli group algebra) *The* ***moduli group algebra*** *of Γ is defined as the image of the $R^*_{flat} := \prod_{[f]\in F} R^*_f$ of the group-theoretic algebra of invariant functions, or, in other words, $\mathcal{N}_\Gamma = R^*_{flat}(\mathrm{Fun}_{inv}(G^{\times E}))$.*

Now let $\Sigma = \Sigma_\Gamma$ be a 2-dimensional compact oriented manifold associated with ribbon graph Γ, and let F_E be the free group on E elements.

Lemma 3.15 *It holds that $\pi_1(\Sigma) := \langle e_1, \ldots, e_E \mid \text{for all } [p] \in F : \ \mathrm{Hol}_{F_E,p}(e_1, \ldots, e_E)\rangle$.*

Proof. This is proven by [12]. ☐

The group G acts on the set $\mathrm{Hom}(\pi_1(\Sigma), G)$ by $(g \cdot \varphi)(x) = g\varphi(x)g^{-1}$.

Definition 3.16 *We denote the set of orbits of $\mathrm{Hom}(\pi_1(\Sigma), G)$ under the above action of G by $\mathrm{Hom}(\pi_1(\Sigma), G)/G$ and we denote an orbit by $[\psi] \in \mathrm{Hom}(\pi_1(\Sigma), G)/G$, where $\psi \in \mathrm{Hom}(\pi_1(\Sigma), G)$.*

Definition 3.17 *We define the function algebra $\mathcal{O}_\Gamma$ by $\mathcal{O}_\Gamma := \mathrm{Fun}(\mathrm{Hom}(\pi_1(\Sigma), G)/G)$.*

The algebra $\mathcal{O}_\Gamma$ is indeed according to Lemma 2.2 an algebra.

Lemma 3.18 *The map*

$$
\begin{aligned}
&H : \mathcal{O}_\Gamma \to \mathrm{Fun}_{inv}(G^{\times E})\\
&\qquad \varphi \mapsto \left(\overbrace{(x_1, \ldots, x_E)}^{x} \mapsto \begin{cases} \varphi\left(\left[\pi_1(\Sigma) \ni e_i \overset{\psi_x}{\mapsto} x_i \in G, 1 \le i \le E\right]\right) & \text{if } \psi_x \in \mathrm{Hom}(\pi_1(\Sigma), G) \\ 0 & \text{else} \end{cases} \right)
\end{aligned}
$$

is an injective morphism of vector spaces that respects multiplication.

Proof. It is clear that H is well-defined, and that H is a linear map that respects multiplication, since the addition, scaling and multiplication are all pointwise operations in the field $\mathbb{F}$. Now suppose that $\varphi \in \ker H$, and that $\sigma \in \mathrm{Hom}(\pi_1(\Sigma), G)$. Denote $s_i := \sigma(e_i)$, $1 \le i \le E$. Then we know that

$$0 = H(\varphi)(s_1, \ldots, s_E) = \varphi\left(\left[\pi_1(\Sigma) \ni e_i \overset{\psi}{\mapsto} s_i \in G, 1 \le i \le E\right]\right) = \varphi([\sigma]),$$

and that implies that $\varphi = 0$, and that gives us that H is injective. ☐

Note that we cannot say that H is a morphism of algebras since the multiplicative unit in $\mathcal{O}_\Gamma$ is not sent to the multiplicative unit in $\mathrm{Fun}_{inv}(G^{\times E})$ by H.

Theorem 3.19 *The algebras $\mathcal{O}_\Gamma$ and $\mathcal{N}_\Gamma$ are isomorphic.*

Consider the map

$$
\begin{aligned}
S : \mathcal{O}_\Gamma &\to \mathcal{N}_\Gamma\\
\varphi &\mapsto R^*_{flat}(H(\varphi)) = (R^*_{flat} \circ H)(\varphi).
\end{aligned}
$$

This map is by construction injective, linear, and respects multiplication.
Now let $x = (x_1, \dots, x_E) \in G^{\times E}$. Then

$$R^*_{flat}(H(\varphi))(x) = \begin{cases} H(\varphi)(x) & \text{if for all } [p] \in F : \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else.} \end{cases}$$

We also know from the definition of H that

$$H(\varphi)(x) = \begin{cases} H(\varphi)(x) & \text{if } \psi_x \in \mathrm{Hom}(\pi_1(\Sigma), G) \\ 0 & \text{else.} \end{cases}$$

Furthermore, we know that $\psi_x \in \mathrm{Hom}(\pi_1(\Sigma), G)$ if and only if for all $[p] \in F$, we have that

$$G \ni e = \psi_x(\mathrm{Hol}_{F_E,p}(e_1, \dots, e_E)) = \mathrm{Hol}_{G,p}(x_1, \dots, x_E).$$

This implies that $R^*_{flat}(H(\varphi)) = H(\varphi)$.
Now let $\phi = \sum_{g \in G^{\times E}} \lambda_g \delta_g \in \mathrm{Fun}_{inv}(G^{\times E})$. Then

$$R^*_{flat}(\phi)(x) = \begin{cases} \lambda_x = \phi(x) & \text{if for all } [p] \in F : \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else.} \end{cases}$$

Denote $\tau = \sum_{[\psi_x] \in \mathcal{O}_\Gamma} \lambda_x \delta_{[\psi_x]}$. We note that τ is well-defined, since $\phi \in \mathrm{Fun}_{inv}(G^{\times E})$, so for a $g \in G$ we have

$$\lambda_{(x_1,\dots,x_E)} = \phi(x_1, \dots, x_E) = \phi(gx_1g^{-1}, \dots, gx_Eg^{-1}) = \lambda_{(gx_1g^{-1},\dots,gx_Eg^{-1})}.$$

Then we can obtain that

$$\begin{aligned} H(\tau)(x) &= \begin{cases} H(\tau)(x) & \text{if for all } [p] \in F : \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \sum_{g \in G^{\times E}} \lambda_g \delta_g(x) & \text{if for all } [p] \in F : \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else} \end{cases} \\ &= \begin{cases} \lambda_x & \text{if for all } [p] \in F : \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else,} \end{cases} \end{aligned}$$

from which we can conclude that $R^*_{flat}(\phi) = H(\tau) = R^*_{flat}(H(\tau)) = S(\tau)$, so S is surjective.
To conclude, let $1_{\mathcal{O}} := \sum_{[\psi_x] \in \mathcal{O}_\Gamma} \delta_{[\psi_x]} \in \mathcal{O}_\Gamma$. We then have that

$$\begin{aligned} S(1_{\mathcal{O}})(x) \cdot R^*_{flat}(\phi)(x) &= \begin{cases} 1 \cdot \phi(x) & \text{if for all } [p] \in F : \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else} \end{cases} = R^*_{flat}(\phi)(x) \\ &= \begin{cases} \phi(x) \cdot 1 & \text{if for all } [p] \in F : \mathrm{Hol}_{G,p}(x) = e \\ 0 & \text{else} \end{cases} = R^*_{flat}(\phi)(x) \cdot S(1_{\mathcal{O}})(x), \end{aligned}$$

so we can conclude that $S(1_{\mathcal{O}})$ is the multiplicative unit of $\mathcal{N}_\Gamma$, which proves that S is an algebra isomorphism. □

Example 3.20 Continuing on Example 3.8, we will find $\mathcal{O}_\Gamma$ in the case $\Gamma = \Gamma_2$ as in Figure 3.2 and $G = S_3$. In order to do this, we need to examine $\mathrm{Hom}(\pi_1(\Sigma), G)$, and, specifically, we start with examining $\pi_1(\Sigma) := \langle e_1, e_2, e_3 \mid \text{for all } [p] \in F : \ \mathrm{Hol}_{F_2,p}(e_1, e_2, e_3)\rangle$. Since we know that $\mathrm{Hol}_{F_2,[(e_3)]}(e_1, e_2, e_3) = e_3$, we already know that $e_3 = e$. This yields that the only other face of Γ_2 is given by p_1 in Example 3.6, and for that face, we know that

$$\mathrm{Hol}_{F_2,p_1}(e_1, e_2, e_3) = e_1 e_2 e_1^{-1} e_3^{-1} e_2^{-1} = e_1 e_2 e_1^{-1} e_2^{-1}.$$

This implies that $\pi_1(\Sigma)$ is the group with two commuting generators of infinite order, ergo, it is isomorphic to $\mathbb{Z}^2$. Note that this is what we expected: the associated surface of Γ_2 is (homeomorphic to) a torus. Since $e_1 e_2 = e_2 e_1$, we know that $f \in \mathrm{Hom}((\pi_1(\Sigma), G)$ if and only if $f(e_1)f(e_2) = f(e_1 e_2) = f(e_2 e_1) = f(e_2)f(e_1)$. Writing $f_{x,y} \in \mathrm{Hom}((\pi_1(\Sigma), G)$ for the unique homomorphism that sends e_1 to x and e_2 to y, one can find that

$$\begin{aligned}\mathrm{Hom}(\pi_1(\Sigma), G)/G = \{&[f_{(1),(1)}], [f_{(1\ 2),(1)}], [f_{(1),(1\ 2)}], [f_{(1\ 2\ 3),(1)}], [f_{(1),(1\ 2\ 3)}], [f_{(1\ 2),(1\ 2)}],\\ &[f_{(1\ 2\ 3),(1\ 2\ 3)}], [f_{(1\ 2\ 3),(1\ 3\ 2)}]\}.\end{aligned}$$

The quantum moduli algebra $\mathcal{M}_{\Gamma_2}$ is thus isomorphic to the function algebra on the eight elements of $\mathrm{Hom}(\pi_1(\Sigma), G)/G$.

Chapter 4

Group algebra gauge theory

Definition 4.1 (Holonomy) *Given a path $p = (e_i^{\epsilon_i})_{i=1}^n$ on Γ, we define the algebra-theoretic **holonomy along** p as the multilinear map $\mathrm{Hol}_{\mathbb{F}[G],p} : \mathbb{F}[G]^{\otimes E} \to \mathbb{F}[G]$ determined by*

$$\begin{aligned} \mathrm{Hol}_{\mathbb{F}[G],p} &: \mathbb{F}[G]^{\otimes E} \to \mathbb{F}[G] \\ g_1 \otimes \cdots \otimes g_E &\mapsto g_{e_n}^{\epsilon_n} \circ \cdots \circ g_{e_1}^{\epsilon_1}. \end{aligned}$$

Definition 4.1 is until some extent a multilinear extention of Definition 3.7. Hence calculation will go accordingly, only with a multilinear extension. This implies, however, that we can make a dual map of the holonomy in the case. Before we do that, we will make an identification that will allow us to think in terms of $\mathrm{Fun}(G)$ with respect to $\mathbb{F}[G]$.

Lemma 4.2 *The spaces $\mathrm{Fun}(G)$ and $\mathbb{F}[G]^*$ are isomorphic as Hopf algebras.*

Proof. Define $U : \mathrm{Fun}(G) \to \mathbb{F}[G]^*$ by $U(\delta_g) = \delta_g$, where $g \in G$. This uniquely defines an isomorphism of Hopf algebras. □

Definition 4.3 *Given a path p, we define $\mathrm{Hol}^*_{\mathbb{F}[G],p} : \mathrm{Fun}(G) \to \mathrm{Fun}(G)^{\otimes E}$ by*

$$\mathrm{Hol}^*_{\mathbb{F}[G],p}(\varphi)(x_1 \otimes \cdots \otimes x_E) = U(\varphi)(\mathrm{Hol}_{\mathbb{F}[G],p}(x_1 \otimes \cdots \otimes x_E)),$$

where $x = (x_1, \ldots, x_E) \in G^{\times E}$.

We note that the notation in this case is actually justified: if one identifies $\mathrm{Fun}(G)$ and $\mathbb{F}[G]^*$ via the isomorphism in Lemma 4.2, then $\mathrm{Hol}^*_{\mathrm{Fun}(G),p}$ is indeed the dual map of $\mathrm{Hol}_{\mathbb{F}[G],p}$.

Lemma 4.4 *Let p be a face path. The map $P_p^* : \mathrm{Fun}(G)^{\otimes E} \to \mathrm{Fun}(G)^{\otimes E}, \alpha \mapsto \mathrm{Hol}^*_{\mathbb{F}[G],p}(\delta_e) \cdot \alpha$ is identical to $P_{p'}^*$ if $p \sim p'$.*

Proof. Unwrapping definitions gives us for $\alpha \in \mathrm{Fun}(G)^{\otimes E}$ and $x = (x_1, \ldots, x_E) \in G^{\times E}$ that

$$P_p^*(\alpha)(x) = \begin{cases} \alpha(x_1 \otimes \cdots \otimes x_E) & \text{if } \mathrm{Hol}_{\mathbb{F}[G],p}(x) = e \\ 0 & \text{else.} \end{cases}$$

Since relations remain relations under cyclic permutations, we find that $\mathrm{Hol}_{\mathbb{F}[G],p}(x) = e \Leftrightarrow \mathrm{Hol}_{\mathbb{F}[G],p'}(x) = e$. □

Note that P_p^* respects multiplication.

Lemma 4.5 *The spaces* $\mathrm{Fun}(G^{\times E})$ *and* $\mathrm{Fun}(G)^{\otimes E}$ *are isomorphic as algebras.*

Proof. Define $T : \mathrm{Fun}(G^{\times E}) \to \mathrm{Fun}(G)^{\otimes E}$ by $T(\delta_g) = \delta_{g_1} \otimes \cdots \otimes \delta_{g_E}$, where $g = (g_1, \ldots, g_E)$. This uniquely defines an isomorphism of algebras. □

This isomorphism plays a crucial role in the identification of $\mathcal{O}_\Gamma$ and the quantum moduli algebra in the group algebra case, and, therefore, has to be investigated more thoroughly. A useful little fact concerning calculation with T is phrased in Corollary 4.6.

Note that for $\omega = \sum_{j\in J} \lambda_j \omega_{1j} \otimes \cdots \otimes \omega_{Ej} \in \mathrm{Fun}(G)^{\otimes E}$, J an index set, we write

$$\omega(x_1 \otimes \cdots \otimes x_E) := \sum_{j\in J} \lambda_j \omega_{1j}(x_1) \otimes \cdots \otimes \omega_{Ej}(x_E),$$

where $x = (x_1, \ldots, x_E) \in G^{\times E}$.

Corollary 4.6 *For* $x = (x_1, \ldots, x_E) \in G^{\times E}$ *and* $\varphi = \sum_{g=(g_1,\ldots,g_E)\in G^{\times E}} \lambda_g \delta_g \in \mathrm{Fun}(G^{\times E})$, *we have that*

$$T(\varphi)(x_1 \otimes \cdots \otimes x_E) = \varphi(x) \cdot (1 \otimes \cdots \otimes 1).$$

Proof. Writing out yields

$$\begin{aligned}
T(\varphi)(x_1 \otimes \cdots \otimes x_E) &= T\left(\sum_{g=(g_1,\ldots,g_E)\in G^{\times E}} \lambda_g \delta_g\right)(x_1 \otimes \cdots \otimes x_E) \\
&= \sum_{g=(g_1,\ldots,g_E)\in G^{\times E}} \lambda_g T(\delta_g)(x_1 \otimes \cdots \otimes x_E) \\
&= \sum_{g=(g_1,\ldots,g_E)\in G^{\times E}} \lambda_g (\delta_{g_1} \otimes \cdots \otimes \delta_{g_E})(x_1 \otimes \cdots \otimes x_E) \\
&= \lambda_x \cdot (1 \otimes \cdots \otimes 1) = \sum_{g=(g_1,\ldots,g_E)\in G^{\times E}} \lambda_g \delta_g(x) \cdot (1 \otimes \cdots \otimes 1) \\
&= \varphi(x) \cdot (1 \otimes \cdots \otimes 1).
\end{aligned}$$

□

Theorem 4.7 *The diagram*

$$\begin{array}{ccc}
\mathrm{Fun}(G^{\times E}) & \xrightarrow{\quad T \quad} & \mathrm{Fun}(G)^{\otimes E} \\
\big\downarrow R_p^* & & \big\downarrow P_p^* \\
R_p^*(\mathrm{Fun}(G^{\times E})) & \xrightarrow{\ T\ } & P_p^*(\mathrm{Fun}(G)^{\otimes E})
\end{array}$$

commutes.

Proof. Plugging in $\varphi \in \mathrm{Fun}(G^{\times E})$ and $x = x_1 \otimes \cdots \otimes x_E$, where $x_i \in G$, $1 \leq i \leq E$, yields

$$P_p^*(T(\varphi))(x) = \begin{cases} T(\varphi)(x) & \text{if } \mathrm{Hol}_{\mathbb{F}[G],p}(x) = e \\ 0 & \text{else.} \end{cases}$$

We also have that

$$T(R_p^*(\varphi))(x) = T(\mathrm{Hol}_{G,p}^*(\eta) \cdot \varphi)(x) = T(\mathrm{Hol}_{G,p}^*(\eta))(x) \cdot T(\varphi)(x),$$

and we know that

$$T(\mathrm{Hol}^*_{G,p}(\eta))(x) = \mathrm{Hol}^*_{G,p}(\eta)(x_1,\ldots,x_E)\cdot(1\otimes\cdots\otimes 1) = \eta(\mathrm{Hol}_{G,p}(x_1,\ldots,x_E))\cdot(1\otimes\cdots\otimes 1)$$
$$= \begin{cases} 1\otimes\cdots\otimes 1 & \text{if } \mathrm{Hol}_{G,p}(x_1,\ldots,x_E)=e \\ 0 & \text{else}\end{cases} = \begin{cases} 1\otimes\cdots\otimes 1 & \text{if } \mathrm{Hol}_{\mathbb{F}[G],p}(x)=e \\ 0 & \text{else.}\end{cases}$$

From this we obtain that

$$T(R^*_p(\varphi))(x) = T(\mathrm{Hol}^*_{G,p}(\eta))(x)\cdot T(\varphi)(x) = \begin{cases} T(\varphi)(x)\cdot(1\otimes\cdots\otimes 1) & \text{if } \mathrm{Hol}_{\mathbb{F}[G],p}(x)=e \\ 0 & \text{else.}\end{cases}$$
$$= \begin{cases} T(\varphi)(x) & \text{if } \mathrm{Hol}_{\mathbb{F}[G],p}(x)=e \\ 0 & \text{else.}\end{cases}$$

□

Definition 4.8 *Let* $\varphi = \varphi_1\otimes\cdots\otimes\varphi_E \in \mathrm{Fun}(G)^{\otimes E}$ *and* $x = (x_1,\ldots,x_E)\in G^{\times E}$*. Then* G *acts on* $\mathrm{Fun}(G)^{\otimes E}$ *as conjugation by*

$$g(\varphi)(x_1\otimes\cdots\otimes x_E) := \varphi_1(gx_1g^{-1})\otimes\cdots\otimes\varphi_E(gx_Eg^{-1}),$$

satisfying the relations $g(\varphi_1+\varphi_2) = g(\varphi_1)+g(\varphi_2)$ *and* $g(\lambda\varphi_1) = \lambda\cdot g(\varphi_1)$ *for* $\varphi_1,\varphi_2\in\mathrm{Fun}(G)^{\otimes E}$ *and* $\lambda\in\mathbb{F}$.

For the remainder of this chapter, assume that Γ has only one vertex.

Definition 4.9 *The* ***algebra-theoretic algebra of invariant functions*** *is defined as*

$$\mathrm{Fun}_{inv}(G)^{\otimes E} := \{\varphi\in\mathrm{Fun}(G)^{\otimes E} \mid \textit{for all } g\in G:\ g(\varphi)=\varphi\}\subset\mathrm{Fun}(G)^{\otimes E}.$$

We note that $\mathrm{Fun}_{inv}(G)^{\otimes E}$ is indeed a subalgebra of $\mathrm{Fun}(G)^{\otimes E}$: as a matter of fact, this is proven in Chapter 5 in general.

Corollary 4.10 *The diagram in Theorem 4.7 induces a commutative diagram*

$$\begin{array}{ccc} \mathrm{Fun}_{inv}(G^{\times E}) & \xrightarrow{T} & \mathrm{Fun}_{inv}(G)^{\otimes E} \\ \downarrow{\scriptstyle R^*_p} & & \downarrow{\scriptstyle P^*_p} \\ R^*_p(\mathrm{Fun}_{inv}(G^{\times E})) & \xrightarrow{T} & P^*_p(\mathrm{Fun}_{inv}(G)^{\otimes E}) \end{array}.$$

Proof. Since $\mathrm{Fun}_{inv}(G^{\times E})\subset\mathrm{Fun}(G^{\times E})$ and $\mathrm{Fun}_{inv}(G)^{\otimes E}\subset\mathrm{Fun}(G)^{\otimes E}$, we only need to show that $T(\mathrm{Fun}_{inv}(G^{\times E}))\subset\mathrm{Fun}_{inv}(G)^{\otimes E}$. So let $\varphi\in\mathrm{Fun}_{inv}(G^{\times E})$ and $x=(x_1,\ldots,x_E)\in G^{\times E}$ and $g\in G$. Using Corollary 4.6, we find that

$$T(\varphi)(x_1\otimes\cdots\otimes x_E) = \varphi(x_1,\ldots,x_E)\cdot(1\otimes\cdots\otimes 1) = \varphi(gx_1g^{-1},\ldots,gx_Eg^{-1})\cdot(1\otimes\cdots\otimes 1)$$
$$= T(\varphi)(gx_1g^{-1}\otimes\cdots\otimes gx_Eg^{-1}) = g(T(\varphi))(x_1\otimes\cdots\otimes x_E),$$

so $T(\varphi)\in\mathrm{Fun}_{inv}(G)^{\otimes E}$. □

Corollary 4.11 *The map* $T:\mathrm{Fun}_{inv}(G^{\times E})\to\mathrm{Fun}_{inv}(G)^{\otimes E}$ *is an algebra isomorphism.*

Proof. In Corollary 4.10 we already showed that $T : \mathrm{Fun}_{inv}(G^{\times E}) \to \mathrm{Fun}_{inv}(G)^{\otimes E}$ is an injective algebra morphism. To conclude it is surjective, we only need to prove that $\mathrm{Fun}_{inv}(G)^{\otimes E} \subset T(\mathrm{Fun}_{inv}(G^{\times E}))$. So let $\rho = \rho_1 \otimes \cdots \otimes \rho_E \in \mathrm{Fun}_{inv}(G)^{\otimes E}$. Then we define $\varphi \in \mathrm{Fun}_{inv}(G^{\times E})$ uniquely by the relation $\rho(x_1 \otimes \cdots \otimes x_E) = \rho_1(x_1) \otimes \cdots \otimes \rho_E(x_E) = \varphi(x) \cdot (1 \otimes \cdots \otimes 1)$, where $x = (x_1, \ldots, x_E) \in G^{\times E}$. To see that $\varphi \in \mathrm{Fun}_{inv}(G^{\times E})$, note that, for $g \in G$,

$$\begin{aligned}\varphi(x_1, \ldots, x_E) \cdot (1 \otimes \cdots \otimes 1) = \rho_1(x_1) \otimes \cdots \otimes \rho_E(x_E) &= \rho_1(gx_1g^{-1}) \otimes \cdots \otimes \rho_E(gx_Eg^{-1}) \\ &= \varphi(gx_1g^{-1}, \ldots, gx_Eg^{-1}) \cdot (1 \otimes \cdots \otimes 1),\end{aligned}$$

so $\varphi(x_1, \ldots, x_E) = \varphi(gx_1g^{-1}, \ldots, gx_Eg^{-1})$, so $\varphi \in \mathrm{Fun}_{inv}(G^{\times E})$. Note then that by Corollary 4.6, $T(\varphi)(x_1 \otimes \cdots \otimes x_E) = \varphi(x) \cdot (1 \otimes \cdots \otimes 1) = \rho_1(x_1) \otimes \cdots \otimes \rho_E(x_E) = \rho(x_1 \otimes \cdots \otimes x_E)$, and, thus, $T(\varphi) = \rho$. □

Corollary 4.12 *The algebras $R_p^*(\mathrm{Fun}_{inv}(G^{\times E})))$ and $P_p^*(\mathrm{Fun}_{inv}(G)^{\otimes E})$ are isomorphic.*

Proof. We know from Corollary 4.10 that $T : R_p^*(\mathrm{Fun}_{inv}(G^{\times E})) \to P_p^*(\mathrm{Fun}_{inv}(G)^{\otimes E})$ is an injective algebra morphism. To see that it is surjective, we only need to note that from Corollary 4.11, $T \circ P_p^*$ is surjective in the diagram in Corollary 4.10 and that the diagram in Corollary 4.10 commutes. □

Now let Γ be of type (3,2).

Definition 4.13 (Moduli algebra) *The **moduli algebra** of Γ is defined as the image of the $P^*_{flat} := \prod_{[f]\in F} P_f^*$ of the algebra-theoretic algebra of invariant functions, or, in other words, $\mathcal{M}_\Gamma = P^*_{flat}(\mathrm{Fun}_{inv}(G)^{\otimes E})$.*

Theorem 4.14 *The algebras $\mathcal{N}_\Gamma$ and $\mathcal{M}_\Gamma$ (and thus $\mathcal{O}_\Gamma$) are isomorphic.*

Proof. This follows immediately from Corollary 4.12 and the fact that E has finite cardinality. □

This identification is vital for the understanding of this chapter, as it provides a simpler approach to the moduli algebra, and helps us therefore to find even more comprehensible examples.

Chapter 5

General construction of moduli algebras

In this chapter, we will give the general construction to create the moduli algebra of an arbitrary ribbon graph Γ and an arbitrary semisimple finite-dimensional Hopf algebra K as was introduced by [2].

In this chapter, we will assume that Γ has only one vertex v. Furthermore, we define the function $\tau : \{1, \ldots, 2E\} \to \{0, 1\}$ by $\tau(i) = 1$ if there exists a $e_j \in E$ such that $c_v(e_j) = i$ and $\tau(i) = 0$ if there exists a $e_j \in E$ such that $c_v(e_j^{-1}) = i$. We can also view τ as the map that sends all of the edge ends around v to 0 if the edge end is incoming, and to 1 if the edge end is outgoing.

In order to define gauge invariance and holonomy on a ribbon graph, we need to define the linear map

$$\begin{aligned} G^* : {K^*}^{\otimes E} &\to {K^*}^{\otimes 2E} \\ (\alpha)_{e_i} &\mapsto (\alpha_{(2)} \otimes \alpha_{(1)})_{c_v(e_i)c_v(e_i^{-1})}. \end{aligned}$$

We will create an algebra structure on ${K^*}^{\otimes 2E}$ in order for this map G^* is to become an injective algebra morphism.

Lemma 5.1 *The multiplication*

$$(\alpha)_i \cdot (\beta)_i = \begin{cases} \langle \beta_{(1)} \otimes \alpha_{(1)}, R \rangle (\beta_{(2)} \alpha_{(2)})_i & \text{if } \tau(i) = 0 \\ (\beta\alpha)_i & \text{if } \tau(i) = 1 \end{cases}$$

$$(\alpha)_i \cdot (\beta)_j = \begin{cases} \langle \beta_{(1+\tau(j))} \otimes \alpha_{(1+\tau(i))}, R \rangle (\alpha_{(2-\tau(i))} \otimes \beta_{(2-\tau(j))})_{ij} & \text{if } i > j \\ (\alpha \otimes \beta)_{ij} & \text{if } i < j \end{cases}$$

on ${K^*}^{\otimes 2E}$ *with* $\alpha, \beta \in K^*$ *defines a multiplication* $\cdot_\Gamma : {K^*}^{\otimes E} \otimes {K^*}^{\otimes E} \to {K^*}^{\otimes E}$ *by*

$$\cdot_\Gamma(\zeta, \theta) = \zeta \cdot_\Gamma \theta := {G^*}^{-1}(G^*(\zeta) \cdot G^*(\theta)),$$

that is, we can pullback the multiplication structure on ${K^*}^{\otimes 2E}$ *to* ${K^*}^{\otimes E}$*. Here,* $R \in K \otimes K$ *is an multiplicative invertible element such that*

- *$R \cdot \Delta(h) \cdot R^{-1} = \tau' \circ \Delta(h)$ for all $h \in K$;*

- *$(\Delta \otimes id)R = (R)_{13} \cdot (R)_{12}$, and*

- *$(id \otimes \Delta)R = (R)_{23} \cdot (R)_{13}$,*

where τ' is the map τ from Figure 2.3 and in which we extend the notation from Chapter 2 in a natural manner.

Proof. The proof of this lemma is given by [2]. Note that ${G^*}^{-1}$ is in general not well defined, but this problem is solved by noting from axiom (2) from Definition 2.5 that comultiplication is always injective. □
For $K = \mathbb{F}[G]$ and $K = \mathcal{D}(\mathbb{F}[G])$, a possible element R is given by Example 2.7 and 2.9 respectively.
Note that $K^{\otimes E}$ has, being a tensor product of Hopf algebras, a Hopf algebra structure (every operation can be applied componentwise), and hence there is an induced comultiplication Δ_{ind} : $K^{\otimes E} \to K^{\otimes E} \otimes K^{\otimes E}$.*

Definition 5.2 (Holonomy) *Given a path $p = (e'^{\epsilon'})$ on Γ, we define the **holonomy along** p as the multilinear map $\mathrm{Hol}_{K,p} : K^{\otimes E} \to K$ determined by*

$$\begin{aligned} \mathrm{Hol}_{K,p} &: K^{\otimes E} \to K \\ k_1 \otimes \cdots \otimes k_E &\mapsto \left(\prod_{f \in E \setminus \{e'\}} \epsilon(k_f) \right) k_{e'}^{\epsilon'} \end{aligned}$$

where $k^{-1} := S(k)$ for all $k \in K$.
*Given a path $p = (e_i^{\epsilon_i})_{i=1}^n$ on Γ, we define the **holonomy along** p as the multilinear map $\mathrm{Hol}_{K,p} : K^{\otimes E} \to K$ determined by*

$$\begin{aligned} &\mathrm{Hol}_{K,p} : K^{\otimes E} \to K \\ &k_1 \otimes \cdots \otimes k_E \mapsto \mathrm{Hol}_{K,(e_n^{\epsilon_n})}((k_1 \otimes \cdots \otimes k_E)_{(n)}) \cdots \mathrm{Hol}_{K,(e_1^{\epsilon_1})}((k_1 \otimes \cdots \otimes k_E)_{(1)}). \end{aligned}$$

The comultiplication used in the above expression is Δ_{ind}.

Again, it is crucial to define the dual of the holonomy, since this will continue to be a necessary ingredient for constructing the algebra of observables.

Definition 5.3 *Given a path p, we define $\mathrm{Hol}^*_{K,p} : K^* \to {K^*}^{\otimes E}$ by*

$$\mathrm{Hol}^*_{K,p}(\varphi)(k_1 \otimes \cdots \otimes k_E) = \varphi(\mathrm{Hol}_{K,p}(k_1 \otimes \cdots \otimes k_E)),$$

where $k = (k_1, \ldots, k_E) \in K^{\times E}$.

It is clear that $\mathrm{Hol}^*_{K,p}$ is the dual of $\mathrm{Hol}_{K,p}$, and, thus, a linear map.

Definition 5.4 *The **Haar integral on** K is the unique element $\eta' \in K^*$ such that $h \cdot \eta' = \eta' \cdot h = \eta^*(h)\eta'$ for all $h \in K^*$ and $\eta^*(\eta') = 1$, where η^* is the counit map on K^*.*

*The multiplication $\cdot_\Gamma$ in Lemma 5.1 defines by the non-degenerate pairing $\langle\ ,\ \rangle : K^* \otimes K \to \mathbb{F}$ another comultiplication structure on $K^{\otimes E}$. This comultiplication structure on $K^{\otimes E}$ results, coincidentally, in an identical definition of the holonomy as the induced comultiplication structure [2].

The fact that the Haar integral is unique is given by [2].

Lemma 5.5 *Let p be a face path. The projecting map $P_p^* : K^{*^{\otimes E}} \to K^{*^{\otimes E}}, \alpha \mapsto \mathrm{Hol}^*_{K,p}(\eta) \cdot \alpha$ is identical to $P_{p'}^*$ if $p \sim p'$.*

Proof. This is proven in [2], together with the fact that P_p^* respects multiplication. □
Note that we sometimes also write $P_{K,p}^*$ instead of P_p^*, in order to prevent confusion.

We define an action on $K^{*^{\otimes 2E}}$, which we can then pullback to find a suited action on $K^{*^{\otimes E}}$ to determine gauge invariance.

Lemma 5.6 *The formula*

$$(\alpha_1 \otimes \cdots \otimes \alpha_{2E}) \triangleleft^* k = \langle S^{\tau(1)}(\alpha_{1_{(1+\tau(1))}}) \cdots S^{\tau(2E)}(\alpha_{2E_{(1+\tau(2E))}}), h\rangle \alpha_{1_{(2-\tau(1))}} \otimes \cdots \otimes \alpha_{2E_{(2-\tau(2E))}}$$

for $k \in K$ defines a K-right module algebra structure on $K^{*\otimes 2E}$, (that is, $\triangleleft^*$ gives a module structure, and for all $\varphi, \varphi' \in K^{*\otimes 2E}$ and $k \in K$, we have that $(\varphi \cdot \varphi') \triangleleft^* k = (\varphi \triangleleft^* k_{(1)}) \cdot (\varphi' \triangleleft^* k_{(2)})$.) We will denote the pulled back module algebra structure by $\triangleleft^*_\Gamma$ or by $\triangleleft^*_{K,\Gamma}$.

Proof. This is given by [2]. □

Definition 5.7 *The* ***algebra of invariant functions*** *is defined as*

$$K_{inv}^{*^{\otimes E}} := \{\varphi \in K^{*^{\otimes E}} \mid \text{for all } k \in K : \varphi \triangleleft^*_\Gamma k = G^{*^{-1}}(G^*(\varphi) \triangleleft^* k) = \epsilon(k)\varphi\} \subset K^{*^{\otimes E}}.$$

We note that $K_{inv}^{*^{\otimes E}}$ is indeed a subalgebra of $K^{*^{\otimes E}}$: since all operations are linear, we only need to check if the algebra if closed under multiplication. For all $\varphi, \varphi' \in K_{inv}^{*^{\otimes E}}$ and $k \in K$, we have that

$$\begin{aligned}
(\varphi \cdot \varphi') \triangleleft^*_\Gamma k &= G^{*^{-1}}(G^*(\varphi) \cdot G^*(\varphi')) \triangleleft^*_\Gamma k = G^{*^{-1}}(G^*(\varphi) \cdot G^*(\varphi') \triangleleft^* k) \\
&= G^{*^{-1}}\left((G^*(\varphi) \triangleleft^* k_{(1)}) \cdot (G^*(\varphi') \triangleleft^* k_{(2)})\right) \\
&= G^{*^{-1}}\left(G^*(\epsilon(k_{(1)})\varphi) \cdot G^*(\epsilon(k_{(2)})\varphi')\right) \\
&= \epsilon(k_{(1)})\varphi \cdot \epsilon(k_{(2)})\varphi' = \epsilon(k_{(1)})\epsilon(k_{(2)})\varphi \cdot \varphi' = \epsilon(k)\varphi \cdot \varphi'
\end{aligned}$$

by axiom (2) of Definition 2.5.
Now let Γ be of type (3,2).

Definition 5.8 (Quantum moduli algebra) *The* ***quantum moduli algebra*** *of Γ belonging to K is defined as the image of the $P_{flat}^* := \prod_{[f]\in F} P_f^*$ of the algebra-theoretic algebra of invariant functions, or, in other words, $\mathcal{M}_\Gamma = P_{flat}^*(K_{inv}^{*^{\otimes E}})$. We also write $\mathcal{M}_\Gamma = \mathcal{M}_{K,\Gamma}$.*

One important feature of these moduli algebras is that they are topologically invariant. The next theorem states this more conretely.

Theorem 5.9 *Let Γ and Γ' be two ribbon graphs of type (3,2). If the associated compact oriented 2-dimensional manifolds Σ_Γ and $\Sigma_{\Gamma'}$ are homeomorphic, then $\mathcal{M}_\Gamma$ and $\mathcal{M}_{\Gamma'}$ are isomorphic as algebras.*

Proof. This is proven in [2].
To motivate that the definition of a moduli algebra is valid, and to connect this definition to the previous chapters, we now prove that in the case $K = \mathbb{F}[G]$, the two given constructions are identical.

Theorem 5.10 *The moduli algebra $\mathcal{M}_\Gamma$ from Definition 4.13 equals $\mathcal{L}_\Gamma = P^*_{flat}(K^{*\otimes E}_{inv})$ from Definition 5.8.*

Proof. We recall from Example 2.7 that $R = e \otimes e \in \mathbb{F}[G] \otimes \mathbb{F}[G]$. Then the comultiplication from Example 2.7 gives us that

$$\Delta_{ind}(k_1 \otimes \cdots \otimes k_E) = (k_1 \otimes \cdots \otimes k_E) \otimes (k_1 \otimes \cdots \otimes k_E)$$

for $k_i \in K$, $i \in \{1, \ldots, E\}$. Using this comultiplication and the fact that $\epsilon(g) = 1$ and $S(g) = g^{-1}$ for $g \in G$, we find that the holonomy along a path $p = (e_i^{\epsilon_i})_{i=1}^n$ is according to Definition 5.2 given by

$$\begin{aligned}
\mathrm{Hol}_{\mathbb{F}[G],p} &: \mathbb{F}[G]^{\otimes E} \to \mathbb{F}[G] \\
g_1 \otimes \cdots \otimes g_E &\mapsto \mathrm{Hol}_{\mathbb{F}[G],(e_n)}((g_1 \otimes \cdots \otimes g_E)_{(n)}) \cdots \mathrm{Hol}_{G,(e_1)}((g_1 \otimes \cdots \otimes k_E)_{(1)}) \\
&= \mathrm{Hol}_{\mathbb{F}[G],(e_n)}(g_1 \otimes \cdots \otimes g_E) \cdots \mathrm{Hol}_{G,(e_1)}(g_1 \otimes \cdots \otimes k_E) \\
&= \left(\prod_{f \in E \setminus \{e_n\}} \epsilon(g_f) \right) g_{e_n}^{\epsilon_n} \cdots \left(\prod_{f' \in E \setminus \{e_1\}} \epsilon(g_{f'}) \right) g_{e_1}^{\epsilon_1} \\
&= \left(\prod_{f \in E \setminus \{e_n\}} 1 \right) g_{e_n}^{\epsilon_n} \cdots \left(\prod_{f' \in E \setminus \{e_1\}} 1 \right) g_{e_1}^{\epsilon_1} \\
&= 1 \cdot g_{e_n}^{\epsilon_n} \cdots 1 \cdot g_{e_1}^{\epsilon_1} = g_{e_n}^{\epsilon_n} \cdots g_{e_1}^{\epsilon_1},
\end{aligned}$$

so we retrieve the same map as in Definition 4.1.
Furthermore, we obtain for $g, h \in G$ and $i \in \{1, \ldots, n\}$ from Example 2.8 that

$$\begin{aligned}
(\delta_g)_i \lhd^*_\Gamma h &= G^{*-1}(G^*((\delta_g)_i) \lhd^* h) \\
&= G^{*-1}\left(\sum_{w \in G} (\delta_w \otimes \delta_{w^{-1}g})_{c_v(e_i^{-1})c_v(e_i)} \lhd^* h \right) \\
&= G^{*-1}\left(\sum_{u,x,w \in G} \langle \delta_u \cdot S(\delta_{x^{-1}w^{-1}g}), h \rangle (\delta_{u^{-1}w} \otimes \delta_x)_{c_v(e_i^{-1})c_v(e_i)} \right) \\
&= G^{*-1}\left(\sum_{u,x,w \in G} \delta_u(h) \cdot \delta_{g^{-1}wx}(h) \delta_{u^{-1}w} \otimes \delta_x)_{c_v(e_i^{-1})c_v(e_i)} \right) \\
&= G^{*-1}\left(\sum_{w \in G} \delta_{h^{-1}w} \otimes \delta_{w^{-1}gh})_{c_v(e_i^{-1})c_v(e_i)}) \quad (u = h, mx = w^{-1}gh \right) \\
&= G^{*-1}\left(\sum_{w \in G} \delta_{h^{-1}w} \otimes \delta_{(h^{-1}w)^{-1}h^{-1}gh})_{c_v(e_i^{-1})c_v(e_i)} \right) = (\delta_{h^{-1}gh})_i,
\end{aligned}$$

and this gives for $h_i, x_i, g \in G$, $i \in \{1, \ldots, n\}$ that

$$\left((\bigotimes_{i=1}^E \delta_{h_i}) \lhd^* g \right) (\bigotimes_{i=1}^E x_i) = \left(\bigotimes_{i=1}^E \delta_{g^{-1}h_i g} \right) (\bigotimes_{i=1}^E x_i) = \bigotimes_{i=1}^E \delta_{g^{-1}h_i g}(x_i) = \bigotimes_{i=1}^E \delta_{h_i}(g x_i g^{-1}),$$

which coincides with Definition 4.8.
If we note that $\epsilon(g) = 1$ for all $g \in G$, and that δ_e is the Haar integral of $\mathbb{F}[G]$, we can conclude that both constructions arise from the same maps, and are hence equal. ☐

Chapter 6

Drinfel'd double gauge theory

Now we move on to a more complex example: the Drinfel'd double. The structure of this type of Hopf algebra is given in Example 2.9. We will construct the algebra of observables using the general construction of moduli algebras as outlined above. For the sake of clear notation, we will write $K = \mathcal{D}(\mathbb{F}[G])$. In our examples, we will show that the condition that every graph needs to have only vertices with a valence larger than two and more faces than one cannot be made redundant. At first, though, some general results about the Drinfel'd double are stated.

Lemma 6.1 *The* ***Haar integral on*** $\mathrm{Fun}(G)$ *is given by* $\ell = \sum_{g \in G} ev_g \in \mathrm{Fun}(G)^*$*, where* $ev_g : \mathrm{Fun}(G) \to \mathbb{F}, \varphi \mapsto \varphi(g)$ *is the evaluation function.*

Proof. This is given by [2]. □

Lemma 6.2 *The Haar integral on* K *is given by* $\ell \otimes \eta$*, where* $\eta = \delta_e$*.*

Proof. This is given by [2]. □

Furthermore, we note that the multiplication on K^* can be found in Appendix A.

6.1 Example 1: the one-edged ribbon graph

In this case, we will be studying the ribbon graph with one edge. The corresponding ribbon graph is given in Figure 6.1.

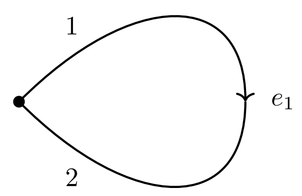

Figure 6.1: *An example of a ribbon graph, which will be called* Γ_0.

In this graph, the set of all faces given by $\{[(e_1)], [(e_1^{-1}]\}$.

Lemma 6.3 (Holonomy) *For the ribbon graph* Γ_0*, the holonomy along* $p = (e_1^{\epsilon_1})$ *is given by*

$$\begin{aligned} \mathrm{Hol}_{K,(e_1^{\epsilon_1})} : K &\to K \\ k_1 &\mapsto k_{e_1}^{\epsilon_1}, \end{aligned}$$

where $k^{-1} := S(k)$ *for all* $k \in K$.

Proof. This is clear from Definition 5.2. □

Lemma 6.4 *Let* p *be a face path (i.e.* $p = (e_1^{\epsilon_1})$*,* $\epsilon_1 \in \{\pm 1\}$*). The projecting map* $P_p^* : K^* \to K^*, \alpha \mapsto \mathrm{Hol}_{K,p}^*(\ell \otimes \eta) \cdot \alpha$ *is identical to* $P_{p'}^*$ *if* $p \sim p'$.

Proof. This is trivial, since the equivalence classes of the face paths only consist out of one face path each. □

Unwrapping definitions gives us for P_p^* and $g, h \in G$ that

$$\begin{aligned} P_p^*(\alpha)(x) &= \mathrm{Hol}_{K,p}^*(\ell \otimes \eta)(\delta_h \otimes g) \cdot \alpha(x) = (\ell \otimes \eta)(\delta_h \otimes g) \cdot \alpha(x) \\ &= \left(\sum_{u \in G} \mathrm{ev}_u(\delta_h) \otimes \delta_e(g) \right) \cdot \alpha(x) \\ &= \left(\sum_{u \in G} \delta_h(u) \otimes \delta_e(g) \right) \cdot \alpha(x) = (1 \otimes \delta_e(g)) \cdot \alpha(x) \end{aligned}$$

if $p = (e_1)$ and

$$\begin{aligned} P_p^*(\alpha)(x) &= \mathrm{Hol}_{K,p}^*(\ell \otimes \eta)(\delta_h \otimes g)\alpha(x) = ((\ell \otimes \eta)(S(\delta_h \otimes g))) \cdot \alpha(x) \\ &= \left(\sum_{u \in G} \mathrm{ev}_u(\delta_{g^{-1}hg}) \otimes \delta_e(g^{-1}) \right) \cdot \alpha(x) \\ &= \left(\sum_{u \in G} \delta_{g^{-1}hg}(u) \otimes \delta_e(g^{-1}) \right) \cdot \alpha(x) = \left(1 \otimes \delta_e(g^{-1})\right) \cdot \alpha(x) \end{aligned}$$

if $p = (e_1^{-1})$. Note that P_p^* respects multiplication in both cases.

In the subsequent part of the example, we make use of the vertex neighbourhood in Figure 6.2.

In our example, we have that $\tau(1) = 1$ and that $\tau(2) = 0$. For the sake of simplifying the argument, we will first do the technical calculation.

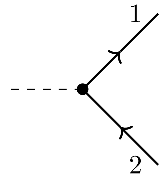

Figure 6.2: *The vertex neighbourhood that we will be using in our calculation.*

Lemma 6.5 *In our setup we have, for $g, h, x_1, x_2, y_1, y_2 \in G$, using the action in Lemma 5.6,*

$$(ev_{x_1} \otimes \delta_{y_1}) \otimes (ev_{x_2} \otimes \delta_{y_2}) \lhd^* (\delta_h \otimes g) \\ = \delta_h(x_2 y_1^{-1} x_1^{-1} y_1)(ev_{x_1} \otimes \delta_{y_1 g}) \otimes (ev_{g^{-1}x_2 g} \otimes \delta_{g^{-1}y_2}), \quad (6.1)$$

where $ev_v \in \mathbb{F}[G]^{**}$ is the evaluation function of $v \in \mathbb{F}[G]$.

Proof. Writing the definitions out using the calculation rules in Section A yields

$$\begin{aligned}
&(\mathrm{ev}_{x_1} \otimes \delta_{y_1}) \otimes (\mathrm{ev}_{x_2} \otimes \delta_{y_2}) \lhd^* (\delta_h \otimes g) \\
&= \langle S((\mathrm{ev}_{x_1} \otimes \delta_{y_1})_{(2)}) \cdot (\mathrm{ev}_{x_2} \otimes \delta_{y_2})_{(1)}, \delta_h \otimes g\rangle((\mathrm{ev}_{x_1} \otimes \delta_{y_1})_{(1)}) \otimes ((\mathrm{ev}_{x_2} \otimes \delta_{y_2})_{(2)}) \\
&= \sum_{(u,v)\in G^{\times 2}} \langle S(\mathrm{ev}_{u^{-1}x_1 u} \otimes \delta_{u^{-1}y_1}) \cdot (\mathrm{ev}_{x_2} \otimes \delta_v), \delta_h \otimes g\rangle(\mathrm{ev}_{x_1} \otimes \delta_u) \otimes (\mathrm{ev}_{v^{-1}x_2 v} \otimes \delta_{v^{-1}y_2}) \\
&= \sum_{(u,v)\in G^{\times 2}} \langle(\mathrm{ev}_{y_1^{-1}uu^{-1}x_1^{-1}uu^{-1}y_1} \otimes \delta_{y_1^{-1}u}) \cdot (\mathrm{ev}_{x_2} \otimes \delta_v), \delta_h \otimes g\rangle(\mathrm{ev}_{x_1} \otimes \delta_u) \otimes (\mathrm{ev}_{v^{-1}x_2 v} \otimes \delta_{v^{-1}y_2}) \\
&= \sum_{(u,v)\in G^{\times 2}} \langle(\mathrm{ev}_{y_1^{-1}x_1^{-1}y_1} \otimes \delta_{y_1^{-1}u}) \cdot (\mathrm{ev}_{x_2} \otimes \delta_v), \delta_h \otimes g\rangle(\mathrm{ev}_{x_1} \otimes \delta_u) \otimes (\mathrm{ev}_{v^{-1}x_2 v} \otimes \delta_{v^{-1}y_2}) \\
&= \sum_{(u,v)\in G^{\times 2}} \langle \mathrm{ev}_{x_2 y_1^{-1}x_1^{-1}y_1} \otimes \delta_{y_1^{-1}u}\delta_v), \delta_h \otimes g\rangle(\mathrm{ev}_{x_1} \otimes \delta_u) \otimes (\mathrm{ev}_{v^{-1}x_2 v} \otimes \delta_{v^{-1}y_2}) \\
&= \sum_{(u,v)\in G^{\times 2}} \delta_h(x_2 y_1^{-1} x_1^{-1} y_1) \otimes \delta_{y_1^{-1}u}(g)\delta_v(g)(\mathrm{ev}_{x_1} \otimes \delta_u) \otimes (\mathrm{ev}_{v^{-1}x_2 v} \otimes \delta_{v^{-1}y_2}) \\
&= \sum_{u\in G} \delta_h(x_2 y_1^{-1} x_1^{-1} y_1) \otimes \delta_{y_1^{-1}u}(g)(\mathrm{ev}_{x_1} \otimes \delta_u) \otimes (\mathrm{ev}_{g^{-1}x_2 g} \otimes \delta_{g^{-1}y_2}) \quad (v = g) \\
&= \delta_h(x_2 y_1^{-1} x_1^{-1} y_1)(\mathrm{ev}_{x_1} \otimes \delta_{y_1 g}) \otimes (\mathrm{ev}_{g^{-1}x_2 g} \otimes \delta_{g^{-1}y_2}). \quad (u = y_1 g)
\end{aligned}$$

□

Using Lemma 6.5, we then find for $x, y, a, b \in G$, using the relations in Appendix A that

$$\begin{aligned}
G^*(\mathrm{ev}_x \otimes \delta_y) \lhd^* (\delta_a \otimes b) &= \sum_{u\in G} (\mathrm{ev}_x \otimes \delta_u) \otimes (\mathrm{ev}_{u^{-1}xu} \otimes \delta_{u^{-1}y}) \lhd^* (\delta_a \otimes b) \\
&= \sum_{u\in G} \delta_a(u^{-1}xuu^{-1}x^{-1}u)(\mathrm{ev}_x \otimes \delta_{ub}) \otimes (\mathrm{ev}_{b^{-1}u^{-1}xub} \otimes \delta_{b^{-1}u^{-1}y}) \\
&= \sum_{v\in G} \delta_a(e)(\mathrm{ev}_x \otimes \delta_v) \otimes (\mathrm{ev}_{v^{-1}xv} \otimes \delta_{v^{-1}y}) = \delta_a(e) G^*(\mathrm{ev}_x \otimes y),
\end{aligned}$$

from which we can conclude that

$$\begin{aligned}
K^{*^{\otimes E}}_{inv} &= \{\varphi \in K^{*^{\otimes E}} \mid \text{for all } a, b \in G : G^{*^{-1}}(G^*(\varphi) \lhd^* (\delta_a \otimes b)) = \epsilon((\delta_a \otimes b))\varphi\} \\
&= \{\varphi \in K^{*^{\otimes E}} \mid \text{for all } a, b \in G : G^{*^{-1}}(G^*(\varphi) \lhd^* (\delta_a \otimes b)) = \delta_e(a)\varphi\} = K^{*^{\otimes E}}.
\end{aligned}$$

This gives us that

$$\begin{aligned}\mathcal{M}_{K,\Gamma_0} &= P^*_{flat}(K^{*\otimes E}_{inv}) = P^*_{flat}(K^{*\otimes E}) = \text{span}\{\delta_h \otimes g \mid h, g \in G, g = e \text{ and } g^{-1} = e\}\\ &= \text{span}\{\delta_h \otimes e \mid h \in G\},\end{aligned}$$

which is the Drinfel'd quantum moduli algebra in the case of Γ_0.

6.2 Example 2: the trivial two-edged graph

In this example, we consider the ribbon graph Γ_1 from Example 3.1.
For the face path $p = (e_1)$ and $g_1, g_2, h_1, h_2 \in G$, we have that

$$\begin{aligned}\text{Hol}^*_{K,p}(\ell \otimes \eta)((\delta_{h_1} \otimes g_1) \otimes (\delta_{h_2} \otimes g_2)) &= (\ell \otimes \eta)\text{Hol}_{K,p}((\delta_{h_1} \otimes g_1) \otimes (\delta_{h_2} \otimes g_2))\\ &= (\ell \otimes \eta)(\delta_{h_2}(e)(\delta_{h_1} \otimes g_1)) = \delta_{h_2}(e)(\ell(\delta_{h_1}) \otimes \delta_e(g_1)) = \delta_{h_2}(e)\delta_e(g_1)\end{aligned}$$

and for the face path $p = (e_2)$ and $g_1, g_2, h_1, h_2 \in G$, we have that

$$\begin{aligned}\text{Hol}^*_{K,p}(\ell \otimes \eta)((\delta_{h_1} \otimes g_1) \otimes (\delta_{h_2} \otimes g_2)) &= (\ell \otimes \eta)\text{Hol}_{K,p}((\delta_{h_1} \otimes g_1) \otimes (\delta_{h_2} \otimes g_2))\\ &= (\ell \otimes \eta)(\delta_{h_1}(e)(\delta_{h_2} \otimes g_2)) = \delta_{h_1}(e)(\ell(\delta_{h_2}) \otimes \delta_e(g_2)) = \delta_{h_1}(e)\delta_e(g_2).\end{aligned}$$

This implies that $P^*_{K,flat}(K^{*\otimes E}) \subseteq \text{span}\{(\text{ev}_e \otimes \delta_e) \otimes (\text{ev}_e \otimes \delta_e)\}$. Therefore, we only have that check if $(\delta_e \otimes e) \otimes (\delta_e \otimes e) \in \mathcal{M}_{K,\Gamma_1}$.
For the face path $p = (e_1^{-1}, e_2^{-1})$ and $g_1, g_2, h_1, h_2 \in G$, we have that

$$\begin{aligned}\text{Hol}^*_{K,p}(\ell \otimes \eta)((\delta_e \otimes e) \otimes (\delta_e \otimes e)) &= (\ell \otimes \eta)\text{Hol}_{K,p}((\delta_e \otimes e) \otimes (\delta_e \otimes e))\\ &= (\ell \otimes \eta)\big(\text{Hol}_{K,(e_2^{-1})}((\delta_e \otimes e)_{(2)} \otimes (\delta_e \otimes e)_{(2)}) \cdot \text{Hol}_{K,(e_1^{-1})}((\delta_e \otimes e)_{(1)} \otimes (\delta_e \otimes e)_{(1)})\big)\\ &= \sum_{u,v \in G} (\ell \otimes \eta)\big(\text{Hol}_{K,(e_2^{-1})}((\delta_u \otimes e) \otimes (\delta_v \otimes e)) \cdot \text{Hol}_{K,(e_1^{-1})}((\delta_{u^{-1}} \otimes e) \otimes (\delta_{v^{-1}} \otimes e))\big)\\ &= \sum_{u,v \in G} (\ell \otimes \eta)\big(\epsilon(\delta_u \otimes e)S(\delta_v \otimes e) \cdot \epsilon(\delta_{v^{-1}} \otimes e)S(\delta_{u^{-1}} \otimes e)\big)\\ &= \sum_{u,v \in G} (\ell \otimes \eta)\big(\delta_u(e)(\delta_{v^{-1}} \otimes e) \cdot \delta_{v^{-1}}(e)(\delta_u \otimes e)\big)\\ &= (\ell \otimes \eta)\big((\delta_e \otimes e) \cdot (\delta_e \otimes e)\big) \ (u = e, v = e)\\ &= (\ell \otimes \eta)\big(\delta_e \otimes e\big) = \ell(\delta_e) \otimes \delta_e(e) = 1 \otimes 1,\end{aligned}$$

so we know that $P^*_{K,flat}(K^{*\otimes E}) = \text{span}\{(\text{ev}_e \otimes \delta_e) \otimes (\text{ev}_e \otimes \delta_e)\}$.
Furthermore, we will derive in Lemma 6.10 that

$$(\text{ev}_e \otimes \delta_e) \otimes (\text{ev}_e \otimes \delta_e) \triangleleft^*_{K,\Gamma_1} = (\text{ev}_e \otimes \delta_e) \otimes (\text{ev}_e \otimes \delta_e),$$

which gives that $\mathcal{M}_{\Gamma_1} = \text{span}\{(\text{ev}_e \otimes \delta_e) \otimes (\text{ev}_e \otimes \delta_e)\}$.
Note that $\mathcal{M}_{\Gamma_1}$ is not isomorphic to $\mathcal{M}_{\Gamma_0}$, while the two ribbon graphs Γ_0 and Γ_1 both have associated surfaces Σ_{Γ_0} and Σ_{Γ_1} homeomorphic to the sphere. Hence, the condition that every graph needs to have only vertices with a valence larger than two and more faces than one cannot be made redundant.

6.3 Isomorphism theorem

In this section, we will prove a final result in the investigation into the moduli algebras of the group algebra and of the Drinfel'd double. We will prove that the quantum moduli algebra of the group algebra is isomorphic to the quantum moduli algebra as created by the Drinfel'd double for certain types of ribbon graphs. The types of ribbon graphs we will study in this section are characterized by the following property.

Definition 6.6 *A ribbon graph Γ is called* ***exclusive*** *if for every edge there is a face path such that the edge is not in the face path: that is, for all $e \in E$, there is a face path $p = (e_i^{\epsilon_i})_{i=1}^n$ such that for all $m \in \{1, \ldots, n\}$, we have that $e \neq e_m^{\epsilon_m} \neq e^{-1}$.*

In order to find the aforementioned isomorphism of moduli algebras, we will first define the algebra morphism that will be in the center of the proof.

Lemma 6.7 *The linear map*

$$\begin{aligned} V &: \mathrm{Fun}(G)^{\otimes E} \to {K^*}^{\otimes E} \\ \delta_{g_1} \otimes \cdots \otimes \delta_{g_E} &\mapsto (ev_e \otimes \delta_{g_1}) \otimes \cdots \otimes (ev_e \otimes \delta_{g_E}), \end{aligned}$$

is injective and respects multiplication (if the multiplication on $\mathrm{Fun}(G)^{\otimes E}$ is pointwise, and if the multiplication on ${K^*}^{\otimes E}$ is that from Lemma 5.1.)

Proof. It is clear that this map is injective. The multiplication in Lemma 5.1 in the case of the Drinfel'd double is shown in Appendix B. If we set $x = a = e \in G$ in those calculations, we obtain that

$$(\mathrm{ev}_e \otimes \delta_y)_i \cdot (\mathrm{ev}_e \otimes \delta_b)_j = \begin{cases} (\mathrm{ev}_e \otimes \delta_b)_j \cdot (\mathrm{ev}_e \otimes \delta_y)_i & \text{if } i \neq j \\ (\mathrm{ev}_e \otimes \delta_b \delta_y) & \text{if } i = j. \end{cases}$$

From these expressions, it is clear that v respects multiplication. □

We will already state the theorem, and then we will prove some necessary lemmas for the sake of clearifying the arguments involved.

Theorem 6.8 *Let Γ be an exclusive ribbon graph of type (3,2) with one vertex. Then $V : P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}(G)^{\otimes E}) \to P^*_{K,flat}({K^*}^{\otimes E})$ is an algebra isomorphism.*

Assume for the remainder of this section our ribbon graph Γ is exclusive.

Continuing the preparations for the proof, we will, as in Chapter 4, construct a commutative diagram that will induce an algebra morphism as a restriction of the linear map V.

Theorem 6.9 *The diagram*

$$\begin{array}{ccc} \mathrm{Fun}(G)^{\otimes E} & \xrightarrow{V} & {K^*}^{\otimes E} \\ \downarrow P^*_{\mathbb{F}[G],flat} & & \downarrow P^*_{K,flat} \\ P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}(G)^{\otimes E}) & \xrightarrow{V} & P^*_{K,flat}({K^*}^{\otimes E}) \end{array}$$

commutes.

Proof. Let $j \in \{1, \dots, E\}$ and denote the corresponding edge by f_j. Since Γ is an exclusive ribbon graph, we know that there exists a face path $p = (e_i^{\epsilon_i})_{i=1}^n$ such that $f_j \neq e_i^{\epsilon_i} \neq f_j^{-1}$ for all $i \in \{1, \dots, n\}$. This implies that the expression $\mathrm{Hol}_p((\delta_{h_1} \otimes g_1) \otimes \cdots \otimes (\delta_{h_n} \otimes g_n))$ $(g_i, h_i \in G$, $i \in \{1, \dots, n\})$ will obtain a factor of $\prod_{i=1}^n \epsilon(\delta_{h_j} \otimes g_j)_{(i)}$. Using axiom (2) from Definition 2.5, we find that

$$\prod_{i=1}^n \epsilon(\delta_{h_j} \otimes g_j)_{(i)} = \epsilon(\delta_{h_j} \otimes g_j) = \delta_{h_j}(e),$$

and this gives that $\prod_{[f]\in F} \mathrm{Hol}^*_{K,f}(\ell \otimes \eta)((\delta_{h_1} \otimes g_1) \otimes \cdots \otimes (\delta_{h_n} \otimes g_n)) \neq 0$ only if $h_j = e$ for all $j \in \{1, \dots, E\}$.

Subsequently, we know for $(g_1, \dots, g_E) \in G^{\times E}$ and for a face path $p = (e_i^{\epsilon_i})_{i=1}^n$ that

$$\begin{aligned}
&\prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{K,p}(\ell \otimes \eta)((\delta_e \otimes g_1) \otimes \cdots \otimes (\delta_e \otimes g_E)) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} (\ell \otimes \eta)(\mathrm{Hol}_{K,p}((\delta_e \otimes g_1) \otimes \cdots \otimes (\delta_e \otimes g_E))) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \sum_u (\ell \otimes \eta) \left(\prod_{i=0}^{n-1} \left(\prod_{f\in E\setminus\{e_{n-i}\}} \epsilon(\delta_{u_{n-i,f}} \otimes g_f) \right) (\delta_{u_{n-i,e_{n-i}}} \otimes g_{n-i}^{\epsilon_{n-i}}) \right) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \sum_u (\ell \otimes \eta) \left(\prod_{i=0}^{n-1} \left(\prod_{f\in E\setminus\{e_{n-i}\}} \delta_{u_{n-i,f}}(e) \right) (\delta_{u_{n-i,e_{n-i}}} \otimes g_{n-i}^{\epsilon_{n-i}}) \right),
\end{aligned}$$

where the summation u is actually a combination of nE summations $u_{i',j'} \in G$, $(i', j') \in \{1, \dots, n\} \times E$ over G such that $\prod_{i'=0}^{n-1} u_{n-i',f} = e \in G$. Note that since Γ is exclusive, we know that for every edge $e' \in E$ there exists a face path p such that e' is not in p. This gives, if the expression is not to vanish, that at $u_{i',e'} = e$ for all $i' \in \{1, \dots, n\}$. (Otherwise the expression $\prod_{i=0}^{n-1} \left(\prod_{f\in E\setminus\{e_{n-i}\}} \delta_{u_{n-i,f}}(e) \right)$, where $p = (e_i^{\epsilon_i})_{i=1}^n$, would be equal to zero.) Continuing the calculation gives us that

$$\begin{aligned}
&\prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{K,p}(\ell\otimes\eta)((\delta_e\otimes g_1)\otimes\cdots\otimes(\delta_e\otimes g_E)) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} (\ell\otimes\eta)\left(\prod_{i=0}^{n-1}(\delta_e\otimes g_{n-i}^{\epsilon_{n-i}})\right) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} (\ell\otimes\eta)\left((\delta_e\otimes g_n^{\epsilon_n}\cdot\cdots\cdot g_1^{\epsilon_1})\right) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \ell(\delta_e)\otimes\eta(g_n^{\epsilon_n}\cdot\cdots\cdot g_1^{\epsilon_1}) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \eta(\mathrm{Hol}_{\mathbb{F}[G],p}(g_1\otimes\cdots\otimes g_E)) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} (\mathrm{Hol}^*_{\mathbb{F}[G],p}(\eta)(g_1\otimes\cdots\otimes g_E)) \\
&= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} V(\mathrm{Hol}^*_{\mathbb{F}[G],p}(\eta))((\delta_e\otimes g_1)\otimes\cdots\otimes(\delta_e\otimes g_E)) \\
&= V\left(\prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{\mathbb{F}[G],p}(\eta)\right)((\delta_e\otimes g_1)\otimes\cdots\otimes(\delta_e\otimes g_E)),
\end{aligned}$$

so we know that

$$\prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{K,p}(\ell\otimes\eta) = V\left(\prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{\mathbb{F}[G],p}(\eta)\right).$$

(It is clear that in the case that one of the delta functions in the argument is not trivial both sides will be equal to zero.) This implies for $\alpha\in\mathrm{Fun}(G)^{\otimes E}$, using the fact V respects multiplication, that

$$\begin{aligned}
P^*_{K,flat}(V(\alpha)) &= \prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{K,p}(\ell\otimes\eta)\cdot V(\alpha) = V\left(\prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{\mathbb{F}[G],p}(\eta)\right)\cdot V(\alpha) \\
&= V\left(\prod_{[p=(e_i^{\epsilon_i})_{i=1}^n]\in F} \mathrm{Hol}^*_{\mathbb{F}[G],p}(\eta)\cdot\alpha\right) = V(P^*_{\mathbb{F}[G],flat}(\alpha)),
\end{aligned}$$

which implies that

$$P^*_{K,flat}\circ V = V\circ P^*_{\mathbb{F}[G],flat}.$$

□

Analogously as in Section 4, this diagram allows us to restrict V to the moduli algebras.

Lemma 6.10 *The diagram in Theorem 6.9 induces a commutative diagram*

$$\begin{array}{ccc} \mathrm{Fun}_{inv}(G)^{\otimes E} & \xrightarrow{\quad V \quad} & K_{inv}^{*\otimes E} \\ \Big\downarrow P^*_{\mathbb{F}[G],flat} & & \Big\downarrow P^*_{K,flat} \\ P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}_{inv}(G)^{\otimes E}) & \xrightarrow{V} & P^*_{K,flat}(K_{inv}^{*\otimes E}) \end{array} .$$

Proof. Since $\mathrm{Fun}_{inv}(G)^{\otimes E} \subset \mathrm{Fun}(G)^{\otimes E}$ and $K_{inv}^{*\otimes E} \subset K^{*\otimes E}$, we only need to show that $V(K_{inv}^{*\otimes E}) \subset K_{inv}^{*\otimes E}$. So let $e' \in E$ and $a, b, y \in G$. Denote the vertex of Γ by v. The action of K for an element $(\delta_e \otimes y)_{e'} \in K^{*\otimes E}$ is given by

$$\begin{aligned}
&G^*((\mathrm{ev}_e \otimes \delta_y)_{e'}) \triangleleft_K^* (\delta_a \otimes b) = \sum_{u \in G} ((\mathrm{ev}_e \otimes \delta_u) \otimes (\mathrm{ev}_e \otimes \delta_{u^{-1}y}))_{c_v(e^{-1})c_v(e)} \triangleleft_K^* (\delta_a \otimes b) \\
&= \sum_{u_1, v_1, v_2 \in G} \langle (\mathrm{ev}_e \otimes \delta_{v_1}) \cdot S(\mathrm{ev}_e \otimes \delta_{v_2^{-1}u^{-1}y}), \delta_a \otimes b \rangle (\mathrm{ev}_e \otimes \delta_{v_1^{-1}u}) \otimes (\mathrm{ev}_e \otimes \delta_{v_2})_{c_v(e^{-1})c_v(e)} \\
&= \sum_{u_1, v_1, v_2 \in G} \delta_a(e)\delta_{v_1}(b)\delta_{(v_2^{-1}u^{-1}y)^{-1}}(b)(\mathrm{ev}_e \otimes \delta_{v_1^{-1}u}) \otimes (\mathrm{ev}_e \otimes \delta_{v_2})_{c_v(e^{-1})c_v(e)} \\
&= \sum_{u \in G} \delta_a(e)(\mathrm{ev}_e \otimes \delta_{b^{-1}u}) \otimes (\mathrm{ev}_e \otimes \delta_{u^{-1}yb})_{c_v(e^{-1})c_v(e)} \; (v_1 = b, v_2 = u^{-1}yb) \\
&= \sum_{u \in G} \delta_a(e)(\mathrm{ev}_e \otimes \delta_{b^{-1}u}) \otimes (\mathrm{ev}_e \otimes \delta_{(b^{-1}u)^{-1}b^{-1}yb})_{c_v(e^{-1})c_v(e)} = \delta_a(e)G^*((\delta_e \otimes \delta_{b^{-1}yb})),
\end{aligned}$$

so this gives that

$$(\mathrm{ev}_e \otimes \delta_{y_1}) \otimes \cdots \otimes (\mathrm{ev}_e \otimes \delta_{y_E}) \triangleleft_{K,\Gamma}^* (\delta_a \otimes b) = \delta_a(e)(\mathrm{ev}_e \otimes \delta_{b^{-1}y_1 b}) \otimes \cdots \otimes (\mathrm{ev}_e \otimes \delta_{b^{-1}y_E b})$$

for $(y_1, \ldots, y_E) \in G^{\times E}$.
Comparing this result with the action in Theorem 5.10, we find for $\varphi \in \mathrm{Fun}_{inv}(G)^{\otimes E}$, $a, b \in G$ that

$$V(\varphi) \triangleleft_{K,\Gamma}^* (\delta_a \otimes b) = \delta_a(e)V(\varphi \triangleleft_{\mathbb{F}[G],\Gamma}^* b) = \epsilon(\delta_a \otimes b)V(\varphi),$$

from which it is obvious that $V(\varphi) \in K_{inv}^{*\otimes E}$. □
With those lemmas, we are now in a position to prove the theorem.
Proof. (**Theorem 6.8**) From Lemma 6.10, we know that $V : P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}_{inv}(G)^{\otimes E}) \to P^*_{K,flat}(K_{inv}^{*\otimes E})$ is an injective linear map that respects multiplication. To prove that the map is surjective, consider $\varphi \in K_{inv}^{*\otimes E}$. We can write φ as

$$\varphi = \sum_{y_i, x_i \in G, i \in 1, \ldots, E} \lambda_{y_1, \ldots, x_E}(\mathrm{ev}_{x_1} \otimes \delta_{y_1}) \otimes \cdots \otimes (\mathrm{ev}_{x_E} \otimes \delta_{y_E})$$

Since we know from Theorem 6.9 that $\varphi((\delta_{h_1} \otimes g_1) \otimes \cdots \otimes (\delta_{h_E} \otimes g_E)) = 0$ for $g_i, h_i \in G$, $i \in \{1, \ldots, E\}$, if there exists a $j \in \{1, \ldots, E\}$ such that $h_j \neq e$, we can rewrite φ as

$$\varphi = \sum_{y_i \in G, i \in 1, \ldots, E} \lambda_{y_1, \ldots, y_E}(\mathrm{ev}_e \otimes \delta_{y_1}) \otimes \cdots \otimes (\mathrm{ev}_e \otimes \delta_{y_E}).$$

Now we define $\alpha = \sum_{y_i \in G, i \in 1, \ldots, E} \lambda_{y_1, \ldots, y_E} \delta_{y_1} \otimes \cdots \otimes \otimes \delta_{y_E}$. It clear from this construction that $V(\alpha) = \varphi$. For V to be surjective, we only need to show that $\alpha \in P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}_{inv}(G)^{\otimes E})$. First, we will show that $\alpha \in \mathrm{Fun}_{inv}(G)^{\otimes E}$. Let $b \in G$. We then have from Lemma 6.10 that

$$V\left(\alpha \triangleleft^*_{\mathbb{F}[G],\Gamma} b\right) = V(\alpha) \triangleleft^*_{K,\Gamma} V(b) = \varphi \triangleleft^*_{K,\Gamma} (\delta_e \otimes b) = \epsilon(\delta_e \otimes b)\varphi = \delta_e(e)\varphi = \varphi = V(\alpha),$$

and since V is injective, we know that $\alpha \triangleleft^*_{\mathbb{F}[G],p} b = \alpha$, thus $\alpha \in \mathrm{Fun}_{inv}(G)^{\otimes E}$.
Using the fact that $P^*_{K,flat}$ is a projection [2], i.e. $P^*_{K,flat} \circ P^*_{K,flat} = P^*_{K,flat}$, that $\varphi \in P^*_{K,flat}(K^{*\otimes E}_{inv})$, and using Lemma 6.10, we also obtain that

$$V(\alpha) = \varphi = P^*_{K,flat}(\varphi) = P^*_{K,flat}(V(\alpha)) = V(P^*_{\mathbb{F}[G],flat}(\alpha)),$$

which by the injectivity of V also implies that $\alpha = P^*_{\mathbb{F}[G],flat}(\alpha)$. This gives us that $\alpha \in P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}_{inv}(G)^{\otimes E})$, which proves that $V : P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}_{inv}(G)^{\otimes E}) \to P^*_{K,flat}(K^{*\otimes E}_{inv})$ is surjective.
There is also a unit in both $P^*_{\mathbb{F}[G],flat}(\mathrm{Fun}_{inv}(G)^{\otimes E})$ and $P^*_{K,flat}(K^{*\otimes E}_{inv})$: in $\mathrm{Fun}(G)^{\otimes E}$, we have that 1 is the constant function $1 = 1 \otimes \cdots \otimes 1$. Since $P^*_{K,flat}$, $P^*_{\mathbb{F}[G],flat}$ and V are algebra morphisms, we know that $P^*_{\mathbb{F}[G],flat}(1)$ is a unit for $\mathrm{Fun}_{inv}(G)^{\otimes E}$, and that $V(P^*_{\mathbb{F}[G],flat}(1))$ is a unit for $K^{*\otimes E}_{inv}$.
It is then clear that $V(P^*_{\mathbb{F}[G],flat}(1))$ is in fact the unit of $P^*_{K,flat}(K^{*\otimes E}_{inv})$, and hence V is an isomorphism of algebras. □
We note that for every ribbon graph Γ of type (3,2), there exists an exclusive ribbon graph Γ' of type (3,2) with one vertex such that the associated compact 2-dimensional manifolds Σ_Γ and $\Sigma_{\Gamma'}$ are homeomorphic. This can be seen by contracting edges of a maximal rooted tree in Γ until only vertex remains, which is always possible [2]. Call the resulting ribbon graph $\tilde{\Gamma}$. If this vertex either has a valence less than three or if the resulting ribbon graph has less than two faces, or if the resulting ribbon graph is not exclusive, one can insert a loop in $\tilde{\Gamma}$, and then double that loop. The resulting graph Γ' is indeed exclusive and of type (3,2), and the associated surfaces are homeomorphic. Hence, the moduli algebras $\mathcal{M}_\Gamma$ and $\mathcal{M}_{\Gamma'}$ are isomorphic. This implies that Theorem 6.8 even holds in greater generality:

Theorem 6.11 *Let Γ be a ribbon graph of type (3,2) and G be a finite group. Then $\mathcal{M}_{\mathcal{D}(\mathbb{F}[G]),\Gamma} = \mathcal{M}_{\mathbb{F}[G],\Gamma}$.* □

Chapter 7

Discussion

As was to be expected, in the classical case of group gauge theory, the moduli algebra of a ribbon graph can be described as the function algebra on moduli space of flat connections $\mathrm{Fun}(\mathrm{Hom}(\pi_1(\Sigma), G)/G)$ [14]. Those results established a connection between former research and this work, validating the approach taken. To transform our classical group gauge theory to a Hopf algebra gauge theory, we replaced our finite group G by a group algebra $\mathbb{F}[G]$. This linearisation process appeared to be equivalent to finding the moduli algebra in the group case. Subsequently, the general construction of the moduli algebra generalised the previous Hopf algebra gauge theory in the case of the group algebra to the case of a general finite-dimensional semisimple Hopf algebra. Theorem 5.10 justifies this construction by identifying the moduli algebra obtained by the linearisation of the group gauge theory and the one obtained by the new general construction. This was done in order to allow other Hopf algebras to be investigated. In particular, we have examined the Drinfel'd double. After providing some motivational examples, Theorem 6.11 completely determines the quantum moduli algebras of $K = \mathcal{D}(\mathbb{F}[G])$ for a finite group G.
Although this thesis provides characterisations of certain types of quantum moduli algebras and sketches the relation between these moduli algebras and their physical applications, many more questions worthwhile delving into have arisen.
As a first, only the moduli algebra of the group algebra and the Drinfel'd double have been examined. Unsurprisingly, there are other Hopf algebras that are interesting to investigate. Examples include the Drinfel'd double for an infinite group, the Drinfeld double of an algebra that is not a group algebra, or the Lie algebra associated to the Heisenberg group, or even Hopf algebras that are not semisimple. Many research nowadays, however, goes out to the case of Hopf algebras that arise form q-deformed universal enveloping algebras at the root of unity. The setup given in this thesis might be utilised to find characterisations of the moduli spaces of such algebras.
Secondly, Hopf algebra gauge theory has been claimed to be a mathematically axiomatic setting to calculate the algebra of observables in Chern-Simons theory on compact orientable 2-dimensional manifolds [2]. However, the quantisation process of Chern-Simons theory lacks a rigorous mathematical derivation. It would lead to substantial amounts of insight if one were to construct such a derivation.
Thirdly, finding the moduli algebra for Hopf algebra gauge theory in the special case of the Drinfel'd double is said to be equivalent to finding the protective space in the Kitaev model [10]. Since this case has now been completely described, one can transfer these results to the Kitaev model and attempt to enhance understanding on that theory.

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Appendices

A Operations in $\mathcal{D}(\mathbb{F}[G])^*$

Some useful relations for calculating in $\mathcal{D}(\mathbb{F}[G])^*$:

If we consider the non-degenerate pairing $\langle\ ,\ \rangle : \mathcal{D}(\mathbb{F}[G])^* \otimes \mathcal{D}(\mathbb{F}[G]) \to \mathbb{F}, \alpha \otimes h = \alpha(h)$, we know from the construction of Hopf algebras that for $g, h, g_1, h_1, g_2, h_2, x, y, x_1, y_1, x_2, y_2 \in G$,

$$\begin{aligned}
\triangle\,(\mathrm{ev}_x \otimes \delta_y)(((\delta_{h_1} \otimes g_1) \otimes (\delta_{h_2} \otimes g_2)) &= \langle \triangle\,(\mathrm{ev}_x \otimes \delta_y), ((\delta_{h_1} \otimes g_1) \otimes (\delta_{h_2} \otimes g_2)\rangle \\
&= \langle \mathrm{ev}_x \otimes \delta_y, (\delta_{h_1} \otimes g_1) \cdot (\delta_{h_2} \otimes g_2)\rangle \\
&= (\mathrm{ev}_x \otimes \delta_y)\,((\delta_{h_1} \otimes g_1) \cdot (\delta_{h_2} \otimes g_2)) \\
&= (\mathrm{ev}_x \otimes \delta_y)(\delta_{g_1^{-1} h_1 g_1}(h_2)\delta_{h_1} \otimes g_1 g_2) \\
&= \delta_{g_1^{-1} h_1 g_1}(h_2)\delta_{h_1}(x) \otimes \delta_y(g_1 g_2),
\end{aligned}$$

$$\begin{aligned}
((\mathrm{ev}_{x_1} \otimes \delta_{y_1}) \cdot (\mathrm{ev}_{x_2} \otimes \delta_{y_2}))(\delta_h \otimes g) &= \langle (\mathrm{ev}_{x_1} \otimes \delta_{y_1}) \cdot (\mathrm{ev}_{x_2} \otimes \delta_{y_2}), \delta_h \otimes g\rangle \\
&= \langle (\mathrm{ev}_{x_1} \otimes \delta_{y_1}), (\delta_h \otimes g)_{(1)}\rangle \langle (\mathrm{ev}_{x_2} \otimes \delta_{y_2}), (\delta_h \otimes g)_{(2)}\rangle \\
&= (\mathrm{ev}_{x_1} \otimes \delta_{y_1})((\delta_h \otimes g)_{(1)})(\mathrm{ev}_{x_2} \otimes \delta_{y_2})((\delta_h \otimes g)_{(2)}) \\
&= \sum_{u \in G} (\mathrm{ev}_{x_1} \otimes \delta_{y_1})(\delta_{u^{-1}h} \otimes g)(\mathrm{ev}_{x_2} \otimes \delta_{y_2})(\delta_u \otimes g) \\
&= \sum_{u \in G} (\delta_{u^{-1}h}(x_1) \otimes \delta_{y_1}(g)) \cdot (\delta_u(x_2) \otimes \delta_{y_2}(g)) \\
&= \delta_{x_2^{-1}h}(x_1) \otimes \delta_{y_1}(g)\delta_{y_2}(g) \\
&= \delta_h(x_2 x_1) \otimes \delta_{y_1}(g)\delta_{y_2}(g), \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
S(\mathrm{ev}_x \otimes \delta_y)(\delta_h \otimes g) &= \langle S(\mathrm{ev}_x \otimes \delta_y), \delta_h \otimes g\rangle \\
&= \langle (\mathrm{ev}_x \otimes \delta_y), S(\delta_h \otimes g)\rangle \\
&= \langle (\mathrm{ev}_x \otimes \delta_y), \delta_{g^{-1}h^{-1}g} \otimes g^{-1}\rangle \\
&= (\mathrm{ev}_x \otimes \delta_y)(\delta_{g^{-1}h^{-1}g} \otimes g^{-1}) \\
&= \delta_{g^{-1}h^{-1}g}(x) \otimes \delta_y(g^{-1})
\end{aligned}$$

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from which it is not difficult to determine that

$$\begin{aligned}\triangle\,(\mathrm{ev}_x\otimes\delta_y) &= \sum_{u\in G}(\mathrm{ev}_x\otimes\delta_u)\otimes(\mathrm{ev}_{u^{-1}xu}\otimes\delta_{u^{-1}y}),\\ (\mathrm{ev}_{x_1}\otimes\delta_{y_1})\cdot(\mathrm{ev}_{x_2}\otimes\delta_{y_2}) &= \mathrm{ev}_{x_2x_1}\otimes\delta_{y_2}\delta_{y_1},\ \text{and}\\ S(\mathrm{ev}_x\otimes\delta_y) &= \mathrm{ev}_{y^{-1}x^{-1}y}\otimes\delta_{y^{-1}}.\end{aligned}$$

B Calculations for the Drinfel'd double

Let $a,b,x,y\in G$ and $i,j\in\{1,\dots,2E\}$, and let $\tau:\{1,\dots,2E\}\to\{0,1\}$ be a function. The multiplication in Lemma 5.1 for $K^*=\mathcal{D}(\mathbb{F}[G])^*$ is given as follows.
If $i\neq j$ and $\tau(i)=1=\tau(j)$, then

$$\begin{aligned}&(\mathrm{ev}_x\otimes\delta_y)_i\cdot(\mathrm{ev}_a\otimes\delta_b)_j = \langle(\mathrm{ev}_a\otimes\delta_b)_{(2)}\otimes(\mathrm{ev}_x\otimes\delta_y)_{(2)},R\rangle((\mathrm{ev}_a\otimes\delta_b)_{(1)})_j\cdot((\mathrm{ev}_x\otimes\delta_y)_{(1)})_i\\ &= \sum_{(u,v,w)\in G^{\times 3}}\langle\mathrm{ev}_{v^{-1}av}\otimes\delta_{v^{-1}b})\otimes(\mathrm{ev}_{u^{-1}xu}\otimes\delta_{u^{-1}y}),(1\otimes w)\otimes(\delta_w\otimes e)\rangle(\mathrm{ev}_a\otimes\delta_v)_j\cdot(\mathrm{ev}_x\otimes\delta_u)_i\\ &= \sum_{(u,v,w)\in G^{\times 3}}\delta_{v^{-1}b}(w)\delta_w(u^{-1}xu)\delta_{u^{-1}y}(e)(\mathrm{ev}_a\otimes\delta_v)_j\cdot(\mathrm{ev}_x\otimes\delta_u)_i\\ &= \sum_{(v,w)\in G^{\times 2}}\delta_{v^{-1}b}(w)\delta_w(y^{-1}xy)(\mathrm{ev}_a\otimes\delta_v)_j\cdot(\mathrm{ev}_x\otimes\delta_y)_i\quad(u=y)\\ &= \sum_{(v)\in G}\delta_{v^{-1}b}(y^{-1}xy)(\mathrm{ev}_a\otimes\delta_v)_j\cdot(\mathrm{ev}_x\otimes\delta_y)_i\quad(w=y^{-1}xy)\\ &= (\mathrm{ev}_a\otimes\delta_{by^{-1}x^{-1}y})_j\cdot(\mathrm{ev}_x\otimes\delta_y)_i.\quad(v=by^{-1}x^{-1}y)\end{aligned}$$

If $i\neq j$ and $\tau(i)=0=\tau(j)$, then

$$\begin{aligned}&(\mathrm{ev}_x\otimes\delta_y)_i\cdot(\mathrm{ev}_a\otimes\delta_b)_j = \langle(\mathrm{ev}_a\otimes\delta_b)_{(1)}\otimes(\mathrm{ev}_x\otimes\delta_y)_{(1)},R\rangle((\mathrm{ev}_a\otimes\delta_b)_{(2)})_j\cdot((\mathrm{ev}_x\otimes\delta_y)_{(2)})_i\\ &= \sum_{(u,v,w)\in G^{\times 3}}\langle\mathrm{ev}_a\otimes\delta_v)\otimes(\mathrm{ev}_x\otimes\delta_u),(1\otimes w)\otimes(\delta_w\otimes e)\rangle(\mathrm{ev}_{v^{-1}av}\otimes\delta_{v^{-1}b})_j\cdot(\mathrm{ev}_{u^{-1}xu}\otimes\delta_{u^{-1}y})_i\\ &= \sum_{(u,v,w)\in G^{\times 3}}\delta_v(w)\delta_w(x)\delta_u(e)(\mathrm{ev}_{v^{-1}av}\otimes\delta_{v^{-1}b})_j\cdot(\mathrm{ev}_xu^{-1}xu\otimes\delta_uu^{-1}y)_i\\ &= \sum_{(v,w)\in G^{\times 2}}\delta_v(w)\delta_w(x)(\mathrm{ev}_{v^{-1}av}\otimes\delta_{v^{-1}b})_j\cdot(\mathrm{ev}_x\otimes\delta_y)_i\quad(u=e)\\ &= \sum_{(v)\in G}\delta_v(x)(\mathrm{ev}_{v^{-1}av}\otimes\delta_{v^{-1}b})_j\cdot(\mathrm{ev}_x\otimes\delta_y)_i\quad(w=x)\\ &= (\mathrm{ev}_{x^{-1}ax}\otimes\delta_{x^{-1}b})_j\cdot(\mathrm{ev}_x\otimes\delta_y)_i.\quad(v=x)\end{aligned}$$

If $i \neq j$ and $\tau(i) = 1$ and $\tau(j) = 0$, then

$$
\begin{aligned}
&(\mathrm{ev}_x \otimes \delta_y)_i \cdot (\mathrm{ev}_a \otimes \delta_b)_j = \langle (\mathrm{ev}_a \otimes \delta_b)_{(1)} \otimes (\mathrm{ev}_x \otimes \delta_y)_{(2)}, (S \otimes \mathrm{id})R \rangle ((\mathrm{ev}_a \otimes \delta_b)_{(2)})_j \cdot ((\mathrm{ev}_x \otimes \delta_y)_{(1)})_i \\
&\sum_{(u,v,w,h)\in G^{\times 4}} \langle (\mathrm{ev}_a \otimes \delta_v) \otimes (\mathrm{ev}_{u^{-1}xu} \otimes \delta_{u^{-1}y}), (\delta_{w^{-1}hw} \otimes w^{-1}) \otimes (\delta_w \otimes e) \rangle (\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b})_j \cdot (\mathrm{ev}_x \otimes \delta_u)_i \\
&= \sum_{(u,v,w,h)\in G^{\times 4}} \delta_{w^{-1}hw}(a)\delta_v(w^{-1})\delta_w(u^{-1}xu)\delta_{u^{-1}y}(e)(\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b})_j \cdot (\mathrm{ev}_x \otimes \delta_u)_i \\
&= \sum_{(v,w,h)\in G^{\times 3}} \delta_{w^{-1}hw}(a)\delta_v(w^{-1})\delta_w(y^{-1}xy)(\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b})_j \cdot (\mathrm{ev}_x \otimes \delta_y)_i \quad (u = y) \\
&= \sum_{(v,h)\in G^{\times 2}} \delta_{y^{-1}x^{-1}yhy^{-1}xy}(a)\delta_v(y^{-1}x^{-1}y)(\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b})_j \cdot (\mathrm{ev}_x \otimes \delta_y)_i \quad (w = y^{-1}xy) \\
&= \sum_{v\in G} \delta_v(y^{-1}x^{-1}y)(\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b})_j \cdot (\mathrm{ev}_x \otimes \delta_y)_i \quad (h = y^{-1}xyay^{-1}x^{-1}y) \\
&= (\mathrm{ev}_{y^{-1}xyay^{-1}x^{-1}y} \otimes \delta_{y^{-1}xyb})_j \cdot (\mathrm{ev}_x \otimes \delta_y)_i. \quad (v = y^{-1}x^{-1}y)
\end{aligned}
$$

If $i \neq j$ and $\tau(i) = 0$ and $\tau(j) = 1$, then

$$
\begin{aligned}
&(\mathrm{ev}_x \otimes \delta_y)_i \cdot (\mathrm{ev}_a \otimes \delta_b)_j = \langle (\mathrm{ev}_a \otimes \delta_b)_{(2)} \otimes (\mathrm{ev}_x \otimes \delta_y)_{(1)}, (\mathrm{id} \otimes S)R \rangle ((\mathrm{ev}_a \otimes \delta_b)_{(1)})_j \cdot ((\mathrm{ev}_x \otimes \delta_y)_{(2)})_i \\
&= \sum_{(u,v,w)\in G^{\times 3}} \langle \mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b}) \otimes (\mathrm{ev}_x \otimes \delta_u), (1 \otimes w) \otimes (\delta_{w^{-1}} \otimes e) \rangle (\mathrm{ev}_a \otimes \delta_v)_j \cdot (\mathrm{ev}_{u^{-1}xu} \otimes \delta_{u^{-1}y})_i \\
&= \sum_{(u,v,w)\in G^{\times 3}} \delta_{v^{-1}b}(w)\delta_{w^{-1}}(x)\delta_u(e)(\mathrm{ev}_a \otimes \delta_v)_j \cdot (\mathrm{ev}_{u^{-1}xu} \otimes \delta_{u^{-1}y})_i \\
&= \sum_{(v,w)\in G^{\times 2}} \delta_{v^{-1}b}(w)\delta_{w^{-1}}(x)(\mathrm{ev}_a \otimes \delta_v)_j \cdot (\mathrm{ev}_x \otimes \delta_y)_i \quad (u = e) \\
&= \sum_{(v)\in G} \delta_{v^{-1}b}(x^{-1})(\mathrm{ev}_a \otimes \delta_v)_j \cdot (\mathrm{ev}_x \otimes \delta_y)_i \quad (w = x^{-1}) \\
&= (\mathrm{ev}_a \otimes \delta_{bx})_j \cdot (\mathrm{ev}_x \otimes \delta_y)_i. \quad (v = bx)
\end{aligned}
$$

If $i = j$ and $\tau(i) = 0$, then we know from the above calculations that

$$
\begin{aligned}
&(\mathrm{ev}_x \otimes \delta_y)_i \cdot (\mathrm{ev}_a \otimes \delta_b)_j = \langle (\mathrm{ev}_a \otimes \delta_b)_{(1)} \otimes (\mathrm{ev}_x \otimes \delta_y)_{(1)}, R \rangle ((\mathrm{ev}_a \otimes \delta_b)_{(2)} \cdot (\mathrm{ev}_x \otimes \delta_y)_{(2)})_i \\
&= \sum_{(u,v,w)\in G^{\times 3}} \langle \mathrm{ev}_a \otimes \delta_v) \otimes (\mathrm{ev}_x \otimes \delta_u), (1 \otimes w) \otimes (\delta_w \otimes e) \rangle ((\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b}) \cdot (\mathrm{ev}_{u^{-1}xu} \otimes \delta_{u^{-1}y}))_i \\
&= \sum_{(u,v,w)\in G^{\times 3}} \delta_v(w)\delta_w(x)\delta_u(e)((\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b}) \cdot (\mathrm{ev}_{u^{-1}xu} \otimes \delta_{u^{-1}y}))_i \\
&= \sum_{(v,w)\in G^{\times 3}} \delta_v(w)\delta_w(x)((\mathrm{ev}_{v^{-1}av} \otimes \delta_{v^{-1}b}) \cdot (\mathrm{ev}_x \otimes \delta_y))_i \quad (u = e) \\
&= ((\mathrm{ev}_{x^{-1}ax} \otimes \delta_{x^{-1}b}) \cdot (\mathrm{ev}_x \otimes \delta_y))_i \quad (v = w = x) \\
&= (\mathrm{ev}_{xx^{-1}ax} \otimes \delta_{x^{-1}b}\delta_y)_i = (\mathrm{ev}_{ax} \otimes \delta_{x^{-1}b}\delta_y)_i.
\end{aligned}
$$

If $i = j$ and $\tau(i) = 1$, then we know that

$$(\mathrm{ev}_x \otimes \delta_y)_i \cdot (\mathrm{ev}_a \otimes \delta_b)_j = ((\mathrm{ev}_a \otimes \delta_b) \cdot (\mathrm{ev}_x \otimes \delta_y))_i = (\mathrm{ev}_{xa} \otimes \delta_y\delta_b).$$

C Vector potentials

The definitions and lemmas in this appendix are from [1] and [15].

Definition C.1 *A* ***vector bundle*** *over a manifold* M *of rank* $n \in \mathbb{N}$ *is a triplet* (E, M, π)*, where* E *and* M *are manifolds, and* $\pi : E \to M$ *is a continuous surjection such that for every* $p \in M$*, the fibre* $E_p := \pi^{-1}(p)$ *is a* n*-dimensional vector space, and that there exists a neighbourhood* U_p *of* $p \in M$ *such that there exists a diffeomorphism* $h_p : \pi^{-1}(U_p) \to U_p \times \mathbb{R}^n$ *such that* $\pi|_{\pi^{-1}(U_p)} = r_p \circ h_p$*, where* $r_p : U_p \times \mathbb{R}^n \to U_p$ *is the projection on the first coordinate, and that* $h_p|_{E_p}$ *is linear.*

It is then straightforward to see that $E = \bigcup_{p \in M} E_p$. We can then define $E^* = \bigcup_{p \in M} E_p^*$ and $\pi^* : E^* \to M$ by sending elements of E_p^* to p for all $p \in M$. Then it can be found in [1] that (E^*, π^*, M) is a also a vector bundle, commonly called the dual vector bundle.
One frequently used type of vector bundle is the tangent bundle.

Example C.2 The tangent bundle of a manifold M is a vector bundle over M given by (TM, π, M), where $TM = \bigcup_{p \in M} T_pM$, and projection map π sends elements of T_pM to p. We denote the dual tangent bundle by T^*M.

Now we are in a position to define a vector potential.

Definition C.3 *Let* (E, M, π) *be a vector bundle. Then a vector potential* A *is a smooth map* $A : M \to E \otimes E^* \otimes T^*M$ *such that* $A(p) \in E_p \otimes E_p^* \otimes T_p^*M$*.*

Note that if we write $A = A_1 \otimes A_2 \otimes A_3$, then A_3 is a 1-form. Next, we will define the trace of a vector potential.

Definition C.4 *Let* V *be a finite-dimensional* $\mathbb{F}$*-vector space. Then the trace tr is a map* $tr : V \otimes V^* \to \mathbb{F}$ *given by* $tr(v \otimes f) = f(v)$*.*
The trace of a vector potential $A = A_1 \otimes A_2 \otimes A_3$ *is given by* $tr(A) = tr(A_1 \otimes A_2)A_3$*, so that* $tr(A)(p) = tr(A_1(p) \otimes A_2(p))A_3(p)$*.*

From this definition, it is clear that $\mathrm{tr}(A)$ is a 1-form, and extending the above definition naturally, it is not difficult to see that the Lagrangian in Equation 1.4 is a 3-form.

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The image on the front page is adapted from [16].