Search for Inflationary Attractors

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Author :	Annet van den Hul
Student ID :	1685465
Supervisor Physics:	Ana Achúcarro
Supervisor Maths:	Arjen Doelman

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Annet van den Hul

Instituut-Lorentz, Leiden University
P.O. Box 9500, 2300 RA Leiden, The Netherlands

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Abstract

In 2018, Achúcarro et al. presented a class of two-field inflationary models known as '*shift-symmetric orbital inflation*'[1]. These models have multi-field behaviour but their predictions remain very close to those of single-field inflation. This thesis will first introduce the topic of inflation, provide background to the field and build the necessary equations and concepts starting at the foundations of cosmology. Then it shall further examine these models and their solutions. Stability will be proven for a small subset of the class. Furthermore, a few sufficient sets of requirements for stability will be presented. It will be argued, but not proven, that stability holds in general.

Contents

Chapter 1

Introduction

The field of cosmology is growing rapidly. While the measurements of the Cosmic Microwave Background (CMB) are becoming more and more precise, the models composed to describe it are also becoming increasingly accurate. The latest data from the Planck collaboration[2] have shown the nature of the primordial perturbations to be very close to Gaussian and adiabatic. These observations are most often explained using single field slow roll inflation, which will be explained in chapter three of this thesis. Although the results of these models compare nicely to the observations, single field inflation has proven to be difficult to embed into an ultraviolet (UV) complete theory[3].

This thesis will focus on the article *'Shift-Symmetric Orbital Inflation: single field or multi-field?'* by Achúcarro et al.[1]. In this article, a class of two-field inflationary models is presented, known as *'shift-symmetric orbital inflation'*. Although these models behave strongly multi-field, their predictions remain remarkably close to those of single-field inflation. This gives a promising new family of models to explore which might possibly bring solutions to some of the problems arising when trying to embed single field inflation into UV complete theories such as string theory.

The aim of this thesis is to further examine the models described above. The goal is to prove stability of the solutions of the models. In order to do so, chapter two will give an introduction to the field of cosmology. It will present some background on the topic, as well as the fundamental equations in play. Chapter three will continue from there and describe the origin and basic principles of inflation, both single- and multi-field, and introduce the slow roll approximation. Then, after these foundations have been set, chapter four will explain the model and the solutions and give proof of stability for certain specific cases.

Chapter 2

Cosmology: the basics

This chapter will provide an introduction to the topics in cosmology necessary to comprehend the background of the mathematical computations provided in the next chapter. To keep the material accessible, it will start at the very basic principles of cosmology and build from that. Cosmology is the study of the origin, nature and development of the universe. It is based upon a few core views and equations, which will be covered briefly in this chapter. Although General Relativity does play a large part in the field of cosmology, it falls outside of the scope of this thesis, and therefore all topics will be approached from a classical viewpoint. This chapter roughly follows the same lines as the book *'Introduction to Cosmology'* by Andrew Liddle[4] complemented by *'The Primordial Density Perturbation'* by David Lyth and Andrew Liddle[5].

The field of cosmology is built around one central idea, the *cosmological priciple*: the universe looks the same wherever you are. This view is specified by saying that the universe is *homogeneous* and *isotropic* in all points. Homogenity implies that there are no preferred locations, every point is the same. An example would be a constant vector field in the x direction: it has a preferred direction, but every point is indistinguishable from all the others. A configuration is called isotropic about a point if there are no preferred directions: an example would be a vector field pointing radially outwards. It definitely has a preferred location, the centre, but all directions are equal. Note that a field that is isotropic in all points, has to be homogeneous as well.

Observations in astronomy show that all far away objects appear to be moving away from us, with a speed ($\vec{v}$) proportional to the distance ($\vec{r}$)

between Earth and the object. This was first realised by George Lemaître, and later observationally confirmed by the famous cosmologist, Edwin Hubble, who composed what was later to be known as *Hubble's law*:

$$\vec{v} = H_0\vec{r}, \tag{2.1}$$

with H_0, *Hubble's constant*, the constant of proportionality. Although it is not an exact law, and especially the peculiar velocities (random motions) of close by galaxies disturb the relationship, it does describe the average behaviour of galaxies extremely well.

One might think this law contradicts the cosmological principle. Since all objects around us move away from us, that must mean that we are in fact in a preferred location in the universe, most likely the centre. However, this is solved by proposing that it is not all galaxies moving away from us through space, but space itself is expanding. In this narrative, it's not just us, but any given observer at any point in the universe would see all objects moving away in all directions. Thus, although expanding, the universe would still be homogeneous and isotropic and the cosmological principle is not violated. This only holds with a linear relation between velocity and distance, any other law would not have worked.

A common way to visualise this is imagining a deflated balloon with a polka-dot pattern. When inflating the balloon, all dots on the surface move apart. From each dot, it would seem as if all other dots are moving away from it. This is true for all dots, hence no one dot is special. The same principle holds for the universe, although more difficult to visualise due to the extra dimension.

With the universe expanding, an interesting question arises. If we turn back time, everything would have to have been much closer together. This is where the *Big Bang Theory* comes in: if we trace the history back long enough everything will come together in one singularity. Calculations using the known laws of physics yield a point of infinite temperature and density, at a finite time in the past. Current models can hence not extrapolate towards this singularity. A model of the universe starting from such a beginning is called a *Big Bang Cosmology*.

2.1 The Friedmann equation

In order to be able to delve into cosmology any further, one of the most important cosmological equations has to be mentioned, the Friedmann equation. This equation describes the expansion of the universe. For a long time, much of the field of cosmology could be reduced to solving the Friedmann equations under different assumptions on the material content of the universe.
Although General Relativity is needed to give a rigorous derivation, the result is exactly the same as when derived using Newtonian mechanics*. This would start with the Newtonian relationship for the force (F) exerted by an object of mass M on an object of mass m at a distance r:

$$F = \frac{GMm}{r^2}, \tag{2.2}$$

with G Newton's gravitational constant. Integrating this equation gives a gravitational potential energy V:

$$V = -\frac{GMm}{r}. \tag{2.3}$$

To derive the Friedmann equation, one requires Newton's Shell Theorem. This is a direct corollary from Gauss' law, and states that a particle in a spherically-symmetric distribution of matter, say at a radius r from the centre, will feel no force from the material positioned at radii larger than r, and the material at smaller radii will exert the exact same force as a point-mass placed at the centre with mass equal to the total integrated mass over these smaller radii.

As a start to the derivation, note that due to the cosmological principle, every point of the universe can be considered the centre. Consider a mass m at a distance r away from the centre. The surrounding medium is expanding uniformly and has mass density ρ. Now, according to the Shell Theorem, the mass only feels attraction from the material in radii smaller than r. This matter has a total mass of $M = 4\pi\rho r^3/3$. Thus, the force and potential (using equations (2.2) and (2.3)), are equal to:

*This is not entirely true as the Newtonian derivation does not account for the existence of the cosmological constant, an extra term that will be discussed further on in this thesis.

$$F = \frac{GMm}{r^2} = \frac{4\pi G\rho r m}{3}, \tag{2.4}$$

$$V = -\frac{GMm}{r} = -\frac{4\pi G\rho r^2 m}{3}. \tag{2.5}$$

To derive an equation describing the behaviour of the separation r of the particle, energy conservation can be used: for a given particle, the sum of the kinetic energy T and potential energy V remains constant. Defining U as the total energy, one finds the following differential equation:

$$U = T + V = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m, \tag{2.6}$$

where the dot signifies a derivative to time.

The distance coordinate in the above equation, r, poses some problems. Since the universe is expanding, it will change over time for two given observers. For this reason, cosmologists often use a different set of coordinates called *comoving coordinates* (as opposed to the *physical coordinates* used above). These coordinates are defined as comoving with the expansion of the universe:

$$\vec{r} = a(t)\vec{x}. \tag{2.7}$$

where a is a function known as *the scale factor*. The homogeneity of space ensures that a, is indeed a function of time alone. It measures the universal expansion rate. If between two times the value of the scale factor has doubled, this implies the universe has expanded by a factor two, i.e. everything is twice as far apart. Also note that, by definition of comoving coordinates, $\dot{x} = 0$, as the distance between the chosen objects is fixed in comoving coordinates.

Substituting physical coordinates with comoving coordinates in equation (2.6) gives:

$$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi}{3}G\rho a^2x^2m. \tag{2.8}$$

The Friedmann equation in its most common form can be found by multiplying each side by $2/ma^2x^2$ and rearranging the terms:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}, \tag{2.9}$$

where $kc^2 = -2U/mx^2$ is a constant independent of position and time. Here c denotes the speed of light and k is a constant related to the curvature of space. For an expanding universe, the value of k is unique, as will be discussed further ahead in this chapter.

It is interesting to now look back at equation (2.1), stating that the recession velocity of far away objects is proportional to their distances. Indeed, since the velocity of recession is in the same direction as $\vec{r}$, we can write:

$$\vec{v} = \frac{|\dot{\vec{r}}|}{|\vec{r}|}\vec{r} = \frac{\dot{a}}{a}\vec{r}, \tag{2.10}$$

where the last step used $\vec{r} = a\vec{x}$ and the fact that these objecta are at a fixed comoving distance. Thus, comparing this to Hubble's law (2.1), we deduce:

$$H = \frac{\dot{a}}{a}. \tag{2.11}$$

The name Hubble's constant is a bit confusing here, since this is not in fact a constant value. Often, cosmologists denote the current value of Hubble's constant with H_0, and call $H = H(t)$ Hubble's parameter, which indeed varies over time.

Sometimes the final term in the Friedmann equation (2.9) is also written as k/a^2, omitting the factor c^2. This is a common practice in some fields of physics, such as cosmology and particle physics: to use so-called natural units in which $c = 1$. In these units, one can use mass density ρ and energy density ϵ interchangably (since mass and energy are related by $E = mc^2$). Common practice is to omit all factors of c and divide or multiply the result with the factors of c needed to recover the correct units. In the same manner, often the constant $\hbar$ will be set to 1. This thesis will uphold these practices, unless it is insightful to make the c or $\hbar$ dependency explicit, to follow the article that will be discussed in further chapters. That paper also chooses to set the reduced Planck mass, $M_{pl} = \sqrt{\frac{\hbar c}{8\pi G}}$ equal to one. The first few chapters of this thesis will not follow this convention, in an attempt to prevent confusion in explanations, but in the fourth chapter this practice will also be upheld. With these points, the Friedmann equation becomes:

$$H^2 = \frac{M_{pl}^2}{3}\rho - \frac{k}{a^2}. \tag{2.12}$$

2.2 The fluid equation

Before the value of k in the Friedmann equation is discussed, it is important to understand the behaviour of the mass density ρ. The most important equation describing the evolution of ρ in the universe is the *fluid* equation. This equation gives a relationship between the mass density ρ of a medium and its pressure.

For the derivation of the fluid equation, look at an expanding volume V* with unit comoving radius and apply the first law of thermodynamics:

$$\mathrm{d}E + p\mathrm{d}V = T\mathrm{d}S, \tag{2.13}$$

where T denotes the temperature and S the entropy. Assuming the expansion is reversible means the entropy will remain constant, $\mathrm{d}S = 0$, thus simplifying equation (2.13). Since the volume has comoving radius 1, the physical radius will be equal to a. We can then set the energy E equal to, using $E = mc^2$:

$$E = \frac{4\pi}{3} a^3 \rho c^2. \tag{2.14}$$

Then, using the chain rule, the change in energy becomes:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 4\pi a^2 \rho c^2 \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{4\pi}{3} a^3 \frac{\mathrm{d}\rho}{\mathrm{d}t} c^2, \tag{2.15}$$

while the change in volume is equal to:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi a^2 \frac{\mathrm{d}a}{\mathrm{d}t}. \tag{2.16}$$

Plugging these both into equation (2.13) and rearranging terms (using $\mathrm{d}S = 0$) gives the fluid equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0. \tag{2.17}$$

The fluid equation shows that the change in density is dependent on two factors. The first factor is the to be expected decrease in density due to the expansion of the volume: the same amount of mass has to be distributed over a larger volume, resulting in a lower density. This corresponds with the first factor in brackets. The second factor changing the density equates

*Note that the V for Volume is a different V than the one previously used for potential energy.

the loss of energy created by the loss of pressure due to the expansion of the universe. This energy is not lost, due to conservation of energy, but transferred to the gravitational potential energy.

2.3 The acceleration equation

With the Friedmann equation (2.9) and the fluid equation (2.17) a third equation can be derived, the acceleration equation, which describes the acceleration of the scale factor. The derivation can be done in three steps. The first step consists of differentiating the Friedmann equation with respect to time, to obtain:

$$2\frac{\dot{a}}{a}\frac{a\ddot{a}-\dot{a}^2}{a^2} = \frac{8\pi G}{3}\dot{\rho} + 2\frac{kc^2\dot{a}}{a^3}. \tag{2.18}$$

Note that a factor $2\dot{a}/a$ vanishes from both sides. Now, $\dot{\rho}$ can be substituted using the fluid equation. This gives the following:

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2}. \tag{2.19}$$

By adding the Friedmann equation, the following result is obtained:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right). \tag{2.20}$$

This equation shows that if the material has positive pressure, it would decelerate the expansion.

2.4 Curvature of space

In cosmology, one usually assumes the density and pressure are uniquely associated to one another: $p = p(\rho)$. This relationship is known as an *equation of state*, which is different for different types of matter and radiation. Once this relationship is known, a cosmologist can solve the Fluid equation (2.17), and with that the Friedmann equation(2.9), to capture the evolution of the universe. In order to do so, first some assumption about the constant k must be made. In the Newtonian derivation of the Friedmann equation (2.9), the final term in the equation $(-k/a^2)$ is introduced as a measure of the energy per particle. General relativity gives a different interpretation for this final term however: it measures *the curvature of space*.

Again, the topic of General Relativity lies beyond the scope and level of this thesis, but the key concepts of curvature of space will be explained briefly.

As written above, the constant k measures the curvature of space. Although homogeneity and isometry must apply, there are still three possible geometries for the universe. k can be either zero, positive or negative, corresponding with a *flat, elliptic or hyperbolic geometry* respectively.

For most, the flat or Euclidean geometry is easiest to visualise. It is based upon a set of five axioms:

1. A straight line may be drawn between any two points.
2. Any finite straight line may be extended indefinitely.
3. A circle may be drawn with any given point as centre and any given radius.
4. All right angles are equal.
5. If two straight lines in a plane are met by another line, and if the sum of the internal angles on one side is less than two right angles, then the straight lines will meet if extended sufficiently on the side on which the sum of the angles is less than two right angles. [6]

Using these five base axioms, one can prove that the angles of a triangle add up to 180° and that the circumference of a circle of radius r adds up to $2\pi r$ exactly.
If the geometry of the universe were indeed flat, homogeneity would state that the universe must be infinite. If not, there would be an edge, which would contradict the notion that all locations are the same.

For a long time, it was believed that the fifth of Euclid's axioms could be proven from the former four. When in the early nineteenth century people started to accept that this was in fact not the case, the field of non-Euclidean geometry was born, a field that lays an important foundation for Einstein's theory of General Relativity, due to the possibilities of curved space. A common example of non Euclidean geometry is elliptic geometry. A universe with an elliptic geometry is called a *closed universe*. A two dimensional closed universe would be most easy to envision as a great sphere, for example the surface of the Earth. Visualising a three dimensional closed space is nearly impossible.
One can see that homogeneity and isotropy do indeed hold for this type of universe: all points look the same, so do all directions. Contrary to the flat universe discussed before, an elliptic universe does not have an edge

and would be finite.
However, a triangle drawn on a sphere does not have angles that add up to 180°. It is even possible to draw a triangle with three 90° angles. Also, for a given circle with radius r, the circumference is less than $2\pi r$.
Note that when drawing a triangle or circle much smaller than the sphere, the Euclidean laws provide a good approximation: the deviation from these laws becomes very small. Thus, if one draws a triangle on the floor, the angles would add up to 180°, even if the surface of the Earth is a sphere.

The third and final option is that of a hyperbolic geometry, where k is negative. Although less known than an elliptic geometry, it is normally represented by a saddle-like surface. Isotropy and homogeneity are less easy to see for oneself on this type of surface, but can be proven to hold. A universe with this type of geometry is often called an *open universe*. It is in many ways the opposite of a closed one: The angles of a triangle sum up to less than 180° and the circumference of a circle is larger than $2\pi r$. In contrast to to the closed universe, parallel lines never meet. This implies that, just like with the flat universe, the open universe has to be infinite in extent, because any edge would violate homogeneity.

Curvature	Geometry	Angles of triangle	Circumference of circle	Type of universe
$k > 0$	Elliptic	$> 180°$	$c < 2\pi r$	Closed
$k = 0$	Euclidean	$= 180°$	$c = 2\pi r$	Flat
$k < 0$	Hyperbolic	$< 180°$	$c > 2\pi r$	Open

Table 2.1: *Summary of the main characteristics of different types of curvature of space.*

Looking back at the Friedmann equation (2.9), one can see that in order to have a flat geometry ($k = 0$), one requires a specific value of the density. This value is called the *critical density* ρ_c, given by :

$$\rho_c(t) = \frac{3H^2}{8\pi G}. \tag{2.21}$$

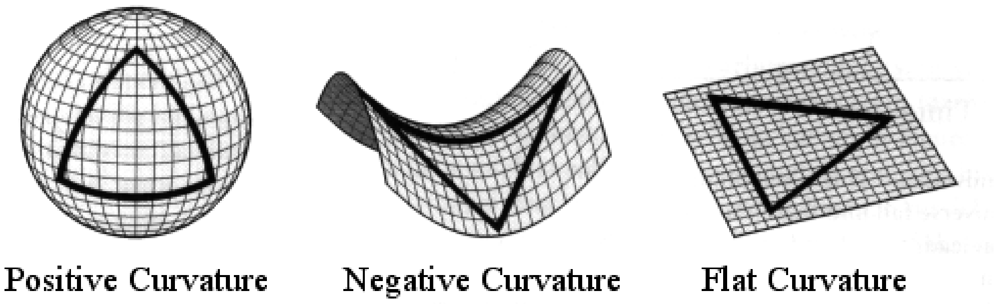

Figure 2.1: *Visualisation of the three possible curvatures of space. Originally posted on [7].*

Although the critical density does not need to be the actual value of the density, since the universe need not be flat, it does provide a natural scale against which to measure any densities. It makes sense to define a *density parameter* Ω:

$$\Omega \equiv \frac{\rho}{\rho_c}. \tag{2.22}$$

With this definition, the Friedmann equation can be rewritten to find:

$$\Omega(t) - 1 = \frac{k}{a(t)^2 H(t)^2}. \tag{2.23}$$

This gives rise to an interesting insight. For $\Omega = 1$, the value of k must be equal to 0. However, since k is a fixed constant, that means the right hand side will remain zero, and thus the value of Ω must be equal to 1 forever. This is known as a *critical density universe*.

2.5 Solutions to the Friedmann equation

It is useful to compare a few different examples of using the fluid and Friedmann equations to solve for H. In order to do so, an equation of state has to be determined. Two possibilities will be discussed: a matter only and a radiation only type universe. For simplicity, a flat geometry will be assumed, corresponding with $k = 0$.

For cosmologists, matter refers to all types of material that exert negligible pressure, so $p = 0$, as opposed to relativistic matter which does exert pressure. A matter-only universe is the textbook example for this topic, as it is the simplest assumption one can make. In fact, once the universe has cooled down after the Big Bang, matter only provides quite a good approximation, since most atoms seldom interact since they are well separated. It is interesting to include a short overview of the thermal history of the universe [8].

- Big Bang Nucleosynthesis (BBN), at $t \sim 10^2$s. The temperature drops below MeV* values. Protons and neutrons combine to form nuclei of light elements (such as hydrogen, helium, lithium, ...). The theoretical predictions on the distribution of elements are in excellent agreement with the observations. There is no direct observational evidence for anything that happened before BBN.
- Matter-Radiation equality, at $t \sim 10^4$. Where at high temperatures radiation dominated the universe, around this time cool matter comes to dominate.
- Recombination, at $t \sim 10^5 yr$, or $T \sim 0.1$ eV. Since the temperature has now dropped below the binding energy, neutral atoms are formed. This increases the free path lenght of photons, turning the universe transparent instead of opaque. Photons from this era are able to reach us unscattered. These photons form the Cosmic Microwave Background (CMB).
- Formation of gravitational bound states/galaxies, at $t \sim 10^9$yr or $T \sim 10^{-3}$ eV.
- Present day, with $T \equiv 2.73\text{K} \sim 10^{-4}$ eV.

For a matter-only universe, the fluid equation (2.17) takes a simple form:

$$\dot{\rho} - 3\frac{\dot{a}}{a}\rho = 0 \tag{2.24}$$

This is a separable differential equation with a simple solution:

$$\rho = \frac{\rho_0 a_0^3}{a^3}, \tag{2.25}$$

where a_0 and ρ_0 are the values of a and ρ at time t_0. Often, $a_0 = 1$ is chosen for t_0 as the current time.

*For distributions of particles, often temperatures are given in eV. It can be converted to Kelvin using $\langle E \rangle = k_B T$. One eV corresponds to about 11,605 Kelvin.

This equation means that the density of the universe falls off along with its volume. No matter can appear, nor can it disappear. Plugging this equation for ρ into the Friedmann equation as in equation (2.12) with $k = 0$ gives:

$$\dot{a}^2 = \frac{8\pi G \rho_0 a_0^3}{3} \frac{1}{a}. \tag{2.26}$$

Again, this is a separable equation, so it can be integrated and solved to find :

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}; \qquad \rho(t) = \frac{\rho_0 a_0^3}{a^3} = \frac{\rho_0 t_0^2}{t^2}. \tag{2.27}$$

Thus, the universe will keep expanding forever. It is however interesting to look at the rate of expansion H:

$$H \equiv \frac{\dot{a}}{a} = \frac{2}{3t}, \tag{2.28}$$

which decreases with time.

This calculation can be repeated analogously for a radiation-only universe. One can prove, using the standard theory of radiation, that the kinetic energy of particle moving at highly relativistic speeds, such as photons, leads to a pressure force equal to $p = \rho c^2/3$. This gives the following form of the fluid equation, very similar to the one solved before:

$$\dot{\rho} - 4\frac{\dot{a}}{a}\rho = 0 \tag{2.29}$$

This equation has a solution similar to the one seen before:

$$\rho = \frac{\rho_0 a_0^4}{a^4}. \tag{2.30}$$

This is different than with matter only and it is an interesting question to see where this proportionality comes from. Just as with 'normal' matter, the density is expected to scale with the volume, providing a factor $\frac{1}{a^3}$. However, for radiation there is another important effect: the wavelength of the light is being stretched by another factor of a. This is because the energy of radiation is proportional to its wavelength ($E = \frac{h}{\lambda}$, and the energy is again proportional to the mass $E = mc^2$.

Again solving the Friedmann equation with the newly found solution for density gives the following final result:

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{1/2}; \qquad \rho(t) = \frac{\rho_0 a_0^4}{a^4} = \frac{\rho_0 t_0^2}{t^2}. \tag{2.31}$$

Here, the universe expands more slowly than in a matter-only universe. The pressure decelerates the expansion, just as the acceleration equation (2.20) predicted.

There are many different possible equations of state. The above computations could be carried out with all kinds of mixtures of radiation and matter, or different states all together. The results will be much more messy, but the procedure will remain the same. The equations as given do provide good approximations for matter and radiation dominated universes, respectively.

2.6 The cosmological constant

By studying distant supernovae different research groups have made convincing measurements of the rate of acceleration of the universe. They found the universe to be accelerating. Neither a matter only universe, radiation only nor a mixture of the two is capable of satisfying this condition, as can be seen directly from the acceleration equation (2.20). However, this simple cosmological model can be extended to account for the acceleration by introducing the *cosmological constant* Λ.

Einstein originally introduced the concept of the cosmological constant in 1917, when first introducing his theory of general relativity. He believed the universe to be static, which his equations did not permit without adding Λ. When however Hubble found the universe to be expanding, he abandoned the concept, calling it 'his greatest blunder'. Around the 1990s, the concept was revisited as a means to explain the acceleration of the universe.

Introducing the cosmological constant adds an extra factor to the Friedmann equation (2.9):

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}, \tag{2.32}$$

where Λ has units $[\text{time}]^{-2}$. Sometimes it is written with an explicit added factor c^2, to transform the units to $[\text{length}]^{-2}$. Although Λ can be either positive or negative, in our present universe it is most often considered to be positive.

The original thought behind adding Λ was to finetune Λ and ρ in such a way that $H(t)$ would be forced equal to 0, thus having a static universe. This would however give a very unstable equilibrium, sensitive to very small perturbations, thus proving very unlikely in practice. As stated before, in current time the constant is more often used in a different context. To better see this, consider the acceleration equation (2.20) with an added term for Λ:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}. \tag{2.33}$$

Although the pressure p and density ρ negatively influence the acceleration, the cosmological constant gives a positive contribution. A large value of Λ would be able to give a positive value for $\ddot{a}$, providing an explanation for the observed acceleration of the universe.

One can define a density parameter for the cosmological constant as:

$$\Omega_\Lambda = \frac{\Lambda}{3H^2}. \tag{2.34}$$

Note that since H varies with time, Ω_Λ is in fact not a constant (while Λ itself is). Using this density parameter, the Friedmann equation as given in equation (2.23) can be rewritten as:

$$\Omega + \Omega_\Lambda - 1 = \frac{k}{a^2H^2}. \tag{2.35}$$

Thus, the condition for a flat universe ($k = 0$) becomes $\Omega + \Omega_\Lambda = 1$, often written as $\Omega_{tot} = 1$.
Λ is often seen as the energy density of 'empty' space. One can interpret Λ as a fluid with density ρ_Λ and pressure p_Λ. By defining:

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}, \tag{2.36}$$

the Friedmann equation (2.32) would transform into:

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{k}{a^2}. \tag{2.37}$$

With this definition of ρ_Λ, one finds $\Omega_\Lambda \equiv \rho_\Lambda / \rho_c$, where ρ_c is the critical density as described before. Note as well that with this definition, ρ_Λ is a constant. Now, revisiting the fluid equation for Λ gives:

$$\dot{\rho}_\Lambda + 3\frac{\dot{a}}{a}\left(\rho_\Lambda + \frac{p - \Lambda}{c^2}\right) = 0. \tag{2.38}$$

Since ρ_Λ is constant, this implies $p_\Lambda = -\rho_\Lambda c^2$. The effective pressure of the cosmological constant is negative. Thus, while the universe is expanding, work is done on the cosmological constant fluid. This allows the constant density in a universe with increasing volume.

Chapter 3

Inflation

In all the topics discussed in the previous chapter, some paradoxes arise. This chapter will describe these and offer the best-established solution to these paradoxes, *inflation*. Although several other possible solutions exist, inflation is by far the most succesful. After a short introduction to the topic, some more in depth aspects necessary will be discussed as well. Much of this chapter again draws from the books *'Introduction to Cosmology'* by Andrew Liddle[4] and *'The Primordial Density Perturbation'* by David Lyth and Andrew Liddle[5]. Besides that, Christopher Hirata [9] and Daniel Baumanns[10] lecture series and the article by Andrew Liddle[11] have been consulted extensively.

3.1 Problems with the Big Bang Theory

One of the problems that arise when discussing the Big Bang is known as the *flatness problem*, which states that the Big Bang Theory is only valid for very specific types of universe. To discuss this problem, the first step is to add absolute values the Friedmann equation in the final form from the previous chapter (2.23), to find:

$$|\Omega(t) - 1| = \frac{|k|}{a^2 H^2}. \tag{3.1}$$

The previous chapter showed that if Ω is equal to 1, it will remain so for all time. Using the examples from last chapter, the behaviour of Ω when it is unequal to zero can be determined. This gives for a matter dominated universe $a^2H^2 \propto t^{-2/3}$ and thus $|\Omega - 1| \propto t^{2/3}$, while for a radiation dominated universe one finds $a^2H^2 \propto t^{-1}$ and thus $|\Omega - 1| \propto t$.

Note that in both cases, the absolute value of the difference between Ω and 1 increases over time. The fixed point $\Omega = 1$ is unstable. Although these solutions were derived under the assumption $k = 0$ and lose validity when the curvature term becomes more influential, they can help to illustrate the flatness problem. Take this (quite conservative) estimate of the value of Ω present day ($t_0 \simeq 4 \times 10^{17}$s) : $0.5 \leq \Omega \leq 1.5$. To reach this value of Ω today, one can, using the approximations above, find limits for the value of Ω around the time of nucleosynthesis* ($t \simeq 1$s):

$$|\Omega - 1| \leq 1 \cdot 10^{-18}.$$

A solution to this problem is to assume the universe does indeed have a flat geometry and Ω is equal to 1. This implies that the Big Bang Theory would only be valid for a flat universe. However, besides this problem, there is no other reason to prefer this choice of geometry. One would prefer the theory to be compatible with other types of geometry.

Another perhaps even more important problem with the Big Bang model is the *horizon problem*. This problem arises from our observations of the Cosmic Microwave Background (CMB). A very prominent fact of the microwave background is that the perturbations are very small; the radiation field as seen from Earth is almost completely isotropic at the same temperature of 2.725 K†.
The best explanation for two regions being at the same temperature is for them to (at some point) have been in thermal equilibrium: able to exchange heat, interact and move towards equilibrium. However, this can not be the case. If we take two opposite points in the sky, the CMB radiation will have been travelling towards us since shortly after the dawn of the universe. Since it has just reached us, it is impossible to have interacted with the area on the other side, the light of which also just got to us. The distance light could have travelled before the CMB was emitted is also much smaller than the current scale of horizon. One can calculate that all regions separated more than two degrees in the night sky are causally disconnected. The Big Bang Theory does not offer any explanation for why the universe appears this homogeneous.

*The period in which the first atoms were formed out of elementary particles

†The actual measured radiation field forms a dipole. However, after compensating the measurements for the movement of our galaxy, the result becomes nearly isotropic

However, irregularities in the CMB might pose an even larger problem. Results from the COBE satellite show irregularities spanning all accesible angular scales, from a few degrees upwards. These perturbations are on scales too large to have been created between the Big Bang and the time the CMB was emitted. Hence, these perturbations must have been part of the initial conditions. Here, the Big Bang Theory does not allow a predictive theory for the origin of structure. Although this is not a requirement for a theory, it would be a disappointment if the Big Bang theory failed to deliver that. Here, inflation will be presented as a solution. After a short introduction and explanation, both problems will be revisited and solved using inflation.

3.2 Inflation

Inflation is the period in the evolution of the universe in which the scale factor was accelerating: $\ddot{a} > 0$. This would only be for a very short period at the earliest times in the evolution of the universe, but does already have rather large implications. For the acceleration equation (2.20), it implies $\rho c^2 + 3p < 0$. With density always being positive, this gives the following constraint for p:

$$p < -\frac{\rho c^2}{3}. \tag{3.2}$$

Although a negative pressure might sound problematic, the field of particle physics has come up with several explanations on how this might have come to be.[12]

However, inflation is most often looked at from a universe with a cosmological constant Λ, as seen at the end of the previous chapter. There, the cosmological constant was seen as a fluid with $p = -\rho c^2$, satisfying the constraint above. With Λ, the Friedmann equation took the following form:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \qquad \text{(2.32, revisited)}$$

This equation is greatly simplified when considering the expansion of the universe, since then the two first terms on the right hand side quickly drop off, while the last term remains constant. This would imply H is a constant. Using the definition of H, one finds:

$$\dot{a} = Ha. \tag{3.3}$$

This can be solved to find:

$$a(t) = \exp(Ht). \tag{3.4}$$

Thus, if the cosmological constant is included, the expansion of the universe becomes exponential, as opposed to polynomial growth in the models discussed before.

3.3 The problems revised

With the new theory of inflation, the problems introduced in the paragraphs before can be revisited. First, recall the flatness problem. The original problem was that in the right hand side of equation (3.1), aH always decreases with time and thus forces Ω towards the value 1 in the distant past. This changes when one looks at inflation, since:

$$\ddot{a} > 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\dot{a}) > 0 \qquad \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}(aH) > 0. \tag{3.5}$$

Hence the condition for inflation actually forces the value of Ω_{tot} towards 1 over time. Considering the case of perfect exponential inflation, as seen above, the effect is especially visible:

$$|\Omega_{tot}(t) - 1| \propto \exp(-2Ht). \tag{3.6}$$

If the period of inflation is long enough, it can even drive Ω so close to 1, that all time since in which Ω_{tot} drifted away from 1 is insufficient to have significant impact on our current day observations. Thus, if it lasts long enough, inflation predicts a universe that is at least very close to spatial flatness. Current observations seem to support this condition.

The horizon problem can also be revisited with the new theory of inflation. We can make a rough estimate of the size of the observable part of the universe, the Hubble length, by taking cH^{-1}, since H^{-1} gives a very rough estimate of the age of the universe. While the universe itself rapidly expands, this length stays about equal. So, a small bit of the universe can during inflation spread out become much larger than the observable universe. Thus, the microwaves coming from opposite ends of the sky can actually originate from the same thermal equilibrium.

It is useful to mention that inflation lasted up until about 10^{-33} seconds after the Big Bang. Because timescales prove difficult, cosmologists prefer measuring time in efolds: the interval in which the universe has expanded a factor e. The number of efolds is often indicated with the letter N.

3.4 Scalar Field theory

Generally, it is supposed that the pressure and density of the cosmological constant are dominated by scalar fields during inflation. Here at first *single-field inflation* will be discussed, where one assumes only one field, ϕ varies. ϕ is more generally called the inflaton. In order to be able to further analyse the inflaton, some aspects of field theory need to be discussed. The following section will heavily draw from [5], [13] and [14]. Some knowledge of classical field theory will be assumed.

When switching to Lagrangian field theory, one replaces the original Lagrangian L* as a function of generalised coordinates by the so called *Lagrangian density* $\mathcal{L}$, a function of the fields in the system, their derivatives and space and time coordinates. $\mathcal{L}$ is Lorentz invariant and has dimensions $[\text{energy}]^4$†. Now, the action S is given as:

$$S = \int \mathrm{d}^4x \mathcal{L}.^{\ddagger} \tag{3.7}$$

For now, a flat spacetime is assumed, but when considering general relativity one would have to include a factor $\sqrt{-g}$ (g being the determinant of the metric $g_{\mu\nu}$ to compensate for the curvature of spacetime).

There are a few restraints on the form of the Lagrangian and the dependence on the coordinates and their derivatives. Besides Lorentz invariance, one would like the theory to have a particle interpretation after quantization. Furthermore, one would like the system to have equations that

*The Lagrangian is a function determining the evolution of a given system. It is most often defined as the difference between the kinetic and potential energies, $L = T - V$.

†This only holds when the convention $c = \hbar = 1$ is used.

‡d^4 because we are working in 4-dimensional spacetime, instead of 'normal' 3-dimensional space.

can be solved with physically reasonable boudary conditions in the classical limit. For the single scalar field ϕ, the simplest form of the Lagrangian density satisfying all these requirements is:

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\nu} - V(\phi). \tag{3.8}$$

The function $V(\phi)$ here is in fact a potential density, but is generally called the scalar field potential, and the first term is called the kinetic term, referring to the classical analogue.

For this theory, the action principle, stating that $\delta S = 0$, meaning that the change in S under small perturbations of the coordinates, δS vanishes, still holds. Thus one has:

$$\delta S = \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta(\partial_{\mu}\phi) \right) = 0. \tag{3.9}$$

This equation can be expanded using integration by parts to find:

$$\delta S = \int \mathrm{d}^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^{\mu}}\left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \right] \right)\delta\phi + \int \mathrm{d}^4x \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}\delta\phi \right) = 0. \tag{3.10}$$

The last term in this equation is just a number, which, according to the action principle, has to vanish. This can be ensured by posing restrictions on ϕ, demanding that it goes to zero 'sufficiently fast' at the boundaries (infinity). The remaining first term can then be written as:

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)} \right) = 0. \tag{3.11}$$

This generalisation of the Euler-Lagrange equations is more generally known as the *field equation*. Plugging in $\mathcal{L}$ from equation (3.8), this gives:

$$\ddot{\phi} - \nabla^2\phi + V'(\phi) = 0. \tag{3.12}$$

Here, the prime is used to indicate a derivative to ϕ. If the field is homogeneous, $\nabla\phi$ is equal to zero and the equation becomes:

$$\ddot{\phi} + V'(\phi) = 0. \tag{3.13}$$

For a homogeneous field, the density and pressure are given by the following equations ([5], equation 13.34):

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi), \qquad P = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{3.14}$$

The first term in both these equations can be considered some sort of potential energy, while the second can be seen as a kinetic term. As mentioned before, this is when considering a flat spacetime. The action principle will still hold in curved spacetime, but the field equation would change slightly. With a Robertson-Walker metric* for example, the scalar field metric becomes:

$$\ddot{\phi} - a^{-2}\nabla^2\phi + 3H\dot{\phi} + V'(\phi) = 0. \tag{3.15}$$

Again, for a homogeneous field, this becomes:

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{3.16}$$

Comparing this to a mechanical analogy, the term $3H\dot{\phi}$ would correspond to friction. It is therefore known as the *Hubble friction*. Note that one would find the same equation by plugging the equations for p and ρ given in (3.14) into the fluid equation (2.17). Using just the equation given for ρ (3.14), the Friedmann equation can be rewritten as:

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \tag{3.17}$$

Here, the curvature term has been ignored ($k = 0$), since when inflation starts, this quickly becomes negligible. Differentiating this equation and using the field equation (3.16) to substitute for some terms, we find the following equation, often known as the *second Friedmann equation*:

$$2\dot{H} = -8\pi G\dot{\phi}^2. \tag{3.18}$$

For inflation to be possible means that the potential part dominates over the kinetic part. After all, we have:

$$\ddot{a} > 0 \iff p < -\frac{\rho}{3} \iff \dot{\phi}^2 < V(\phi). \tag{3.19}$$

In order for the potential to dominate, it should be 'flat enough', so that the scalar field would be expected to 'roll slowly' [11]. This lead to the

*$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right] = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right)$, the second equation only holding if $k = 0$.

strategy most commonly used for solving the above equations, the *slow-roll approximation* (SRA). Summarized, the SRA assumes that inflation is almost exponential:

$$\epsilon_H = \frac{|\dot{H}|}{H^2} \ll 1. \tag{3.20}$$

Here, ϵ_H is one form of the *first slow-roll parameter*, which will be discussed in a bit in its other form. This assumption of near exponentiality implies that a term can be neglected in each equation, leaving a much simpler set of equations. By ignoring the $\dot{\phi}$ term in the Friedmann equation (3.17) and the $\ddot{\phi}$ term in the field equation; (3.16), these equations become, respectively

$$H^2 \simeq \frac{8\pi G}{3} V, \tag{3.21}$$

$$3H\dot{\phi} \simeq -V'. \tag{3.22}$$

To ensure these approximations are allowed, the *slow-roll parameters* can be defined:

$$\epsilon_V(\phi) = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 \quad ; \quad \eta_V(\phi) = \frac{1}{8\pi G}\frac{V''}{V}. \tag{3.23}$$

The first slow-roll parameter, ϵ, measures the slope of the potential, while the second parameter, η, measures curvature. With these definitions, necessary conditions for the SRA to hold are:

$$\epsilon_V \ll 1 \quad ; \quad |\eta_V| \ll 1. \tag{3.24}$$

These conditions are necessary, but not yet sufficient for the approximation to be allowed. Liddle et al. offer a more elaborate version of the SRA, based on the Hamilton-Jacobi formulation of inflation, which is both sufficient and necessary [15]. In this paper it is also proven that the general solution to the equations has an attractor property. This property allows for the elimination of one parameter, necessary for the use of the SRA, since after using the approximation the order of the system of equations is reduced by one.

3.5 Multi-field inflation

So far, only one field has been allowed to vary. Although single-field inflation gives some promising results, it is found to be difficult to incorporate in an UV complete theory such as string theory [3]. It is therefore interesting to study the case where not one, but multiple fields are involved. [5] and [16] have been extensively consulted for this section.

Let $\vec{\phi}$ be a vector of n components (n finite). Then, just as in the single field case, the first and second Friedmann equations (3.17) and (3.18) can be written as:

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}|\dot{\vec{\phi}}^2| + V\left(\phi\right)\right) \tag{3.25}$$

and:

$$2\dot{H} = -8\pi G|\dot{\vec{\phi}}^2|. \tag{3.26}$$

For the multi-field case as well, the SRA can be applied. Again the assumption is made that the inflation is nearly exponential:

$$\epsilon_H = \frac{-\dot{H}}{H^2} \ll 1, \quad \eta_H = \frac{\dot{\epsilon_H}}{H\epsilon_H} \ll 1. \tag{3.27}$$

Now, the trick is to again use an approximation to form a set of equations. For the rest of this section, we will focus on two field inflation $n = 2$.

To find the equations, one first has to use the Frenet-Serret formalism to describe the turning trajectories:

$$\vec{T} = \frac{\dot{\vec{\phi}}}{|\dot{\vec{\phi}}|}, \quad \dot{\vec{T}} = -\omega\vec{N}, \quad \dot{\vec{N}} = \omega\vec{T}, \tag{3.28}$$

Here $\vec{T}$ is a unit vector tangent to the trajectory, $\vec{N}$ is a unit vector normal to the trajectory and ω is the trajectory's rate of turning. $\vec{T}$ and $\vec{N}$ are orthonormal, ie $\vec{T}\cdot\vec{T} = \vec{N}\cdot\vec{N} = 1$, $\vec{T}\cdot\vec{N} = 0$.

The Euler-Lagrange equations for $\vec{\phi}$ are formed by equations (3.25), (3.26) and the following equation:

$$\ddot{\vec{\phi}} + 3H\dot{\vec{\phi}} + \nabla V = 0. \tag{3.29}$$

Now, defining $V_T = \vec{T} \cdot \nabla V$ and $V_N = \vec{N} \cdot \nabla V$, one can project equation (3.29) onto the vectors tangent and normal to the trajectory. This will lead to two different equations, which will function as the two SRA equations from the previous section. For the first equation, we will project onto the tangent vector:

$$0 = \vec{T} \left[\ddot{\vec{\phi}} + 3H\dot{\vec{\phi}} + \nabla V \right] = \ddot{\varphi} + 3H\dot{\varphi} + V_T, \tag{3.30}$$

where φ is defined as $\varphi = \int |\dot{\vec{\phi}}| \mathrm{d}t$. Note that we consider the displacement along the trajectory, so $\dot{\varphi} = |\dot{\vec{\phi}}| = \vec{T} \cdot \dot{\vec{\phi}}$, and that using integration by parts, one finds (since $\dot{\vec{T}} \cdot \dot{\vec{\phi}} = 0$):

$$\vec{T} \cdot \ddot{\vec{\phi}} = \left(\vec{T} \cdot \dot{\vec{\phi}} \right)^{\bullet} - \dot{\vec{T}} \cdot \dot{\vec{\phi}} = \ddot{\varphi}. \tag{3.31}$$

So, equation (3.30) gives the first equation:

$$\ddot{\varphi} + 3H\dot{\varphi} + V_T = 0. \tag{3.32}$$

The second equation is derived by projecting on the vector normal to the trajectory:

$$0 = \vec{N} \cdot \left[\ddot{\vec{\phi}} + 3H\dot{\vec{\phi}} + \nabla V \right]. \tag{3.33}$$

Here, we again use integration by parts to find:

$$\vec{N} \cdot \ddot{\vec{\phi}} = \left(\vec{N} \cdot \dot{\vec{\phi}} \right)^{\bullet} - \dot{\vec{N}} \cdot \dot{\vec{\phi}} = -\omega \vec{T} \cdot \dot{\vec{\phi}} = -\omega \dot{\varphi}. \tag{3.34}$$

This gives, using the fact that $\vec{N}\dot{\vec{\phi}} = 0$:

$$\omega \dot{\varphi} = V_N. \tag{3.35}$$

This equation, together with equation (3.32), forms the two-field equivalent of equations (3.21) and (3.22).

Chapter 4

Mathematical solutions

With the physical foundations laid out, we can dive into the mathematical reasoning. This chapter will be largely based on Achúcarro et al.[1]. Remember that from this chapter on, the value of the reduced Planck mass $M_{pl} = \sqrt{\frac{\hbar c}{8\pi G}}$ will be set equal to 1.

The original paper investigates potentials of the form:

$$V = 3H(\theta)^2 - 2\frac{H_\theta^2}{f(\rho)}, \tag{4.1}$$

where $H = H(\theta)$ is a function of θ alone and $f = f(\rho) > 0$ is a strictly positive function of ρ. We can now use the Euler-Lagrange equations of motion as seen in (3.11), compensated for the curvature of space by adding a factor $\sqrt{-g}$ for the metric g[17], to find the following exact solutions:

$$\rho = \rho_0, \;\; \dot{\theta} = -2\frac{H_\theta}{f}. \tag{4.2}$$

This can eb related back to the previous chapter. We now study trajectories $\phi = (\rho, \theta)$ with $\rho = \rho_0$ constant and $\dot{\theta} = \omega$. Thus, we have the following identities, seen in section (3.5):

$$\nabla = \left(\frac{\partial}{\partial \rho}, \frac{1}{\sqrt{f}} \frac{\partial}{\partial \theta} \right), \tag{4.3}$$

$$V_T = \frac{V_\theta}{\sqrt{f(\rho)}} \tag{4.4}$$

$$\dot{\varphi} = \sqrt{f(\rho)} \dot{\theta}. \tag{4.5}$$

The action used to generate the equations of motion takes the following form:

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[(\partial_\mu \rho) (\partial^\mu \rho) + f(\rho) \partial_\mu \theta \partial^\mu \theta - V[\rho \cdot \theta] \right]. \tag{4.6}$$

The form of the chosen potential is derived from the application of the Hamilton-Jacobi formalism. The thought here is to express the input in terms of the Hubble parameter instead of the potential. For the single field case this can be derived by replacing $H(t)$ with $H(\phi)$ in the second Friedmann equation (3.18), which gives:

$$\dot{H} = \dot{\phi} H_\phi = -\frac{\dot{\phi}^2}{2} \tag{4.7}$$

which, assuming $\dot{\phi} \neq 0$, can be simplified to find:

$$-2H_\phi = \dot{\phi}. \tag{4.8}$$

With this expression, the first Friedmann equation (3.17) transforms into:

$$V = 3H^2 - 2H_\phi^2 \tag{4.9}$$

Note that all functions here are explicitly dependent on ϕ. This equation allows one to quickly find the potential if the Hubble parameter is known.

The multi-field case is very similar, where the potential is now expressed as a function of ϕ^α. Repeating the same steps gives:

$$\dot{H} = \dot{\phi}^\alpha H_\alpha = -\frac{\dot{\phi}^a \dot{\phi}^b G_{ab}}{2} \longrightarrow H_\alpha = -\frac{G_{ab} \dot{\phi}^b}{2}, \tag{4.10}$$

This is not the most general solution, since an extra term normal to H could also be included, but this term will be neglected. Then, this equality can be used to transform the first Friedmann equation into:

$$3H^2 = V + 2H^\alpha H_\alpha. \tag{4.11}$$

The authors wanted to achieve shift-symmetric orbital inflation. To achieve this, it is important that the inflation trajectory is at any radius along the isometry direction. When looking at a field space (θ, ρ) with metric $G_{ab} = \text{diag}\{f(\rho), 1\}$, this means that for any value of ρ, the inflation should move in the θ direction. In the equations, this means that $H^\alpha H_\alpha = \frac{H_\theta^2}{f(\rho)}$. This is how the final form of the potential (4.1) is constructed.

4.1 Proof of stability

Achúcarro et al. first study the specific choices of $H = \sqrt{\frac{1}{6}} m\theta$ and $f = \rho^2$, since for these specific choices, the potential and solutions take the following form:

$$V = \frac{1}{2} m^2 \left(\theta^2 - \frac{2}{3\rho^2} \right), \quad \rho = \rho_0, \quad \dot{\theta} = \pm \sqrt{\frac{2}{3}} \frac{m}{\rho_0^2}. \tag{4.12}$$

These solutions correspond to trajectories with a set radius ρ_0 along the θ direction, as shown in Fig. 4.1.

Achúcarro et al. proved *neutral stability* for the specific solutions for the model in (4.12), and for the more general model 4.1 for functions H of the form $H \propto \theta^n$. This proof deserves some additional remarks and observations, thus it will be replicated over the next few pages. *Neutral stability* implies that small perturbations orthogonal to a given orbital trajectory will shift the system to a trajectory with a different radius, where it will remain, i.e. $\dot{\rho} = 0$.

The simplest way to prove this would be to look at the eigenvalues corresponding to these perturbations. However, these all turn out to be equal to zero, providing us with no new information on the stability.

One can make the following observation about the potential where H is linear in θ. Indeed, for $H \sim \theta$, we find that the potential in (4.1) meets the following relation:

$$\theta V_\theta - 2\frac{f}{f_\rho} V_\rho = 2V \tag{4.13}$$

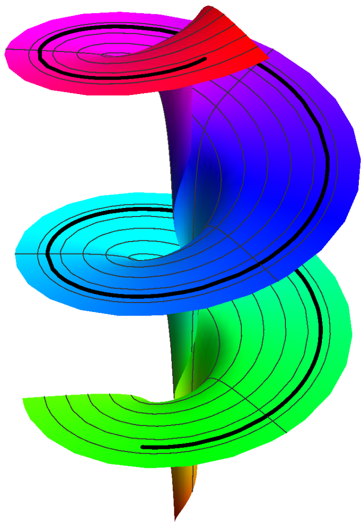

Figure 4.1: *Visualisation of the model potential $V(\rho,\theta)$ given in equation 4.1, with the black line depicting an example trajectory. Originally posted in [1].*

The first term on the left hand size of this equation is solely dependent on θ, while the second term solely depends on ρ. Thus, the potential splits, as can be seen in equation (4.12). This observation was cause for the following transformation in coordinates:

$$x(\theta,\rho,\theta',\rho') = \frac{fH}{H_\theta}\theta' - 2\frac{f}{f_\rho}\rho' + 2, \tag{4.14}$$

$$y(\theta,\rho,\theta',\rho') = \frac{fH}{H_\theta}\theta' + 2, \tag{4.15}$$

$$z(\theta,\rho,\theta',\rho') = \frac{fH^2}{H_\theta^2} - 2/3. \tag{4.16}$$

Here, a prime was used to denote a derivative with respect to the number of efolds, $(..)' = \frac{d}{dN}(..)$ as explained in the previous chapter (just before section 3.4). For mathematical purposes, this has no significant impact and can be regarded as the more 'common' derivative to time t.

Proving neutral stability of the system now comes down to showing the existence of an attractor at $(x,y) = (0,0)$ [1, Appendix B].

In order to prove anything about stability, one first has to rewrite the system in terms of the new variables. The result is as follows:

$$x' + (3-\epsilon)x + \left(2\left(\frac{f}{f_\rho}\right)_\rho - g(\theta)\right)(\rho')^2$$
$$+ \frac{2(z+2/3)}{z} g(\theta)(\epsilon - \epsilon_0) = 0, \quad (4.17)$$
$$y' + (3-\epsilon)y + \frac{2}{z}\left(-\frac{1}{3}(\rho')^2 - \frac{1}{2}y^2 + 2y\right)$$
$$- g(\theta)(\rho')^2 + \frac{2(z+2/3)}{z} g(\theta)(\epsilon - \epsilon_0) = 0, \quad (4.18)$$
$$z' = 2(y-2)(1-g(\theta)) + \left(\frac{f_\rho}{f}\right)^2 \frac{y-x}{2}\left(z + \frac{2}{3}\right), \quad (4.19)$$
$$\rho' = \frac{f_\rho}{f}\frac{y-x}{2}, \quad (4.20)$$
$$\epsilon = \frac{1}{2}\frac{(y-2)^2}{z+2/3} + \frac{f_\rho^2}{f^2}\frac{(x-y)^2}{8}. \quad (4.21)$$

Here we have introduced two other new variables:

$$\epsilon_0 = \frac{2}{z+2/3} \quad \text{and} \quad g(\theta) \equiv \frac{H H_{\theta\theta}}{H_\theta^2}. \quad (4.22)$$

Remember that we were looking to establish the stability of the plane $(x,y) = (0,0)$. This amounts to the point $(x,y,z',\rho') = (0,0,-4(1-g(\theta)),0)$, found by substituting x and y in the equations above. To study this system, we linearly perturb the system of equations to obtain:

$$\delta x' + \left(3 - \frac{2(z+2/3)}{z}\right)\delta x - \frac{4g(\theta)}{z}\delta y = 0, \quad (4.23)$$
$$\delta y' + \left(3 - \frac{2(z+2/3)}{z} + \frac{4(1-g(\theta))}{z}\right)\delta y = 0, \quad (4.24)$$
$$\delta z' = 2(1-g(\theta))\delta y + \left(\frac{f_\rho}{f}\right)^2 \frac{\delta y - \delta x}{2}\left(z + \frac{2}{3}\right), \quad (4.25)$$
$$\delta\rho' = \frac{f_\rho}{f}\frac{\delta y - \delta x}{2}. \quad (4.26)$$

Now, Achúcarro et al. note that 'this linearized system is very simple for any $g(\theta)$. They find that for $H \propto \theta^n$, $g(\theta)$ reduces to a constant, $\frac{n-1}{n}$,

allowing them to solve the system explicitly. Substituting this constant and $(y, \rho') = (0, 0)$ into equation (4.19), gives $z' = -\frac{4}{n}$.
To find a solution for δy one has to integrate the δy equation 4.24, using $z = z_0 - \frac{4}{n}N$. The resulting expression for δy can be used to solve for δx to find the following set of equations:

$$\delta x = \delta x_0 \left(\frac{2+3z_0}{2+3z}\right)^{n/2} e^{-3N} + \delta y_0 \frac{4(n-1)N}{n} \left(\frac{2+3z_0}{2+3z}\right)^{n/2} e^{-3N}, \tag{4.27}$$

$$\delta y = \delta y_0 \frac{z}{z_0} \left(\frac{2+3z_0}{2+3z}\right)^{n/2} e^{-3N}. \tag{4.28}$$

From the negative exponent e^{-3N} we can conclude that all perturbations will over time fade out. For this specific choice of H, one can thus conclude that $(x, y) = (0, 0)$ is indeed an attracting solution, and thus that the system is neutrally stable.

Since $g(\theta)$ contains both a function and its derivative, a natural choice for H would be an exponential. Indeed, note that for the option $H \propto e^{c\theta}$, the function $g(\theta)$ also becomes equal to a constant, 1 to be more specific. One can then repeat the integration we saw before. This time, since $g(\theta) = 1$ and $(y, \rho') = (0, 0)$, equation (4.19) becomes equal to $z' = 0$. Thus, integrating the δy equation with $z = z_0$ gives the following solution:

$$\delta y = \delta y_0 \exp -\frac{9z_0}{3z_0 + 2} \tag{4.29}$$

This solution can in the same way be used to obtain a similar expression for δx.
Note that this solution shows (especially for z_0 large) similar behaviour. This choice of H will also lead to neutrally stable solutions. More generally, we can solve the equation $g(\theta) = \frac{HH_{\theta\theta}}{H_\theta^2} =$ constant, to find an entire class of functions for which we can explicitly solve the system and prove stability (albeit with slightly different values for z). The second order differential equation $HH_{\theta\theta} = cH_\theta^2$ can be solved by setting $v = H_\theta$ and using $H_{\theta\theta} = \frac{dv}{d\theta} = \frac{dv}{dH}\frac{dH}{d\theta} = v\frac{dv}{dH}$. This gives a differential equation we can solve to find $v = kH^c$, with k a free variable. This can again be solved to find:

$$H = \begin{cases} \alpha e^{\beta\theta} & c = 1 \\ \sqrt[1-c]{(c-1)(\alpha - \beta\theta)} & c \neq 1 \end{cases}, \tag{4.30}$$

where α and β are free variables. This gives a class of functions H for which the original system is neutrally stable. Checking $c = \frac{n-1}{n}$ indeed gives the solution $H \propto \theta^n$.

We can look at our system of equations in a more general sense. We can rewrite the system of linearized equations 4.23 and 4.24 in the following manner:

$$\delta x' + \xi(z)\delta x + \nu(\theta, z)\delta y = 0 \tag{4.31}$$

$$\delta y' + \mu(\theta, z)\delta y = 0. \tag{4.32}$$

We can now see that the y variable depends on the behaviour of the function μ. It would be an interesting subject of study to prove stability for functions μ with $\lim_{N\to\infty} \mu < 0$. Similarly, we need $\lim_{N\to\infty} \xi < 0$ and $\lim_{N\to\infty} \nu < 0$. Do note that the physical relevance of this result is limited, because inflation might have ended before the limits become negative.

4.2 Slow roll analysis

It is also insightful to look at the problem in light of the Slow Roll Approximation (SRA), as presented by Achúcarro et al.[1] as well. One can calculate the slow roll parameters:

$$\epsilon = \frac{2H_\theta^2}{fH^2}, \;\; \eta = -\frac{4H_{\theta\theta}}{fH} + \frac{4}{f}\left(\frac{H_\theta}{H}\right)^2. \tag{4.33}$$

Note that the function $g(\theta)$ (4.22) can also be expressed in terms of these parameters:

$$g(\theta) = (2\epsilon - \eta)\sqrt{\frac{f}{8\epsilon}}. \tag{4.34}$$

If the slow-roll approximation and the condition $\eta \ll \sqrt{\epsilon}$ both hold, the function $g(\theta)$ becomes very small. Then, the linearized perturbation equations (4.23-4.26) can be written in the following form:

$$\delta x' + (3-\epsilon)\delta x - \frac{2\epsilon - \eta}{1-\epsilon/3}\sqrt{\frac{\epsilon f}{2}}\delta y = 0, \tag{4.35}$$

$$\delta y' + \left(3 - \epsilon + \frac{1}{1-\epsilon/3}\left(2\epsilon - (2\epsilon - \eta)\right)\sqrt{\frac{\epsilon f}{2}}\right)\right)\delta y = 0, \tag{4.36}$$

$$\delta z' = 2\left(1 - (2\epsilon - \eta)\sqrt{\frac{f}{8\epsilon}}\right)\delta y + \left(\frac{f_\rho}{f}\right)^2 \frac{\delta y - \delta x}{\epsilon}, \tag{4.37}$$

$$\delta \rho' = \frac{f_\rho}{f}\frac{\delta y - \delta x}{2}. \tag{4.38}$$

If the SRA ($\epsilon \ll 1, \eta \ll 1$) is upheld, these equations simplify greatly. The behaviour of the solutions then comes down to:

$$\delta x = \delta x_0 \exp -3N, \tag{4.39}$$

$$\delta y = \delta y_0 \exp -3N, \tag{4.40}$$

which is exponentially decaying. Thus, in the case of slow-roll inflation, the system is indeed neutrally stable.

This chapter has explicitly proven stability for a certain category of the models, namely models with H of the form

$$H = \begin{cases} \alpha e^{\beta\theta} & c = 1 \\ \sqrt[1-c]{(c-1)(\alpha - \beta t} & c \neq 1 \end{cases}. \tag{4.30}$$

Besides that, models have been found to be stable if they fulfill the requirements described below equation (4.31-4.32). Finally, stability has been found if the slow roll approximation holds.
As has been noted in [1] as well, stability is expected to hold in general. Given the promising predictions generated by this class of models, it would indeed be worthwhile to try to prove this general stability.

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