Fermion-Boson Dualities in 2+1 Dimensions and Higher

THESIS

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Abstract

An overview of Quantum Field Theory dualities is given, highlighting the tools physicists have been using to derive them and the importance of symmetries in searching for such dualities. Most duality derivations take place in 2+1d where one may use flux attachment to realise dualities between fermionic and bosonic theories. The phase transition method for finding dualities is then discussed in 2+1d and 3+1d, and a novel derivation of the Montonen-Olive duality is given using this method.

To my parents

"*Que seria de nós se não existisse o deleatur, suspirou o revisor.*"
José Saramago - História do Cerco de Lisboa

Contents

Notation and Conventions

In this thesis, I will be closely following the usual notation adopted in Quantum Field Theory texts, unless otherwise stated. Furthermore, we will only be dealing with Minkowski and Euclidean metrics, and for the former we adopt the "mostly minus" convention. For instance, in 3-dimensional Minkowski spacetime, we have $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}[+1, -1, -1]$. Greek indices ($\mu, \nu, \sigma$, etc) imply summation over all components (space and time) while Latin characters (i, j, k, etc) imply summation over only the spatial components. Hence,

$$a_\mu b^\mu = g_{\mu\nu} a^\mu b^\nu = g_{00} a^0 b^0 + g_{10} a^1 b^0 + g_{01} a^0 b^1 + g_{11} a^1 b^1 + ... \tag{1}$$

where $g_{\mu\nu}$ is the metric tensor of the d-dimensional manifold in question.

Furthermore, we shall be working in natural units, which has $\hbar = c = 1$ and thus units of time and length are the same and are the inverse of units of energy and mass. This has as consequence, for instance, that the square of the 4-momentum of a particle with rest mass m and energy E reads

$$p^2 = p_\mu p^\mu = E^2 - |\mathbf{p}|^2 = m^2 \tag{2}$$

where bold letters denote 3-vectors.

Furthermore, in this thesis, an important role is played by the totally antisymmetric Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$, and it is such that even permutations of its d indices yield +1 and odd permutations yield -1. Hence we have for instance, in 4 spacetime dimensions, $\epsilon^{0123} = -\epsilon_{0123}$ and also swapping any two indices inverts a sign, e.g. $\epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\rho\nu\mu\sigma}$. In this notation, Maxwell's equations of relativistic electrodynamics read

$$\begin{cases} \epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0 \\ \partial_\mu F^{\mu\nu} = e j^\nu \end{cases} \tag{3}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor, A_μ the vector potential and j^μ the current density.

Finally, at some points in this discussion it will prove useful to switch into differential form notation. As a quick reminder of the properties of such objects, a p-form f living in a d-dimensional manifold is defined as

$$f = \frac{1}{p!} f_{\mu_1 \mu_2 ... \mu_p} dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_p} \tag{4}$$

where $\wedge$ represents the antisymmetric wedge product. Furthermore, we define the exterior derivative d such that

$$df = \frac{1}{p!}\partial_\nu f_{\mu_1\mu_2...\mu_p} dx^\nu \wedge dx^{\mu_1} \wedge dx^{\mu_2} ... \wedge dx^{\mu_p} \tag{5}$$

The last operation we will need is the *Hodge Dual*, denoted by the $*$ operator which acts on a p-form and returns a $(d-p)$-form. In Minkowski space,

$$*\, f = \frac{1}{p!(d-p)!} f^{\mu_1\mu_2...\mu_p} \epsilon_{\mu_1\mu_2...\mu_p...\mu_d} dx^{\mu_{p+1}} \wedge dx^{\mu_{p+2}} ... \wedge dx^{\mu_d} \tag{6}$$

In this language, Maxwell's field equations read:

$$\begin{cases} dF = 0 \\ d * F = *j \end{cases} \tag{7}$$

With these conventions in place, we are now ready to dive right into it.

Chapter 1

Introduction

In Quantum Field Theory (QFT), we often regard quantum fields as being the most basic ontological entities in existence [13]. From the Lagrangian (or equivalently the Hamiltonian) of a theory, one may *in principle* derive all of its possible interactions, cross-sections, some physical constants and anomalies [60]. This does not imply that those are simple tasks, however. In fact, a large portion of objects and systems one deals with when doing QFT remains mathematically obscure or debatable, for instance the mathematical adequateness of the path integral [14, 43]. Celebrated examples of systems which are difficult to deal with in QFT are non-renormalisable theories like that of gravity as well as theories with strong coupling, like the strong force acting in the confinement of quarks or superconductivity in materials with high T_c [4, 28]. Because of their strong coupling, the usual perturbative approach in QFT does not capture the entire behaviour of the theory. In this thesis, we shall not be dealing with renormalisation, but the main topic at hand (dualities) may indeed have some bearing on theories which present strong coupling, for reasons which will be discussed in more detail later on.

This thesis has two objectives. Firstly, it will shed light on some of the developments, both old and new, in the area of QFT dualities, and attempt to explain duality derivations using several different methodologies in a way that is comprehensible to someone who has a background in QFT but may never have seen dualities before. The mathematical steps are stated as clearly as possible and often accompanied by images for aid in visualising the relevant processes. A second objective of this thesis is to use one of such methods for deriving dualities, namely the phase transition method, to discuss dualities in 3d and 4d, with an emphasis on opening potential doors for bosonization in 4d, which has been so far rather difficult to achieve. What exactly 4d bosonization is and why it is so elusive will be explained in the following sections. Then, the structure of this thesis is organised as follows: firstly, an introduction to what dualities are is given, as well as a discussion on bosonization. Before delving into the dualities themselves, we devote a chapter to laying out all the tools that we will need in this thesis to understand the derivations which come next. Specifically, we will give a brief introduction to the topics of Chern-Simons terms, Flux Attachment, Anyons, Vortices and Monopoles and the Parity Anomaly. All of these are aimed at the duality derivations that we will carry out in 3d but they are not all necessarily confined to 3d manifolds. Once we have all the tools in place, we can then move onto the dualities themselves. The following chapters will see dualities in 3d being derived and discussed by using distinct arguments, and the latter chapters contain derivations using the phase transition method, which is the one we will use to achieve duality in 4 dimensions. Then, we introduce the Montonen-Olive duality in 4d by the phase transition method, which is a well-known duality but to the best of our knowledge has never been derived using the phase transition method. Finally, we summarise and discuss the results and lay out possibilities for the future, like how this method might help us achieve 4d bosonization.

Thus, let us at last step into the world of dualities.

1.1 Dualities and their World of Symmetry

First, it is important to lay out what we mean by *duality*. Generally, saying two descriptions are dual to each other means that they exhibit some kind of complementarity or equivalence. This is a very general philosophy that does not only apply to Physics but also to Mathematics[1] and many other areas of Science [7]. In Physics, this often involves changing what one regards as the fundamental entities in the theory and the interactions that arise as a result, for instance in the historically celebrated *wave-particle duality*, in which the ontological nature of light (and more generally of any quantum object) might be that of a particle or a wave, depending on the situation at hand [56]. In principle, two theories that are dual might be seemingly very different to each other, but they can be shown to describe the same physical system and structure, as long as one changes the framework. Let us now make these statements more specific.

Here, whenever we talk about *theories*, it should be clear that in essence what is meant is *Lagrangians*, or equivalently *Hamiltonians* (since these are simply related by a Legendre transformation), and their consequent partition

[1]Notably, this is the guiding principle behind the area in Mathematics known as *Category Theory* [35].

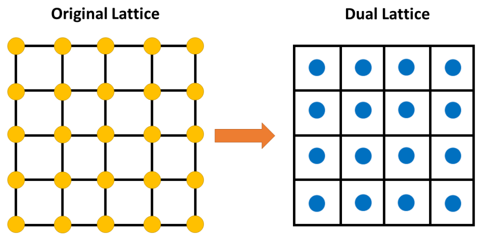

Figure 1.1: *The original and dual lattices considered when finding T_c in the Ising model.*

functions [29, 40]. Thus, whenever it is claimed that two theories are dual, what is meant by this is that we have initially two Lagrangians giving rise to partition functions which appear different by having distinct elementary fields, couplings and potentials, but may be manipulated in some way to show that these two functions are in fact equivalent to each other.

Historically, many of the earliest dualities derived in this sense were shown throughout the 20th century, some of which we will discuss in this thesis, for instance the Electric-Magnetic duality, in section 5.1. A classical and visual example of one of older-known dualities is present in the Ising model [33, 34], which has spins on a square lattice interacting each with their nearest neighbours. This model's partition function reads

$$\mathcal{Z}_{\text{Ising}} = \sum_{\langle S \rangle} e^{K \sum_{(ij)} s_i s_j} \tag{1.1}$$

where $K \equiv \frac{J}{k_B T}$, with J being a constant defining the coupling strength, T the system's temperature, $\langle S \rangle$ denotes all possible configurations of spins in the lattice and (ij) are nearest-neighbour pairs. Even before this system was solved exactly [58], is was possible to find its critical temperature T_c by means of a simple duality transformation. As shown on Figure 1.1, instead of dealing with this in the original lattice, we translate this problem into the dual lattice, which has lattice points in the gaps of where the original lattice was. Now, it is possible to show that the new theory has a coupling constant K' which relates to the original K as follows:

$$(\sinh 2K')(\sinh 2K) = 1 \tag{1.2}$$

From this inverse relationship, we note that high temperatures in the original lattice are mapped to low temperatures in the dual lattice, and vice-versa. Then, if the Ising model has a unique phase transition, it must happen when these two descriptions cross, which is known as the *self-dual point*, meaning it happens when $K' = K$ and briefly the original description and its dual are the same. From (1.2) we easily compute the condition to find the critical temperature:

$$\sinh \frac{2J}{k_B T_c} = 1 \tag{1.3}$$

A helpful analogy to bear in mind when dealing with dualities is that of *transforms*, the most common example of which is the Fourier Transform, which allows us to translate a physical problem between coordinate space and reciprocal space [6]:

$$f(x) \leftrightarrow \hat{f}(k) \tag{1.4}$$

Often, a system that is rather untreatable in coordinate space is rendered considerably simpler in reciprocal space, and the results can then be translated back into coordinate space by the inverse transform. Dualities work similarly. Since we can recast a partition function into another one, it might be that the new partition function exhibits an action functional that has known solutions or is much more treatable than the original. Then, one can solve the system in this new language and then, if necessary, translate back into the original system. An important example of this process taking place is research in Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence, which aims to map the complex QFT of many bodies that are strongly correlated into the dynamics of a theory of gravity in one extra dimension [37], as part of a wider area of research known as the *holographic principle* [59].

A point to be emphasised as to why dualities may make a problem simpler is, as already noted, in our Ising model example the dual coupling constant in (1.2) is connected to the original one by an inverse relationship, meaning that a strong coupling regime in the original lattice would map to a weak coupling regime in the dual lattice and vice-versa. There are many systems in which strongly coupled regimes make the problem impervious to perturbative techniques, notably as explained before, in some superconducting systems and when dealing with the strong force in high-energy Physics. Such problems can benefit from having a dual theory whose richness is captured by standard perturbative techniques.

A point which will emerge in this thesis is that the duality transformations involve changing the fields that describe a theory, but the elementary *structure* of that theory remains intact. This means that the symmetries of the problem are left untouched. For now let it be clear that if a theory exhibits for example a global U(1) symmetry, then its dual theory must also have a global U(1) symmetry. In our Ising model example, all the symmetries are matched from the original to the dual problem because both take place on a square lattice.

As a final thought for introducing one to the idea of QFT dualities, one of the most interesting aspects of it is that it does not only map bosonic theories to other bosonic theories and fermionic theories to their fermionic duals. It is also possible to find fermion theories which have boson theories as their dual counterparts. To this is given the name *bosonization*, and it will be crucial in this thesis, since bosonization involves the statistical transmutation of particles from one kind into another, and this can be much simpler to do in certain dimensionalities than others.

1.2 The Elusive Hunt for 4d Bosonization

Historically, it was through the work of Tomonaga [50] that the scientific community started taking steps towards a rigorous proof of bosonization in 1d. Previously, it had been only implied by Bloch that it might be possible to describe a fermionic system using quantized sound waves obeying bosonic statistics. This prediction turned out to be right, and with subsequent improvements on the method [30, 31] it is well-known now that in 1d one may write $\psi_\eta(x) \sim F_\eta e^{-i\phi_\eta(x)}$, where $\psi_\eta(x)$ represents the wavefunctions of η fermions, $\phi_\eta(x)$ are the corresponding boson wavefunctions and F_η is the so-called *Klein factor*, which has the effect of lowering the number of the η fermions by one[2][55].

The reasons why bosonization has proven to be a helpful tool are usually related to simplicity: it is considerably easier to deal with bosonic than fermionic fields in QFT, and thus finding a relationship between the two can prove rather useful. This allows one to go back and forth between the boson and fermion frameworks and thus potentially turn extremely difficult problems into much simpler ones. Furthermore, in Monte-Carlo (MC) simulations, the computation time for fermionic theories scales exponentially with $\beta = 1/T$, as opposed to polynomially as is the case with usual MC methods. Thus, simulating low-temperature fermions quickly becomes too computationally expensive. This is related to an as-of-yet mostly unresolved issue in MC for fermions known as the *fermionic sign problem*[3] [36, 54].

[2]Some authors even treat the Klein factor as being Majorana fermions, which is not exactly correct. The reason for this is simply that F_η^2 does not reduce to the identity precisely because it removes two fermions.

[3]For the computationally inclined, the fermionic sign problem has been shown to be

The dualities present in this thesis are mostly going to be in 2+1d and 3+1d, but for completeness we note that it is possible to reach bosonization as described here by manipulating the partition function of a system in 1+1d [12]. Here, we go through a simple example beginning with a fermionic theory as follows:

$$\mathcal{Z}_F = \int D\psi D\bar{\psi} \exp\left\{ i \int d^2x \, \mathcal{L}_F \right\} \tag{1.5}$$

where

$$\mathcal{L}_F \equiv -\bar{\psi}\partial\!\!\!/\psi + ia_\mu\bar{\psi}\gamma^\mu\psi + ib_\mu\bar{\psi}\gamma^\mu\gamma^3\psi \tag{1.6}$$

with a_μ and b_μ representing external interactions (which need not be of this exact form, this is rather an illustrative example). Then, one may gauge the fermions by changing the partition function to

$$\mathcal{Z}_F = \int D\psi D\bar{\psi} DA_\mu D\Lambda \exp\left\{ i \int d^2x \left(\mathcal{L}_F + iA_\mu\bar{\psi}\gamma^\mu\psi + \frac{1}{2}\Lambda\epsilon^{\mu\nu}F_{\mu\nu} \right) \right\} \tag{1.7}$$

up to a gauge-fixing factor. Now, the duality is achieved by noting that one arrives at seemingly different points by integrating out the fields in different orders. If we start by integrating out the auxiliary field Λ (which is warranted since it is linear in the action), it serves as a Lagrange multiplier for the constraint $F_{\mu\nu} = 0$ (which has $A_\mu = 0$ as a possible solution), and then we return to the original fermionic Lagrangian. If, however, we choose to first integrate out the fermion field and then the gauge field, we will end up with a bosonic Lagrangian, up to an arbitrary constant:

$$\mathcal{L}_B = -\frac{\pi}{2}\partial_\mu\Lambda\,\partial^\mu\Lambda + b^\mu\partial_\mu\Lambda + \epsilon^{\mu\nu}\partial_\mu\Lambda\,a_\nu \tag{1.8}$$

Since these two theories have been reached simply by integrating out the fields in different orders, they must be in fact equivalent, and thus we claim a bosonization duality between (1.5) and (1.8). Hence it is possible to achieve bosonization through dualization of a QFT, at least for 2 spacetime dimensions.

Once bosonization had been established for 2 spacetime dimensions, it was natural to ask whether it would be possible to take this into higher dimensions, with an eye especially for 4d, for it is the dimensionality wherein most of our QFTs reside. The 3d case will be treated in more depth in the next chapters so we shall not go into it right now, so suffice it to say that it is possible to achieve 3d bosonization and it may be done by employing the

NP-hard [53].

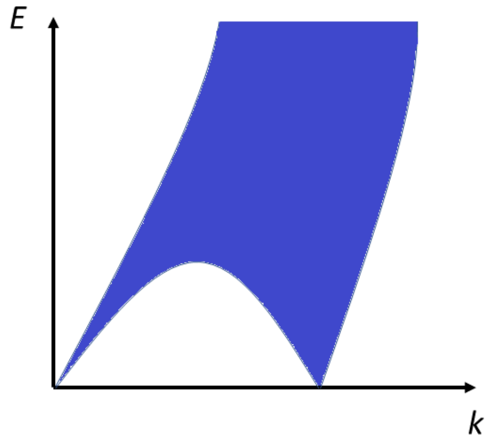

Figure 1.2: *Spectrum of a particle-hole pair in one spatial dimension. The shaded region is the allowed region for the dispersion relation, and at low energies the pair propagates coherently as a result of the narrowing.*

help of the statistical transmutation properties of particles which behave as neither bosons nor fermions. However, back to the idea of 4d bosonization, this has turned out to be a rather herculean task, and there have been many attempts at obtaining a complete and rigorous derivation of it, with varying degrees of success [8, 11, 20, 49]. The explanations for why it is easier to perform bosonization in lower dimensions are varied, and here we will go over two of them, in a heuristic way. The first arises in the context of Condensed Matter Theory and the second is more general, related to rotations in space.

If we picture a one-dimensional electron gas spectrum, we may ask ourselves whether electrons being pulled out of the Fermi sea and the consequent holes might form a pair that behaves just like a boson, much like Bloch asked himself [46]. In fact, this process is presented in Figure 1.2. In it, we see the allowed region for the dispersion relation of a particle-hole pair creation in our system. As one may note, at low energies, this region becomes narrower and narrower, tending to a linear one-particle dispersion relation. As a result, the particle and hole created by this excitation will propagate within the medium coherently, which here is to say they will have the same group velocity. Since they are moving through the system together, in the presence of any kind of attractive interaction between them they will act as one particle which can then be shown to obey Bose statistics. From this conceptual point of view, in one dimension it is very natural that one might be able to find a description in which translating between fermions and bosons is possible. This phenomenon of a narrowing dispersion relation region does not take place in higher dimensions, where the Fermi surface may take on

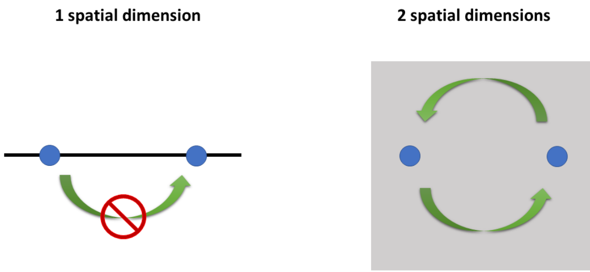

Figure 1.3: *In one spatial dimension it is not possible to rotate space, whereas on a 2D plane it is.*

more complex shapes, and thus these do not lend themselves so easily to bosonization.

The second argument has to do with the possibility of rotating space and evaluating how a particle's wavefunction changes under this process, since this is one of the defining features of bosons and fermions. When we have only one spatial dimension, i.e. our world is but a line with particles as excitations in different parts of it, there is no way to revolve this world and swap two particles without entering a higher dimension, and in this sense their statistics under such a rotation do not matter, as shown on Figure 1.3. Further, if we were to try to swap them by moving them continuously, they would have to cross and overlap at some point, which could become a problem in the presence of some kind of interaction between them. As a result, at least in this case we constrain some liberties we would usually have when trying to distinguish fermions from bosons, and as result finding equivalent descriptions between them is a simpler task. Note that these arguments do not apply to 2+1d and above, where you can rotate the plane without barging into higher dimensions and also swap the particles without overlapping them. The 3d bosonization case relies on certain special terms which shall be discussed in the next chapter.

With this, we have seen that there are conceptual arguments as to why giving descriptions of fermions in terms of bosons is possible in lower dimensions, but it is more challenging in 3d and above. In the next chapter we are going to lay out the tools we will need to derive dualities in 3d and 4d.

Chapter 2

Life in Flatland: Planar Physics and Chern-Simons Theories in 2+1d

Studying lower dimensions can be helpful because the phenomena taking place in them are often simpler to calculate, predict and simulate, and in many cases the concepts can be extrapolated to our usual 4 dimensions, even if solving equations of motion or computing Feynman Diagrams does not extrapolate as straightforwardly. Furthemore, it is not uncommon in Condensed Matter Physics, as we will see in this chapter, for some physical systems embedded in 4d spacetime to behave like 3d systems, usually because of some degree of freedom that has been constrained by a symmetry of the system and as a result reduces the effective number of dimensions of the theory.

In this chapter, we will give a brief overview of some features of QFT in 2+1d and focus on a specific kind of term which may show up in odd-dimensional Lagrangians: the *Chern-Simons* and its variants. This will prove a very useful term when searching for dualities in 3d and 4d and talking about the phenomenon of *flux attachment*. However, lest we spoil the interesting discussions that are yet to come, let us begin with the life of bosons, fermions and in-betweens in Flatland [4].

2.1 Vortices and Monopoles

The first features of space we shall explore is that of vortices and monopoles, which are part of a larger family of objects in certain QFTs which includes

[4]Rather than the geometrical shapes which roamed such a world in Edwin Abbott's 1884 novel.

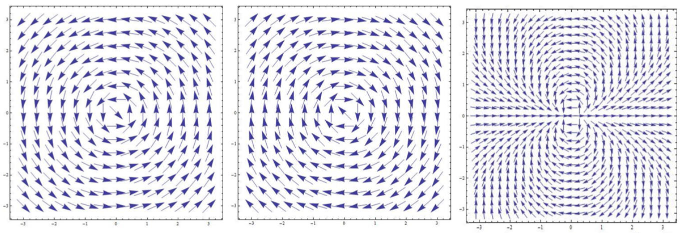

Figure 2.1: *Examples of vortices with different winding numbers k.*

members like instantons and domain walls. These are known as *topological defects*. They are not necessarily limited to three dimensions, and in fact we shall use them to derive dualities using the phase transition method in 3d and 4d, and hence a brief introduction to them is in order. Both vortices and monopoles are present in several different areas of physics. For instance, vortices will appear in certain phases of Condensed Matter systems. They can be carriers of magnetic flux in Quantum Hall fluids as well as arise as cosmic strings in cosmology [51].

We begin by talking about vortices. These are an order parameter of the system that winds around a specific point. In mathematical terms this could be, for instance, the phase of a complex quantum field [10]. Let $\phi = |\phi| e^{i\theta}$. Then, if we allow this complex phase θ to wind around a singular point while being otherwise smooth, we will encounter a vortex, as shown on Figure 2.1. One important characteristic of such a solution to the desired QFT is that these vortices are *topological*, meaning that their effects will be present beyond the bulk of the system and spread towards infinity or the boundaries of the system, if they exist. This happens because the smoothness condition for θ (except for the singular point) ensures that the phase winds spatially around the singularity an integer number of times, i.e.

$$\frac{1}{2\pi} \oint_\gamma dx^i \, \partial_i \theta = k \in \mathbb{Z} \tag{2.1}$$

for a loop γ enclosing the singularity. Then, even if one is probing at the boundary of the system, without having access to its bulk, in principle it is possible to detect the existence of such a vortex inside. Usually, when the *vorticity* k is a positive integer, the solution is called a *vortex* and when the vorticity is negative, it is called an *anti-vortex*. A path γ that encloses more than one vortex or anti-vortex will compute a vorticity that is the sum of the vorticities of the individual vortices and anti-vortices enclosed by the path. Thus, if the bulk of a system contains N (anti-)vortices, the total vorticity measured at the boundary will be the sum of the vorticities of all (anti-)vortices in the system:

$$k_{\text{Total}} = \sum_{n=1}^{N} k_n \tag{2.2}$$

In a slightly more rigorous way, we may say that the vorticity of a system is a topological invariant, because systems with different vorticities do not belong to the same homotopy group, meaning they cannot be continuously deformed into one another. Notationally, we say that in a manifold $\mathcal{M}$, which in our example is 3d, we have[5] $\pi_1(\mathcal{M}) \cong \mathbb{Z}$ [39].

Back to equation (2.1), we may go to 3 spatial dimensions so that we have a vector normal to the plane of the vortex. Then, we apply Stoke's Theorem and find that

$$\oint_\gamma dx^j\, \partial_j \theta = \int_\Gamma dS^k\, \epsilon_{ijk} \partial^i \partial^j \theta = 2\pi k \tag{2.3}$$

where dS represents an oriented area element of the region Γ enclosed by the path γ. Because of the singularity at the centre of the vortex, our field is no longer single-valued everywhere, and thus our partial derivatives are no longer commuting, so we cannot take this to be zero identically. In the next chapter, our first duality will involve separating the smooth part of the vortex field from its multi-valued part so they may be dealt with in their own terms. For now, it is important to highlight as far as vortices are concerned that the integrand in (2.3) may be interpreted as a vortex current density flowing through the infinitesimal oriented area dS. Thus, we may write

$$J^k_{\text{Vortex}} = \epsilon^{ijk} \partial_i \partial_j \theta \tag{2.4}$$

Now, we will do a short incursion in the topic of monopoles. These will be a central piece of the puzzle when deriving dualities by the phase transition method in chapters 4 and 5. Magnetic monopoles have never been observed experimentally and Maxwell's equations continue to insist that $\nabla \cdot B = 0$. However, 't Hooft and Polyakov [41, 48] have shown that magnetic monopoles

[5] Here, π_1 represents the fundamental group, or all the ways in which a circle S^1 may be mapped onto $\mathcal{M}$. More generally, $\pi_n(\mathcal{M})$, or the nth homotopy group has as its elements all the n-spheres S^n which may be continuously mapped onto one another in the space $\mathcal{M}$.

arise naturally in non-abelian gauge theories. In fact, in any gauge theory which possesses a compact, unbroken U(1) group they may exist. Thus, the interest on these objects was revived as the search for a Grand Unified Theory gained momentum. In our discussion of dualities, their main purpose will be to break a symmetry of a theory and thus remove the lines between formerly different phases. For now, it is important mentioning that it is possible to have magnetic monopoles which are consistent with a gauge field A_μ, as long as we define two of such fields.

It is not difficult to show that having a single magnetic field $\mathbf{B}$ which satisfies the condition $\nabla \cdot \mathbf{B} = 4\pi g \delta^3(\mathbf{r})$ (where g stands for the magnetic charge of the monopole) incurs in a line singularity known as the *Dirac String* [1]. However, we may avoid this issue, as was noted by Wu and Yang [57], by setting two different vector potentials $\mathbf{A^N}$ and $\mathbf{A^S}$ in the North and South hemispheres of a sphere surrounding the monopole, respectively. They may be expressed, in spherical coordinates, as

$$\begin{cases} \boldsymbol{A^N}(\mathbf{r}) = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \\ \boldsymbol{A^S}(\mathbf{r}) = -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{\boldsymbol{e}}_{\boldsymbol{\phi}} \end{cases} \tag{2.5}$$

where $\hat{\mathbf{e}}_\phi$ stands for the unit vector in the ϕ direction. This definition has the effect that the difference between the gauge fields in the North and South hemispheres is $\boldsymbol{A^N} - \boldsymbol{A^S} = 2g\nabla\phi$, and since this is nothing but a gauge transformation, it should not affect physical observables, thus being compatible with a gauge field. However, there is one important consequence if we are to accept this. A charged quantum particle whose electromagnetic field undergoes a gauge transformation has the phase of its wavefunction changed accordingly. Thus, a particle with wavefunctions ψ^N and ψ^S in the North and South hemispheres will have to obey the following relation:

$$\psi^S(\mathbf{r}) = e^{-2ieg\phi}\psi^N(\mathbf{r}) \tag{2.6}$$

where e is the electric charge of our particle.

Now if we desire our wavefunction to be well-defined in the equator where we are patching the vector potentials together, then it must remain the same as we go around it from $\phi = 0$ to $\phi = 2\pi$, which in turn incurs in the following condition:

$$2eg = m \in \mathbb{Z} \tag{2.7}$$

This innocuous-looking expression is in fact the famous *Dirac quantisation condition*, which forces all magnetic monopoles, if they exist, to be quantised. Furthermore, because of the presence of the electric charge of the particle in the expression, it also implies that if at least one magnetic monopole exists in the universe, then all electric charges are *also* quantised. This quantisation condition will be part of some of our discussions in the following chapters.

Finally, we note that topological defects like these are robust. They arise because in a certain theory the vacuum solution is degenerate, and thus at spatial infinity in different directions the field might choose different vacua. As a result, at some point in space these different choices will clash and the transition between them will act as a particle (or more precisely a *quasiparticle*) of its own. Destroying such a state would require changing the field up to spatial infinity, and this would incur in spending infinite energy, hence their robustness. In equivalent fashion as done for the vortex current (2.4), we may define a more general topological current $j = dA$ for a field A which is conserved as a result of the robustness of these quasiparticles. Since A will be in most cases a U(1) gauge field, then dA will be the current of a global U(1) symmetry which arises when we have degenerate vacua allowing for such solutions [27]. A Maxwell theory of pure gauge and without any sources will have a Lagrangian proportional to $F \wedge *F$. This gives rise to the field equation of motion and the Bianchi identity, respectively:

$$\begin{cases} d * F = 0 \\ dF = 0 \end{cases} \tag{2.8}$$

These are then the equations exhibiting the conservation of the currents discussed. In 4d, the two symmetries associated are the global electric and magnetic symmetries, respectively $\mathrm{U}_E(1)$ and $\mathrm{U}_M(1)$, and in the language of generalised global symmetries [25], these are 1-form[6] symmetries associated with the conservation of dA and $*dA$. Adding electric or magnetic monopoles breaks them as a result of having a nonzero RHS to one of the equations in 2.8. In 3d the situation is analogous, although there is only one such symmetry, associated with dA, which may be broken by the introduction of a magnetic monopole.

2.2 Properties of Abelian Chern-Simons Terms

The Chern-Simons (CS) term in 3d spacetime reads [16]:

[6] The reason why they are called 1-form symmetries is because the charged objects for such currents are Wilson Lines and t' Hooft Lines, which inhabit 1-manifolds. Then, the conserved charge may be computed by simply integrating the current j over space: $Q(\mathcal{M}^{d-1}) = \oint_{\mathcal{M}^{d-1}} j$.

$$\mathcal{L}_{\text{CS}} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{2.9}$$

where A_μ is the U(1) gauge potential, and k a constant. We will only focus on 3d CS terms in this discussion, but for completeness here is the CS term in an arbitrary 2n+1-dimensional manifold:

$$\mathcal{L}_{\text{CS}} = \kappa\epsilon^{\mu_1\mu_2\ldots\mu_{2n+1}}A_{\mu_1}\partial_{\mu_2}A_{\mu_3}\partial_{\mu_4}A_{\mu_5}\ldots\partial_{\mu_{2n}}A_{\mu_{2n+1}} \tag{2.10}$$

for $n \in \mathbb{N}$ and where κ is a normalisation constant. Now, let us compute the Euler-Lagrange (EL) equation of motion for such a term in 3d. It reads:

$$\frac{k}{4\pi}\epsilon^{\mu\nu\rho}(\partial_\nu A_\rho - \partial_\rho A_\nu) \equiv \frac{k}{4\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho} = 0 \tag{2.11}$$

where $F_{\nu\rho}$ is the electromagnetic field tensor, and furthermore if we compute its stress-energy tensor[7][18],

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{\eta}}\frac{\partial\mathcal{L}_{\text{CS}}}{\partial\eta_{\mu\nu}} = 0 \tag{2.12}$$

it does not seem at first to give rise to any interesting physics. However, what makes CS terms special is what happens when they couple to other Lagrangian terms. For instance, if we add a Maxwell term to our Lagrangian, we have now

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \tag{2.13}$$

Before computing the equation of motion for this Lagrangian, let us note that as a result of having as many derivatives and fields as there are dimensions in a specific manifold, CS terms will always bear dimensionless coupling constants. This is not true for example with Maxwell terms, which because of their dimensionful constant will be infrared (IR) irrelevant in 3d and lower. This is yet another hint that bosonization in spacetime dimensions higher than 3 will prove to be harder, since then Maxwell terms become marginal or relevant. Now let us return to the equation of motion for A_μ in this new Lagrangian. It has changed in a small but rather important way:

$$\partial_\mu F^{\mu\nu} = -\frac{ke^2}{4\pi}\epsilon^{\nu\rho\sigma}F_{\rho\sigma} \tag{2.14}$$

[7] In fact it is quite straightforward to see that CS terms are metric-independent and thus have a vanishing stress-energy tensor using differential form notation. In such notation, the 3d CS term reads $\mathcal{L}_{\text{CS}} = \frac{k}{4\pi}A \wedge dA$.

If we consider the field dual to $F_{\mu\nu}$, i.e. we set $F^{\mu\nu} = \epsilon^{\rho\mu\nu}(*F)_\rho$, we may act on this relationship on both sides with $\epsilon_{\nu\alpha\beta}\partial^\alpha\partial_\mu$ and substitute (2.14) in it twice to show that this reduces to a massive Klein-Gordon Equation for the dual field:

$$\left[\Box + \left(\frac{ke^2}{2\pi}\right)^2\right](*F)_\beta = 0 \tag{2.15}$$

This is a rather intriguing phenomenon: in spite of there being no explicit mass term in the original Lagrangian, we were able to algebraically manipulate the electromagnetic field and show that it obeys a massive Klein-Gordon equation (the fact that the dual field obeys it implies that so does the original field). This can also be seen by computing the quantum propagator for A_μ. In momentum space it reads

$$\Delta_{\mu\nu} = e^2\left(\frac{p^2 g_{\mu\nu} - p_\mu p_\nu - i\frac{k}{2\pi}e^2\epsilon_{\mu\nu\rho}p^\rho}{p^2(p^2 - (\frac{ke}{2\pi})^2)} + \xi\frac{p_\mu p_\nu}{p^4}\right) \tag{2.16}$$

where ξ is a gauge-fixing parameter. We see this expression has a pole when $p^2 = (ke/2\pi)^2$, and we have a massive propagator. Our system was thus gapped by the addition of a CS term in the Lagrangian. Hence, in spite of it being topological (in other words, independent of the metric) and thus not contributing to the Energy-Stress tensor, it still couples to the field in such a way as to generate the so-called *topological mass*. This special kind of massive excitation will be central in our discussion of the phase transition method to derive QFT dualities in the next chapters.

For now, let us turn to another noteworthy feature of these terms: their relationship with parity and time reversal transformations. We may show that they break parity invariance and invert sign as time is reversed. These facts will also play an important role in our discussion in later chapters. Let us quickly derive these features. First, a parity tranformation in 2 spatial dimensions is not like one in 3 spatial dimensions. We are used to seeing

$$\mathcal{P}(\mathbf{x}) = -\mathbf{x} \tag{2.17}$$

when performing a parity transformation. However, in 2 spatial dimensions, such a transformation is nothing but a rotation, a transform belonging to the SO(2) group and hence whose matrix representation has unit determinant. This is not what we want from a parity inversion. Therefore, to ensure that our tranformation belongs to O(2) but not SO(2), we define the coordinates to transform as follows:

$$\begin{cases} x^0 \to x^0 \\ x^1 \to -x^1 \\ x^2 \to x^2 \end{cases} \tag{2.18}$$

Since there is no preferred coordinate between the spatial ones (x^1 and x^2), the choice of which one is going to be inverted is arbitrary and thus subject to convention. This does not affect the final result. In a CS term, between the two A factors and the derivative, one of them will invert its sign, and therefore so will the entire term. Thus,

$$\mathcal{P}(\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho) = -\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \tag{2.19}$$

A similar argument holds for time reversal. In this case, we have $\mathcal{T}(\mathbf{A}) = -\mathbf{A}$ and $\mathcal{T}(A^0) = A^0$, while the derivative changes sign only for the 0 index. Then, we have two possibilities: either the derivative contains a time index or a spatial index. In the former case, the derivative and the two A factors will swap sign, amounting to a total minus sign. In the other case, only one of the A factors will be spatial and swap sign, again amounting to a total minus sign. Thus,

$$\begin{cases} \mathcal{T}(\epsilon^{i0j} A_i \partial_0 A^j) = -\epsilon^{i0j} A_i \partial_0 A_j \\ \mathcal{T}(\epsilon^{0ij} A_0 \partial_i A^j) = -\epsilon^{0ij} A_0 \partial_i A_j \end{cases} \tag{2.20}$$

Hence we have shown that CS terms change sign under $\mathcal{P}$ and $\mathcal{T}$ operations. This, as stated previously, will not only be useful when we derive dualities containing such terms, but also more generally can be desirable features if one is seeking to write an effective field theory for a system which breaks parity or time-reversal invariance.

Finally, we will briefly mention a kind of term which can be seen as the CS term's sibling: the Background Field (BF) term. This will show up as a topological coupling between fields in our theories from which we shall derive dualities and will also be responsible for the creation of topological mass in some systems. The BF term is of the form

$$\mathcal{L}_{BF} = \frac{k}{2\pi} \epsilon^{\mu\nu\rho} A_\mu \partial_\nu B_\rho \tag{2.21}$$

where B_μ is usually a background gauge field (i.e. it is nondynamical and hence not integrated over in the partition function path integral).

2.3 Chern-Simons and Gauge Invariance: A Turbulent Relationship

We have so far avoided the question of whether CS terms are actually gauge-invariant, which is usually a *sine qua non* condition for physicists. Then, let us now evaluate it. We vary the electromagnetic potential such that $A_\mu \to A_\mu + \partial_\mu \lambda$. Then, our Lagrangian changes by

$$\begin{aligned}\mathcal{L}_{\text{CS}} \to \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu\partial_\rho\lambda + \partial_\mu\lambda\partial_\nu A_\rho + \partial_\mu\lambda\partial_\rho\lambda) \\ = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\partial_\mu\lambda \\ = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \partial_\mu\left(\frac{k}{4\pi}\lambda\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\right)\end{aligned} \tag{2.22}$$

Where terms symmetric in their indices contracting with the Levi-Civita tensor are taken to be identically zero. One would normally note that the Lagrangian changes by a total derivative and thus conclude that the CS term is gauge-invariant. However, this is not necessarily so. When we deal with systems which possess boundaries, which are important for the effective field theories in which CS terms are normally used, then we cannot simply disregard boundary effects [52]. Firstly, let us note that the easier path to evaluate the gauge invariance (or lack thereof) of Abelian CS terms is to place our system in a finite temperature, meaning we perform a Wick rotation into Euclidean spacetime and take time to be periodic. In other words, our time coordinate is now S^1. For the sake of simplicity, we focus our attention to what happens to the zero component of our A field under such a transformation. Because of the new geometry in the time direction, now our factor λ cycles around a circle parameterised by the Euclidean time τ, as follows:

$$\lambda = \frac{2\pi\tau}{e\beta} \tag{2.23}$$

with $\beta \equiv 1/T$ being the inverse temperature. This means that when we perform the gauge transformation, we have

$$A_0 \to A_0 + \partial_0\left(\frac{2\pi\tau}{e\beta}\right) = A_0 + \frac{2\pi}{e\beta} \tag{2.24}$$

As one can readily notice, this is a constant shift in the value of A_0, and cannot be continuously deformed into the identity (i.e. a map such that

$A_0 \to A_0$). Such a transformation is usually referred to as a *large gauge transformation*. Now, the Chern-Simons term itself has a value which we can compute. Again for simplicity, we consider a system with constant A_μ, such that we may disregard time derivatives. Let A_0 be also uniform. It is customary to perform this calculation in $S^1 \times S^2$, i.e. a spherical shell in euclidean time enclosing a magnetic monopole, such that the magnetic field on the surface is a flux flowing through the sphere. We then have the CS action:

$$\begin{aligned} &\frac{k}{4\pi}\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \\ &= \frac{k}{4\pi}\int d^3x [A_0(\partial_1 A_2 - \partial_2 A_1) + A_1\partial_2 A_0 - A_2 \partial_1 A_0] \\ &= \frac{k}{4\pi}\int d^3x [A_0(\partial_1 A_2 - \partial_2 A_1) - A_0\partial_2 A_1 + A_0 \partial_1 A_2] \\ &\qquad = \frac{k}{2\pi} A_0 \int d^3x F_{12} = \frac{kA_0\beta}{2\pi}\int_{S^2} d^2x F_{12} \end{aligned} \tag{2.25}$$

where we used integration by parts and we have already integrated the time coordinate in the last step. As long as the charged particles in our system obey the Dirac quantisation condition, as discussed in section 2.1, we know that $\frac{1}{2\pi}F_{12}$, representing the magnetic field, will integrate to an integer multiple of $1/e$. With one quantum of charge, our CS action (2.25) yields

$$\frac{k}{4\pi}\int d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho = \frac{kA_0\beta}{e} \tag{2.26}$$

Thus, we are now finally ready to see how this CS term action changes under large gauge transformations. We insert the change (2.24) into (2.26) to find

$$\frac{kA_0\beta}{e} \to \frac{kA_0\beta}{e} + \frac{2\pi k}{e^2} \tag{2.27}$$

Hence, when we consider boundary effects, it starts to look like gauge invariance is broken. However, there is still one way to salvage Abelian CS terms, and that is by considering that the physical object which must be gauge invariant is the partition function. Thus, if we have

$$\mathcal{Z}_{CS}[A] = e^{iS_{CS}} \to e^{iS_{CS} + i\frac{2\pi k}{e^2}} \tag{2.28}$$

this can still be invariant as long as $\frac{2\pi k}{e^2}$ is a multiple of 2π, which amounts to constraining $k/e^2 = \nu \in \mathbb{Z}$. In that way, we have quantised the values that our coupling constant k can take on, by insisting that gauge invariance be respected even for systems with non-negligible boundary effects, as well as requiring that the Dirac quantisation condition be upheld. This result is remarkable in that CS terms naturally offer a framework to design effective field theories for phenomena like the Quantum Hall Effect (QHE). While the QHE is not the focus of this thesis and we will not explore this topic further, let us just briefly mention that CS terms with quantised $k = e^2\nu$ emulate the basic behaviour of the integer[8] Quantum Hall Effect, and give rise to the Hall conductivity $\sigma_{xy} = \frac{e^2\nu}{2\pi}$, which is precisely the one observed experimentally.

2.4 Anyons and Flux Attachment

CS terms lend themselves very usefully to the discussion of *flux attachment*, a phenomenon which will play a central role in one of the dualities explored in the next chapter. To understand its origin, let us have the CS term now coupled to some charge current j^μ:

$$\mathcal{L} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu j^\mu \tag{2.29}$$

Now, if we compute the EL equation of motion for A_μ, we obtain

$$\frac{k}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu A_\rho = j^\mu \tag{2.30}$$

If we now place a stationary particle with charge q at the origin, we have $j^\mu = (q\delta^2(x), 0, 0)$. Thus, replacing this in the derived equation of motion (2.30), we obtain

$$\frac{k}{2\pi}F_{12} = q\delta^2(x) \tag{2.31}$$

In a 2+1 dimensional world, the element F_{12} of the electromagnetic field tensor represents a scalar magnetic field. Thus, what equation (2.31) is telling us is that wherever we have an electric charge density, there will also be a flux of magnetic field attached to it (hence the name). This is a remarkable result, not only because it is unusual to see a magnetic flux always accompanying stationary charges, but also because this has important consequences on the statistics of such particles when we consider quantum-mechanical effects. As

[8] In fact, even the behaviour of the fractional Quantum Hall Effect may be captured with the help of CS and BF terms. Let our action have N emergent gauge fields a^i and our usual background gauge field A. Then we may write a Lagrangian $\mathcal{L} = \frac{1}{2\pi}t_i\epsilon^{\mu\nu\rho}A_\mu\partial_\nu a^i_\rho + \frac{1}{4\pi}K_{ij}\epsilon^{\mu\nu\rho}a^i_\mu\partial_\nu a^j_\rho$, where K_{ij} and t_i are matrices containing the relevant coupling constants between the various terms. Then, the Hall conductivity may be straightforwardly computed from $\sigma_{xy} = (K^{-1})^{ij}t_it_j$.

a charged quantum particle adiabatically traverses a closed path enclosing a region of nonzero electromagnetic potential, it picks up a geometric phase [2]:

$$|\psi\rangle \to e^{i\gamma}|\psi\rangle \tag{2.32}$$

with

$$\gamma = q' \int_{\mathcal{C}} F_{12}\, dS = q' \int_{\mathcal{C}} F \tag{2.33}$$

where q' is the particle's charge. Suppose, then, that we have two particles in our system, with charges q_1 and q_2. If we adiabatically move particle 1 around particle 2 until it is back where it started, it will have picked up a phase factor of

$$\gamma_1 = q_1 \int F_{12}\, dS = q_1 \int \frac{2\pi}{k} q_2\, \delta^2(x) dS = q_1 q_2 \frac{2\pi}{k} \tag{2.34}$$

Because we have some liberty when choosing the level k of our CS term (as long as we are careful enough to preserve gauge invariance), this means we also have freedom in choosing what will be the phase factor picked up by the particles when they are swapped once, or twice, or any number of times. This clearly shows a departure from the usual kind of physics we are used to in dimensionalities higher than 3, where particles have to either be bosons or fermions, characterised by their statistics. In 4d and above, when we swap two fermions, the total wavefunction picks up a factor of -1, whereas when we swap two bosons, the factor is 1. This simple fact has far-reaching consequences, like the Pauli Exclusion Principle and Bose-Einstein or Fermi-Dirac statistics [21]. However, in 3d one may show generally that it is possible to have the wavefunction pick up *any* factor θ as we swap around two particles. To these intriguing inhabitants of Flatland we give the name *anyons*. Their unconventional statistical properties that we have briefly outlined opens the doors for several interesting aplications in Quantum Computing, notably the possibility of *braiding* these particles, i.e. swapping several of them in a specific order to control the phase factors that appear in their wavefunctions [42], although here we will be interested in using them to turn fermions into bosons.

2.5 The Parity Anomaly and Induced Chern-Simons Terms

We now turn to a feature of 3d fermionic Lagrangians known as the *Parity Anomaly*. In QFT, something bearing the title of *anomaly* constitutes a symmetry that is classically present in the Lagrangian of the theory, but is broken as we compute quantum corrections to it. Here, we are interested in invariance under parity transformations. As we shall see, because of the parity anomaly, in certain cases we may have induced CS terms in our Lagrangian. This anomaly is only present in odd-dimensional spacetimes, and its even-dimensional counterpart, the *Chiral Anomaly*, is discussed in Appendix A.

We begin with the theory of a charged massive fermion in 2+1d [18]:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu)\psi - m\bar{\psi}\psi = \bar{\psi}(i\not{D})\psi - m\bar{\psi}\psi \tag{2.35}$$

We have defined the parity transformation as per (2.18), and under this the fermion field changes as $\psi \to \gamma^1\psi$. Let us see the consequences of this in our Lagrangian. When we parity-transform it and adjust ψ accordingly, we get

$$\mathcal{L} \to \mathcal{L}' = (\gamma^1\psi)^\dagger\gamma^0(i\gamma^\mu D'_\mu)\gamma^1\psi - m(\gamma^1\psi)^\dagger\gamma^0\gamma^1\psi \tag{2.36}$$

where $D'_\mu \equiv (-1)^{\delta_{1\mu}} D_\mu$ (the μ indices are not being contracted). Simplifying, the first term yields

$$\begin{aligned}(\gamma^1\psi)^\dagger\gamma^0(i\gamma^\mu D'_\mu)\gamma^1\psi &= \psi^\dagger(\gamma^1)^\dagger\gamma^0(i\gamma^\mu D'_\mu)\gamma^1\psi \\ &= \psi^\dagger(\gamma^1)^\dagger\gamma^0(i\gamma^0\gamma^1 D_0 - i\gamma^1\gamma^1 D_1 + i\gamma^2\gamma^1 D_2)\psi \\ &= -\psi^\dagger(\gamma^1)^\dagger\gamma^0\gamma^1(i\not{D})\psi = \psi^\dagger(\gamma^1)^\dagger\gamma^1\gamma^0(i\not{D})\psi\end{aligned} \tag{2.37}$$

Now, using the Dirac representation, we have that $(\gamma^1)^\dagger\gamma^1 = \mathbb{1}$ and hence our first term comes back to what it was: $\bar{\psi}(i\not{D})\psi$. Now we move onto the second term:

$$-m(\gamma^1\psi)^\dagger\gamma^0\gamma^1\psi = m\psi^\dagger(\gamma^1)^\dagger\gamma^1\gamma^0\psi = m\bar{\psi}\psi \tag{2.38}$$

As we can see, under parity conjugation, the sign of the mass has changed. However, in such a Lagrangian, the overall sign of the mass term does not matter. Nonetheless, once we choose the sign of the mass we must stay with it, because as we shall see now if the sign changes there will be an induced term as a consequence. This happens due to 1-loop corrections, and hence it is a purely quantum effect, thus constituting an anomaly. With this in mind, let us calculate the 1-loop quantum correction to this Lagrangian. The uncorrected effective action for massive fermions in standard Quantum Electrodynamics (QED) reads:

$$\mathcal{S}_{\text{Eff}} = N_f \log[\det(i\not{\partial} + \not{A} + m)] \tag{2.39}$$

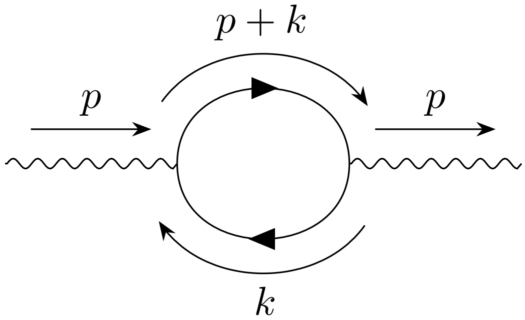

Figure 2.2: *The 1-loop diagram being considered in the calculation of the parity anomaly induced term.*

Where N_f stands for the number of fermion flavours in the theory. In our case this will be $N_f = 1$. Let us rewrite this in order to perform the next step.

$$\mathcal{S}_{\text{Eff}} = \log\left\{\det\left[(i\partial\!\!\!/ + m)(\mathbb{1} + (i\partial\!\!\!/ + m)^{-1}A\!\!\!/)\right]\right\} \tag{2.40}$$

Now, to evaluate the corrections perturbatively, we use the following identity for a matrix M:

$$\det M = e^{\text{tr}(\log M)} \tag{2.41}$$

Thus, applying this to (2.40),

$$\begin{aligned}\mathcal{S}_{\text{Eff}} = \log\left\{\det\big(i\partial\!\!\!/ + m\big)e^{\text{tr}\log\left(\mathbb{1}+(i\partial\!\!\!/+m)^{-1}A\!\!\!/\right)}\right\} \\ = \log\det\big(i\partial\!\!\!/ + m\big) + \text{tr}\log[\mathbb{1} + (i\partial\!\!\!/ + m)^{-1}A\!\!\!/] \\ = \log\det\big(i\partial\!\!\!/ + m\big) + \text{tr}\left[(i\partial\!\!\!/ + m)^{-1}A\!\!\!/\right] + \frac{1}{2}\text{tr}\left[(i\partial\!\!\!/ + m)^{-1}A\!\!\!/(i\partial\!\!\!/ + m)^{-1}A\!\!\!/\right] + ...\end{aligned} \tag{2.42}$$

where in the last line we have expanded the logarithm. Since we are here looking for a 1-loop correction which may contribute to the mass, we focus on the quadratic term. There, we have the diagram represented by Figure 2.2, whose effective action is given by the following expression:

$$\mathcal{S}_{\text{Eff}}^{\text{Quad}} = \frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3k}{(2\pi)^3}A_\mu(-p)\,\text{tr}\left[\gamma^\mu\frac{p\!\!\!/ + k\!\!\!/ - m}{(p+k)^2+m^2}\gamma^\nu\frac{k\!\!\!/ - m}{k^2+m^2}\right]A_\nu(p) \tag{2.43}$$

If we focus on the trace, we have

$$\operatorname{tr}\left[\gamma^\mu \frac{\not{p}+\not{k}-m}{(p+k)^2+m^2}\gamma^\nu \frac{\not{k}-m}{k^2+m^2}\right] \tag{2.44}$$

Using the trace identities in 3d, one can show that the only terms left after expanding are the ones of the form $-m\,p_\rho\,\operatorname{tr}[\gamma^\mu\gamma^\nu\gamma^\rho]$. All other terms will either cancel out or be identically zero. In 3d, the following relation holds:

$$\operatorname{tr}(\gamma^\mu\gamma^\nu\gamma^\rho) = -2\epsilon^{\mu\nu\rho} \tag{2.45}$$

Hence, the k integral in 2.43 reduces to

$$\begin{aligned} 2m\epsilon^{\mu\nu\rho}p_\rho \int \frac{d^3k}{(2\pi)^3}[(p+k)^2+m^2]^{-1}(k^2+m^2)^{-1} \\ = \epsilon^{\mu\nu\rho}p_\rho \frac{1}{2\pi}\frac{m}{|p|}\arcsin\left(\frac{|p|}{\sqrt{p^2+4m^2}}\right) \end{aligned} \tag{2.46}$$

Taking now the low energy and large mass limit (i.e. $p \to 0$ and $m \to \infty$), we may take the linear term of the arcsin function to be the dominant one, and thus (2.46) becomes

$$\backsimeq \frac{1}{4\pi}\frac{m}{|m|}\epsilon^{\mu\nu\rho}p_\rho = \frac{1}{4\pi}sgn(m)\epsilon^{\mu\nu\rho}p_\rho \tag{2.47}$$

where $sgn(m)$ denotes the sign function. Then, putting this back into (2.43), we have that

$$\mathcal{S}_{\text{Eff}}^{\text{Quad}} = \frac{1}{8\pi}sgn(m)\int d^3p\,\epsilon^{\mu\nu\rho}A_\mu(-p)p_\rho A_\nu(p) \tag{2.48}$$

Finally, back into coordinate space, our induced term in the effective action becomes

$$\mathcal{S}_{\text{Eff}}^{\text{CS}} = -i\frac{1}{8\pi}sgn(m)\int d^3x\,\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{2.49}$$

In our theory, we may define a fermion mass sign to begin with, since as argued previously the overall sign of the mass does not matter for the Dirac Lagrangian. However, what (2.49) tells us is that if at some point our m changes sign, then this will induce an effective CS term in our Lagrangian. This is the Parity Anomaly, and it will be used in one of the duality derivations in Chapter 4.

Chapter 3

The 3d Duality Hunter: Tools of the Trade

Throughout the last decades, we have seen a large web of field-theoretical dualities in various dimensionalities be found. Usually, a new method or insight is developed which allows for some new kind of duality to be shown and then this opens the floodgates of entire arrays of dualities which can then be derived by this method [32, 45]. As it stands then, there is no unique systematic way of finding them and we rely on navigating uncharted territory or recycling old methods in novel ways. The purpose of this chapter is to discuss some dualities derived in the past by different methods in order to showcase some of the ways in which we may go about looking for dualities. In the next chapter, we will introduce the most important derivations, which involve the phase transition method due to Seiberg, Senthil, Wang & Witten [45], which we compute for boson-boson and boson-fermion duality in 3d and is our method of choice to generalise into 4d manifolds afterwards. Let us then explore our first duality.

3.1 Superfluid - Maxwell Electromagnetism

The Bose-Hubbard Model [26], which describes interacting bosonic particles with no spin on a lattice has the following Hamiltonian:

$$H = -\sum_{ij} t_{ij} \hat{b}_i^\dagger \hat{b}_j - \mu \sum_i \hat{n}_i + \frac{1}{2} U \sum_i \hat{n}_i (\hat{n}_i - 1) \tag{3.1}$$

where t_{ij} is a matrix of couplings, μ and U constants, $\hat{b}$ and $\hat{b}^\dagger$ the annihilation and creation operators and $\hat{n} = \hat{b}^\dagger \hat{b}$ the number operator. The sums run over

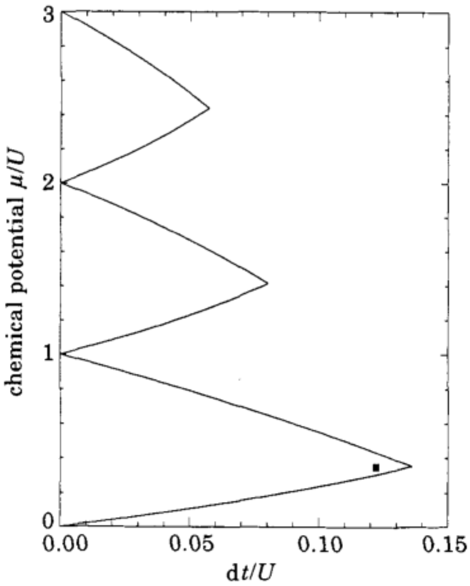

Figure 3.1: *The phase diagram of the Bose-Hubbard model in 2 spatial dimensions. On the right of the transition line we have the Superfluid phase, whereas on the left we have the Mott Insulator phase.*

the lattice points. This system has two phases which depend on the tuning of the constants, as shown on Figure 3.1 (taken from [23]). The phases are the Superfluid and the Mott Insulator.

As we approach the transition line from the Superfluid side, we will start to see the formation of vortices in the system, until they proliferate into the Mott Insulator phase. Thus, we shall here derive a duality for the Superfluid phase in which we treat such vortices as the elementary particles of the system. Let ϕ be the phase of the creation and annihilation operators $\hat{b}^\dagger$ and $\hat{b}$ of the Bose-Hubbard model. In that case, one may write the Euclidean Lagrangian in the continuum limit as [22]

$$\mathcal{L}_{\text{Superfluid}} = \frac{1}{2g}\partial_\mu^{(\text{ph})}\phi\partial^{\mu(\text{ph})}\phi \tag{3.2}$$

where g is a coupling constant and we define $\partial_\mu^{(\text{ph})} \equiv (\frac{1}{c_{\text{ph}}}\partial_\tau, \nabla)$ with c_{ph} being a constant for the superfluid and τ a time parameter. g and c_{ph} depend on the initial constants μ, U and t_{ij}. The *ph* sub- and superscripts will from now on be suppressed for clarity. We can see from this that fluctuations in ϕ start to become very energetically expensive for a small coupling constant g. Thus, when g is small the system will naturally suppress such fluctuations, and we enter the zero sound mode for ϕ. Then, we may apply a trick to the partition function and linearise this theory simply by introducing an auxiliary vector field ω_μ and expanding the path integral so that it also integrates over our new field [47]. Thus, we rewrite:

$$\mathcal{Z}_{\text{Superfluid}} = \int D\phi e^{-\int d^4x \frac{1}{2g}\partial_\mu\phi\partial^\mu\phi} = \int D\phi D\omega_\mu e^{-\int d^4x\left[\frac{g}{2}\omega_\mu\omega^\mu - \omega_\mu\partial^\mu\phi\right]} \tag{3.3}$$

At first, it may not seem obvious that these two partition functions are equivalent, however we may verify that by noting that the second action in the exponential is quadratic in the ω_μ field. Thus, we are warranted in integrating this field out through its EL equation of motion. When we compute the EL equation for ω_μ, we arrive at

$$\omega_\mu = \frac{1}{g}\partial_\mu\phi \tag{3.4}$$

now if we substitute this back into the modified partition function, we recover precisely the original partition function. Let us pause for a second here and point out that the ω_μ field is in fact the canonical momentum of the ϕ field, as we can see from equation (3.4). Furthermore when we do this transformation, we note that the coupling constant has inverted, in the sense that in the original Lagrangian it is being divided by our initial field and in the dual Lagrangian it is multiplying the dual field. This strong-weak coupling mapping will prove a very common trait in further duality analyses. Following the overview given in section 2.1, we may now separate our phase field into its smooth and multivalued components: $\phi = \phi_{\text{Smooth}} + \phi_{\text{MV}}$ in such a way that line integration through a closed path around a topological defect will yield $2\pi N$, with N representing the winding number around this path. Then substituting this in, our dual action has become

$$\mathcal{S}_{\text{Dual}} = \int d^4x \left[\frac{1}{2}g\,\omega_\mu\omega^\mu - \omega_\mu\partial^\mu\phi_{\text{MV}} - \omega_\mu\partial^\mu\phi_{\text{Smooth}}\right] \tag{3.5}$$

we may now perform a partial integration on the last term, and throwing out boundary terms in ω_μ, which we assume to be well-behaved, we have now

$$\mathcal{S}_{\text{Dual}} = \int d^4x \left[\frac{1}{2}g\,\omega_\mu\omega^\mu - \omega_\mu\partial^\mu\phi_{\text{MV}} - (\partial_\mu\omega^\mu)\phi_{\text{Smooth}}\right] \tag{3.6}$$

ϕ_{Smooth} is now linear in the action and may be integrated out. In fact, as we perform such an integration, we may consider it to be the Lagrange multiplier for the constraint $\partial_\mu\omega^\mu = 0$, which expresses the conservation of current for ω. We can insist that this be the case by introducing a U(1) gauge field b_ρ such that

$$\omega^\mu = \epsilon^{\mu\nu\rho}\partial_\nu b_\rho \tag{3.7}$$

If we decide to perform the path integral over b_ρ instead of ω_μ in the partition function, we will have

$$\mathcal{Z}_{\text{Dual}} = \int D\phi_{\text{MV}} Db_\rho e^{-\int d^4x\left[\frac{1}{2}g(\epsilon^{\mu\nu\rho}\partial_\nu b_\rho)^2 - \epsilon^{\mu\nu\rho}(\partial_\nu b_\rho)(\partial_\mu \phi_{\text{MV}})\right]} \tag{3.8}$$

where we have omitted a gauge-fixing factor in the path integral. Because our gauge field is smooth, we may then integrate the second term in the dual Lagrangian by parts, which leaves us with

$$\mathcal{L}_{\text{Dual}} = \frac{1}{2}g(\epsilon^{\mu\nu\rho}\partial_\nu b_\rho)^2 - \epsilon^{\mu\nu\rho}b_\rho(\partial_\nu\partial_\mu\phi_{\text{MV}}) \tag{3.9}$$

The contraction of the antisymmetric Levi-Civita tensor with two partial derivatives, which usually commute, would in most cases vanish identically. However, here we must remember that we are dealing with a multivalued field. Thus, the order of the derivatives does matter and as a result the last term does not vanish. Instead, we can define the vortex current, as done in the discussion of section 2.1:

$$\epsilon^{\mu\nu\rho}\partial_\nu\partial_\mu\phi_{\text{MV}} \equiv J^{\rho(\text{V})} \tag{3.10}$$

Now, there is only one more step before arriving at the final duality. Turning our attention to the first term in the dual Lagrangian, we note that

$$\epsilon^{\mu\nu\rho}\partial_\nu b_\rho = \frac{1}{2}\epsilon^{\mu\nu\rho}(\partial_\nu b_\rho - \partial_\rho b_\nu) = \frac{1}{2}\epsilon^{\mu\nu\rho}f_{\nu\rho} \tag{3.11}$$

where we have defined $\partial_\nu b_\rho - \partial_\rho b_\nu \equiv f_{\nu\rho}$. Finally, if we insert this into the dual Lagrangian, we are left with

$$\begin{aligned}\mathcal{L}_{\text{Dual}} &= \frac{g}{8}\epsilon^{\mu\nu\rho}\epsilon_{\mu\alpha\beta}f_{\nu\rho}f^{\alpha\beta} - b_\rho J^{\rho(\text{V})} \\ &= \frac{g}{4}f_{\nu\rho}f^{\nu\rho} - b_\rho J^{\rho(\text{V})}\end{aligned} \tag{3.12}$$

At last, we have arrived at our final dual Lagrangian. We note that it looks exactly like an electromagnetic theory in 2+1d, where the dual gauge fields b_μ are fulfilling the role of photons and the vortex current $J^{\rho(\text{V})}$ plays that of charged sources. We have thus established a duality between our initial Superfluid Lagrangian (3.2) and that of Coulomb theory (3.12) in the weak-coupling regime (i.e. for small g). In this case, we have employed a brute-force method to show that the particles in one theory are related to the vortices of another. We have encountered thus a *particle-vortex duality*. In the next section we will derive another such duality and subsequently analyse in more depth its symmetry properties.

3.2 Free Scalar - Scalar QED

After seeing how to derive dualities by manipulating a specific partition function until one reaches another desired (dual) partition function, it is strategic to point out that new dualities can be (and very often are) derived from known ones, which gives rise to a web of dualities. In this section, we will assume a previous duality and with relatively simple manipulations we may derive others, although our starting point here will be derived later on using the phase transition method. The duality we assume to begin with is as follows [32]:

$$\begin{aligned}&\int D\psi\ \exp\left\{i\int d^3x\ \left[i\bar{\psi}\not{D}_A\psi-\frac{1}{8\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho\right]\right\}\\&\quad=\int D\phi Da\,\exp\left\{i\int d^3x\ \left[|D_a\phi|^2+\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho+\frac{1}{2\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho\right]\right\}\end{aligned} \tag{3.13}$$

Firstly, let us note that both A_μ and a_μ are gauge fields, and A_μ is taken to be a background field whereas a_μ represents an emergent, dynamical field. The left-hand side of the equation has a CS term coupled to the massless fermion. On the other side of this duality we see a bosonic theory with a complex scalar ϕ which couples to A_μ and a_μ through a BF term. Because our background field is on one side coupled to a fermion field and on the other side of the duality to a bosonic field, this is an example of 3d bosonization. It is also an example of particle-vortex duality, since the A field is coupled on one side to the fermion current $\bar{\psi}\gamma^\mu\psi$ and on the other side to the flux density of the dynamical field $\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$ (after integration by parts in the last term of the right-hand side). In fact, the CS and BF terms present on both sides are crucial to the bosonization and particle-vortex duality since they ensure that there will be flux attachment present.

Now, let us assume this duality to be true and see what comes from it. We now take A_μ to be a dynamical field (which means it is now also integrated over) just like a_μ and add to both sides a BF coupling between A_μ and a new background field C_μ:

$$\int D\psi DA\,\exp\Big\{i\int d^3x\,\Big[i\bar{\psi}\not{D}_A\psi-\frac{1}{8\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho-\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu C_\rho\Big]\Big\}$$
$$=\int D\phi DaDA\,\exp\Big\{i\int d^3x\,\Big[|D_a\phi|^2+\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho+\frac{1}{2\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu A_\rho-\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu C_\rho\Big]\Big\} \tag{3.14}$$

On the right-hand side, we have the field A_μ appearing linearly in the Lagrangian density, and thus it may be integrated out. Its equation of motion is simply $da = dC$. As long as we have no holonomy, i.e. parallel-transporting a_μ and C_μ around closed loops preserves their initial states, then we may simply take $a = C$ and we end up with

$$\int D\psi DA\,\exp\Big\{i\int d^3x\,\Big[i\bar{\psi}\not{D}_A\psi-\frac{1}{8\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho-\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu C_\rho\Big]\Big\}$$
$$=\int D\phi\,\exp\Big\{i\int d^3x\,\Big[|D_C\phi|^2+\frac{1}{4\pi}\epsilon^{\mu\nu\rho}C_\mu\partial_\nu C_\rho\Big]\Big\} \tag{3.15}$$

And this is another known duality [3, 9, 15]. However, we may take this duality a bit further, which will also introduce another tool that proves useful when deriving dualities: time reversal. We now divide both sides of (3.15) by the CS term involving C_μ, add to both sides a BF coupling to a gauge background field $\tilde{A}_\mu$ and promote the C_μ field to a dynamical one, meaning it is now also integrated over. Now we have:

$$\int D\psi DADC\,\exp\{i\int d^3x\,[i\bar{\psi}\not{D}_A\psi-\frac{1}{8\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho-\frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu C_\rho$$
$$-\frac{1}{4\pi}\epsilon^{\mu\nu\rho}C_\mu\partial_\nu C_\rho+\frac{1}{2\pi}\epsilon^{\mu\nu\rho}C_\mu\partial_\nu\tilde{A}_\rho]\}$$
$$=\int D\phi DC\,\exp\Big\{i\int d^3x\,\Big[|D_C\phi|^2+\frac{1}{2\pi}\epsilon^{\mu\nu\rho}C_\mu\partial_\nu\tilde{A}_\rho\Big]\Big\} \tag{3.16}$$

Now, the right-hand side has become simply the partition function for Scalar QED, and since we will not tamper with it anymore, let us define it to be $\mathcal{Z}_{\text{Scalar-QED}}[\tilde{A}]$. As for the left-hand side, since it is quadratic in C_μ, we may integrate out this field through its EL equations of motion. They then read $dC = d(\tilde{A}-A)$. Once again, if there are no holonomies present we may directly substitute $C = \tilde{A}-A$, leaving us with

$$\int D\psi DA \exp\{i \int d^3x\,[i\bar{\psi}\not{D}_A\psi + \frac{1}{8\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu\tilde{A}_\rho + \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\tilde{A}_\mu\partial_\nu\tilde{A}_\rho]\} = \mathcal{Z}_{\text{Scalar-QED}}[\tilde{A}] \tag{3.17}$$

If we briefly leave this particular equation and remind ourselves of the duality (3.15) which we reached, we may use that to our advantage. We wish to time-reverse that duality. In fact, as discussed in section 2.2, CS and BF terms pick up a minus sign when time-reversed. Thus, under such a transformation, and relabeling $C_\mu \to \tilde{A}_\mu$ for convenience, (3.15) becomes

$$\int D\psi DA \exp\{i \int d^3x\,[i\bar{\psi}\not{D}_A\psi + \frac{1}{8\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \frac{1}{2\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu\tilde{A}_\rho]\} = \int D\phi \exp\{i \int d^3x\,[|D_A\phi|^2 - \frac{1}{4\pi}\epsilon^{\mu\nu\rho}\tilde{A}_\mu\partial_\nu\tilde{A}_\rho]\} \tag{3.18}$$

Now, we notice that the first three terms of the left-hand side of (3.17) are the same as the time-reversed duality we have just derived, equation (3.18) (the sign on the BF term sign does not matter because taking $\tilde{A} \to -\tilde{A}$ leaves the CS term unchanged and swaps the BF term sign on the other side of the duality, and with $\tilde{A}$ being a background field this is of little matter). Thus, substituting (3.18) into (3.17), the CS terms in $\tilde{A}_\mu$ cancel out and we are left with simply

$$\mathcal{Z}_{\text{Scalar-QED}}[\tilde{A}] = \int D\phi \exp\left\{i \int d^3x\,|D_{\tilde{A}}\phi|^2\right\} \tag{3.19}$$

We may then say

$$\mathcal{Z}_{\text{Scalar-QED}}[\tilde{A}] = \mathcal{Z}_{\text{Scalar}}[\tilde{A}] \tag{3.20}$$

Thus, by making use of the time-reversal and previously known dualities we are able to derive a range of new dualities. Here, we have arrived at a boson-boson duality. In fact, from here we may tune the background field, add a symmetry-breaking potential $V(\phi)$ to both sides and an IR-irrelevant Maxwell term to one side and with that we obtain the XY Model - Abelian Higgs Model duality, which is the main subject of dsicussion in the next section.

3.3 XY Model - Abelian Higgs Model

In this section, instead of deriving a new duality we shall use one that is a consequence of the last duality in the previous chapter and analyse it in more depth. Particularly, we are interested in the symmetry properties of this duality, since this will highlight some arguments which shall be invoked in the next chapter when the phase transition method is introduced. Firstly, the XY Model in its continuum form reads [52]:

$$\mathcal{L}_{XY} = |\partial_\mu \phi_1|^2 - V_1(|\phi_1|^2) \tag{3.21}$$

now, we present the Abelian Higgs Model:

$$\mathcal{L}_{AH} = |D_\mu \phi_2|^2 - \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - V_2(|\phi_2|^2) \tag{3.22}$$

where in these models, we have symmetry-breaking potentials of the form $V_i(|\phi|^2) = \alpha_i|\phi|^2 + \beta_i|\phi|^4$, where α_i and β_i are parameters which will help us establish in which phase of the theory we find ourselves. We wish to come forth with supporting arguments for mapping regimes of one theory into another. This process will emphasize an important aspect of QFT dualities, which is that of symmetries. For two theories (or regimes within these theories) to be dual to each other, they must possess the same symmetries, otherwise one Lagrangian could not possibly be mapped into the other. This points to a deeper message which is that symmetries are a more fundamental aspect to a theory's structure than its constituent fields themselves, since between dual theories what we call fields, particles and vortices are subject to algebraic manipulations in the partition function, whereas their underlying symmetries are not.

Let us now consider the Lagrangians (3.21) and (3.22). Firstly, for us to have stable theories, the β_i parameters within the potentials in both theories must be greater than zero. This thus constrains our evaluation to only the value of the α_i. When we have $\alpha_1 > 0$, the Vacuum Expectation Value (VEV) of the ϕ_1 field remains zero and the XY model theory keeps its U(1) symmetry intact. Furthermore, the excitations in ϕ_1 are massive. This is the *Coulomb Phase* of the theory. Let us then look at the Coulomb Phase of the Abelian Higgs model. Here, we will have to tread carefully since the symmetries are not so obvious. First, we notice that there is a U(1) gauge symmetry which finds no equivalent in the XY model, since it has no gauge fields. This is not going to be a problem because gauge symmetries are not, strictly speaking, symmetries. They are merely redundancies in the way we describe our fields. Furthermore, the Abelian Higgs model might allow for a multitude of vacuum solutions, and thus we have a global U(1) symmetry associated with the topological current $j = dA$, as discussed in section 2.1. Nonetheless, when $\alpha_2 > 0$ and hence $\langle \phi \rangle = 0$, this symmetry will be absent since the vacuum state is nondegenerate. Since this symmetry has been broken, there must be a corresponding Goldstone boson, and we can show that indeed this happens by integrating out the field ϕ_2 since it is now massive. We are left with only the Maxwell term for the gauge field, and we perform the following transformation:

$$\begin{aligned} \mathcal{Z}_{AH}^{\text{Coulomb}} &= \int DA e^{-i\int d^3x \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu}} \\ &= \int DF D\sigma \, e^{i\int d^3x \left(-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\pi}\sigma \epsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho}\right)} \end{aligned} \tag{3.23}$$

The role of the scalar field σ here is, for now, only to serve as a Lagrange multiplier which enforces the Bianchi identity, $\partial_{[\mu}F_{\nu\rho]} = 0$. However, as we shall soon see, σ is an important field in this duality. Now we can integrate out $F_{\mu\nu}$, since it is quadratic in the theory at hand, and our equation of motion yields

$$F = -\frac{e^2}{\pi} d\sigma \tag{3.24}$$

Furthermore, our Lagrangian has now become

$$\mathcal{L}_{AH}^{\text{Coulomb}} \sim e^2 \partial_\mu \sigma \partial^\mu \sigma \tag{3.25}$$

Which describes a free massless excitation, and this is our Goldstone boson. Furthermore, integration of both sides of (3.24) shows, if we decide to obey the Dirac quantisation condition, that σ must have a periodicity of 2π, which sounds reminiscent of the phase of some order parameter around a vortex. This, as we shall see soon, is no coincidence.

Now, we may move onto the *Higgs Phase* of both theories, which is when $\alpha_i < 0$. In the XY Model, we can see that when the ϕ_1 field acquires a nonzero VEV, its gauge field becomes massive by the Higgs mechanism and, very importantly, has its global U(1) symmetry spontaneously broken. As such, we may rewrite the field $\phi_1 = (r - r_0)e^{i\sigma}$, where r_0 is the VEV of ϕ_1, r represents the massive fluctuations around this value and the massless field standing for the phase of ϕ_1 has been suggestively called σ. This is the Goldstone boson of the theory in this regime, and because of the degeneracy of the vacua in this regime, the phase is free to wind around certain points, as long as it does so an integer number of times. Hence,

$$\frac{1}{2\pi} \oint dx^i \, \partial_i \sigma = N \in \mathbb{Z} \tag{3.26}$$

This feature represents vortex solutions of the theory in the Higgs phase. On the Abelian Higgs side, the ϕ_2 field also acquires an expectation value but its global symmetry is preserved, since the vacuum is degenerate and we have the conservation of topological current. This regime also yields vortex solutions, which in this case are charged under dA and are massive.

Thus, we are now ready to identify the two theories with each other. For a summary of our findings so far, we may turn to tables 3.1 and 3.2.

	Global symmetry	Massless excitations	Vortices
XY Model	Preserved	Absent	Absent
Abelian Higgs	Broken	Present	Absent

Table 3.1: *Features of the XY and Abelian Higgs Model in their Coulomb phases.*

	Global symmetry	Massless excitations	Vortices
XY Model	Broken	Present	Present
Abelian Higgs	Preserved	Absent	Present

Table 3.2: *Features of the XY and Abelian Higgs Model in their Higgs phases.*

As we can see, the Coulomb phase of the XY Model matches in symmetry properties with those of the Higgs phase of the Abelian Higgs model, and the Higgs phase of the XY model matches that of the Coulomb phase of the Abelian Higgs model. Hence, if we are to identify these two theories as dual to each other, we must match them in a way that the underlying symmetries are unaffected by the duality. The upshot of this is that we have

$$\begin{cases} \text{XY Model}_{Coulomb} \leftrightarrow \text{Abelian Higgs}_{Higgs} \\ \text{XY Model}_{Higgs} \leftrightarrow \text{Abelian Higgs}_{Coulomb} \end{cases} \tag{3.27}$$

Furthermore, moving on from the global symmetry properties, since both Higgs phases accept vortex solutions and these are mapped onto particle excitations of dual regimes, this is also an example of particle-vortex duality.

In this discussion, the opportunity has been taken to discuss the role of symmetries and their resilience as the ultimate structure that must be upheld when seeking dualities. Now we move onto the phase transition method, where these ideas will be put into practice.

Chapter 4

The Phase Transition Method in 3d

In this chapter, we will focus our attention on one specific method for hunting dualities in 3d: the phase transition method. This is a powerful tool for deriving dualities, and a noteworthy feature of it is the fact that it may be generalisable to higher dimensions. We shall explore this in the next chapter. Here, we will go through two dualities found by Seiberg et al. [45], the first is a boson-boson duality and the second consists of 3d bosonization. This way, we intend to build some working knowledge for the next step, which is to try and find dualities in 4d using the same procedure.

4.1 Wilson-Fisher Boson + BF - Wilson-Fisher Boson

The first duality we will derive through the phase transition method was first proposed by Peskin [19] and later verified by Monte-Carlo simulations by Dasgupta and Halperin [17]. It is also the quintessential example of a particle-vortex duality in 3d, since as we shall see it dualizes two rather simple Lagrangians in the IR regime.

Consider the following Lagrangian:

$$\mathcal{L} = |D_b\phi|^2 + |D_{\hat{b}}\hat{\phi}|^2 - V(|\phi|, |\hat{\phi}|) + \frac{1}{2\pi} b \wedge d\hat{b} + \frac{1}{2\pi} b \wedge dB \tag{4.1}$$

Here, b_μ and $\hat{b}_\mu$ are U(1) gauge fields, B_μ represents a background gauge field $V(|\phi|, |\hat{\phi}|)$ is a symmetry-breaking potential for the complex scalar fields ϕ and $\hat{\phi}$, and D_b represents the gauge covariant derivative using the b_μ field for connection. The ground state of such a Lagrangian has four phases which depend on the VEV of ϕ and $\hat{\phi}$. What characterises them as different phases

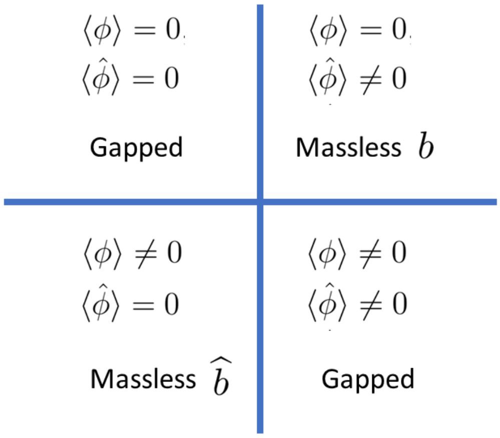

Figure 4.1: *The 4 phases of the vacuum state of the Lagrangian (4.1), and their respective massive or massless fields.*

are the global symmetries which are present or broken. Particularly, in this section we will be interested in the global symmetries [25] of the different phases. We have already discussed those symmetries in sections 2.1 and 3.3 under the guise of the conserved topological current $j = dA$. In this case, such underlying symmetries manifest themselves by giving rise or not to massless excitations in the different phases.

The four phases present in this Lagrangian are:

$$\begin{cases} 1.\ \langle\phi\rangle = 0, \langle\hat{\phi}\rangle = 0 \\ 2.\ \langle\phi\rangle \neq 0, \langle\hat{\phi}\rangle = 0 \\ 3.\ \langle\phi\rangle \neq 0, \langle\hat{\phi}\rangle \neq 0 \\ 4.\ \langle\phi\rangle = 0, \langle\hat{\phi}\rangle \neq 0 \end{cases} \tag{4.2}$$

These four phases are depicted in Figure 4.1. Firstly, let us look at phase 2. In it, the VEV of ϕ is nonzero, and as a result its gauge U(1) symmetry is broken. b_μ is then Higgsed and becomes massive, while $\hat{b}_\mu$ remains massless. Due to the presence of such a massless mode, this phase is gapless. A similar process happens in phase 4, where $\hat{b}_\mu$ is Higgsed as a result of the spontaneous breaking of the gauge U(1) symmetry for $\hat{\phi}$, and b_μ remains massless. Thus the phase is also gapless. Phase 3 is also straightforward: the gauge U(1) symmetries for both ϕ and $\hat{\phi}$ are broken and as a result both gauge fields become massive and the system is gapped.

Now, it may seem at first that phase 1 would then simply have its U(1) gauge symmetries preserved and keep both gauge fields massless. However, the gauge fields do not, as we shall see, remain massless. The reason for this is that the BF coupling terms added to the Lagrangian will cause a topological mass to arise in the system. To see this, we rewrite (4.1) in its effective form for phase 1. We have not defined specifically what $V(|\phi|^2, |\hat{\phi}|^2)$ is, but it will generically contain a mass term for the two fields. As a result, they will both be massive and they may be integrated out. This only leaves us with:

$$\mathcal{L}_{\text{Eff}} = \frac{1}{2\pi} b \wedge d\hat{b} + \frac{1}{2\pi} b \wedge dB \tag{4.3}$$

In order to see the topological mass arising from this, we briefly add now to this effective Lagrangian Maxwell terms for b and $\hat{b}$. In 2+1d these are irrelevant terms because of the dimension of the coupling constant which they bear. As a result, in the IR they will not play a role and we may add them for free. Thus, we study

$$\mathcal{L}_{\text{Eff}} = -\frac{1}{4e}(db)^2 - \frac{1}{4e}(d\hat{b})^2 + \frac{1}{2\pi} b \wedge d\hat{b} + \frac{1}{2\pi} b \wedge dB \tag{4.4}$$

We are ready then to compute the equations of motion for b and $\hat{b}$. They read, respectively,

$$\begin{cases} \frac{1}{2\pi}(d\hat{b} + dB) = -\frac{1}{e} d * db \\ \frac{1}{2\pi} db = -\frac{1}{e} d * d\hat{b} \end{cases} \tag{4.5}$$

Our background field in this system acts as a source term. To study the masses, we need only consider the homogeneous linear equation, i.e. with $B = 0$. Then, we perform the $d*$ operation on both sides of the first equation, which leaves us with

$$d * d\hat{b} = -\frac{2\pi}{e} d * d * db \tag{4.6}$$

Substituting this into the second equation, we have

$$\frac{1}{2\pi} db = \frac{2\pi}{e^2} d * d * db \tag{4.7}$$

Then, we rearrange this equation to yield

$$\frac{1}{e^2} d * d * db - \frac{1}{(2\pi)^2} db = 0 \tag{4.8}$$

This is now the Klein-Gordon equation for a massive field. In order to make this fact clearer, we may go back to index notation by using the identity $*d*d \equiv -\Box$, and we now have

$$\left(\frac{1}{e^2}\Box + \frac{1}{(2\pi)^2}\right) b_\rho = 0 \tag{4.9}$$

Now, because equations (4.5) for $B = 0$ are mapped back onto themselves under the transformation $b \to -\hat{b}$, $\hat{b} \to b$, then we may perform this action on (4.9) to find that the same topological mass equation holds for $\hat{b}$. Thus, the presence of the BF couplings in the original Lagrangian qualitatively changes its physics at low energies. The system is hence gapped in this phase.

Now, we wish to gap phase 2 by breaking the global symmetry whose topological current is $d\hat{b}$. This, as discussed previously in section 2.1, is a U(1) global symmetry and may be broken by adding a monopole to the theory. This monopole, or vortex term, has the form $\phi^\dagger M_{\hat{b}}$. Because $M_{\hat{b}}$ has charge 1 under $U_b(1)$, we add to it $\phi^\dagger$, which ensures gauge invariance. From the point of view of mass, now phase 2 is gapped in both hatted and unhatted fields, whereas phases 1 and 3 were already gapped to begin with and phase 4 still has a massless b_μ, since the unhatted fields emerge unscathed from this addition of a monopole operator. If we now assume that the theory has no further structure to it, the new system consists then of a single fixed point. If we now find that the critical limits on either side of this phase transition line lead to different Lagrangians, then we can claim that these Lagrangians are dual to each other. This is precisely what we will do now.

We analyse one limit of the line, as depicted in Figure 4.2 by taking $\langle\hat{\phi}\rangle \to \infty$. There, because the higgsing has caused $\hat{b}$ to become very massive, we may integrate it out together with $\hat{\phi}$, and its covariant derivative disappears since the field ϕ has frozen and lost its dynamics. Thus, it leaves us with

$$\mathcal{L}_{\langle\hat{\phi}\rangle\to\infty} = |D_b\phi|^2 - V(|\phi|) + \frac{1}{2\pi} b \wedge dB \tag{4.10}$$

The phase transition line is characterised by a critical point. We may reach this from the Lagrangian by tuning the potential to Wilson-Fisher fixed point as the VEV of ϕ changes between being zero and nonzero. Thus,

$$\mathcal{L}_{\langle\hat{\phi}\rangle\to\infty} = |D_b\phi|^2 - |\phi|^4 + \frac{1}{2\pi} b \wedge dB \tag{4.11}$$

On the other side of the transition line, we must take $\langle\phi\rangle = 0$. Here, ϕ becomes massive and we may integrate it out. We then have

$$\mathcal{L}_{\langle\phi\rangle=0} = |D_{\hat{b}}\hat{\phi}|^2 - V(|\hat{\phi}|) + \frac{1}{2\pi} b \wedge d(\hat{b} + B) \tag{4.12}$$

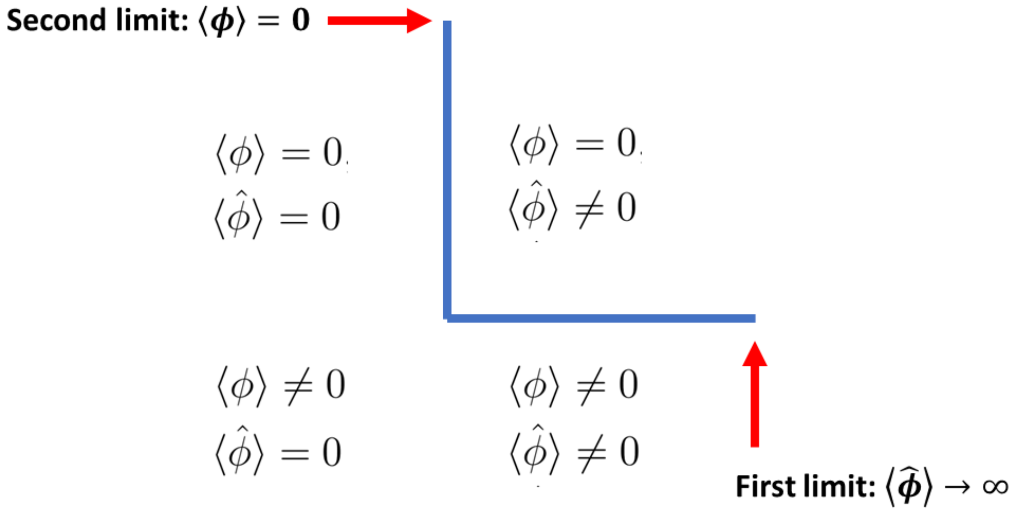

Figure 4.2: *The two limits taken for the bosonic duality derivation.*

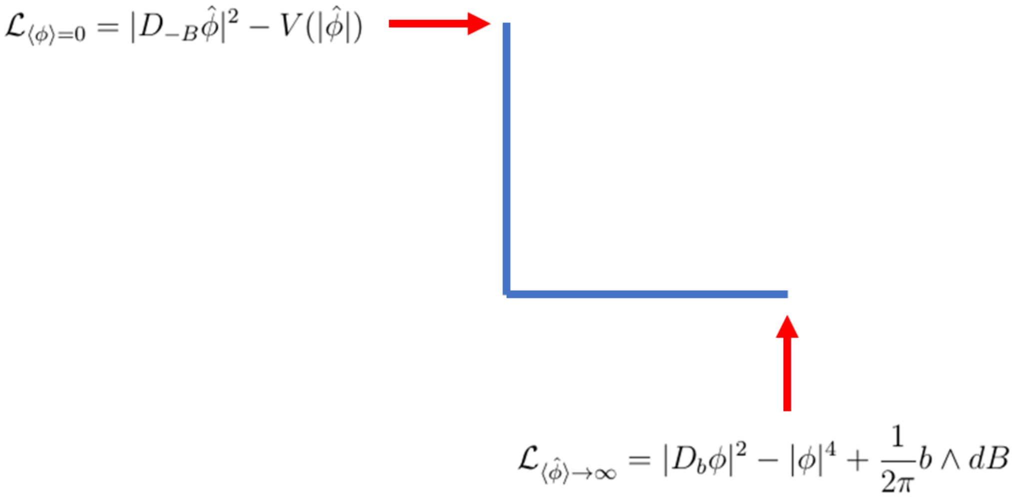

Figure 4.3: *In the absence of further structure, the theories on the endpoints of the transition line are dual to each other.*

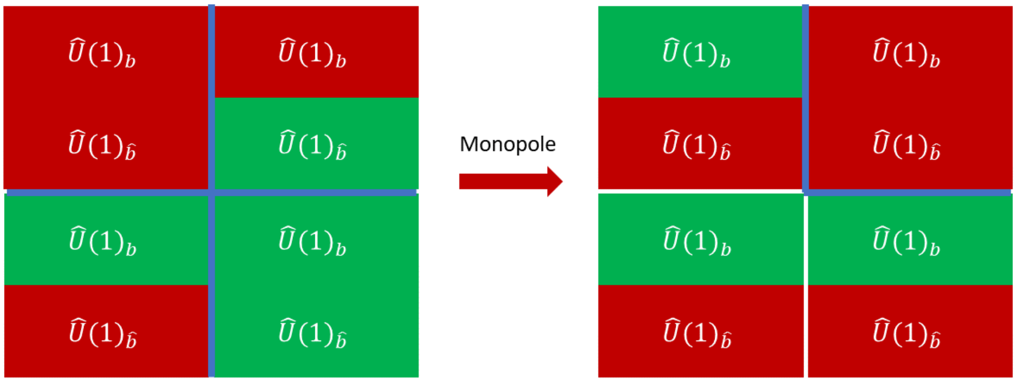

Figure 4.4: *The underlying global symmetries of each of the phases of the original Lagrangian and how they are affected by the addition of the monopole operator. Green stands for a preserved symmetry while red stands for an absent or broken symmetry.*

Now we have a linear theory in b and as such it may be integrated out, yielding $\hat{b} = -B$, in the absence of holonomies. Thus, we are left with

$$\mathcal{L}_{\langle\phi\rangle=0} = |D_{-B}\hat{\phi}|^2 - V(|\hat{\phi}|) \tag{4.13}$$

Finally, we again tune the potential to the Wilson-Fisher fixed point on this part of the transition line, to evaluate the critical behaviour as the VEV of $\hat{\phi}$ changes. Thus,

$$\mathcal{L}_{\langle\phi\rangle=0} = |D_{-B}\hat{\phi}|^2 - |\hat{\phi}|^4 \tag{4.14}$$

Thus, in the absence of further structure to (4.1) we can claim that (4.11) and (4.14) are dual to each other, as shown on Figure 4.3. While this derivation was mostly focused on the gapping of the phases, as discussed previously this but a manifestation of the underlying global symmetries present in each. On Figure 4.4 we can see how the addition of the monopole operator breaks the $\hat{b}$ global U(1) symmetry in the entire system and thus matches the symmetries of 3 of the former phases, causing them to merge into one, thus giving rise to a single transition line. It is not clear whether the monopole operator manages to restore the $U_b(1)$ symmetry for the topologically gapped phase or if this gapping itself causes both symmetries to be preserved to begin with, and the addition of the monopole only breaks $U_{\hat{b}}(1)$, thus matching the symmetries in all phases.

4.2 Free Dirac Theory - Massless Boson + Chern-Simons

Now we can use this same tool to consider other possible dualities. In particular, we can use it to derive the 2+1d bosonization duality that we assumed as a starting point in section 3.2. The Lagrangian we start with is:

$$\mathcal{L} = i\bar{\chi}\not{D}_{b+A}\chi + m\bar{\chi}\chi + |D_b\phi|^2 - V(|\phi|^2) \tag{4.15}$$

where $\not{D}_{b+A} \equiv \gamma^\mu[\partial_\mu - i(b_\mu + A_\mu)]$. This Lagrangian has four independent phases, and in which phase the system is depends on the value of $\langle\phi\rangle$, as well as the sign of m. We have four possibilities:

$$\begin{cases} 1.\ m > 0, \langle\phi\rangle = 0 \\ 2.\ m > 0, \langle\phi\rangle \neq 0 \\ 3.\ m < 0, \langle\phi\rangle \neq 0 \\ 4.\ m < 0, \langle\phi\rangle = 0 \end{cases} \tag{4.16}$$

It is important to note that the Lagrangian (4.15) has two global symmetries. The first is the U(1) symmetry for A associated with fermion number and the second is the symmetry associated with the conservation of the topological current db. The breaking of these symmetries is what controls the phase changes, like discussed in the end of the last section.

Firstly, we deal with the cases in which $\langle\phi\rangle$ is nonzero, in which case the $U_b(1)$ gauge symmetry is higgsed, the b field acquires a mass by the Higgs mechanism and thus may be integrated out. Hence, for $\langle\phi\rangle \neq 0$ the covariant derivative acting on the fermions $\not{D}_{b+A}$ becomes simply $\not{D}_A$. This is important because as was discussed in section 2.5, we can always set our reference so as to not include the CS term induced by the parity anomaly in the original Lagrangian, but once we have made that choice, we need to stay with it, which implies that if the mass of our fermion changes sign, integrating out the fermions will induce quantum correction terms. As a result, for $m < 0$ we have an induced term $-\frac{1}{4\pi}(b + A) \wedge d(b + A)$ for the $\langle\phi\rangle = 0$ side of the phase diagram (since there the covariant derivative still has A and b), and a $-\frac{1}{4\pi}A\wedge dA$ for the $\langle\phi\rangle \neq 0$ side (since there the b field has been integrated out and the covariant derivative only has A left), as shown on Figure 4.5. As for gapping, the $\langle\phi\rangle \neq 0$ phases are gapped by the Higgs mechanism, while the phase with $\langle\phi\rangle = 0$ and $m < 0$ acquires topological mass because of the induced Chern-Simons terms, as was discussed in Section 2.2.

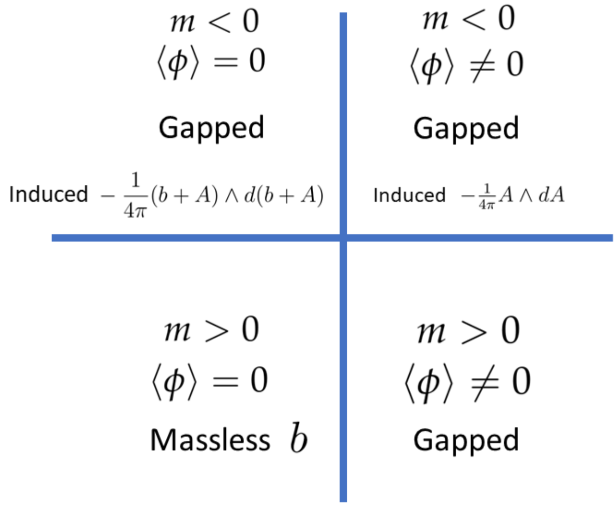

Figure 4.5: *The 4 phases of the vacuum state of the Lagrangian (4.15), and their respective massive or massless fields and induced terms.*

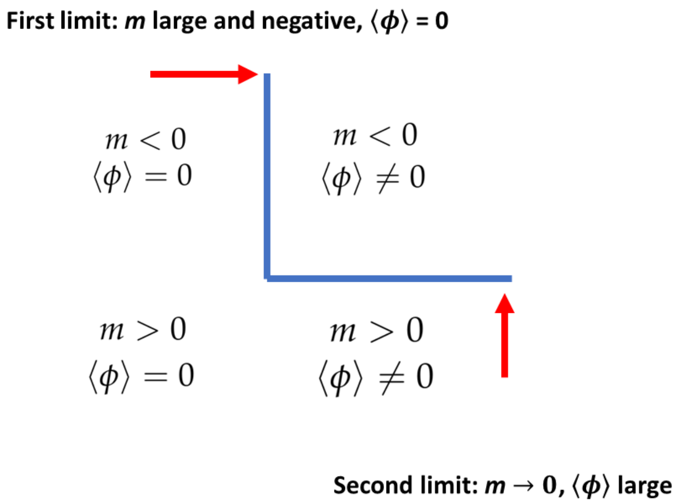

Figure 4.6: *The two limits taken on the duality derivation after adding the monopole operator.*

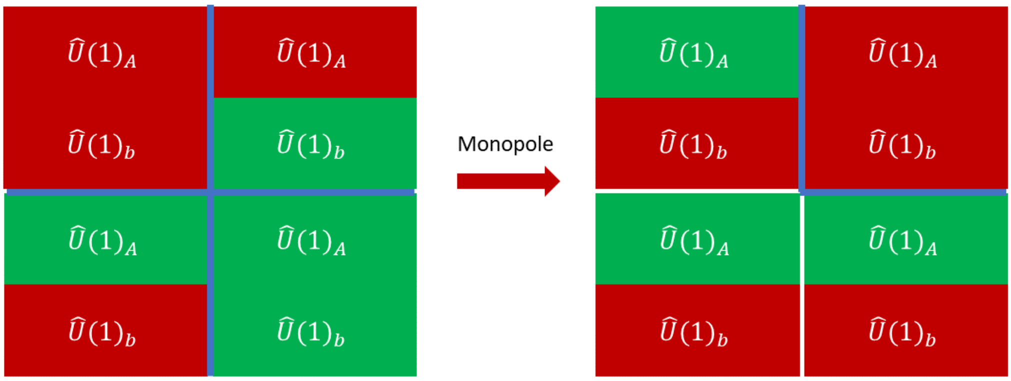

Figure 4.7: *The underlying global symmetries of each of the phases of the original Lagrangian and how they are affected by the addition of the monopole operator. Green stands for a preserved symmetry while red stands for an absent or broken symmetry.*

Now, like in the previous section, we may choose to add to the theory a monopole operator for b. This explicitly breaks the global U(1) symmetry which had a conserved topological current db, for the reasons discussed in the end of Section 2.1. Like before, we can also look at the underlying global symmetries that were broken or preserved after adding the monopole operator. These are shown on Figure 4.7, although the role of the monopole in restoring symmetries is unclear, like before. Then, we now argue that there is a single transition line described by a unique fixed point. The limits of the transition line we are going to consider are shown on Figure 4.6.

Let us explore the first limit of the transition line. As argued previously, we'll have an induced CS term as we take m to be very large and negative and integrate the massive χ field. The Lagrangian is now

$$\mathcal{L}_B = |D_b\phi|^2 - V(|\phi|^2) - \frac{1}{4\pi}(b+A)\wedge d(b+A) \tag{4.17}$$

Now, following the same argument as in the previous section, we tune the potential to the WF fixed point. Thus,

$$\mathcal{L}_B = |D_b\phi|^2 - |\phi|^4 - \frac{1}{4\pi}(b+A)\wedge d(b+A) \tag{4.18}$$

which describes a massless boson with a Chern-Simons term. Now we focus our attention on the other limit of one of the transition line, as pictured in figure 4.6. Now we take $\langle\phi\rangle \to \infty$. As argued previously, on this side of the diagram b has been integrated out. Furthermore, as a result of the limit we

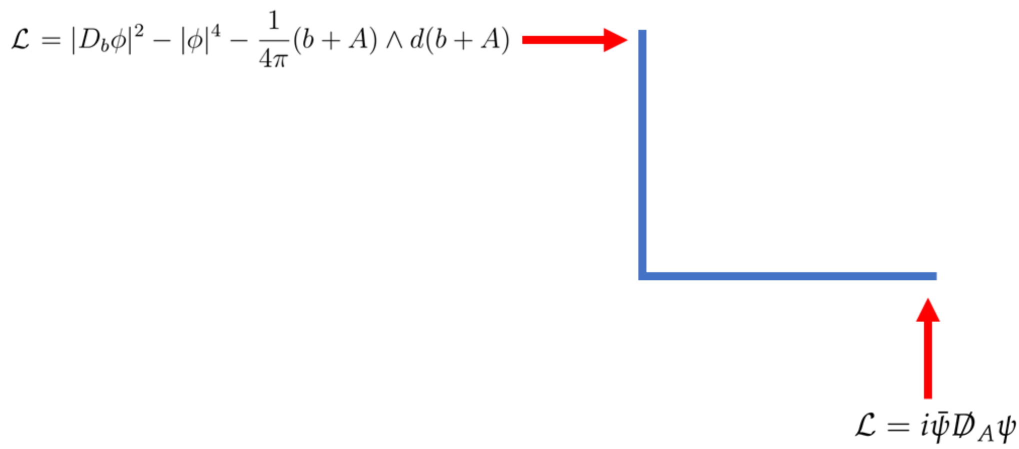

Figure 4.8: *The resulting theory after adding the monopole operator and computing the limits. There is a single fixed point and this we argue that the theories on either side of the transition line are dual to each other.*

take, ϕ loses its dynamics and its derivative vanishes. We are then left with

$$\mathcal{L}_F = i\bar{\chi}\not{D}_A\chi + m\bar{\chi}\chi - V(|\phi|^2) \tag{4.19}$$

Now, since the fermion has decoupled from ϕ, we may ignore the potential. Furthermore, we tune the system to the transition line by taking $m \to 0$. Hence, we have only

$$\mathcal{L}_F = i\bar{\chi}\not{D}_A\chi \tag{4.20}$$

This describes a free fermion. In order to preserve gauge invariance, here we need to define the operator $\psi = \phi^\dagger\chi$. This operator carries charge 1 under the global $\mathrm{U}_A(1)$ symmetry, and is invariant under the gauge $\mathrm{U}_b(1)$ symmetry. With this, our Lagrangian has become

$$\mathcal{L}_F = i\bar{\psi}\not{D}_A\psi \tag{4.21}$$

which describes a free, massless fermion.

The result is pictured in Figure 4.8. Now finally, given the lack of additional structure to the theory, we can say that theories (4.18) and (4.21) are dual to each other. Then, we have carried out bosonization in 2+1d. Further, if we set the background field B to zero, we recover the duality which served as a starting point in section 3.2.

Chapter 5

The Phase Transition Method in 4d

Now that we have seen plenty of physical dualities taking place in lower-dimensional spaces, we would like to know whether we can translate some of the methods we have used into four-dimensional manifolds. As an incursion into 4d, let us rederive a well-known duality, the *Electric-Magnetic Duality* in this framework.

5.1 Electric-Magnetic Duality

The E-M Duality, or *Montonen-Olive Duality* is one of the oldest known dualities [38]. It has its roots in a very simple observation: Maxwell's Equations *look* very symmetric. To understand this, let us have a look at them, in tensorial notation [40]. In the absence of sources, they read

$$\begin{aligned} \partial_\mu F^{\nu\mu} &= 0 \\ \partial_\mu * F^{\nu\mu} &= 0 \end{aligned} \tag{5.1}$$

Here, the dual electromagnetic field strength is defined as $*F^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$. Then, these equations are taken into one another if we perform the duality transformations

$$\begin{cases} F^{\nu\mu} \to *F^{\nu\mu} \\ *F^{\nu\mu} \to -F^{\nu\mu} \end{cases} \tag{5.2}$$

Under this, electric fields in the electromagnetic tensor are taken to magnetic fields and vice-versa [5]. Thus, at least for a sourceless theory, we have established that electric or magnetic formulations of it are dual to each other and therefore equivalent [44]. A much longer and in-depth discussion could be made if we were to consider theories with sources, and we wish to do that in the context of the phase transition method. We shall now rederive the Electric-Magnetic Duality so as to obtain some working knowledge of how to derive dualities in 4d.

We start with a similar setup as the one in section 4.2: we need a theory with two scalar fields, a symmetry-breaking potential and a term which couples the gauge fields, creating thus topological mass. We suggest the following Lagrangian density

$$\mathcal{L} = -f^2 - \hat{f}^2 + |D_b\phi|^2 + |D_{\hat{b}}\hat{\phi}|^2 - V(|\phi|, |\hat{\phi}|) + 2db \wedge \hat{b} \wedge B \tag{5.3}$$

Here, f^2 is shorthand for $\frac{1}{2}f \wedge *f$, and equivalently for $\hat{f}$. In the mixing term, a factor of 2 has been included for later convenience in the equations of motion. As done before, B here is a nondynamical gauge field. We added an extra field to the term considering that Chern-Simons terms are only defined in odd-dimensional spaces and we need all terms to have the same rank, which is 4 in this case. Like before, we have four phases depending on the vacuum expectation values of ϕ and $\hat{\phi}$. Let us go quickly through the trivial phases:

$$\begin{cases} \langle\phi\rangle, \langle\hat{\phi}\rangle \neq 0 \to b \text{ and } \hat{b} \text{ higgsed, gapped phase} \\ \langle\phi\rangle \neq 0, \langle\hat{\phi}\rangle = 0 \to b \text{ higgsed, massless } \hat{b} \\ \langle\phi\rangle = 0, \langle\hat{\phi}\rangle \neq 0 \to \hat{b} \text{ higgsed, massless } b \end{cases} \tag{5.4}$$

The phase diagram that will be generated from this is exactly the same as the one in the 3d boson-boson duality, as well as the limits that we shall consider. Thus, we will not repeat the images but for reference one may go back to Figures 4.1 to 4.4.

The phase that is left for us to verify is the one in which $\langle\phi\rangle = \langle\hat{\phi}\rangle = 0$. We need to show that this phase is gapped because the term mixing db, $\hat{b}$ and B in (5.3) incurs in the generation of topological mass for b and $\hat{b}$. In index notation, the EL equations of motion for the b and $\hat{b}$ fields will read:

$$\begin{cases} -\partial_\mu f^{\mu\sigma} = \epsilon^{\sigma\mu\nu\rho}\hat{f}_{\mu\nu}B_\rho \\ -\partial_\mu \hat{f}^{\mu\sigma} = \epsilon^{\sigma\mu\nu\rho}f_{\mu\nu}B_\rho \end{cases} \tag{5.5}$$

To get to this, we have used the fact that B is nondynamical, and thus any derivative acting on it will yield zero. Now, we will act on the first equation with $\epsilon_{\sigma\lambda\alpha\beta}\partial^\lambda$. It becomes

$$-\epsilon_{\sigma\lambda\alpha\beta}\partial^\lambda\partial_\mu f^{\mu\sigma} = \epsilon_{\sigma\lambda\alpha\beta}\epsilon^{\sigma\mu\nu\rho}\left(\partial^\lambda\hat{f}_{\mu\nu}\right)B_\rho \tag{5.6}$$

On the left-hand side, we use the fact that $f^{\mu\nu} = \partial^\mu b^\nu - \partial^\nu b^\mu$, to rewrite it as

$$-\epsilon_{\sigma\lambda\alpha\beta}\partial^\lambda\partial_\mu f^{\mu\sigma} = -\epsilon_{\sigma\lambda\alpha\beta}\partial^\lambda\partial_\mu(\partial^\mu b^\sigma - \partial^\sigma b^\mu) \tag{5.7}$$

We can use the antisymmetry properties of the Levi-Civita tensor and note that the second term will vanish because the exchange $\sigma \leftrightarrow \lambda$ when contracting will cause the swapped terms to be negative and cancel out. We are left with

$$\begin{aligned} &-\epsilon_{\sigma\lambda\alpha\beta}\partial^\lambda\partial_\mu(\partial^\mu b^\sigma - \partial^\sigma b^\mu) = -\partial_\mu\partial^\mu\epsilon_{\sigma\lambda\alpha\beta}\partial^\lambda b^\sigma \\ &= \Box\left(\epsilon_{\lambda\sigma\alpha\beta}\partial^\lambda b^\sigma\right) = \frac{1}{2}\Box\left(\epsilon_{\lambda\sigma\alpha\beta}\partial^\lambda b^\sigma - \epsilon_{\sigma\lambda\alpha\beta}\partial^\lambda b^\sigma\right) \\ &= \frac{1}{2}\epsilon_{\lambda\sigma\alpha\beta}\Box\left(\partial^\lambda b^\sigma - \partial^\sigma b^\lambda\right) = \frac{1}{2}\epsilon_{\lambda\sigma\alpha\beta}\Box f^{\lambda\sigma} \end{aligned} \tag{5.8}$$

Inserting this back into (5.6), we have

$$\frac{1}{2}\epsilon_{\lambda\sigma\alpha\beta}\Box f^{\lambda\sigma} = \epsilon_{\sigma\lambda\alpha\beta}\epsilon^{\sigma\mu\nu\rho}\left(\partial^\lambda\hat{f}_{\mu\nu}\right)B_\rho \tag{5.9}$$

Now we need to take care of the RHS. We note that the contraction of Levi-Civita tensors will yield several terms of Kronecker deltas with permuted indices. Thus,

$$\epsilon_{\sigma\lambda\alpha\beta}\epsilon^{\sigma\mu\nu\rho}\left(\partial^\lambda\hat{f}_{\mu\nu}\right)B_\rho = \delta^{\mu\nu\rho}_{\lambda\alpha\beta}\left(\partial^\lambda\hat{f}_{\mu\nu}\right)B_\rho \tag{5.10}$$

where $\delta^{\mu\nu\rho}_{\lambda\alpha\beta} \equiv (3!)\delta^{[\mu}_\lambda\delta^\nu_\alpha\delta^{\rho]}_\beta$ is the generalised Kronecker delta and the square brackets denote antisymmetrisation of indices. Here, for each of the cyclic terms there will be another with a minus sign where the two indices contracting with $\hat{f}$ will be swapped. However, we can take advantage of the antisymmetry of $\hat{f}$ and swap its indices too, causing the minus sign to disappear and the odd and even terms to sum. As a result, we are left with but three terms:

$$\begin{aligned} \delta^{\mu\nu\rho}_{\lambda\alpha\beta}\left(\partial^\lambda\hat{f}_{\mu\nu}\right)B_\rho &= 2\partial^\lambda\left(\hat{f}_{\lambda\alpha}B_\beta + \hat{f}_{\alpha\beta}B_\lambda + \hat{f}_{\beta\lambda}B_\alpha\right) \\ &= 2\left[B_\lambda\partial^\lambda\hat{f}_{\alpha\beta} + B_\beta\partial^\lambda\hat{f}_{\lambda\alpha} - B_\alpha\partial^\lambda\hat{f}_{\lambda\beta}\right] \end{aligned} \tag{5.11}$$

Now, we use (5.5) and substitute the second and third term in brackets. Thus,

$$\begin{aligned} &2\left[B_\lambda\partial^\lambda\hat{f}_{\alpha\beta} + B_\beta\partial^\lambda\hat{f}_{\lambda\alpha} - B_\alpha\partial^\lambda\hat{f}_{\lambda\beta}\right] \\ &\quad = 2B^\lambda\partial_\lambda\hat{f}_{\alpha\beta} - 2f^{\mu\nu}B^\rho\left(B_\beta\epsilon_{\alpha\mu\nu\rho} - B_\alpha\epsilon_{\beta\mu\nu\rho}\right) \end{aligned} \tag{5.12}$$

Inserting this back into the RHS of (5.9), we get:

$$\frac{1}{2}\epsilon_{\lambda\sigma\alpha\beta}\Box f^{\lambda\sigma} = 2B^{\lambda}\partial_{\lambda}\hat{f}_{\alpha\beta} - 2f^{\mu\nu}B^{\rho}\left(B_{\beta}\epsilon_{\alpha\mu\nu\rho} - B_{\alpha}\epsilon_{\beta\mu\nu\rho}\right) \tag{5.13}$$

Now, to break the antisymmetry in the α and β indices and get rid of one term, let us contract the whole equation with B^{α}. The first term in brackets on the right-hand side will vanish, because contracting a totally symmetric tensor ($B^{\rho}B^{\alpha}$) with a totally antisymmetric one ($\epsilon_{\alpha\mu\nu\rho}$) gives zero identically. Hence, we are left with

$$\frac{1}{2}\epsilon_{\lambda\sigma\alpha\beta}B^{\alpha}\Box f^{\lambda\sigma} = 2B^{\alpha}B^{\lambda}\partial_{\lambda}\hat{f}_{\alpha\beta} + 2B^{2}f^{\mu\nu}B^{\rho}\epsilon_{\beta\mu\nu\rho} \tag{5.14}$$

where we have defined $B_{\alpha}B^{\alpha} \equiv B^{2}$. Relabelling the dummy indices and rearranging the equation, we get

$$\epsilon_{\beta\mu\nu\rho}B^{\rho}(\Box + 4B^{2})f^{\mu\nu} = -4B^{\alpha}B^{\lambda}\partial_{\lambda}\hat{f}_{\alpha\beta} \tag{5.15}$$

This looks tantalasingly similar to a massive Klein-Gordon Equation for $f^{\mu\nu}$, were it not for the pesky $\hat{f}$ factor on the RHS. Hence, we have no option but to brute-force our way into seeing the topological mass. Let us simplify the equation by taking the limit $\mathbf{k} \to \mathbf{0}$. In other words, we take the system to zero 3-momentum, where the topological mass should still be present. Switching this between real and reciprocal space, this means all spatial derivatives of f and $\hat{f}$ will vanish. Hence,

$$\epsilon_{\beta\mu\nu\rho}B^{\rho}(\partial_0^2 + 4B^{2})f^{\mu\nu} = -4B^{\alpha}B^{0}\partial_{0}\hat{f}_{\alpha\beta} \tag{5.16}$$

Because B is a background field, we may define it to be $B^{\mu} = (0, B_x, B_y, B_z)$ such that $B_jB^j > 0$. Thus, we are left with

$$\epsilon_{\beta\mu\nu\rho}B^{\rho}(\partial_0^2 + 4B^{2})f^{\mu\nu} = 0 \tag{5.17}$$

And finally now this looks more like a massive Klein-Gordon equation at zero momentum. We can then see that the topological mass is given by $\sqrt{2B_jB^j}$. Because the equations of motion (5.5) are completely symmetric under the exchange $f \leftrightarrow \hat{f}$, an identical equation will hold for $\hat{f}$, and thus we have proven that this phase, in spite of no higgsing of the fields, is gapped.

Now we follow on with our derivation of the electric-magnetic duality. Just like was done before in section 4.2, we now add a monopole term to the Lagrangian with the intent of gapping the system in the phase where $\hat{b}$ is massless. We add the term $\phi^{\dagger}M_{\hat{b}}$ and this lifts the massless $\hat{b}$ boson into a gapped excitation. Further, it breaks the topological symmetry whose current is $d\hat{b}$. If we assume that there is no further structure to the theory, this now means that the three gapped phases are in fact one phase, separated from the massless b phase by a single fixed point, which we will now probe. For this, we first take the limit $\langle\hat{\phi}\rangle \to \infty$. Because $\hat{\phi}$ becomes frozen its derivatives yield zero and because of the higgs mechanism $\hat{b}$ becomes very massive and may be integrated out. Our initial Lagrangian (5.3) then becomes

$$\lim_{\langle\hat{\phi}\rangle\to\infty} \mathcal{L} = -f^2 + |D_b\phi|^2 - V(|\phi|, |\hat{\phi}|) \tag{5.18}$$

By tuning the potential to the fixed point, we get

$$\lim_{\langle\hat{\phi}\rangle\to\infty} \mathcal{L} = -f^2 + |D_b\phi|^2 - |\phi|^4 \tag{5.19}$$

This theory looks simply like a scalar field minimally coupled to a gauge field. In other words, here ϕ is charged under b, which can be seen from the equation of motion for b:

$$\partial_\nu f^{\mu\nu} = i(\phi\partial^\mu\phi^\dagger - \phi^\dagger\partial^\mu\phi) + 2b^\mu|\phi|^2 \tag{5.20}$$

The LHS is the usual derivative of the field strength for Maxwell's equations in tensorial notation, while the RHS represents the current for the ϕ field. Now that we have shown that one end of this phase transition line corresponds simply to 4d scalar electrodynamics, let us have a look at what happens at the other end. Now we instead take the limit $\langle\phi\rangle \to 0$. Since ϕ is massive, we integrate it out and our initial Lagrangian (5.3) reduces to

$$\lim_{\langle\phi\rangle\to 0} \mathcal{L} = -f^2 - \hat{f}^2 + |D_{\hat{b}}\hat{\phi}|^2 - V(|\hat{\phi}|) + 2db \wedge \hat{b} \wedge B \tag{5.21}$$

Since we are approaching the transition line from the gapped phase, both b and $\hat{b}$ are massive. If we work in the linear coupling regime, meaning we do not consider terms quadratic in b, the EL equations of motion for $\hat{b}$ and b then read:

$$\begin{cases} \partial_\mu \hat{f}^{\rho\mu} = iJ^\rho_{\hat{\phi}} - \frac{1}{2}\epsilon^{\rho\mu\nu\sigma} f_{\mu\nu} B_\sigma \\ \partial_\mu f^{\rho\mu} = \frac{1}{2}\epsilon^{\rho\mu\nu\sigma} \hat{f}_{\mu\nu} B_\sigma \end{cases} \tag{5.22}$$

where $J^\mu_{\hat{\phi}} \equiv \hat{\phi}^\dagger \overleftrightarrow{\partial^\mu} \hat{\phi} = \hat{\phi}\partial^\mu\hat{\phi}^\dagger - \hat{\phi}^\dagger\partial^\mu\hat{\phi}$. Particularly, in the low energy regime the derivatives vanish and the first equation reduces to

$$J^\rho_{\hat{\phi}} = -\frac{i}{2}(*f)^{\rho\sigma} B_\sigma \tag{5.23}$$

where $*f_{\rho\gamma}$ has been defined like in (5.1). As we can see, on this end of the transition line the $\hat{\phi}$ current has become charged under the dual of f. The duality operation on the field strength has the effect of taking electric fields to magnetic fields and vice-versa. Therefore, here the current is magnetically charged, as opposed to electrically charged on the other side of the transition line, as shown.

Because after the addition of the monopole operator the theory has no further structure, we claim that the theories on either side of the transition line are dual to each other. On one end, the scalar field is charged electrically, whereas on the other end it is charged magnetically, thus establishing an electric-magnetic duality in the low-energy regime, which has benefitted from the background field B for mediation.

Chapter 6

Conclusion

In this thesis, we have discussed dualities in QFT, which have been shown to arise as different descriptions of the same quantum partition function. With this, we discussed why these can be of use to physicists by comparing them to transforms, in that they can translate a problem between two frameworks and render it much simpler to solve. Particularly, we emphasized that it is a common attribute of dual theories that they have inverse relationships between their coupling constants, opening the doors to mapping strong coupling regimes to weak coupling regimes and thus applying standard perturbative methods. This has applications from condensed matter theory to particle physics and string theory. However, there are no systematic methods to find dualities, and we often rely on finding creative ways to manipulate the path integral or using symmetry arguments.

We also discussed a specific kind of duality, between bosonic and fermionic theories, and explored the possibilities of this in different dimensionalities. Namely, in one spatial dimension bosonization is well established, but it starts to become harder to achieve in higher dimensions. In 2 spatial dimensions one may take advantage of flux attachment which allows for fractional statistics. However, achieving bosonization in 3+1d has been much more difficult.

Then, in order to build a working knowledge as to how dualities are derived, we first discussed the tools that would be needed to understand these derivations in 2+1d, namely the topological defects of vortices and monopoles, and the various properties of CS terms, which often give rise to flux attachment and topological mass.

We started discussing the dualities themselves by making use of the Bose-Hubbard model and describing it in terms of its vortices, thus establishing a particle-vortex duality between its superfluid phase and Maxwell electromagnetism. Then, we proceeded into a web of dualities in 2+1d which can be derived by assuming one specific duality to start with. We only visited a small part of this web, but several further dualities may be derived from there, and partly thanks to the terms causing flux attachment, it is possible to find boson-boson, fermion-fermion and boson-fermion dualities. We showed that with simple substitutions and by making use of the time-reversal properties of CS ad BF terms we may arrive at the duality between scalar QED and a scalar theory, which may be readily turned into the XY/Abelian Higgs duality. Using the latter as an example, we then discussed the role of symmetry in dualities, more specifically how global symmetries must be preserved in a duality transformation. In the example discussed, we had global U(1) symmetries which were broken or preserved depending on whether the system was in its Coulomb or Higgs phase, and thus we were able to map the Coulomb phase of the Abelian Higgs model to the Higgs phase of the XY model and the Higgs phase of the Abelian Higgs model to the Coulomb phase of the XY model. We also took the opportunity to discuss the global U(1) symmetry whose current is the topological current dA. This symmetry would prove important for the next dualities.

We then proceeded to discuss a specific method for deriving dualities, namely the phase transition method. Its advantage is that it may be more readily generalised to higher dimensions than other methods for searching dualities. The phase transition method makes use of a Lagrangian with four phases which depend on a parameter of the action (in our examples, the VEV of the fields and the sign of a fermion mass). By adding a monopole operator to the theory, we break certain symmetries and argue that there is now a single transition line and thus only two phases. The endpoints of such transition line describe dual theories, since they lie in the single fixed point of the system. We used this method to show the Wilson-Fisher/Wilson-Fisher+BF duality and a 2+1d bosonization which made use of the parity anomaly. Then, we stepped one dimension up and gave arguments in support of the Montonen-Olive duality in a novel way using this framework. Still, there remains a puzzle regarding the symmetry-matching of these theories, which is related to the role of the monopole operator in restoring U(1) symmetries in certain phases. Alternatively, these symmetries might already be there from the beginning, although it is not clear why.

Future developments in this area will possibly look further into 4d bosonization, and potentially in even higher dimensionalities. It might be the case that just like the parity anomaly may be employed to derive bosonization in 2+1d, the chiral anomaly might be of help for 3+1d, and it is discussed in appendix A. It could also be the case that one may derive bosonization in higher dimensions and then project it back onto 3+1d.

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Appendices

Appendix A

The Chiral Anomaly

Since in section 2.5 we saw an example of a field-theoretical anomaly arising on a three-dimensional manifold, it is natural to ask whether similar phenomena take place in a different number of spacetime dimensions. Indeed, they do. In fact, it is possible to show that the parity anomaly will be present in any odd-dimensional manifold. In even-dimensional manifolds, we can study its cousin: the *Chiral Anomaly.* Here, we follow the derivation as given in [40], which is based on the work of Fujikawa [24]. Let us begin with a massless fermionic theory in four dimensions. We know that its Lagrangian density is given by

$$\mathcal{L} = \bar{\psi}(i\not{D})\psi \tag{A.1}$$

This Lagrangian has a global symmetry given by chiral rotations, i.e. if we perform the following transformations,

$$\begin{cases} \psi(x) \to \psi'(x) = (1 + i\alpha(x)\gamma^5)\psi(x) \\ \bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)(1 + i\alpha(x)\gamma^5) \end{cases} \tag{A.2}$$

its action will change according to

$$\int d^4x \bar{\psi}'(i\not{D})\psi' = \int d^4x[\bar{\psi}(i\not{D})\psi - (\partial_\mu \alpha(x))\bar{\psi}\gamma^\mu\gamma^5\psi] \tag{A.3}$$

From here, we can read off directly that if this global symmetry is to hold, then its conserved Noether current must be given by

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{A.4}$$

such that

$$\partial_\mu j^{\mu 5} = 0 \tag{A.5}$$

$j^{\mu 5}$ is known as the *chiral* or *axial* current, a pseudovector which gives the net flux of chirality-oriented (right- and left-handed) currents into a region of space. To see this, we can decompose the chiral current:

$$j^{\mu 5} = \bar{\psi}\gamma^\mu \left(\frac{1+\gamma^5}{2}\right)\psi - \bar{\psi}\gamma^\mu \left(\frac{1-\gamma^5}{2}\right)\psi \tag{A.6}$$

where in the first term one can see the right-handed projection operator acting on ψ and in the second term we have the left-handed projection operator. Thus, the conservation of the chiral currents given by (A.5) implies that the total number of right-handed and left-handed particles are independently conserved, or at the very least they vary together. We will now set out to show that in four dimensions (and more generally in any even number of dimensions), the right-hand side of (A.5) is in fact nonzero for quantum theories. Showing that $j^{\mu 5}$ is conserved by varying the action under the transformation (A.2) assumes that the measure of the path integral giving the partition function of this theory does not change. In other words, that the structure

$$Z = \int\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\int d^4x \bar{\psi}(i\not{D})\psi} \tag{A.7}$$

will be unaffected by (A.2) such that

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = \mathcal{D}\bar{\psi}\mathcal{D}\psi \tag{A.8}$$

after the chiral rotation. Then, our objective now is to show that, if we wish to preserve gauge invariance (and we do, since gauge invariance is very dear to physicists), (A.8) does not hold.

To carry this out, we will firstly expand the fermionic field in a basis of eigenstates of $\not{D}$. Then, let us define its right and left eigenvectors:

$$\begin{aligned} (i\not{D})\phi_n &= \lambda_n \phi_n \\ \hat{\phi}_n(i\not{D}) &= \lambda_n \hat{\phi}_n \end{aligned} \tag{A.9}$$

When we have a fixed A_μ background field, the eingenvalues λ_n will tend asymptotically to their free form ($A_\mu = 0$) as the momentum grows larger. Hence, in this limit,

$$\lambda_n^2 = k_\mu k^\mu = (k^0)^2 - (\vec{k})^2 \tag{A.10}$$

We now expand the fermionic fields in this eigenstate basis:

$$\begin{array}{l} \psi(x) = \sum_n a_n \phi_n \\ \bar{\psi}(x) = \sum_n \hat{a}_n \hat{\phi}_n \end{array} \tag{A.11}$$

Here, a_n and $\hat{a}_n$ are Grassmann coefficients. Hence, they are anticommuting. The measure of our partition function (A.7) may be then recast in the form

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_n da_n d\hat{a}_n \tag{A.12}$$

What we then need to do is obtain the a'_n and $\hat{a}'_n$ as a function of a_n and $\hat{a}_n$, and from there we can see how the total measure changes. If we apply the tranformation (A.2) to the expanded fermion fields and take $\alpha(x)$ to be infinitesimal, we obtain:

$$\psi'(x) = \sum_n a'_n \phi_n = (1 + i\alpha(x)\gamma^5)\psi(x) \tag{A.13}$$

Now we multiply this equation by $\phi^\dagger_m$ from the left and integrate, which yields

$$\int d^4x \sum_n \phi^\dagger_m a'_n \phi_n = \int d^4x\, \phi^\dagger_m (1 + i\alpha(x)\gamma^5) \sum_n a_n \phi_n \tag{A.14}$$

where we have also expanded the $\phi(x)$ field. Then, the orthonormality of basis eigenstates leads to

$$\sum_n a'_n \delta_{mn} = \sum_n \int d^4x\, \phi^\dagger_m (1 + i\alpha(x)\gamma^5)\phi_n a_n \tag{A.15}$$

and

$$a'_m = \sum_n \int d^4x\, \phi^\dagger_m (1 + i\alpha(x)\gamma^5)\phi_n a_n \equiv \sum_n (\delta_{mn} + C_{mn}) a_n \tag{A.16}$$

where C_{mn} is an infinitesimal matrix. Thus, we have found the coefficients of the Jacobian determinant of this transformation

$$\mathcal{D}\bar{\psi}'\mathcal{D}\psi' = J^{-2}\mathcal{D}\bar{\psi}\mathcal{D}\psi \tag{A.17}$$

where $J \equiv det(1 + C)$. We evaluate it using the identity (2.41). Hence,

$$J = det(1 + C) = e^{tr[log(1+C)]} = e^{tr(C - \frac{1}{2}C^2 + \frac{1}{3}C^3 - \ldots)} \cong e^{tr(C)} \tag{A.18}$$

Here, we have used the fact that C is infinitesimal and hence neglected all orders higher than one. Using (A.16) and (A.18), we are left with

$$\log J = i \int d^4x \left[\alpha(x) \sum_n \phi_n^\dagger(x) \gamma^5 \phi_n(x) \right] \tag{A.19}$$

The sum within the integral is not convergent, and thus we need to use some regularisation procedure. Because the squared eigenvalues of ϕ_n are negative at high momenta (after a Wick rotation, which we will perform shortly), we can do as follows:

$$\sum_n \phi_n^\dagger \gamma^5 \phi_n = \lim_{M\to\infty} \sum_n \phi_n^\dagger \gamma^5 \phi_n e^{\lambda_n^2/M^2} \tag{A.20}$$

Or, in operator form,

$$\begin{aligned} \sum_n \phi_n^\dagger \gamma^5 \phi_n &= \lim_{M\to\infty} \sum_n \phi_n^\dagger \gamma^5 e^{(i\not{D})^2/M^2} \phi_n \\ &= \lim_{M\to\infty} \langle x| \, tr \left[\gamma^5 e^{(i\not{D})^2/M^2} \right] |x\rangle \end{aligned} \tag{A.21}$$

Where the trace is taken over Dirac indices. Now, we use the commutation properties of the covariant derivative and gamma matrices and rewrite $(i\not{D})^2$ as

$$\begin{aligned} (i\not{D})^2 &= -\frac{1}{2}(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]) D_\mu D_\nu \\ &= -\eta^{\mu\nu} D_\mu D_\nu - \frac{1}{2}[\gamma^\mu, \gamma^\nu] D_\mu D_\nu \\ &= -D^2 - \frac{1}{4}([\gamma^\mu, \gamma^\nu] D_\mu D_\nu - [\gamma^\nu, \gamma^\mu] D_\mu D_\nu) \\ &= -D^2 - \frac{1}{4}[\gamma^\mu, \gamma^\nu][D_\mu, D_\nu] \\ &= -D^2 + \frac{e}{2}\sigma^{\mu\nu} F_{\mu\nu} \end{aligned} \tag{A.22}$$

where we have defined $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ and used the fact that $[D_\mu, D_\nu] = -ieF_{\mu\nu}$. Now, if we insert this into (A.21) and ignore the background gauge field A_μ (since it is in principle controllable and its will not affect our result), we obtain

$$\lim_{M\to\infty} \langle x| \, \mathrm{tr} \left[\gamma^5 e^{(-\partial^2 + \frac{e}{2}\sigma^{\mu\nu} F_{\mu\nu})/M^2} \right] |x\rangle \tag{A.23}$$

If we expand the second term of the exponential (which does not act on $|x\rangle$) in a power series, we see that the first order term will have γ^5 traced with two gamma matrices, which in 3+1d gives zero. The second order will give a nonzero contribution, and all other orders will be killed by the limit of M taken to infinity. Hence, we obtain:

$$\begin{aligned} &\lim_{M\to\infty} \langle x|\, tr\left[\gamma^5 e^{(-\partial^2+\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu})/M^2}\right]|x\rangle \\ &\cong \lim_{M\to\infty} tr\left[\gamma^5 \frac{1}{2!}\left(\frac{e}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right]\langle x|\, e^{-\partial^2/M^2}\,|x\rangle \end{aligned} \tag{A.24}$$

The operator acting on $|x\rangle$ may be evaluated in reciprocal space by Wick rotating, as promised. Let us take the eigenfunctions to be plane waves in momentum space. Thus,

$$\begin{aligned} \langle x|\, e^{-\partial^2/M^2}\,|x\rangle &= \lim_{x\to y}\int \frac{d^4k}{(2\pi)^4} e^{-ik_\mu x^\mu} e^{k^2/M^2} e^{ik_\mu y^\mu)} \\ &= i\int \frac{d^4k_E}{(2\pi)^4} e^{-k_E^2/M^2} = i\frac{M^4}{16\pi^2} \end{aligned} \tag{A.25}$$

where k_E stands for 4-momentum in Euclidean space. Therefore, our regularised sum becomes

$$\begin{aligned} &\lim_{M\to\infty} \mathrm{tr}\left[\gamma^5 \frac{1}{2!}\left(\frac{e}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right]\langle x|\, e^{-\partial^2/M^2}\,|x\rangle \\ &= \lim_{M\to\infty} \frac{-ie^2}{128\pi^2}\frac{M^4}{(M^2)^2} F_{\mu\nu}F_{\lambda\sigma}\,\mathrm{tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma\right] \\ &= -\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \end{aligned} \tag{A.26}$$

At long last, going back to (A.19), our Jacobian has become

$$J = e^{-i\int d^4x\alpha(x)\left(\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}\right)} \tag{A.27}$$

Thus, this has to be included in the partition function when we perform the chiral rotation, yielding:

$$Z = \int\int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{\int d^4x\left[\bar{\psi}(i\not{D})\psi + \alpha(x)\left(\partial_\mu j^{\mu 5} + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma}\right)\right]} \tag{A.28}$$

Therefore, should we insist that the Lagrangian keep the invariance under global chiral rotations, we can read off from the transformed partition function that, finally,

$$\partial_\mu j^{\mu 5} = -\frac{e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}F_{\lambda\sigma} \tag{A.29}$$

Thus, we have shown that the chiral current is not conserved in our spacetime. In fact, we could have adjusted our regularisation to have the right-hand-side of (A.29) equal zero. However, this would have come at the cost of making the right-hand-side of

$$\partial_\mu j^\mu = \partial_\mu \bar{\psi}\gamma^\mu \psi = 0 \tag{A.30}$$

nonzero, and this is an exchange most physicists are not willing to accept, for it would imply the violation of gauge invariance. Alas, in even-dimensional spacetimes, gauge invariance is incompatible with chiral current conservation. It is possible to show, through an entirely equivalent procedure as done here, that the general result for even d spacetime dimensions with $n = d/2$ is

$$\partial_\mu j^{\mu 5} = (-1)^{n+1} \frac{2e^n}{n!(4\pi)^n} \epsilon^{\mu_1 \mu_2 \dots \mu_{2n}} F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} \dots F_{\mu_{2n-1} \mu_{2n}} \tag{A.31}$$