



---

# Supersymmetry, or solving difficult potentials easily

---

THESIS

submitted in partial fulfillment of the  
requirements for the degree of

BACHELOR OF SCIENCE

in

MATHEMATICS AND PHYSICS

Author :

M.G. Wellens

Student ID :

s1329367

Supervisor :

Prof. Dr. K.E. Schalm

2<sup>nd</sup> corrector :

Dr. M.F.E. de Jeu

Leiden, The Netherlands, July 13, 2018



# Supersymmetry, or solving difficult potentials easily

**M.G. Wellens**

Huygens-Kamerlingh Onnes Laboratory, Leiden University  
P.O. Box 9500, 2300 RA Leiden, The Netherlands

July 13, 2018

## **Abstract**

We first introduce the concept of partner potentials in non-relativistic quantum mechanics, i.e. a pair of potentials with the same spectrum, possibly except for a zero-energy ground state. We use this to define a family of partner potentials, giving us a technique to calculate the entire spectrum of a potential. The mechanism of partner potentials is then used for a quantum mechanical model of supersymmetry. It turns out that a special class of potentials exists where the spectrum can be determined very quickly using the techniques developed. We explore some of these potentials, called shape invariant potentials or SIPs and discover some simple properties of them. Finally, we take a quick look at a Hamiltonian with a  $p^4$  term in it, discovering that for a small class of potentials, we can make a supersymmetric quantum mechanical model out of it.



# Contents

<b>1</b>	<b>Introduction</b>	<b>7</b>
<b>2</b>	<b>Supersymmetry</b>	<b>11</b>
2.1	Simple supersymmetry	11
2.1.1	Partner potentials in quantum mechanical systems	11
2.1.2	Example: infinite square well	20
2.1.3	Families of partner potentials	21
2.2	Supersymmetric model	24
2.3	Singular potentials	28
<b>3</b>	<b>Shape Invariant Potentials</b>	<b>31</b>
3.1	Easily solvable potentials	31
3.1.1	Concept of shape invariant potentials	31
3.1.2	Example: Morse potential	36
3.1.3	Shape invariance in multiple steps	39
3.2	Translational SIPs	42
3.3	Multiplication SIPs	45
<b>4</b>	<b>Further Topics</b>	<b>49</b>
4.1	Non-linear oscillators	49
4.2	Reflection and transmission	52
4.3	Omitted topics	55
<b>A</b>	<b>Mathematics</b>	<b>57</b>
A.1	Super linear algebra	57



# Introduction

Symmetries of systems are studied in all parts of physics. They give insight in the structure of a problem, such as in the study of crystals. Equations also become easier with them, for example those that describe the more-dimensional oscillator. Noether's theorem makes explicit that continuous symmetries in a system give rise to conserved quantities, such as conservation of energy from the translational symmetry in time and conservation of angular momentum from rotational symmetry along an axis. In quantum mechanics symmetries give rise to degenerate eigenvalues, as is seen in the treatment of the quantum mechanical hydrogen atom. Symmetries are thus an integral part of physics.

The first subject of this thesis is one particular symmetry, namely supersymmetry, and a model of it in quantum mechanics. The prefix *super* is from the fact that this symmetry behaves different than other symmetries we encounter. Normally, when symmetries are studied, they are studied in the context of Lie algebras, i.e. vector spaces equipped with a multiplication operator, the Lie bracket, which is the commutator of two elements<sup>1</sup>. For supersymmetries, we generalise this so we may use the anti-commutator. For example, there are no longer only commuting generators of the Lie algebra, i.e.  $[A, B] = -[B, A]$ , with  $A$  and  $B$  commuting generators and  $[\cdot, \cdot]$  a bilinear and alternating multiplication. Anti-commuting generators are now allowed as well, i.e.  $[C, D] = [D, C]$ , with  $C$  and  $D$  anti-commuting generators and  $[\cdot, \cdot]$  the same bilinear bracket, but no longer alternating for every two generators<sup>2</sup>. Furthermore, where commuting generators commute with themselves,  $[A, A] = -[A, A] = 0$ , anti-commuting generators anti-commute with themselves,  $[C, C] = [C, C] = 0$ . This is the concept behind supersymmetry.

The reason we use supersymmetries is because of one important group of continuous symmetries, the Poincaré group, and its connection to discrete symmetries, also called internal symmetries. The Poincaré group is the Lie group of continuous symmetries of Minkowski spacetime [28], i.e. the three boosts and three rotations along the space axes and the four translations along all the axes. The Hamiltonian and momentum

---

<sup>1</sup>Other choices are possible, but normally the commutator is chosen.

<sup>2</sup>Details are found in Appendix A.1.

operators are for example part of a representation of this group<sup>3</sup>. Discrete symmetries are for example time reversal, the laws of nature are the same if you go forward in time or backward, or charge symmetry, changing the electric charges in the hydrogen atom for example should not alter the behaviour of system. Internal symmetries may depend on the coordinates, but they only change the physical system, not the space the system lives in [29].

One question one could ask is if there is a connection between these symmetries, i.e. if one can generate the total group of symmetries by only using the discrete one, or only using the Poincaré group. This was tried, but in 1967 it was proven by Coleman and Mandula that, under some mild assumptions, a symmetry group containing both the Poincaré group and other symmetries will be a direct product of both parts [5]. So, there does not exist an interesting connection between these types of symmetries.

In 1975 by Haag, Lopuszanski and Sohnius a loophole was discovered [18]. Instead of only using commuting symmetries, they allowed anti-commuting symmetries as well. This made it possible to have a group of anti-commuting discrete symmetries to generate a group of symmetries including the Poincaré group. One application of this discovery is a description of a quantum gravity by Freedman, Ferrara and Van Nieuwenhuizen [30].

Physically, anti-commuting symmetries can be seen as symmetries between bosons and fermions. The idea is as follows. An anti-commuting symmetry anti-commutes with itself, as we already saw, so for a symmetry  $C$  we have  $CC + CC = 0$ . If we look at a representation of this Lie algebra, so we look at the operators on states that correspond to these symmetries, we thus have an operator  $Q$  corresponding with  $C$ , such that  $QQ + QQ = 0$ . However, multiplication in the linear maps is given, it is just the composition. This means we have  $Q^2 = 0$ .

For example, imagine a system with two non-interacting bosons in the same state. If we change one boson into a fermion, while preserving all the other properties of the particle, we expect the total energy to be the same. This is because in both cases we have two almost equal particles that behave the same in the system. However, if we let this symmetry act again on the system, we get an invalid system, because we end up with two fermions in the same state. It is therefore only possible to let the symmetry act on the system once.

The second subject of this thesis are partner Hamiltonians. In the model of quantum mechanical supersymmetry, a two-dimensional non-relativistic Hamiltonian is generated by a pair of real operators as  $H = (Q + Q^*)^2$ , making the ground state energy of the Hamiltonian non-negative. Looking at the one-dimensional sub-Hamiltonians,  $H_0$  and  $H_1$ , we see that they are given by the linking operators  $A$  and its adjoint,  $A^*$ , by the relations  $H_0 = A^*A$  and  $H_1 = AA^*$ . These relations makes it possible to reduce the differential equation for the zero-energy ground state from a second order equation (calculating the kernel of  $H_0$ ) to a first order equation (calculating the kernel of  $A$ ).

With these operators it is also possible to calculate the eigenstates of  $H_0$  from the eigenstates of  $H_1$ , hence the name partner Hamiltonians. As we can easily calculate the part-

---

<sup>3</sup>A representation of a group is a map from this group to the linear maps on a vector space.



ner of the partner of a Hamiltonian, we get a sequence of Hamiltonians, each with the same spectrum of the previous Hamiltonian, but with one state less. This sequence can then be applied to calculate the complete spectrum of the first Hamiltonian. Chapter 2 is dedicated to this model and the main concepts of partner Hamiltonians.

The last subject of this text is shape invariance. It turns out that the technique of partner Hamiltonians is especially useful for a special, and luckily large and interesting, class of potentials that have the shape invariance property. In short, shape invariance means that the partner potential of a given potential is just a parameter change of the original potential. Since we already calculated the partner Hamiltonian for every parameter, the partner of the partner of this first Hamiltonian is already known in terms of the first Hamiltonian, only with a parameter change. This allows us to easily calculate the entire sequence of Hamiltonians and therefore to calculate the complete spectrum of the first Hamiltonian. The details are found in Chapter 3.

There are a couple of interesting potentials that are shape invariant. The Morse potential for example, which is used to model the binding force between two atoms [10]. This potential is also given as an example in Subsection 3.1.2. Other interesting shape invariant potentials are the Pöschl-Teller potential, used to study non-linear behaviour such as second and third harmonic generation [24], or the Scarf potential, for example used to describe photonic crystals [27].

This thesis is for a large part based on the review of supersymmetric quantum mechanics written by F. Cooper, A. Khare and U. Sukhatme [8].



# Supersymmetry

In this chapter we will examine the basic mechanics of the supersymmetric model we will use. We start with the definition of partner potentials, which are potentials with (almost) the same spectrum as their partner. These partner potentials and Hamiltonians are the main topic of Chapter 3 and in a lesser degree of Chapter 4, because they can be used to easily solve the Schrödinger equation for the given potential. At the end of the chapter, we show the supersymmetric model that can be build with these partner potentials and linking operators, thus explaining why this method is called supersymmetric quantum mechanics.

Normally, Lie algebras are used to describe systems of a Hamiltonian and its symmetries, but our symmetries do not fit into this description. To accommodate these anti-commuting operators, we use instead super Lie algebras. In Appendix A.1 a brief introduction into this topic is given. For a rigorous treatment of this subject, which also goes into the mathematics of supergravity, we recommend Varadarajan's *Supersymmetry for mathematicians: an introduction* [31].

We strive for mathematically rigorous proofs, but for some it is simply not possible to do this in the scope of thesis. Therefore, sometimes we give a more intuitive proof, leaving the details to the references. In these cases we point out were the main problem lies and directions to solve them.

## 2.1 Simple supersymmetry

### 2.1.1 Partner potentials in quantum mechanical systems

In quantum mechanics the space of all possible states consists of all square integrable functions on the space the particles are in. Then, through the Schrödinger equation the Hamiltonian of the system, the operator related to the total energy, determines how the system evolves in time. The definitions of these objects we will use are as follows.

**Definition 2.1.1** (State Space). Let  $\mathcal{H} := L^2(\mathbb{R})$  be the square integrable, complex-valued functions on  $\mathbb{R}$  with inner product

$$\langle f|g \rangle := \int_{\mathbb{R}} \overline{f(x)}g(x) dx.$$

Then we call  $\mathcal{H}$  the one-dimensional, one-particle state space.

**Definition 2.1.2** (Time-independent Hamiltonian). Let  $H : \text{dom}(H) \rightarrow \mathcal{H}$ , with  $\text{dom}(H) \subset \mathcal{H}$  a linear subspace, be an operator given by

$$H := -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x),$$

where  $V : \mathbb{R} \rightarrow \mathbb{R}$ , called the potential, is an everywhere defined smooth function,  $\hbar$  is the reduced Planck constant and  $m$  is the mass of the particle.<sup>1</sup>

Note that the domain is not as rigorously defined as should be, because we are differentiating in a space where the derivative is normally not well defined. To solve this, one can use Sobolev-space to construct a weaker definition of differentiation, based distribution rather than functions, to rigorously define the domain of the Hamiltonians and all other operators we use in quantum mechanics. However, although interesting, this process is highly technical and beyond the scope of the text. We refer to Fackler's text [13] *Mathematical Foundations of Quantum Mechanics* for further detail.

In this text, we will focus on time-independent Hamiltonians and ignore the much larger class of time-dependent Hamiltonians. The solutions of the time-independent Hamiltonian will therefore be time-independent as well. These solutions can be divided into two categories, *bound states* and *scattered states*. Bound state solutions are the square integrable solutions of the Schrödinger equation. These solutions can be understood as being the probability density functions of the position<sup>2</sup>. Scattered states are unbound states and generally describe quantum waves being scattered by the potential. Most of this thesis is only concerned with bound states, Section 4.2 being the exception.

For one-dimensional quantum mechanical systems, there is a nice and simple result that the eigenvalues of the Hamiltonian are non-degenerate<sup>3</sup>. This result will be used in subsequent results, so we state it here.

**Lemma 2.1.1.** *Let  $H$  be a one-dimensional Hamiltonian. Then the eigenvalues of the normalisable eigenfunctions are non-degenerate.*

<sup>1</sup>Note that we use the notation  $V(x)$  both as the function  $V : x \mapsto V(x)$  and as the linear operator  $V : \phi(x) \mapsto V(x)\phi(x)$ .

<sup>2</sup>It is entirely possible to use the momentum instead of the position, thus using as the basic wave function the momentum wave function. We however only use the position wave functions.

<sup>3</sup>This only holds as long as we only look at the Schrödinger equation itself. If we take for example symmetries on spins or electrical charges into account, then it is possible to have degeneracy in the one-dimensional case.

*Proof.* Let  $E$  be an eigenvalue of  $H$  and  $\phi$  and  $\psi$  be two eigenfunctions of  $H$  with eigenvalue  $E$ . Using the Schrödinger equations for both eigenfunctions

$$E\phi = -\frac{\hbar^2}{2m} \frac{d^2\phi}{dx^2} + V(x)\phi, \quad E\psi = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi,$$

we multiply the left equation with  $\psi$ , the right equation with  $\phi$  and equate both  $E\phi\psi$  terms to get

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} \phi + V(x)\psi\phi = -\frac{\hbar^2}{2m} \frac{d^2\phi}{dx^2} \psi + V(x)\phi\psi.$$

Subtracting the potential parts, dividing out the factors and using the product rule we get

$$-E \frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} \phi = -E \frac{\hbar^2}{2m} \frac{d^2\phi}{dx^2} \psi \Rightarrow \frac{d}{dx} \left( \phi \frac{d\psi}{dx} \right) - \frac{d\psi}{dx} \cdot \frac{d\phi}{dx} = \frac{d}{dx} \left( \frac{d\phi}{dx} \psi \right) - \frac{d\phi}{dx} \cdot \frac{d\psi}{dx}.$$

Subtracting again the common terms  $(\frac{d\phi}{dx} \cdot \frac{d\psi}{dx})$  and integrating both sides gives

$$\phi \frac{d\psi}{dx} = \psi \frac{d\phi}{dx} + c,$$

with  $c \in \mathbb{C}$  a constant. Now use the fact that solutions of the Schrödinger equation go to zero for  $x \rightarrow \infty$  to show that  $c = 0$ . This gives us

$$\frac{1}{\phi} \frac{d\phi}{dx} = \frac{1}{\psi} \frac{d\psi}{dx}.$$

From this we conclude that both eigenfunctions are multiples of each other, therefore proving the eigenvalue is non-degenerate.  $\square$

A more rigorous prove can be given by using the observation that the time-independent Schrödinger equation is in fact a Sturm-Liouville equation. If the space the functions are defined on is a compact interval, it can even be proven that every excited state has one more zero than its previous state (the state with the next lower eigenvalue) [3]. To show this result for non-compact intervals, we have to look at singular Sturm-Liouville problems, which are generally harder to solve. A reference is Krall's book on analysis, *Applied Analysis*, chapter 12 [19].

In this proof we also used a second simplification, namely the assumption that bounded solutions of the Schrödinger equation go to zero in both the limits  $x \rightarrow \pm\infty$ . Physically, this is clear, because a bound state represents a trapped particle in some region, so it should not be able to escape to infinity that easily [17]. If it would escape with a large probability, the particle would have more energy than possible to be trapped in the first place. This assumption can be proven for large classes of potentials and holds for almost every potential that we normally encounter. For further information see Agmon's paper on exponential bounds [1].

It is now time to define the main tools we will use to calculate the spectrum of  $H_0$ . The idea is to define for every Hamiltonian  $H_0$  a partner Hamiltonian  $H_1$  with the same spectrum, except the ground state energy of  $H_0$ , such that every excited eigenstate of

$H_0$  can be calculated from an eigenstate of  $H_1$  using the adjoint of the linking operator  $A$ . If one then knows the spectrum of  $H_1$ , the excited spectrum of  $H_0$  can be easily calculated. It turns out that the best way of defining the operator  $A$  and the partner Hamiltonian  $H_1$  is by  $H_0 =: A^*A$  and  $H_1 := AA^*$ . The relation between the spectra of the two can then be easily proved, as we will do in Theorem 2.1.1. The next definition makes the linking operators, as we will call  $A$  and  $A^*$  from now on, precise.

**Definition 2.1.3** (Linking Operators and Superpotential). *Let  $\mathcal{H}$  be a state space and  $H : \text{dom}(H) \rightarrow \mathcal{H}$  a Hamiltonian, with  $V(x)$  its potential. Then we define the linking operators*

$$A : \text{dom}(H) \rightarrow \mathcal{H}; \quad \psi \mapsto \frac{\hbar}{\sqrt{2m}} \frac{d\psi}{dx} + W(x)\psi,$$

$$A^* : \text{dom}(H) \rightarrow \mathcal{H}; \quad \psi \mapsto -\frac{\hbar}{\sqrt{2m}} \frac{d\psi}{dx} + W(x)\psi,$$

with  $V(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x)$ . The real function  $W(x)$  is called the superpotential<sup>4</sup>.

This is the main definition of the linking operators. However, we often like our Hamiltonian to have a ground state at zero energy, especially when we are going to calculate the partner of the partner of a Hamiltonian. Therefore, for a given Hamiltonian  $H$  with potential  $V(x)$  and ground state energy  $E_0$  we often write

$$H = \frac{\hbar^2}{2m} \frac{d}{dx^2} + V_0(x) + E_0 = A^*A + E_0,$$

with  $V(x) = V_0(x) + E_0$  and  $V_0(x) = W^2(x) - W'(x)$ . This essentially is shifting the potential down to a ground state energy of zero. This also ensures that the ground state, although not of energy zero, is in the kernel of  $A$ .

Before we take a look at the superpotential, we first have to show the linking operators are well defined and are actually each other's adjoint. This is done in the following lemma.

**Lemma 2.1.2.** *The linking operators are each other adjoints.*

*Proof.* We use the following definition of adjoint:  $\langle Af|g \rangle = \langle f|A^*g \rangle$ . Let  $f, g$ , bounded solutions of  $H$ , be arbitrary given. Then we have

$$\begin{aligned} \langle Af|g \rangle &= \int_{\mathbb{R}} \overline{Af(x)}g(x) dx \\ &= \int_{\mathbb{R}} \left( \overline{\frac{\hbar}{\sqrt{2m}} \frac{df}{dx}(x) + W(x)f(x)} \right) g(x) dx \\ &= \int_{\mathbb{R}} \frac{\hbar}{\sqrt{2m}} \frac{d\bar{f}}{dx}(x)g(x) + \overline{W(x)f(x)}g(x) dx \end{aligned}$$

---

<sup>4</sup>Although we speak of *the* superpotential of  $V(x)$ , it can be shown there are many possible superpotentials for a given potential. For further detail see section 2.3. The solution we will use will be the one without nodes, i.e. the one that is derived from the ground state. We will explain this issue later in more detail.

and

$$\begin{aligned}
\langle f|A^*g\rangle &= \int_{\mathbb{R}} \overline{f(x)} A^*g(x) \, dx \\
&= \int_{\mathbb{R}} \overline{f(x)} \left( -\frac{\hbar}{\sqrt{2m}} \frac{dg}{dx}(x) + W(x)g(x) \right) \, dx \\
&= \int_{\mathbb{R}} -\frac{\hbar}{\sqrt{2m}} \overline{f(x)} \frac{dg}{dx}(x) + \overline{f(x)} W(x)g(x) \, dx.
\end{aligned}$$

Note that  $W(x) = \overline{W(x)}$  ( $W(x)$  is real), thus if we subtract the second inner product from the first, we will lose the term with the superpotential. This means we get

$$\langle Af|g\rangle - \langle f|A^*g\rangle = \frac{\hbar}{\sqrt{2m}} \int_{\mathbb{R}} \frac{df}{dx}(x)g(x) + \overline{f(x)} \frac{dg}{dx}(x) \, dx.$$

Using the product rule we get

$$\langle Af|g\rangle - \langle f|A^*g\rangle = \frac{\hbar}{\sqrt{2m}} \left[ \overline{f(x)}g(x) \right]_{-\infty}^{\infty}.$$

Now note that both limits go to zero<sup>5</sup>, because both functions are solutions of the Hamiltonian. This proves that  $A$  and  $A^*$  are each others adjoint.  $\square$

The mathematical problem with this proof is in the definition of adjoints. Formally, the adjoint of a bounded linear operator  $T : X \rightarrow Y$ , with  $X$  and  $Y$  normed spaces is a bounded operator  $T^* : Y^* \rightarrow X^*$ , with  $X^*$  and  $Y^*$  the dual spaces of  $X$  and  $Y$ , which satisfies for every  $x \in X$  and  $y^* \in Y^*$  the equality  $y^*(Tx) = (T^*y^*)(x)$  [23]. The form of  $A^*$  is thus dependent on the domain of  $A$ . In our case however, these operators are each others adjoint when the domain is well defined using Sobolev-space. This also solves another problem, because we did not clarify if  $im(A^*) \subset dom(A)$  and  $im(A) \subset dom(A^*)$ .

Now this is settled, we can look at the other object defined, the superpotential. First look at the equation to get the superpotential: this equation comes from the chain rule when we calculate the product of  $A$  with  $A^*$ . This is shown in the following calculation:

$$\begin{aligned}
A^*A\psi &= \left( -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right) \left( \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} + W(x) \right) \psi \\
&= -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + W(x) \frac{\hbar}{\sqrt{2m}} \frac{d\psi}{dx} - W(x) \frac{\hbar}{\sqrt{2m}} \frac{d\psi}{dx} - \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx} \psi + W^2(x) \psi \\
&= -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \left( W^2(x) - \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx} \right) \psi
\end{aligned}$$

Equating this to  $H$  means  $V(x) = W^2(x) - \frac{\hbar}{\sqrt{2m}} W'(x)$ , which therefore gives the relation in Definition 2.1.3.

<sup>5</sup>See our assumption after Lemma 2.1.1.

The partner Hamiltonian we were talking about is now easily calculated by reversing the order of  $A$  and  $A^*$ , which is

$$AA^*\psi = -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \left( W^2(x) + \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx} \right) \psi.$$

As potential of this new Hamiltonian we take  $\left( W^2(x) + \frac{\hbar}{\sqrt{2m}} \frac{dW(x)}{dx} \right)$ . This potential is creatively called the partner potential.

**Definition 2.1.4** (Partner potential). *Let  $\mathcal{H}$  be a state space and let  $H_0 = A^*A + E_0$  be a Hamiltonian, with  $E_0$  its ground state energy<sup>6</sup>, written as a product of linking operators. Then we call the operator  $H_1 = AA^* + E_0$  its partner Hamiltonian and the potential*

$$V_1(x) = W^2(x) + \frac{\hbar}{\sqrt{2m}} W'(x),$$

with  $W(x)$  the superpotential taken from  $V_0(x)$ , the partner potential of  $V_0(x)$ .

It should be noted that often we do not factorise the Hamiltonian  $H$  directly, but first subtract its ground state energy from the potential, as in  $H_0 = A^*A + E_0$ , and thus  $V(x) = V_0(x) + E_0$ . The most important use of this is that the ground state is in the kernel of  $A$ , because for adjoint we have that  $\ker(A) = \text{im}(A^*)^\perp$  and  $\ker(A^*) = \text{im}(A)^\perp$ .

A very interesting mathematically corollary is that if we know the exact ground state wave function of a potential we know the potential and its superpotential as well.

**Lemma 2.1.3** (Potentials and ground states). *Let  $\mathcal{H}$  be a state space with  $H_0 = A^*A + E_0^{(0)}$  and  $H_1 = AA^* + E_0^{(0)}$  partner Hamiltonians. Then the potentials and superpotentials can be written, with  $n \in \{0, 1\}$ , in the form*

$$V_n(x) = \frac{\hbar^2}{2m} \frac{1}{\psi_0^{(n)}} \frac{d^2\psi_0^{(n)}}{dx^2}, \text{ respectively } W_n(x) = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln \left( \psi_0^{(n)} \right).$$

*Proof.* We prove this for  $n = 0$ . The other case follows straight forward from this one. For the potential we use the Schrödinger equation of the ground state  $\psi_0^{(0)}$

$$E_0^{(n)} \psi_0^{(0)} = -\frac{\hbar^2}{2m} \frac{d^2\psi_0^{(0)}}{dx^2} + V_0(x) \psi_0^{(0)} + E_0^{(0)} \psi_0^{(0)} \Rightarrow V_0(x) = \frac{\hbar^2}{2m} \frac{1}{\psi_0^{(0)}} \frac{d^2\psi_0^{(0)}}{dx^2}.$$

For the superpotential we note that

$$E_0^{(0)} \psi_0^{(0)} = H_0 \psi_0^{(0)} = A^* A \psi_0^{(0)} + E_0^{(0)} \psi_0^{(0)},$$

---

<sup>6</sup>As with the potential, we use the notation  $E_m^{(n)}$  both as the energy of the  $m$ -th bound state of  $V_n(x)$  and as the operator  $E_m^{(0)} : \phi \mapsto E_m^{(0)} \phi$ .



thus  $A^* A \psi_0^{(0)} = 0$  and thus  $A \psi_0^{(0)} = 0$ , because  $A$  and  $A^*$  are adjoints<sup>7</sup>. From Definition 2.1.3 we find

$$W_n(x) = -\frac{\hbar}{\sqrt{2m}} \frac{1}{\psi_0^{(n)}} \frac{d\psi_0^{(n)}}{dx} = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} \ln(\psi_0^{(n)}).$$

This gives the result. □

Note that if we know the superpotential of a system, we can directly calculate the ground state of it by using the proof of Lemma 2.1.3. This is because the ground state is in the kernel of  $A$ , which gives us the first-order linear differential equation

$$0 = \frac{\hbar}{\sqrt{2m}} \frac{d\psi}{dx} + W(x)\psi.$$

The solution is simply

$$\psi_0(x) = N_0 e^{-\frac{\sqrt{2m}}{\hbar} W_I(x)}, \quad (2.1)$$

where  $W_I'(x) = W(x)$  and where  $N_0$  is a normalisation constant, if the solution is normalisable. If it is not normalisable,  $H_0$  does not have a zero-energy ground state. What this physically means is described in Section 2.2. Mathematically, it means that every eigenstate of  $H_0$  can be calculated from the eigenstates of  $H_1$ , even the ground state.

The above equation also gives the reason why the superpotential is often used in contrast to the normal potential: when we know the superpotential, we do not only know the potential but also its ground state. These two results are interesting, because it shows that the existence of a bound ground state is equivalent with the existence of a superpotential. This can also be shown in a different way. The equation of the superpotential is a Riccati equation and by a careful change of coordinates it can be shown this equation is equivalent to the time-independent Schrödinger equation. Now we have learned enough about the linking operators and the superpotential, we can finally show why we want these partner Hamiltonians. It turns out that if  $\phi$  is an eigenfunction of  $H_0$  with eigenvalue  $E \neq 0$ , then  $A\phi$  is an eigenfunction of  $H_1$  with the same eigenvalue,  $E$ . Noting that  $H_1$  has no ground state at energy  $E_0^{(0)}$ , it follows from that that the ground state of  $H_1$  corresponds with the first excited state of  $H_0$ , the first excited state of  $H_1$  corresponds with the second excited state of  $H_0$ , etc. In this way, every state of  $H_1$  corresponds to an excited state of  $H_0$ , with the same energy. In Figure 2.1 you can see a graphical representation of this degeneracy. The degeneracy is proven in Theorem 2.1.1.

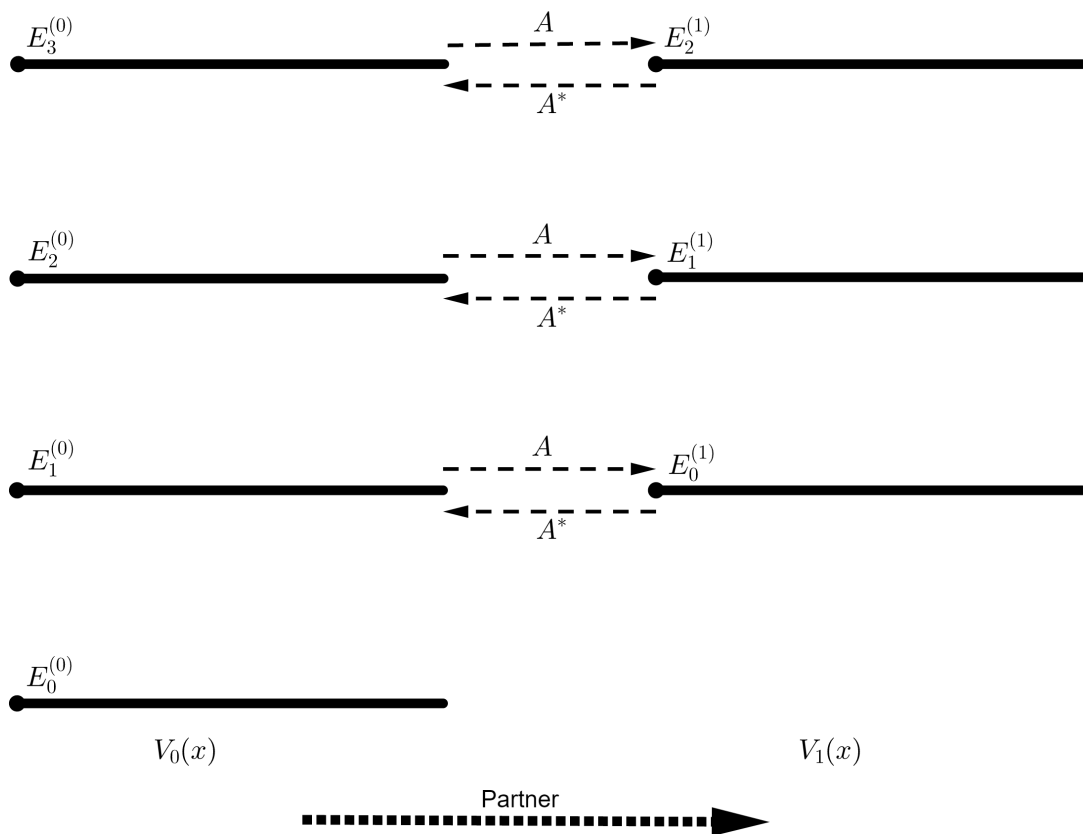
**Theorem 2.1.1.** *Let  $\mathcal{H}$  be a state space and let  $H_0$  and  $H_1$  be partner Hamiltonians. Then their normalised, nonzero-energy eigenfunctions for  $\psi_{n+1}^{(0)}$  and  $\psi_n^{(1)}$ ,  $n \in \mathbb{N}_0$ , are related by*

$$\begin{aligned} \psi_n^{(1)} &= \left(E_{n+1}^{(0)} - E_0^{(0)}\right)^{-\frac{1}{2}} A \psi_{n+1}^{(0)}, \\ \psi_{n+1}^{(0)} &= \left(E_n^{(1)} - E_0^{(0)}\right)^{-\frac{1}{2}} A^* \psi_n^{(1)}, \end{aligned}$$

---

<sup>7</sup>Remember:  $Im(A) = Ker(A^*)^\perp$ .

**Figure 2.1:** Left we have the potential  $V_0(x)$ , with right its partner potential  $V_1(x)$ . A partner potential has the same energy levels as its partner, except a possible zero energy ground state. In this figure, we thus have  $E_0^{(0)} = 0$ . The linking operators  $A$  and  $A^*$  are also seen, where  $A$  transforms an eigenstate of  $V_0(x)$  into an eigenstate of  $V_1(x)$  and  $A^*$  transforms an eigenstate of  $V_1(x)$  into an eigenstate of  $V_0(x)$ . This figure is based on a figure from [8].



where  $\psi_{n+1}^{(0)}$  and  $\psi_n^{(1)}$  have the same eigenvalue.

*Proof.* This proof follows Coopers explanation [7]. Let  $\psi_0^{(0)}$  be the ground state. First, let  $\psi_{n+1}^{(0)}$ ,  $n \in \mathbb{N}_0$ , with  $E_{n+1}^{(0)} > 0$  be an arbitrary given eigenfunction of  $H_0$ . Then  $A\psi_{n+1}^{(0)}$  is an eigenfunction of  $H_1$ , as we have

$$\begin{aligned} H_1 \left( A\psi_{n+1}^{(0)} \right) &= AA^* A\psi_{n+1}^{(0)} + E_0^{(0)} A\psi_{n+1}^{(0)} \\ &= A \left( H_0 - E_0^{(0)} \right) \psi_{n+1}^{(0)} + E_0^{(0)} A\psi_{n+1}^{(0)} \\ &= A \left( E_{n+1}^{(0)} - E_0^{(0)} \right) \psi_{n+1}^{(0)} + E_0^{(0)} A\psi_{n+1}^{(0)} \\ &= E_{n+1}^{(0)} \left( A\psi_{n+1}^{(0)} \right). \end{aligned}$$

The proof for  $A^*\psi_n^{(1)}$  goes the same.

To show  $\psi_{n+1}^{(0)} \propto A^*\psi_n^{(1)}$  and  $\psi_n^{(1)} \propto A\psi_{n+1}^{(0)}$ , note that the proof above also states that every eigenfunction of  $H_1$  corresponds to an excited eigenfunction of  $H_0$  with the same energy and that every excited eigenfunction of  $H_0$  corresponds to an eigenfunction of  $H_1$ , also with the same energy. From Lemma 2.1.1 we know that the energies of both Hamiltonians are non-degenerate and from the proof of Lemma 2.1.3 we know that the kernel of  $A$  is the ground state of  $H_0$  and the kernel of  $A^*$  is empty (we only work with bounded solutions), so  $A$  is a one-to-one function from the excited eigenfunctions of  $H_0$  to the eigenfunctions of  $H_1$  and  $A^*$  is a one-to-one function from the eigenfunctions of  $H_1$  to the excited eigenfunctions of  $H_0$ . Both  $A$  and  $A^*$  preserve the energy, so using the fact that the energies are increasing in  $n$ , we find that  $\psi_{n+1}^{(0)} \propto A^*\psi_n^{(1)}$  and  $\psi_n^{(1)} \propto A\psi_{n+1}^{(0)}$ .

For the normalisation constant we exploit the fact that  $A$  and  $A^*$  are each others adjoint, as stated in Lemma 2.1.2. Using the inner product we have

$$\begin{aligned} \langle A\psi_n^{(0)} | A\psi_n^{(0)} \rangle &= \langle \psi_n^{(0)} | A^* A\psi_n^{(0)} \rangle \\ &= \langle \psi_n^{(0)} | \left( H_0 - E_0^{(0)} \right) \psi_n^{(0)} \rangle \\ &= \langle \psi_n^{(0)} | \left( E_n^{(0)} - E_0^{(0)} \right) \psi_n^{(0)} \rangle \\ &= \left( E_n^{(0)} - E_0^{(0)} \right) \langle \psi_n^{(0)} | \psi_n^{(0)} \rangle = E_n^{(0)} - E_0^{(0)}. \end{aligned}$$

For  $E_n^{(0)} \neq 0$ . Taking the square root of this energy gives the factor by which  $A\psi_n^{(0)}$  has to be divided to be normalised again.  $\square$

Theorem 2.1.1 holds for every state, except the ground state of  $H_0$ . To understand this, remember that the ground state of  $H_0$  is given by the superpotential of equation 2.1, which was derived from the fact that the ground state is in the kernel of  $A$ . The simple answer is thus that by multiplying with  $A$ , the ground state goes to zero, which is not normalisable.

A better explanation is the following. The eigenstate of  $H_1$  with the same energy as the ground state of  $H_0$  should be in the kernel of  $A^*$ , because we have  $H_1 = AA^* + E_0^{(0)}$

and  $\text{im}(A^*) = \text{ker}(A)^\perp$ . This means that this state, we call it  $\phi$ , is of the form

$$\phi = Ne^{\frac{\sqrt{2m}}{\hbar}W_I(x)}. \quad (2.2)$$

We also know that the ground state of  $H_0$  goes to zero for  $x \rightarrow \pm\infty$  (remember the assumption). This means that  $W_I(x) \rightarrow \infty$  for  $x \rightarrow \pm\infty$ , if we want the ground state of  $H_0$ , given by Equation 2.1 to be normalisable. Forcing Equation 2.1 to be normalisable thus forces that  $\phi \rightarrow \infty$  for  $x \rightarrow \pm\infty$ . This means  $\phi$  is not normalisable and not a valid bound solution.

In this subsection we have shown the basics of the mechanics we will use in this text, especially how one can construct a partner potential with almost the same spectrum of its predecessor. In the next subsection, we will give an example of a factorisation of a Hamiltonian and the partner potential you get by this process.

## 2.1.2 Example: infinite square well

In this example we will work out a simple system, namely the infinite square well. The state space of this system is  $\mathcal{H}_0 = L^2([0, L])$ ,  $L \in \mathbb{R}_{>0}$ , with the potential  $V(x) = 0$ . The particle is thus contained in a small, compact 'container' or well. This quantum mechanical system is often the first potential encountered in the study of quantum mechanics, thus we can safely quote the eigenstates and energies of this system [17]:

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} (n+1)^2, \quad \psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}(n+1)x\right), n \in \mathbb{N}_0.$$

Separating the ground state energy out from  $V(x)$  and renumbering we get  $V_0(x) = V(x) - E_0^{(0)} = -E_0^{(0)} = -\frac{\hbar^2 \pi^2}{2mL^2}$ , thus our Hamiltonian becomes

$$H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x) + E_0^{(0)}.$$

Now  $V_0(x)$  has a zero-energy ground state. Using Lemma 2.1.3 on the ground state of  $V_0(x)$ , which is the same as the ground state of  $V(x)$ , we get the superpotential

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\pi \cos(\frac{\pi}{L}x)}{L \sin(\frac{\pi}{L}x)} = -\sqrt{E_0^{(0)}} \frac{\cos(\frac{\pi}{L}x)}{\sin(\frac{\pi}{L}x)}.$$

The linking operators therefore become

$$A = \frac{\hbar}{\sqrt{2m}} \frac{d}{dx} - \sqrt{E_0^{(0)}} \frac{\cos(\frac{\pi}{L}x)}{\sin(\frac{\pi}{L}x)}, \quad A^* = -\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} - \sqrt{E_0^{(0)}} \frac{\cos(\frac{\pi}{L}x)}{\sin(\frac{\pi}{L}x)},$$

and the partnerpotentials (using Definitions 2.1.3 and 2.1.4):

$$\begin{aligned}
V_0(x) &= W^2(x) - \frac{\hbar}{\sqrt{2m}}W'(x) \\
&= E_0^{(0)} \frac{\cos^2(\frac{\pi}{L}x)}{\sin^2(\frac{\pi}{L}x)} - E_0^{(0)} \frac{1}{\sin^2(\frac{\pi}{L}x)} = -E_0^{(0)}, \\
V_1(x) &= W^2(x) + \frac{\hbar}{\sqrt{2m}}W'(x) \\
&= E_0^{(0)} \frac{\cos^2(\frac{\pi}{L}x)}{\sin^2(\frac{\pi}{L}x)} + E_0^{(0)} \frac{1}{\sin^2(\frac{\pi}{L}x)} = E_0^{(0)} \frac{1 + \cos^2(\frac{\pi}{L}x)}{\sin^2(\frac{\pi}{L}x)}.
\end{aligned}$$

We now have two Hamiltonians on  $[0, L]$ ,

$$\begin{aligned}
H_0 &= A^*A + E_0^{(0)} = \frac{\hbar^2}{2m} \frac{d}{dx^2} + V_0(x) + E_0^{(0)} \\
\text{and } H_1 &= AA^* + E_0^{(0)} = \frac{\hbar^2}{2m} \frac{d}{dx^2} + V_1(x) + E_0^{(0)},
\end{aligned}$$

that have, apart from the ground state of  $H_0$ , the same energies. Their eigenstates are also related to each other, because multiplying the excited eigenstates of  $H_0$  gives the eigenstates of  $H_1$ . By Theorem 2.1.1 for  $n \in \mathbb{N}_0$  we have:

$$\begin{aligned}
\psi_n^{(1)} &= \left(E_{n+1}^{(0)} - E_0^{(0)}\right)^{-\frac{1}{2}} A\psi_{n+1}^{(0)} \\
&= \left(\frac{\hbar^2 \pi^2}{2mL^2}(n+1)(n+3)\right)^{-\frac{1}{2}} \left(\frac{\hbar}{\sqrt{2m}} \frac{d}{dx} - \sqrt{E_0^{(0)}} \frac{\cos(\frac{\pi}{L}x)}{\sin(\frac{\pi}{L}x)}\right) \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L}(n+2)x\right) \\
&= \sqrt{\frac{2}{(n+1)(n+3)L}} \left((n+2) \cos\left(\frac{\pi}{L}(n+2)x\right) - \cos\left(\frac{\pi}{L}x\right) \frac{\sin\left(\frac{\pi}{L}(n+2)x\right)}{\sin\left(\frac{\pi}{L}x\right)}\right).
\end{aligned}$$

This example shows that the spectrum of one of the partner Hamiltonians can be used to calculate the spectrum of the other quite easily. However, you have to be lucky to have a partner you know everything about. To really benefit from this technique, we like to factorise  $H_1$  again, so we have another pair of linking operators,  $A_1$  and  $A_1^*$ , such that  $H_1 = A_1^*A_1 + E_0^{(1)}$ . In this way we can solve more easily for the ground state of  $H_1$  using the kernel of  $A_1$ , and then calculate the first excited state of  $H_0$ . How we then go on to calculate the other excited states of  $H_0$  is the topic of Subsection 2.1.3.

### 2.1.3 Families of partner potentials

In Subsection 2.1.1, we saw how we can define partner potentials for a given Hamiltonian, such that the partner Hamiltonians have the same eigenvalues and the eigenstates are related by linking operators. In this section we will take this idea one step further. Instead of just one partner Hamiltonian we define a family of them having,

excluding some initial states, the same eigenvalues and related eigenstates.

The idea is the same as in Subsection 2.1.1, we only go further and calculate a sequence of partner Hamiltonians  $H_n$ , where every two adjacent Hamiltonians are each other's partner. So if  $H_n = A_n^* A_n + E_0^{(n)}$ , then  $H_{n+1} = A_n A_n^* + E_0^{(n)} = A_{n+1}^* A_{n+1} + E_0^{(n+1)}$ , with  $A_n$  and  $A_n^*$  the linking operators between  $H_n$  and  $H_{n+1}$ .

For example, if we use  $A_2$  to calculate the ground state of  $H_2$ , i.e.  $\psi_0^{(2)}$  (remember Equation 2.1), we can go on and calculate the first excited state of  $H_n$  by multiplying with  $A_n^*$ , as is given by Theorem 2.1.1:

$$\psi_1^{(1)} = \left(E_0^{(2)} - E_0^{(1)}\right)^{-\frac{1}{2}} A_1^* \psi_0^{(2)}.$$

If we now use the linking operator between  $H_0$  and  $H_1$ , namely  $A_0^*$ , on  $\psi_1^{(1)}$  then Theorem 2.1.1 gives us the second excited state of  $H_0$ :

$$\begin{aligned} \psi_2^{(0)} &= \left(E_1^{(1)} - E_0^{(0)}\right)^{-\frac{1}{2}} A_0^* \psi_1^{(1)} \\ &= \left(E_1^{(1)} - E_0^{(0)}\right)^{-\frac{1}{2}} A_0^* \left(E_0^{(2)} - E_0^{(1)}\right)^{-\frac{1}{2}} A_1^* \psi_0^{(2)} \\ &= \left(E_2^{(0)} - E_0^{(0)}\right)^{-\frac{1}{2}} \left(E_2^{(0)} - E_1^{(0)}\right)^{-\frac{1}{2}} A_0^* A_1^* \psi_0^{(2)}. \end{aligned}$$

Of course, we could go on, starting with  $H_3$ ,  $H_4$  etc, but the main idea is clear from this calculation. In this subsection, we explore the details of this method and prove this idea works, but the above calculation basically shows what is going on. In Figure 2.2 the eigenstates and energies of a family of partner potentials is drawn.

**Definition 2.1.5** (Family of partner Hamiltonians). *Let  $\mathcal{H}$  be a state space and  $H_0$  a Hamiltonian on this space. Then the  $n^{\text{th}}$  partner Hamiltonian  $H_n$ ,  $n \in \mathbb{N}$ , of  $H_0$  is, if the ground state of  $H_{n-1}$  is normalizable, recursively defined by  $H_n := A_{n-1} A_{n-1}^* + E_0^{(n-1)}$ , where  $H_{n-1} = A_{n-1}^* A_{n-1} + E_0^{(n-1)}$  is the previous partner Hamiltonian with  $E_0^{(n-1)}$  its ground state energy. Otherwise  $H_n$  is not defined.*

If the Hamiltonian has a finite number of normalised states, then the family of partner Hamiltonians consists of only a finite number of Hamiltonians. Note that the partner Hamiltonian  $H_n$  misses its first  $n$  eigenstates with respect to  $H_0$ . From this we get a useful lemma, where we see that the potential and superpotential can be written in terms of their ground state.

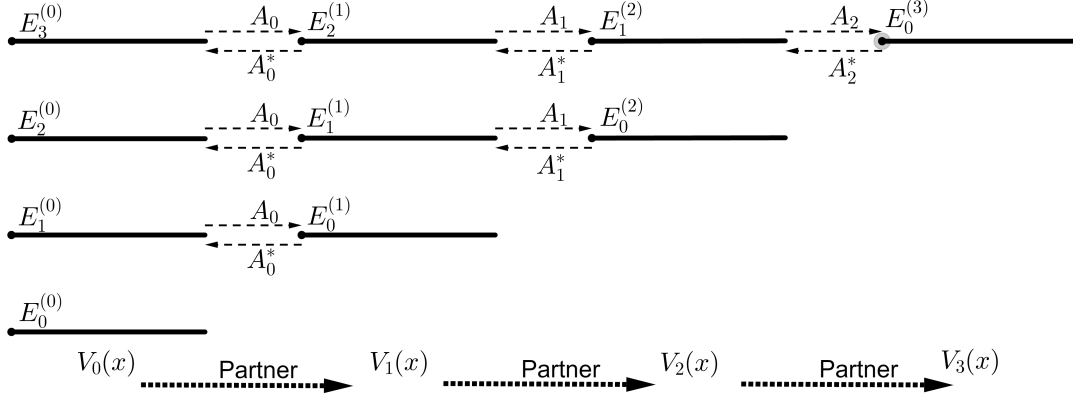
The following result can be useful for calculations.

**Corollary 2.1.1** (Family of Partner Potentials). *Let  $\mathcal{H}$  be a state space and  $H_n$ ,  $n \in \mathbb{N}_0$ , a family of partner potentials, possibly finite. Then the potentials satisfy the relations*

$$V_{n+1}(x) = V_n(x) - 2 \frac{\hbar}{\sqrt{2m}} \frac{d^2}{dx^2} \ln \left( \psi_0^{(n)} \right),$$

with  $n \in \mathbb{N}_0$  and  $\psi_0^{(n)}$  the ground state of  $H_n$ .

**Figure 2.2:** We see here the family of the potential  $V_0(x)$ . The linking operators are drawn between the eigenstates. Note that every partner has one bound eigenstate less than the previous one. The last partner (not drawn),  $V_4(x)$  would have no bound states left. This figure is based on a figure from [8].



*Proof.* Let  $n \in \mathbb{N}_0$  be arbitrarily given. Let  $W_n(x)$  be the superpotential with  $V_n(x) = W_n^2(x) - \frac{\hbar}{\sqrt{2m}}W_n'(x) + E_0^{(n)}$  and  $V_{n+1}(x) = W_n^2(x) + \frac{\hbar}{\sqrt{2m}}W_n'(x) + E_0^{(n)}$ . Then we have

$$\begin{aligned}
 V_{n+1}(x) &= W_n^2(x) + \frac{\hbar}{\sqrt{2m}}W_n'(x) + E_0^{(n)} \\
 &= W_n^2(x) - \frac{\hbar}{\sqrt{2m}}W_n'(x) + E_0^{(n)} + 2\frac{\hbar}{\sqrt{2m}}W_n'(x) \\
 &= V_n(x) + 2\frac{\hbar}{\sqrt{2m}}W_n'(x) \\
 &= V_n(x) - 2\frac{\hbar}{\sqrt{2m}}\frac{d^2}{dx^2}\ln\left(\psi_0^{(n)}\right).
 \end{aligned}$$

Where we used Lemma 2.1.3 in the last equation. □

The relations between eigenvalues and eigenstates are equivalent with those of a single partnership, as the following theorem states.

**Theorem 2.1.2** (Degeneracy of a Family). *Let  $\mathcal{H}$  be a state space and  $H_n$ ,  $n \in \mathbb{N}_0$ , a family of partner Hamiltonians, possibly finite. Then the eigenvalues satisfy the relations*

$$E_m^{(n+l)} = E_{m+l}^{(n)} \quad n, m, l \in \mathbb{N}_0,$$

and the eigenstates the relations

$$\begin{aligned}\psi_m^{(n+l)} &= \prod_{i=1}^l \left( \left( E_{m+i}^{(n+l-i)} - E_0^{(n+l-i)} \right)^{-\frac{1}{2}} A_{n+l-i} \right) \psi_{m+l}^{(n)}, \\ \psi_{m+l}^{(n)} &= \prod_{i=1}^l \left( \left( E_{m+l-i}^{(n+i)} - E_0^{(n+i)} \right)^{-\frac{1}{2}} A_{n+i-1}^* \right) \psi_m^{(n+l)},\end{aligned}$$

with  $n, m \in \mathbb{N}_0$  and  $l \in \mathbb{N}$ , for all defined Hamiltonians.

*Proof.* Let  $n, m \in \mathbb{N}_0$  be arbitrarily given. Note that if this formula holds for  $l = 1$ , it also holds for  $l > 1$ , because  $l > 1$  is just using multiple times the case  $l = 1$  with every time different values for  $n$  and  $m$ . We thus have to prove that  $E_m^{(n+1)} = E_{m+1}^{(n)}$  and

$$\begin{aligned}\psi_m^{(n+1)} &= \left( E_{m+1}^{(n)} - E_0^{(n)} \right)^{-\frac{1}{2}} A_n \psi_{m+1}^{(n)}, \\ \psi_{m+1}^{(n)} &= \left( E_m^{(n+1)} - E_0^{(n+1)} \right)^{-\frac{1}{2}} A_n^* \psi_m^{(n+1)}.\end{aligned}$$

However, this follows directly from Theorem 2.1.1. □

Theorem 2.1.2 tells us that if we know the ground states of every Hamiltonian in the family, we know all the (bounded) eigenstates of every Hamiltonian in the family. The ground states can be calculated from the linking operators, because the ground states of the Hamiltonians in the family are in the kernels of the operators  $A_n$ . This is a good result, because one does not have to solve a second order differential equation with this technique.

The downside of this method is that you have to solve a non-linear first order differential equation to get the linking operators, as Definition 2.1.3 requires a superpotential for the linking operators. In most cases, this slows down the calculation, so we would like to have a shortcut, to not have to calculate the superpotential or potential again every time we add another partner to the family. A possible shortcut would be to write the partner potential of a given potential in terms of the given potential, such that we only have to solve for the superpotential ones. This idea is explored in Chapter 3.

## 2.2 Supersymmetric model

For some background information on super linear algebra and super Lie algebras, we refer to Appendix A.1.

We are now at a point where we can introduce a supersymmetric quantum mechanical model. Normally, a symmetry group is described by a Lie group, a manifold that also has a group structure. However, because there is a close correspondence between Lie groups and Lie algebras (the latter can mathematically be seen as the tangent space of the identity element or physically as the logarithm around the identity element),



we can also say that this Lie group generates the Lie algebra. In the general case, this algebra contains both the Poincaré group of Minkowski symmetries and a group of internal symmetries. The question asked by Coleman and Mandula was if there were internal symmetries that could generate the Lie algebra containing the Poincaré group. Their answer was no, because they only used commuting symmetries. However, when using the loophole, i.e. allowing anti-commuting symmetries, discovered by Haag, Lopuszanski and Sohnius, it is possible to have some internal symmetries generating the total Lie algebra, or now better called the total super Lie algebra.

In our quantum mechanical case, we only use time translation in our Poincaré group of symmetries, so the representation of our symmetry group will only contain the Hamiltonian, not the momenta or other continuous symmetries. As anti-commuting symmetries, we will use two supersymmetries, which will be represented by an operator and its adjoint. Our super Lie algebra will therefore contain only three base vectors, which represent the Hamiltonian and the two supersymmetries. The supersymmetries will be the odd vectors, named  $B$  and  $C$ , and the Hamiltonian the even vector,  $A$ . The commutation and anti-commutation relations are therefore given by

$$[A, B] = [A, C] = 0, [B, C] = A, [A, A] = [B, B] = [C, C] = 0.$$

Note that, because  $B$  and  $C$  are odd, we have  $[B, C] = [C, B]$ , in contrary to  $A$ , where we have  $[A, B] = -[B, A]$  and  $[A, C] = -[C, A]$ . As the commutation relations show, our super Lie algebra is generated by only the anti-commuting symmetries,  $B$  and  $C$ , because we have  $[B, C] = A$ . This is the super Lie algebra we will use.

The only task we have to do now is to define a representation of this super Lie algebra in terms of quantum operators, which is done in Definition 2.2.1, and to show this actually is a representation of our super Lie algebra. As operators we will use a two-dimensional Hamiltonian composed of a one-dimensional Hamiltonian and its partner, and two charges, composed of a matrix multiplied with one of the two linking operators. It turns out that this choice is a perfect representation of our little super Lie algebra and is therefore a nice model for supersymmetric quantum mechanics.

**Definition 2.2.1** (Charge operators). *Let  $\mathcal{H}_0$  be a state space and let  $H_0, H_1$  be partner Hamiltonians given by linking operators  $H_0 = A^*A$  and  $H_1 = AA^*$ . Define a new state space  $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$  with Hamiltonian  $H := H_0 \oplus H_1$ . Then the charge operators  $Q$  and  $Q^*$  and the Hamiltonian  $H$  are given by*

$$Q := \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^* := \begin{pmatrix} 0 & A^* \\ 0 & 0 \end{pmatrix} \text{ and } H := \begin{pmatrix} A^*A & 0 \\ 0 & AA^* \end{pmatrix}.$$

First notice that these charges are each others adjoint. Second, notice that  $Q^*Q + QQ^* = H$  and  $Q^2 = (Q^*)^2 = 0$ . These two relations are in line with the anti-commuting relations, because  $QQ + QQ = 2Q^2 = 0$ ,  $Q^*Q^* + Q^*Q^* = 2(Q^*)^2 = 0$  and  $Q^*Q + QQ^* = H$  hold. We have to use the anti-commutator, because  $B$  and  $C$  are odd, and thus  $Q$  and  $Q^*$  should also be odd. The commutators of these charges with

the Hamiltonian are given by

$$\begin{aligned}
HQ - QH &= Q^*QQ + QQ^*Q - QQ^*Q - QQQ^* \\
&= Q^*QQ - QQQ^* = 0 - 0 = 0, \\
HQ^* - Q^*H &= Q^*QQ^* + QQ^*Q^* - Q^*Q^*Q - Q^*QQ^* \\
&= QQ^*Q^* - Q^*Q^*Q = 0 - 0 = 0.
\end{aligned}$$

The operators comply with the relations of the super Lie algebra, if we use the standard commutator and anti-commutator as super Lie bracket. Note that we have used commuting relations and super Lie bracket almost as the same concept. They are however not. Commuting relations are only defined when there is a normal multiplication. The super Lie bracket is not a commutation relation, it is actually the multiplication in the super Lie algebra. Luckily for us, the commutation/anti-commutation relations can be proven to comply with the axioms for a super Lie bracket. We will come back to this later in Theorem 2.2.1.

Going back to our operators, we see that our Hamiltonian is two-dimensional, but we would rather describe it as having a bosonic and a fermionic part. The  $n$ th excited state can be interpreted as the  $n$ -particle state, where the degeneracy determines if one of the particles is a boson or fermion. The  $Q$ -charge alters a boson into a fermion and  $Q^*$  alters a fermion into a boson. For the energy this change does not matter, so there is a symmetry between bosons and fermions. As the commutation relations between  $Q$ ,  $Q^*$  and  $H$  already showed,  $Q$  and  $Q^*$  commute with  $H$  and are therefore symmetries of  $H$ . The exact degeneracy is showed in Lemma 2.2.1.

**Lemma 2.2.1 (Degeneracy).** *Let  $\mathcal{H}$ ,  $H$ ,  $Q$  and  $Q^*$  as in Definition 2.2.1. Then every non-zero eigenvalue of  $H$  is two-fold degenerate. The zero-eigenvalue, if it exists, is nondegenerate.*

*Proof.* First, let  $E \neq 0$  and  $\psi \in \text{dom}(H)$ <sup>8</sup> with  $H\psi = E\psi$  be arbitrarily given. Then we have for its components that:

$$\begin{pmatrix} E\psi^{(0)} \\ E\psi^{(1)} \end{pmatrix} = E\psi = H\psi = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} \begin{pmatrix} \psi^{(0)} \\ \psi^{(1)} \end{pmatrix} = \begin{pmatrix} H_0\psi^{(0)} \\ H_1\psi^{(1)} \end{pmatrix},$$

thus  $H_0\psi^{(0)} = E\psi^{(0)}$  and  $H_1\psi^{(1)} = E\psi^{(1)}$ . Looking at the first equality, this only exists if  $E = E_{n+1}^{(0)}$ ,  $n \in \mathbb{N}_0$ , thus  $\psi^{(0)} = c_1\psi_{n+1}^{(0)}$ ,  $c_1 \in \mathbb{C}$ . Then using Theorem 2.1.1 we get that  $\psi^{(1)} = c_2\psi_n^{(1)}$ ,  $c_2 \in \mathbb{C}$ . This shows that

$$\psi = c_1 \begin{pmatrix} \psi_{n+1}^{(0)} \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ \psi_n^{(1)} \end{pmatrix},$$

thus it lies in the span of  $\begin{pmatrix} \psi_{n+1}^{(0)} \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ \psi_n^{(1)} \end{pmatrix}$ . This completes the first part.

For the second part, notice that if  $\psi$  is in the kernel,  $H_0\psi^{(0)} = 0$  and  $H_1\psi^{(1)} = 0$ . We already

<sup>8</sup>See the assumption in subsection 2.1.1.

saw that if a non-zero  $\psi_0^{(0)}$  exists, that the kernel of  $H_1$  is empty. Thus the only way for a non-zero  $\psi$  to be in the kernel of  $H$  is to be of the form

$$\psi = c_1 \begin{pmatrix} \psi_0^{(0)} \\ 0 \end{pmatrix}.$$

This makes the kernel one-dimensional and thus is the eigenvalue 0 non-degenerate.  $\square$

To complete our model, we only have to prove our system is a super Lie algebra and therefore a true representation of a supersymmetric system.

**Theorem 2.2.1.** *Let  $\mathcal{H}_0$  be a state space with partner potentials  $H_0$  and  $H_1$ . Let  $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$  and  $H = H_0 \oplus H_1$ . Then  $\mathcal{H}$  is a super vector space and the super Lie algebra generated by the charges contains  $H$ .*

*Proof.* The fact that  $\mathcal{H}$  is a super vector space is trivial, because of its definition.

The space spanned by  $Q$ ,  $Q^*$  and  $H$  is a super vector space  $V$ , with  $V_0 := \text{span}(H)$  and  $V_1 := \text{span}(Q, Q^*)$ . Note that from  $Q^2 = (Q^*)^2 = 0$  and  $Q^*Q + QQ^* = H$  we see  $Q$  and  $Q^*$  are odd, because the composition of two odd function should, and is, even.  $H$  is obviously even.

We thus only have to show that this space is a super Lie algebra generated by the charges. We define the bracket of the super Lie algebra by  $[A, B] = AB - (-1)^{p(A)p(B)}BA$ , where the multiplication is the composition of maps. This definition clearly is bilinear. The charges are odd, meaning the bracket of the charges equals  $[Q^*, Q] = Q^*Q + QQ^* = H$ .  $H$  is thus generated by the charges. Note that the super vector space is closed under the bracket, because the bracket of a charge with the Hamiltonian is equal to the commutator of the two and therefore equal to zero.

We only have to show our commutator/anti-commutator is a super Lie bracket. For the super anti-linearity we have ( $A, B \in V$ ):

$$\begin{aligned} [A, B] &= AB - (-1)^{p(A)p(B)}BA \\ &= -(-1)^{p(A)p(B)}(BA - (-1)^{p(A)p(B)}AB) \\ &= -(-1)^{p(A)p(B)}[B, A]. \end{aligned}$$

For the last property, notice that we only have three generators, so we may just calculate the identity with these three:

$$\begin{aligned} [H, [Q, Q^*]] + (-1)^{p(H)p(Q)+p(H)p(Q^*)}[Q, [Q^*, H]] + (-1)^{p(H)p(Q^*)+p(Q)p(Q^*)}[Q^*, [H, Q]] \\ = [H, H] + [Q, 0] - [Q^*, 0] = 0. \end{aligned}$$

The other permutations are similar, because every inner bracket is either the zero bracket or  $H$ . This means that the outer bracket is given by  $[H, H]$ ,  $[Q^*, H]$  or  $[Q, H]$  (the elements in the brackets can be switched). All these brackets are zero, so the identity always holds. This means  $V$  is a super Lie algebra.  $\square$

We therefore have a super Lie algebra of a Hamiltonian and charges, generated by the charges, that is constructed from the linking potentials and partner Hamiltonians of Subsection 2.1.1. This is the reason why the method described in that section is called supersymmetric.

Now we have the supersymmetric quantum mechanical model, there is only one issue to address: what is the difference between having a zero-energy ground state and not having such state? The answer lies in the difference between *broken* and *unbroken* supersymmetry. As with every other symmetry, you would want to know if there are any states that are invariant under the symmetry. For example, the even states (ground state, second excited state etc.) of the infinite square well are invariant under the symmetry  $x \mapsto L - x$ , the symmetry of mirroring the potential around  $x = \frac{1}{2}L$ . For supersymmetry, there only is one candidate, the zero-energy state. This would directly be the ground state, because we can write

$$H = Q^*Q + QQ^* = (Q + Q^*)^2,$$

thus every eigenvalues is non-negative<sup>9</sup>. This means that the vacuum state, the physical interpretation of the ground state, is invariant under this supersymmetry, if it has zero energy. This is called *unbroken* supersymmetry. If the ground state has non-zero energy, then it is not invariant under supersymmetry, so we call this case *broken* supersymmetry. The physical significance lies in the fact that in the first case, bosons and fermions have the same mass [32]. If the vacuum energy is not invariant under this symmetry, it can be proven a massless fermion exists, the so-called Goldstone fermion [25]. Thus, having a zero-energy ground state is physically significant.

## 2.3 Singular potentials

Until now, we used that  $V(x)$  is regular, i.e. it has no singular points. This allowed us to build up a theory where we can proof a degeneracy theorem, Theorem 2.1.1. It would be interesting to see if this theorem would hold even if  $V(x)$  has one or more singularities. This question turns out to be related to another topic, namely the definition of the linking operators in Definition 2.1.4. Here we took the linking operators  $A$  and  $A^*$  such that  $H_0 = A^*A + E_0$ , with  $E_0$  the ground state energy. This meant that the ground state of this Hamiltonian is part of the kernel of  $A$ . We could however chose another term, in stead of  $E_0$ . This choice determines the behaviour of our system and gives rise to singular potentials<sup>10</sup>.

The main idea is to look for solutions of the Schrödinger equation without caring for normalisability. With  $\epsilon$  the parameter we use to modify our choices and  $\phi$  a possible solution, the Schrödinger equation becomes

$$-\frac{d^2\phi}{dx^2} + V(x)\phi = \epsilon\phi.$$

<sup>9</sup>This is also the case in quantum field theory [14].

<sup>10</sup>For more information on singular potentials in general, we recommend the review written by Frank, Land and Spector [15].

Here we have taken  $\hbar = 2m = 1$  for clarity. For  $\epsilon = E_0$  we get our ground state solution back. Writing  $V_0(x) = V(x) - E_0$  then gives the situation of Definition 2.1.4. This case is called *unbroken supersymmetry*, as was discussed in the last section. This is also the assumed case for this text (except this section).

The case  $\epsilon < E_0$  is called *broken supersymmetry*. Note that in this case  $V_0(x)$  has a ground state energy larger than zero, because it is just an increased  $V(x)$ . Although a simple translation, this displacement has large consequences. This was also discussed in the previous section.

The case  $\epsilon > E_0$  is the case where singular potentials come into play. If we write  $\phi_\epsilon$  for the solution with energy  $\epsilon$ , then we have

$$\begin{aligned} V(x) - \epsilon &= \frac{1}{\phi_\epsilon} \frac{d^2 \phi_\epsilon}{dx^2} = \frac{1}{\phi_\epsilon^2} \left( \frac{d^2 \phi_\epsilon}{dx^2} \phi_\epsilon - \left( \frac{d\phi_\epsilon}{dx} \right)^2 + \left( \frac{d\phi_\epsilon}{dx} \right)^2 \right) \\ &= \left( -\frac{1}{\phi_\epsilon} \frac{d\phi_\epsilon}{dx} \right)^2 - \frac{d}{dx} \left( -\frac{1}{\phi_\epsilon} \frac{d\phi_\epsilon}{dx} \right) \\ &= W_\epsilon^2(x) - W'_\epsilon(x), \end{aligned}$$

where the first step is directly seen from the Schrödinger equation. This shows us that  $W_\epsilon(x) = -\frac{1}{\phi_\epsilon} \frac{d\phi_\epsilon}{dx}$  is a superpotential that gives  $V(x) - \epsilon$ . Note that the ground state does not have zeros [4], but solutions after this state do. Therefore,  $W_\epsilon(x)$  has at least one singularity, and consequently the partner potential  $V_1(x)$  has at least one too.

The question remains if Theorem 2.1.1 holds. This is not the case. The problem is that the superpotential is part of the linking operator, making the linking operator singular at particular values of  $x$ . To have meaningful wave functions, every wave function should be zero at that particular point, so not to have singular wave functions. In general, this is not possible, as the example of the infinite square well shows.

However, partial degeneracy can be possible, if the potentials are symmetric around their singularity [9]. This can be done if we use the first excited state of  $V_0(x)$ , because it only has one zero<sup>11</sup>. In this case some wave functions arise that are symmetric around the singularity and have a zero there. These wave functions can nullify the singularity in the linking operator, making it possible to have a meaningful partner function. An example is the infinite square well from Example 2.1.2. If we take  $L = \pi$  the first excited state of the infinite square well becomes

$$\psi_1^{(0)}(x) = \sqrt{\frac{2}{\pi}} \sin(2x).$$

The superpotential is then

$$W(x) = -\frac{1}{\psi_1^{(0)}} \frac{d\psi_1^{(0)}}{dx} = -\frac{2 \cos(2x)}{\sin(2x)}.$$

The partner potentials are

$$V_0(x) = W^2(x) - W'(x) = -4, \quad V_1(x) = W^2(x) + W'(x) = 4 \frac{\cos^2(2x) + 1}{\sin^2(2x)}.$$

---

<sup>11</sup>For solutions with  $\epsilon > E_1$ , this will not be possible, because they have at least two singularities, making this argument a lot harder.

The second potential definitely has a strong singularity at  $x = \frac{1}{2}\pi$ . Now note that  $V_1(x)$  is just twice the standard partner potential of  $V_0(x)$ , one at  $[0, \frac{1}{2}\pi)$  and one at  $(\frac{1}{2}\pi, \pi]$ . At  $x = \frac{1}{2}\pi$  it forces the wave functions to zero, as the 'doubled' potential shows. Now look how the eigenfunctions of  $V_0(x)$  are transformed to 'eigenfunctions' of  $V_1(x)$  with the help of the linking operators:

$$\begin{aligned} A\psi_{n+1}^{(0)}(x) &\propto \left( \frac{d}{dx} + \frac{2\cos(2x)}{\sin(2x)} \right) \sin((n+1)x) \\ &= (n+1)\cos((n+1)x) - \frac{2\cos(2x)}{\sin(2x)}\sin((n+1)x) \quad \propto \psi_n^{(1)}(x). \end{aligned}$$

Remember that  $n = 1$  is the ground state of the infinite square well (convention), and we basically subtracted the first excited energy of the potential, so the energy of the first excited state should be zero. This means the first excited state should be nullified by the linking operator  $A$ , as is the case. Evaluating  $A\psi_n^{(0)}(x)$  at  $x = \frac{1}{2}\pi$  gives the limit

$$\lim_{x \rightarrow \frac{1}{2}\pi} n \cos(nx) - 2 \cos(2x) \frac{\sin(nx)}{\sin(2x)}.$$

For  $n$  odd we have the first cosine zero, the second equal to 1, but the fractal becomes  $\pm\infty$ , depending on  $n$ , thus the singularity of the linking operators does not vanish. However, for  $n$  even, this limit does go to zero. In this case, the infinite square well can just be described as two infinite square wells combined, just as the partner potential is described. There are some sign problems, but that can be fixed by the right factor for  $A$ . This means there is a partial degeneration. However, as we already stated, this is an exception, because in general, degeneracy is broken.

# Shape Invariant Potentials

In this chapter we will look at a class of potentials that can be solved easily with algebraic methods, called *shape invariant potentials*. First we describe the basic case of this class, how it is defined and how it is used. Then we give an example to illustrate a way of using them by calculating the excited states of a slightly modified Morse potential. Using a generalisation of shape invariance, called multiple-step shape invariance, we open up a huge class of potentials, only limited by the fact that those potentials can be rather exotic. We end the chapter with a short introduction of ways to find shape invariant potentials and with a remark on how some shape invariant potentials are related to each other by coordinate changes.

## 3.1 Easily solvable potentials

### 3.1.1 Concept of shape invariant potentials

In Chapter 2 we saw the definition of partner potentials and how they can be used to solve for the spectrum of a Hamiltonian. It became clear that this method, although useful, has its drawbacks, because we would still have to solve for the superpotential every time we add a new partner to the sequence. This chapter is about a class of potentials where this problem does not occur. These potentials, called *shape invariant potentials* (SIPs), have the property that their partner are of the same form. This means that when you have to solve for the superpotential to add a new state, the equation becomes a lot easier, because you already solved it for a similar potential. Before we go into details, we start with an example.

Shape invariant potentials, take  $V_0(x, a)$ , are dependent on one or more parameters,  $a \in \mathbb{R}^n$ , such that their partner has the same form as  $V_0(x, a)$ , but only with a different parameter:  $V_1(x, a) = V_0(x, f(a)) + R(a)$ <sup>1</sup>. Here is  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  a parameter change and  $R : \mathbb{R}^n \rightarrow \mathbb{R}$  a vertical shift. This parameter change is why this class is called

<sup>1</sup>Here we use  $R(a)$  both as a *function*  $R : a \mapsto R(a)$  and as an *operator*  $R : \phi \mapsto R(a)\phi$ .

shape invariant, because we only change the settings of the potential, not the overall shape. We allow the partner potential to be vertically translated, because it turns out that we can calculate the eigenvalues using  $R(a)$ .

Note that we do not have to stop here, the entire family of this potential can easily be calculated. This is because we can just separate the ground state out of the potential: if we start with a Hamiltonian of the form<sup>2</sup>

$$H_0 = -\frac{\hbar^2}{2m} \frac{d}{dx^2} + V_0(x, a) + E_0^{(0)}(a) = A_0^*(x, a)A_0(x, a) + E_0^{(0)},$$

the next two Hamiltonians in the family of  $H_0$  become

$$\begin{aligned} H_1 &= A_0(x, a)A_0^*(x, a) + E_0^{(0)}(a) \\ &= -\frac{\hbar^2}{2m} \frac{d}{dx^2} + V_1(x, a) + E_0^{(0)}(a) \\ &= -\frac{\hbar^2}{2m} \frac{d}{dx^2} + V_0(x, f(a)) + R(a) + E_0^{(0)}(a) \\ &= A_0^*(x, f(a))A_0(x, f(a)) + R(a) + E_0^{(0)}(a); \\ H_2 &= A_0(x, f(a))A_0^*(x, f(a)) + R(a) + E_0^{(0)}(a) \\ &= -\frac{\hbar^2}{2m} \frac{d}{dx^2} + V_1(x, f(a)) + R(a) + E_0^{(0)}(a) \\ &= -\frac{\hbar^2}{2m} \frac{d}{dx^2} + V_0(x, f^2(a)) + R(f(a)) + R(a) + E_0^{(0)}(a) \\ &= A_0^*(x, f^2(a))A_0(x, f^2(a)) + R(f(a)) + R(a) + E_0^{(0)}(a) \\ H_n &= -\frac{\hbar^2}{2m} \frac{d}{dx^2} + V_0(x, f^n(a)) + \sum_{i=1}^n R(f^{i-1}(a)) + E_0^{(0)}(a) \\ &= -\frac{\hbar^2}{2m} \frac{d}{dx^2} + V_0(x, f^n(a)) + E_0^{(n)}(a). \end{aligned}$$

The last step will be proven in Corollary 3.1.1.

As an example, take the Hamiltonian<sup>3</sup>

$$H_0 = -\frac{d}{dx^2} + V_0(x, a) = -\frac{d}{dx^2} + \frac{a(a-1)}{\cos^2(x)} - a^2,$$

with  $a \geq 0$ , defined on  $[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ . This is a special case of the trigonometric Scarf 1 potential [9] and has zero ground state energy. The superpotential of this potential is  $W(x, a) = a \tan(x)$ . The linking operators of Definition 2.1.3 are given by

$$A(x, a) = \frac{d}{dx} + a \tan(x), \quad A^*(x, a) = -\frac{d}{dx} + a \tan(x).$$

<sup>2</sup>As with  $R(a)$ ,  $E_m^{(n)}(a)$  is a function as well as an operator, giving the  $m$ -th energy level of the  $n+1$ -th potential in the family, because we started with  $V_0(x, a)$ .

<sup>3</sup>We set  $\hbar = 2m = 1$  for clarity.



The partner Hamiltonians become

$$\begin{aligned}
H_0 &= A^*(x, a)A(x, a) \\
&= -\frac{d}{dx^2} + W^2(x, a) - W'(x, a) \\
&= -\frac{d}{dx^2} + \frac{a^2 \sin(x)^2}{\cos(x)^2} - \frac{a}{\cos(x)^2} \\
&= -\frac{d}{dx^2} + \frac{a(a-1)}{\cos^2(x)} - a^2; \\
H_1 &= A(x, a)A^*(x, a) \\
&= -\frac{d}{dx^2} + W^2(x, a) + W'(x, a) \\
&= -\frac{d}{dx^2} + \frac{a^2 \sin(x)^2}{\cos(x)^2} + \frac{a}{\cos(x)^2} \\
&= -\frac{d}{dx^2} + \frac{a(a+1)}{\cos^2(x)} - a^2,
\end{aligned}$$

so we have

$$V_1(x, a) = \frac{a(a+1)}{\cos^2(x)} - a^2.$$

These two definitely look like each other. In fact, we can write  $V_1(x, a)$  as

$$\begin{aligned}
V_1(x, a) &= \frac{a(a+1)}{\cos^2(x)} - a^2 \\
&= \frac{a(a+1)}{\cos^2(x)} - (a+1)^2 + (2a+1) \\
&= V_0(x, a+1) + R(a).
\end{aligned}$$

Here we took  $R(a) = 2a + 1$ . Now we separate out the ground state energy, such that

$$H_1 = -\frac{(d)}{dx^2} + V_0(x, a+1) + R(a) = A^*(x, a+1)A(x, a+1) + R(a).$$

The next Hamiltonian in the family now becomes

$$\begin{aligned}
H_2 &= A(x, a+1)A^*(x, a+1) + R(a) \\
&= -\frac{d}{dx^2} + W^2(x, a+1) + W'(x, a+1) + R(a) \\
&= -\frac{d}{dx^2} + \frac{(a+1)^2 \sin(x)^2}{\cos(x)^2} + \frac{a+1}{\cos(x)^2} + 2a+1 \\
&= -\frac{d}{dx^2} + \frac{(a+1)(a+2)}{\cos^2(x)} - a^2.
\end{aligned}$$

Again, this potential is just a parameter shift (or two) away from  $V_0(x, a)$ :

$$\begin{aligned}
V_2(x, a) &= \frac{(a+1)(a+2)}{\cos^2(x)} - a^2 \\
&= \frac{(a+1)(a+2)}{\cos^2(x)} - a^2 - 2a - 1 + 2a + 1 \\
&= \frac{(a+1)(a+2)}{\cos^2(x)} - (a+1)^2 + 2a + 1 \\
&= V_1(x, a+1) + R(a) \\
&= \frac{(a+1)(a+2)}{\cos^2(x)} - (a+1)^2 - 2a - 3 + 2a + 3 + 2a + 1 \\
&= \frac{(a+1)(a+2)}{\cos^2(x)} - (a+2)^2 + 2a + 3 + 2a + 1 \\
&= V_0(x, a+2) + R(a+1) + R(a).
\end{aligned}$$

As we can see in this example, the family of a SIP is very easily calculated. In Definition 3.1.1 and Lemma 3.1.1 we will make this idea precise.

**Definition 3.1.1** (Shape Invariant Potential). *Let  $\mathcal{H}$  be a state space,  $H_0 = A^*A + E_0^{(0)}$  and  $H_1 = AA^* + E_0^{(0)}$  partner Hamiltonians and  $V_0(x, a)$  and  $V_1(x, b)$  their potentials with parameters  $a, b \in \mathbb{R}^n$ . Then  $V_0(x, a)$  is shape invariant if it satisfies the relation*

$$V_1(x, a) = V_0(x, f(a)) + R(a),$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $R : \mathbb{R}^n \rightarrow \mathbb{R}$  are continuous functions.

Note that we assume  $V_0(x, a)$  has a zero-energy ground state, because we split the Hamiltonian into a factorisation term and a ground state energy term. We will therefore often use  $E_0^{(0)}(a) = 0$ . Later in Subsection 3.1.3 this will be used again, but more explicitly. In Section 2.3 we have gone deeper into this topic, studying potentials with non-zero ground state energy.

If we use this shape invariant condition multiple times, we get a family of partner potentials that all have the same form. This is the topic of Lemma 3.1.1.

**Lemma 3.1.1** (Family of SIPs). *Let  $\mathcal{H}$  be a state space and  $V_0(x, a)$ ,  $a \in \mathbb{R}^n$ , a shape invariant potential with zero ground state energy for all  $a \in \mathbb{R}^n$ , thus  $E_0^{(0)}(a) = 0$ . Then the family of Hamiltonians  $H_n$ ,  $n \in \mathbb{N}_0$ , given by*

$$H_n = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x, f^n(a)) + \sum_{i=1}^n R(f^{i-1}(a)), n \in \mathbb{N}$$

and  $H_0 = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x, a)$  is a family of partner Hamiltonians.

*Proof.* The case  $n = 0$  is obvious. So let  $n \in \mathbb{N}$  be arbitrarily given and assume the statement holds for every  $0 \leq m \leq n - 1$ . First, write the Hamiltonians  $H_n$  and  $H_{n+1}$  as

$$\begin{aligned} H_n &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x, f^n(a)) + \sum_{i=1}^n R(f^{i-1}(a)) \\ &=: -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \tilde{V}_n(x, a) + \sum_{i=1}^n R(f^{i-1}(a)), \\ H_{n+1} &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \left( V_0(x, f^{n+1}(a)) + R(f^n(a)) \right) + \sum_{i=1}^n R(f^{i-1}(a)) \\ &=: -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \tilde{V}_{n+1}(x, a) + \sum_{i=1}^n R(f^{i-1}(a)). \end{aligned}$$

We have to prove  $H_n$  and  $H_{n+1}$  are partner Hamiltonians, thus  $\tilde{V}_n(x, a)$  and  $\tilde{V}_{n+1}(x, a)$  are partner potentials. Note that for  $V_0(x, f^n(a))$  we have  $E_0^{(0)}(f^n(a)) = 0$ , thus the ground state energy of  $H_n$  equals  $E_0^{(n)}(a) = \sum_{i=1}^n R(f^{i-1}(a))$ . This means we can write  $H_n$  as

$$H_n = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \tilde{V}_n(x, a) + E_0^{(n)}(a) = A_n^*(a)A_n(a) + E_0^{(n)}(a).$$

Using the shape invariant condition on  $V_0(x, f^n(a))$  we get

$$\begin{aligned} H_{n+1}' &= A_n(a)A_n^*(a) + E_0^{(n)}(a) \\ &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_0(x, f^{n+1}(a)) + R(f^n(a)) + E_0^{(n)}(a) \\ &= -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \tilde{V}_{n+1}(x, a) + E_0^{(n)}(a) = H_{n+1}. \end{aligned}$$

This means  $H_n$  and  $H_{n+1}$  are partner Hamiltonians. □

**Corollary 3.1.1** (Energies of a SIP family). *The eigenvalues of a family of SIP-partner Hamiltonians are given by*

$$E_m^{(n)} = \sum_{i=1}^{n+m} R(f^{i-1}(a)), \quad n, m \in \mathbb{N}_0, n + m \geq 1,$$

and  $E_0^{(0)} = 0$ .

*Proof.* This follows directly from the proof of Lemma 3.1.1. □

From the previous discussion it is clear that we can very easily generate the family of partner Hamiltonians of a SIP. The reason why the spectrum of a SIP is very easily calculated becomes clear when we note that if one solves for the ground state of  $V_0(x, a)$ , for every  $a$ , one directly gets the ground state of every other potential in the

family: if  $\psi_0^{(0)}(x, a)$  is the ground state of  $V_0(x, a)$ , then  $\psi_0^{(0)}(x, f(a))$  is the ground state of  $V_0(x, f(a))$ . Shifting the potential up with a constant does not change the behaviour of the eigenstates, so  $\psi_0^{(0)}(x, f(a))$  also is the ground state of  $V_0(x, f(a)) + R(a)$ , only with a different energy. Using the shape invariant condition we therefore see that  $\psi_0^{(0)}(x, f(a))$  is the ground state of  $V_1(x, a)$ .

Now we have the ground state of every Hamiltonian in the family of partner Hamiltonians, getting the complete spectrum of  $H_0$  is easy. The linking operators are all clear because of the shape invariance condition on the superpotentials. So taking the ground state of a Hamiltonian, say  $H_2$ , and multiplying it with the appropriate linking operators, here  $A^*(x, a)A^*(x, f(a))\psi_0^{(0)}(x, f(a))$ , we can get every excited state of  $H_0$ , here the second excited state. Theorem 3.1.1 makes this idea precise.

**Theorem 3.1.1** (Degeneracy of a SIP-family). *Let  $\mathcal{H}$  be a state space and  $H_n$ ,  $n \in \mathbb{N}_0$ , a family of SIP-partner Hamiltonians, possibly finite. Then the eigenvalues satisfy the relations*

$$E_m^{(n+l)} = E_{m+l}^{(n)} = \sum_{i=1}^{n+m+l} R(f^{i-1}(a)), \quad n, m, l \in \mathbb{N}_0, n + m + l \geq 1,$$

$E_0^{(0)}(a) = 0$  and the eigenstates the relations

$$\psi_m^{(n)}(x, a) = \prod_{i=1}^m \left( \left( \sum_{j=n+i}^{n+m} R(f^{j-1}(a)) \right)^{-\frac{1}{2}} A^*(x, f^{n+i-1}(a)) \right) \psi_0(x, f^{n+m}(a)),$$

with  $n \in \mathbb{N}_0$  and  $m \in \mathbb{N}$ , for all defined Hamiltonians.

*Proof.* Clear from Theorem 2.1.2, Corollary 3.1.1 and the observation that  $\psi_0^{(n)}(x, a) = \psi_0^{(0)}(x, f^n(a))$ .  $\square$

Although the condition on this class is strong, some rich families of potentials have been found in this class. Simple examples are the radial Coulomb potential and the harmonic oscillator. It is not known if all families of SIPs are found, nor if shape invariance is necessary for a potential to be analytically solvable. There are results that may indicate there are analytically solvable potentials that are not shape invariant [6].

### 3.1.2 Example: Morse potential

To show the power of SIP's we will now calculate the bounded eigenstates of an important potential, namely the Morse potential. This potential, normally given by [10]

$$V(x) = D(1 - e^{-ax})^2, \quad (3.1)$$

with  $D > 0$  the depth of the potential,  $\alpha > 0$  an inverse width of the well and  $x = 0$  the equilibrium point of the potential, is used as an approximation of the binding force between two atoms in a molecule. Here one particle is at  $-\infty$ , where the other is in the potential. The behaviour of this particle is as follows: for  $x < 0$ , it gives a repulsive force, because the potential goes to infinity. This models strong repulsive forces between two atoms coming to close to each other. Going in the other direction, the potential gradually increases to a constant. This means that for large  $x > 0$  the Morse potentials describes a decreasing attractive force. The finite limit at  $x \rightarrow \infty$  ensures there are only a finite number of bounded states<sup>4</sup>. However, to use our newly developed techniques on this system we have to slightly modify the potential to make it shape invariant:

$$V_0(x, a) = (a - be^{-\alpha x})^2 - \alpha be^{-\alpha x}. \quad (3.2)$$

We still have the qualitative behaviour of the potential, the parameters are only a little less physically obvious. If we take as superpotential<sup>5</sup>  $W(x, a) = a - be^{-\alpha x}$ , this change becomes clear:

$$\begin{aligned} V_0(x, a) &= (a - be^{-\alpha x})^2 - \alpha be^{-\alpha x} \\ &= W^2(x, a) - W'(x, a) \\ &= a^2 - 2b(a + \frac{1}{2}\alpha)e^{-\alpha x} + b^2e^{-2\alpha x}. \end{aligned}$$

We thus used the square in the Morse potential in our advantage to make the potential shape invariant. Furthermore, we can give physical meaning to the parameters by using  $D = a^2 = b^2$ , where  $a, b > 0$  to ensure the right physical behaviour. Of course, it would also be possible to first derive the ground state by another method and then calculate the superpotential using Lemma 2.1.3. This would however lead to a more complicated superpotential. We are only interested in the qualitative behaviour of this model, so we use this way. The partner potential of  $V_0(x, a)$  can now be calculated:

$$\begin{aligned} V_1(x, a) &= W^2(x, a) + W'(x, a) \\ &= a^2 - 2b(a - \frac{1}{2}\alpha)e^{-\alpha x} + b^2e^{-2\alpha x} \end{aligned}$$

The function  $f(a)$  is clear from the factor  $(a - \frac{1}{2}\alpha)$ :  $f(a) = a - \alpha$ . This gives us  $R(a)$  using Definition 3.1.1:

$$R(a) = V_1(x, a) - V_0(x, f(a)) = a^2 - (a - \alpha)^2$$

Now we have  $R(a)$  and  $f(a)$  it is possible to calculate the energy eigenvalues, without even using the eigenstates. The eigenvalues are given by Theorem 3.1.1:

$$E_m^{(0)} = \sum_{k=1}^m (a - (k-1)\alpha)^2 - (a - k\alpha)^2 = a^2 - (a - m\alpha)^2.$$

---

<sup>4</sup>That this is actually the case becomes clear later.

<sup>5</sup>Again, we use  $\hbar = 2m = 1$  for clarity.

This equation already gives an indication there are only a finite number of bounded states, because for

$$m > \frac{a}{\alpha}$$

the energies go down again, which is physically impossible. To calculate the eigenstates, we need the ground state, because even the spectra of shape invariant potentials do not come for free. We will quote this function here from [7].

$$\psi_0^{(0)}(x, a) = \left( \frac{2b}{\alpha} e^{-\alpha x} \right)^{\frac{a}{\alpha}} e^{-\frac{1}{2} \frac{2b}{\alpha} e^{-\alpha x}} = y(x)^{\frac{a}{\alpha}} e^{-\frac{1}{2} y(x)},$$

where we used  $y(x) = \frac{2b}{\alpha} e^{-\alpha x}$ . The first excited state is then calculated by just multiplying  $\psi_0^{(0)}(x, f(a))$  with  $(E_1^{(0)} - E_0^{(0)})^{-\frac{1}{2}} A^*(x, a)$  (we assume  $\frac{a}{\alpha} > 1$ ):

$$\begin{aligned} \psi_1^{(0)}(x, a) &= (E_1^{(0)} - E_0^{(0)})^{-\frac{1}{2}} A^*(x, a) \psi_0^{(0)}(x, f(a)) \\ &= (2a\alpha - \alpha^2)^{-\frac{1}{2}} \left( -\frac{d}{dx} + a - be^{-\alpha x} \right) y(x)^{\frac{a-\alpha}{\alpha}} e^{-\frac{1}{2} y(x)} \\ &= (2a\alpha - \alpha^2)^{-\frac{1}{2}} \\ &\quad \cdot \left( (a - \alpha) y(x)^{\frac{a}{\alpha}} e^{-\frac{1}{2} y(x)} - \frac{1}{2} \alpha y(x)^{\frac{a}{\alpha}+1} e^{-\frac{1}{2} y(x)} + a y(x)^{\frac{a}{\alpha}} e^{-\frac{1}{2} y(x)} - \frac{\alpha}{2} y(x)^{\frac{a}{\alpha}+1} e^{-\frac{1}{2} y(x)} \right) \\ &= (2a\alpha - \alpha^2)^{-\frac{1}{2}} y(x)^{\frac{a}{\alpha}} e^{-\frac{1}{2} y(x)} (2a - \alpha - \alpha y(x)) \\ &= \left( \frac{2a}{\alpha} - 1 \right)^{-\frac{1}{2}} y(x)^{\frac{a}{\alpha}} e^{-\frac{1}{2} y(x)} \left( \frac{2a}{\alpha} - 1 - y(x) \right). \end{aligned}$$

Using Theorem 3.1.1 for  $\psi_m^{(0)}$ ,  $m \in \mathbb{N}$  we get excited states in the following form:

$$\begin{aligned} \psi_m^{(0)}(x, a) &= \prod_{i=1}^m \left( \left( \sum_{j=i}^m R(f^{j-1}(a)) \right)^{-\frac{1}{2}} A^*(x, f^{i-1}(a)) \right) \cdot \psi_0(x, f^m(a)) \\ &= \prod_{i=1}^m \left( (E_m^{(0)} - E_{i-1}^{(0)})^{-\frac{1}{2}} \left( -\frac{d}{dx} + W(x, f^{i-1}(a)) \right) \right) \cdot \psi_0^{(0)}(x, a - \alpha m). \end{aligned}$$

With this equation we can get every excited state of the Morse potential by just multiplying the appropriate ground state by a number of slightly different operators. This identity is already calculated into an explicit form, namely [9]

$$\psi_m^{(0)}(x, a) = y(x)^{\frac{a}{\alpha} - m} e^{-\frac{1}{2} y(x)} L_m^{2(\frac{a}{\alpha} - m)}(y(x)),$$

with  $y(x) = \frac{2b}{\alpha} e^{-\alpha x}$  and  $L_n^\alpha(x)$  the Laguerre polynomials<sup>6</sup>. With shape invariance, we have therefore solved the entire spectrum of the Morse potential.

<sup>6</sup>The Laguerre polynomial  $L_n^\alpha(x)$  is defined as the polynomial solution of the equation  $xy'' + (\alpha + 1 - x)y' + ny = 0$ ,  $n \in \mathbb{N}_0$  [26].

### 3.1.3 Shape invariance in multiple steps

In Subsection 3.1.1 we saw the basic definition of shape invariance, namely  $V_1(x, a) = V_0(x, f(a)) + R(a)$ , with  $a \in \mathbb{R}^n$  a parameter and  $f(a)$  and  $R(a)$  some functions. This equality, which gives a relation between every two subsequent potentials in a family of partner potentials, was then used to calculate the ground states of these potentials. With multiple-step shape invariant potentials we generalise this equality to open up the possibility of even more solvable potentials. However, this generalisation has its costs: to know the complete spectrum of a multi-step SIP, only solving for the ground state of the first potential is not enough anymore, we need more.

So how does this work? As an example of the concept, we begin with a special case of a  $N$ -step SIP,  $N \in \mathbb{N}$ . We start with a family of partner potentials  $V_n(x, a)$ ,  $n \in \mathbb{N}_0$  and  $a \in \mathbb{R}^n$ . The  $N$ -step shape invariance condition is now  $V_N(x, a) = V_0(x, f(a)) + R_N(a)$ . The partner potentials in between are connected as usual with superpotentials  $W_{n_1}(x, a)$ ,  $0 \leq n_1 \leq N - 1$ . To make it a family, we take  $V_N(x, a)$  and treat it like  $V_0(x, a)$ . The superpotentials needed for this new  $N$ -step are already given, because of the  $N$ -step shape invariance condition. Therefore, we can just go along building up the family.

In the normal case of shape invariance in multiple steps, we allow for the intermediate potentials to be vertically translated. This means that  $W_0(x, a)$  connects  $V_0(x, a)$  with a 'twin' potential  $V_1^+(x, a)$ , which is translated down with a displacement of  $R_1(a)$ , such that its 'twin',  $V_1^-(x, a) = V_1^+(x, a) - R_1(a)$ , becomes the partner potential of  $V_2^+(x, a)$  with superpotential  $W_1(x, a)$ . The definition of  $N$ -step shape invariance is therefore the following.

**Definition 3.1.2** ( $N$ -step shape invariance). *Let  $N \in \mathbb{N}_{\geq 2}$  (the case  $N = 1$  is the standard shape invariance). Let  $V_0(x, a)$ ,  $V_{n_1}^\pm(x, a)$ ,  $1 \leq n_1 \leq N - 1$ , and  $V_N(x, a)$  be potentials such that  $V_0(x, a)$  and  $V_1^+(x, a)$ ,  $V_{n_1}^-$  and  $V_{n_1+1}^+$ , with  $1 \leq n_1 \leq N - 2$ , and  $V_{N-1}^-(x, a)$  and  $V_N(x, a)$  are partner potentials with superpotentials  $W_{n_1}(x, a)$ ,  $0 \leq n_1 \leq N - 1$ , and  $V_{n_1}^+(x, a) = V_{n_1}^-(x, a) + R_{n_1}(a)$  hold for every  $1 \leq n_1 \leq N - 1$ , for some  $R_{n_1} : \mathbb{R}^n \rightarrow \mathbb{R}$ . Let  $W_{n_1}(x, a)$  be the superpotential of  $V_{n_1}^-(x, a)$  if  $n_1 \geq 1$  and of  $V_0(x, a)$  if  $n_1 = 0$ . Then  $V_0(x, a)$  is called  $N$ -step shape invariant if*

$$V_N(x, a) = V_0(x, f(a)) + R_N(a)$$

for some  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $R_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ .

To simplify the notation a little bit, we often use instead of  $n$  the double index  $n = Nn_N + n_1$ . Here  $n_N \in \mathbb{N}_0$  is the index that gives the number of shape invariant 'steps' and  $n_1$  is the index that for  $1 \leq n_1 \leq N - 1$  gives the potentials in between and for  $n_1 = 0$  the shape invariant potentials. The in-between potentials are thus written as  $V_{Nn_N+n_1}^\pm(x, a)$ , where the plus and minus signs denote the upper or lower variants,  $1 \leq n_1 \leq N - 1$ . The shape invariant potentials do not have plus and minus variants. Note that the  $N$ -step shape invariance condition also connects the intermediate potentials: if  $V_{N+n_1}^+(x, a)$  is an intermediate potential after  $V_N(x, a)$ , then it is given by  $V_{N+n_1}^+(x, a) = V_{n_1}^+(x, f(a)) + R_N(a)$ , because the superpotentials between  $V_0(x, f(a))$  and  $V_{n_1}^+(x, f(a))$  are the same to those between  $V_N(x, a)$  and  $V_{N+n_1}^+(x, a)$ . This is the

subject of Lemma 3.1.2.

**Lemma 3.1.2** (Shape invariance for intermediate potentials). *Let  $V_0(x, a)$  be a  $N$ -step shape invariant potential, with its family as in Definition 3.1.2. Then for every two intermediate potentials we have*

$$V_{Nn_N+n_1}^\pm(x, a) = V_{n_1}^\pm(x, f^{n_N}(a)) + \sum_{i=1}^{n_N} R_N(f^{n_N-1}(a)),$$

with  $n_N \in \mathbb{N}$  and  $1 \leq n_1 \leq N - 1$ .

*Proof.* Clear by using the  $N$ -step shape invariance condition to recognise that  $W_{Nn_N+n_1}(x, a) = W_{n_1}(x, f^{n_N}(a))$ , because the terms involving  $R_{n_1}(a)$  are not used in the determination of the superpotentials, as is in the case of normal SIPs.  $\square$

As was the case in the standard shape invariance, we will assume the ground state energies of  $V_{n_1}^-(x, a)$  are zero. This assumption is important, because it allows us to calculate all the excited energies for  $V_0(x, a)$ . For example, the first excited energy of  $V_0(x, a)$  is equal to the ground state energy of  $V_1^+(x, a)$ . From  $V_1^+(x, a) = V_1^-(x, a) + R_1(a)$  and the assumption that  $V_1^-(x, a)$  has a zero-energy ground state we thus find that  $E_1^{(0)} = E_0^{(1)} = R_1(a)$ . Going further, for the first  $N - 1$  energies we have

$$E_{n_1}^{(0)}(a) = \sum_{i=1}^{n_1} R_i(a),$$

with  $1 \leq n_1 \leq N - 1$ . Note that we stated that the energies depend on  $a$ . In the normal shape invariance we did not state it, although those energies do depend on  $a$  too. In the multiple-step shape invariance we state it, because we need it to calculate the energies from  $N$  and above. Using the  $N$ -step shape invariance condition and the previous result we get the equation

$$\begin{aligned} E_{n_1}^{(Nn_N)}(a) &= E_{n_1}^{(0)}(f^{n_N}(a)) + \sum_{i=1}^{n_N} R_N(f^{i-1}(a)) \\ &= \sum_{i=1}^{n_1} R_i(f^{n_N}(a)) + \sum_{i=1}^{n_N} R_N(f^{i-1}(a)). \end{aligned}$$

To get to  $E_{Nn_N+n_1}^{(0)}(a)$ , note that

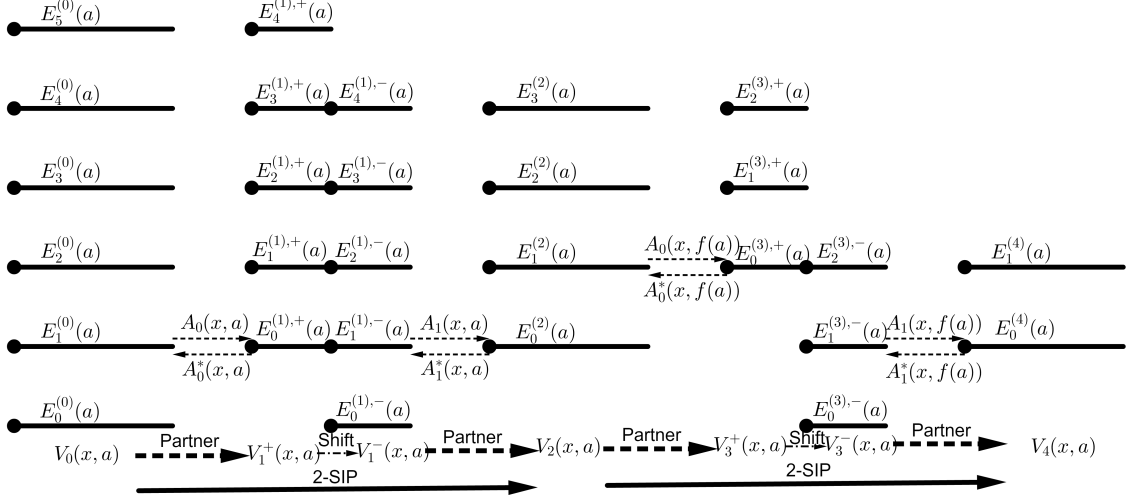
$$E_{n_1+1}^{(Nn_N-1),+}(a) = E_{n_1}^{(Nn_N)}(a) + R_{N-1}(f^{n_N-1}(a)),$$

because  $V_{Nn_N-1}^-(x, a)$  and  $V_{Nn_N}(x, a)$  are partner potentials. Going further we get

$$E_{N+n_1}^{(Nn_N-N)}(a) = E_{n_1}^{(Nn_N)}(a) + \sum_{i=1}^{N-1} R_i(f^{n_N-1}(a)).$$



**Figure 3.1:** Figure of a 2-step SIP family of  $V_0(x, a)$ . The figure shows two 2-steps, from  $V_0(x, a)$  to  $V_4(x, a)$ . Note that, although the energies between two partner potentials are equal, the energies between a plus and a minus potential are possibly not equal, so the equality between the energies of  $V_0(x, a)$  and  $V_2(x, a)$  should not be taken for granted. It is shown that the linking operator from  $V_2(x, a)$  to  $V_3^-(x, a)$  is  $A_0(x, f(a))$ . This is because of the shape invariance property.



Solving recursively we arrive at the energies of  $V_0(x, a)$ :

$$\begin{aligned}
 E_{Nn_N+n_1}^{(0)}(a) &= E_{n_1}^{(Nn_N)}(a) + \sum_{i=1}^{n_N} \sum_{j=1}^{N-1} R_j(f^{i-1}(a)) \\
 &= \sum_{i=1}^{n_1} R_i(f^{n_N}(a)) + \sum_{i=1}^{n_N} R_N(f^{i-1}(a)) + \sum_{i=1}^{n_N} \sum_{j=1}^{N-1} R_j(f^{i-1}(a)) \\
 &= \sum_{i=1}^{n_1} R_i(f^{n_N}(a)) + \sum_{i=1}^{n_N} \sum_{j=1}^N R_j(f^{i-1}(a)).
 \end{aligned}$$

Even with multiple-step shape invariance, we can still calculate all the energies with only our knowledge about the shape invariance conditions.

Calculating the eigenfunctions of  $V_0(x, a)$  is similar to the standard SIP case. For the  $Nn_N + n_1$ th excited state we look at the ground state of  $V_{Nn_N+n_1}^+(x, a)$ , which is known, because it is by Lemma 3.1.2 equal to the ground state of  $V_{n_1}^+(x, f^{n_N}(a))$ . This ground state is known through the superpotential  $W_{n_1}(x, a)$ . Using the various linking operators and noting that an eigenfunction of  $V_{Nn_N+n_1}^-(x, a)$  is equal to a corresponding eigenfunction of  $V_{Nn_N+n_1}^+(x, a)$  (only its energy is different), we can calculate every excited state of  $V_0(x, a)$ . The result of this construction is also shown in Figure 3.1. The results are given in Theorem 3.1.2.

**Theorem 3.1.2** (Degeneracy of a  $N$ -SIP family). *Let  $V_0(x, a)$  be a potential and let its family*

given by Definition 3.1.2. Then the eigenvalues are given by

$$\begin{aligned} E_{n_1}^{(0)}(a) &= \sum_{i=1}^{n_1} R_i(a), \\ E_{Nn_N}^{(0)}(a) &= \sum_{i=1}^{n_N} \sum_{j=1}^N R_j(f^{i-1}(a)), \\ E_{Nn_N+n_1}^{(0)}(a) &= \sum_{i=1}^{n_N} \sum_{j=1}^N R_j(f^{i-1}(a)) + \sum_{i=1}^{n_1} R_i(f^{n_N}(a)), \end{aligned}$$

with  $1 \leq n_1 \leq N - 1$  and  $n_N \in \mathbb{N}$ . The eigenfunctions are given by

$$\begin{aligned} \psi_{n_1}^{(0)}(x, a) &\propto \left( \prod_{i=1}^{n_1} A_{i-1}^*(a) \right) \psi_0^{(n_1)}(x, a), \\ \psi_{Nn_N}^{(0)}(x, a) &\propto \left( \prod_{i=1}^{n_N} \prod_{j=1}^N A_{j-1}^*(x, f^{i-1}(a)) \right) \psi_0^{(0)}(x, f^{n_N}(a)), \\ \psi_{Nn_N+n_1}^{(0)}(x, a) &\propto \left( \left( \prod_{i=1}^{n_N} \prod_{j=1}^N A_{j-1}^*(x, f^{i-1}(a)) \right) \left( \prod_{i=1}^{n_1} A_{i-1}^*(f^{n_N}(a)) \right) \right) \psi_0^{(n_1)}(x, f^{n_N}(a)), \end{aligned}$$

again with  $1 \leq n_1 \leq N - 1$  and  $n_N \in \mathbb{N}$ .

*Proof.* Clear from the discussion. □

Although artificially looking, there actually is a huge class of  $N$ -step SIPs, as is shown by Barclay et al. [2]. However, the solutions are only given as Taylor expansions of the superpotentials in the parameter, with the coefficients being functions of  $x$ . This simply means these potentials are not practical to work with. In contrast, for theoretical purposes these potentials are of use, because they represent a new class of completely solvable potentials.

## 3.2 Translational SIPs

Now we know how SIPs work and what we can do with it, a natural question to ask is: how do we find them<sup>7</sup>? If SIPs are so useful, we should want to find more of them. This however is not an easy task, as one has to solve the equation

$$W^2(x, a) + W'(x, a) = W^2(x, f(a)) - W'(x, f(a)) + R(a),$$

which we get from the shape invariance condition and the formulas for partner potentials. Here, differentiation has been done with respect to  $x$ . There has not been a general solution for this equation yet, but there are large classes of potentials that are

---

<sup>7</sup>Their existence is already proven in Example 3.1.2.

governed by it. In this section we will show one of such classes.

A method that is used to find such classes is to first assume a parameter change  $f(a)$ . Second, one guesses a suitable and general form for the superpotential. Last, this form is substituted into the equation to get a shape invariant potential. If there is a solution, it has to be shown that the ground state is normalisable, so that there exists at least one bound state. This is also the method we will use.

Our assumption for the parameter change will be that it is translational, so  $f(a) = a + \alpha$  for some  $\alpha \in \mathbb{R}$ . In contrast, in the next section we will use as assumption that  $f(a)$  is a multiplication,  $f(a) = qa$ , where we have  $q \in (0, 1)$ . As our assumption for the superpotential we will use

$$W(x, a) = f(x) + (a + c_1)g(x) + \frac{h(x)}{a + c_2}.$$

Here  $f(x)$ ,  $g(x)$  and  $h(x)$  are functions to be determined and  $c_1, c_2 \in \mathbb{R}$  constants. Of course, this assumption can be made more general, for example by using more parameters for the parameter change. For two parameters, this gives two families of SIPs, but more parameters seem to be too much [9].

The assumed form comes from the observation that the eigenvalues of non-relativistic potentials seem to follow a certain law for large  $n$  [22],

$$E_n \propto n^{-\gamma},$$

with  $\gamma \in [-2, 2]$ . For example, the infinite square well of Example 2.1.2 goes like  $E_n \propto n^2$ , where the Coulomb potential goes like  $E_n \propto n^{-2}$ . 'Differentiating'  $E_n$  with respect to  $n$ , because of the energy relation of Theorem 3.1.1, gives  $R(f^n(a)) \propto n^{\gamma-1}$ . If we restrict  $\gamma$  to integers, the powers of  $n$  will correspond to the powers of  $n$  (from  $f^n(a) = a + n\alpha$ ) in the shape invariance equation for superpotentials: if we look at the shape invariant condition for the potentials, the equation reads

$$V_n(x, a) = V_0(x, f^n(a)) + \sum_{k=0}^{n-1} R(f^k(a)) = V_0(x, f^n(a)) + E_n.$$

If  $E_n \propto n^\gamma$  with  $\gamma \in [-2, 2]$ , then it would be a good guess there are terms in  $V_0(x, f^n(a))$  that also go like  $n^\gamma$ . This then translates to  $n^\delta$ ,  $\delta \in [-1, 1]$  for the superpotential, because of the square in  $V_0(x, a) = W^2(x, a) - W'(x, a)$ .

These are all observations, assumptions and heuristics, but if it helps us to find a large class of SIPs, it is worth the effort. Substituting our assumption in the equation for

superpotentials, we get the new equation

$$\begin{aligned}
& f(x)^2 + (a + c_1)^2 g(x)^2 + \frac{h(x)^2}{(a + c_2)^2} + 2(a + c_1)f(x)g(x) + 2\frac{f(x)h(x)}{a + c_2} + 2\frac{a + c_1}{a + c_2}g(x)h(x) \\
& + f'(x) + (a + c_1)g'(x) + \frac{h'(x)}{a + c_2} \\
= & f(x)^2 + (a + \alpha + c_1)^2 g(x)^2 + \frac{h(x)^2}{(a + \alpha + c_2)^2} \\
& + 2(a + \alpha + c_1)f(x)g(x) + 2\frac{f(x)h(x)}{a + \alpha + c_2} + 2\frac{a + \alpha + c_1}{a + \alpha + c_2}g(x)h(x) \\
& - f'(x) - (a + \alpha + c_1)g'(x) - \frac{h'(x)}{a + \alpha + c_2} + R(a).
\end{aligned}$$

The best way to tackle this equation is by looking at special solutions, i.e. where one or more of the functions  $f(x)$ ,  $g(x)$  and  $h(x)$  are zero. Physically, this would not be a big problem, because it is practically the same as saying that the energy goes like  $E_n \propto n^2$  for example and not  $E_n \propto n^{-2} + n^2$ . For example, take  $g(x) = h(x) = 0$ , the simplest choice, then we get

$$2f'(x) = R(a).$$

This states that both sides have to be constant, say  $\omega > 0$ , so that  $R(a) = \omega$  and  $f(x) = \frac{1}{2}\omega x + c$ . This also means that  $W(x, a) = \frac{1}{2}\omega x + c$ , so the superpotential is independent of  $a$ . The potential corresponding with this solution is

$$V_0(x, a) = \frac{1}{4}\omega^2 x^2 + c\omega x + c^2 - \frac{1}{2}\omega = \frac{1}{4}\omega^2 \left(x + \frac{2c}{\omega}\right)^2 - \frac{1}{2}\omega,$$

which is the harmonic oscillator (with zero energy ground state)! Note that in this case the linking operators do not depend on  $a$ , because the superpotential does not either. This means Theorem 3.1.1 takes a simple form and the linking operators become the very known raising and lowering operators.

In another case, if we take  $g(x) = 1$  and  $c_1 = 0$ , in contrast to  $g(x) = 0$ , we get the equation for the Morse potential:

$$\begin{aligned}
a^2 + 2af(x) + f'(x) &= (a + \alpha)^2 + 2(a + \alpha)f(x) - f'(x) + R(a) \\
\Rightarrow f'(x) &= \alpha f(x) + \frac{R(a) - a^2}{2}.
\end{aligned}$$

If we want that  $f(x)$  does not depend on  $a$ , we have to set<sup>8</sup>  $R(a) = a^2 - 2c\alpha$ , so we get

$$f(x) = c \pm be^{\alpha x}.$$

The superpotential is therefore  $W(x, a) = f(x) + ag(x) = a + c \pm be^{\alpha x}$ . We can drop  $c$ , because it is just a translation of  $a$ . This already looks like the Morse superpotential, as

---

<sup>8</sup>The form of the term  $-2c\alpha$  is for convenience and without loss of generality. We could have taken every other term independent of  $a$ .

it should be To make the identification easier, we mirror the  $x$ -axis and only use  $\alpha > 0$ . This gives us

$$W(x, a) = a \pm be^{-\alpha x}, \alpha > 0.$$

To get the sign right we look at the ground state, given by equation 2.1:

$$\psi_0^{(0)}(x, a) \propto e^{-ax \pm \frac{1}{a}e^{-\alpha x}}.$$

For  $x \rightarrow \infty$  the ground state goes in both cases to zero. For  $x \rightarrow -\infty$ , this does not happen for the plus sign. This causes both terms in the exponent of  $e$  to go to  $\infty$ , making the ground state unnormalisable. For the minus sign we do not have this problem, because the limit of the exponent always goes to  $-\infty$ . This means we need the minus sign, as the Morse potential has already shown<sup>9</sup>.

The method we used is already extensively studied, and probably all possible potentials from this equation are solved by [12, 20].

There is however something interesting about these potentials: the harmonic oscillator, Coulomb potential and the Morse potential can be transformed into each other by a suitable coordinate change. This is also true for the other potentials. However, pairs of potentials between these two classes, i.e. a Morse potential and a Scarf potential, can not be transformed into each other with a simple coordinate change. They can be transformed, but this needs some more generalised transformation [16].

These connections turn out to be caused by the fact that the Schrödinger equation can often be reduced to two general equations, the hypergeometric and the confluent hypergeometric equations [11]. This simply means that, apart from a (exotic) coordinate change, the Schrödinger equations for these potentials are the same. This is not something special of translational SIPs, as some other potentials get their Schrödinger equation reduced to the hypergeometric equation as well. These potentials are called Natanzon potentials [6]. It just happens that the translational SIPs are special cases of the Natanzon potential.

The connections between all the potentials are nicely illustrated in Figure 3.2, which is taken from Gangopadhyaya, Panigrahi and Sukhatme [16].

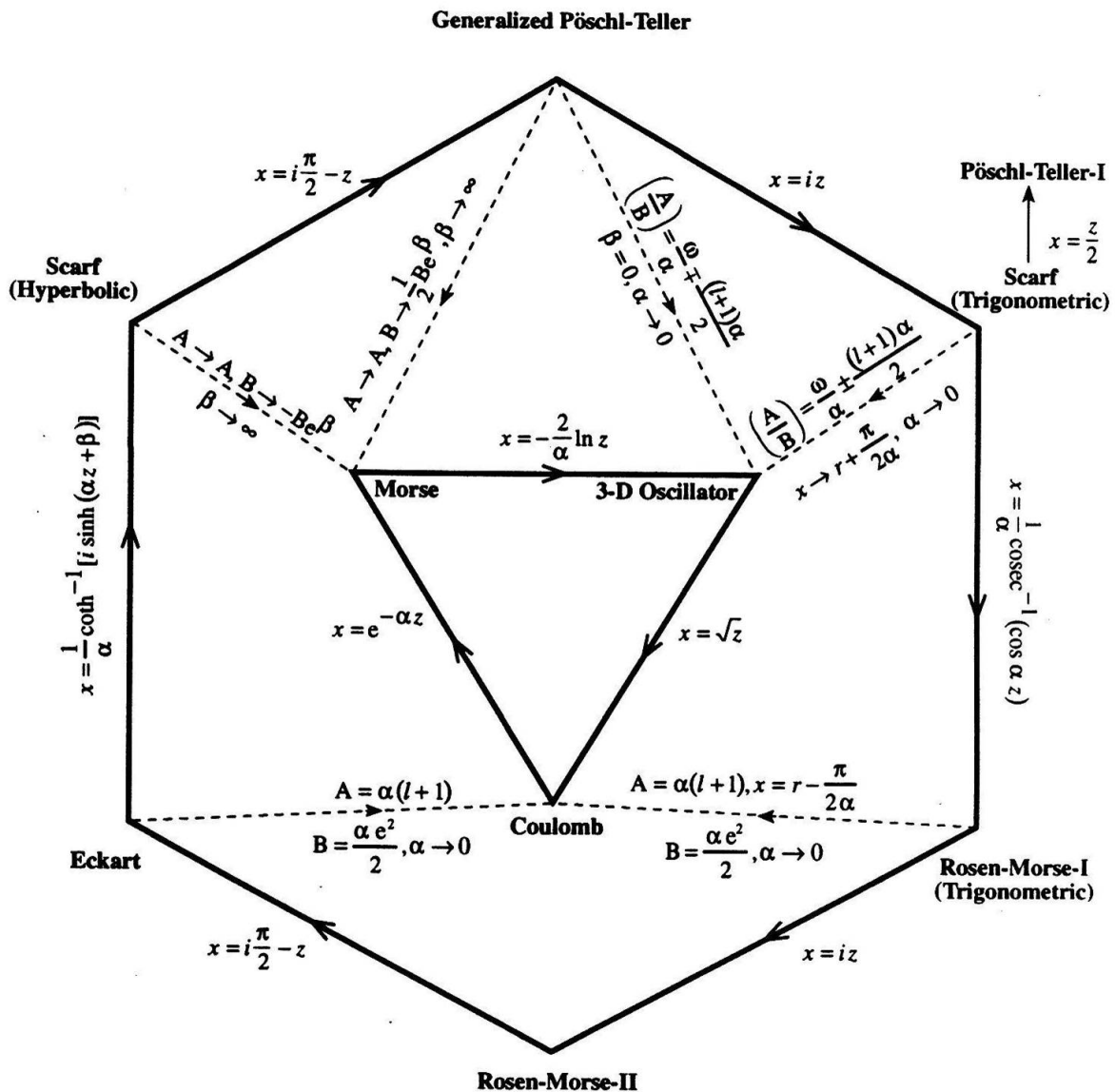
### 3.3 Multiplication SIPs

In Section 3.2 we saw a method to find translational SIPs by guessing the shape of the superpotentials using assumptions on the energies. This method worked fine for the type of SIPs it was looking for: translational SIPs. However, there are more classes of SIPs. Multiplicational SIPs for example, where the parameter change is given by  $f(a) = qa, q \in (0, 1)$ . The superpotentials for these are harder to find with the translational method, therefore we need something else. In this section, we will show another method of finding SIPs, specially made for multiplicational SIPs, but also useful

---

<sup>9</sup>Physically, this choice is obvious, because the plus sign corresponds to a non-increasing potential. This means that it is energetically always better for a particle to go to  $x = \infty$ . This behaviour can not correspond to a bound state, were the particle has a finite, non-zero probability to be in a compact space.

**Figure 3.2:** The different translational SIPs are depicted in this figure, as well as the coordinate transformations between them. For example, if we take the Coulomb potential and substitute  $x = e^{-\alpha z}$ , we get a (mirrored) Morse potential. The bold arrows give the point-canonical transformations, the dashed ones the transformations with limiting procedure. The figure is taken from Gangopadhyaya, Panigrahi and Sukhatme [16].



for other parameter changes like  $f(a) = qa^p$ ,  $q \in (0, 1)$  and  $p \in \mathbb{N}$ . Some interesting properties of these SIPs are shown.

The idea is to use a Taylor expansion of the superpotential  $W(x, a)$  around  $a = 0$ , with the coefficients functions of  $x$ , such that  $W(x, a)$  is given by

$$W(x, a) = \sum_{n=0}^{\infty} W_n(x) a^n.$$

The first question is if this sum converges at all, and if yes, what its radius of convergence is. Furthermore, it is also necessary that the ground state, given by Equation 2.1, is normalisable, if we want bound states to happen. We will not go into detail to solve these problems, the goal of the section is to show how we can find multiplicative SIPs.

Using this equation (and the relation between partner potentials and their superpotential) in the shape invariance condition for superpotentials gives us

$$W^2(x, a) + W'(x, a) = W^2(x, f(a)) - W'(x, f(a)) + R(a),$$

where the differentiation is done with respect to  $x$ . This gives the following equation in terms of the power series:

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{k=0}^n W_k(x) W_{n-k}(x) a^n + \sum_{n=0}^{\infty} W'_n(x) a^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n W_k(x) W_{n-k}(x) (qa)^n - \sum_{n=0}^{\infty} W'_n(x) (qa)^n + \sum_{n=0}^{\infty} R_n a^n. \end{aligned}$$

To solve this equation, we note that the coefficients of equal powers of  $a$  have to be the same, so we can write the following equations

$$\begin{aligned} & \sum_{k=0}^n W_k(x) W_{n-k}(x) + W'_n(x) = \sum_{k=0}^n W_k(x) W_{n-k}(x) q^n - W'_n(x) q^n + R_n \\ \Rightarrow W'_n(x) &= \frac{R_n}{1+q^n} - \frac{1-q^n}{1+q^n} \sum_{k=0}^n W_k(x) W_{n-k}(x). \end{aligned}$$

This equation can be solved recursively, by first solving for  $W_0(x)$ :

$$W'_0(x) = \frac{R_0}{2} \quad \Rightarrow \quad W_0(x) = \frac{R_0}{2} x + c_0.$$

After this, the equations for  $W'_n(x)$  just become first order linear differential equations, although difficult to solve, because of their non-homogeneous part. If we take for example  $W_0(x) = 0$ , we get a direct formula for  $W_n(x)$ ,

$$W_n(x) = \int \frac{R_n}{1+q^n} - \frac{1-q^n}{1+q^n} \sum_{k=1}^{n-1} W_k(x) W_{n-k}(x) dx,$$

because the  $W_n(x)$  terms are multiplied by the zero of  $W_0(x)$ .

Basically, if one defines  $\{R_n\}_{n \in \mathbb{N}_0}$ , we can calculate the complete superpotential, test it

for convergence and normalisability of the ground state. Physically, defining  $\{R_n\}_{n \in \mathbb{N}_0}$  means defining the energy values of the potential, because of Theorem 3.1.1. In this way, one can find shape invariant potentials with normalisable ground states, such as done by Barclay et al. [2]. In this paper, they also used this method to construct multiple-step SIPs, showing the diversity of this method's uses.

Using this method gives SIPs with interesting behaviour, depending on  $W_0(x)$ . Look for example at  $V_0(x, 0)$ , which is given by<sup>10</sup>

$$V_0(x, 0) = \sum_{n=0}^{\infty} \sum_{k=0}^n W_k(x) W_{n-k}(x) 0^n - \sum_{n=0}^{\infty} W'_n(x) 0^n = W_0^2(x) - W'_0(x).$$

We use  $q \in (0, 1)$ , so  $\lim_{n \rightarrow \infty} f^n(a) = 0$ . This means that, if the potentials are continuous, we get the following limit from the shape invariance condition

$$\lim_{n \rightarrow \infty} V_n(x, a) = \lim_{n \rightarrow \infty} V_0(x, f^n(a)) + \sum_{k=0}^{n-1} R(f^k(a)) = V_0(x, 0) + \sum_{k=0}^{\infty} R(f^k(a)),$$

by using the shape invariance condition multiple times. If we substitute  $W_0(x)$  instead of  $V_0(x, 0)$  and use Theorem 3.1.1 for the energies we get

$$\begin{aligned} \lim_{n \rightarrow \infty} V_n(x, a) &= W_0^2(x) - W'_0(x) + \lim_{n \rightarrow \infty} E_n(a) \\ &= \frac{R_0^2}{4} x^2 + R_0 c_0 x + c_0^2 + \lim_{n \rightarrow \infty} E_n(a). \end{aligned}$$

There are now two cases. If  $E_\infty < \infty$ , then  $R_0 = 0$  and the limiting potential is equal to empty space. If  $E_\infty = \infty$ , the limiting potential goes to the harmonic oscillator, but elevated to infinity. This result was to be expected, because taking the limit  $n \rightarrow \infty$  is qualitatively the same as  $q \rightarrow 0$ . If  $q = 0$ , there is no parameter change, and the only solution of no parameter change is, as we saw in Section 3.2, the harmonic oscillator. Free space is just a limiting case of the harmonic oscillator, with  $R_0 = 0$ . In Section 4.2 it is also shown that multiplicative SIPs are reflectionless.

---

<sup>10</sup>We use the convention  $0^0 = 1$ .



## Further Topics

### 4.1 Non-linear oscillators

Until now, we used a classical Hamiltonian, a Hamiltonian with the classical kinetic energy. A question that arises is does this method also work for other Hamiltonians. The technique of using linking operators does not depend on the form of the Hamiltonian, so one would expect that it also works for non-classical examples. This is the case, as the proofs could be easily modified to suit non-classical Hamiltonians, but it becomes quite an effort to use it, and it will not even always work, as the following example will show.

In this example, we use the following Hamiltonian:

$$H_0 = V_2^{(0)} \frac{d}{dx^4} - V_1^{(0)} \frac{d}{dx^2} + V_0^{(0)}(x).$$

We have  $V_1^{(0)}, V_2^{(0)} \in \mathbb{R}_{>0}$ ,  $V_0(x) : \mathbb{R} \rightarrow \mathbb{R}$  a potential and  $\hbar = 2m = 1$ . The linking operators we will use are of the form

$$A = a_2(x) \frac{d}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x), \quad B = b_2(x) \frac{d}{dx^2} + b_1(x) \frac{d}{dx} + b_0(x),$$

with  $a_2(x), a_1(x), a_0(x), b_2(x), b_1(x), b_0(x) : \mathbb{R} \rightarrow \mathbb{R}$  functions to be determined. Note that this is a quite general assumption. We do not assume from the start that  $A = B^*$ , as we will see later that this would limit us dearly. If we want these three operators to be part of the construction of a representation of the super Lie algebra of operators of Section 2.2, we need to have  $H_0 = AB$  and for the partner Hamiltonian  $H_1 = BA$ . This

gives us the following equations:

$$\begin{aligned}
H_0 = AB &= \left( a_2(x) \frac{d}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x) \right) \cdot \left( b_2(x) \frac{d}{dx^2} + b_1(x) \frac{d}{dx} + b_0(x) \right) \\
&= a_2(x) b_2(x) \frac{d}{dx^4} \\
&\quad + (a_2(x) b_1(x) + a_1(x) b_2(x) + 2a_2(x) b_2'(x)) \frac{d}{dx^3} \\
&\quad + (a_0(x) b_2(x) + a_2(x) b_0(x) + a_1(x) b_1(x) + a_1(x) b_2'(x) + 2a_2(x) b_1'(x) + a_2(x) b_2''(x)) \frac{d}{dx^2} \\
&\quad + (a_0(x) b_1(x) + a_1(x) b_0(x) + a_1(x) b_1'(x) + 2a_2(x) b_0'(x) + a_2(x) b_1''(x)) \frac{d}{dx} \\
&\quad + a_0(x) b_0(x) + a_1(x) b_0'(x) + a_2(x) b_0''(x); \\
H_1 = BA &= \left( b_2(x) \frac{d}{dx^2} + b_1(x) \frac{d}{dx} + b_0(x) \right) \cdot \left( a_2(x) \frac{d}{dx^2} + a_1(x) \frac{d}{dx} + a_0(x) \right) \\
&= b_2(x) a_2(x) \frac{d}{dx^4} \\
&\quad + (b_2(x) a_1(x) + b_1(x) a_2(x) + 2b_2(x) a_2'(x)) \frac{d}{dx^3} \\
&\quad + (b_0(x) a_2(x) + b_2(x) a_0(x) + b_1(x) a_1(x) + b_1(x) a_2'(x) + 2b_2(x) a_1'(x) + b_2(x) a_2''(x)) \frac{d}{dx^2} \\
&\quad + (b_0(x) a_1(x) + b_1(x) a_0(x) + b_1(x) a_1'(x) + 2b_2(x) a_0'(x) + b_2(x) a_1''(x)) \frac{d}{dx} \\
&\quad + b_0(x) a_0(x) + b_1(x) a_0'(x) + b_2(x) a_0''(x).
\end{aligned}$$

Note that solving this system of equations for a given Hamiltonian is possible, but not very practical. We will therefore look at some special cases of this system.

Furthermore, note that it is already clear why we did not use just  $a_2(x) = -a_1(x) = b_2(x) = b_1(x) = 1$ ,  $a_0(x) = b_0(x)$ , as we would expect from the linking operators from Definition 2.1.3: the Hamiltonian  $H_0$  would become

$$H_0 = \frac{d}{dx^4} + (2a_0(x) - 1) \frac{d}{dx^2} + (2a_0'(x)) \frac{d}{dx} + a_0^2(x) - a_0'(x) + a_0''(x).$$

However, we do not want odd powers of the differential operator (it would be the momentum operator with a factor  $i$  in front of it), because this would mean the Hamiltonian does not obey time parity. Disobeying time parity makes the energy dependent on the *direction* of the velocity, which we do not want. Therefore we require that the coefficients in front of the odd powers of the differential operators of the Hamiltonian to become zero. In this particular case it would mean the Hamiltonian is the Hamiltonian of empty space. The eigenstates for the eigenvalue  $E$  are

$$\psi_E^{\pm 1, \pm 2}(x) = A e^{\pm 1 i \sqrt{\frac{1}{2} (1 \pm 2 \sqrt{1 - 4(a^2 - E)})} x}, A \in \mathbb{R}.$$

If configuration space is  $\mathbb{R}$ , we do not have bound solutions. This is the reason we have introduced the coefficients in front of the differential operators.

If we would force  $A = B^*$ , we get three equations, one for each coefficient in front of the differential operator (here we use the rule  $(CD)^* = D^*C^*$  on the differential operator and the coefficients):

$$A^* = a_2(x) \frac{d}{dx^2} + (2a_2(x) - a_1(x)) \frac{d}{dx} + (a_2''(x) - a_1'(x) + a_0(x)).$$

It turns out that, if we also want time parity for  $H_0$ ,  $V_2$  and  $V_1$  have to be constant, so our example is the only Hamiltonian, i.e.  $H = V_2(x)p^4 + V_1(x)p^2 + V_0(x)$ , where it is possible to have  $A = B^*$ . The full solution is

$$\begin{aligned} V_2^{(0)}(x) &= V_2, & V_1^{(0)}(x) &= V_1, \\ V_0^{(0)}(x) &= \frac{V_1^2}{4V_2} + \frac{V_1}{\sqrt{V_2}}a_1'(x) + \frac{V_1}{2V_2}a_1^2(x) + \frac{7}{4}(a_1'(x))^2 + \frac{2}{\sqrt{V_2}}a_1^2(x)a_1'(x) + \frac{1}{4V_2}a_1^4(x), \\ &+ \frac{3}{2}a_1(x)a_1''(x) + \frac{\sqrt{V_2}}{2}a_1'''(x), \\ V_2^{(1)}(x) &= V_2, & V_1^{(1)}(x) &= V_1 + 4\sqrt{V_2}a_1'(x), & \tilde{V}(x) &= 2\sqrt{V_2}a_1''(x) \\ V_0^{(1)}(x) &= \frac{V_1^2}{4V_2} + \frac{V_1}{\sqrt{V_2}}a_1'(x) + \frac{V_1}{2V_2}a_1^2(x) + \frac{7}{4}(a_1'(x))^2 + \frac{1}{4V_2}a_1^4(x), \\ &- \frac{1}{2}a_1(x)a_1''(x) + \frac{3\sqrt{V_2}}{2}a_1'''(x), \\ b_2(x) &= a_2(x) = \sqrt{V_2}, & b_1(x) &= -a_1(x), & b_0(x) &= \frac{V_1}{2\sqrt{V_2}} + \frac{1}{2}a_1'(x) + \frac{1}{2\sqrt{V_2}}a_1^2(x), \\ a_0(x) &= \frac{V_1}{2\sqrt{V_2}} + \frac{3}{2}a_1'(x) + \frac{1}{2\sqrt{V_2}}a_1^2(x). \end{aligned}$$

Here we used

$$H_1 = V_2^{(1)}(x) \frac{d}{dx^4} + V_1^{(1)}(x) \frac{d}{dx^2} + \tilde{V}(x) \frac{d}{dx} + V_0^{(1)}(x),$$

with  $\tilde{V}(x)$  the 'potential' for the odd differential term. The cubed differential term always is zero. First note that it is practically impossible to solve for  $a_1(x)$ , so using this equation will mainly involve choosing a function for  $a_1(x)$ . Secondly, if  $a_1(x) = ax + b$ , i.e. if  $a_1(x)$  is linear, the partner Hamiltonian will both have time parity and a constant  $V_1^{(1)}$ , thus enabling us to calculate its partner while retaining the adjoint condition on  $A$ !

Plugging  $a_1(x) = ax + b$  into the equations gives us

$$\begin{aligned}
V_2^{(0)}(x) &= V_2, & V_1^{(0)}(x) &= V_1, \\
V_0^{(0)}(x) &= \frac{V_1}{4\sqrt{V_2}} + \frac{V_1 a}{\sqrt{V_2}} + \frac{7}{4}a^2 + \left(\frac{V_1}{2V_2} + \frac{2a}{\sqrt{V_2}}\right)b^2 + \frac{b^4}{4V_2} \\
&\quad + \left(\left(\frac{V_1}{2V_2} + \frac{2a}{\sqrt{V_2}}\right)2ab + \frac{ab^3}{V_2}\right)x \\
&\quad + \left(\left(\frac{V_1}{2V_2} + \frac{2a}{\sqrt{V_2}}\right)a^2 + \frac{5a^2b^2}{4V_2}\right)x^2 \\
&\quad + \left(\frac{a^3b}{V_2}\right)x^3 + \left(\frac{a^4}{4V_2}\right)x^4, \\
V_2^{(1)}(x) &= V_2, & V_1^{(1)}(x) &= V_1 + 4\sqrt{V_2}a, \\
V_0^{(1)}(x) &= \frac{V_1}{4\sqrt{V_2}} + \frac{V_1 a}{\sqrt{V_2}} + \frac{7}{4}a^2 + \frac{V_1}{2V_2}b^2 + \frac{b^4}{4V_2} \\
&\quad + \left(\frac{V_1}{2V_2}2ab + \frac{ab^3}{V_2}\right)x \\
&\quad + \left(\frac{V_1}{2V_2}a^2 + \frac{5a^2b^2}{4V_2}\right)x^2 \\
&\quad + \left(\frac{a^3b}{V_2}\right)x^3 + \left(\frac{a^4}{4V_2}\right)x^4.
\end{aligned}$$

However, there does not exist a  $(a_1(x), V_1, V_2)$  triple which gives the second Hamiltonian when using the linking operators, because  $a_1(x)$  and  $V_2$  have to be the same in both cases, but  $4\sqrt{V_2}a$  has to be both added and subtracted from  $V_1$  to give the potential. This means that, although we did succeed in defining a supersymmetric system with a non-classical Hamiltonian, we did not succeed in defining a family of partner potentials.

## 4.2 Reflection and transmission

Until now we only looked at bounded states, i.e. solutions that can be normalised to represent real physical solutions. However, scattering solutions are also of interest. These solutions do not represent real physical particles (not in quantum mechanics), but wave solutions. In these solutions, we let a wave of the form  $e^{ikx}$ , with  $k$  the wave number, travel from  $x = -\infty$  to  $x = \infty$ . Due to the presence of a potentials, which can simulate a particle, a electromagnetic field of some kind or something else, a part of this wave reflects back and a part deforms and is transmitted. The main question in this system is which probability the initial wave has to reflect and go back to  $x = -\infty$  and which probability it has to go through to  $x = \infty$ .

The coefficients that answer this question are called the reflection  $R(k)$  and transmission  $T(k)$  coefficients. As with the state functions we require  $|R(k)|^2 + |T(k)|^2 = 1$ ,

because the wave has to go somewhere. To calculate them, we assume that the wave function (a solution of the Schrödinger equation) has the following limits:

$$\begin{aligned}\lim_{x \rightarrow -\infty} \psi_{wave}(x) &\propto e^{ikx} + R(k)e^{-ikx}, \\ \lim_{x \rightarrow \infty} \psi_{wave}(x) &\propto T(k)e^{ik'x},\end{aligned}$$

with  $k'$  the wave number of the transmitted wave. This wave function is a solution of the Schrödinger equation, with a particular energy  $E$ . Therefore,  $k$  can be calculated as a function of  $E$ .

So why are partner potentials and shape invariance interesting with respect to waves? The linking operators are not defined on the space of unbound functions, nor is the Hamiltonian. We still want to say something about these solutions, so we regard them as generalised solutions<sup>1</sup>. In this way, we can use the ideas of Theorem 2.1.1 for these solutions, because if  $\psi_E^{(0)}$  is an eigenfunction with eigenvalue  $E$  for the Hamiltonian  $H_0 = A^*A + E_0$ , then  $A\psi_E^{(0)}$  is an eigenfunction with eigenvalue  $E$  for its partner Hamiltonian  $H_1 = AA^* + E_0$ . The same is true for Lemma 2.1.1, the proof did not use boundedness. Therefore, it is plausible to use these generalised solutions.

Now we go to supersymmetry. If  $\psi_E^{(0)}(x)$  is a solution of energy  $E$  of the Hamiltonian  $H_0 = A^*A + E_0$ , with  $A^*$  and  $A$  the linking operators, then  $A\psi_E^{(0)}(x)$  is a solution of  $H_1 = AA^* + E_0$ . Therefore, we need, using Lemma 2.1.1, that  $\psi_E^{(0)}(x) = NA^*\psi_E^{(1)}(x)$ , with  $N$  a constant. This means, when we take limits to  $x = \pm\infty$ , we get [9]

$$\begin{aligned}e^{ikx} + R_0(k)e^{-ikx} &= N \left( (-ik + W(x))e^{ikx} + (ik + W(x))R_1(k)e^{-ikx} \right), \\ T_0(k)e^{ik'x} &= N \left( (-ik' + W(x))T_1(k)e^{ik'x} \right)\end{aligned}$$

Using<sup>2</sup>  $W_{\pm} := \lim_{x \rightarrow \pm\infty} W(x)$ , we get from the first equation that  $N = \frac{1}{W_- - ik}$ , because the terms with  $e^{ikx}$  have to be equal. From the term  $e^{-ikx}$  we therefore get

$$R_0(k) = \frac{W_- + ik}{W_- - ik} R_1(k). \quad (4.1)$$

Using the second equation we also get

$$T_0(k) = \frac{W_+ - ik'}{W_- - ik} T_1(k). \quad (4.2)$$

This means the amplitudes of the reflection and transmission coefficients are equal for partner potentials! Note that there is a problem: what if the limits of the superpotential are infinite? Notice that we have that  $V_0(x) = W^2(x) - W'(x)$ . If  $W_{\pm} = \pm\infty$ , then also

<sup>1</sup>Another way of saying this is that we use a different space of functions as solution space. We thus have a trade-off where we gain extra solutions at the expense of boundedness.

<sup>2</sup>Requiring that the limits of the superpotential exist means we rule out for example a  $W(x) \propto \sin(x), x \rightarrow \infty$ . This is not a great loss, because you can see those cases as an infinite number reflection and transmission cases. We choose to only solve one of them.

$\lim_{x \rightarrow \pm\infty} V_0(x) = \infty$ . The mathematics is rescued by physics. Scattering is physically nonsensical if the wave comes from a side where  $V_0$  is infinite, because that wave would have infinite energy. Therefore, only superpotentials are chosen where at most one of these limits would be infinite. If one is infinite, we would directly know on which side it is: at the  $x \rightarrow \infty$  side! In that case we need  $T_2(k)$  to be zero, thus  $|R_1(k)| = 1$ . Therefore  $|R_2(k)| = 1$  and thus  $T_1(k) = 0$ . In case of an infinite mountain, we therefore always have full reflection.

Also note that if one potential in a given family is reflectionless, every potential in that family will be reflectionless as well. For example, look at a special case of the Scarf 2 potential [8]:

$$V_0(x, a) = a^2 - a(a - \alpha) \operatorname{sech}^2(\alpha x),$$

with  $f(a) = a - \alpha$ . This is a translational SIP with a finite number of bounded states. If  $A$  is a multiple of  $\alpha$ , at some point a potential in the family of partner potentials will become a shifted empty space. Empty space has a reflection of zero, thus every potential in this family has reflection zero too. This is thus a simple way of defining reflectionless potentials.

We get the wave numbers by noticing that this wave equation is an eigenfunction of the Hamiltonian, giving us the equations

$$\begin{aligned} E \left( e^{ikx} + R_0(k)e^{-ikx} \right) &= k^2 e^{ikx} + k^2 R_0(k)e^{-ikx} + V_0(x) \left( e^{ikx} + R_0(k)e^{-ikx} \right), \\ ET_1(k)e^{ik'x} &= -(k')^2 T_1(k)e^{ik'x} + V_0(x) T_1(k)e^{ik'x} \end{aligned}$$

The first equation only holds if  $E = \lim_{x \rightarrow -\infty} V_0(x) + k^2$ . Noting our remark on the limits of these potentials we can write  $k = \sqrt{E - W_-^2}$  and  $k' = \sqrt{E - W_+^2}$ .

For a shape invariant potential,  $V_0(x, a)$ , we notice that the reflection and transmission coefficients depend on  $a$ , thus we have  $R_0(k, a)$  and  $T_0(k, a)$ . Its partner potential is simply given by  $V_1(x, a) = V_0(x, f(a)) + R(a)$ . Reflections and transmissions do not depend on a possible vertical translation of a potential, they only depend on the shape of a potential. Therefore we have  $R_1(x, a) = R_0(x, f(a))$  and  $T_1(x, a) = T_0(x, f(a))$ . This means we get

$$\begin{aligned} R_0(k, a) &= \frac{W_- + ik}{W_- - ik} R_0(k, f(a)), \\ T_0(k, a) &= \frac{W_+ - ik'}{W_- - ik} T_0(k, f(a)). \end{aligned}$$

This result is interesting when use it with multiplicative SIPs. If we assume  $R_0(k, a)$  is continuous in  $a$ , a fairly general assumption, we have

$$\lim_{a \rightarrow 0} R_0(k, a) = \lim_{n \rightarrow \infty} R_0(k, f^n(a)) = R_0(k, 0),$$

for all  $k \in \mathbb{R}$ . However, for multiplicative SIPs we have that  $V_0(x, 0)$  is free space or the harmonic oscillator. In the first case its reflection coefficient is zero, in the second scattering is impossible, thus the reflection coefficient is definitely zero. From the fact that  $|R_0(k, a)| = |R_0(k, f^n(a))|$  for all  $n \in \mathbb{N}$  and the limit above we thus find  $R_0(k, f^n(a)) = 0$ . Therefore  $R_n(x, a) = 0$  for all  $n \in \mathbb{N}$ . If the reflection coefficient is continuous around  $a = 0$  and  $q \in (0, 1)$ , we thus have that multiplicative SIPs are reflectionless.

### 4.3 Omitted topics

A lot is written about supersymmetric quantum mechanics and its applications, such as the family of partner Hamiltonians. It is therefore impossible to give a thorough review of the whole subject, it is simply too much. We therefore tried to show the basics of supersymmetric quantum mechanics, while sometimes adding an application. In this section we therefore point to some topics that, although omitted from this thesis, are too interesting to not have a mention here.

First, the techniques of partner potentials can be used in the theory of inverse spectral problems, i.e. to find for a given potential a potential with the same eigenvalues. This problem can be tackled in at least two ways with the use of partner potentials. One method of doing this is from [21]. In this method the partner of  $V_0(x)$ , the potential we were given, the superpotential is calculated to get its partner,  $V_1(x)$ . Then, we solve the equation

$$V_1(x) = (W(x) + \phi(x))^2 + W'(x) + \phi'(x),$$

i.e. we search for the most general superpotential which gives  $V_1(x)$ . Using this general superpotential we can then calculate a family of potentials, all a partner of  $V_1(x)$ , such that every potential in this family has the same spectrum. Of course, this method can be generalised by using a potential further in the family of partner potentials to get a larger family of isospectral potentials.

Isospectral also connects to topics in pure mathematics. There are some non-linear partial differential equations, like the KdV-equation, that have as solutions families of isospectral potentials. The variable defining different potentials in the family can then be taken as a function of a variable from the equation, such that the family of iso-spectral potentials can be seen as an evolution of one potential, while retaining its spectrum [9].

Second, the concepts of supersymmetric quantum mechanics can be generalised as well. In our case, we used as symmetries anti-commuting symmetries, such that the operators followed  $Q^2 = (Q^*)^2 = 0$ . In this way we could define a supersymmetric system with a symmetry between bosons and fermions, because if we would try to change two bosons into the same fermionic state, the symmetries would prohibit it. Generalising this concept, we can look at operators that a higher degree, i.e.  $Q^n = (Q^*)^n = 0$ . In this case, we do not talk about fermions anymore, but about parafermions [9]. These parafermions have the property that at most  $p - 1$  of them can be in the same state at once and are therefore studied for their statistical properties, as they are different from the standard bosonic and fermionic statistics.

Lastly, we look at a more mathematical topic of interest, looking more at the super linear algebra. Normally, a smooth differentiable manifold is defined to be locally diffeomorphic to Euclidean space. However, this can be taken further by allowing the manifold to be locally diffeomorphic to a super vector space [31]. The concept of supermanifolds is for example used in some theories for quantum gravity [30].

## Acknowledgements

A thesis is never written alone, there are always other people helping behind the scenes. This thesis is not different. First of all I want to thank my supervisors Koenraad Schalm and Marcel de Jeu for their time and effort in helping writing me this thesis. Secondly, I want to thank Kevin van Helden and Georgy Gomon for their valuable comments on earlier versions of the thesis. Lastly, I want to thank my fellow students for their company and advice.

Michael Wellens, July 13, 2018.



# Mathematics

## A.1 Super linear algebra

This section is based on Chapter 3.1 of V. Varadarajan's book on supersymmetry [31].

Our model of supersymmetry is based on some concepts from super linear algebra. In super linear algebra elements of vector spaces and algebras are divided into even and odd elements. We start with the definition of a super vector space.

**Definition A.1.1** (Super vector space). *Let  $V_0$  en  $V_1$  be (possibly infinite dimensional) vector spaces. Then the space  $V$ , defined by*

$$V := V_0 \oplus V_1,$$

*is called a super vector space. The elements of  $V_0$  are called the even vectors with parity 0 and the elements of  $V_1$  are the odd vectors with parity 1. If  $V$  is finite dimensional, we denote it by instead  $V^{p|q}$ , with  $p$  the dimension of the even subspace and  $q$  of the odd subspace.*

This means a super vector space is just another vector space, only with the notion of even and uneven vectors. The parity of a vector  $v \in V$  will be denoted as  $p(v)$ . Note that the parity of a vector of the form  $v = v_0 + v_1$ ,  $v_0 \in V_0$ ,  $v_1 \in V_1$ , is in general not properly defined. The parity of the zero map can be seen as both even and odd, so we define it as even.

The homomorphisms between the super vector spaces are the linear maps between these two vector spaces. The vector space of linear maps is again a super vector space, because we can divide it in parity preserving homomorphisms, the even ones, and the parity reversing homomorphisms, the odd ones. The next Lemma will prove this statement.

**Lemma A.1.1** (Hom( $V, W$ ) is a super vector space). *Let  $V$  and  $W$  be super vector spaces and Hom( $V, W$ ) be the vector space of all linear maps between  $V$  and  $W$ . Then Hom( $V, W$ ) is a super vector space.*

*Proof.* Let  $V_0$  and  $W_0$  be the even subspaces and  $V_1$  and  $W_1$  be the odd ones of  $V$  and  $W$ . Notice that for a linear map  $f \in \text{Hom}(V, W)$  we can write it as a matrix

$$f = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where  $A : V_0 \rightarrow W_0$ ,  $B : V_1 \rightarrow W_0$ ,  $C : V_0 \rightarrow W_1$  and  $D : V_1 \rightarrow W_1$  are linear maps. This can be done, because we have  $V = V_0 \oplus V_1$  and  $W = W_0 \oplus W_1$ . If  $B = 0$  and  $C = 0$  we thus have an even map and if  $A = 0$  and  $D = 0$  we have an odd map. It is thus clear we can write  $\text{Hom}(V, W)$  as

$$\begin{aligned} \text{Hom}(V, W) &= (\text{Hom}(V_0, W_0) \oplus \text{Hom}(V_1, W_1)) \oplus (\text{Hom}(V_0, W_1) \oplus \text{Hom}(V_1, W_0)) \\ &=: \text{Hom}(V, W)_0 \oplus \text{Hom}(V, W)_1, \end{aligned}$$

where  $\text{Hom}(V, W)_0$  are the even functions and  $\text{Hom}(V, W)_1$  are the odd ones. □

It is clear why we choose the used parities, because the product of two linear endomorphisms on a super vector space is even both when both maps are even and when both maps are odd. Defining a parity preserving multiplication on a super vector space gives a superalgebra.

**Definition A.1.2 (Superalgebra).** Let  $A$  be a super vector space, then we call  $A$  a superalgebra if it has an associative multiplication preserving parity, i.e. for two  $a, b \in A$  we have

$$p(ab) = p(a) + p(b) \pmod{2},$$

if  $p(a)$  and  $p(b)$  are both defined.

As we already expected, the maps between two super vector spaces constitute a superalgebra.

**Corollary A.1.1.**  $\text{Hom}(V)$  is a superalgebra under composition.

*Proof.* From Lemma A.1.1 we already know that  $\text{Hom}(V)$  is a super vector space. From linear algebra we already know the composition of linear maps is associative, so we only have to show parity preservation. Using the matrix from the proof of Lemma A.1.1, we see for example that for  $f, g \in \text{Hom}(V)$ , with  $f$  even and  $g$  odd, that:

$$f \cdot g = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & AB \\ DC & 0 \end{pmatrix},$$

which is odd. The cases even  $\cdot$  even, odd  $\cdot$  even and odd  $\cdot$  odd are similar. From this we conclude that composition is parity preserving, thus  $\text{Hom}(V)$  is a superalgebra. □

The last concept we need is that of a super Lie algebra, because the operators in our supersymmetric model will be a super Lie algebra.

**Definition A.1.3** (Super Lie Algebra). *Let  $\mathfrak{a}$  be a super vector space. It is called a super Lie algebra if it has a bilinear bracket operation  $[\cdot, \cdot] : \mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathfrak{a}$  which satisfies the following properties for  $a, b, c \in \mathfrak{a}$ :*

- $[a, b] = -(-1)^{p(a)p(b)} [b, a];$
- $[a, [b, c]] + (-1)^{p(a)p(b)+p(a)p(c)} [b, [c, a]] + (-1)^{p(a)p(c)+p(b)p(c)} [c, [a, b]] = 0.$

With this last definition, we have enough super linear algebra for our purposes.



# Bibliography

- [1] Shmuel Agmon. *Bounds on exponential decay of eigenfunctions of Schrödinger operators*. Springer, Berlin, Heidelberg, 1984.
- [2] D. T. Barclay, R. Dutt, A. Gangopadhyaya, Avinash Khare, A. Pagnamenta, and U. Sukhatme. New exactly solvable Hamiltonians: Shape invariance and self-similarity. *Physical Review A*, 48(4):2786–2797, oct 1993.
- [3] Martin Braun. *Differential Equations and Their Applications*. Springer-Verlag New York, fourth edition, 1993.
- [4] Michael Cohen. *Excitations of Liquid Helium*. Thesis, California Institute of Technology, 1956.
- [5] Sidney Coleman and Jeffrey Mandula. All possible symmetries of the S matrix. *Physical Review*, 159(5):1251–1256, jul 1967.
- [6] Fred Cooper, Joseph N Ginocchio, and Avinash Khare. Relationship between supersymmetry and solvable potentials. *Physical Review D*, 36(8):2458–2473, 1987.
- [7] Fred Cooper, Avinash Khare, and Uday Sukhatme. Supersymmetry and quantum mechanics, jan 1995.
- [8] Fred Cooper, Avinash Khare, and Uday Sukhatme. *Supersymmetry in Quantum Mechanics*. WORLD SCIENTIFIC, jun 2001.
- [9] Fred Cooper, Avinash Khare, and Uday Sukhatme. *Supersymmetry in Quantum Mechanics*, volume 28. IOP Publishing, aug 2001.
- [10] Jens Peder Dahl and Michael Springborg. The Morse oscillator in position space, momentum space, and phase space. *The Journal of Chemical Physics*, 88(7):4535–4547, 1988.
- [11] Jan Dereziński and Michał Wrochna. Exactly Solvable Schrödinger Operators. *Annales Henri Poincaré*, 12(2):397–418, 2011.
- [12] Ranabir Dutt, Avinash Khare, and Uday Sukhatme. Supersymmetry, shape invariance, and exactly solvable potentials. *American Journal of Physics*, 56(2):163, feb 1988.

- [13] Stephan Fackler. *Mathematical Foundations of Quantum Mechanics*, 2015.
- [14] William M. Frank, David J. Land, and Richard M. Spector. Singular potentials. *Reviews of Modern Physics*, 43(1):36–98, oct 1971.
- [15] William M. Frank, David J. Land, and Richard M. Spector. Singular potentials. *Reviews of Modern Physics*, 43(1):36–98, jan 1971.
- [16] Asim Gangopadhyaya, Prasanta K. Panigrahi, and Uday P. Sukhatme. Inter-Relations of Solvable Potentials. *Helvetica Physica Acta*, 67(4):363—368, 1993.
- [17] David J. Griffiths. *Introduction to Quantum Mechanics*. Pearson Education Limited, Essex, second edition, 2014.
- [18] Rudolf Haag, Jan T. Łopuszański, and Martin Sohnius. All possible generators of supersymmetries of the S-matrix. *Nuclear Physics, Section B*, 88(2):257–274, jul 1975.
- [19] Allan M. Krall. The Singular Sturm-Liouville Problem. In *Applied Analysis*, pages 308–363. Springer Netherlands, Dordrecht, 1986.
- [20] G Levai. A search for shape-invariant solvable potentials. *Journal of Physics A: Mathematical and General*, 22(6):689–702, mar 1989.
- [21] Michael Martin Nieto. Relationship between supersymmetry and the inverse method in quantum mechanics. *Physics Letters B*, 145(3-4):208–210, sep 1984.
- [22] Michael Martin Nieto and L. M. Simmons. Limiting spectra from confining potentials. *American Journal of Physics*, 47(7):634–635, jul 1979.
- [23] Walter Rudin. *Functional Analysis*. McGraw-Hill, Inc, second edition, 1991.
- [24] S. Sakiroglu, F Urgan, U Yesilgul, M E Mora-Ramos, C A Duque, E Kasapoglu, H Sari, and I. I. S??kmen. Nonlinear optical rectification and the second and third harmonic generation in Pöschl-Teller quantum well under the intense laser field. *Physics Letters, Section A: General, Atomic and Solid State Physics*, 376(23):1875–1880, 2012.
- [25] A. Salam and J. Strathdee. On goldstone fermions. *Physics Letters B*, 49(5):465–467, may 1974.
- [26] Neeraj Sarna. Laguerre polynomials. *Encyclopedia of mathematics*, 10(2):283–284, 2016.
- [27] S. Sree Ranjani, A K Kapoor, and P K Panigrahi. Bound states and band structure - A unified treatment through the quantum Hamilton-Jacobi approach. *Annals of Physics*, 320(1):164–174, 2005.
- [28] Ticciati. *Quantum Field Theory for Mathematicians*. pages 1–48, 1999.
- [29] David Tong. *Quantum Field Theory*. Cambridge, 2006.

- [30] P. van Nieuwenhuizen. *Supergravity*, 1981.
- [31] V S Varadarajan. *Supersymmetry for mathematicians: an introduction*. By the author, first edition, 2004.
- [32] Edward Witten. Supersymmetry and Morse theory. *Journal of Differential Geometry*, 17(4):661–692, 1982.