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Fibre Bundles in General Relativity

Bachelor's thesis

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Abstract

In this thesis, we introduce the language of smooth manifolds, which is the natural setting for general relativity, and show how the restricted Lorentz group is related to the complex special linear group in two dimensions, and argue how this relation shows that spinors come up naturally in general relativity. We then consider fibre bundles and how they come up in general relativity, and how they are necessary to define what a spin structure is, and examine under which assumptions it exists. We conclude with a proper definition of Einstein's field equation.

Contents

Introduction

In Einstein's theory of general relativity, the mathematical model of our universe is a *spacetime manifold* $\mathcal{M}$, defined as a 4-dimensional smooth manifold which is connected, non-compact and space- and time-oriented. Moreover, it has the property that the tangent space at each point of the manifold is isomorphic to Minkowski space, so the metric is represented by the matrix $\eta = \mathrm{diag}(1, -1, -1, -1)$. This is (more or less) the mathematical expression of Einstein's postulates that "physics is locally governed by special relativity", and that gravity is a manifestation of the geometry of $\mathcal{M}$, more specifically the curvature, which is in turn influenced by the matter which is present in the universe. To quote John A. Wheeler: "Spacetime tells matter how to move; matter tells spacetime how to curve".

Consider an event P in spacetime, which is just a point p in $\mathcal{M}$, and an observer A, which is equipped with a local frame, i.e. a basis for the tangent space at each point in a some neighbourhood of p. Suppose for the moment that there is no gravity, then $\mathcal{M}$ can be identified with Minkowski space and the observer A can actually be equipped with a global frame, i.e. with a basis at each point in $\mathcal{M}$. It is an axiom of physics that any (meaningful) physical theory should be Lorentz covariant, meaning that the equations which A writes down should be of the same form for any other *admissible* observer, whose frame is connected to the frame of A by some restricted Lorentz transformation, i.e. some element of the restricted Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$. Since we consider all admissible observers, and since each admissible observer's frame is connected to that of A by a unique Lorentz transformation, we can equivalently say (that is, we have an isomorphism $\{\text{Admissible observers at } p\} \cong \mathrm{SO}^{\uparrow}(1,3)$, but this isomorphism depends on the chosen observer A) that we consider the whole Lorentz group. We do this at each point (since we have global frames), which can be expressed formally as forming the Cartesian product $\mathcal{M} \times \mathrm{SO}^{\uparrow}(1,3)$. Since $\mathrm{SO}^{\uparrow}(1,3)$ is a Lie group, this is again a smooth manifold, which we consider as spacetime together with its group of symmetries. Taking gravity into account, the only thing that changes, which is in fact the crucial thing, is that we can only hold the foregoing argument *locally*. This will then result in a "twisted product", which is locally a simple product, but whose topology can globally be different. It should be noted that the group plays an important role here.

What we have described just now is really an informal definition of a principal $\mathrm{SO}^{\uparrow}(1,3)$-bundle over M. We see that this object comes up fairly naturally, and we will see that this notion generalises and formalises much of what we already know, using the language of fibre bundles and principal bundles. It is however not, simply a generalisation for generalisation's sake. Much of the standard model, which incorporates the weak, strong, and electromagnetic interaction, is formulated using this framework. But the main reason for studying these objects in relation to general relativity is because using spinors to reformulate problems in general relativity has turned out be very useful. Spinors were first introduced by Paul A.M. Dirac and Wolfgang E. Pauli in quantum mechanics when studying the electron. It was Roger Penrose who primarily introduced and advocated the use of spinors in general relativity [1,2], and two notable results which are still used today are the *spin-coefficient formalisms* introduced by Roger Penrose, Ezra T. Newman and Robert Geroch [3,4].

However, there is a natural question which one might ask: under what circumstances spinors can be defined properly on a manifold? To even be able to address this question, one has to properly set up and define the aforementioned language of fibre bundles and principal bundles, and this is what this thesis will be concerned with. We start by developing some manifold theory, and show the relation between $\mathrm{SL}(2,\mathbb{C})$ and the Lorentz group, where the former comes into play since it is the group under which spinors transform. After this we will develop some theory on fibre bundles, which allows us to properly define what a spin structure is, which is necessary to have spinors, and we will mention the results on the existence of spin structures on a non-compact manifold. Lastly, we will define what a connection on the tangent bundle of a manifold is, which will enable us to write down the Einstein field equation locally.

The motivation to study these subjects arose from a simple question, which asked whether it was possible to learn more about the structure of the electromagnetic Hopf field [5] by trying to find (exact) solutions for Einstein's equation when this field is taken as a source, which was tried by one of my supervisors, Jan Willem Dalhuisen [6]. His approach proved unsuccessful (so far), and he has suggested to use spinors to have a better chance of tackling this problem. It was then my personal mathematical interest which has led me down the road taken and outlined above.

Notations and Conventions

The natural numbers are defined as $\mathbb{N} := \mathbb{Z}_{\geq 0}$, and for any $n \in \mathbb{N}_{\geq 1}$, we define $[n] := \{1, \ldots, n\}$. For a map $f : A_1 \times \ldots \times A_n \longrightarrow A$, where $A_1, \ldots, A_n$ and A are sets and $n \in \mathbb{N}_{\geq 1}$, we write $f(a_1, \ldots, a_n)$ instead of $f((a_1, \ldots, a_n))$, for all $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$. For $i \in [n]$, we define $\mathrm{Proj}_i : A_1 \times \ldots \times A_n \longrightarrow A_i$, $(a_1, \ldots, a_n) \longmapsto a_i$. A topological space $(X, \mathcal{T})$ is denoted by X, and any non-empty subset U of X is equipped with the subspace topology, unless otherwise stated. A group $(G, \cdot, e)$ is denoted by G and we write gh for $g \cdot h$, for all $g, h \in G$. A right (left) group action of a group G on a set X is referred to as a right (left) action of G on X. For $n, m \in \mathbb{N}$, a map $f : U \to V$ between open subsets U and V of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, is said to be *smooth* if it is of class $\mathcal{C}^\infty$. For $K \in \{\mathbb{R}, \mathbb{C}\}$, we define $K_* := K \setminus \{0\}$, and for $n \in \mathbb{N}_{\geq 1}$, we define $\mathrm{Mat}(n, K)$ to be the set of all $n \times n$ matrices over K, and $\mathbb{I}_n$ is the $n \times n$ identity matrix. The subset $\mathrm{GL}(n, K)$ is the group of all invertible $n \times n$ matrices over K, and $\mathcal{H}(2, \mathbb{C}) = \{H \in \mathrm{GL}(n, \mathbb{C}) \mid H = H^\dagger\}$ is the set of all Hermitian matrices, where $A^\dagger$ denotes the conjugate transpose of $A \in \mathrm{GL}(n, \mathbb{C})$. Any $g \in \mathrm{GL}(n, K)$ and its inverse g^{-1} will be written as

$$g = \begin{pmatrix} g^1{}_1 & \cdots & g^1{}_n \\ \vdots & & \vdots \\ g^n{}_1 & \cdots & g^n{}_n \end{pmatrix}, \quad g^{-1} = \begin{pmatrix} g_1{}^1 & \cdots & g_n{}^1 \\ \vdots & & \vdots \\ g_1{}^n & \cdots & g_n{}^n \end{pmatrix},$$

so that $g^i{}_j$ denotes the (i, j)-th entry of g and $g_j{}^i$ denotes the (i, j)-th entry of g^{-1}. Throughout this thesis, we will employ the *Einstein summation convention*, meaning that in an expression of the form $\lambda^i e_i$, there is implied a summation over the index i, whose range will be clear from the context and will usually be the dimension of the space under consideration. Finally, the Hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

are called the *Pauli matrices*.

CHAPTER 1

Smooth Manifolds

In this chapter we will introduce the category of smooth manifolds, whose objects (the smooth manifolds) and morphisms (the smooth maps between them) will play an important role throughout this thesis. It provides the natural setting for Einsteins's theory of general relativity which models spacetime as a 4-dimensional smooth manifold, and underlines the departure from the Newtonian description of gravity as a force in Euclidean space, to Einstein's description of gravity as a property of spacetime. Furthermore, we will mention some basic properties of the tensor product, and we will discuss Minkowski space and the relation between the restricted Lorentz transformations and the group $\mathrm{SL}(2,\mathbb{C})$, for which we will borrow some theory on Lie groups.

1.1 SMOOTH MANIFOLDS

Intuitively, a manifold is a space which locally looks ordinary Euclidean space. An example is the earth, which is (ignoring the flattening at the poles) a sphere, and locally looks like a plane. The following definitions will make this precise. Let $n \in \mathbb{N}$.

Definition 1.1. Let M be a topological space. An *n-dimensional chart for M* is a pair (U, φ), where $U \subset M$ is open and φ is a homeomorphism onto an open subset $\varphi(U)$ of $\mathbb{R}^n$. The (continuous) map $x^i := \mathrm{Proj}_i \circ \varphi$ is called the *i-th coordinate function of φ*, for each $i \in [n]$, and we refer to the maps $x^1, \ldots, x^n$ as *local coordinates on U*.

Definition 1.2. A topological space M is *locally n-Euclidean* if for each $m \in M$ there exists an n-dimensional chart (U_m, φ_m) for M with $m \in U_m$.

Remarks. Let M be a locally n-Euclidean topological space, let $m \in M$, and let (U_m, φ_m) be an n-dimensional chart for M with $m \in U_m$. We say that (U_m, φ_m)

is an n-dimensional chart for M *at* m. For any other n-dimensional chart $(\tilde{U}_m, \tilde{\varphi}_m)$ for M at m, the map

$$(\varphi_m \circ \tilde{\varphi}_m^{-1})|_{\tilde{\varphi}(U_m \cap \tilde{U}_m)} : \tilde{\varphi}(U_m \cap \tilde{U}_m) \longrightarrow \varphi(U_m \cap \tilde{U}_m) \tag{1.1}$$

is a homeomorphism between two open subsets of $\mathbb{R}^n$, and is called an *overlap function.*

Definition 1.3. Let M be a topological space. An *n-dimensional topological atlas for M* is a set $\mathcal{A} = \{(U_i, \varphi_i) \mid i \in I\}$, where I is some indexing set, such that (U_i, φ_i) is an n-dimensional chart for M for each $i \in I$, and $M = \bigcup_{i \in I} U_i$.

Definition 1.4. A pair $(M, \mathcal{A})$ is an *n-dimensional topological manifold* if M is a topological space which is Hausdorff and second countable, and $\mathcal{A}$ is an n-dimensional topological atlas for M.

Definition 1.5. Let M be a topological space. A *smooth n-dimensional atlas $\mathcal{A}$* for M is an n-dimensional topological atlas $\mathcal{A}$ for M such that all overlap functions are smooth. An n-dimensional chart (U, φ) for M is *admissible* to a smooth n-dimensional atlas $\mathcal{A}$ for M if $\mathcal{A} \cup \{(U, \varphi)\}$ is a smooth n-dimensional atlas for M, and $\mathcal{A}$ is *maximal* if there are no n-dimensional charts $(U, \varphi) \notin \mathcal{A}$ which are admissible to $\mathcal{A}$. A *smooth structure on M* is a maximal smooth n-dimensional atlas $\mathcal{A}$ for M.

It is easy to see that a smooth n-dimensional atlas $\mathcal{A}$ for a topological space M determines a unique maximal smooth n-dimensional atlas $\mathcal{M}$ for M; for a proof, see Proposition 1.17 in [7]. However, two n-dimensional atlases $\mathcal{A}$ and $\mathcal{A}'$ for M need not be *smoothly compatible*, i.e. there can exist a chart $(U, \varphi) \in \mathcal{A}$ such that (U, φ) is not admissible to $\mathcal{A}'$. If this is the case, then $\mathcal{A}$ and $\mathcal{A}'$ define two different smooth structures $\mathcal{M}$ and $\mathcal{M}'$ on M, and the resulting smooth n-dimensional manifolds $(M, \mathcal{M})$ and $(M, \mathcal{M}')$ may or may not be "the same", i.e. there may or may not exist a diffeomorphism (see Definition 1.9) between them.

This brings up the question of how many "inequivalent" smooth structures can be defined on an n-dimensional topological manifold M, which has been addressed by, among others, Simon K. Donaldson, Michael H. Freedman and John W. Milnor (see the discussion on page 40 of [7] and the references mentioned there). In this thesis we will not be concerned with this question, but it is worth mentioning the result by Donaldson on the so-called *fake* $\mathbb{R}^4$*'s*, which states that there is an uncountable set of 4-dimensional smooth manifolds which are all homeomorphic to $\mathbb{R}^4$, but pairwise not diffeomorphic to each other[1]. This result supports the claim "dimension four is different", and while it may seem rather far-fetched to look for

[1]Incidentally, it is nice to note that key ideas in some of the proofs of these and other related results originated from the *Yang-Mills theories* developed in theoretical physics. See the preface in [8].

something physically significant in constructions of this kind, there has been an interest in how these concepts could be used to gain a better understanding of gravity [9–12].

Definition 1.6. A pair $(M, \mathcal{M})$ is an n-dimensional smooth manifold if M is a topological space which is Hausdorff and second countable, and $\mathcal{M}$ is a maximal smooth n-dimensional atlas for M. They are the objects of the category $\mathbf{Man}^\infty$ of smooth manifolds.

Henceforth, we will refer to an n-dimensional smooth manifold $(M, \mathcal{M})$ as a smooth manifold M and to an n-dimensional chart for M as a chart for M. If we say that something holds for each chart for M, we mean that it holds for each $(U, \varphi) \in \mathcal{M}$, where $\mathcal{M}$ is the smooth structure on M. From our definition it follows that every smooth manifold has the property of being *paracompact*; see Theorem 1.15 in [7] for a proof.

Examples 1.7. We list some examples of smooth manifolds which we will need later on.

1. The pair $(\mathbb{R}^n, \mathcal{M}_{\mathbb{R}^n})$, where $\mathcal{M}_{\mathbb{R}^n}$ is the *standard smooth structure on* $\mathbb{R}^n$ defined by $\mathcal{A}_{\mathbb{R}^n} = \{\mathbb{R}^n, \mathbb{I}_{\mathbb{R}^n}\}$, is an n-dimensional smooth manifold. Identifying $\mathrm{Mat}(n, \mathbb{R})$ with $\mathbb{R}^{n^2}$, we see that $\mathrm{Mat}(n, \mathbb{R})$ is an n^2-dimensional smooth manifold.
2. The n-dimensional sphere $(\mathbb{S}^n, \mathcal{M}_{\mathbb{S}^n})$ is an n-dimensional smooth manifold, conform Example 1.31 in [7]; this smooth structure on $\mathbb{S}^n$ is called the *standard smooth structure on* $\mathbb{S}^n$.
3. For any two smooth manifolds M and M', the product $M \times M'$ (equipped with the product topology) is clearly an $(n + n')$-smooth manifold, whose smooth structure is defined[2] by the smooth structures on M and M'.
4. Any non-empty open subset U of a smooth manifold M is a smooth manifold of the same dimension as M, whose smooth structure is the restriction[3] to U of the smooth structure on M.
5. An n-dimensional real vector space V is an n-dimensional smooth manifold, conform Example 1.24 in [7].
6. The general linear group $\mathrm{GL}(n, \mathbb{R}) = \det^{-1}(\mathbb{R}_*)$ is an open subset of $\mathrm{Mat}(n, \mathbb{R})$ since the determinant function is continuous, so $\mathrm{GL}(n, \mathbb{R})$ is an n^2-dimensional smooth manifold.

[2] If $\mathcal{M}$ and $\mathcal{M}'$ are the smooth structures on M and M', respectively, then

$$\mathcal{A}_\times := \{(U_M \times U_{M'}, \varphi_M \times \varphi_{M'}) \,|\, (U_M, \varphi_M) \in \mathcal{A}_M \wedge (U_{M'}, \varphi_{M'}) \in \mathcal{A}_{M'}\}$$

is an $(n+n')$-dimensional smooth atlas for $M \times M'$ which defines the smooth structure on $M \times M'$.

[3] If $\mathcal{M} = \{(U_i, \varphi_i) \,|\, i \in I\}$ is the smooth structure on M, then the restriction of $\mathcal{M}$ to U is the set $\mathcal{M}_U := \{(U \cap U_i, \varphi_i|_{U \cap U_i}) \,|\, i \in I\}$, which is clearly a smooth structure on U.

1.2 Smooth maps

Now that we know what a manifold is, we want to know if and how we can generalise the concept of a smooth function defined on Euclidean space to a smooth function on a manifold. Let M and M' be smooth manifolds.

Definition 1.8. Let $k \in \mathbb{N}$. A map $f : M \longrightarrow \mathbb{R}^k$ is *smooth* if for each chart (U, φ) for M the map $f \circ \varphi^{-1} : \varphi(U) \longrightarrow \mathbb{R}^k$ is smooth. The set of all smooth functions from M to $\mathbb{R}$ is denoted by $\mathcal{C}^\infty(M)$, and for any non-empty open subset U of M, the set of all smooth functions from U to $\mathbb{R}$ is denoted by $\mathcal{C}^\infty(M|_U)$.

Remark. The set $\mathcal{C}^\infty(M)$ is naturally a real vector space and a commutative ring, where the constant map $\mathbf{1} : M \longrightarrow \mathbb{R}$, $m \longrightarrow 1$ is the identity.

Definition 1.9. A continuous map $f : M \longrightarrow M'$ is *smooth* if the map

$$(\varphi' \circ f \circ \varphi^{-1})|_{\varphi(U \cap f^{-1}(U'))} : \varphi(U \cap f^{-1}(U')) \longrightarrow \tilde{\varphi}(U') \tag{1.2}$$

is smooth for each chart (U, φ) for M and for each chart (U', φ') for M'. A *diffeomorphism* is a smooth bijective map $f : M \longrightarrow M'$ such that f^{-1} is smooth, and M and M' are called *diffeomorphic* if there exists a diffeomorphism between them.

Remark. The identity map $\mathbb{I}_M : M \longrightarrow M$ is clearly smooth.

Proposition 1.10. Let M'' be a smooth manifold, and let $f : M \longrightarrow M'$ and $g : M' \longrightarrow M''$ be smooth maps. Then the composition $g \circ f : M \longrightarrow M''$ is smooth.

Proof. See Proposition 2.10 in [7]. □

The smooth maps are the morphisms in $\mathbf{Man}^\infty$, and by the previous remark and proposition, this indeed defines a category.

Definition 1.11. Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open cover of M. A *smooth partition of unity subordinate to* $\mathcal{U}$ is a set $\mathscr{P}_\mathcal{U} = \{p_i \mid i \in I\}$, where

- each $p_i \in \mathscr{P}_\mathcal{U}$ is a smooth map $p_i : U_i \longrightarrow \mathbb{R}$ such that $0 \leq p_i(m) \leq 1$ holds for all $m \in M$,
- for all $i \in I$ it holds that $\mathrm{supp}(p_i) := \overline{\{m \in U_i \mid p_i(m) \neq 0\}} \subset U_i$, and $\{\mathrm{supp}(p_i) \mid i \in I\}$ is locally finite, i.e. for each $m \in M$ there exist and open subset U of M with $m \in U$ such that U has non-empty intersection with only finitely many elements of $\{\mathrm{supp}(p_i) \mid i \in I\}$, and
- $\sum_{i \in I} p_i(m) = 1$ holds for all $m \in M$.

Theorem 1.12. *For any open cover $\mathcal{U}$ of M there exists a smooth partition of unity $\mathscr{P}_\mathcal{U}$ subordinate to $\mathcal{U}$.*

Proof. See Theorem 2.23 in [7]. □

1.2.1 *Lie groups*

Lie groups come up often in physics, as they are groups *and* manifolds, and can thus properly represent the smooth symmetries so important in physics.

Definition 1.13. A group G is a *Lie group* if G is a smooth manifold such that the multiplication $G \times G \longrightarrow G$, $(g,h) \longmapsto gh$ and inversion $G \longrightarrow G$, $g \longmapsto g^{-1}$ on G are smooth.

Definition 1.14. Let G and G' be Lie groups. A *Lie group homomorphism* is a group homomorphism $\lambda : G \longrightarrow G'$ which is also smooth.

Examples 1.15. We list some examples of Lie groups which we will encounter later on.

1. The matrix group $\mathrm{GL}(n, \mathbb{R})$ is a Lie group, since matrix multiplication and inversion (by Cramer's rule) are both smooth.
2. The circle $\mathbb{S}^1$, viewed as a subgroup of $\mathbb{C}_*$, is a compact Lie group called the *circle group*. We will also denote it by $\mathrm{U}(1)$.
3. The group of orthogonal matrices $\mathrm{O}(n, \mathbb{R}) = \det^{-1}(\{-1, 1\})$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ of dimension $\frac{1}{2}n(n-1)$, as is the indentity component $\mathrm{SO}(n, \mathbb{R}) = \det^{-1}(1)$. By the closed subgroup theorem (Theorem 20.12 in [7]), these groups are both Lie groups.
4. The special linear group $\mathrm{SL}(2, \mathbb{C}) = \{A \in \mathrm{Mat}(2, \mathbb{C}) \mid \det(A) = 1\}$ in two dimensions is a simply connected Lie group of dimension 6.

1.3 The tangent bundle

Let M be a smooth manifold, and let $m \in M$. We assume that $\dim(M) \in \mathbb{N}_{\geq 1}$.

Definition 1.16. A *tangent vector at* m is an element $v \in \mathrm{Hom}_{\mathbb{R}}(\mathcal{C}^\infty(M), \mathbb{R})$ such that $v(fg) = g(p)v(f) + f(p)v(g)$ holds for all $f, g \in \mathcal{C}^\infty(M)$. The *tangent space of* M *at* m is the real vector space of all tangent vectors at m, and is denoted by T_mM.

Let (U, φ) be a chart for M at m. Define for each $i \in [n]$ the map

$$\begin{aligned} \partial_i|_m : \mathcal{C}^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto D_i(f \circ \varphi^{-1})(\varphi(m)), \end{aligned} \tag{1.3}$$

where $D_i(f \circ \varphi^{-1})(\varphi(m))$ is the i-th partial derivative. By the chain rule this map is a tangent vector at m, for each $i \in [n]$. As the following proposition shows, and as makes sense intuitively, the tangent space is n-dimensional and is spanned by the maps defined above. We can thus view the tangent vectors as being the generalisation of the operation of taking directional derivatives.

Proposition 1.17. The set $\{\partial_1|_m, \ldots, \partial_n|_m\}$ is basis for T_mM, so T_mM is of dimension n.

Proof. See Proposition 3.15 in [7]. $\square$

Definition 1.18. The *cotangent space of M at m* is the dual space of T_mM, and is denoted by T_m^*M. For any chart (U, φ) for M at m, the basis dual to $\{\partial_1|_m, \ldots, \partial_n|_m\}$ is denoted by $\{dx^1|_m, \ldots, dx^n|_m\}$.

Definition 1.19. The *tangent bundle of M* is the disjoint union

$$TM := \coprod_{m \in M} T_mM, \tag{1.4}$$

and the *cotangent bundle of M* is the disjoint union

$$T^*M := \coprod_{m \in M} T_m^*M \tag{1.5}$$

There are natural *projections* $\pi_t : TM \longrightarrow M$ and $\pi_c : T^*M \longrightarrow M$.

Note that the fibres $\pi_t^{-1}(m)$ and $\pi_c^{-1}(m)$ are both isomorphic (as real vector spaces) to $\mathbb{R}^n$, so considering TM and T^*M as sets, they can both be viewed as $M \times \mathbb{R}^n$. To however be able to generalise the notion of a smooth vector field on $\mathbb{R}^n$ to a smooth vector field on M, we need a way of smoothly assigning to each point on the manifold an element of the tangent space at that point (i.e. in the fibre of π over that point). That is, we need a map $V : M \longrightarrow TM$ such that $V(m') \in \pi_t^{-1}(m')$ holds for all $m' \in M$, and this map should be smooth, so the tangent bundle should be a smooth manifold. Let (U, φ) be a chart for M. Any $v \in \pi_t^{-1}(U)$ can be written as $v = v^i\partial_i|_{m'}$ for some $(v^1, \ldots, v^n) \in \mathbb{R}^n$, where $m' \in U$ is such that $\pi_t(v) = m'$, so define a map

$$\begin{aligned} \psi_\varphi : \pi^{-1}(U) &\longrightarrow \varphi(U) \times \mathbb{R}^n \\ v &\longmapsto (\varphi(m), v^1, \ldots, v^n), \end{aligned} \tag{1.6}$$

which is clearly a bijection. Let $(\tilde{U}, \tilde{\varphi})$ be a chart for M such that $\tilde{U} \cap U \neq \emptyset$. Then

$$\begin{aligned} (\tilde{\psi}_{\tilde{\varphi}} \circ \psi_\varphi^{-1})(\varphi(m'), v^1, \ldots, v^n) &= \tilde{\psi}_{\tilde{\varphi}}(v^i\partial_i|_{m'}) \\ &= \tilde{\psi}_{\tilde{\varphi}}(v^iD_{1i}\tilde{\partial}_1|_{m'} + \ldots + v^iD_{ni}\tilde{\partial}_n|_{m'}) \\ &= (\tilde{\varphi}(m'), v^iD_{1i}, \ldots, v^iD_{ni}) \end{aligned} \tag{1.7}$$

holds by the chain rule for all $(\varphi(m'), v^1, \ldots, v^n) \in \varphi(U \cap \tilde{U}) \times \mathbb{R}^n$, where D_{ij} is the (i, j)-th entry of the Jacobian matrix $D(\tilde{\varphi} \circ \varphi^{-1})(\varphi(m'))$, for each $i, j \in [n]$, so $\tilde{\psi} \circ \psi^{-1}$ is smooth. We can thus define a topology on TM by declaring a subset V of TM to be open if $\psi_\varphi(V \cap \pi^{-1}(U))$ is open for each chart (U, φ) for M, and the smooth structure on TM is determined by the smooth atlas $\{(\pi^{-1}(U), \psi_\varphi) \,|\, (U, \varphi) \in \mathcal{A}\}$, where $\mathcal{A}$ is a smooth atlas for M. A similar procedure works for the cotangent bundle, and we have the following proposition.

Proposition 1.20. The sets TM and T^*M are $2n$-dimensional smooth manifolds such that π_t and π_c are smooth.

Proof. See Proposition 3.18 and Proposition 11.9 in [7]. □

In general, the tangent bundle of M will not be trivial, i.e. there won't be a diffeomorphism[4] $\Phi : TM \longrightarrow M \times \mathbb{R}^n$. Note that for any two charts (U, φ), $(\tilde{U}, \tilde{\varphi})$ for M, we can define a smooth map

$$\begin{aligned} g_{U\tilde{U}} : U \cap \tilde{U} &\longrightarrow \mathrm{GL}(n, \mathbb{R}) \\ m' &\longmapsto D(\tilde{\varphi} \circ \varphi^{-1})(\varphi(m')). \end{aligned} \tag{1.8}$$

As we will see in the next chapter, these maps actually *define* the tangent bundle, and the way in which they do determines how "non-trivial" the tangent bundle is. Note that if we consider a manifold which is covered by a single chart (M, φ), such as $\mathbb{R}^n$ or some finite-dimensional vector space, then $\varphi : M \longmapsto \varphi(M)$ is a diffeomorphism, so $\psi : TM \longmapsto \varphi(M) \times \mathbb{R}^n$ defines a diffeomorphism between TM and $M \times \mathbb{R}^n$. Now that we have a smooth structure on TM, we can define what a smooth vector field is on a manifold.

Definition 1.21. A *smooth vector field on* M is a smooth map $V : M \longrightarrow TM$ such that $\pi \circ V = \mathbb{I}_M$. The set of all smooth vectorfields on M is denoted by $\Gamma(TM)$. A *smooth covector field on* M is a smooth map $\omega : M \longrightarrow T^*M$ such that $\pi \circ \omega = \mathbb{I}_M$. The set of all smooth covector fields on M is denoted by $\Gamma(T^*M)$.

The first part of this definition indeed amounts to a smooth assignment of a tangent vector at each point $m' \in M$, and what is important, at the tangent space $T_{m'}M$ at m'. As we know from calculus, in $\mathbb{R}^n$ any vector field can be written as a linear combination of the vector fields determined by the standard basis, i.e. the smooth functions $E_i : \mathbb{R}^n \longmapsto \mathbb{R}^n$, $m \longmapsto (0, \ldots, 1, \ldots, 0)$ for all $i \in I$, where the 1 is in the i-th slot. Similarly, we can define global coordinates in Minkowski space, i.e. spacetime without gravity, since this is also just a vector space. In general, however, this won't be possible, which forces us to work locally in a chart (U, φ), where we have the coordinate vector fields $\partial_i : U \longrightarrow TM$, $m \longmapsto \partial_i|_m$. This leads to the following definition.

Definition 1.22. The tangent bundle of M is *parallelisable* if there exist smooth vector fields $V_1, \ldots, V_n$ such that $\{V_1(m'), \ldots, V_n(m')\}$ is a basis for $T_{m'}M$, for all $m' \in M$.

[4] This diffeomorphism should in fact also satisfy some other property, which we will discuss later on.

When we deal with $\mathbb{R}^n$, we are used to taking the standard basis $\{e_1, \ldots, e_n\}$, which we refer to as right-handed. Since there are many things in physics where some sort of "right-hand rule" comes up, we tend to forget that taking basis is still *only* a choice. To formalise what we mean by this choice, let V be a finite-dimensional vector space, and let $\mathcal{B}(V)$ be the set of all bases for V. We can define an equivalence relation on $\mathcal{B}(V)$, by letting $B, B' \in \mathcal{B}(V)$ be equivalent if and only if $\det(T_{BB'}) > 0$, where $T_{BB'} : V \longrightarrow V$ is the $\mathbb{R}$-linear isomorphism sending $e_i \in B$ to $e_i' \in B'$ for each $i \in [\dim(V)]$. Since for any $B, B' \in \mathcal{B}(V)$ it holds that $T_{BB} = \mathbb{I}_V$ and $T_{BB'} = T_{B'B}^{-1}$, and for any $B'' \in \mathcal{B}(V)$ it holds that $T_{BB''} = T_{B'B''} \circ T_{BB'}$, it follows from the multiplicative property of the determinant that this is indeed an equivalence relation. The set $\mathcal{B}(V)/\sim$ clearly has two elements, and an orientation in V is defined as a choice of an element $\mathcal{O} \in \mathcal{B}(V)$.

We also have the notion of orientability in a smooth manifold, which comes down to a way of consistently choosing an orientation in each tangent space.

Definition 1.23. A smooth manifold M' is said to be *orientable* if there exists an atlas $\{(U_i, \varphi_i) \,|\, i \in I\}$ for M' such that for each $i, j \in I$ with $U_i \cap U_j \neq \emptyset$, it holds that $\det(D(\varphi_j \circ \varphi^{-1})(\varphi(m'))) > 0$ for all $m' \in U_i \cap U_j$.

The classical example of a non-orientable manifold is the *Möbius strip*.

1.4 The tensor product

Let R be a commutative ring with unity, and let M and N be R-modules[5].

Definition 1.24. The *tensor product of M and N over R* is an R-module $M \otimes_R N$ equipped with an R-bilinear map $T : M \times N \longrightarrow M \otimes_R N$, $(m, n) \longmapsto m \otimes n$ satisfying the universal property

- *(Universal property of the tensor product)* Let P be an R-module. For each R-bilinear map $B : M \times N \longrightarrow P$, there exists a unique R-linear map $\tilde{B} : M \otimes_R N \longrightarrow P$ such that the diagram

$$\begin{tikzcd} M \times N \arrow[r, "T"] \arrow[dr, "B"'] & M \otimes_R N \arrow[d, "\tilde{T}"] \\ & P \end{tikzcd} \tag{1.9}$$

  commutes.

[5] Any R-module is assumed to be unital. The dual of M is $M^* := \mathrm{Hom}_R(M, R)$.

Proposition 1.25. The tensor product $M \otimes_R N$ exists and is unique, up to isomorphism.

Proof. See Proposition 2.12 in [13]. $\square$

Remarks. The R-module $M \otimes_R N$ is generated by elements of the form $m \otimes n$ with $m \in M$ and $n \in N$, and it follows from the definition that the equalities

$$\begin{aligned} (m+m') \otimes n &= m \otimes n + m' \otimes n, \\ m \otimes (n+n') &= m \otimes n + m \otimes n', \\ r(m \otimes n) &= (rm) \otimes n) \\ &= m \otimes (rn) \end{aligned} \tag{1.10}$$

hold for all $m, m' \in M$, $n, n' \in N$, and $r \in R$.

Proposition 1.26. Let P be an R-module. The maps

$$\begin{aligned} M \otimes_R N &\longrightarrow N \otimes_R M \\ m \otimes n &\longmapsto n \otimes m, \\ (M \otimes_R N) \otimes_R P &\longrightarrow M \otimes_R N \otimes_R P \\ (m \otimes n) \otimes p &\longmapsto m \otimes n \otimes p, \\ M \otimes_R N \otimes_R P &\longrightarrow M \otimes_R (N \otimes_R P) \\ m \otimes n \otimes p &\longmapsto m \otimes (n \otimes p), \end{aligned} \tag{1.11}$$

are R-module isomorphisms.

Proof. These maps and their inverses are easily constructed using the universal property of the tensor product. $\square$

Proposition 1.27. For any R-module P, the R-modules $\mathrm{Hom}_R(M \otimes_R N, P)$ and $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ are isomorphic. In particular, there is an isomorphism $(M \otimes_R N)^* \cong \mathrm{Hom}_R(M, N^*)$.

Proof. See the remarks before Proposition 2.18 in [13]. $\square$

Let M be a smooth manifold. The real vector spaces $\Gamma(TM)$ and $\Gamma(T^*M)$ are naturally $\mathcal{C}^\infty(M)$-modules[6], and are in fact both reflexive $\mathcal{C}^\infty(M)$-modules. To see this, we will argue that $\Gamma(T^*M) \cong \Gamma(TM)^*$ and $\Gamma(T^*M)^* \cong \Gamma(TM)$, from which the statement then follows immediately. Define a map

$$\begin{aligned} f : \Gamma(T^*M) &\longrightarrow \Gamma(TM)^* \\ \omega &\longmapsto \tilde{\omega}, \end{aligned} \tag{1.12}$$

[6]The multiplications are defined by $(fX)(m) := f(m)X(m)$ and $(f\omega)(m) := f(m)\omega(m)$ respectively, for all $f \in \mathcal{C}^\infty(M)$, $X \in \Gamma(TM)$, $\omega \in \Gamma(T^*M)$ and $m \in M$.

where $f(\omega)(V) = \tilde{\omega}(V) := \omega V$ is defined as $\omega V(m) := \omega(m)(V(m)) \in \mathbb{R}$ for all $\omega \in \Gamma(T^*M), V \in \Gamma(TM)$ and for all $m \in M$. Then f is well-defined, since for all $\omega \in \Gamma(T^*M), V \in \Gamma(TM)$ and for each chart (U, φ) for M, there are smooth functions $\omega_1, \dots, \omega_n \in \mathcal{C}^\infty(M|_U)$ and $V^1, \dots, V^n \in \mathcal{C}^\infty(M|_U)$ such that[7] $\omega = \omega_i dx^i$ and $V = V^i \partial_i$, and

$$\begin{aligned} \omega V(m) &= \omega_i(m) dx^i|_m (V^j(m) \partial_j|_m) \\ &= \omega_i(m) V^j(m) dx^i|_m (\partial_j|_m) \\ &= \omega_i(m) V^i(m) \end{aligned} \tag{1.13}$$

holds for all $m \in U$, so $\omega V \in \mathcal{C}^\infty(M)$. It is clear that f is $\mathcal{C}^\infty(M)$-linear and thus a $\mathcal{C}^\infty(M)$-module homomorphism. Define a map

$$\begin{aligned} f^{-1} : \Gamma(TM)^* &\longrightarrow \Gamma(T^*M) \\ \varphi &\longmapsto \omega_\varphi \end{aligned} \tag{1.14}$$

and define $\omega_\varphi(m)(v_m) := \varphi(V)(m)$ for all $\varphi \in \Gamma(TM)^*, m \in M$ and for all $v_m \in T_m M$, where $V \in \Gamma(TM)$ is such that $V(m) = v_m$, which always exists by Proposition 8.7 in [7]. Then $\omega_\varphi(m) \in T_m^* M$ for all $\varphi \in \Gamma(TM)$ and for all $m \in M$, and from the proof on pages 265-266 in [14] it follows that this map is well-defined (i.e. it does not depend on the choice of V in the definition of φ_m). Since $\varphi(V) \in \mathcal{C}^\infty(M)$ for all $\varphi \in \Gamma(TM)^*$ and $V \in \Gamma(TM)$, the map f^{-1} indeed maps into $\Gamma(T^*M)$. It is easily checked using the definitions that f^{-1} is $\mathcal{C}^\infty(M)$-linear and that f^{-1} is the inverse of f, so $\Gamma(TM)^* \cong \Gamma(T^*M)$. Mimicking this proof, we find that $\Gamma(T^*M)^*$ and $\Gamma(TM)$ are also isomorphic, so $\Gamma(TM)$ and $\Gamma(T^*M)$ are both reflexive, enabling us to identify the double dual $(\Gamma(TM)^*)^*$ (respectively $(\Gamma(T^*M)^*)^*$) with $\Gamma(TM)$ (respectively $\Gamma(T^*M)$).

Finally, in most physics textbooks, tensors are introduced in a somewhat different manner [15,16], and it's worth to take a moment to see how it corresponds to the formal definition given here. Let $p, q \in \mathbb{N}$ and $m \in M$. A (p,q)-*tensor* T is a $\mathbb{R}$-multilinear map[8] $T : T_m^* M^{\times p} \times T_m M^{\times q} \longrightarrow \mathbb{R}$, which descends to a linear map $\tilde{T} : T_m^* M^{\otimes p} \otimes_{\mathbb{R}} T_m M^{\otimes q} \longrightarrow \mathbb{R}$, and since $T_m M$ is finite-dimensional, this corresponds to an element of $T_m M^{\otimes p} \otimes_{\mathbb{R}} T_m^* M^{\otimes q}$.

[7] Here $dx^i : U \longrightarrow T^*M$, $m \longmapsto dx^i|_m$ are the coordinate covector fields on U, for each $i \in [n]$. Note that the coordinate vector and covector fields constitute a basis for $\Gamma(TM|_U)$ and $\Gamma(T^*M|_U)$ respectively, where $\Gamma(TM|_U)$ is just the set of smooth vector fields on U, and similarly for $\Gamma(T^*M|_U)$.

[8] This is just the Cartesian product $T_m^* M \times \ldots \times T_m^* M \times T_m M \times \ldots \times T_m M$, where there are p copies of $T_m^* M$ and q copies of $T_m M$, and similarly for $T_m^* M^{\otimes p} \otimes_{\mathbb{R}} T_m M^{\otimes q}$. Of course, all tensor products in $T_m^* M^{\otimes p} \otimes_{\mathbb{R}} T_m M^{\otimes q}$ are taken over the same ring, in this case over $\mathbb{R}$.

1.5 Minkowski space and $\mathrm{SL}(2,\mathbb{C})$

Minkowski space serves as the model for spacetime in the absence of gravity. One of Einstein's postulates was that spacetime should "locally look like Minkowski space", a fact which is mathematically expressed by the fact (as we will see later) that each tangent space to the spacetime manifold is (isomorphic to) Minkowski space. We should define then, what Minkowski space is.

Definition 1.28. The real vector space $\mathbb{R}^4$ equipped with a non-degenerate symmetric bilinear form $B : \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$ of signature $(1,3)$ is called *Minkowski space* and is denoted by $\mathscr{M}$.

Define the matrix

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1.15}$$

and let $\mathcal{B} = \{e_0, e_1, e_2, e_3\}$ be a basis for $\mathscr{M}$ such that[9] $B(v,w) = v \cdot \eta w$, for all $v, w \in \mathscr{M}$. Choose the equivalence class of $\mathcal{B}$ as an orientation in $\mathscr{M}$. The *homogeneous Lorentz group* $\mathcal{L}$ is the group consisting of all linear transformations (called *Lorentz transformations*) of $\mathscr{M}$ which preserve the quadratic form Q induced by B, i.e. all linear maps $\Lambda : \mathscr{M} \longrightarrow \mathscr{M}$ such that $Q(\Lambda(v)) = Q(v)$ holds for all $v \in \mathscr{M}$. Its matrix representation (with respect to this basis) is the group

$$\mathrm{O}(1,3) := \{L \in \mathrm{GL}(4,\mathbb{R}) \mid L^\top \eta L = \eta\}. \tag{1.16}$$

For each $L \in \mathrm{O}(1,3)$, it follows from $L^\top \eta L = \eta$ that $(\det(L))^2 = 1$ and thus $\det(L) = \pm 1$. Moreover, it follows that $(L^1{}_1)^2 - (L^2{}_1)^2 - (L^3{}_1)^2 - (L^4{}_1)^2 = 1$, which implies that $|L^1{}_1| \geq 1$. This group thus splits up into four connected components, according to the sign of the determinant and the sign of $L^1{}_1$.

In physics, we often only want to consider Lorentz transformations which reverse neither time nor parity. Mathematically, this means that we have to consider the *restricted Lorentz group*, which is the subgroup

$$\mathrm{SO}^\uparrow(1,3) := \{L \in \mathrm{O}(1,3) \mid \det(L) = 1 \wedge L^1{}_1 \geq 1\} \tag{1.17}$$

and the identity component of $\mathrm{O}(1,3)$.

We will now show how $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}^\uparrow(1,3)$ are related. Define $\sigma_0 := \mathbb{I}_2$, and note that $\mathcal{H}(2,\mathbb{C}) = \mathcal{L}_\mathbb{R}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ (the $\mathbb{R}$-linear span of $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$). Indeed, it is clear that any matrix in $\mathcal{L}_\mathbb{R}\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ is Hermitian. To establish

[9] Note that such a basis exists by Sylvester's law of inertia. Also, $v \cdot w$ denotes the regular inner product on $\mathbb{R}^4$, for all $v, w \in \mathscr{M}$.

the other inclusion, using the Hermitian condition $H = H^\dagger$ for any $H \in \mathcal{H}(2,\mathbb{C})$ it is easily shown that H can be written as

$$H = \tfrac{1}{2}\mathrm{Tr}(H)\sigma_0 + \Re({H^2}_1)\sigma_1 + \Im({H^1}_2)\sigma_2 + \tfrac{1}{2}({H^1}_1 - {H^2}_2)\sigma_3. \tag{1.18}$$

The following proposition serves as the starting point to establish the relation between $\mathrm{SO}^\uparrow(1,3)$ and $\mathrm{SL}(2,\mathbb{C})$.

Proposition 1.29. The $\mathbb{R}$-linear extension φ of the assignment

$$\forall\, i \in \{0,1,2,3\} : \mathscr{M} \ni e_i \longmapsto \sigma_i \in \mathcal{H}(2,\mathbb{C}) \tag{1.19}$$

defines a linear isomorphism between $\mathscr{M}$ and $\mathcal{H}(2,\mathbb{C})$.

Proof. The map φ is clearly a bijection and thus a linear isomorphism. It is nevertheless useful to write down the inverse, which is given by $\varphi^{-1} : \mathcal{H}(2,\mathbb{C}) \longrightarrow \mathscr{M}$, $H \to \sum_{i=0}^{3} \mathrm{Tr}(H\sigma_i)e_i$. This map is linear because the trace is linear, and because $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$ holds for all $i,j \in \{0,1,2,3\}$, it is indeed the inverse of φ. $\square$

The reason why this isomorphism is useful is made clear by the following observation. Let $v = v^i e_i \in \mathscr{M}$. Then

$$\begin{aligned} \det(\varphi(v)) &= \det\left(\begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix}\right) \\ &= (v^0)^2 - (v^1)^2 - (v^2)^2 - (v^3)^2 \\ &= Q(v), \end{aligned} \tag{1.20}$$

so we may equally well work with Hermitian matrices instead of elements of $\mathscr{M}$. Let $S \in \mathrm{SL}(2,\mathbb{C})$ and $v = v^i e_i \in \mathscr{M}$. For all $H \in \mathcal{H}(2,\mathbb{C})$, the matrix $SHS^\dagger$ is Hermitian and $\det(S\varphi(v)S^\dagger) = \det(\varphi(v))$, so $\varphi^{-1}(S\varphi(v)S^\dagger) = L_S v$ for some $L_S \in \mathrm{O}(1,3)$. Note that $L_S = L_{-S}$. For each $j \in \{0,1,2,3\}$, we have

$$\begin{aligned} (L_S v)_j &= (\varphi^{-1}(S\varphi(v)S^\dagger))_j \\ &= \tfrac{1}{2} v^i \mathrm{Tr}(S\sigma_i S^\dagger \sigma_j), \end{aligned} \tag{1.21}$$

and we find[10]

$$\begin{aligned} 2{(L_S)^1}_1 &= \alpha\overline{\alpha} + \beta\overline{\beta} + \gamma\overline{\gamma} + \delta\overline{\delta}, & 2{(L_S)^1}_2 &= \alpha\overline{\beta} + \beta\overline{\alpha} + \gamma\overline{\delta} + \delta\overline{\gamma}, \\ 2{(L_S)^2}_1 &= \alpha\overline{\gamma} + \gamma\overline{\alpha} + \beta\overline{\delta} + \delta\overline{\beta}, & 2{(L_S)^2}_2 &= \alpha\overline{\delta} + \delta\overline{\alpha} + \gamma\overline{\beta} + \beta\overline{\gamma}, \\ 2{(L_S)^3}_1 &= i(\alpha\overline{\gamma} - \gamma\overline{\alpha} + \beta\overline{\delta} - \delta\overline{\beta}), & 2{(L_S)^3}_2 &= i(\alpha\overline{\delta} - \delta\overline{\alpha} + \beta\overline{\gamma} - \gamma\overline{\beta}), \\ 2{(L_S)^4}_1 &= \alpha\overline{\alpha} + \beta\overline{\beta} - \gamma\overline{\gamma} - \delta\overline{\delta}, & 2{(L_S)^4}_2 &= \alpha\overline{\beta} + \beta\overline{\alpha} - \gamma\overline{\delta} - \delta\overline{\gamma}, \end{aligned} \tag{1.22}$$

[10] We spare the reader the explicit calculations. Here $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ are such that $S = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

$$
\begin{aligned}
2(L_S)^1{}_3 &= i(-\alpha\overline{\beta}+\beta\overline{\alpha}-\gamma\overline{\delta}+\delta\overline{\gamma}), & 2(L_S)^1{}_4 &= \alpha\overline{\alpha}-\beta\overline{\beta}+\gamma\overline{\gamma}-\delta\overline{\delta},\\
2(L_S)^2{}_3 &= i(-\alpha\overline{\delta}+\delta\overline{\alpha}+\beta\overline{\gamma}-\gamma\overline{\beta}), & 2(L_S)^2{}_4 &= \alpha\overline{\gamma}+\gamma\overline{\alpha}-\beta\overline{\delta}-\delta\overline{\beta},\\
2(L_S)^3{}_3 &= \alpha\overline{\delta}+\delta\overline{\alpha}-\beta\overline{\gamma}-\gamma\overline{\beta}, & 2(L_S)^3{}_4 &= i(\alpha\overline{\gamma}-\gamma\overline{\alpha}-\beta\overline{\delta}+\delta\overline{\beta}),\\
2(L_S)^4{}_3 &= i(-\alpha\overline{\beta}+\beta\overline{\alpha}+\gamma\overline{\delta}-\delta\overline{\gamma}), & 2(L_S)^4{}_4 &= \alpha\overline{\alpha}-\beta\overline{\beta}-\gamma\overline{\gamma}+\delta\overline{\delta}.
\end{aligned}
\tag{1.23}
$$

It is clear that $(L_S)^i{}_j \in \mathbb{R}$ for all $i,j \in [4]$, and that $(L_S)^1{}_1 > 0$ and thus $(L_S)^1{}_1 \geq 1$ holds, and another explicit calculation shows that $\det(L_S) = (\alpha\delta - \beta\gamma)(\overline{\alpha}\overline{\delta} - \overline{\beta}\overline{\gamma}) = 1$, so $L_S \in \mathrm{SO}^{\uparrow}(1,3)$. We can thus define a map

$$
\begin{aligned}
\rho : \mathrm{SL}(2,\mathbb{C}) &\longrightarrow \mathrm{SO}^{\uparrow}(1,3)\\
S &\longmapsto L_S,
\end{aligned}
\tag{1.24}
$$

which is smooth, as follows from the explicit expressions in (1.22). Since $\rho(\sigma_0) = \mathbb{I}_4$ and

$$
\begin{aligned}
L_{S_1S_2}(v) &= \rho(S_1S_2)(v)\\
&= S_1S_2\varphi(v)S_2^{\dagger}S_1^{\dagger}\\
&= S_1(L_{S_2}v)S_1^{\dagger}\\
&= L_{S_1}(L_{S_2}(v)),
\end{aligned}
\tag{1.25}
$$

holds for all $S_1, S_2 \in \mathrm{SL}(2,\mathbb{C})$, it is a Lie group homomorphism. Its image $\rho(\mathrm{SL}(2,\mathbb{C}))$ is connected in $\mathrm{SO}^{\uparrow}(1,3)$ since $\mathrm{SL}(2,\mathbb{C})$ is simply connected, and $\ker(\rho) \cong \{-\sigma_0, \sigma_0\} =: Z$. It is also surjective[11], so ρ descends to a group isomorphism $\overline{\rho} : \mathrm{SL}/Z \longmapsto \mathrm{SO}^{\uparrow}(1,3)$, which is in fact a Lie group isomorphism by Theorem 21.27 in [7]. This in fact shows that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{SO}^{\uparrow}(1,3)$. Namely, consider the action of Z on $\mathrm{SL}(2,\mathbb{C})$ defined by matrix multiplication. This action is smooth, free and thus proper, as follows from Corollary 21.6 in [7], since Z is a compact Lie group. Theorem 21.23 then guarantees that the quotient map $\pi : \mathrm{SL}(2,\mathbb{C}) \longmapsto \mathrm{SL}(2,\mathbb{C})/Z$ is a (smooth) covering map, which is clearly a double covering as $Z \cong \mathbb{Z}/2\mathbb{Z}$. The diagram

$$
\begin{tikzcd}
\mathrm{SL}(2,\mathbb{C}) \arrow[d, "\pi"'] \arrow[dr, "\rho"] & \\
\mathrm{SL}(2,\mathbb{C})/Z \arrow[r, "\overline{\rho}"'] & \mathrm{SO}^{\uparrow}(1,3)
\end{tikzcd}
\tag{1.26}
$$

[11] This is proven explicitly in section 1.7 in [17].

commutes and $\overline{\rho}$ is a Lie group isomorphism, so $\mathrm{SL}(2,\mathbb{C})$, is the (since $\mathrm{SL}(2,\mathbb{C})$ is simply connected) double cover of $\mathrm{SO}^{\uparrow}(1,3)$. Incidentally, this also shows that the Lorentz group is not simply connected, as it follows that $\pi_1(\mathrm{SO}^{\uparrow}(1,3)) = \mathbb{Z}/2\mathbb{Z}$.

What follows from the above observations is that the Lorentz group is isomorphic to the *projective special linear group* $\mathbb{P}\mathrm{SL}(2,\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})/Z(\mathrm{SL}(2,\mathbb{C}))$, where $Z(\mathrm{SL}(2,\mathbb{C})) := \{\lambda\sigma_0 \,|\, \lambda \in \mathbb{C} : \det(\lambda\sigma_0) = 1\} = Z$ is the (normal) subgroup consisting of all scalar multiples of σ_0 with unit determinant, which naturally acts on the complex projective line $\mathbb{P}^1(\mathbb{C})$. This group can in turn be identified with the Möbius group, which is the automorphism group $\mathrm{Aut}(\mathbb{C}_\infty)$ of the Riemann sphere $\mathbb{C}_\infty$. This observation was key for Penrose to introduce spinors in general relativity (see chapter 1 in [2]).

We can shortly and informally discuss how 2-spinors arise naturally from the conclusion that $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{SO}^{\uparrow}(1,3)$, since to say of all this properly, one must really turn to representation theory and study the representations of $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{SO}^{\uparrow}(1,3)$. One can, loosely speaking, define 2-spinors as elements of $\mathbb{C}^2$, which is the representation space of the regular matrix representation of the special linear group, since $\mathrm{SL}(2,\mathbb{C})$ acts on $\mathbb{C}^2$ by matrix multiplication. Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{C}^2$, and let $\kappa = (\zeta, \eta) \in \mathbb{C}^2$. The matrix

$$\begin{aligned} H_\kappa :&= \kappa\kappa^\dagger \\ &= \begin{pmatrix} \zeta \\ \eta \end{pmatrix} \begin{pmatrix} \overline{\zeta} & \overline{\eta} \end{pmatrix} \\ &= \begin{pmatrix} \zeta\overline{\zeta} & \zeta\overline{\eta} \\ \eta\overline{\zeta} & \eta\overline{\eta} \end{pmatrix} \end{aligned} \tag{1.27}$$

is clearly Hermitian, and thus defines an element $\varphi^{-1}(H_\kappa) \in \mathscr{M}$. This vector is *null*, i.e. $Q(\varphi^{-1}) = \det(H_\kappa) = \zeta\overline{\zeta}\eta\overline{\eta} - \zeta\overline{\eta}\overline{\zeta}\eta = 0$. Note also that $-\kappa$ defines the same matrix, i.e. $H_{-\kappa} = H_\kappa$, and thus the same element of $\mathscr{M}$. Now let $A \in \mathrm{SL}(2,\mathbb{C})$. Then $H_{A\kappa} = A\kappa(A\kappa)^\dagger = A\kappa\kappa^\dagger A^\dagger = AHA^\dagger = L_A(\varphi^{-1}(H))$, so we can equivalently consider the action of $\mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2$ or the action of the restricted Lorentz group on $\mathscr{M}$, except for the sign-ambiguity which exists since κ and $-\kappa$ define the same element of $\mathscr{M}$[12]. This sign ambiguity is then of course precisely the potential reason why spinors cannot be defined properly on a manifold; more on this can be found in chapter 1 of [2].

[12] As it is written now, any element $e^{i\theta}\kappa$ with $\theta \in \mathbb{R}$ defines the same element of $\mathscr{M}$. However, when this is all defined properly, this freedom essentially disappears and we are only left with $\pm\kappa$. See chapter 1 in [2].

Chapter 2

Fibre Bundles

In this chapter we will introduce the notion of fibre bundles, objects which come up naturally in almost any physical theory that has some group of symmetries associated to[1] it which encodes the symmetries associated to the specific theory, called the *gauge group* in physics. As mentioned in section 1.5, a group which is of interest in the theory of both special and general relativity is the restricted Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$, the group of "proper" symmetries of Minkowski space. The demand (which is only there because of physical reasons) that we should have the freedom to consider all observers which are connected to some initially chosen proper frame of reference (i.e. a basis of the tangent space to the spacetime manifold) by a restricted Lorentz transformation can be roughly translated to the mathematical demand that there should be a principal $\mathrm{SO}^{\uparrow}(1,3)$-bundle over the spacetime manifold. To define this notion properly, and to see how we can extend this to a description of spacetime which allows for spinors, we have to start with the frame bundle of a manifold, which has the bigger group $\mathrm{GL}(n,\mathbb{R})$ as its symmetry group.

2.1 Fibre bundles

Let M and M' be manifolds.

Definition 2.1. A *smooth fibre bundle over* M is a triple $\mathcal{F} = (E, \pi, F)$, where

- E and F are manifolds, called the *total space* and *typical fibre of* $\mathcal{F}$, respectively, and

[1] Each of the four fundamental forces known in physics has associated to it a group of symmetries. For example, the unitary group $\mathrm{U}(1)$ consisting of all complex numbers of norm 1 is the symmetry group for electrodynamics, the strong interaction has the special unitary group $\mathrm{SU}(3)$ in 3 dimensions, and the weak interaction has $\mathrm{SU}(2)$. The principal bundle approach to incorporating the particular symmetry group "of interest" into a formal mathematical formulation of a physical theory has led to the advent of the earlier mentioned Yang-Mills theories, which have so far proved to be very successful in describing nature.

- $\pi : E \longrightarrow M$ is a smooth surjective map, called the *projection of* $\mathcal{F}$, such that for each $m \in M$ there exists an open subset U of M containing m and a diffeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times F$, such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi|_{\pi^{-1}(U)}"'] & U \times F \arrow[dl, "\mathrm{Proj}_1"] \\ U & \end{tikzcd} \tag{2.1}$$

  commutes.

Remarks. It follows from the definition that $E_m := \pi^{-1}(m)$, the *fibre of* π *over* m, is diffeomorphic to F, for each $m \in M$. Any pair (U, φ), where $U \subset M$ is open and $\varphi : \pi^{-1}(U) \longrightarrow U \times F$ is a diffeomorphism, for which (2.1) commutes, is called a *local trivialisation of* $\mathcal{F}$. Any set $\mathcal{C} = \{(U_i, \varphi_i) \,|\, i \in I\}$, where I is some indexing set and (U_i, φ_i) is a local trivialisation of $\mathcal{F}$ for all $i \in I$, such that $M = \bigcup_{i \in I} U_i$ holds, is called a *trivialising cover of* M. A smooth fibre bundle $\mathcal{F} = (E, \pi, F)$ over M will be referred to as a fibre bundle over M, and will be written as $F \longrightarrow E \xrightarrow{\pi} M$ or simply $E \xrightarrow{\pi} M$.

Example 2.2. Let F be a manifold. The triple $(M \times F, \mathrm{Proj}_1, F)$ is a fibre bundle over M, called the *trivial bundle over* M. A trivialising cover is given by $\{(M, \mathrm{Proj}_1 \times \mathbb{I}_F)\}$.

Definition 2.3. Let $\mathcal{F}$ and $\mathcal{F}'$ be fibre bundles over M and M', respectively. A *bundle map from* $\mathcal{F}$ *to* $\mathcal{F}'$ is a pair (Φ, φ), where $\Phi : E \longrightarrow E'$ and $\varphi : M \longrightarrow M'$ are smooth maps, such that the diagram

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\varphi"'] & M' \end{tikzcd} \tag{2.2}$$

commutes.

Remarks. The map φ in Definition 2.3 is uniquely and completely determined by Φ, since π is surjective; the map Φ is said to *cover* φ. Two fibre bundles $\mathcal{F}$ and $\mathcal{F}'$ over M are *equivalent* if there exists a bundle map $(\Phi, \mathbb{I}_M)$ from $\mathcal{F}$ to $\mathcal{F}'$ with Φ a diffeomorphism; such a bundle map is called a *bundle equivalence* between $\mathcal{F}$ and $\mathcal{F}'$, and $\mathcal{F}$ is *trivial* if it is equivalent to $(M \times F, \mathrm{Proj}_1, F)$. It follows that there is a category $\mathbf{Bun}$, where any object is a fibre bundle over some manifold, and a morphism from a fibre bundle $\mathcal{F}$ over M to a fibre bundle $\mathcal{F}'$ over M' is a bundle map (Φ, φ) from $\mathcal{F}$ to $\mathcal{F}'$.

2.1.1 *Transition functions*

Definition 2.4. Let $\mathcal{F}$ be a fibre bundle over M. A *section* of $\mathcal{F}$ is a smooth map $s : U \to E$ such that $\pi \circ s = \mathbb{I}|_U$, where $U \subset M$ is open and non-empty. A section is called *local* if U is a proper subset, and *global* if $U = M$. For any proper open subset U of M the set of all local sections is denoted by $\Gamma(E|_U)$, and the set of all global sections is denoted by $\Gamma(P)$.

Let $F \longrightarrow E \stackrel{\pi}{\longrightarrow} M$ be a fibre bundle over M and $\mathcal{C} = \{(U_i, \varphi_i) \mid i \in I\}$ a trivialising cover of M. Let $i, j, k \in I$ be such that $U_{ij} := U_i \cap U_j \neq \emptyset$ and $U_{ijk} := U_i \cap U_j \cap U_k \neq \emptyset$. Then

$$(\varphi_i \circ \varphi_j^{-1})|_{U_{ij} \times F} : U_{ij} \times F \longrightarrow U_{ij} \times F \tag{2.3}$$

is a diffeomorphism, so for each $m \in U_{ij}$, the map

$$\begin{aligned} \varphi_{i,m} \circ \varphi_{j,m}^{-1} : F &\longrightarrow F \\ f &\longmapsto \mathrm{Proj}_{2,m} \circ \varphi_i \circ \varphi_j^{-1} \circ \mathrm{Proj}_{2,m}^{-1} \end{aligned} \tag{2.4}$$

is a diffeomorphism. We can thus define a smooth map[2]

$$\begin{aligned} t_{ij} : U_{ij} &\longrightarrow \mathrm{Diff}(F) \\ p &\longmapsto \varphi_{i,m} \circ \varphi_{j,m}^{-1}, \end{aligned} \tag{2.5}$$

called a *transition function*. Note that for any $m \in U_{ij}$ and for any $e \in \pi^{-1}(m)$, the elements $\varphi_i(e) = (m, f_i) \in \{p\} \times F$ and $\varphi_j(e) = (m, f_j) \in \{m\} \times F$ are related as $(p, f_i) = (p, t_{ij}(p) f_j)$. The set $\{t_{ij} : i, j \in I\}$ of transition functions induced by $\mathcal{C}$ is denoted by $\mathscr{C}_{\mathcal{C}}$. The transition functions satisfy certain “compatibility conditions”, as expressed by the following lemma.

Lemma 2.5. *Let $F \longrightarrow E \stackrel{\pi}{\longrightarrow} M$ be a fibre bundle over M, let $\mathcal{C}$ be a trivialising cover of M, and let $\mathscr{C}_{\mathcal{C}}$ be the induced set of transition functions. For all $i, j, k \in I$, the conditions*

- $\forall\, m \in U_i : t_{ii}(m) = \mathbb{I}_F$,
- $\forall\, m \in U_{ij} : t_{ij}(m) = (t_{ji}(m))^{-1}$,
- $\forall\, m \in U_{ijk} : t_{ij}(m) \circ t_{jk}(m) = t_{ik}(m)$, *(Čech cocycle condition)*

[2]The group $\mathrm{Diff}(F)$ is an open submanifold $\mathcal{C}^\infty(M, M)$, conform Theorem 7.1 in [18]. What the smooth structure is in this case is beyond the scope of this thesis.

hold. The set $\mathscr{C}_C$ *is called a cocycle on* M *associated to the open covering* $\{U_i : i \in I\}$ *of* M*.*

Proof. This follows easily from the definition. □

We thus see that any fibre bundle with a chosen trivialising cover of the base space determines a set of transition functions which take values in $\mathrm{Diff}(F)$. There is the following converse to this statement.

Theorem 2.6. *Let* $\{U_i : i \in I\}$ *be an open cover of* M*, let* F *be a manifold, and let* $\{t_{ij} : U_{ij} \longrightarrow \mathrm{Diff}(F)\}$ *be a set of smooth maps satisfying the conditions of Lemma 2.5. These data determine a unique (up to equivalence) fibre bundle over* M *with typical fibre* F*.*

Proof. See Theorem 3 in Chapter 16 of [19]. □

The transition functions determine how the fibre F is "glued" onto the base manifold, and therefore how "non-trivial" the fibre bundle is (we have of course seen this before, with the tangent bundle). If all transition functions can be taken to be the identity, then the fibre bundle is clearly equivalent to the trivial bundle. The group $\mathrm{Diff}(F)$ is often be too large to be of interest, but as we will see, all fibre bundles in which we are interested will have a trivialising cover of the base space such that the corresponding transition functions take values in some Lie group. This observation leads us to the concept of a smooth principal G-bundle.

2.2 Principal bundles

Let G and G' be Lie groups.

Definition 2.7. A *smooth principal* G*-bundle over* M is a triple $\mathcal{P} = (P, \pi, \sigma)$, where (P, π, G) is fibre bundle over M, and $\sigma : P \times G \longrightarrow P$ is a smooth right action of G on P such that the fibres of π are G-invariant. In addition, there exists for each $m \in M$ a *local* G*-trivialisation* of $\mathcal{P}$, which is a local trivialisation (U, φ) of $\mathcal{P}$ such that $\varphi(p) = (\pi(p), \tilde{\varphi}(p))$ for all $p \in U$, where $\tilde{\varphi} : \pi^{-1}(U) \longrightarrow G$ is G-equivariant.

Remarks. The Lie group G is called the *structure group* of $\mathcal{P}$; in physics, it is called the gauge group. A smooth principal G-bundle will be referred to as a G-bundle over M and, if no confusion can arise, will be written as $G \longrightarrow P \xrightarrow{\pi} M$ or $P \xrightarrow{\pi} M$. The action of G will be written as $p \triangleleft g$, for all $p \in P$ and for all $g \in G$. Each fibre of π is now diffeomorphic to G, and is thus a G*-torsor.*

Example 2.8. Consider the trivial fibre bundle over M, and define a right action $\sigma_{\blacktriangleleft} : (M \times G) \times G \longrightarrow G$ by $\sigma_{\blacktriangleleft}((p, g), h) := (p, g) \blacktriangleleft h := (p, gh)$ for all $p \in M$ and for all $g, h \in G$. Then $(M \times G, \mathrm{Proj}_1, \sigma_{\blacktriangleleft})$ is a G-bundle over over $M \times G$, called the *trivial* G*-bundle over* M.

Example 2.9. Define

$$\begin{aligned} \sigma : \mathbb{S}^3_{\mathbb{C}} \times \mathrm{U}(1) &\longrightarrow \mathbb{S}^3_{\mathbb{C}} \\ ((\alpha_1, \alpha_2), g) &\longmapsto (\alpha_1 g, \alpha_2 g), \end{aligned} \tag{2.6}$$

then σ is a smooth right action, and since $(\alpha_1 g, \alpha_2 g) = (\alpha_1, \alpha_2)$ implies that $g = 1$ for all $(\alpha_1, \alpha_2) \in \mathbb{S}^3_{\mathbb{C}}$, this action is free. Since $\mathrm{U}(1)$ is compact, this is also a proper action, so[3] the orbit space $\mathbb{S}^3_{\mathbb{C}}/\mathrm{U}(1)$ is a manifold such that the quotient map

$$\begin{aligned} q : \mathbb{S}^3_{\mathbb{C}} &\longrightarrow \mathbb{S}^3_{\mathbb{C}}/\mathrm{U}(1) \\ (\alpha_1, \alpha_2) &\longmapsto [\alpha_1, \alpha_2]_q \end{aligned} \tag{2.7}$$

is smooth. Note that $\mathbb{S}^3/\mathrm{U}(1)$ is diffeomorphic to $\mathbb{P}^1(\mathbb{C})$, via the map

$$\begin{aligned} f : \mathbb{S}^3_{\mathbb{C}}/\mathrm{U}(1) &\longrightarrow \mathbb{P}^1(\mathbb{C}) \\ [\alpha_1, \alpha_2]_q &\longmapsto [\alpha_1, \alpha_2], \end{aligned} \tag{2.8}$$

and define $\pi := f \circ q$. For $k \in \{1, 2\}$, define $U_k := \{[\alpha_1, \alpha_2] \in \mathbb{P}(\mathbb{C}^2) \mid \alpha_k \neq 0\}$ and

$$\begin{aligned} \varphi_k : \pi^{-1}(U_k) &\longrightarrow U_k \times \mathrm{U}(1) \\ (\alpha_1, \alpha_2) &\longmapsto \left([\alpha_1, \alpha_2], \frac{\alpha_k}{|\alpha_k|}\right). \end{aligned} \tag{2.9}$$

Then U_k is open and φ_k is a diffeomorphism, and

$$\varphi_k(\alpha_1, \alpha_2) = (\pi(\alpha_1, \alpha_2), \tilde{\varphi}_k(\alpha_1, \alpha_2)) \tag{2.10}$$

holds for all $(\alpha_1, \alpha_2) \in \pi^{-1}(U_k)$, where the map

$$\begin{aligned} \tilde{\varphi}_k : \pi^{-1}(U_k) &\longrightarrow \mathrm{U}(1) \\ (\alpha_1, \alpha_2) &\longmapsto \frac{\alpha_k}{|\alpha_k|} \end{aligned} \tag{2.11}$$

is $\mathrm{U}(1)$-equivariant. It follows that $\mathcal{C} := \{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a trivialising cover of $\mathbb{P}^1(\mathbb{C})$, and thus that

$$\mathrm{U}(1) \longrightarrow \mathbb{S}^3_{\mathbb{C}} \longrightarrow \mathbb{P}^1(\mathbb{C}) \tag{2.12}$$

is a $\mathrm{U}(1)$-bundle over $\mathbb{S}^3_{\mathbb{C}}$, called the *complex Hopf bundle*.

Remark. Since $\mathbb{P}^1(\mathbb{C})$ is diffeomorphic to $\mathbb{S}^2$ and $\mathbb{S}^3_{\mathbb{C}}$ is diffeomorphic to $\mathbb{S}^3$, the

[3] See Corollary 21.6 and Theorem 21.10 in [7].

complex Hopf bundle is sometimes also written as $\mathbb{S}^1 \longrightarrow \mathbb{S}^3 \longrightarrow \mathbb{S}^2$, to emphasize the involvement of the spheres.

Lemma 2.10. *Let $\mathcal{P}$ be a G-bundle over M. The action of G on P is free, and transitive on the fibres of π.*

Proof. Let $p \in P$, and suppose there exists some $g \in G$ such that $p = p \triangleleft g$. Let (U, φ) be a local G-trivialisation of $\mathcal{P}$ such that $\pi(p) \in U$. Then

$$\begin{aligned}(\pi(p), \tilde{\varphi}(p)) &= (\pi(p \triangleleft g), \tilde{\varphi}(p \triangleleft g)) \\ &= (\pi(p), \tilde{\varphi}(p)g))\end{aligned} \tag{2.13}$$

and thus $g = e$, so the action is free. Let $m \in M$, let $p, q \in P_m$, and let (U, φ) be a local G-trivialisation of $\mathcal{P}$ such that $m \in U$. Then $\tilde{\varphi}(p), \tilde{\varphi}(q) \in G$, so there exists some $g \in G$ such that $\tilde{\varphi}(q) = \tilde{\varphi}(p)g = \tilde{\varphi}(p \triangleleft g)$, and

$$\begin{aligned}\varphi(q) &= (\pi(q), \tilde{\varphi}(q)) \\ &= (\pi(p \triangleleft g), \tilde{\varphi}(p \triangleleft g)) \\ &= \varphi(p \triangleleft g),\end{aligned} \tag{2.14}$$

so since φ is injective, it follows that $q = p \triangleleft g$, so the action is transitive on the fibres of π. $\square$

Let $\mathcal{P}$ be a G-bundle over M, and let $\mathcal{C}$ be a trivialising cover of M consisting of local G-trivialisations. The transition functions for a G-bundle $\mathcal{P}$ over M are readily recovered from $\mathcal{C}$. Let $(U_i, \varphi_i), (U_j, \varphi_j) \in \mathcal{C}$ be such that $U_i \cap U_j \neq \emptyset$. Let $m \in U_i \cap U_j$ and $p, p' \in \pi^{-1}(m)$. Then $p' = p \triangleleft g$ for some $g \in G$, so $\varphi_i(p')(\varphi_j(p'))^{-1} = \varphi_i(p)gg^{-1}(\varphi_j(p))^{-1} = \varphi_i(p)(\varphi_j(p))^{-1}$, and we can a define a map $g_{ij} : U_{ij} \longrightarrow G$, $m \longmapsto \varphi_i(p)(\varphi_j(p))^{-1}$, where p is any element in the fibre of π over m, and $\mathscr{C}_{\mathcal{C}} = \{g_{ij} : U_{ij} \longrightarrow G \mid i, j \in I\}$.

Definition 2.11. Let $\mathcal{P}$ be a G-bundle over M, and let $\mathcal{P}'$ be a G'-bundle over M'. A *principal bundle map from $\mathcal{P}$ to $\mathcal{P}'$* is a triple (Φ, φ, λ), where (Φ, φ) is a bundle map from $\mathcal{P}$ to $\mathcal{P}'$ and $\lambda : G \longrightarrow G'$ is a Lie group homomorphism, such that the diagram

$$\begin{tikzcd} P \times G \arrow[r, "\Phi \times \lambda"] \arrow[d, "\sigma"'] & P' \times G' \arrow[d, "\sigma'"] \\ P \arrow[r, "\Phi"'] & P' \end{tikzcd} \tag{2.15}$$

commutes.

Remarks. Two G-bundles $\mathcal{P}$ and $\mathcal{P}'$ over M are *equivalent* if there exists a bundle equivalence $(\Phi, \mathbb{I}_M)$ between $\mathcal{P}$ and $\mathcal{P}'$ such that $(\Phi, \mathbb{I}_M, \mathbb{I}_G)$ is a principal bundle map from $\mathcal{P}$ to $\mathcal{P}'$, called a *G-bundle equivalence between $\mathcal{P}$ and $\mathcal{P}'$*, and $\mathcal{P}$ is *trivial* if it is equivalent to $(M \times G, \mathrm{Proj}_1, \sigma)$. There is thus a category $\mathbf{P}$-$\mathbf{Bun}$, where any object is an H-bundle $\mathcal{P}$ over N, for some Lie group H and some manifold N, and a morphism from an $\mathcal{P}$ to a G-bundle $\mathcal{P}'$ over M is a principal bundle map (Φ, φ, λ) from $\mathcal{P}$ to $\mathcal{P}'$. If we fix the Lie group G and the manifold M, we get a subcategory $\mathbf{P}G$-$\mathbf{Bun}(M)$. As the following lemma shows, this last category is quite restrictive.

Lemma 2.12. *Let $\mathcal{P}, \mathcal{P}' \in \mathbf{P}G$-$\mathbf{Bun}(M)$. If $(\Phi, \mathbb{I}_M, \mathbb{I}_G)$ is a principal bundle map from $\mathcal{P}$ to $\mathcal{P}'$, then Φ is a diffeomorphism.*

Proof. Let $p, q \in P$ be such that $\Phi(p) = \Phi(q)$. Then p and q are elements of the same fibre of π, since $\pi(p) = \pi'(\Phi(p)) = \pi'(\Phi(q)) = \pi(q)$, so there exists a unique $g \in G$ such that $q = p \triangleleft g$. Then

$$\begin{aligned} \Phi(q) &= \Phi(p \triangleleft g) \\ &= \Phi(q) \triangleleft' g, \end{aligned} \tag{2.16}$$

which implies that $g = e$ and thus that $p = q$, so Φ is injective. Let $p' \in P'$, and let $p \in \pi^{-1}(\pi'(p'))$. Then $\Phi(p)$ and p' are elements of the same fibre of π', since $\pi'(\Phi(p)) = \pi(p) = \pi'(p')$, so there exists a unique $g \in G$ such that $p' = \Phi(p) \triangleleft' g$. Then

$$\begin{aligned} \Phi(p \triangleleft g) &= \Phi(p) \triangleleft' g \\ &= p', \end{aligned} \tag{2.17}$$

so Φ is bijective. The inverse is given by

$$\begin{aligned} \Phi^{-1} : \mathcal{P}' &\longrightarrow P \\ p' &\longmapsto p \triangleleft g_{pp'}, \end{aligned} \tag{2.18}$$

where $p \in \pi^{-1}(\pi'(p'))$, and $g_{pp'} \in G$ is the unique group element such that $p' = \Phi(p) \triangleleft' g_{pp'}$ holds. Then Φ^{-1} is a smooth map which preserves the fibres of π' such that $\Phi^{-1}(p' \triangleleft' g) = \Phi^{-1}(p') \triangleleft g$ holds for all $p' \in P$ and for all $g \in G$, so F^{-1} is a principal bundle map. □

The following lemma illustrates another important property of principal bundles.

Lemma 2.13. *A G-bundle $\mathcal{P}$ over M is trivial if and only if there exists a global section.*

Proof. Suppose that $\mathcal{P}$ is trivial, and let $(\Phi, \mathbb{I}_G)$ be a principal bundle map. Define

$$\begin{aligned} s : M &\longrightarrow P \\ m &\longmapsto \Phi^{-1}(m, e), \end{aligned} \tag{2.19}$$

then s is smooth and $\pi \circ s = \mathbb{I}_M$, so $s \in \Gamma(P)$. Suppose that $\Gamma(P)$ is non-empty and let $s \in \Gamma(P)$. Define

$$\begin{aligned} \Phi_s : M \times G &\longrightarrow P \\ (m, g) &\longmapsto s(m) \triangleleft g, \end{aligned} \tag{2.20}$$

then $(\Phi_s, \mathbb{I}_M)$ is a bundle map from $M \times G$ to P, and

$$\begin{aligned} \Phi_s((m, g) \blacktriangleleft h) &= \Phi_s(m, gh) \\ &= s(m) \triangleleft gh \\ &= (s(m) \triangleleft g) \triangleleft h \\ &= \Phi_s(m, g) \triangleleft h \end{aligned} \tag{2.21}$$

holds for all $m \in M$ and for all $g, h \in G$, so $(\Phi_s, \mathbb{I}_M, \mathbb{I}_G)$ is a principal bundle map from $M \times G$ to P. By Lemma 2.12 this map is an equivalence from $M \times G$ to P, so P is trivial. □

2.2.1 *The frame bundle*

Definition 2.14. Let $m \in M$. A *frame at* m is an ordered basis $e_m = (e_1, \ldots, e_n)$ for T_mM. The set of all frames at m is denoted by L_mM, and $LM := \dot{\bigcup}_{m \in M} L_mM$ is the set of all frames at all points in M.

Since any element of LM is a frame at some point in the manifold M, there is a natural projection $\pi_{LM} : LM \longrightarrow M$ sending each $e_m \in LM$ to the point $m \in M$ at which e_m is a frame.

Lemma 2.15. *The set LM is an $(n + n^2)$-dimensional manifold such that*

$$\begin{aligned} \pi_{LM} : LM &\longrightarrow M \\ e_m &\longmapsto m \end{aligned} \tag{2.22}$$

is a smooth map.

Proof. See section 3.3 in [20]. □

Let $e_m = (e_1, \ldots, e_n) \in L_mM$, and let $g = (g^i{}_j) \in \mathrm{GL}(n, \mathbb{R})$. Define

$$e \triangleleft g := (e_i g^i{}_1, \ldots, e_i g^i{}_n), \tag{2.23}$$

then $e \triangleleft g$ is again a frame at m. Now write e as a column vector, i.e. as $(e_1 \cdots e_n)$; then $e \triangleleft g$ can be viewed as the column vector

$$(e_i {g^i}_1 \cdots e_i {g^i}_n) = (e_1 \cdots e_n) \begin{pmatrix} {g^1}_1 & \cdots & {g^1}_n \\ \vdots & & \vdots \\ {g^n}_1 & \cdots & {g^n}_n \end{pmatrix} \tag{2.24}$$

which makes it clear that $(e \triangleleft g) \triangleleft h = e \triangleleft (gh)$ holds for all $h \in \mathrm{GL}(n, \mathbb{R})$, so that

$$\begin{aligned} \sigma_{LM} : LM \times \mathrm{GL}(n, \mathbb{R}) &\longrightarrow LM \\ (e, g) &\longmapsto e \triangleleft g \end{aligned} \tag{2.25}$$

is a right action of $\mathrm{GL}(n, \mathbb{R})$ on LM. It is clear that this action is free, and that it is transitive when restricted to $L_m M$, for each $m \in M$.

Lemma 2.16. *The triple $\mathcal{FM} := (LM, \pi_{LM}, \sigma_{LM})$ is a principal $\mathrm{GL}(n, \mathbb{R})$-bundle over M.*

Proof. See section 3.3 in [20]. $\square$

Remark. The $\mathrm{GL}(n, \mathbb{R})$-bundle $\mathcal{FM}$ is called the *frame bundle of M*.

Lemma 2.17. *The tangent bundle is parallelisable if and only if the frame bundle $\mathcal{FM}$ over M is trivial.*

Proof. The existence of n linearly independent sections of the tangent bundle is clearly equivalent to the existence of a global section of $\mathcal{FM}$. By Lemma (2.13), the result follows. $\square$

2.3 Associated fibre bundles

Let F be a manifold equipped with a smooth left action[4] $\tau : F \times G \longrightarrow F$, let $\mathcal{P}$ be a G-bundle over M, and consider the action

$$\begin{aligned} \Delta : (P \times F) \times G &\longrightarrow P \times F \\ ((p, f), g) &\longmapsto (p \triangleleft g, g^{-1} \triangleright f). \end{aligned} \tag{2.26}$$

The action Δ is smooth since σ and τ are, and $P_F := (P \times F)/G$ is a topological space equipped with the quotient topology. Denote by $[p, f]$ the equivalence class in P_F of $(p, f) \in P \times F$. Since

$$\begin{aligned} \tilde{\pi} : P \times F &\longrightarrow M \\ (p, f) &\longmapsto \pi(p) \end{aligned} \tag{2.27}$$

[4]In analogy to the right action defined on P, this action will be written as $g \triangleright f$, for all $g \in G$ and for all $f \in F$.

is a G-equivariant map with respect to Δ, it descends to a continuous map

$$\begin{aligned} \pi_F : P_F &\longrightarrow M \\ [p,f] &\longmapsto \pi(p). \end{aligned} \tag{2.28}$$

Lemma 2.18. *The triple $\mathcal{P}_F := (P_F, \pi_F, F)$ is a fibre bundle over M.*

Proof. See Theorem 6.87 in [21]. □

The bundle $\mathcal{P}_F$ is called the *fibre bundle associated to $\mathcal{P}$ via τ*, or simply an *associated fibre bundle of $\mathcal{P}$*. There is a slight abuse of notation here, since the associated fibre bundle depends on the specific action τ and the notation does not reflect this. However, there will be no possibility for confusion due to this. As we will see now, many fibres bundles which come up naturally, are associated to the frame bundle.

Example 2.19. Consider the frame bundle of M, and the left action τ_1 of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ defined by[5] matrix multiplication. Note that τ_1 is smooth. Then

$$\begin{aligned} \tilde{\Phi} : LM \times \mathbb{R}^n &\longrightarrow TM \\ (e,f) &\longmapsto f^i e_i \end{aligned} \tag{2.29}$$

is a well-defined $\mathrm{GL}(n,\mathbb{R})$-equivariant map, since

$$\begin{aligned} (g^{-1} \triangleright f)^i (e \triangleleft g)_i &= {g_j}^i f^j e_k {g^k}_i \\ &= {g^k}_i {g_j}^i f^j e_k \\ &= \delta^k_j f^j e_k \\ &= f^j e_j \end{aligned} \tag{2.30}$$

holds for all $g \in \mathrm{GL}(n,\mathbb{R})$ and for all $(e,f) \in LM \times \mathbb{R}^n$, so it descends to a map $\Phi : LM_{\mathbb{R}^n} \longrightarrow TM$ such that the diagram

$$\begin{tikzcd} LM_{\mathbb{R}^n} \arrow[r, "\Phi"] \arrow[d, "\pi_{\mathbb{R}^n}"'] & TM \arrow[d, "\pi_t"] \\ M \arrow[r, "\mathbb{I}_M"'] & M \end{tikzcd} \tag{2.31}$$

commutes; it follows that $(\Phi, \mathbb{I}_M)$ is a bundle map from $LM_{\mathbb{R}^n}$ to the tangent bundle $\mathcal{T}_M := (TM, \pi_t, \mathbb{R}^n)$. In fact, it is a bundle equivalence. Let $m \in M$, let $X \in T_m M$, and let (U, φ) be a chart for M around m with local coordinates

[5]The elements of $\mathbb{R}^n$ are viewed as column vectors, and we write $f = (f^1, \ldots, f^n) \in \mathbb{R}^n$.

$x^1, \dots, x^n$ on U. Then $e_m^U := (\partial_1|_m, \dots, \partial_n|_m)$ is a frame at m, so $V = f^i \partial_i|_m$ for some $f_m^U := (f^1, \dots, f^n) \in \mathbb{R}^n$, and the element $[e_m^U, f_m^U] \in LM_{\mathbb{R}^n}$ is such that $\Phi([e,f]) = X$, so Φ is surjective. Let $[e,f], [\tilde{e}, \tilde{f}] \in LM_{\mathbb{R}^n}$ be such that $\Phi([e,f]) = \Phi([\tilde{e}, \tilde{f}])$. Then $\pi_{LM}(e) = \pi_{LM}(\tilde{e})$, so there exists a unique $g \in \mathrm{GL}(n, \mathbb{R})$ such that $\tilde{e} = e \triangleleft g$. Then $f^i e_i = \tilde{f}^i (e \triangleleft g)_i$ implies that $\tilde{f} = g^{-1} \triangleright f$, so Φ is injective. The inverse map is given by

$$\begin{aligned} \Phi^{-1} : TM &\longrightarrow LM_{\mathbb{R}^n} \\ X &\longmapsto [e_m^U, f_m^U], \end{aligned} \tag{2.32}$$

where $m \in M$ is such that $X \in T_m M$. Note that Φ^{-1} is well-defined, since if $(\tilde{U}, \tilde{\varphi})$ is another chart for M around m, then $e_{\tilde{U}} = e_U \triangleleft g$ and $f_{\tilde{U}} = g^{-1} \triangleright f_U$, where g is the Jacobian at $\tilde{\varphi}(m)$ of the overlap function $\varphi \circ \tilde{\varphi}^{-1}$. We thus see that the tangent bundle can be viewed as an associated fibre bundle of $\mathcal{FM}$.

The above description of the tangent bundle reveals how the formal definition of tangent vectors and vector fields corresponds to the way in which they are usually presented in the physics literature, namely as a set of components (or component functions) with respect to some basis, such that if the basis transforms by a basis transformations Λ, then the components transform with the inverse transformation Λ^{-1}.

Example 2.20. Consider again the frame bundle of M, and the action τ of $\mathrm{GL}(n, \mathbb{R})$ on[6] $(\mathbb{R}^n)^*$ defined by $(g \triangleright f)_i = f_j (g^{-1})^j{}_i$ for all $f \in (\mathbb{R}^n)^*$ and for all $g \in \mathrm{GL}(n, \mathbb{R})$. From the previous example, it is clear that the associated fibre bundle $\mathcal{FM}_{(\mathbb{R}^n)^*}$ is equivalent to the cotangent bundle $\mathcal{T}_M^* := (T^*M, \pi_c, \mathbb{R}^n)$.

To close this section, we make an informal remark which brings up the point made about representations in section 1.5. Namely, any linear representation R of the Lie group G defines a smooth left action of G on the representation space of R, and thus also an associated fibre bundle, and most associate bundles which are important in physics come from a representation of a Lie group.

2.4 Altering the structure group

As mentioned before, the relevant group in general relativity is the restricted Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$, so the frame bundle, with its structure group $\mathrm{GL}(n, \mathbb{R})$, is not the structure we need, as it would allow physically for many non-admissible observers. We would therefore like to *reduce* the structure group. The first most obvious choice is to reduce $\mathrm{GL}(n, \mathbb{R})$ to the subgroup $\mathrm{O}(n, \mathbb{R})$, which means that we are left with only orthormal frames. Eventually, we need to restrict to the Lorentz group $\mathrm{SO}^{\uparrow}(1,3)$. Once we have done that, we can consider the

[6] The elements of $(\mathbb{R}^n)^*$ are viewed as row vectors, and we write $f = (f_1, \dots, f_n) \in (\mathbb{R}^n)^*$.

question whether we can "lift" the structure group[7] to $\mathrm{SL}(2,\mathbb{C})$, since having this group is essentially what allows us to define spinors on a spacetime. To define what orthogonality means, we need an inner product in each tangent space: a Riemannian metric. We first need a definition. Define the set

$$T^*M \otimes T^*M := \coprod_{m\in M} T^*_m M \otimes T^*_m M. \tag{2.33}$$

This set inherits in an analogous way as for the tangent and cotangent bundle a smooth structure from M. Since the dimension of the tensor product of two vector spaces is the product of their respective dimensions, it is a $(n+n^2)$-dimensional smooth manifold.

Definition 2.21. A *Riemannian metric* is an element $g \in \Gamma(T^*M \otimes T^*M)$ such that $g(m) \in T^*_m M \otimes T^*_m M$ is symmetric and positive-definite for each $m \in M$. If g is a Riemannian metric on M, we say that (M,g) is a Riemannian manifold.

Lemma 2.22. *A Riemannian metric exists.*

Proof. Let $\{(U_\alpha,\varphi_\alpha)\,|\,\alpha \in I\}$ be an atlas for M, and let $\mathscr{C} = \{p_\alpha\,|\,\alpha \in J\}$ be a smooth partition of unity subordinate the covering $\{U_\alpha\,|\,\alpha \in I\}$ of M. Let $(U_\alpha,\varphi_\alpha)$ be a chart with local coordinates $x^1_\alpha,\ldots,x^n_\alpha$ on U_α, and define

$$\begin{aligned} g^\alpha : U &\longrightarrow T^*M \otimes T^*M \\ m &\longmapsto \sum_{j=1}^n dx^j_\alpha|_m \otimes dx^j_\alpha|_m \end{aligned} \tag{2.34}$$

Then g^α is symmetric and positive-definite, and the map

$$\begin{aligned} g : M &\longrightarrow T^*M \otimes T^*M \\ m &\longmapsto \sum_{\alpha\in I} p_\alpha(m) g^\alpha(m) \end{aligned} \tag{2.35}$$

defines a Riemannian metric on M. $\square$

If (M,g) is a Riemannian manifold, then a construction completely analogous to the construction of the frame bundle allows us to construct the orthonormal frame bundle $\mathcal{OM} = (OM, \pi_{OM}, \sigma_{OM})$ over M, which has $\mathrm{O}(n,\mathbb{R})$ has its structure group. Similarly, if the manifold is orientable, we may construct the oriented orthonormal frame bundle, whose structure group is $\mathrm{SO}(n,\mathbb{R})$[8]. We see that refining the group is in a way equivalent to introducing more structure to the manifold, and to get to the Lorentz group, we need to have not a Riemannian metric, but a Lorentzian metric.

[7] We say lift because $\mathrm{SL}(2,\mathbb{C})$ is the double cover of $\mathrm{SO}^\uparrow(1,3)$.

[8] See also pages 156-159 in [20]

Definition 2.23. A *Lorentzian metric on* M is an element $g_L \in \Gamma(T^*M \otimes T^*M)$ such that $g_L(m)$ is symmetric of signature $(1, n-1)$ for each $m \in M$. If M is equipped with a Lorentzian metric g_L, we say that (M, g_L) is a *Lorentzian manifold*.

From now on, we will restrict to our attention to connected, non-compact and 4-dimensional smooth manifold $\mathcal{M}$, and assume that there is a Lorentzian manifold g_L defined on $\mathcal{M}$. This assumption if of course physically motivated, as compact manifolds have certain unphysical properties; see section 1.5 in [2] and references therein, and because it is a postulate of general relativity that we have a Lorentzian metric.

The Lorentzian metric g_L allows us to classify the tangent vectors to $\mathcal{M}$ as follows. For any $m \in \mathcal{M}$ and $V_m \in T_m\mathcal{M}$, we say that V_m is *spacelike* if $g_L(m)(V_m, V_m) < 0$, *timelike* if $g_L(m)(V_m, V_m) = 0$, and *null* if it holds that $g_L(m)(V_m, V_m) = 0$. Then we say that $(\mathcal{M}, g_L)$ is *time oriented* if there exists an everywhere non-vanishing smooth vector field such that $g_L(m)(V(m), V(m)) > 0$ for all $m \in \mathcal{M}$. If $(\mathcal{M}, g)$ is time oriented also oriented, then $(\mathcal{M}, g)$ is called *spacetime oriented*. For the same reasons as discussed above, we will assume that $\mathcal{M}$ is oriented and time oriented[9]. From this it follows that we may further reduce the group to $\mathrm{SO}^{\uparrow}(1,3)$, as follows from corollary 1 in [22] and the discussion on page 171 in [20], and we thus get a $\mathrm{SO}^{\uparrow}(1,3)$-bundle over M, which will be denoted by $\mathrm{SO}^{\uparrow}(\mathcal{M})$.

2.4.1 *Spin structure*

We can now define what a spin structure on $\mathcal{M}$ is.

Definition 2.24. A *spin structure on* $\mathcal{M}$ consists of

- a principal $\mathrm{SL}(2,\mathbb{C})$-bundle $\mathrm{SL}(2,\mathbb{C}) \longrightarrow \mathrm{S}(\mathcal{M}) \longrightarrow \mathcal{M}$ over $\mathcal{M}$, and
- a smooth map $\Phi : \mathrm{S}(\mathcal{M}) \longmapsto \mathrm{SO}^{\uparrow}(\mathcal{M})$ such that the diagram

$$\begin{tikzcd}
\mathrm{S}(\mathcal{M}) \times \mathrm{SL}(2,\mathbb{C}) \arrow[rr, "\Phi\times\rho"] \arrow[d, "\sigma_s"'] & & \mathrm{SO}^{\uparrow}(M) \times \mathrm{SO}^{\uparrow}(1,3) \arrow[d, "\sigma_l"] \\
\mathrm{S}(\mathcal{M}) \arrow[rr, "\Phi"'] \arrow[dr, "\pi_s"] & & \mathrm{SO}^{\uparrow}(\mathcal{M}) \arrow[dl, "\pi_l"] \\
& \mathcal{M} &
\end{tikzcd} \tag{2.36}$$

[9]See sections 1-2 in [22] for more on the (mathematical) justification of these assumptions.

commutes, where $\pi_s : \mathrm{S}(\mathcal{M}) \longmapsto \mathcal{M}$ and $\pi_l : \mathrm{SO}^{\uparrow}(\mathcal{M}) \longmapsto \mathcal{M}$ are the bundle projections, and σ_s and σ_l are the actions of respectively $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{SO}^{\uparrow}(1,3)$ on $\mathcal{M}$.

Since $\mathrm{SL}(2, \mathbb{C})$ is not a subgroup of $\mathrm{SO}^{\uparrow}(1,3)$ but its double cover, we also say that a spin structure is a lift of the structure group $\mathrm{SO}^{\uparrow}(1,3)$ to $\mathrm{SL}(2, \mathbb{C})$. A spin structure does not always exist. A result by Robert Geroch [23] states that a spin structure exists on $\mathcal{M}$ if and only if there exist four smooth vector fields e_1, e_2, e_3 and e_4 on $\mathcal{M}$, such that $\{e_1(m), e_2(m), e_3(m), e_4(m)\}$ forms a basis for $T_m\mathcal{M}$ for which it holds that the value of $g_L(m)(e_i(m), e_j(m))$ is 1 if $i = j = 1$, -1 if $i = j \in \{2,3,4\}$, and 0 if $i, j \in [4]$ and $i \neq j$. Another way to express the obstruction to having a spin structure is the following. There is a topological invariant called the *second Stiefel-Whitney class*, which is the element $w_2(\mathcal{M}) \in H^2(\mathcal{M}, \mathbb{Z}/2\mathbb{Z})$ in the *second Čech cohomology group* of $\mathcal{M}$ with coefficients in $\mathbb{Z}/2\mathbb{Z}$ (see section 11.6 in [24]).

2.5 Einstein's field equations

In this section, we will introduce the concept of a connection on the tangent bundle of a manifold, which is necessary in order to write Einstein equation. One specific type of connection, namely the Levi-Civita connection, will turn out to be the appropriate choice connection in general relativity. Having done this, it becomes a fairly straightforward matter to write down the field equations in a local chart for the manifold. Let M be a smooth manifold.

Definition 2.25. A connection on $\mathcal{T}_M$ is a $\mathbb{R}$-linear map

$$D : \Gamma(TM) \longrightarrow \Gamma(TM) \underset{\mathcal{C}^\infty(M)}{\otimes} \Gamma(T^*M) \tag{2.37}$$

such that $D(f \cdot V) = f \cdot D(V) + V \otimes df$ holds for all $V \in \Gamma(TM)$ and for all $f \in \mathcal{C}^\infty(M)$.

Remark. Since for any $V \in \Gamma(TM)$ it holds that $D(V) = \sum_{i=1}^n f_i V_i \otimes \alpha_i$ for some $n \in \mathbb{N}_{\geq 1}$, where $f_i \in \mathcal{C}^\infty(M)$, $V_i \in \Gamma(TM)$ and $\alpha_i \in T^*M$ for all $i \in [n]$, we can define

$$D(V)(U) := \sum_{i=1}^n \alpha_i(U) f_i V_i \in \Gamma(TM) \tag{2.38}$$

for all $U \in \Gamma(TM)$.

We recognize some sort of "product rule" in Definition 2.25. As the terminology in the following definition suggests, this is because the connection and subsequently the covariant derivative associated to it are supposed to generalise the notion of a directional derivative of a vector field on a manifold, which allows us to consider the change of one vector field on a manifold with respect to another vector field. Let D be a connection on $\mathcal{T}_M$. This motivates the following definition.

Definition 2.26. The *covariant derivative induced by D* is the $\mathbb{R}$-bilinear map

$$\begin{aligned} \mathcal{C} : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (U, V) &\longmapsto D_U V, \end{aligned} \tag{2.39}$$

where $D_U V := D(V)(U)$ for all $U, V \in \Gamma(TM)$.

Remarks. Note that $D_{fU}V = fD_U V$ and $D_U(fV) = fD_U V + U(f)V$ holds for all $f \in \mathcal{C}^\infty(M)$ and for all $U, V \in \Gamma(TM)$, as follows immediately from the definition of D. It is clear that for any open $U \subset M$, the map $\mathcal{C}$ restricts to a map $\mathcal{C}|_U : \Gamma(TM|_U) \times \Gamma(TM|_U) \longrightarrow \Gamma(TM|_U)$.

Fix a chart (U, φ) for M with local coordinates $x^1, \dots, x^n$ on U (for the remainder of this section; whenever we write locally, we mean that we are working in the chart (U, φ).). Since the image of two vector fields under $\mathcal{C}$ is again a vector field, we can express the result in terms of the basis local frame $\{\partial_1, \dots, \partial_n\}$ on U. So for any $i, j \in [n]$, there are smooth functions ${\Gamma^1}_{ij}, \dots, {\Gamma^n}_{ij} \in \mathcal{C}^\infty(M|_U)$ such that

$$D_{\partial_i}\partial_j = {\Gamma^k}_{ij}\partial_k. \tag{2.40}$$

The elements $\{{\Gamma^k}_{ij} : i, j, k \in [n]\}$ are called the *Christoffel symbols* associated to D, and are in the phyisics literature often introduced axiomatically in order to define a "new" kind of derivative, which is supposed to replace the ordinary derivative. Here we see how that works formally.

Definition 2.27. The *torsion of D* is the $\mathbb{R}$-bilinear map

$$\begin{aligned} T : \Gamma(TM) \times \Gamma(TM) &\longrightarrow \Gamma(TM) \\ (U, V) &\longmapsto D_U V - D_V U - [U, V] \end{aligned} \tag{2.41}$$

Here $[U, V]$ is the *Lie bracket* of the vector fields U and V, defined for all $f \in \mathcal{C}^\infty(M)$ by $[U, V]f = U(V(f)) - V(U(f))$.

Lemma 2.28. *The torsion is $\mathcal{C}^\infty(M)$-bilinear.*

Proof. Let $U, V \in \Gamma(TM)$ and $f \in \mathcal{C}^\infty(M)$. Then it holds that $D_{fU}V = fD_U V$ and $D_V(fU) = fD_V U + V(f)U$, and

$$\begin{aligned} [fU, V]g &= (fU)(V(g)) - V(fU(g)) \\ &= f(U(V(g)) - V(f)U(g) - fV(U(g)) \\ &= f[U, V]g - V(f)U(g) \end{aligned} \tag{2.42}$$

for all $g \in \mathcal{C}^\infty(M)$, so $D_{fU}V - D_V(fU) - [fU, V] = f(D_U V - D_V U - [U, V]$ and thus $T(fU, V) = fT(U, V)$. The torsion is clearly antisymmetric, so T is $\mathcal{C}^\infty(M)$-bilinear ☐

By the previous lemma, the torsion map T descends to a $\mathcal{C}^\infty(M)$-linear map $\hat{T} : \Gamma(TM) \otimes \Gamma(TM) \longmapsto \Gamma(TM)$. The image of two vector fields under this map $\hat{T}$ can locally be expressed in terms of the Christoffel symbols: using equation 2.40 and the definition, we find that

$$\begin{aligned} T(\partial_i, \partial_j) &= D_{\partial_i}\partial_j - D_{\partial_j}\partial_i - [\partial_i, \partial_j] \\ &= \Gamma^k{}_{ij}\partial_k - \Gamma^k{}_{ji}\partial_k, \end{aligned} \tag{2.43}$$

since the partial derivates commute when acting on smooth functions. What is maybe the most important in object in general relativity (besides the metric), is the curvature of D.

Definition 2.29. The *curvature of D* is the $\mathbb{R}$-trilinear map

$$\begin{aligned} F : \Gamma(TM)^{\times 3} &\longrightarrow \Gamma(TM) \\ (U, V, W) &\longmapsto D_U(D_V W) - D_V(D_U W) - D_{[U,V]}W \end{aligned} \tag{2.44}$$

Lemma 2.30. *The curvature is $\mathcal{C}^\infty(M)$-trilinear.*

Proof. Let $U, V, W \in \Gamma(TM)$ and $f \in \mathcal{C}^\infty(M)$. Then

$$\begin{aligned} D_{fU}(D_V W) &= fD_U(D_V W), \\ D_V(D_{fU}W) &= D_V(fD_U W) \\ &= fD_V(D_U W) + V(f)D_U W, \\ D_{[fU,V]}W &= D_{f[U,V]-V(f)U}W \\ &= fD_{[U,V]}W - V(f)D_U W, \end{aligned} \tag{2.45}$$

so

$$F(fU, V, W) = fF(U, V, W) \tag{2.46}$$

Since $F(U, V, W) = -F(V, U, W)$, it follows that $F(U, fV, W) = fF(U, V, W)$. Finally, it holds that

$$\begin{aligned} D_U(D_V(fW)) &= D_U(fD_V W + V(f)W) \\ &= fD_U(D_V W) + U(f)D_V W + V(f)D_U W + \\ &\qquad U(V(f))W, \\ D_V(D_U(fW)) &= fD_V(D_U W) + V(f)D_U W + U(f)D_V W + \\ &\qquad V(U(f))W, \\ D_{[U,V]}(fW) &= fD_{[U,V]}W + [U, V](f)W, \end{aligned} \tag{2.47}$$

and thus $F(U, V, fW) = fF(U, V, W)$. □

From Lemma 2.30 it follows that F descends to a $\mathcal{C}^\infty(M)$-linear map

$$\hat{F} : \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) \longrightarrow \Gamma(TM), \tag{2.48}$$

and thus that F corresponds to a $\mathcal{C}^\infty(M)$-linear map

$$R : \Gamma(T^*M) \otimes \Gamma(TM) \otimes \Gamma(TM) \otimes \Gamma(TM) \longrightarrow \mathcal{C}^\infty(M), \tag{2.49}$$

called the *Riemann curvature tensor.* Again, we can locally express the components of the map $\hat{F}$ in terms of the Christoffel symbols. A quick calculation shows that

$$\hat{F}(\partial_i, \partial_j, \partial_k) = (\partial_i \Gamma^l{}_{jl})\partial_l - (\partial_j \Gamma^l{}_{il})\partial_l + \Gamma^l{}_{jk}\Gamma^{l'}{}_{il}\partial_{l'} - \Gamma^l{}_{ik}\Gamma^{l'}{}_{jl}\partial_{l'}. \tag{2.50}$$

For physical reasons, it turns out be interesting to consider a very specific type of connection, namely the Levi-Civita connection, which always exists, and is unique.

Lemma 2.31. *There exists a unique connection $\mathcal{D}$ on (M, g), called the* Levi-Civita connection*, such that $\mathcal{D}$ is torsion-free, i.e. $T_{\mathcal{D}} = 0$, and g-compatible, i.e. for all $U, V, W \in \Gamma(TM)$ it holds that $U(g(V, W)) = g(D_U V, W) + g(V, D_U W)$.*

Proof. See Theorem 13.9 in [21]. □

Now let $(\mathcal{M}, g_L)$ be a *spacetime manifold*, i.e. a non-compact and connected Lorentzian manifold, and let $\mathcal{D}$ be the Levi-Civita connection associated to g_L. Since the modules $\Gamma(T\mathcal{M}|_U)$ and $\Gamma(T^*\mathcal{M}|_U)$ are now free of rank 4 with bases given by $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ and $\{dx^1, dx^2, dx^3, dx^4\}$ respectively, we have the isomorphism

$$\mathrm{Hom}(\Gamma(T^*\mathcal{M}|_U) \otimes \Gamma(T\mathcal{M}|_U)^{\otimes 3}) \cong \Gamma(T\mathcal{M}|_U) \otimes \Gamma(T^*\mathcal{M}|_U)^{\otimes 3}, \tag{2.51}$$

so R corresponds to an element $\mathscr{R} \in \Gamma(T\mathcal{M}|_U) \otimes \Gamma(T^*\mathcal{M}|_U)^{\otimes 3}$. This means that there exist smooth functions $\{\mathscr{R}^\alpha{}_{\beta\gamma\delta} \in \mathcal{C}^\infty(U) : \alpha, \beta, \gamma, \delta \in [4]\}$ such that $\mathscr{R} = \mathscr{R}^\alpha{}_{\beta\gamma\delta}\partial_\alpha \otimes dx^\beta \otimes dx^\gamma \otimes dx^\delta$, which is the local representation of the Riemann curvature tensor. The *Ricci tensor* Ric is defined as the contraction

$$\begin{aligned} \mathrm{Ric} &= \mathscr{R}^\alpha{}_{\beta\alpha\delta} dx^\alpha(\partial_\alpha) dx^\beta \otimes dx^\gamma \\ &= \mathscr{R}^\alpha{}_{\beta\alpha\delta} dx^\beta \otimes dx^\delta, \end{aligned} \tag{2.52}$$

where for each $\alpha' \in [4]$, $dx^{\alpha'}(\partial_{\alpha'}) : \mathcal{M} \longmapsto \mathcal{C}^\infty(\mathcal{M})$ sends any $m \in U$ to[10] $dx^{\alpha'}|_m(\partial_{\alpha'}|_m) = 1$, and is thus the constant function with value 1 on U, so that we may rightfully leave it out. We write $\mathrm{Ric} = \mathrm{Ric}_{\alpha\beta} dx^\alpha \otimes dx^\beta$, where

[10]Note that there is now, confusingly enough, *no* summation, as we consider one specific index.

$\mathrm{Ric}_{\alpha\beta} = \mathscr{R}^{\gamma}{}_{\beta\gamma\delta}$, for all $\alpha, \beta \in [4]$. We can make one more object out of $\mathscr{R}$. Since the metric $g_L \in \Gamma(T^*\mathcal{M} \otimes T^*\mathcal{M})$ can locally be written as $g_L = g_{\alpha\beta} dx^\alpha \otimes dx^\beta$, where $g_{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{M}|_U)$ for all $\alpha, \beta \in [4]$, we can consider its image $\tilde{g}_L$ in $\Gamma(T\mathcal{M}|_U) \otimes \Gamma(T\mathcal{M}|_U)$, which we we write as $\tilde{g}_L = g^{\alpha\beta}\partial_\alpha \otimes \partial_\beta$. Note that $g_{\alpha\beta} = g^{\alpha\beta}$ for each $\alpha, \beta \in [4]$ (we have "raised the indices"). This allows us to define the *scalar curvature* $\mathcal{R}$, which is defined as the further contraction

$$\begin{aligned} \mathcal{R} &= g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta} dx^\alpha(\partial_\alpha) dx^\beta(\partial_\beta) \\ &= g^{\alpha\beta}\mathrm{Ric}_{\alpha\beta} \end{aligned} \tag{2.53}$$

Note that this is a smooth function on U. The *stress-energy tensor* is a symmetric element $T \in \Gamma(T^*\mathcal{M}|_U) \otimes \Gamma(T^*\mathcal{M}|_U)$, which contains all information about any present mass or energy in the region U of spacetime, and we can write $T = T_{\alpha\beta}\, dx^\alpha \otimes dx^\beta$, where $T_{\alpha\beta} \in \mathcal{C}^\infty(\mathcal{M}|_U)$ for all $\alpha, \beta \in [4]$. We can now write down Einstein's field equation, which expresses mathematically how the presence of mass influences the geometry of spacetime, and vice versa.

Definition 2.32. The equation

$$\mathrm{Ric} + (\Lambda - \tfrac{1}{2}\mathcal{R})g_L = T, \tag{2.54}$$

is called *Einstein's field equation*, and is valid locally on U. Here $\Lambda \in \mathbb{R}_{>0}$ is called the *cosmological constant*.

These equations govern the dynamics of spacetime, and the main goal in the study of general relativity is to solve these equations. The cosmological constant has to account for the expansion of the universe. The fact that $\mathcal{D}$ is g-compatible can be used to express the Christoffel symbols and thus the Ricci tensor and the scalar curvature in terms of the metric and its first and second derivatives (with respect to the coordinates in the chart), so that equation (2.54) becomes a complicated set of partial differential equations (for all components), which are in general very hard to solve.

We end this discussion with an outlook. As we mentioned shortly before, the spinors appear as elements of a representation space of a representation of $\mathrm{SL}(2, \mathbb{C})$, which are for general relativity usually taken to be the two-spinors, i.e. elements of $\mathbb{C}^2$, on which $\mathrm{SL}(2, \mathbb{C})$ by matrix multiplication. Then one can proceed to reformulate all relevant vector and tensor equations in terms of the spinors, which as we mentioned in the introduction, has proved to be useful and is still used today.

Conclusions

In this thesis, we have discussed the basic language of manifolds without which we cannot sensibly speak of anything relating to general relativity. We have seen how the Lorentz group is intimately related to $\mathrm{SL}(2, \mathbb{C})$, and how this relation is the reason for being able to consider spinors in a meaningful way in general relativity. In the second chapter, we some theory on fibre bundles, which allowed to define a spin structure, and stated the results on its existence. We have seen that under conditions which are physically desirable, a spin structure indeed exists. Finally, we concluded by demonstrating how to properly define Einstein's field equation.

The mathematics involved in general relativity and the theory on fibre bundles is vast and has many applications, and this thesis has (naturally) only uncovered a small part of this area of mathematics. We pass on the suggestion which triggered this thesis, to use spinors to try and solve Einstein's equation for the Hopf field.

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