Multi-Component Oscillons

Fabio van Dissel[1]

[1] *Institute Lorentz of Theoretical Physics, Leiden University, 2333 CA Leiden, The Netherlands*

(Dated: July 20, 2020)

Oscillons, localized oscillating configurations in nonlinear field theories, have been known to exist since the mid 1990's. Since then a lot of research has been done to understand these exotic solutions. They can emerge under rather general conditions and have been found to exist in well-motivated physical models. Although they have been studied extensively in single-field models, not a lot is known about them in theories where fields interact. In this thesis I try to gain insights into the complicated questions surrounding multi-field oscillons. In the first chapters I start by reviewing what oscillons are in the context of single-field models. I also show why it is generally expected that oscillons might have an impact on early Universe cosmology. In the last chapters I tackle multi-field oscillons in the context of two coupled scalar fields. I manage to find stable oscillons in a model with a specific "exchange" symmetry and find a criterion to assess their stability by extending the Vakhitov-Kolokolov criterion. Oscillons in this system can both exist in in-phase and out-of-phase configurations, highlighting an interesting characteristic of oscillons in multi-field theories. Finally, I analyse oscillons in more general models of scalar fields, showing the influence of a mass mismatch on the oscillon solution and discussing its influence on stability. I conclude with some expected differences between symmetric and asymmetric couplings between the fields that need to be tested in the future.

CONTENTS

I. INTRODUCTION

What do fundamental particles, atoms and molecules have in common? All of them represent extremely stable configurations containing high concentrations of localized energy. It is therefore obvious that localization of energy is an all-encompassing concept that is fundamental to understanding the natural world. In fact, in the last 50 years, physicists and mathematicians alike have found a plethora of nonlinear field theories capable of supporting a wide variety of localized structures. The family of such structures that is expected to be most relevant to physics are those that have a stable character. Intuitively this simply means that, once these structures are formed, they will continue to exist for long periods of time. Only when this is the case can we expect that they have a role to play in cosmic history.

Although many of such configurations exist, their characteristics vary quite a bit. In some cases they are completely static, like the Sine-Gordon solitons; while in other cases they oscillate in time, like Q-balls. They can be either stable because the nonlinear potential introduces some sort of conserved charge in the system, or because of specific interactions between nonlinearities in the potential and dispersive effects. In this thesis I will be investigating those objects that are called oscillons. These are oscillating (hence the name) localized structures that can exist in nonlinear scalar field theories if the potential satisfies certain conditions. Since a large plethora of stable and static configurations have been discovered in nonlinear field theories it might now be wise to spend some time outlining the properties of oscillons, what makes them different from other localized structures, and what their relevance might be in the physical world.

A. What are oscillons?

Oscillons are part of a large family of soliton-like structures. Solitons in nonlinear field theories refer to localized wave-packets that retain their spatial form while propagating through space. However, the term is effectively used to signify any configuration that is stable for very long periods of time. The stability of these solutions are always due to nonlinearities in the theory, since a linear theory will in general force rapid dispersion. So where exactly among all these solutions lies the place of the oscillon? In general, solitons can be classified according two characteristics. They are either completely static (ignoring propagation) or oscillate during their lifetimes. Furthermore, their stability is either due to some conserved charge or to interactions between nonlinearities and dispersive effects. Oscillons are oscillating (hence the name) and have no known conserved charges.

Examples of solutions falling into other categorizations are Q-balls and Sine-Gordon solitons [1, 2], although a lot more have been found. Q-balls are also oscillating but have a conserved charge that causes them to be stable. The same can be said of Sine-Gordon soliton kinks although they are also completely static. To see how a conserved charge can lead to stable configurations let's consider a typical Q-ball configuration. Starting from a potential that has $U(1)$ symmetry

$$\mathcal{L} = |\partial_\mu \phi|^2 - U(|\phi|) \tag{1}$$

The Q-ball is then defined as a region where the field is not equal to the global minimum of $V(|\phi|)$ (or vacuum) and oscillating with frequency ω: $\phi = \Phi_0 e^{i\omega t}$. The energy within the region is then simply

$$E = (\omega^2 \Phi_0^2 + U(\Phi_0))V \tag{2}$$

Where V is the volume of the region. Now, because of the $U(1)$ symmetry of the system there is a conserved charge in the region given by $Q = \omega \Phi_0^2 V$. Plugging this back in and minimizing the energy with respect to the volume we obtain a new expression for E

$$E = Q\sqrt{\frac{2U(\Phi_0)}{\Phi_0^2}} \tag{3}$$

It becomes clear that, since Q is conserved the Q-ball configuration is stable if $\frac{2U(\Phi_0)}{\Phi_0^2}$ is minimized by $\Phi_0 \neq \phi_{vac}$, since any decay would then be energetically unfavorable. Clearly, conserved charges can cause structures in nature to become stable in this way. What's interesting about oscillons is that they exist solely due to non-trivial interactions between nonlinearities and dispersion. They do not require any additional symmetries in order to exist which makes them much more general.

Both numerical and analytical arguments show that oscillons are in fact not exactly stable; they consist of an oscillating localized core and a radiating tail. The radiation slowly destroys the oscillon; it is exponentially suppressed however so that the oscillon can exist for periods of time that are far greater than the natural time scales of the system (like the oscillation time), making its lifetime extremely long. This, together with the fact that the restrictions on the (scalar) potential are limited (although there are some as we'll see), makes it that they can have very real implications for the physical world. For this reason, they have received quite some attention from the scientific community in the past twenty years. In the next section I will give a short overview of the work that has been done, and the questions that still require answers.

B. History

The first indication for the existence of oscillating, localized structures in nonlinear scalar field theories, came in the 70's [3]. They were initially dubbed pulsons and were found to emerge naturally after bubble collapse in a very constrained nonlinear scalar field model. These findings didn't gain much traction initially. With the emergence of Cosmology as a data-driven science in the following decades however, and the potential importance of phase transitions and symmetry breaking in the early Universe became apparent, interest in relativistic nonlinear field theories piqued. Solitons became a central theme in scientific debate because of their supposed natural formation under cosmological circumstances. This lead to the (re)discovery of a wide range of stable field configurations like Q-balls, Peakons and Solitonic kinks [1, 4, 5]. Eventually it was shown in 1994 that extremely stable, oscillating structures emerge under very general conditions during bubble collapse in nonlinear field theories. These were subsequently named oscillons by the author, M. Gleiser of the original paper. This is where the history of oscillons, albeit short, begins [6].

Although a lot is still unclear about the precise workings of oscillons, steady progress in understanding them has been made in the last twenty years. Initially, most work focused on simplistic models, where the oscillon is approximated as a pure Gaussian oscillating harmonically: $\phi_{osc} \sim e^{-r^2} \cos \omega t$ [7, 8]. The whole oscillon profile is essentially reduced to two parameters: its width and its amplitude. These two can then be taken as probes for measuring the degree in which the oscillon "feels" the nonlinearities in the potential. It was quite clear that these weren't the real oscillon solutions, since numerical simulations showed that the Gaussian first relaxed into a configuration with a different asymptotic behavior: $\phi(r) \sim e^{-r}$ as $r \to \infty$, before reaching stability. However, since the model was simple, one was able to make predictions about the stability and eventual collapse of the oscillon. With this ansatz it was shown that oscillons are present in a wide variety of models. They exist in the Sine-Gordon potential, they electroweak sector of the Standard Model, in the $SU(2)$ gauged Higgs model and all sort of nonlinear scalar field theories [9–11].

Although numerical arguments showed that oscillons exist under very general conditions, a lot of questions remained. Most of these related to the uniqueness and attractiveness of the oscillon solution. Namely, do models allow for the existence of multiple stable oscillon configurations and do the fields flow naturally to these solutions from generic initial conditions. Both points were tackled early on using the Gaussian models highlighted above. Numerical and analytical techniques led some scientists to conclude that the oscillon is indeed unique and loosely attractive. However, the simplicity of these models carries an inherent imprecision and rigorous mathematical arguments to tackle these problems are still needed to this day. The issue did inspire the use of a perturbative technique called the two-timing analysis in order to analytically construct the oscillon spatial profile. There now was a way to systematically find an equation, often called the profile equation, that the spatial envelope of the oscillon should solve. The problem of finding oscillons was now reduced to finding zero-mode solutions of this equation which resolved the uniqueness problem somewhat [12].

Application of perturbation theories in this way was an important step in understanding oscillons. Oscillon solutions could now be constructed, either analytically or numerically, which made it possible to study their stability more precisely. Through the rediscovery of older mathematical work, it became apparent that the oscillon solution is merely an asymptotic solution and that it will always lose energy through outgoing radiation. It was shown that this radiating tail was exponentially suppressed however, leading to the first clear mathematical explanation for the longevity of oscillons. It was also shown that this might not be the case for quantum mechanical radiation. Now that the spatial profiles of oscillons were known physicist could assess the stability of oscillons against small perturbations. This can be done by application of Floquet theory, but is in general very complicated. The Vakhitov-Kolokolov criterion has since been used to address specific initial conditions, but the problem has never been solved for generic perturbations [13, 14].

From the start it was suspected that oscillons could have real cosmological implications. Initially, they were shown to form both in symmetric and asymmetrical bubble collapses[15]. They might play a role in axion dynamics near the QCD phase transition [16]. In later years it also became apparent that they emerge naturally after slow-roll Inflation, thereby capturing and freezing some of the entropy in the Universe (even up to 50%) [17, 18]. In this way they might delay thermalization in the early Universe during reheating. Oscillons should in principle leave an imprint on cosmological observables [19, 20]. In general, the consensus is that since the restrictions on models that support oscillons are limited, they should play an important role in nonlinear field theories. In nature, fields rarely come in isolated form, and in recent years small advancements in studying oscillons in multi-field systems have been made.

Oscillons in models with multiple interacting fields are more complicated to study [9, 11, 21, 22]. The amount of literature on this topic is therefore somewhat lacking which is a problem since fields in the physical world tend to interact with each other. Oscillons do in fact exist in these multi-component systems, and have even been shown to form in hybrid Inflation models. A better understanding of their properties is needed in the future. The goal of this thesis is to take the first steps in the right direction, outlining where these type of oscillons might differ from single-field oscillons, but also what similarities they share. A lot of research is still needed but in this work are the first indications as to where potential solutions can be found.

C. This Thesis

The thesis is organized as follows. In chapter II I analyse oscillons in a single-component theory. This will help the reader to build intuition for the more complicated multi-component oscillons of later chapters. In chapter III I focus on the role that oscillons play in cosmic history, mainly focusing on their formation during preheating after Inflation. These first chapters are essentially a review of the most important work that has been done on oscillons in the past 20 years or so, although I try to give new insights where possible. The last chapters focus on oscillons in nonlinear field theories comprising two fields. The main goal of these chapters is to understand the properties of oscillons in these more complicated systems, both regarding stability and structure. In chapter IV I analyse a toy model characterized by an "exchange" symmetry. I construct oscillons in this system and analyse their stability analytically. The Vakhitov-Kolokolov criterion is extended to apply to this system, making this the first instance where this criterion is applied to multi-component oscillons. The lessons learned from this toy model are then used to discuss more complicated models in chapter V. I also use the last chapter to highlight areas that still require investigation to completely solve the question of multi-field oscillons.

II. SINGLE-FIELD OSCILLONS

In general it should not be expected that isolated scalar fields exist in nature. Only if a system has very specific symmetries in place, or certain symmetries are broken, it is impossible for (quantum) fields to couple. In all other cases, the general field theoretic approach dictates that all possible couplings between the fields are present. The study of oscillons in these types of isolated systems is therefore most likely a purely mathematical exercise and I aim to study more realistic models in later sections. However, for the uninitiated reader it can be helpful to understand the exact workings of oscillons in these simplified systems. Furthermore, since these type of oscillons are understood relatively well, they might give intuition about the more complicated models I will consider in later sections. So in this chapter we will investigate the world of oscillons in isolated scalar fields. First, we'll have to understand what conditions the system must satisfy in order to support stable oscillons.

A. Dispersion

In section I it was already mentioned that not all models support oscillons. To illustrate this, let's consider a free field theory. The potential of such a theory contains no nonlinear terms by definition. The Lagrangian of the field ϕ can then be written

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 \tag{4}$$

Resulting in an equation of motion that is in fact the Klein Gordon equation

$$(\partial^2 + m^2)\phi = 0 \tag{5}$$

Oscillons, by definition are localized, oscillating configurations in the field that remain stable for timescales that are orders of magnitude bigger than the natural timescales of the system (like the oscillation time). However, any configuration that evolves with equation (5) will tend to spread out over space, a phenomenon probably familiar to the reader as dispersion. Ultimately this is due to the fact that in a free theory all Fourier modes are decoupled and therefore all k-modes travel independently at different speeds during the time evolution of the system. This will delocalize the original configuration. Let's see how this works in more detail. Switching to Fourier space, equation (5) becomes

$$\ddot{\phi}_k = -(k^2 + m^2)\phi_k \tag{6}$$

Clearly, each Fourier mode evolves independently as a plane wave $\phi_k \propto e^{i(kx-\omega t)}$, where ω is given by the dispersion relation

$$\omega(k) = \sqrt{k^2 + m^2} \tag{7}$$

Let's now see what happens to an initially localized configuration. We consider a wave-packet in one spatial dimension such that initially

$$\phi(x,0) = e^{-\frac{1}{2}\left(\frac{x-x_0}{\sigma_x}\right)^2} \tag{8}$$

The field is a Gaussian centered around x_0 with width σ_x. Fourier transforming this initial condition is straightforward

$$\phi(k,0) = \frac{\sigma_x}{2\sqrt{\pi}} e^{-\frac{\sigma_x^2}{2}k^2} e^{ikx_0} \tag{9}$$

Now, using equations (6) and (7) the general solution at some later time can be readily found

$$\phi(x,t) = \int dk \frac{\sigma_x}{2\sqrt{\pi}} e^{-\frac{\sigma_x^2}{2}k^2} e^{ikx_0} e^{i(kx-\sqrt{k^2+m^2}t)} \tag{10}$$

From equation (9) it becomes evident that large k-modes are exponentially suppressed in this approximation. This allows us to Taylor expand the dispersion relation in equation (10) to second order and obtain an exact solution

$$\begin{aligned}
\phi(x,t) &= \int dk \frac{\sigma_x}{2\sqrt{\pi}} e^{-\frac{\sigma_x^2}{2}k^2} e^{ikx_0} e^{i(kx-m^2t-\frac{k^2}{2m}t)} \\
\phi(x,t) &= \exp\left[-\frac{1}{2}\left(\frac{(x-x_0)}{\sqrt{\sigma_x^2 - i\frac{1}{m}t}}\right)^2\right] e^{-im^2t} \\
\phi(x,t) &= \exp\left[-\frac{1}{2}\left(\frac{(x-x_0)}{\sigma(t)}\right)^2\right] e^{i\gamma(t)t}
\end{aligned} \tag{11}$$

Clearly, the solution remains Gaussian, but now with a time-evolving width

$$\sigma(t) = \sigma_x \sqrt{1 + \frac{1}{m^2\sigma_x^4}t^2} \tag{12}$$

And modulated by a phase

$$\gamma(t) = -m^2 t - \frac{tm}{t^2 + \sigma_x^4 m^2} \tag{13}$$

We are mainly interested in the implications of equation (12) which tells us that the solution will spread out over time

$$\sigma(t)^2 - \sigma_x^2 = \frac{t^2}{m^2\sigma_x^2} \tag{14}$$

So $\sigma(t) \to \infty$ as $t \to \infty$. The spreading happens over timescales that are comparable to the natural timescales of the theory $\sim m\sigma_x^2$. So the initial localized packet will spread out and become delocalized.

The argumentation listed above can be generalized to any theory with a clearly defined dispersion relation for $\omega(k)$, since we can always just Taylor expand the relation for small k and obtain similar results. In fact, if there is a clearly defined relation for $\omega(k)$ that is not purely linear, the speed of the different k-modes $v_k = \frac{\omega(k)}{k}$ will vary, which makes it impossible for a localized solution to stay exactly that: localized. Clearly, any linear field theory (like the free example discussed above) will have a well-defined dispersion relation. Also, if we're considering massive relativistic theories the dispersion relation will never be purely linear. From these considerations we conclude that we must look for nonlinear field theories in order to find oscillons. In nonlinear field theories the different k-modes of the field couple in non-trivial ways. There is therefore no clear way to write down a dispersion relation. Simply adding nonlinearities to the theory is not enough however. We'll explore the relevant conditions in the next section.

B. Restrictions on the nonlinear potential

We are looking for scalar potentials that support localized, oscillating solutions. Oscillons don't require conserved charges to exist and therefore there are no symmetry restrictions on the potential. A scalar field in an expanding background will evolve with the equation of motion

$$\ddot{\phi} + H\dot{\phi} - \frac{\nabla^2}{a^2}\phi + V'(\phi) = 0 \tag{15}$$

Where H is the Hubble constant. Plugging in an ansatz for the oscillon solution as $\phi_{osc} = \Phi(x)\cos\omega t$, equation (15) becomes

$$-\omega^2\Phi(x) - H\omega\Phi(x)\tan\omega t - \frac{\nabla^2}{a^2}\Phi + \frac{V'(\phi)}{\cos\omega t} = 0 \tag{16}$$

Since we're looking for localized solutions of $\Phi(x)$, we must have that $\Phi(x) \to \infty$ as $x \to \infty$. We can therefore throw away all nonlinear terms

$$-\omega^2\Phi(x) - H\omega\Phi(x)\tan\omega t - \frac{\nabla^2}{a^2}\Phi(x) + m^2\Phi(x) = 0 \tag{17}$$

Since this equation should be valid at all times, and a localized solution decays as $x \to \infty$ (leading to the conclusion that $\nabla^2\Phi > 0$ in the asymptotic region), it is straightforward to conclude

$$\omega^2 < m^2 \tag{18}$$

Plugging in (18) into equation (16) we obtain

$$\kappa^2\Phi(x) - (H\omega\tan(\omega t) + \frac{\nabla^2}{a^2})\Phi(x) + \frac{(V'(\phi) - m^2\Phi\cos\omega t)}{\cos\omega t} = 0 \tag{19}$$

Where $\kappa^2 = m^2 - \omega^2 > 0$. Again this should be valid at all times; setting $t = \frac{2\pi}{\omega}$, the equation becomes

$$\kappa^2\Phi(x) - \frac{\nabla^2}{a^2}\Phi(x) + (V'(\Phi) - m^2\Phi) = 0 \tag{20}$$

If smooth localized solutions exist than necessarily there is a region near the center of the solution where $\frac{\nabla^2}{a^2}\Phi(x) < 0$. This imposes a heuristic condition on the potential of the scalar field. Namely,

$$V'(\Phi) - m^2\Phi < 0 \tag{21}$$

For at least some region of the potential. Furthermore the region should be probed by field values that are of the same order of magnitude as the oscillon itself. Although the most straightforward way to satisfy (21) is to add nonlinear terms to the potential, it has been shown that the same effect can be produced in a system with non-canonical kinetic terms [23].

Many physically motivated potentials that satisfy condition (21) exist. In particular, oscillons have been found in several systems similar to the Standard Model [9, 11]. Furthermore, slow-roll Inflation can be sourced by a scalar field that supports oscillons as well. This idea will be investigated in the next chapter. In the next two sections I will focus on the analytical construction of the oscillon. First, I'll find the spatial profile of oscillons in a particular scalar field model via a perturbative technique called the two-timing analysis. Then we'll focus on the asymptotic behavior of this oscillon in the far distance regime, outlining why the oscillon can be so extremely long-lived. The analysis is done for a particular model but the techniques and conclusions can be easily generalized to other systems.

C. Oscillon analytical construction

Let's consider the action of a scalar field minimally coupled to gravity

$$S_{scalar} = \int d^{d+1}x\sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)] \tag{22}$$

With a potential given by

$$V(\phi) = \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 + \frac{1}{6}g\phi^6 \tag{23}$$

Where $m, \lambda, g > 0$ and $m, \lambda, g \sim O(1)$. In what follows I will assume the Universe to be static ($H = 0$) and set $a = 1$ without loss of generality. Clearly, the potential in (23) satisfies condition (21) for some range of ϕ. The sextic term is added to assure that the field remains bounded from below. The equation of motion of the scalar field can be readily obtained by extremizing the action (22). It is

$$\ddot{\phi} - \nabla^2\phi + m^2\phi - \lambda\phi^3 + g\phi^5 = 0 \tag{24}$$

Assuming the field is spherically symmetric

$$\ddot{\phi} - \partial_r^2\phi - \frac{(d-1)}{r}\partial_r\phi + m^2\phi - \lambda\phi^3 + g\phi^5 = 0 \tag{25}$$

Since equation (25) is nonlinear it isn't obvious how to find localized field configurations that behave like oscillons. There exists a perturbative approach to this problem however, often referred to as the Two-timing analysis in the literature. The idea is that the oscillon has relevant behavior on two distinct time scales. The technique can be understood intuitively by imagining a completely uniform field oscillating in a purely quadratic potential. The theory is then linear and the field simply oscillates at its natural frequency m. Adding nonlinear terms to the potential as in (23) will alter this frequency, but if the amplitude of the configuration is small, the nonlinearities are only probed perturbatively. Two different time scales can therefore be identified: a short time scale associated with the natural frequency of the system and a long time scale over which the nonlinearities of the potential become significant. Localized, oscillating solutions (oscillons) can therefore be found by perturbatively moving away from a uniform field moving in a quadratic potential. The solution is then also expected to be wide, so that the gradient terms also only enter perturbatively. These considerations inspire the following change of variables

$$\tau = \epsilon^2 t \tag{26}$$

$$\rho = \epsilon r \tag{27}$$

where $\epsilon \ll 1$. Here τ is the "slow" time scale associated with the deviation from the natural frequency and ρ with the "long" spatial width that the oscillon has. Since the oscillon has behavior on both the slow and natural time scales, we send $\phi(x,t) \to \phi(\rho,\tau,t)$. The time derivatives in (25) thus become full derivatives

$$\begin{aligned} \partial_t \to d_t =& \partial_t + \epsilon^2\partial_\tau \\ \partial_t^2 \to d_t^2 =& \partial_t^2 + 2\epsilon^2\partial_\tau\partial_t + O(\epsilon^4) \end{aligned} \tag{28}$$

Finally, plugging all of this into the equation of motion in (25) we obtain

$$\partial_t^2\phi + 2\epsilon^2\partial_\tau\partial_t\phi - \epsilon^2\partial_\rho^2\phi - \epsilon^2\frac{(d-1)}{\rho}\partial_\rho\phi + m^2\phi - \lambda\phi^3 + g\phi^5 + O(\epsilon^4) = 0 \tag{29}$$

This is all we need to perturbatively find our oscillon solution. As in standard perturbation theory we introduce an ansatz $\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + ...$ and solve the equations order by order. However, in this case we should not allow for a zero-order term since by assumption the nonlinearities in the potential can only by probed perturbatively in order for the two-timing analysis to be valid. So

$$\phi(\rho,\tau,t) = \sum_{n=1}^{\infty}\epsilon^n\phi_n(\rho,\tau,t) \tag{30}$$

Inserting the expansion in (30) into (29) we can write down the first three order equations

$O(\epsilon)$:

$$\partial_t^2\phi_1 + m^2\phi_1 = 0 \tag{31}$$

$O(\epsilon^2)$:

$$\partial_t^2\phi_2 + m^2\phi_2 = 0 \tag{32}$$

$O(\epsilon^3)$:

$$\partial_t^2\phi_3 + m^2\phi_3 = -2\partial_\tau\partial_t\phi_1 + \partial_\rho^2\phi_1 + \frac{(d-1)}{\rho}\partial_\rho\phi_1 + \lambda\phi_1^3 \tag{33}$$

The first and second order equations, (31) and (32), are immediately identified as the equations for a harmonic oscillator with mass m. It is straightforward to write down a general solution, requiring the result to be real

$$\begin{aligned}\phi_1(\rho,\tau,t) &= \mathrm{Re}\{A(\rho,\tau)e^{imt}\}\\ \phi_2(\rho,\tau,t) &= \mathrm{Re}\{B(\rho,\tau)e^{imt}\}\end{aligned} \tag{34}$$

Where $A(\rho,\tau)$ and $B(\rho,\tau)$ can be imaginary functions. These functions capture the behavior of the solution on long time- and spatial scales, and can thus be thought of as encompassing the oscillon behavior. They are therefore often referred to as the envelope functions of the oscillon in the literature. To find equations governing these envelopes we plug in the solution $\phi_1(\rho,\tau,t) = \mathrm{Re}\{A(\rho,\tau)e^{imt}\} = \frac{Ae^{imt}+A^*e^{-imt}}{2}$ into equation (33)

$$\partial_t^2\phi_3 + m^2\phi_3 = \left(-im\partial_\tau A + \frac{1}{2}\partial_\rho^2 A + \frac{(d-1)}{2\rho}\partial_\rho A + \frac{3}{8}\lambda|A|^2A\right)e^{imt} + c.c. + h.o. \tag{35}$$

This is nothing more than the equation of a harmonic oscillator of mass m with a parametric driving force. Notice that the term between brackets is then a resonant term since it oscillates with the natural frequency m. This term will amplify ϕ_3 due to resonance. However, by assumption this can not be the case since the oscillon remains small. The term between brackets has to equate to 0, resulting in the envelope equations of the oscillon

$$-im\partial_\tau A + \frac{1}{2}\partial_\rho^2 A + \frac{(d-1)}{2\rho}\partial_\rho A + \frac{3}{8}\lambda |A|^2 A = 0 \tag{36}$$

Equation (219) is of the Non-linear Schrodinger type and governs the behavior of the oscillon on long time- and spatial scales. Since we're interested in finding the spatial profiles of the oscillon we perform a separation of variables: $A(\rho,\tau) = a(\rho)e^{ic\tau}$, where c is some arbitrary constant and $a(\rho)$ a real function of the spatial profile of the oscillon. Inserting this in the envelope equation gives us the profile equation

$$-m^2 a + \partial_\rho^2 a + \frac{(d-1)}{\rho}\partial_\rho a + \frac{3}{4}\lambda a^3 = 0 \tag{37}$$

Where we set $c = -\frac{m}{2}$. A few remarks about the parameter c are in order. c is in essence a free parameter, as long as it is not too large. This is simply because it can be absorbed into a redefinition of ϵ, since $\tau = \epsilon^2 t$. We don't lose any generality by setting $c = -\frac{m}{2}$. We also require $c < 0$ for the solutions to decay as $a \to 0$ and $\rho \to \infty$. In this asymptotic regime (221) reduces to $\partial_\rho^2 a = -cma$, and localized solutions must therefore have $c < 0$.

The oscillon spatial profile are localized solutions of (221). Finding the solutions $a_{loc}(\rho)$ allows us to write down the analytical form of the oscillon up to first order in ϵ

$$\boxed{\phi_{oscillon}(x,t) \approx \epsilon\phi_1(x,t) = \epsilon\,\mathrm{Re}\{a(\rho)e^{im(t-\frac{\tau}{2})}\} = \epsilon a_{loc}(\epsilon x)\cos\left(mt(1-\frac{\epsilon^2}{2})\right)} \tag{38}$$

The frequency of the oscillon is therefore $\omega = m(1-\frac{\epsilon^2}{2})$. It is important to realize that this is only an asymptotic solution, since a full solution would require finding profile equations for all the ϕ_n in the perturbative expansion. This can be done with exactly the same procedure. For example, to find $\phi_2(\rho,\tau,t) = \mathrm{Re}\{B(\rho,\tau)e^{imt}\}$ we simply write down the $O(\epsilon^4)$ equation of motion and eliminate resonant terms as before. However, the procedure only cares about eliminating the resonant terms in the equation of motion. In general there will always be a remainder, including the higher harmonics. This causes the oscillon to lose energy through outgoing radiation. This problem will be tackled in section II D. For now, let's focus on finding the localized solutions of the profile equation (221).

1. *Localized solutions of the profile equation*

We managed to reduce the problem of finding oscillons to finding localized solutions of the profile equation (221). This can in some cases be done analytically but will in general require numerical methods. Before we do this, we might want to ask ourselves how many of such solutions exist. Here we focus strictly on the zero-mode solutions, meaning that they are strictly positive (or negative). This is best understood by first examining the one-dimensional case for which $d = 1$. The profile equation reduces to

$$-m^2 a + \partial_\rho^2 a + \frac{3}{4}\lambda a^3 = 0 \tag{39}$$

If we interpret ρ as a temporal instead of spatial variable, equation (39) looks like the equation of motion of a point particle of unit mass, moving in zero-dimensional space under influence of a conservative potential $V(a) = -\frac{1}{2}m^2a^2 + \frac{3}{16}\lambda a^4$. The equation therefore has a "conserved energy"

$$E_\rho = \frac{1}{2}(\partial_\rho a)^2 + V(a) \tag{40}$$

For localized solutions as $\rho \to \infty$ we have that $\partial_\rho a \to 0$ and $a \to 0$. We conclude that localized solutions must have 0 energy. Requiring the solution to be smooth at the origin, $\partial_\rho a(0) = 0$, and using (40), completely fixes the localized solution of the second order PDE in (39)

$$V(a(0)) = 0 \to a(0) = 0 \vee a(0) = \pm m\sqrt{\frac{8}{3\lambda}} \tag{41}$$

We conclude that there are exactly two localized solutions of the profile equation in one dimension, since $a(0) = 0$ just gives $a(\rho) = 0$ everywhere, which is not very interesting. These two solutions satisfy $a_1(\rho) = -a_2(\rho)$. In fact, in one dimension the profile equation can be solved analytically. The solution is a sech function, and for this specific model reads $a(\rho) = m\sqrt{\frac{8}{3\lambda}}\sec(m\rho)$. The situation is a little more tricky in higher dimensions since there is now a friction term in the "equation of motion" ($\propto \partial_\rho a$) and the energy of the solution is no longer conserved. We can rewrite the profile equation using the energy defined in (40)

$$\frac{dE_\rho}{d\rho} = -\frac{d-1}{\rho}\left(\partial_\rho a(\rho)\right)^2 \tag{42}$$

Since $\rho \geq 0$ for $d > 1$, the r.h.s. of this equation is strictly negative (or zero). The energy of any solution will decrease as $\rho \to \infty$. Since the energy of a localized solution must be 0 in the far-distance regime we conclude that $E_\rho \geq 0$ at $\rho = 0$. We again require the solution to be smooth at the origin; this puts a bound on $a(0)$

$$|a(0)| > m\sqrt{\frac{8}{3\lambda}} \tag{43}$$

So what kind of solutions that satisfy this condition exist, and how many of them are localized ($a \to 0$ as $\rho \to \infty$)? It is best to think about this in terms of the trajectories of solutions in phase space $(a, \partial_\rho a)$. Since we're interested in functions that are smooth at the origin, all trajectories start on the $\partial_\rho a = 0$ axis. Any trajectory will continuously intersect $E_\rho = k$ curves, and due to relation (42), the k-value of these curves must decrease as $\rho \to \infty$. The $k = 0$ curve then defines the boundary between localized and non-localized solutions. Namely, a localized solution must intersect this curve in the origin.

It turns out that only a countable set $a_n(\rho)$ of these kind of trajectories can be drawn in phase space [24]. Here n is the amount of nodes of the solution. We conclude that there is exactly one zero-mode solution of the profile equation (really two since again $-a_0(\rho)$ is also a solution), even in three dimensions. This simply means that for every choice of ϵ there exists exactly one zero-mode oscillon. However, since the only requirement for ϵ is that it is a small number, the model defined in (22) still has an infinite amount of oscillon solutions, all oscillating at frequencies defined by the specific choice of ϵ: $\omega = m(1 - \frac{\epsilon^2}{2})$. To find the exact spatial profiles for $d \neq 1$ we are restricted to numerical methods. These were found for $d = 3$ using the shooting method.

2. *Scaling dependence of ϵ and free parameters*

In our quest of finding the profile equations through the two-timing analysis, we've implicitly made some assumptions about the scalings of our perturbative approach. To be completely general we should write

$$\begin{aligned} \tau &= \delta_\tau t \\ \rho &= \delta_\rho r \\ \phi &= \sum_{n=1}^{\infty} \delta_\phi^n \phi_n \end{aligned} \tag{44}$$

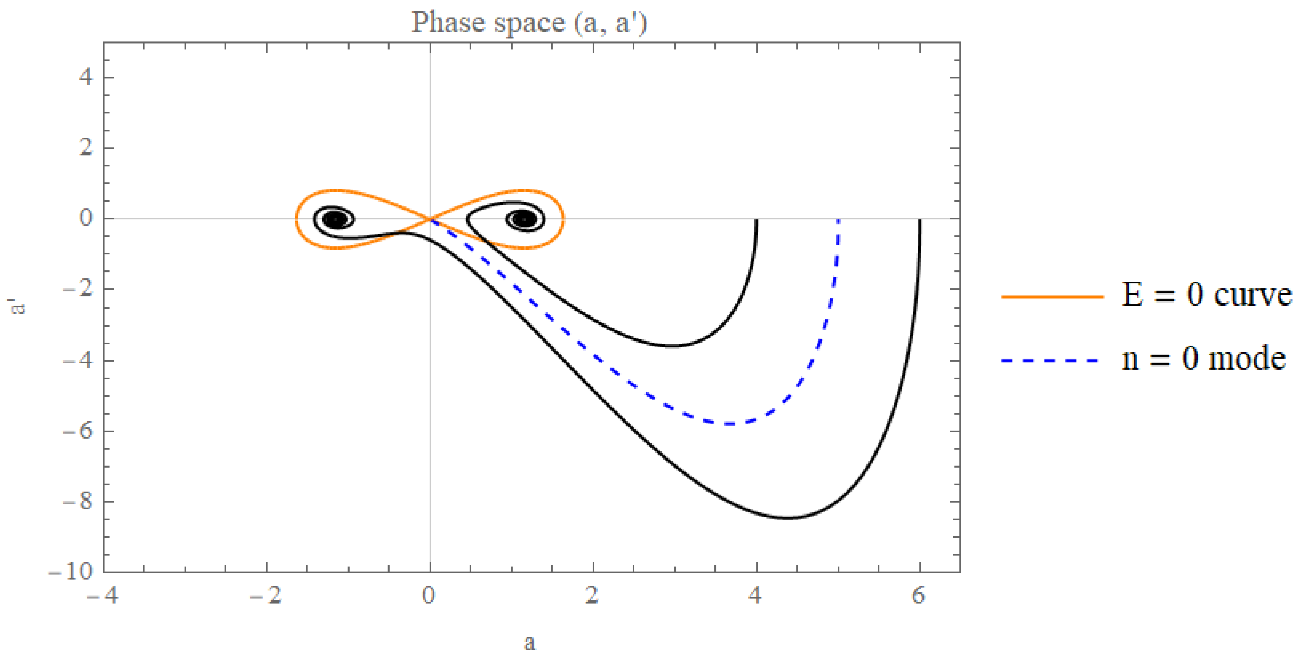


FIG. 1: The path in phase space (a, a') of three different solutions of the profile equations that are smooth at the origin. Once the path intersects the $E_\rho = 0$ curve it can never leave the region enclosed by it due to relation (42). A localized solution must intersect this curve in the origin. It turns out there are a countable set of such solutions.

Where δ_τ, δ_ρ and δ_ϕ are just small numbers. In our derivation we set $\delta_\tau = \epsilon^2$ and $\delta_\rho = \delta_\phi = \epsilon$; but there is no a-priori reason to believe that this is the only option. To see why these scalings are correct consider a schematic form of the equation of motion after inserting (44) (in $d = 1$ for simplicity's sake)

$$\partial_t^2 \phi + 2\delta_\tau \partial_t \partial_\tau \phi + \delta_\tau^2 \partial_\tau^2 \phi - \delta_\rho^2 \partial_\rho^2 \phi + m^2 \phi + [nonlinear] = 0 \tag{45}$$

To find well-defined profile equations the slow time scale δ_τ needs to enter at the same order in perturbation theory as the long space scale δ_ρ. From inspection of (45) we immediately conclude $\delta_\tau = \alpha \delta_\rho^2$, with $\alpha \sim O(1)$. Furthermore, to have well defined localized solutions, the first nonlinear term in (45) should also enter at this order. Since in the potential under consideration this is a cubic term we conclude $\delta_\phi^3 = \beta^2 \delta_\phi \delta_\rho^2$, with $\beta^2 \sim O(1)$, and thus $\delta_\phi = \beta \delta_\rho$. Setting $\delta_\rho = \epsilon$ completely fixes the other scalings up to two free parameters α and β. To see how the introduction of these parameters influences the oscillon spatial profile, consider the envelope equation under these scalings (in one dimension for simplicity)

$$-im\alpha \partial_\tau A + \frac{1}{2} \partial_\rho^2 A + \lambda \frac{3}{8} \beta^2 |A|^2 A = 0 \tag{46}$$

Using separation of variables and plugging in $A = a(\rho)e^{ic\tau} = a(\rho)e^{ic\alpha\epsilon^2 t}$, gives us the modified profile equation of the oscillon

$$2m\alpha c a(\rho) + \partial_\rho^2 a(\rho) + \lambda\frac{3}{4}\beta^2 a(\rho)^3 = 0 \tag{47}$$

Naively, We must conclude that for each choice of the the small parameter ϵ, which fixes the width of the oscillon through the parameter ρ, there is a family of profile equations, all with different zero-mode solutions, that when solved might yield stable oscillons. However, this is in fact not true, and these are "phantom" parameters. To see this suppose we have a solution $a_1(\rho)$ for specific choices β_1 and $\kappa_1 = \alpha c$. Let's now suppose we're looking for a solution $a_2(\rho)$ corresponding to choices β_2 and κ_2. This solution should off course solve

$$2m\kappa_2 a_2(\rho) + \partial_\rho^2 a_2(\rho) + \lambda\frac{3}{4}\beta_2^2 a_2(\rho)^3 = 0 \tag{48}$$

Then plugging in the ansatz solution $a_2(\rho) = Ba_1(A\rho)$, we obtain (after some rewriting)

$$2m\frac{\kappa_2}{A^2}a_2(A\rho) + \partial_{A\rho}^2 a_1(A\rho) + \frac{B^2}{A^2}\lambda\frac{3}{4}\beta_2^2 a_1(A\rho)^3 = 0 \tag{49}$$

So we reach the conclusion that $A = \sqrt{\kappa_2/\kappa_1}$ and $B = \frac{\beta_1}{\beta_2}\sqrt{\kappa_2/\kappa_1}$. So how does the oscillon solution look like in real space for the choices β_2 and κ_2. Using the solution we found for $a_2(\rho)$ and (38)

$$\phi_{osc}(x,t) = \epsilon\beta_1\sqrt{\kappa_2/\kappa_1}a_1(\sqrt{\kappa_2/\kappa_1}\epsilon x)\cos\big((m+\kappa_2\epsilon^2)t\big) \tag{50}$$

Now it becomes evident that the actual oscillon solution is exactly the same as the solution we found for parameters β_1 and κ_1, just for a different ϵ. Namely, write $\tilde{\epsilon} = \sqrt{\kappa_2/\kappa_1}\epsilon$ to obtain

$$\phi_{osc}(x,t) = \tilde{\epsilon}\beta_1 a_1(\tilde{\epsilon}x)\cos\big((m+\kappa_1\tilde{\epsilon}^2)t\big) \tag{51}$$

Which is exactly our original solution. So all choices for β and κ result in the same oscillon, only with slightly different values of ϵ. We don't lose any relevant behavior by choosing specific values for κ and β. Notice for example that the parameter κ here can be identified with c in our explicit derivation from the previous section. We have thus implicitly set $\kappa = -\frac{m}{2}$ and $\beta = 1$ before. We didn't miss out on any solutions by doing this. As a final remark notice that the above derivation is not possible in general when the profile equation contains more nonlinear terms. In that case the free parameters β and κ can not be removed so trivially as we'll also see in later chapters.

The two-timing analysis reduces the problem of finding oscillon solutions to that of finding zero-mode solutions of the profile equation. The amount of free parameters might differ between models, but the technique remains robust in general. It is in essence a special application of perturbation theory, which means that to any order solutions can only be asymptotically correct. Since oscillons have been observed to be extremely long lived, the corrections to the perturbative expansion can not be too large. In fact, in the next section we'll see that they produce outgoing radiation that is exponentially suppressed.

D. Radiation

The two-timing analysis highlighted in the previous section is a clear-cut procedure that allows one to find localized oscillating solutions of a given theory. One simply chooses a small scaling parameter ϵ and solve the equations of motion perturbatively. Spatial profiles are then found by requiring that resonant terms cancel at each order, resulting in the so-called profile equations from the previous section. It was understood quite early on by mathematicians that this procedure will in general not converge to a perfect breather solution [25]. Breathers are localized oscillating solutions of a nonlinear theory [26**?**]. It misses an exponentially small radiating tail $\sim e^{-\frac{1}{\epsilon}}$ that is beyond all orders in perturbation theory. An important exception is the Sine-Gordon breather. In recent years physicists realized that this principle should also apply to oscillons, and it was shown that in fact they lose energy via radiation. In this section we'll see why the oscillon solution must in fact have a small radiating tail, and how to find it.

The fact that (most) oscillons have a radiating tail is ultimately tied to the fact that the series expansion in ϵ is only asymptotically correct. To any order in perturbation theory the series expansion will not exactly solve the equation of motion: there will always be a remainder term that acts as a source for radiation. To see this, consider a scalar field theory with potential

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{1}{6}\phi^6 \tag{52}$$

Clearly this potential satisfies condition (21) and this theory supports oscillons. It is just the theory considered in the previous section, with all couplings set to 1.

$$\ddot{\phi} - \nabla^2\phi + \phi - \phi^3 + \phi^5 = 0 \tag{53}$$

Now consider the oscillon solution truncated to order N

$$\phi(x,t) = \sum_{n=1}^{N} \epsilon^n \phi_n(x,t) \tag{54}$$

where all individual ϕ_n are found by perturbatively solving the equation of motion as explained in the previous section. At each order the equation of motion is a (forced) harmonic oscillator equation of which the solution can be found exactly. Each ϕ_n can therefore be written in the form $\phi_n \sim a(\rho)\cos(n\omega t) + ...$ where $\omega = 1 - \frac{\epsilon^2}{2}$ and $n = 1, 3, 5$ since the potential is symmetric. Plugging this solution into the equation of motion it is clear that all terms up to $\sim O(\epsilon)$ cancel. Furthermore, all terms proportional to $\cos(\omega t)$ also cancel by construction. However, there now clearly remain terms that are of higher order in ϵ and don't oscillate at frequency ω. In this specific potential the remainder term can schematically be written as

$$J(x,t) = \epsilon^{N+2} j(x) cos(3\omega t) + h.h. + O(\epsilon^{N+3}) \tag{55}$$

which will in general not be 0 (albeit very small). The only way to solve this discrepancy is to add a perturbation to the oscillon solution. Writing $\phi(x,t) = \phi_{osc} + \delta(x,t)$ and linearizing the equation of motion we see that $J(x,t)$ acts as a source

$$\ddot{\delta} - \nabla^2 \delta + \delta = J(x,t) \tag{56}$$

Where I've ignored a parametric resonance term $\propto V_{\phi\phi}(\phi_{osc})\delta$ that is suppressed relative to $J(x,t)$. The remainder of the asymptotic oscillon expansion thus sources a perturbation and we'll see that this results in outgoing radiation. The full derivation involves complex analysis and some readers might satisfy themselves with the hand-wavy argumentation presented here. All they should know is that the solution of $\delta(x,t)$ is exponentially small $\delta(x,t) \sim e^{-\frac{1}{\epsilon}}$ making oscillons extremely long-lived. In what follows I present a more nuanced analysis[27–29].

The perturbative expansion of the field $\phi(x,t) = \sum_{n=1} \epsilon^n \phi_n$ will at each order in ϵ give a forced oscillator equation. The forced oscillator will oscillate with frequency $\omega = 1 + \kappa\epsilon^2$. In a symmetric potential only the terms for odd n can give non-zero results. This is because in a symmetric potential all ϕ_{2n} will enter linearly in the perturbation equations. This will always lead to the conclusion that $\phi_{2n} = \phi_{2n-1}$. This corresponds to a simple rescaling of the solution, so that we might as well set $\phi_{2n} = 0$. For the potential given in (52) the first non-zero terms in the expansion are given by

$$\begin{aligned} \phi_1 &= f_1(x)\cos(\omega t) \\ \phi_3 &= f_3(x)\cos(\omega t) + F(x)\cos(3\omega t) \\ \phi_5 &= f_5(x)\cos(\omega t) + G(x)\cos(3\omega t) + H(x)\cos(5\omega t) \end{aligned} \tag{57}$$

Where all f_n can be found by cancelling resonant terms and the functions $F, G, H...$ can be found by solving the forced harmonic oscillator equations at each order

$$\begin{aligned} F(x) &= \frac{1}{32} f_1^3 \\ G(x) &= \frac{1}{8}\left(\frac{3}{4} f_1^2 f_3 - \frac{5}{16} f_1^5 + \frac{6}{128} f_1^5\right) \\ H(x) &= \frac{1}{24}\left(\frac{3}{128} f_1^5 - \frac{1}{16} f_1^5\right) \end{aligned} \tag{58}$$

All ϕ_n in the peturbative expansion are time-periodic in ω; and all spatial functions in (58) are uniquely defined. It is therefore natural to search for a solution in terms of Fourier modes and use the functions in (58) as boundary conditions near the central part of the oscillon. We look for solutions

$$\phi(x,t) = \sum_{n=0}^{\infty} \Phi_n(x)\cos(n\omega t) \tag{59}$$

Plugging these into the equations of motion and gathering terms oscillating at the same frequency, we obtain equations for the spatial part of the Fourier modes (working in $d = 1$ and assuming spherical symmetry)

$$-n^2\omega^2\Phi_n - \partial_r^2\Phi_n + \Phi_n = F_n \tag{60}$$

Where F_n is a nonlinear function of the (other) Fourier modes. In a symmetric potential $F_n = 0$ for even n and there are no localized solutions for the corresponding modes. The small amplitude expansion actually gives expressions for the different Fourier modes.

$$\begin{aligned} \Phi_1 &= \epsilon f_1 + \epsilon^3 f_3 + \epsilon^5 f_5 + ... \\ \Phi_3 &= \epsilon^3 F + \epsilon^5 G + ... \\ \Phi_5 &= \epsilon^5 H + ... \end{aligned} \tag{61}$$

The problem is that, as we've seen, this series won't satisfy (60) at any finite order in ϵ. We will thus have to add a correction term that is beyond all orders in perturbation theory but agrees with (61) in a "matching region". Next, we'll define this region.

We've seen that with the potential in (52), $f_1(x) = \sqrt{\frac{8}{3}}\,\mathrm{sech}(\epsilon r)$. We can analytically continue this function to the complex plane. Here the sec function has a pole at $\epsilon x = \frac{i\pi}{2}$. Similarly there is one at $\epsilon x = -\frac{i\pi}{2}$. The sech-function can be expanded around its upper pole $\mathrm{sech}\big(x = R + \frac{i\pi}{2}\big) = -\frac{i}{R} + i\frac{R}{6} + O(R^3)$ for small $|R|$. The Laurent series around the upper pole of $f_1(x)$ can then be written as

$$f_1(y) = -\sqrt{\frac{8}{3}}\frac{i}{\epsilon y} + i\sqrt{\frac{8}{3}}\frac{\epsilon y}{6} + O(\epsilon^3 y^3) \tag{62}$$

Where $R = \epsilon y = \epsilon r - \frac{i\pi}{2}$. The matching region is defined as $y \to \infty$. For R to remain small we also send $\epsilon \to 0$. From (61) we learn that to lowest order in ϵ the Fourier modes Φ_n are proportional to $\epsilon^n f_1^n$. In the matching region they satisfy

$$\begin{aligned}
\Phi_1 &= -\sqrt{\frac{8}{3}}\frac{i}{y} + O(\epsilon^2, 1/y^2) \\
\Phi_3 &= \frac{8}{3}\sqrt{\frac{8}{3}}\frac{i}{y^3} + O(\epsilon^2, 1/y^4) \\
\Phi_5 &= -\frac{64}{9}\sqrt{\frac{8}{3}}\frac{i}{y^5} + O(\epsilon^2, 1/y^6)
\end{aligned} \tag{63}$$

In the matching region the mode-equations (60) can be written as

$$-n^2\Phi_n - \partial_y^2\Phi_n + \Phi_n = F_n \tag{64}$$

We need to find solutions of (64) that can be matched upon the boundary conditions in (63). The asymptotic expansion converges nowhere and it is not clear how to find a correction to the series. Notice however that for $\mathrm{Re}(y) = 0$ the imaginary part of the series converges trivially; namely $\mathrm{Im}(\Phi_n) = 0$. In order to find a correction to the series that satisfies this condition we split the modes into real and imaginary parts $\Phi_n = \Psi_n + i\Omega_n$. Plugging this into (64)

$$-n^2\Omega_n - \partial_y^2\Omega_n + \Omega_n = \mathrm{Im}(F_n) \tag{65}$$

In the matching region we see that all $\Phi_n \to 0$ and we can approximate the solution by ignoring the nonlinear terms $\propto F_n$. To first order the solutions are then

$$\begin{aligned}
\Omega_1 &= \nu_1 y \\
\Omega_3 &= \nu_3 \exp\Big(-i\sqrt{8}y\Big) \\
\Omega_5 &= \nu_5 \exp\Big(-i\sqrt{24}y\Big)
\end{aligned} \tag{66}$$

It becomes clear that only the solutions for $n \geq 3$ can be matched to the ϵ expansion on the real axis as $\mathrm{Im}(y) \to -\infty$. Switching back to the coordinate $y = r - \frac{i\pi}{2\epsilon}$ we find corrections of the ϵ expansion beyond all orders

$$\begin{aligned} \delta\Phi_1 &= 0 \\ \delta\Phi_3 &= i\nu_3 \exp\left(-i\sqrt{8}r\right) \exp\left(-\sqrt{8}\frac{\pi}{2\epsilon}\right) \\ \delta\Phi_5 &= i\nu_5 \exp\left(-i\sqrt{24}r\right) \exp\left(-\sqrt{24}\frac{\pi}{2\epsilon}\right) \end{aligned} \tag{67}$$

So that the correction to the real part of the expansion is

$$\begin{aligned} \delta\Phi_1 &= 0 \\ \delta\Phi_3 &= \nu_3 \sin\left(\sqrt{8}r\right) \exp\left(-\sqrt{8}\frac{\pi}{2\epsilon}\right) \\ \delta\Phi_5 &= \nu_5 \sin\left(\sqrt{24}r\right) \exp\left(-\sqrt{24}\frac{\pi}{2\epsilon}\right) \end{aligned} \tag{68}$$

Since these corrections are exponentially suppressed we can approximate the full oscillon solution as a superposition of these corrections and the standard ϵ expansion. We get a final expression for the oscillon

$$\phi_{osc} = \sum_{n=1} \epsilon^n \phi_n + \nu_3 \sin\left(\sqrt{8}r\right) \exp\left(-\sqrt{8}\frac{\pi}{2\epsilon}\right) \cos(3\omega t) + h.h. \tag{69}$$

The corrections will only dominate the solution as $r \to \infty$ since in this regime all $\phi_n \to 0$. We can then obtain an approximate expression for the radiation tail of the oscillon

$$\phi_{rad} \approx \sum_{n=3} \nu_n \sin\left(\sqrt{n^2-1}r + nt\right)) \exp\left(-\sqrt{n^2-1}\frac{\pi}{2\epsilon}\right) \tag{70}$$

Where the sum only runs over odd n. Since the radiation tail is suppressed, the energy loss in the oscillon is extremely slow. Since the energy in the radiation is proportional to the squared amplitude of the tail, it becomes clear that

$$\boxed{\frac{dE}{dt} \propto \exp\left(-\sqrt{n^2-1}\frac{\pi}{\epsilon}\right)} \tag{71}$$

And the asymptotic ϵ expansion resulting in our localized oscillon solution can thus be extremely long lived. It has been shown that the same conclusions can be found for oscillon in higher dimensions [27]. Here a similar analysis was performed, although the poles of the function $f_1(x)$ had to be found numerically. A analysis using Borel summation to calculate the values of the constants ν_n was also performed, finding that they were of $O(1)$ in general [28]. The results were also confirmed numerically. The central conclusion is unanimous however. The two-timing analysis leads to solutions that live for extremely long times. In general they will not decay naturally (on conceivable time scales). In the next section we'll investigate how stable these solutions are when perturbed however.

E. Linear Stability Analysis

In the previous sections we've constructed oscillon solutions perturbatively using the two-timing analysis. These solutions lose energy via a small radiating tail that will dominate the solution in the far-distance regime. The rate of energy loss is very small however, which results in an extremely long-lived oscillon. In this section we'll be interested in the behavior of small perturbations living atop the 'core' part of the oscillon, which means we won't be interested in the small radiating tail. The main question we would like to answer is what conditions an oscillon should satisfy in order to be stable against small perturbations. This is an important question since it is expected that perturbations arise naturally, both through radiation and coupling to other fields. If the oscillon is not robust to such perturbations, it might decay rapidly.

To analyse the stability of oscillon solutions we again work in a scalar field model with potential

$$V(\phi) = \frac{1}{2}\phi^2 - \frac{1}{4}\phi^4 + \frac{1}{6}\phi^6 \tag{72}$$

Resulting in equations of motion

$$\ddot{\phi} - \nabla^2\phi + \phi - \phi^3 + \phi^5 = 0 \tag{73}$$

Although the analysis performed in this section can easily be extended to other models supporting oscillons. In a first step we perturb the field around the oscillon solution $\phi_{osc} \sim a(r)\cos(\omega t)$ found via the two-timing analysis

$$\phi(x,t) = \phi_{osc}(x,t) + \delta(x,t) \tag{74}$$

We assume that the field $\delta(x,t)$ is small. Plugging this ansatz into the equation of motion and linearizing the result we obtain the equation that $\delta(x,t)$ satisfies

$$\ddot{\delta} - \nabla^2\delta + \delta - 3\phi_{osc}^2\delta + 5\phi_{osc}^4\delta = 0 \tag{75}$$

The terms containing the oscillon solutions $\sim \phi_{osc}$ act as a driving force for the field $\delta(x,t)$. For some initial conditions $\delta(x,0)$ (obviously if $\delta(x,0) = 0$ it will remain 0) this driving force can cause the perturbation to grow giving rise to instabilities. The equation is linear in the field $\delta(x,t)$ and the first reflex of any good physicist should be to to solve this equation in Fourier space. However, because the oscillon has a spatial structure all Fourier modes are coupled to one another and an analysis for generic initial conditions is difficult. However, a great simplification can be made by separating the analysis for perturbations in two categories. On the one side we analyse perturbations of very short wavelengths (relative to the oscillon), while on the other side we analyse perturbations that are about the same size as the oscillons. For short wavelengths the oscillon only varies very slowly with respect to the perturbation itself. Near the center of the oscillon we can approximate the oscillon as a constant oscillating background. The equation of motion (75) can then be solved in Fourier space using Floquet theory. Perturbations that are about the same size of the oscillon can be analysed using a criterion discovered by mathematicians in the 70's, called the Vakhitov-Kolokolov criterion. Both analyses will be performed separately.

1. Short Wavelengths: Floquet theory

The two-timing analysis automatically gives oscillons that are quite wide via the introduction of the variable $\rho = \epsilon r$. The width of the oscillon is thus of $O(1/\epsilon)$ in real space, meaning that in Fourier space it has a width of $O(\epsilon)$ (since $k \propto \lambda^{-1}$). When speaking of short wavelength perturbations we mean those k-modes for which $k \gg \epsilon$. These modes don't "feel" the large scale spatial variation of the oscillon solution and it can therefore be approximated as a constant oscillating background. For perturbations living near the center of the oscillon we write ϕ_{osc} as

$$\phi_{osc}(x,t) = \epsilon a(0)\cos(\omega t) = \epsilon a_0 \cos(\omega t) \tag{76}$$

In this approximation the oscillon has no complicated spatial structure, and there is nothing stopping us from analysing the problem in Fourier space. Plugging in the approximation (76) into the equation of motion (75) and switching to Fourier space we obtain

$$\ddot{\delta}_k + \left(1 + k^2 - 3\epsilon^2 a_0^2 \cos(\omega t)^2 + 5\epsilon^4 a_0^4 \cos(\omega t)^4\right)\delta_k = 0 \tag{77}$$

Setting $A(t) = 1 + k^2 - 3\epsilon^2 a_0^2 \cos(\omega t)^2 + 5\epsilon^4 a_0^4 \cos(\omega t)^4$, equation (77) becomes

$$\ddot{\delta}_k + A(t)\delta_k = 0 \tag{78}$$

Now clearly the function $A(t)$ is periodic. Namely, $A(t + \frac{2\pi}{\omega}) = A(t)$. Equation (78) is a particular formulation of what is known as "Hill's equation", named after mathematician George Hill, who introduced it in 1886. The equation can be solved using Floquet theory. Set $x_1 = \delta_k$ and $x_2 = \dot{\delta}_k$, then equation (78) can be written in matrix form

$$\dot{\mathbf{x}} = \mathbf{A(t)x} \tag{79}$$

Where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A(t)} = \begin{bmatrix} 0 & 1 \\ -A(t) & 0 \end{bmatrix} \tag{80}$$

Floquet theory can be applied to systems which satisfy equation of the form (79) when $\mathbf{A}$ is periodic, which clearly is the case here. Floquet theory then tells us that the solutions of $\mathbf{x}$ are of the form $\mathbf{x} \propto \mathbf{e}^{\mu \mathbf{t}}\mathbf{p(t)}$ where $\mathbf{p(t)}$ is a periodic function with the same period as $\mathbf{A(t)}$. μ is referred to as the Floquet exponent and controls whether the solution grows or oscillates. This prompts us to seach for solutions of the form

$$\delta_k = \sum_{n=0}^{\infty} c_n(t)\cos(n\omega t) + b_n(t)\sin(n\omega t) \tag{81}$$

Where it is assumed that the function $c_n(t)$ and $b_n(t)$ are slowly varying with respect to the period $T \sim \frac{2\pi}{\omega}$ since the solutions of (77) are purely oscillatory for $\epsilon \to 0$. To solve this system using Floquet theory we rewrite equation (77)

$$\ddot{\delta_k} + \left(\Omega_k^2 + \epsilon^2(\beta\cos(2\omega t) + \gamma\cos(4\omega t)\right)\delta_k = 0 \tag{82}$$

Where

$$\begin{aligned} \Omega_k^2 &= 1 + k^2 - \epsilon^2(\frac{3}{2}a_0^2 + \epsilon^2\frac{15}{8}a_0^4) \\ \beta &= -\frac{3}{2}a_0^2 + \epsilon^2\frac{5}{2}a_0^4 \\ \gamma &= \epsilon^2\frac{5}{8}a_0^4 \end{aligned} \tag{83}$$

The Floquet exponent for this set of equations system was found approximately in [18] in the context of oscillon formation during preheating. The derivation of the solution will be given precisely in the next chapter, when we discuss the role of oscillons in cosmic history. For now, the reader must content himself with simply the result:

$$\boxed{\mu_k = \pm\frac{1}{4\omega}\sqrt{\beta^2 - 4(\omega^2 - \Omega_k^2)^2}} \tag{84}$$

In the region in phase-space (a_0, k) where the Floquet exponent μ_k is real perturbations will grow, and have a chance of destroying the oscillon. Thus we want to find the region where

$$\boxed{\beta^2 - 4(\omega^2 - \Omega_k^2)^2 > 0 \wedge k \gg \epsilon} \tag{85}$$

Where the second condition comes from the initial assumption that the width of the perturbation is smaller than the oscillon itself. It has been suggested in [10] that the instability band for the conditions in (85) are very narrow, scaling with ϵ^2. As $\epsilon \to 0$ these instability bands can be neglected.

2. Long Wavelengths: the V-K criterion

Applying Floquet theory to perturbations that live on the same spatial scale as the oscillon is a lot more complicated since we now have to consider the spatial structure of the oscillating background. This can in general not be done analytically. Luckily, Vakhitov and Kolokolov developed a clever way to assess the stability of localized configurations in nonlinear field theories. This criterion was later extended to assess the stability of oscillons to exactly these type of large scale perturbations. In what follows I will show how to apply the V-K criterion in this context, and how to derive it exactly.

Since we're interested in perturbations that are about the same width as the oscillon itself, we again switch variables

$$r \to \rho = \epsilon r \tag{86}$$

Assuming that the perturbation $\delta(x,t)$ is spherically symmetric and plugging in the oscillon solution $\phi_{osc}(x,t) = \epsilon a(\epsilon x)\cos(\omega t)$, the equation of motion (75) becomes

$$\ddot{\delta} - \epsilon^2 \partial_\rho^2 \delta - \epsilon^2 \frac{d-1}{r} \partial_\rho \delta + \delta - 3\epsilon^2 a^2 \cos(\omega t)^2 \delta + O(\epsilon^4) = 0 \tag{87}$$

Where $\omega = 1 + \kappa\epsilon^2$ (since $m = 1$ in this specific theory). Remember that κ is a free parameter of $O(1)$ that is negative and enters during the two-timing analysis. Since the perturbations are small. Now, intuitively it is expected that the perturbation will have behavior on two time scales. On the one hand it will oscillate with the oscillon on time scales of order $1/\omega$ while it might also start to grow slowly. This is analogous to the two time scales found in the two-timing analysis and prompts the introduction of two new variables

$$\begin{aligned} T &= \omega t \\ \tau &= \epsilon^2 t \end{aligned} \tag{88}$$

So in the equations of motion we must change $\partial_t \to d_t$ and therefore $\partial_t^2 \to \omega^2 \partial_T^2 + 2\epsilon^2 \omega \partial_T \partial_\tau + \epsilon^4 \partial_\tau^2 = \partial_T^2 + \epsilon^2 \partial_\tau \partial_T + 2\kappa\epsilon^2 \partial_T^2 + O(\epsilon^4)$. Finally we have

$$\partial_T^2 \delta + \epsilon^2 \partial_\tau \partial_T \delta + 2\kappa\epsilon^2 \partial_T^2 \delta - \epsilon^2 \partial_\rho^2 \delta - \epsilon^2 \frac{d-1}{r} \partial_\rho \delta + \delta - 3\epsilon^2 a^2 \cos(T)^2 \delta + O(\epsilon^4) = 0 \tag{89}$$

As the attentive reader might at this point realise, we would like to solve the equation perturbatively. We perform an ϵ expansion of the perturbation δ and investigate the equation of motion order by order. So we write

$$\chi = \delta_0 + \epsilon\delta_1 + ... \tag{90}$$

The zeroth and second order equation then become respectively

$$\partial_T^2 \delta_0 + \delta_0 = 0 \tag{91}$$

$$\partial_T^2 \delta_2 + \delta_2 = -\left(\partial_T \partial_\tau - \partial_\rho^2 - \frac{d-1}{\rho}\partial_\rho - 3a^2\cos(T)^2 + 2\kappa\partial_T^2\right)\delta_0 \tag{92}$$

The most general solution of the zeroth order equation can be written as

$$\delta_0(T,\tau,\rho) = u(\rho,\tau)\cos(T) + v(\rho,\tau)\sin(T) \tag{93}$$

As expected the perturbation oscillates with the oscillon itself at frequency ω. The potential growth of the perturbations is entirely encoded in the functions $u(\tau,\rho)$ and $v(\tau,\rho)$. The question is thus there are solutions for $u(\tau,\rho)$ and $v(\tau,\rho)$ that grow on timescales $\sim \tau$. By assumption, the perturbation $\delta(x,t)$ is small, and there can therefore be no resonant terms at each order in perturbation. Plugging in (93) into (92) and gathering secular terms on the right hand side

$$\begin{aligned} \partial_T^2 \delta_2 + \delta_2 = &-\left(\partial_\tau v - \partial_\rho^2 u - \frac{d-1}{\rho}\partial_\rho u - \frac{9}{4}a^2 - 2\kappa u\right)\cos(T) \\ &+\left(\partial_\tau u + \partial_\rho^2 v + \frac{d-1}{\rho}\partial_\rho v + \frac{3}{4}a^2 v + 2\kappa v\right)\sin(T) + h.h. \end{aligned} \tag{94}$$

And cancelling the secular terms gives us two linear equations for $u(\tau,\rho)$ and $v(\tau,\rho)$

$$\partial_\tau u = -Lv \tag{95}$$

$$\partial_\tau v = Mu \tag{96}$$

Where $L = \partial_\rho^2 + \frac{d-1}{\rho}\partial_\rho + \frac{3}{4}a^2 + 2\kappa$ and $M = \partial_\rho^2 + \frac{d-1}{\rho}\partial_\rho + \frac{9}{4}a^2 + 2\kappa$ are linear operators. Since these equations are linear we can perform a separation of variables and write $u(\tau,\rho) = u(\rho)e^{\Omega\tau}$ and $v(\tau,\rho) = v(\rho)e^{\Omega\tau}$. With this assumption equations (95) and (96) reduce to one "master" equation

$$-\Omega^2 u = LMu \tag{97}$$

The issue is now reduced to a simple linear eigenvalue problem. In fact, the only thing we now need to know is whether the operator LM has a (real) negative eigenvalue. In that case Ω will be real; the perturbation will grow and eventually force the oscillon away from its stable configuration. The Vakhitov-Kolokolov criterion can tell us if $\max(\Omega^2) > 0$. I will derive it in what follows.

To find the smallest eigenvalue of LM we would like to invert the operator L. Notice however that L acting on the oscillon spatial profile function $a(\rho)$ is actually nothing more than the profile equation itself. The function $a(\rho)$ is an eigenvector of L with eigenvalue 0. This means that L can not be inverted in the full vector space. However, from (95) and (96) it follows that

$$\Omega \langle u|a\rangle = \langle -Lv|a\rangle = -\langle v|La\rangle = 0 \tag{98}$$

Where I've made use of the fact that L is a real operator. Since we're interested in functions $u(\rho)$ for which $\Omega \neq 0$, it is sufficient to consider the subspace of functions that is orthogonal to $a(\rho)$. Since $a(\rho)$ is non-vanishing L is non-negative and has exactly one eigenvector with eigenvalue 0, namely $a(\rho)$. This means that on the subspace orthogonal to this function L is positive definite and we can rewrite (97) as

$$-\Omega^2 = \frac{\langle u| M |u\rangle}{\langle u| L^{-1} |u\rangle} \tag{99}$$

So we need to find out if there is a function $u(\rho)$ orthogonal to $a(\rho)$ for which the right hand side (99) is negative. In particular, since L^{-1} is positive definite, this is the case when

$$G = \min(\langle u| M |u\rangle) < 0 \tag{100}$$

So all we need to do is to check this condition on G. To find G we use the method of indefinite Lagrange multipliers. Namely, if we write

$$M |u\rangle = \lambda |u\rangle + \alpha |a\rangle \tag{101}$$

Then G is the smallest value of λ for which there exists a function $u(\rho)$ orthogonal to $a(\rho)$ and an α such that (101) is solved. To find the exact value of this particular λ one would need to check this for a complete set of functions $u(\rho)$. However we are only interested in the sign of λ. Expanding $|u\rangle$ and $|a\rangle$ in a complete set of orthogonal eigenfunctions $|\psi_n\rangle$ (with corresponding eigenvalues λ_n) of M, we obtain an expression for $|u\rangle$

$$|u\rangle = \alpha \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \lambda} |\psi_n\rangle \tag{102}$$

Where $c_n = \langle\psi_n|a\rangle$. However, since we require $\langle a|u\rangle = 0$ and $\langle a| = c_n \langle\psi_n|$, we must also have

$$\alpha \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n - \lambda} = \alpha f(\lambda) = 0 \tag{103}$$

So either $\alpha = 0$ or $f(\lambda) = 0$. From numeric investigations it turns out that M has exactly one negative eigenvalue and that the corresponding eigenvector is non-vanishing. Since for $\alpha = 0$, the function $u(\rho)$ is an eigenfunction of M and since $a(\rho)$ is also non-vanishing we conclude that $\lambda \geq 0$ in order for $\langle u|a\rangle = 0$. The situation where $\alpha \neq 0$ might therefore be a bit more interesting. In this cases λ is determined by the smallest root of the equation

$$f(\lambda) = \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n - \lambda} = 0 \tag{104}$$

Since we just concluded that $c_1 \neq 0$ corresponding to the only eigenvalue for which $\lambda_n < 0$, we conclude that the smallest root must lie between $\lambda_1 < \lambda_{min} < \lambda_2$. Since the function $f(\lambda)$ only increases between these two values of λ, it suffices to know the value at $\lambda = 0$ in order the know if the smallest root of $f(\lambda)$ is negative. In particular, if $f(0) > 0$ then $\lambda_{min} < 0$. We can then write

$$f(0) = \sum_{n=1}^{\infty} \frac{c_n^2}{\lambda_n} = \sum_{n=1}^{\infty} \frac{\langle a|\psi_n\rangle \langle\psi_n|a\rangle}{\langle\psi_n| M |\psi_n\rangle} = \langle a| M^{-1} |a\rangle \tag{105}$$

So we are essentially interested in $\mathrm{sgn}(\langle a| M^{-1} |a\rangle)$. To find this let's derive the full profile equation with respect to the free parameter κ. So

$$\frac{\partial La}{\partial \kappa} = 0 = M\frac{\partial a}{\partial \kappa} + 2a \tag{106}$$

Thus we conclude that $M^{-1}|a\rangle = -\frac{1}{2}\frac{\partial a}{\partial \kappa}$. PLugging this into equation (105) and rewriting we come to the conclusion

$$\langle a| M^{-1} |a\rangle = -\frac{1}{2}\left\langle a \middle| \frac{\partial a}{\partial \kappa} \right\rangle = -\frac{1}{4}\frac{\partial}{\partial \kappa}\langle a|a\rangle \tag{107}$$

Finally we have derived the illustrious Vakhitov-Kolokolov criterion

$$\boxed{G = -\,\mathrm{sgn}\,\frac{\partial I}{\partial \kappa}} \tag{108}$$

Where $I = \int d\rho^d a(\rho)^2$ is the definition of the inner product in our vector space. In particular, if $G < 0$, there are perturbative modes of about the same size as the oscillon that will grow and force the field configuration away from the oscillon solution.

All of this might sound as a lot of mathematical mumbo-jumbo, which it is in a sense. To end this section I will try to give some intuition about the meaning of the criterion. Remember that κ was a free parameter that controls the frequency of the oscillon solution through $\omega = 1 + \kappa\epsilon^2$. Increasing κ will thus increase the frequency of the oscillon. Thinking about this from the particle perspective, this simply means that κ controls the effective mass of the particles in the oscillon. Since κ must be negative for oscillons, this also explains why oscillons are stable under the V-K criterion: the particles in the oscillon can lower their mass by becoming part of it. Therefore the criterion tells us that, if the oscillon loses particles (approximately governed by I) by decreasing the mass of the individual particles, the solution will be unstable. The individual particles will tend to decrease their own effective mass, which decreases the total amount of particles in the oscillon. The oscillon decays thus decays. Off course, this explanation isn't perfect, it is just meant to give some intuition about the interpretation of the criterion.

F. Summary

In this chapter we've thoroughly investigated the characteristics of oscillons in a single-field nonlinear scalar theory. First of all, we've seen what characteristics the nonlinear potential should possess in order to support oscillons in the first place. The only requirement being that the potential is "shallower than quadratic" for some range of the field values. Next, we've seen how to analytically construct oscillons in a model that contained up to sextic terms in the potential, although the method can easily be generalized to more complicated models. Via the two-timing analysis we found oscillons with a $\sec(x)$ like profile, differing from the Gaussian packets that are often taken as an ansatz for oscillons in the literature. We also showed that the two-timing analysis will in general not give an exact solution. The oscillon will have a small radiating tail that is exponentiall suppressed. This also explains why oscillons can be so long-lived. Finally, we investigated the stability of oscillons to small perturbations. It was found nearly impossible to solve this problem completely, which is why the problem was separated into small- and large wavelength regimes. For small wavelengths we could use Floquet theory, while the Vakhitov-Kolokolov criterion was derived for long wavelength perturbations. Although the analysis is not complete, these derivations can give us a good idea of what oscillon solutions are stable. We thus have completely dissected single-field oscillons: their analytical construction, asymptotic behavior and linear stability were all analysed. In later chapters we'll use similar methods to analyse oscillons in more complicated models, and also perform numerical simulations in order to support our conclusions.

III. OSCILLONS IN THE PHYSICAL WORLD

The reader should at this point have a solid grasp of (single-component) oscillons. Namely, how to construct them, under what conditions they are stable and why they are so extremely long-lived. The reader would at this point be excused in thinking that oscillons are nothing more than mathematical oddities, arising as non-trivial solutions of nonlinear PDE's. Although it has been shown that oscillons are supported in certain sections of the Standard Model, and it is a real possibility that the Inflaton supports oscillons as well, there is no a priori reason to assume that there are natural processes that create them. At this point it is therefore not at all clear if they have a role to play in cosmic history.

Since this thesis is being written in the context of Physics and not Mathematics it might be wise to spend some time outlining the role of oscillons in the physical world. Namely, during what processes they are formed and what empirical imprints they might leave behind. While much work has been done in this regard, the most interesting claims come from their potential formation during the parametric resonance phase of (p)reheating. In fact, (Slow-Roll) Inflation is one of the most famous examples where nonlinear scalar field dynamics play an important role (together with the Higgs mechanism of course). It is therefore to be expected that oscillons are an eminent figure during this period. In this chapter I will mainly focus on this period, briefly outlining some other cases where oscillons become important in later sections. It will now be useful to shortly recap what Slow-Roll Inflation and (p)reheating entail exactly.

A. Slow-Roll Inflation

There are several cosmological observations that can be explained by a period of accelerated (exponential) expansion during the early Universe, called Inflation. The most important of these observations are often referred to as the Horizon problem, the Flatness problem and the Magnetic monopole problem. The Horizon problem is the fact that different regions of the sky share the same thermal imprint, although there should not have been enough time for these regions to come into thermal equilibrium. This problem is solved by Inflation since the Hubble radius shrinks during exponential expansion meaning that regions in thermal equilibrium can get out of causal contact. The Flatness problem is the simple fact that we seem to live in an exceedingly flat Universe. This fact is very hard to explain without exponentially fine-tuned initial conditions. Inflation flattens the curvature of the Universe to such an extent that the initial conditions of the Universe become in fact irrelevant to the late time observation of cosmological curvature. Finally, the monopole problem highlights the contradiction between the theoretical formation of magnetic monopoles at some GUT-scale and the lack of observation of such particles. Accelerated expansion in the early Universe would dilute these particles, making it nearly impossible to detect them in our time. Besides the resolution of these problems, Inflation also offers a natural way to introduce scalar (and tensor) perturbations in the hot Big Bang model. I will not go further into explaining the appeal of Inflation here and instead refer the reader to the many pages of literature dedicated to this topic.

Although many theoretical models exist that explain how an era of accelerated expansion could occur in the early Universe, I will focus here on the simplest one. This model, referred to as Slow-roll Inflation, involves a single scalar field, called the Inflaton, that is minimally coupled to Gravity. The action is given by

$$S = \int d^4x\sqrt{-g}[\frac{1}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)] \tag{109}$$

Assuming a FRW background geometry and an homogeneous scalar field, the dynamics of the system are given by the Friedmann equation

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right) \tag{110}$$

and the equation of motion of the scalar field

$$\ddot{\phi} + 3H\dot{\phi} + V_\phi(\phi) = 0 \tag{111}$$

Finally, the energy momentum tensor, defined as $\frac{\delta S}{\delta \phi}$ shows that the scalar field behaves as a perfect fluid with

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \tag{112}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \tag{113}$$

It is now important to realize that all of the above equations are very general statements about homogeneous scalar fields in a FRW geometry. We will need certain restrictions on our potential in order to cause a period of accelerated expansion. Namely, the acceleration equation in this system becomes

$$\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_\phi + 3P_\rho) \tag{114}$$

In order to have a period of accelerated expansion, the right hand side of this equation needs to be positive and we therefore require

$$\frac{\rho_\phi}{P_\rho} = \frac{\frac{1}{2}\dot{\phi}^2 + V(\phi)}{\frac{1}{2}\dot{\phi}^2 - V(\phi)} > -3 \tag{115}$$

From (115) we obtain a heuristic argument for Slow-roll Inflation: if there is a region of the potential that is shallow, so that the scalar field kinetic energy is small and the potential energy dominates, equation (115) will always be satisfied. The field is than "slowly rolling" towards the potential minimum. In this case, where $\dot{\phi} \ll V(\phi)$, the Hubble parameter given by equation (110) becomes approximately constant

$$H^2 \approx \frac{1}{3}V(\phi) \tag{116}$$

Such that the expansion of the Universe is quasi-exponential

$$a(t) \sim e^{Ht} \tag{117}$$

And H is slowly varying. The fact that the accelerated expansion of the Universe needs to persist for a certain amount of e-folds in order to solve the aforementioned empirical problems puts even more heuristic restrictions on the Inflaton potential. These restrictions are summarized in the slow-roll parameters

$$\epsilon_V = \frac{M_{pl}^2}{2}\left(\frac{V_\phi}{V}\right)^2 \tag{118}$$

$$\eta_V = M_{pl}^2\left(\frac{V_{\phi\phi}}{V}\right)^2 \tag{119}$$

Where subscripts signify derivatives with respect to ϕ. In the region of the potential where slow-roll inflation occurs we must then have

$$\epsilon_V, |\eta_V| \ll 1 \tag{120}$$

In order to understand oscillons in the context of Inflation it is not necessary to understand exactly how these parameters are derived. The main message for the reader to take away is that it is widely believed that a period of accelerated expansion occurred in the Early Universe. This type of expansion can be sourced by a single scalar field "slowly rolling" down a potential. Inflation ends as the Inflaton nears some local or global minimum. As the Inflaton oscillates around this minimum it starts to shed its energy into other fields. Due to the probabilistic nature of this decay (it is

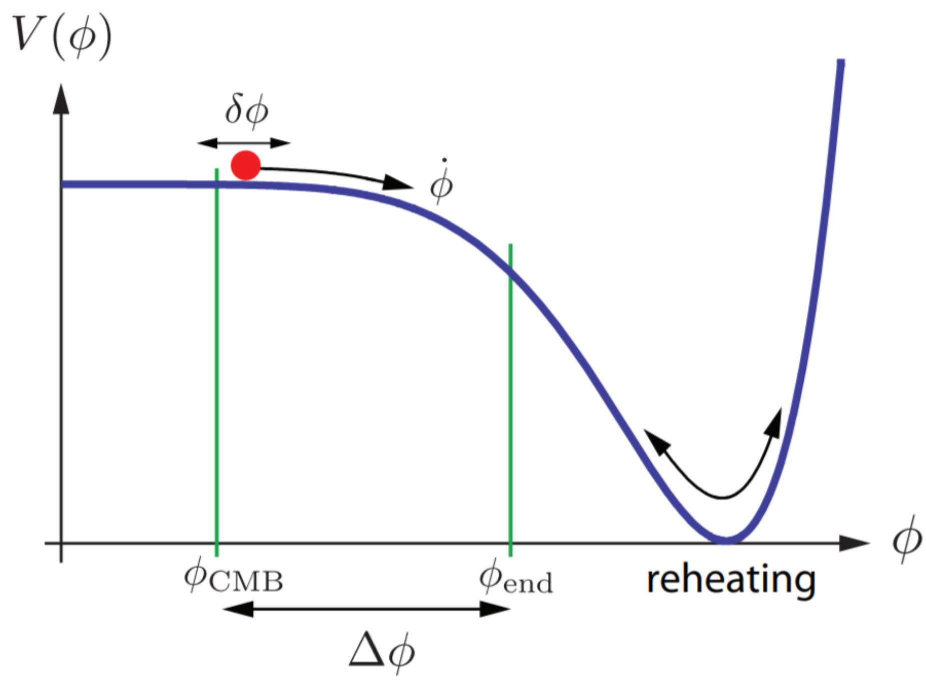


FIG. 2: The slow-roll Inflation scenario: the scalar Inflaton slowly rolls down a potential hill sourcing accelerated expansion. As the field reaches its minimum, it starts to oscillate and deposit energy into other fields. This is the start of reheating.

governed by Quantum Mechanics), the scalar field loses homogeneity and starts to "fragment". This stage in cosmic history, referred to as (p)reheating, is the birthplace of the particles of the Standard Model and the start of the Hot Big Bang. In the next sections we'll see that if the Inflaton supports oscillons, we should expect a significant amount of energy to be frozen in the form of oscillons as well.

B. Reheating

As the Inflaton slowly rolls down the potential, sourcing accelerated expansion of the Universe, there will be a point where the conditions (120) are no longer met and Inflation stops. The Inflaton ϕ then starts to oscillate around some global or local minimum of the potential $V(\phi)$. The Inflaton ϕ then satisfies approximately

$$\phi(t) = \phi_0 + \Phi(t)\cos(\omega t) \tag{121}$$

Where ϕ_0 is the vacuum value of the field and $\Phi(t_0) = \phi_c$, the field value where the conditions (120) fail. During this epoch the Inflaton equation of state parameter is effectively $\omega_{eff} = 0$, and the

Universe evolves as if it was matter dominated. The Hot Big Bang model requires thermal initial conditions however. In standard Inflationary lore this happens through coupling of the Inflaton to gravity and other (Standard Model) fields [30]. This can be modelled approximately by adding an extra friction term to the equation of motion of the scalar Inflaton

$$\ddot{\phi} + H\dot{\phi} + \Gamma\dot{\phi} + V_{\phi}(\phi) = 0 \tag{122}$$

Where Γ is the decay rate of the Inflaton into other fields. This equation can be interpreted as follows: the Inflaton loses energy both through coupling to other fields (through Γ) and through expansion (through H). Since expansion slows down as the Universe expands (we're in an matter dominated phase), there will be a moment in time where $\Gamma = H$. This fixes the temperature scale at which the Hot Big Bang start, since most energy in the Inflaton will decay into particles after this moment in time.

In general, using equation (122) will not give very accurate results. In particular, it neglects fluctuations in both the Inflaton and the metric. Furthermore, finding the decay rate of the Inflaton through the creation of particles from some vacuum state requires a full quantum mechanical calculation. Both of these points lead to an inhomogeneous decay of the Inflaton where particles and fluctuations are created locally. This epoch of "local" decay is referred to as preheating. Only after interactions between the locally created particles, does the Universe thermalize. In what follows I'll show how fluctuations in the field can be enhanced during preheating, forming oscillons.

C. Oscillon Formation During Preheating

As the Inflaton oscillates at the bottom of its potential during preheating it can excite fluctuations that eventually become oscillons. To analyse this process we approximate the Inflaton as a homogeneous background field. For oscillons to form the Inflaton potential should satisfy condition (21). Near the vacuum of the field we therefore approximate it as

$$V(\phi) = \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4}\phi^4 + \frac{g}{6}\phi^6 \tag{123}$$

Where we follow the analysis in [18] and [17] by choosing $\lambda, m \sim O(1)$ and $g \gg 1$. The equation of motion of the background oscillating field $\bar{\phi}$ is then given by

$$\ddot{\bar{\phi}} + H\dot{\bar{\phi}} + V_{\phi}(\bar{\phi}) = 0 \tag{124}$$

We can find an approximate solution for the homogeneous oscillating background if we assume the field to be small. Looking at our ansatz in (121) we see that the energy density of the background scales with $V(\bar{\Phi}(t)) \approx \frac{1}{2}m^2\bar{\Phi}(t)^2$. Since the background is effectively in a matter-phase we conclude $\bar{\Phi}(t) = \frac{\bar{\Phi}_i}{\sqrt{a(t)^3}}$. This allows us to write down an approximate form for our background oscillating Inflaton

$$\bar{\phi}(t) = \frac{\bar{\Phi}_i}{\sqrt{a(t)^3}} \cos(\omega t) \tag{125}$$

Furthermore

$$H = \frac{H_i}{\sqrt{a(t)^3}} \tag{126}$$

Since we're in a matter-dominated epoch. Since the background field $\bar{\Phi(t)}$ is small by assumption we can plug in $\bar{\Phi(t)} = \epsilon\Phi_0(t)$ (where $\epsilon \ll 1$) into (124) and conclude conclude $\omega^2 = m^2 - \epsilon^2(\frac{3\lambda}{4}\Phi_0^2 + \kappa\frac{5}{8}\Phi_0^4) + O(\epsilon^3)$. Here I've used the fact that $g \gg 1$ and set $g = \frac{\kappa}{\epsilon^2}$. The question now is how fluctuations of the field are amplified in this oscillating background. The analysis is closely related to the linear stability analysis performed in the last chapter where a key difference is that we are now dealing with a homogeneous background. Adding a fluctuation atop of the background field $\bar{\phi}(t) \to \bar{\phi}(t) + \delta\phi(x,t)$. The equations of motion for $\delta\phi(x,t)$ then are

$$\ddot{\delta\phi} - \frac{\nabla^2}{a^2}\delta\phi + 3H\dot{\delta\phi} + V_{\phi\phi}(\bar{\phi})\delta\phi = 0 \tag{127}$$

Since this equation is linearized we can best switch to Fourier space

$$\delta\ddot{\phi}_k + \frac{k^2}{a^2}\delta\phi_k + 3H\delta\dot{\phi}_k + V_{\phi\phi}(\bar{\phi})\delta\phi_k = 0 \tag{128}$$

Finding the solutions for the different modes $\delta\phi_k$ can thus be done using Floquet theory. We'll first solve the problem in a static background ($H = 0$ and $a = 1$) and extend the solution to understand the behavior in an expanding background afterwards. In a static background

$$\delta\ddot{\phi}_k + k^2\delta\phi_k + V_{\phi\phi}(\bar{\phi})\delta\phi_k = 0 \tag{129}$$

If the background is static the Inflaton oscillates as

$$\bar{\phi} \approx \epsilon\Phi_0\cos(\omega t) \tag{130}$$

Where $\omega^2 \approx m^2 - \epsilon^2(\frac{3\lambda}{4}\Phi_0^2 + \kappa\frac{5}{8}\Phi_0^4)$. We thus need to solve the following equation

$$\delta\ddot{\phi}_k + (m^2 + k^2 - \epsilon^2(3\lambda\Phi_0^2\cos(\omega t)^2 - 5\kappa\Phi_0^4\cos(\omega t)^4)\delta\phi_k = 0 \tag{131}$$

Where again I've set $g = \frac{\kappa}{\epsilon^2}$. This equation can clearly be solved using Floquet theory. An approximate solution for the Floquet exponents of this equation will be found in the next section. These define instability bands in which fluctuations will grow (and maybe even form oscillons). A final remark regarding equation (127) is now in order. A complete analysis of the growth of fluctuations should also consider fluctuations in the metric $\sim \delta g^{\mu\nu}$. This will add an extra term to the equation of motion. We choose here to focus solely on growth through parametric resonance of the background field and ignore these extra terms. Some interesting work regarding the effect of metric fluctuations during preheating has been done in [31, 32].

1. *Finding instability bands: Floquet again*

Finding the instability bands of equation (131) can be done using Floquet theory. In fact, it can be rewritten into equation (132), and thus has similar solutions for the Floquet exponents. We are now not limited by the assumption that $k \gg \epsilon$, so that fluctuations can grow on all scales. We will now see how to find the exact form of the Floquet exponents μ_k. We rewrite the equation in (131) as we did before in chapter II. It becomes

$$\delta\ddot{\phi}_k + \left(\Omega_k^2 + \epsilon^2(\beta\cos(2\omega t) + \gamma\cos(4\omega t)\right)\delta\phi_k = 0 \tag{132}$$

Where

$$\begin{aligned}
\Omega_k^2 &= m^2 + k^2 - \epsilon^2(\lambda\frac{3}{2}a_0^2 + \kappa\frac{15}{8}a_0^4) \\
\beta &= -\lambda\frac{3}{2}a_0^2 + \kappa\frac{5}{2}a_0^4 \\
\gamma &= \kappa\frac{5}{8}a_0^4
\end{aligned} \tag{133}$$

In [18] the instability bands were found by looking for solutions of the form

$$\delta\phi_k = \sum_{n=1,2,3} a_n(t)\cos(\Omega_k t) + b_n(t)\sin(\Omega_k t) \tag{134}$$

Where the $a_n(t)$ and $b_n(t)$ were assumed to be slowly varying so that $\ddot{a}_n = \ddot{b}_n \approx 0$. Since the $n = 2k$ and $n = 2k+1$ terms decouple in the equations of motion the even n terms were set to zero without loss of generality. Plugging this ansatz into the equations of motion the authors were able to reduce the problem to a Floquet analysis with infinite oscillation period (i.e. constant coefficients).

$$\dot{\mathbf{x}} = \frac{1}{4\omega}\mathbf{A}\cdot\mathbf{x} + \mathbf{O}(\epsilon^{\mathbf{2}}) \tag{135}$$

Where

$$\mathbf{x} = \begin{bmatrix} a_1 \\ b_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & \beta + 2(\omega^2 - \Omega_k^2) \\ -\beta + 2(\omega^2 - \Omega_k^2) & 0 \end{bmatrix} \tag{136}$$

$\mathbf{A}$ has an infinite period (it is constant). This tells us that the solutions of this equation are given by $\mathbf{x} = \mathbf{e^{\mu_k t}p}$ where $\mathbf{p}$ is a constant vector. Inserting this into equation (135) we obtain the following expression

$$\mu_k\mathbf{x} = \frac{\mathbf{1}}{\mathbf{4}\omega}\mathbf{A}\mathbf{x} \tag{137}$$

We conclude that the Floquet exponents are simply the eigenvalues of the matrix $\frac{1}{4\omega}\mathbf{A}!$. They are now readily found to be

$$\boxed{\mu_k = \pm\frac{1}{4\omega}\sqrt{\beta^2 - 4(\omega^2 - \Omega_k^2)^2}} \tag{138}$$

Finally we obtain an analytic expression for the (growing) Floquet exponent to lowest order in ϵ

$$\mu_k = \frac{k}{2}\sqrt{\frac{3\lambda}{2}\left(\frac{\bar{\phi}}{m}\right)^2\left(1 - \left(\frac{\bar{\phi}}{\phi_i}\right)^2\right) - \left(\frac{k}{m}\right)^2} \tag{139}$$

Where I've set $\bar{\phi} = \epsilon\Phi_0$ and $\phi_i = \sqrt{\frac{3\lambda}{5g}}$. When the expression in (139) is real, modes will be amplified. Using it we can find instability bands in the $k - \bar{\phi}$ plane. In a static background modes will forever stay inside the instability bands and grow until they become nonlinear. However the situation is a bit more complex during preheating. In fact during preheating the Universe continues to expand, which we've ignored so fat. Both the physical length scale of a perturbation and the amplitude of the Inflaton change. Co-moving modes can therefore enter and an instability band, get amplified for some time, and stop growing once it exits the band again. We'll now see how amplification works in an expanding background.

2. Including expansion

To study the effect of an expanding background on the amplifications of fluctuations in the scalar field, we'll make the assumption that the location of the instability bands in the $k - \bar{\Phi}$ plane is not altered as the scale factor of the Universe grows. We've seen however that in an expanding background the Inflaton $\bar{\phi} \propto a^{-3/2}$. Furthermore the scale of any perturbation will grow with a and therefore $k \propto a^{-1}$. These considerations allow us to determine a trajectory in the $k - \bar{\Phi}$ plane, making the identifications

$$\begin{aligned} k &\to k_p = k_{co} a_i a^{-1} \\ \bar{\phi} &\to \bar{\phi} = \phi_i a_i^{3/2} a^{-3/2} \end{aligned} \tag{140}$$

Plugging these trajectories into (139) we get an expression for $\mu_k(a)$, the Floquet exponent for a perturbation with co-moving wavenumber k when the scale of the Universe is $a(t)$. Setting $a_i = 1$

$$\mu_k(a) = \frac{k}{2a}\sqrt{\frac{9\lambda^2}{10g}\left(\frac{1}{m^2 a^3}\right)(1 - a^{-3}) - \left(\frac{k}{ma}\right)^2} \tag{141}$$

The total amplification of a mode can now be estimated. The amplification of a mode during a time interval dt is given by

$$d\delta\phi_k = e^{\mu_k dt} = e^{\mu_k \frac{d\ln(a)}{H}} \tag{142}$$

The total amplification of a mode can now be obtained by integrating this expression along the trajectory in the $k - \phi$ plane. The boundaries of integration are obtained by requiring that the mode finds itself inside the principal instability zone (or in other words, scale factors for which μ_k is real). The total (linear) amplification of a mode is thus given by

$$A_{\delta\phi_k} = \exp\left[\int_\sigma d\ln(a) \frac{\frac{k}{2a}\sqrt{\frac{9\lambda^2}{10g}\left(\frac{1}{m^2a^3}\right)(1 - a^{-3}) - \left(\frac{k}{ma}\right)^2}}{H}\right] \tag{143}$$

From this expression it isn't yet clear why oscillons would emerge from this resonant phase of preheating. To fully answer this question we need to include nonlinear dynamics which is non-trivial. It is clear however that different modes get amplified differently, resulting in gradients in the field that can eventually settle into oscillons. In fact a mode is only amplified if

$$k^2 < \frac{9\lambda^2}{10g}\frac{1}{a}\left(1 - \frac{1}{a^3}\right) \tag{144}$$

This puts a bound on modes that can grow during expansion

$$k^2 < \frac{9\lambda^2}{10g}\max\left(\frac{1}{a}\left(1 - \frac{1}{a^3}\right)\right) \approx 0.42\frac{\lambda^2}{g} \tag{145}$$

From (145) we see that if $k \approx 0.65\frac{\lambda}{\sqrt{g}}$, no significant amplification can take place. Since by assumption in this model $g = \kappa/\epsilon^2$ we conclude that only modes of $O(\epsilon)$ can grow. As we've seen the width of oscillon is of $O(1/\epsilon)$ and thus only k-modes of $O(\epsilon)$ should be important in their

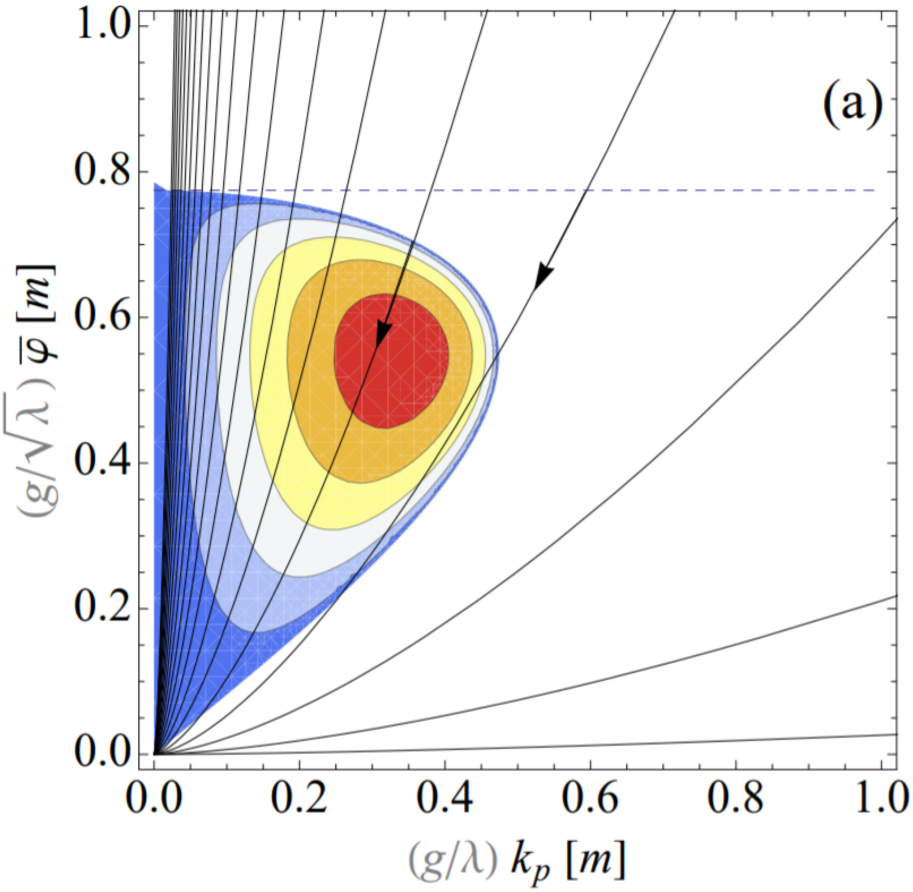


FIG. 3: The journey of different co-moving Fourier modes through phase space. As the Universe expands the Inflaton $\bar{\phi}$ slowly loses energy to the expansion of the Universe. Furthermore, the physical scale of the perturbation also increases. This causes different modes to get amplified by different amounts. Taken from [18].

formation. This makes it plausible that these modes, as they enter the nonlinear regime first, settle into an oscillon solution. Since oscillons can be extremely stable, even in an expanding background [10], these configurations will stick around and influence the post-Inflationary Universe. Next, we'll estimate the number density of oscillons that are formed in this way during parametric resonance.

D. Estimating the number density of oscillons

Although it is in practice impossible to understand all the complicated nonlinear dynamics that is involved in the formation of oscillons during preheating, an heuristic argument for estimating the number density of oscillons was given in [18]. I will repeat the argumentation here, and show some of the results obtained in the paper of Amin. As we've just seen, if a mode becomes nonlinear it means that its (co-moving) wavenumber is of $O(\epsilon)$. It therefore seems fair to assume that, through some complicated dynamics, the now nonlinear mode will settle into an oscillon. Its length scale is then approximately

$$R_{osc} \sim a\left(\frac{2\pi}{k}\right) \tag{146}$$

As an approximation we can assume that as soon as a mode becomes nonlinear it will fill out real space with oscillons of this length scale. The first mode that becomes nonlinear will dominate the landscape. The total amount of oscillons is then given by

$$N_{osc} \sim V\left(\frac{k_{nl}}{2\pi a}\right)^3 \tag{147}$$

Where V is the volume of the Universe and k_{nl} is the mode that becomes nonlinear first. The number density of oscillons is then estimated as

$$n_{osc}a^3 \sim \left(\frac{k_{nl}}{2\pi}\right)^3 \tag{148}$$

The prescription is now clear. We need to use the expression for the amplification (143) to calculate k_{nl}. To do this we need to specify initial conditions. In [18] these where chosen to be Gaussian

$$\delta\phi_{k,i} \approx \frac{1}{\sqrt{2(k^2+m^2)}} \tag{149}$$

So that the fluctuation at scale factor a is given by

$$\delta\phi_k = \frac{A_{\delta\phi_k}}{\sqrt{a^3}}\frac{1}{\sqrt{2(k^2+m^2)}} \tag{150}$$

The mode enters the nonlinear regime when $\delta\phi_k \sim \bar{\phi}$. At this point different modes will become coupled, and there is significant backreaction on the homogeneous evolution of the Inflaton. In essence, all assumptions needed to obtain the Floquet exponent break down. k_{nl} (and therfore n_{osc} can then be computed numerically. A rough analytic estimation that was somewhat consistent with simulations ($\sim 15\%$) was also given. The main point of the previous few sections is to show how parametric resonance can force small fluctuations in the Inflation into oscillon configurations. The analysis performed here was specific to one formulation of model that supports oscillons, but similar arguments will also work in more general situations. The fact that oscillons emerge under general conditions in simulations of Inflation seems to support this fact [17, 21]. In the last section of this chapter we'll see how the emerging oscillons can influence the early Universe.

E. Role of oscillons after Inflation

Although the analysis presented here is specific to a particular model, I hope that the reader has some intuition regarding the emergence of oscillons during the Early Universe. The fact that oscillons seem to emerge in simulations of preheating makes this fact even more plausible. In fact, most investigations into models that support oscillons reach the conclusion that a significant portion of the energy density of the Universe must be stored in the form of oscillons. The exact figure is dependent on the complicated oscillon dynamics specific to the model, but most figures have an estimate of $\rho_{osc}/\rho_{tot} \sim O(10\%)$. Once the oscillons are formed they can influence the Universe in a variety of ways. I will use this section to highlight the most important imprints they can leave behind.

Decay into particles We've seen that after preheating the Universe is essentially filled with oscillons. Although these radiate very slowly in a classical sense, there is literature suggesting that this is not true in a full quantum mechanical description. In [29] it was shown that oscillons can create particles very efficiently. In practice, it means that although the post-inflationary Universe might be filled with oscillons, they can very quickly decay into other particles. In this way oscillons would create localized regions in space, where the density of particles is high. This delays thermalization of the Universe and therefore the start of the Hot Big Bang scenario [19].

Gravitational Wave signals A general consequence of Inflation is the creation of tensor and scalar perturbations. The tensor perturbations are expected to be the source of a distinct Gravitational Wave signal. In recent years the GW imprint of oscillons has become a prevalent theme in research. In particular, they have been shown to leave an imprint on the GW power spectrum, generated during preheating, in symmetric potentials. This signal is mainly associated with their formation during preheating but it has also been suggested that in asymmetrical potentials oscillons can serve as sources for GW production. In these systems oscillons don't settle into spherically symmetric configurations as quickly, which is needed for the production of GWs [20, 33–35].

Seeds for structure formation Oscillons are localized regions of high energy density. Through interaction with the metric they create gravitational potential wells that can serve as the seeds for structure formation. If the oscillons formed during preheating have long enough lifetimes, and stick around until matter-radiation equality, they can leave an imprint on the power spectrum of the Universe, as they are the seeds of structure formation on small scales. It has even been suggested that oscillons can take on the role of Dark Matter if they are extremely long-lived [36, 37].

Emergence after thermalization In this chapter we've focused on oscillon formation during Preheating. However it has been suggested in [12] that oscillons also emerge in a thermal Universe as it cools down and expands. This speaks to the attractiveness of the oscillon solution in the space of possible field configurations.

F. Summary

In this chapter we've seen how oscillons emerge naturally in the early Universe. The analysis was performed for a specific model but the methods and ideas can easily be generalized. The main idea is the following: as the Inflaton enters its oscillatory phase, fluctuations can get amplified by the homogeneous background. The total amplification is dependent on the scale of the perturbation itself, meaning that different scales get amplified by different amounts. This eventually creates gradients in the field as different scales become nonlinear at different times. If the potential of the field allows for it there is a chance that the perturbations eventually settle into oscillons. This qualitative description seems to be supported by numerical simulations of Inflation where a significant portion of the energy density of the Universe is locked in the form of oscillons [17, 18, 21]. As we've seen oscillons are long-lived configurations and can thus significantly influence the physics of the early Universe. We've discussed how they can lead to an enhanced decay of the Inflaton, be a source of a GW signal, and serve as the seeds for structure formation. I hope I've convinced the reader that oscillons are not simply a mathematical oddity but can have real consequences in the natural world. Everything we've done until now has been in the context of single-field scalar models. In general, fields in nature tend to interact with each other. Because of the generality of oscillons and their potential role in cosmic history, it is important to understand them in the context of multi-field models. The rest of this thesis will therefore focus on oscillons in scalar field models involving two interacting fields.

IV. SYMMETRIC MULTI-FIELD OSCILLONS

In the first few chapters of this thesis we've seen the characteristics of oscillons in single-field models, and how they form after Inflation. In physical theories it is rare for fields to be decoupled. In fact, only if a system has specific symmetries it is possible for fields to be isolated. In the chapters that follow I'll therefore focus on oscillons in more complicated models involving two fields. The literature on this topic is rather scarce. Analysing the stability of oscillons in these models is difficult, and we'll be limited to numerical analysis most of the time. However, in this chapter we discuss a model with a specific exchange symmetry in which oscillons exist whose stability can be analysed exactly using an extension of the V-K criterion. To the best of my knowledge this is the first time that the V-K criterion is applied to multi-field systems. In the next chapter I'll consider a more general model with both cubic and quartic interactions.

A. The Model

In [10] stable oscillons were found to exist in a single-field model with a large sextic term. The Lagrangian in that model was given by

$$S = \int d^d x dt \sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4 - \frac{g}{6\epsilon^2}\phi^6] \tag{151}$$

Where $m, \lambda, g \sim O(1)$ and $\epsilon \ll 1$. This was the inspiration for the model that we'll consider in this chapter. The two-field model we'll consider here is

$$S = \int d^3x dt \left[\sum_{I=1,2}\left(\frac{1}{2}\partial_\mu\phi^I\partial^\mu\phi^I - \frac{1}{2}m^2\left(\phi^I\right)^2 + \frac{\lambda}{4}\left(\phi^I\right)^4 - \frac{g}{6}\left(\phi^I\right)^6\right) + \frac{\Lambda}{2}(\phi^1)^2(\phi^2)^2\right] \tag{152}$$

Where again $g \sim k/\epsilon^2$ is very large. This seems somewhat like a trivial model, but as we'll see the coupling between the fields greatly influences the stability of the oscillon solutions.

Before proceeding, it is worth noting both the restrictions that the action of Eq. (152) has as well as some theoretical motivation for considering it. We will draw our example from the well-motivated area of α-attractors, specifically the T-model potential. Since metrics on hyperbolic spaces can be formulated in many different forms, following Möbius transformations [38]. Following the metric used in Refs. [39–41]

$$ds^2 = d\chi^2 + e^2 b(\chi) s\phi^2 \tag{153}$$

where $b(\chi) = \log\left(\cosh(\beta\chi)\right)$. The two-field potential for the T-model in this field basis is

$$V(\phi,\chi) = \alpha\mu^2\left(\frac{\cosh(\beta\phi)\cosh(\beta\chi) - 1}{\cosh(\beta\phi)\cosh(\beta\chi) + 1}\right)\left(\cosh(\beta\chi)\right)^{2/\beta^2} \tag{154}$$

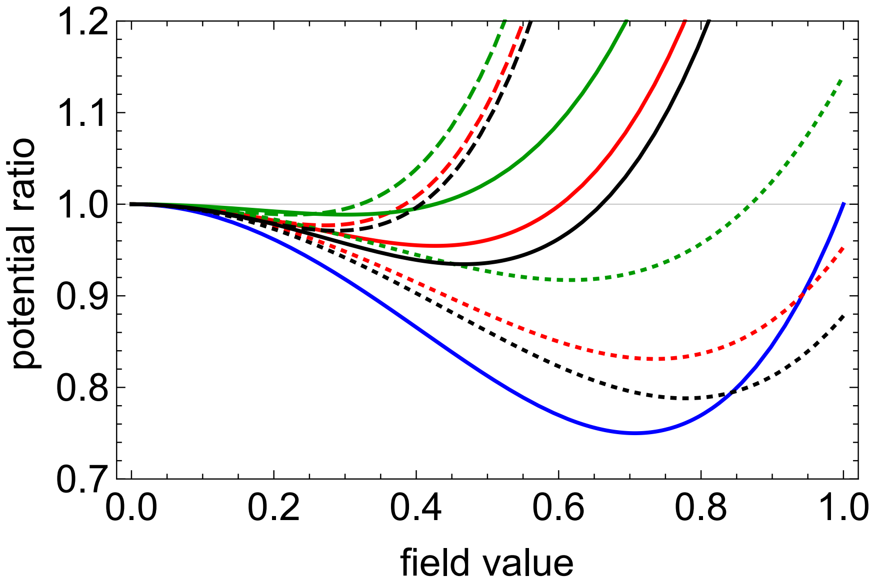


FIG. 4: The ratio of the potential of Eq. (152) over the quadratic term $\frac{V(\phi,\chi)}{m^2(\phi^2+\chi^2)/2}$ for ...

where $\beta = \sqrt{2/3\alpha}$.

For small value of α, the potential around the origin is expanded as

$$V(\phi,\chi) \simeq \frac{\mu^2}{6}\chi^2 - \frac{\mu^2}{54\alpha}\chi^4 + \frac{17\mu^2}{9720\alpha^2}\chi^6 + \frac{\mu^2}{6}\phi^2 - \frac{\mu^2}{54\alpha}\phi^4 + \frac{17\mu^2}{9720\alpha^2}\phi^6$$
$$+\frac{\mu^2}{6}\phi^2\chi^2 - \frac{\mu^2}{16\alpha^2}\phi^2\chi^2(\phi^2+\chi^2) \tag{155}$$

where we included terms up to sextic order in the fields and neglected higher order contributions in $1/\alpha$ in each term. We see that this expansion has the characteristics that we consider in our model: two scalar fields with identical potential parameters, a negative quartic term and a very large sextic term (since $\alpha \ll 1$, as well as a quartic coupling. Of course, the true T-model expansion differs from the idealized action of Eq. (152), in that it contains sextic interaction terms and it provides less freedom for choosing the various potential coupling strengths. Furthermore, the analysis of the T-model must also take into account the non-canonical kinetic structure, leading to the couplings

$$\mathcal{L}_{\rm kin} \supset \frac{\chi^2}{3\alpha}(\partial\phi)^2 + \frac{2\chi^4}{27\alpha^2}(\partial\phi)^2 + ... \tag{156}$$

Oscillons in systems with non-canonical kinetic terms have been considered in the literature, but only for single-field systems [**?**]. In the present work, we will only focus on the symmetric sextic potential of Eq. (152) and consider canonical kinetic terms for the two fields. An extension of this work to inherently two-field systems, such as α-attractors is left for future work.

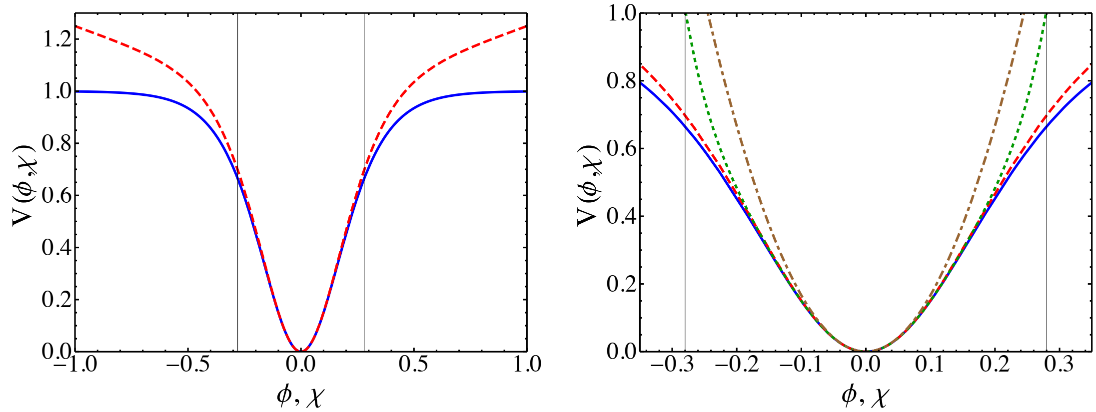


FIG. 5: The T-model potential for $\chi = 0$ (blue solid) and $\chi = 0$ (red dashed), along with the sextic Taylor expansion (green dotted). The brown dot-dashed curve shows the quadratic term, which is steeper than the total potential, allowing in principle for oscllilon formation in both ϕ and χ fields. The vertical lines show the field value of ϕ at the end of inflation.

B. Two-timing analysis

The T-model is an example of a well-motivated inflationary potential that possesses a similar Taylor-expansion as the model described by the action of Eq. (152). In what follows we'll focus on analysing oscillons in the somewhat more idealised system whose action is given by Eq. (152), leaving the inclusion of the kinetic terms of α-attractors for future work. In this section we show how to construct approximate profiles of two-field oscillons and relate them to the single-field oscillons found in Ref. [10].

Eq. (152) contains several parameters, including λ, m, Λ, g. It is convenient to work with dimensionless space-time variables and fields. This is done by the rescalings $x^\mu \to \tilde{x^\mu} = x^\mu m$, $\phi^I \to \tilde{\phi}^I = m^{-1}\lambda^{1/2}\phi^I$, $g \to \tilde{g} = (m/\lambda)^2 g$ and $\Lambda \to \tilde{\Lambda} = (1/\lambda)\Lambda$. The Lagrangian is then re-written as

$$\mathcal{L} = \frac{\lambda}{m^4}\sum_{I=1,2}\left(\frac{1}{2}\partial_\mu\tilde{\phi}^I\partial^\mu\tilde{\phi}^I + \frac{1}{2}\left(\tilde{\phi}^I\right)^2 - \frac{1}{4}\left(\tilde{\phi}^I\right)^4 + \frac{\tilde{g}}{6}\left(\tilde{\phi}^I\right)^6\right) + \frac{\tilde{\Lambda}}{2}(\tilde{\phi}^1)^2(\tilde{\phi}^2)^2 \tag{157}$$

For simplicity, in what follows we'll write $\{\tilde{\phi}^1, \tilde{\phi}^2\} = \{\tilde{\phi}, \tilde{\chi}\}$ and drop all tildes. The Lagrangian of Eq. (157) leads to a system of two coupled equations of motion for ϕ and χ. We are interested in spherically symmetric solutions so that in 3 spatial dimensions $\partial_x^2 + \partial_y^2 + \partial_z^2 \to \partial_r^2 + \frac{2}{r}\partial_r$. The equations of motion then reduce to

$$\partial_t^2\phi - (\partial_r^2 + \frac{2}{r}\partial_r)\phi + \phi = \phi^3 - g\phi^5 + \Lambda\phi\chi^2\,, \tag{158}$$

$$\partial_t^2\chi - (\partial_r^2 + \frac{2}{r}\partial_r)\chi + \chi = \chi^3 - g\chi^5 + \Lambda\chi\phi^2\,. \tag{159}$$

The attentive reader will immediately notice that these equations are symmetrical under exchange of the fields, meaning that sending $\chi \to \phi$ in either of the equations gives back the other. This greatly simplifies the analytical search for oscillons which can be difficult or even impossible in more general multi-component systems.

It is well established in the literature that oscillons live on long time- and large spatial scales. This suggests a perturbative approach to extract oscillon solutions from the equations of motion, known as the two-timing analysis. The idea is that oscillons exhibit behavior on two time scales, one capturing the natural frequency of the free field theory and one capturing the correction to this frequency characteristic for a non-linear potential. This behavior is found by introducing a new time variable

$$\tau = \alpha\epsilon^2 t \tag{160}$$

where $\alpha \sim O(1)$ and $\epsilon \ll 1$. Finally, to capture the broadness of the oscillon we perform a change of variable

$$r \to \rho = \epsilon r \tag{161}$$

Note that the (double) time derivatives in Eqs. (158) and (159) must now be interpreted as full time derivatives. So in the equations of motion we must change $\partial_t \to d_t$ and therefore $\partial_t^2 \to \partial_t^2 + 2\alpha\epsilon^2\partial_t\partial_\tau + O(\epsilon^4)$. The equations become

$$\partial_t^2\phi + 2\alpha\epsilon^2\partial_t\partial_\tau\phi - \epsilon^2(\partial_\rho^2 + \frac{2}{\rho}\partial_\rho))\phi + \phi = \phi^3 - g\phi^5 + \Lambda\phi\chi^2 + O(\epsilon^4) \tag{162}$$

$$\partial_t^2\chi + 2\alpha\epsilon^2\partial_t\partial_\tau\chi - \epsilon^2(\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)\chi + \chi = \chi^3 - g\chi^5 + \Lambda\chi\phi^2 + O(\epsilon^4) \tag{163}$$

Lastly, we assume that the oscillon profile is proportional to the expansion parameter ϵ, allowing us to compute the effects of the non-linearities order-by-order. We therefore consider solutions of the form

$$\phi(x,t) = \sum_{n=1} \epsilon^n \phi_n(\rho, t, \tau) \tag{164}$$

$$\chi(x,t) = \sum_{n=1} \epsilon^n \chi_n(\rho, t, \tau) \tag{165}$$

Inserting this into Eqs. (162) and (163), the lowest order equations in ϵ are

$$\begin{aligned}\partial_t^2\phi_1 + \phi_1 =& 0\\ \partial_t^2\chi_1 + \chi_1 =& 0\end{aligned} \tag{166}$$

which are identical to the harmonic oscillator and capture the main oscillatory behavior of the oscillon. The equations in the next non-trivial order in ϵ are

$$\begin{aligned}\partial_t^2\phi_3 + \phi_3 =& (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)\phi_1 - 2\alpha\partial_t\partial_\tau\phi_1 + \phi_1^3 - k\phi_1^5 + \Lambda\phi_1\chi_1^2\\ \partial_t^2\chi_3 + \chi_3 =& (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)\chi_1 - 2\alpha\partial_t\partial_\tau\chi_1 + \chi_1^3 - k\chi_1^5 + \Lambda\chi_1\phi_1^2\end{aligned} \tag{167}$$

Notice that we used the fact that g is large and have therefore written $g = \frac{k}{\epsilon^2}$, with $k \sim O(1)$. The solutions to Eqs. (166) are trivial since these are nothing more than the equations for a harmonic oscillator. Therefore we can write $\phi_1 = \mathrm{Re}\{A(\rho,\tau)e^{-it}\}$ and $\chi_1 = \mathrm{Re}\{B(\rho,\tau)e^{-it}\}$; where $A(\rho,\tau)$ and $B(\rho,\tau)$ are complex functions. Inserting this into the equations of (167) and eliminating secular terms gives us the envelope equations of the system of oscillons.

$$2i\alpha\partial_\tau A + (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)A + \frac{3}{4}|A|^2A + \frac{\Lambda}{2}|B|^2A + \frac{\Lambda}{4}A^*B^2 - k\frac{5}{8}|A|^4A = 0 \tag{168}$$

$$2i\alpha\partial_\tau B + (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)B + \frac{3}{4}|B|^2B + \frac{\Lambda}{2}|A|^2B + \frac{\Lambda}{4}B^*A^2 - k\frac{5}{8}|B|^4B = 0 \tag{169}$$

These equations are of the Nonlinear Schrodinger type. They control the behavior of the oscillon on long time- and large spatial scales. To find solutions of these equations that are localized in space we insert the "oscillon ansatz". Since by assumption the oscillon is just some localized structure oscillating in time we should look for solutions of the form, $A(\rho,\tau) \sim a(\rho)e^{ic_1\tau}$ and $B(\rho,\tau) \sim b(\rho)e^{ic_2\tau}$; where a and b are real functions determining the spatial character of the oscillon. Due to the symmetry of the potential we can also set $c_1 = c_2 = c$. Inserting this into equations (168) and (169), we obtain the profile equations of the oscillons.

$$-\alpha a + (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)a + \frac{3}{4}a^3 + \frac{3\Lambda}{4}ab^2 - k\frac{5}{8}a^5 = 0 \tag{170}$$

$$-\alpha b + (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)b + \frac{3}{4}b^3 + \frac{3\Lambda}{4}ba^2 - k\frac{5}{8}b^5 = 0 \tag{171}$$

Where we set $c = \frac{1}{2}$. A few remarks about the parameter c are now in order. c is in essence a free parameter, as long as it is not too large. This is because it can always be absorbed into a redefinition of α. We do however require it to be positive, since we need decay of the solution as

$a, b \to 0$ for there to be localized solutions. We can therefore safely set $c = \frac{1}{2}$.
An interesting property given to us by the symmetry of the system now becomes apparent. Namely, any localized solution $a(\rho)$ of the equation

$$-\alpha a + (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho)a + \frac{3}{4}(1+\Lambda)a^3 - k\frac{5}{8}a^5 = 0 \tag{172}$$

directly solves the system of equations (170) and (171) if we set $a = b$. The attentive reader will now also notice that finding a solution of (172) will also solve both profile equations if we set $a = -b$. Our two-timing analysis thus predicts the existence of oscillons that are oscillating both in phase and out of phase with each other. That the phase difference between oscillons can have a role in two-field systems was to be expected but is now made explicit by the two-timing analysis. The difficulty of finding oscillons in this coupled system is greatly reduced due to the "exchange" symmetry present in this model. The profiles can be related to the solutions for single-field oscillons in a quartic-sextic potential that was studied in Ref. [10].

C. Localized solutions and dependence on α

First of all, I should comment on the fact that in our two-timing analysis I've kept the scaling parameter of the slow time variable τ explicit. In the single-field model we tackled in chapter II this scaling parameter, together with the scaling β of the field itself, could be absorbed in a redefinition of the small expansion parameter ϵ. This meant that the only relevant parameter that had to be chosen was this small number ϵ. This was possible simply because the solutions of the profile equations were related via the simple relation $a_2(\rho) = \frac{\beta_1}{\beta_2}\sqrt{\alpha_1/\alpha_2}a_1(\sqrt{\alpha_1/\alpha_2}\rho)$. However, this type of relation doesn't exist for the equation in (172) and only the free parameter β can be removed in this way. The parameter α must thus be kept explicit, and we'll see that it influences the shapes of the profiles considerably.

In one dimension the equation in (172) has a conserved first energy

$$E_\rho = \frac{1}{2}\left(\frac{\partial a}{\partial \rho}\right)^2 - \frac{1}{2}\alpha a^2 + \frac{3}{16}(1+\Lambda)a^4 - k\frac{5}{48}a^6 \tag{173}$$

So that the equation in three dimensions can be written as

$$\frac{dE}{d\rho} = -\frac{2}{\rho}\left(\frac{\partial a}{\partial \rho}\right)^2 \tag{174}$$

Since localized solutions have $E_\rho \to 0$ as $\rho \to \infty$ we must have $E_\rho \geq 0$ at the origin. Since we're looking for solutions that are smooth at the origin we obtain a bound on α

$$\alpha_c = \frac{27}{160} \frac{(1+\Lambda)^2}{k} \tag{175}$$

If α is larger than this value E_ρ can not be positive at the origin. As $\alpha \to \alpha_c$ the solution for $a(\rho)$ approaches the homogeneous solution for which $a(\rho) = a_c = \sqrt{\frac{9(1+\Lambda)}{10k}}$ uniformly. It turns out that equation (172) has a countable set of localized solutions $a_n(\rho)$, where n is the number of nodes of the solutions. Of these we identify the $n = 0$ mode solution with our oscillons. This fact is best understood in phase space $(a, \partial_\rho a)$ and we refer the reader to figure 1 for a clear picture. The zero-mode solutions were found numerically using a shooting method. They are plotted in figures 6 and 7. We see that, as the value of α increases, the amplitude of the solution increases, while

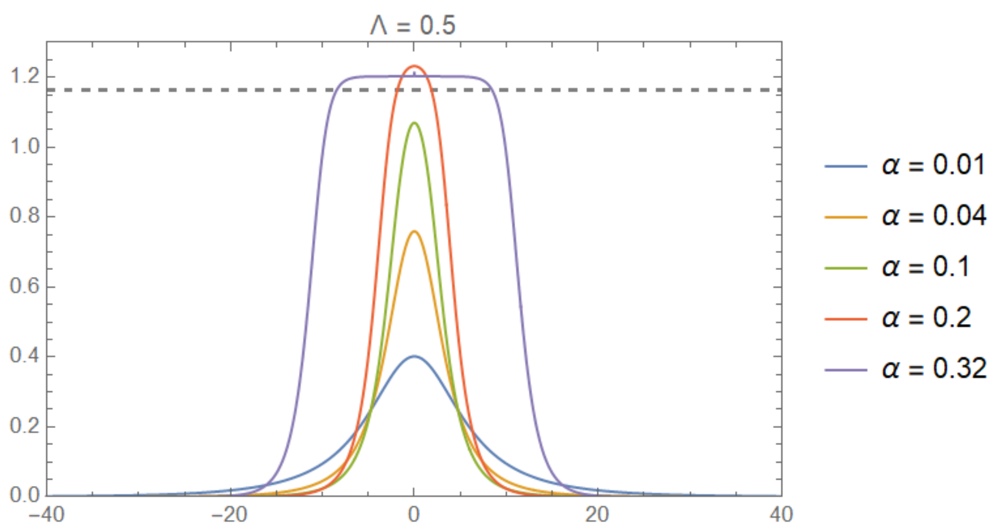


FIG. 6: The localized zero-mode solutions of the profile equations in this model for different values of the parameter α, and $k = 1$. Although the profiles initially become slimmer and taller, they eventually become wider again as $\alpha \to \alpha_c$. They obtain increasingly flat tops as they approach the homogeneous solution $a(\rho) = \sqrt{\frac{9(1+\Lambda)}{10k}}$ from above (the dashed line).

its width decreases. This type of behavior is also typical for the sech-like oscillons we saw in earlier chapters. However, as α approaches the critical value α_c the solutions become wider again with very flat tops. The width will continue to increase until $\alpha = \alpha_c$ at which point we've reached the homogeneous solution. It is important to realise that this non-trivial behavior of the solutions is entirely due to the large sextic term in our original action. In the next section we'll see that this can type of behavior can grant the oscillons stability to small perturbations.

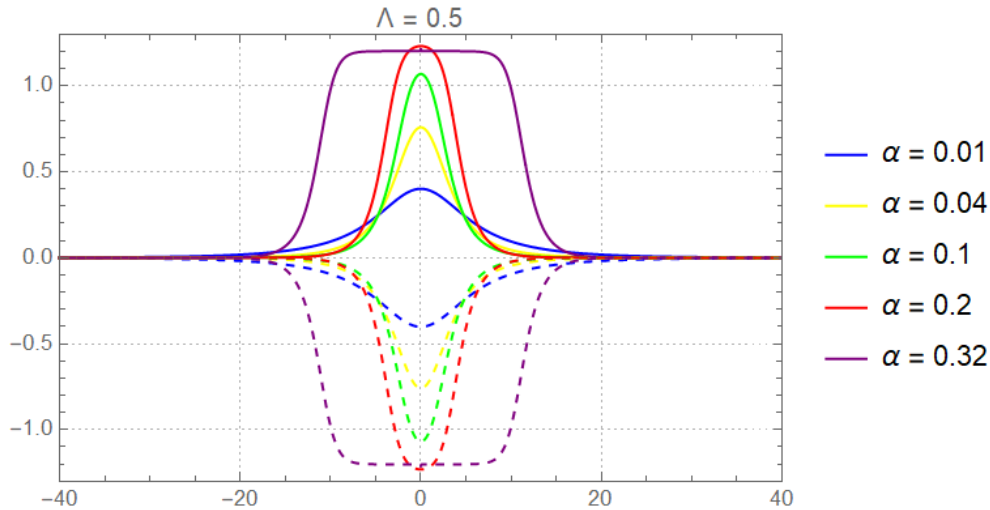


FIG. 7: The oscillon solutions in this model for different values of α. We have solutions for which the two fields overlap exactly, but solutions for which the fields oscillate out of phase (dashed lines). This is an interesting consequence of oscillons in multi-field models with symmetric potentials.

D. Stability Analysis

It has been shown repeatedly that no true breather solutions like the oscillon can exist in nonlinear systems. In general there will always be (classical) outgoing radiation in the tails of the oscillon. Classically, this radiation is exponentially suppressed, which is why oscillons can be extremely long-lived. Quantum Mechanical radiation might even play a more important role in real physical systems. Although these statements have been explicitly proven for single-component oscillons, it is expected that similar behavior can be found in more complicated systems like the one that is the subject of this paper. Explicit proofs are left for future work.

Radiation will in general perturb the oscillon system and it is therefore worthwhile to assess the stability of oscillons with respect to small perturbations. In the model we are investigating let

$$\begin{aligned} \phi(x,t) &= \phi_{osc}(x,t) + \delta(x,t) \\ \chi(x,t) &= \chi_{osc}(x,t) + \Delta(x,t) \end{aligned} \tag{176}$$

Where it is assumed $\delta(x,t), \Delta(x,t) \ll O(\epsilon)$. Plugging these into the equations of motion (158) and (159) and linearizing the result

$$\begin{aligned} \partial_t^2\delta + \delta - (\partial_r^2 + \frac{2}{r}\partial_r)\delta - 3\phi_{osc}^2\delta + 5g\phi_{osc}^4\delta - \Lambda\chi_{osc}^2\delta - 2\Lambda\phi_{osc}\chi_{osc}\Delta &=0 \\ \partial_t^2\Delta + \Delta - (\partial_r^2 + \frac{2}{r}\partial_r)\Delta - 3\chi_{osc}^2\Delta + 5g\chi_{osc}^4\Delta - \Lambda\phi_{osc}^2\Delta - 2\Lambda\chi_{osc}\phi_{osc}\delta &=0 \end{aligned} \tag{177}$$

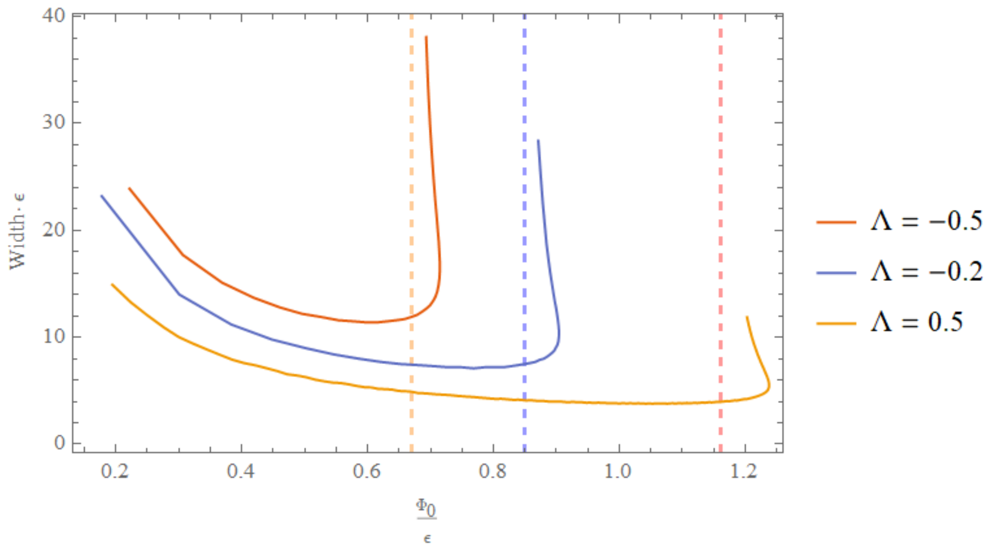


FIG. 8: The unexpected relation between the width and height of the localized solutions in this model. As as $\alpha \to \alpha_c$, the oscillon solution regain width and approach the homogeneous $a(\rho) = \sqrt{\frac{9(1+\Lambda)}{10k}}$ from above (the dashed line). We will see how this non-trivial relation grants the oscillons stability under the V-K criterion.

Since the exact shape of the initial perturbation is in essence unpredictable, a full linear stability analysis would require solving (177) for arbitrary initial conditions. This requires solving the full Floquet matrix for the coupled nonlinear system which is in principle difficult. It turns out that, for this system, there is a useful simplification if we only consider perturbations that are about the same size as the oscillon itself. This is in essence an extension of the Vakhitov-Kolokolov criterion that has been used in the past to assess stability of single-component oscillons.

E. Extension of the V-K criterion

In earlier sections we have already shown that this system supports oscillons of the form

$$\begin{aligned} \phi_{osc}(x,t) &= \epsilon\,\mathrm{Re}\Big\{a(\epsilon x)e^{it(1-\frac{\epsilon^2}{2})}\Big\} = a(\epsilon x)\cos\omega t \\ \chi_{osc}(x,t) &= \epsilon\,\mathrm{Re}\Big\{b(\epsilon x)e^{it(1-\frac{\epsilon^2}{2})}\Big\} = b(\epsilon x)\cos\omega t \end{aligned} \tag{178}$$

Where $\omega = 1 - \alpha\frac{\epsilon^2}{2}$. The symmetry of the system dictates that we either have $a = b$ or $a = -b$. This allows us to turn equations 24 in a single variable system and then follow the procedure in [10] and derive the V-K criterion. We tackle the two cases separately.

1. $a(\epsilon x) = b(\epsilon x)$

In this case the equations in (177) become

$$
\begin{aligned}
\partial_t^2\delta + \delta - (\partial_r^2 + \frac{2}{r}\partial_r)\delta - \epsilon^2 a^2(3+\Lambda)cos(\omega t)^2\delta + \epsilon^4 5ga^4 cos(\omega t)^4\delta - \epsilon^2 2\Lambda a^2 cos(\omega t)\Delta =&0\\
\partial_t^2\Delta + \Delta - (\partial_r^2 + \frac{2}{r}\partial_r)\Delta - \epsilon^2 a^2(3+\Lambda)cos(\omega t)^2\Delta + \epsilon^4 5ga^4 cos(\omega t)^4\Delta - \epsilon^2 2\Lambda a^2 cos(\omega t)\delta =&0
\end{aligned}
\tag{179}
$$

Now enters the true power of this model. Adding the equations and introducing the new variable $\xi(x,t) = \delta(x,t) + \Delta(x,t)$ reduces (179) to

$$
\partial_t^2\xi + \xi - (\partial_r^2 + \frac{2}{r}\partial_r)\xi - \epsilon^2 3a^2(1+\Lambda)cos(\omega t)^2\xi + \epsilon^4 5ga^4 cos(\omega t)^4\xi = 0 \tag{180}
$$

This equation only depends on the variable $\xi(x,t)$, greatly simplifying our calculations. Since we are interested in perturbations that are about the same size as the oscillon itself we perform the same change of variable as before

$$
r \to \rho = \epsilon r \tag{181}
$$

Next, we expect there perturbations to oscillate near the oscillon frequency. To capture the growth or decay of the perturbation we should also introduce a new "slow" time variable.

$$
\begin{aligned}
T &= \omega t\\
\tau &= \epsilon^2 t
\end{aligned}
\tag{182}
$$

As before we expect behavior on both time scales, so $\xi(x,t) \to \xi(\rho,T,\tau)$, and partial time-derivatives turn into full derivatives. Looking for solutions

$$
\xi(\rho,T,\tau) = \sum_{n=0} \epsilon^n \xi_n(\rho,T,\tau) \tag{183}
$$

Allows a perturbative analysis of equation (180). The first and second order equations are respectively

$$
\partial_T^2\xi_0 + \xi_0 = 0 \tag{184}
$$

$$
\partial_T^2\xi_1 + \xi_1 = -\left(\partial_T\partial_\tau - \alpha\partial_T^2 - (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho) - 3a^2(1+\Lambda)\cos T^2 + 5ka^4\cos T^4\right)\xi_0 \tag{185}
$$

Equation (184) gives the most general result for ξ_0

$$
\xi_0(\rho,T,\tau) = u(\rho,\tau)\cos T + v(\rho,\tau)\sin(T) \tag{186}
$$

$u(\rho,\tau)$ and $v(\rho,\tau)$ capture the potential growth of the perturbation on timescales of order $\sim O(\tau)$. The V-K criterion can tells us whether there are perturbation that can grow exponentially based on the form of the oscillon itself. Namely, inserting (186) into equation (185) and eliminating secular terms on the right hand side gives us two equations for u and v

$$\partial_\tau u = H_1 v \tag{187}$$

$$\partial_\tau v = -H_2 u \tag{188}$$

Where H_1 and H_2 are Hermitian, linear operators defined as

$$H_1 = \alpha - (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho) - \frac{3}{4}(1+\Lambda)a^2 + k\frac{5}{8}a^4 = 0 \tag{189}$$

$$H_2 = \alpha - (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho) - \frac{9}{4}(1+\Lambda)a^2 + k\frac{25}{8}a^4 = 0 \tag{190}$$

Separating variables as $u = e^{\Omega\tau}$ and $v = e^{\Omega\tau}$ the problem reduces to the eigenvalue problem.

$$\Omega^2 u = -H_1 H_2 u \tag{191}$$

The question of whether the oscillon is stable to general long-wavelength perturbation is now reduced to an eigenvalue problem. If the operator $-H_1H_2$ has an eigenvalue $\Omega^2 > 0$, perturbations can grow and the oscillon will in general be unstable. If not, perturbations simply oscillate. The problem can be solved using a similar procedure as Vakhitov and Kolokolov, which we'll not derive in its entirety here, but was performed for the single-field model in chapter II. The criterion for stability states that $max(\Omega^2) < 0$ if and only if $dN/d\alpha > 0$, where

$$N = \int a^2(\rho)d^3\rho \tag{192}$$

A final remark about the validity of this derivation might now be in order however. The criterion only states whether perturbation of the form $\xi(x,t) = \delta(x,t) + \Delta(x,t)$ will grow. In principle, the growth of the perturbation might only be present in δ or Δ. However, since the system is completely symmetric under exchange of the fields, we can conclude that, if $\xi(x,t)$ grows, both $\delta(x,t)$ and $\Delta(x,t)$ will grow exponentially. The criterion is therefore valid for the whole system.

2. $a(\epsilon x) = -b(\epsilon x)$

Analogously, equations (177)

$$\begin{aligned}
\partial_t^2\delta + \delta - (\partial_r^2 + \frac{2}{r}\partial_r)\delta - \epsilon^2 a^2(3+\Lambda)cos(\omega t)^2\delta + \epsilon^4 5ga^4 cos(\omega t)^4\delta + \epsilon^2 2\Lambda a^2 cos(\omega t)^2\Delta =&0 \\
\partial_t^2\Delta + \Delta - (\partial_r^2 + \frac{2}{r}\partial_r)\Delta - \epsilon^2 a^2(3+\Lambda)cos(\omega t)^2\Delta + \epsilon^4 5ga^4 cos(\omega t)^4\Delta + \epsilon^2 2\Lambda a^2 cos(\omega t)^2\delta =&0
\end{aligned} \tag{193}$$

Now, instead of looking for unstable modes in the field $\xi = \delta + \Delta$, we introduce $\psi = \delta - \Delta$. Subtracting the second equation in (193) from the first, we obtain the equation

$$\partial_t^2\psi + \psi - (\partial_r^2 + \frac{2}{r}\partial_r)\psi - \epsilon^2 3a^2(1+\Lambda)cos(\omega t)^2\psi + \epsilon^4 5ga^4 cos(\omega t)^4\psi = 0 \tag{194}$$

This is exactly the same equation as in (180). The rest of the derivation thus follows exactly the same steps as before so we won't repeat them here. We end up with the eigenvalue equation

$$\Omega^2 u = -H_1 H_2 u \tag{195}$$

Where H_1 and H_2 are Hermitian, linear operators defined as

$$H_1 = \alpha - (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho) - \frac{3}{4}(1+\Lambda)a^2 + k\frac{5}{8}a^4 = 0 \tag{196}$$

$$H_2 = \alpha - (\partial_\rho^2 + \frac{2}{\rho}\partial_\rho) - \frac{9}{4}(1+\Lambda)a^2 + k\frac{25}{8}a^4 = 0 \tag{197}$$

Resulting in exactly the same V-K criterion as before. Namely, there are unstable perturbations of the oscillon if and only if $dN/d\alpha > 0$, where

$$N = \int a^2(\rho)d^3\rho \tag{198}$$

The fact that the criterion also holds for the out-of-phase oscillons is somewhat to be expected due to the symmetry in the original model.

In this section we have derived a nontrivial extension of the V-K criterion for assessing the stability of oscillons in this coupled system to long wavelength perturbations. It is, to our knowledge, the first time that the criterion has been derived for multi-component oscillons. In the next sections we will check these results numerically.

F. Numerical analysis

The V-K criterion tells us that oscillon solution is stable to perturbations that are about the same size as the oscillon itself if and only if N, which can be associated with the amount of particles in the oscillon increases if we increase the small parameter α. If the profile equation in (172) only contained cubic terms this would never be the case in three dimensions. However, in this model the oscillon profiles have a non-trivial height-width relation as highlighted in figure 8. This stabilises the oscillon as $\alpha \to \alpha_c$. The value of N with respect to the parameter α for various couplings Λ are shown in figure 10. As $\alpha \to \alpha_c$ the value of the integral $N \to \infty$. The non-trivial

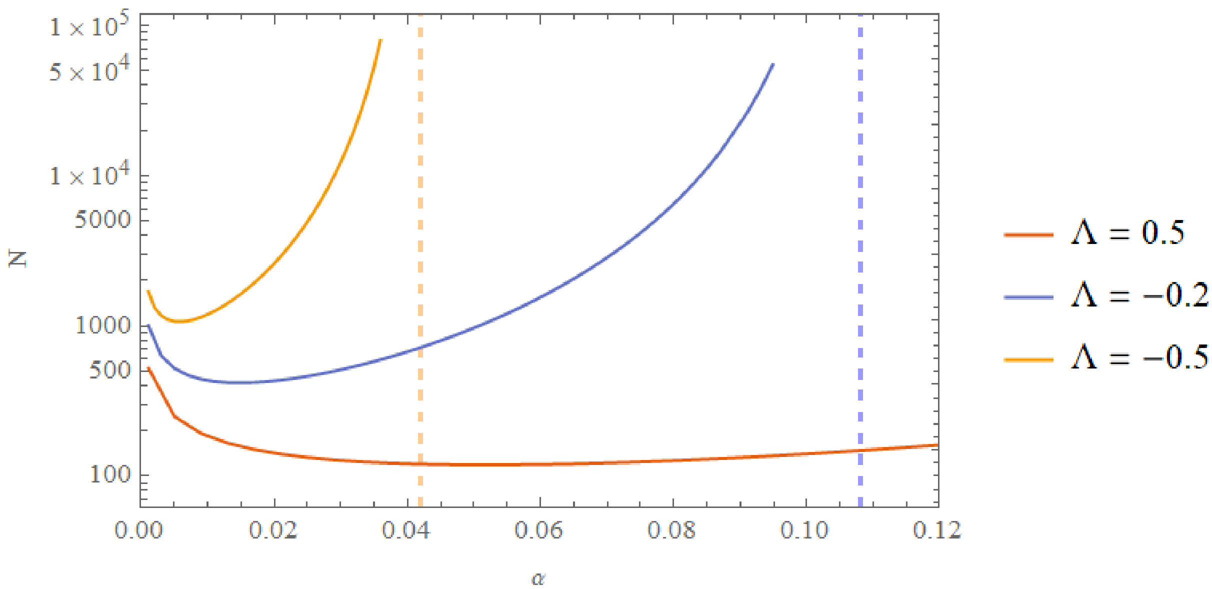


FIG. 9: The value of N as defined in equation (198) for a wide range of α and Λ values. Each point on the graph corresponds to a specific oscillon solution, oscillating with frequency $\omega = 1 - \alpha\frac{\epsilon^2}{2}$. The solution is believed to be stable when the gradient of the graph is positive. This is clearly the case for large values of α, but will not be true for models that have sech-like oscillons. The large sextic term in this model thus stabilises the oscillons solutions.

height-width relation stabilises the oscillons in this model to perturbations that are about the same size as the oscillon itself. Interestingly the influence of the coupling Λ is clear: the graphs are essentially shifted versions of each other. We conclude that although the model is somewhat trivial, the coupling between the fields influences the stability of the solutions, which is a general characteristic of multi-component oscillons.

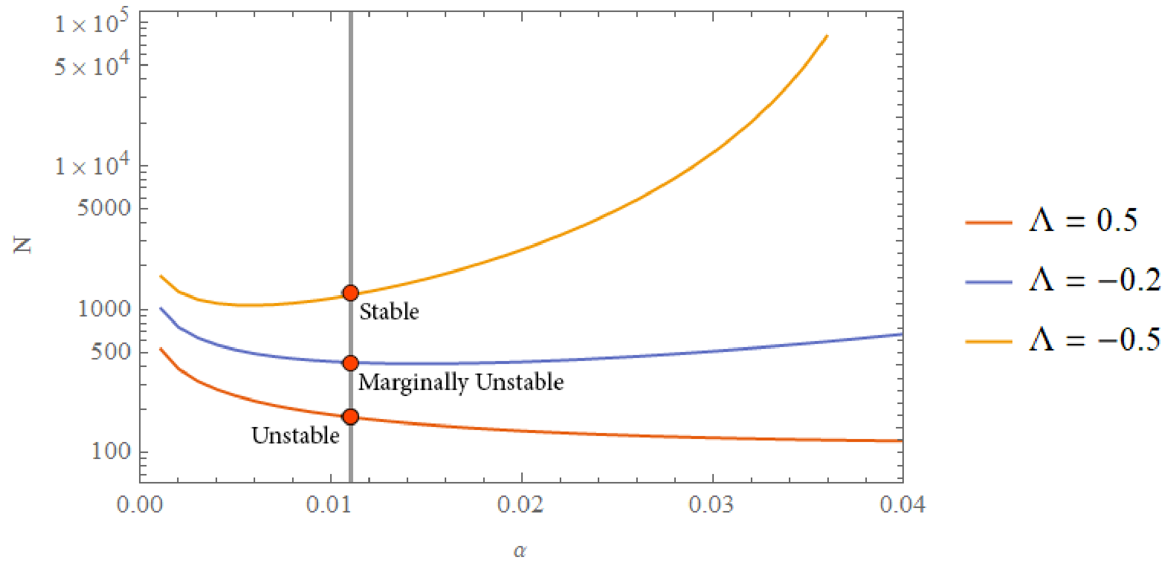


FIG. 10: The effect of the coupling strength Λ on the stability of the oscillon solutions. We see that although the model is somewhat of a toy model, the coupling between the fields can influence the stability of an oscillon. For the same value of α we see here oscillons that are both stable and unstable.

In a last step we checked the implications of the V-K criterion numerically. We did this evolving the full equations of motion of the system as given in (158) and (159). As initial conditions we chose $\phi(x,0) = \chi(x,0) = \epsilon a(\epsilon x) + \epsilon\delta(x)$, $\dot{\phi}(x,0) = \dot{\chi}(x,0) = 0$ for in-phase oscillons and $\phi(x,0) = -\chi(x,0) = \epsilon a(\epsilon x) + \epsilon\delta(x)$, $\dot{\phi}(x,0) = \dot{\chi}(x,0) = 0$ for out-of-phase oscillons. Here $a(\epsilon x)$ solves the profile equation for a chosen value of α and $\delta(x)$ is the unstable mode corresponding to that specific solution (essentially the first eigenvector u of the operator $H_1 H_2$ in (191) and (195)). The oscillon was defined as the region in space where 95% of the initial energy resided. During the evolution of the system the energy in this region was tracked. The oscillon is said to decay when only 20% of the initial energy is left inside the region. We ran the simulations for $\epsilon = 0.01, \epsilon = 0.05$ and $\epsilon = 0.08$, obtaining similar results for all three cases. The conclusions of these numerical simulations can be summarized in the following points:

- The stability of the oscillon solution does indeed seem to be decided by the V-K criterion. During an initial oscillatory phase the perturbation either grows or oscillates, depending on the criterion. Eventually, if the perturbation reaches a certain threshold the oscillon decays and starts losing all its energy. The decay happens via dispersion.

- There is virtually no difference between the in-phase and out-of-phase oscillons. Both leave almost exactly the same signature in our simulations.

- The initial amplitude of the perturbation $\delta(x)$ doesn't seem to influence our conclusions. A larger initial amplitude does lead to a quicker decay of the oscillon however. This make sense since the threshold at which the oscillon decays is reached earlier.

- We do not yet know how to predict the exact lifetime of an unstable oscillon. The lifetime seems to inversely scale with the parameter α: if α is smaller the unstable oscillons persists for a longer period of time. This makes sense since the natural timescale of the oscillon scales with $\tau = \alpha\epsilon^2 t$. This is not the whole story however since the V-K criterion predicts a larger Lyapunov exponent Ω for smaller α. This is related to the fact that the value of $dN/d\alpha$ becomes more negative as $\alpha \to 0$ (also see the full derivation of the V-K criterion in chapter II). Thus the amplitude of the perturbation $|\delta| \propto e^{\Omega\epsilon^2 t}$ grows more rapidly in this case. The decay of the oscillon is probably an interplay of these two counteracting principles. A full analysis of the oscillon decay is left for future work.

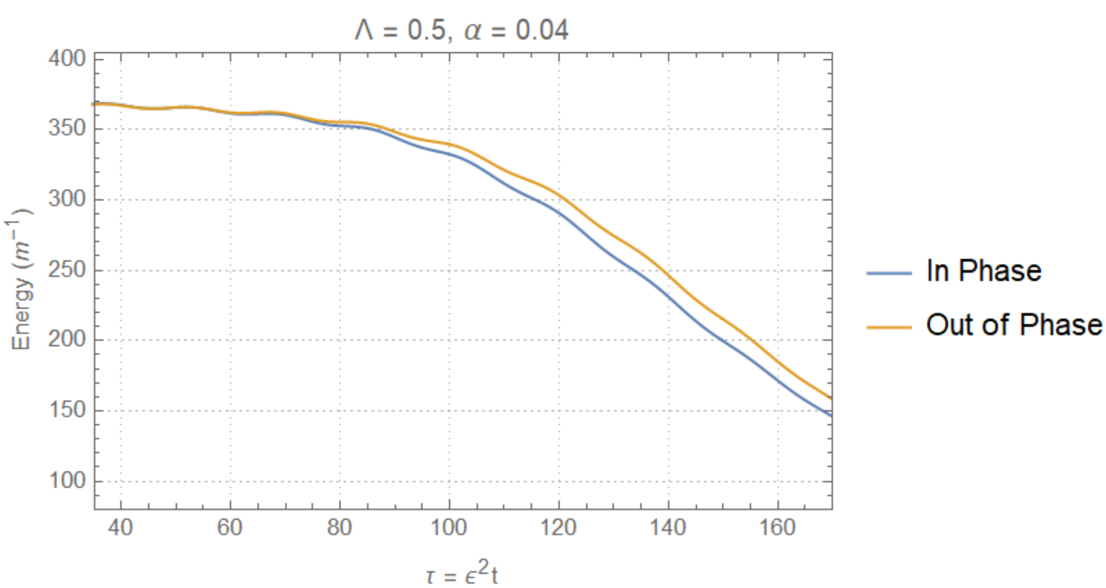


FIG. 11: The evolution of the localized energy of the oscillon solution for $\Lambda = 0.5$ and $\alpha = 0.04$, both for in-phase and out-of-phase initial conditions. The oscillon starts to disperse around $\tau \approx 100$ with no real difference between the in-phase and out-of-phase solution.

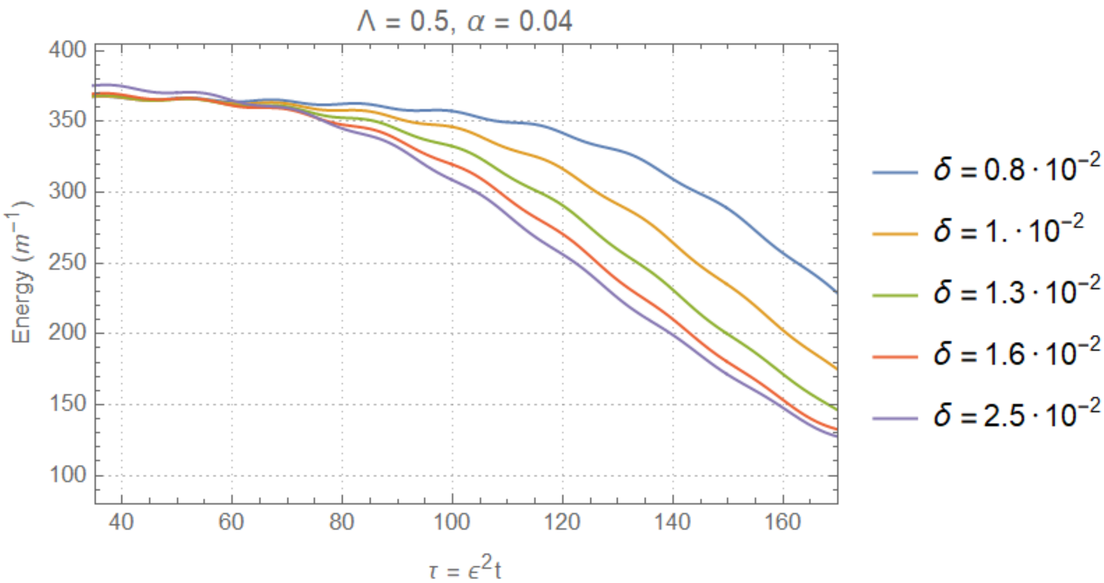


FIG. 12: The evolution of the localized energy of the oscillon solution for $\Lambda = 0.5$ and $\alpha = 0.04$, for the in-phase solutions with different amplitudes of the initial perturbation $\delta(x)$. We again observe dispersion, although the onset of decay is dependent on the initial value of $|\delta|$. This hints at the fact that the amplitude of the perturbation needs to grow to a certain threshold before dispersion begins.

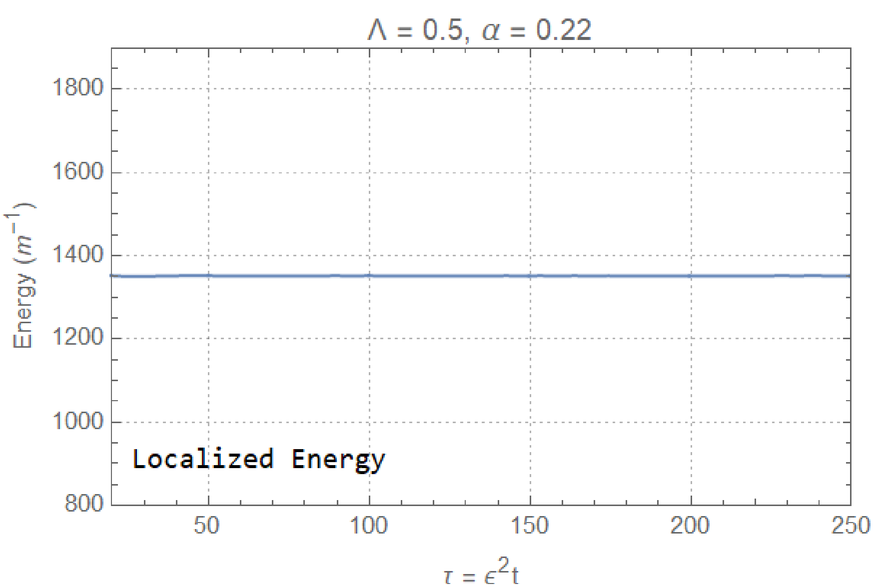
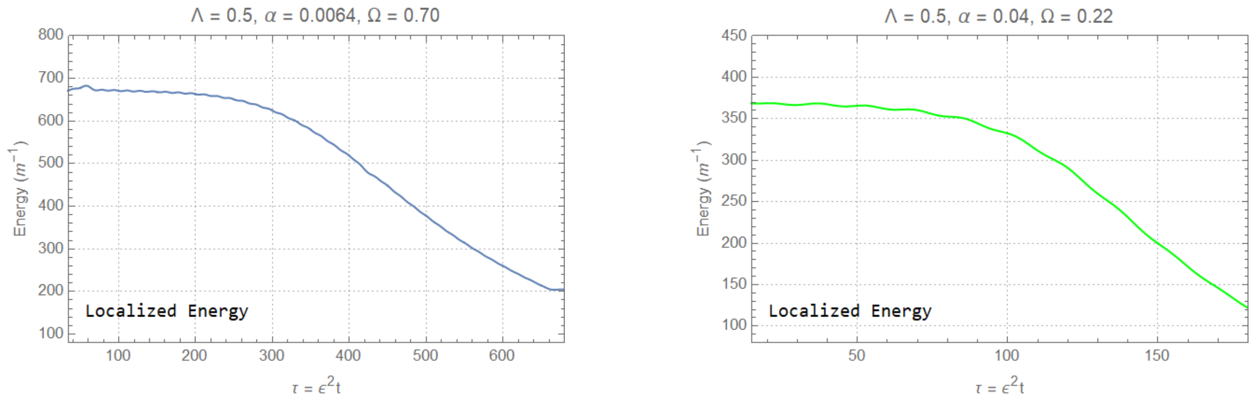


FIG. 13: The evolution of the localized energy of the oscillon solution for $\Lambda = 0.5$ and $\alpha = 0.22$, for the in-phase solutions. This solution doesn't fail the V-K criterion and we don't see decay as expected.



(a) The evolution of the localized energy of the oscillon solution for $\Lambda = 0.5$ and $\alpha = 0.0064$. The solution theoretically has a Lyapunov exponent $\Omega = 0.7$. The onset of dispersion is approximately at $\tau = 350$

(b) The evolution of the localized energy of the oscillon solution for $\Lambda = 0.5$ and $\alpha = 0.04$. The solution theoretically has a Lyapunov exponent $\Omega = 0.22$. The onset of dispersion is approximately at $\tau = 100$.

FIG. 14: Both of these solutions fall in the unstable section of solution space (following the V-K) criterion but disperse on very different timescales. The decay time is probably an interplay between the Lyapunov exponent Ω and the slow time scaling α.

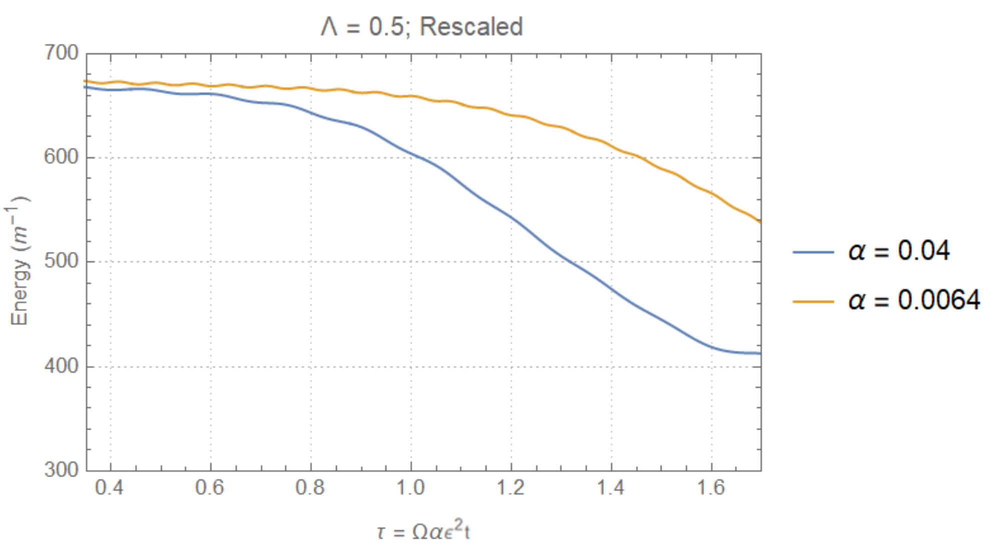


FIG. 15: The energy evolution of the two oscillon solutions as given in figure 14 but with their respective time axis rescaled by $\epsilon^2 t \to \alpha\Omega\epsilon^2 t$. The graphs now show significantly more overlap. The exact lifetime of the solution is probably an interplay between the parameter α associated with the solution and the Lyapunov exponent Ω given by the V-K criterion.

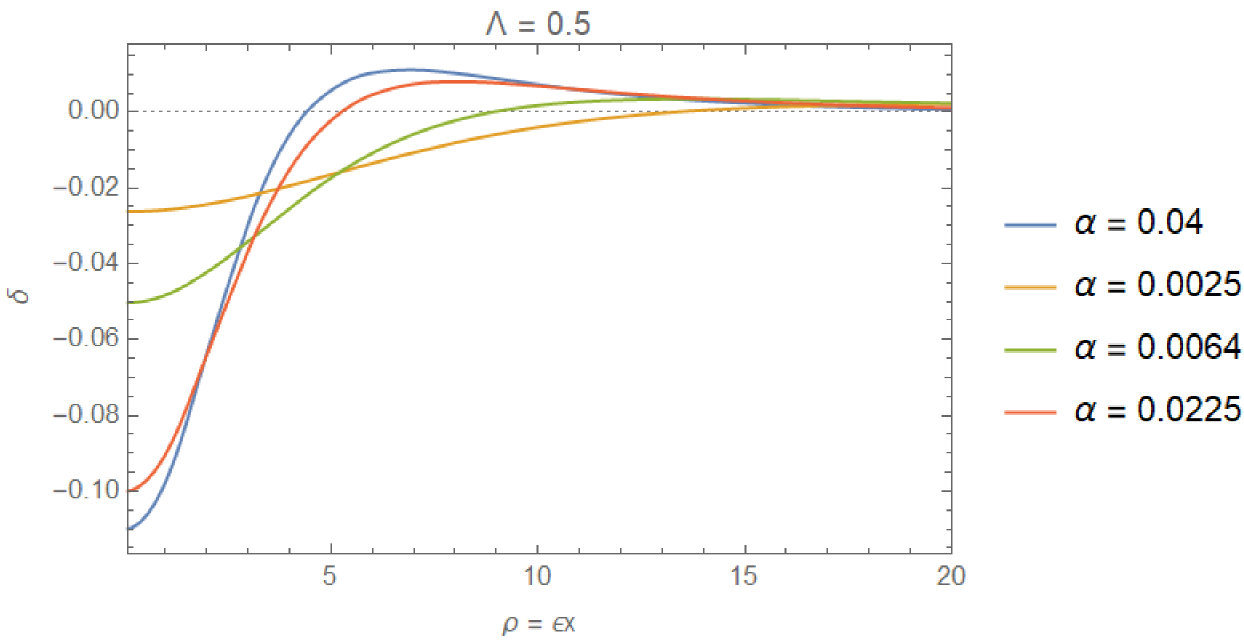


FIG. 16: The profiles of the unstable modes of the oscillons with values of α that fail the V-K criterion for a coupling $\Lambda = 0.5$. These were calculated numerically using the variational principle.

G. Summary and discussion

In this chapter we've done a full analytic analysis of oscillons in a nonlinear scalar field model containing two fields. The model had a specific "exchange" symmetry that made a full analytical analysis possible. Stable oscillons were found to exist in this model. The stable solutions will in general have two branches of solutions, one in which the oscillons oscillate exactly in phase, while in the other they oscillate out of phase. It is expected that this is a general consequence of oscillons in symmetric multi-field potentials: if a stable in-phase oscillon exists, the out-of-phase solution will in general also be stable. This is simply because the equations get an overall minus sign in symmetric potentials when the initial configurations are out-of-phase. This fact will be highlighted again in the next chapter. It is worth mentioning that the stability of the oscillons that were found was only tested for perturbations that are about the same size as the oscillon itself. Although this restriction allows for an analytical approach using the Vakhitov-Kolokolov criterion, the criterion is not enough to check stability against all perturbations. An analysis of short wavelength perturbations can be done by approximating the oscillon as a constant oscillating background. The analysis was performed in [10] and it was suggested that the instability bands were very small and narrow. This would mean that in general these short wavelength perturbations can not destroy the oscillon. Adding a very long wavelength perturbation to our solution might be thought of as an overall shift to our solution. This can be reabsorbed in a redefinition of ϵ and α, and might in some cases destroy the oscillon (by effectively moving to another section of the V-K graph). This is still conjecture and a full stability analysis must be performed in the future. It has been suggested in [17] that it is in fact the V-K criterion that is the deciding factor in whether oscillon solutions will form during preheating, making it plausible that this is in fact the most important assessment of oscillon stability.

V. GENERAL MULTI-FIELD OSCILLONS

In the previous chapter, we were able to solve a multi-field system of oscillons exactly by working in a model that had a specific symmetry. Namely, the model was symmetric under exchange of the fields. Although we somewhat motivated this system by drawing a comparison to the T-model, it still seems a little bit contrived. In general, we do not expect the masses of the fields to be exactly the same. Furthermore, in most models the specific symmetry present in (152) will not hold. Therefore, in this chapter, we'll focus on a more complicated model involving again two scalar fields. We'll investigate oscillons in a symmetric potential with quartic interactions. The two-timing analysis can still be applied to this more complicated model to obtain profile equations, although the structure of the equations will be less symmetric than the ones discussed in the last chapter. The analytical analysis of the stability of the oscillon solutions found in this chapter will be left for future work. In this chapter we'll focus on a numerical analysis, and try to give qualitative interpretations of the results where possible. I will also give some considerations for oscillons in asymmetric potentials (with cubic interactions), based on preliminary investigations.

A general two-field system wit canonical kinetic terms will have an action given by

$$S = \int d^dx dt \sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\phi,\chi)] \tag{199}$$

Where ϕ and χ are two scalar fields. As before, we'll be looking for oscillon solutions that look like $\phi = \phi_0 \cos(\omega t)$ and $\chi = \chi_0 \cos(\omega t)$. Using the same argumentation as in chapter II we can again conclude that such solutions can only be found in potentials that are shallower than quadratic for some region of the potential. In particular

$$\begin{aligned} V_\phi(\phi,\chi) - m_1^2\phi < 0 \\ V_\chi(\phi,\chi) - m_2^2\chi < 0 \end{aligned} \tag{200}$$

Where m_1 is the mass of the ϕ-field and m_2 is the mass of the χ-field, and subscripts indicate derivatives with respect to that field. Notice now that we do not require the masses of the field to be equal. Looking at the conditions (200), it now seems plausible that oscillons can exist solely through interactions with another field. We'll investigate both quartic and cubic interactions and see the effect that the difference in mass of the fields can have.

A. Symmetric potentials with quartic interactions

In this section we consider a model with the following Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}\partial_\mu\chi\partial^\mu\chi - V(\phi,\chi) \tag{201}$$

and where the potential is given by

$$V(\phi,\chi) = \frac{1}{2}m_1^2\phi^2 + \frac{1}{2}m_2^2\chi^2 - \frac{\lambda}{4}\phi^4 + \frac{g}{6}\phi^6 - \frac{\Lambda}{2}\chi^2\phi^2 \tag{202}$$

Where $m_1, m_2, \lambda, g, \Lambda \sim O(1)$. Notice that the potential (202) satisfies the conditions in (200) but that the condition for the field χ is solely satisfied because of the interaction term. This model gives rise to two equations of motion

$$\partial_t^2\phi - \nabla^2\phi + m_1^2\phi - \lambda\phi^3 + g\phi^5 - \Lambda\chi^2\phi = 0 \tag{203}$$

$$\partial_t^2\chi - \nabla^2\chi + m_2^2\chi - \Lambda\chi\phi^2 = 0 \tag{204}$$

It is again convenient to work with dimensionless space-time variables and fields. This is done by rescaling $x^\mu \to \tilde{x}^\mu = x^\mu m_1$, $\phi \to \tilde{\phi} = m_1^{-1}\lambda^{\frac{1}{2}}\phi$, $\chi \to \tilde{\chi} = m_1^{-1}\lambda^{\frac{1}{2}}\chi$, $g \to \tilde{g} = (m_1/\lambda)^2 g$ and $\Lambda \to \tilde{\Lambda} = (1/\lambda)\Lambda$. In these new variables, dropping all the tildes and assuming spherically symmetric solutions, the equations of motion become

$$\partial_t^2\phi - \partial_r^2\phi - \frac{d-1}{r}\partial_r\phi + \phi - \phi^3 + g\phi^5 - \Lambda\chi^2\phi = 0 \tag{205}$$

$$\partial_t^2\chi - \partial_r^2\chi - \frac{d-1}{r}\partial_r\chi + \left(\frac{m_2}{m_1}\right)^2\chi - \Lambda\chi\phi^2 = 0 \tag{206}$$

We can now use the two-timing analysis to find localized oscillon solutions of these equations. The term proportional to $\left(\frac{m_2}{m_1}\right)$ is convenient to analyse the solutions for different masses of the field.

As before we introduce a new time and spatial variable

$$\begin{aligned} \tau &= \epsilon^2 t \\ \rho &= \epsilon x \end{aligned} \tag{207}$$

And assume that the oscillon has relevant behavior both on time scales of $O(1)$ and of $O(1/\epsilon^2)$. The equations of motion in (205) and (206) then become

$$\begin{aligned} &\partial_t^2\phi + 2\epsilon^2\partial_t\partial_\tau\phi + \epsilon^4\partial_\tau^2\phi - \epsilon^2\partial_\rho^2\phi - \epsilon^2\frac{d-1}{\rho}\partial_\rho\phi + \phi - \phi^3 + g\phi^5 - \Lambda\chi^2\phi = 0 \\ &\partial_t^2\chi + 2\epsilon^2\partial_t\partial_\tau\xi + \epsilon^4\partial_\tau^2\chi - \epsilon^2\partial_\rho^2\chi - \epsilon^2\frac{d-1}{\rho}\partial_\rho\chi + \left(\frac{m_2}{m_1}\right)^2\chi - \Lambda\chi\phi^2 = 0 \end{aligned} \tag{208}$$

We solve these equations now perturbatively by using the expansion

$$
\begin{aligned}
\phi(x,t) &= \phi_{osc}(\rho,t,\tau) = \sum_{n=1} \epsilon^n \phi_{n,osc}(\rho,t,\tau) \\
\chi(x,t) &= \chi_{osc}(\rho,t,\tau) = \sum_{n=1} \epsilon^n \chi_{n,osc}(\rho,t,\tau)
\end{aligned}
\tag{209}
$$

Writing down the equations up to $O(\epsilon^3)$

$O(\epsilon)$:

$$\partial_t^2 \phi_{1,osc} + \phi_{1,osc} = 0 \tag{210}$$

$$\partial_t^2 \chi_{1,osc} + \left(\frac{m_2}{m_1}\right)^2 \chi_{1,osc} = 0 \tag{211}$$

$O(\epsilon^2)$:

$$\partial_t^2 \phi_{2,osc} + \phi_{2,osc} = 0 \tag{212}$$

$$\partial_t^2 \chi_{2,osc} + \left(\frac{m_2}{m_1}\right)^2 \chi_{2,osc} = 0 \tag{213}$$

$O(\epsilon^3)$:

$$\partial_t^2 \phi_{3,osc} + \phi_{3,osc} = \partial_\rho^2 \phi_{1,osc} + \frac{d-1}{\rho}\partial_\rho \phi_{1,osc} - 2\partial_t\partial_\tau \phi_{1,osc} + \phi_{1,osc}^3 + \Lambda \chi_{1,osc}^2 \phi_{1,osc} \tag{214}$$

$$\partial_t^2 \chi_{3,osc} + \left(\frac{m_2}{m_1}\right)^2 \chi_{3,osc} = \partial_\rho^2 \chi_{1,osc} + \frac{d-1}{\rho}\partial_\rho \chi_{1,osc} - 2\partial_t\partial_\tau \chi_{1,osc} + \Lambda \chi_{1,osc} \phi_{1,osc}^2 \tag{215}$$

From first inspection it is clear that the first two equations constrain the temporal part of the configuration while the last equation might be used to find actual constraints on the spatial part of the oscillon configuration, by canceling secular terms. Since the fields can now have different masses, finding profile equations is less straightforward. This is due to the extra factor of $\left(\frac{m_2}{m_1}\right)^2$ that is present in the perturbative equations. We thus limit our search to a few specific mass ratios.

1. $m_1 = m_2$

When the masses of the fields are equal the first order equations give the simple solutions $\phi_1 = \mathrm{Re}\{A(\rho,\tau)e^{-it}\}$ and $\chi_1 = \mathrm{Re}\{B(\rho,\tau)e^{-it}\}$. The second order equations also give similar results for ϕ_2 and χ_2 but since these are suppressed by ϵ in the final oscillon configuration we'll ignore them here. The third order equations can be rewritten using the solutions for ϕ_1 and χ_1 we just found and give a remarkable result:

$$\partial_t^2\phi_3 + \phi_3 = \frac{e^{-it}}{2}(\partial_\rho^2 A + \frac{d-1}{\rho}\partial_\rho A + 2i\partial_\tau A + \frac{3}{4}|A|^2 A + \frac{\Lambda}{2}|B|^2 A + \frac{\Lambda}{4}A^* B^2) + c.c. + h.h. \quad (216)$$

$$\partial_t^2\chi_3 + \chi_3 = \frac{e^{-it}}{2}(\partial_\rho^2 B + \frac{d-1}{\rho}\partial_\rho B + 2i\partial_\tau B + \frac{\Lambda}{2}B|A|^2 + \frac{\Lambda}{4}A^2 B^*) + c.c. + h.h. \quad (217)$$

The attentive reader will immediately notice that these equations will give nonphysical results if the factors between parentheses are non-zero. As before, we need to cancel all resonant terms. This leaves us with two equations for $A(\rho,\tau)$ and $B(\rho,\tau)$, which are our envelope equations since they describe the oscillon behavior on large (time) scales. They are given in equations 167 and (219).

$$\partial_\rho^2 A + \frac{d-1}{\rho}\partial_\rho A + 2i\partial_\tau A + \frac{3}{4}|A|^2 A + \frac{\Lambda}{2}|B|^2 A + \frac{\Lambda}{4}A^* B^2 = 0 \quad (218)$$

$$\partial_\rho^2 B + \frac{d-1}{\rho}\partial_\rho B + 2i\partial_\tau B + \frac{\Lambda}{2}B|A|^2 + \frac{\Lambda}{4}A^2 B^* = 0 \quad (219)$$

Since oscillons oscillate we can write without loss of generality: $A(\rho,\tau) = a(\rho)e^{ic_1\tau}$ and $B(\rho,\tau) = b(\rho)e^{ic_2\tau}$ where $a(\rho)$ and $b(\rho)$ are both real functions. We end up with the final profile equations for the oscillons in this system, describing its spatial configuration.

$$\partial_\rho^2 a + \frac{d-1}{\rho}\partial_\rho a - a + \frac{3}{4}a^3 + \frac{3\Lambda}{4}b^2 a = 0 \quad (220)$$

$$\partial_\rho^2 b + \frac{d-1}{\rho}\partial_\rho b - b + \frac{3\Lambda}{4}a^2 b = 0 \quad (221)$$

In these equations a value of $c_1 = c_2 = \frac{1}{2}$ was chosen. This choice is again free since as before the introduction of the slow time variable τ effectively introduces a free parameter. Any choice of c_1, c_2 can thus be reabsorbed. They are chosen positive so that the asymptotic behavior of the oscillon is correct (i.e. decaying at infinity).

2. $m_1^2 - m_2^2 = \Delta^2 \cdot \epsilon$

Here it is assumed that Δ^2 is of $O(1)$. Inserting this ansatz in the perturbation equations effectively adds an extra term proportional to Δ^2 at the next order in perturbation theory. This immediately makes it impossible to cancel resonant terms. In particular the $O(\epsilon^2)$ equations are

$$\partial_t^2\phi_{2,osc} + \phi_{2,osc} = 0 \quad (222)$$

$$\partial_t^2\chi_{2,osc} + \chi_{2,osc} = \Delta^2\chi_{1,osc} \quad (223)$$

Now clearly the solution $\xi_1 = \mathrm{Re}\{B(\rho,\tau)e^{-it}\}$ will always resonate at second order in perturbation theory. This means that $b(\rho) = 0$ is the only viable solution of the system of equations. We conclude that for this mass ratio there can be no stable oscillons (at least those that can be found perturbatively).

3. $m_1^2 - m_2^2 = \Delta^2 \cdot \epsilon^2$

In this case the same thing happens essentially. However, now the term proportional to Δ^2 enters two orders later in perturbation theory. This means that the resonant terms can now be cancelled and there is now a-priori problem. This results in the modified envelope equations

$$\partial_\rho^2 A + \frac{d-1}{\rho}\partial_\rho A + 2i\partial_\tau A + \frac{3}{4}|A|^2 A + \frac{\Lambda}{2}|B|^2 A + \frac{\Lambda}{4}A^* B^2 = 0 \tag{224}$$

$$\partial_\rho^2 B + \frac{d-1}{\rho}\partial_\rho B + 2i\partial_\tau B + \Delta^2 B + \frac{\Lambda}{2}B|A|^2 + \frac{\Lambda}{4}A^2 B^* = 0 \tag{225}$$

Leading to modified profile equations

$$\partial_\rho^2 a + \frac{d-1}{\rho}\partial_\rho a - a + \frac{3}{4}a^3 + \frac{3\Lambda}{4}b^2 a = 0 \tag{226}$$

$$\partial_\rho^2 b + \frac{d-1}{\rho}\partial_\rho b - b + \Delta^2 b + \frac{3\Lambda}{4}a^2 b = 0 \tag{227}$$

We see that an extra linear term enters the profile equation for $b(\rho)$. This will modify the asymptotic behavior of the localized solutions (as $b(\rho) \to 0$). The localized solutions will be found (numerically) in the next section.

4. Localized solutions

Solving the profile equations found in the previous section must be done numerically in general. A remarkable simplification takes place if we look for solutions $b(\rho) = ka(\rho)$ for the case $m_1 = m_2$ however. In particular the equations (221) become

$$\partial_\rho^2 a + \frac{d-1}{\rho}\partial_\rho a - a + \left(\frac{3}{4} + k^2\frac{3\Lambda}{4}\right)a^3 = 0 \tag{228}$$

$$\partial_\rho^2 a + \frac{d-1}{\rho}\partial_\rho a - a + \frac{3\Lambda}{4}a^3 = 0 \tag{229}$$

Clearly, if we find a localized solution of (229), we've found a localized solution of the whole system if $k = \pm\sqrt{1 - \frac{1}{\Lambda}}$. We clearly need $\Lambda > 1$, which is simply a reflection of our conditions

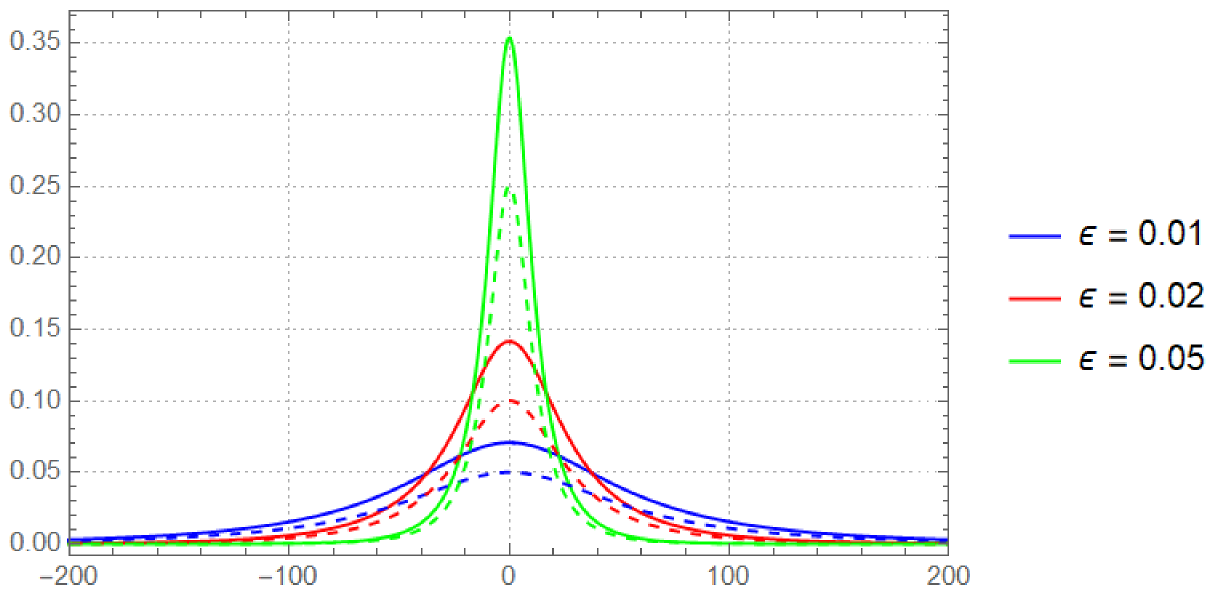


FIG. 17: The solutions of a (thick) and b (dashed) of the the profile equations (in real space) for the case $m_1 = m_2$. Since there are only cubic nonlinear terms in our profile equation ϵ takes on the role of the free parameter α from the previous chapter. As ϵ becomes larger, the solutions become slimmer and taller. The solutions are given for $\Lambda = 2$.

(200). Notice that again, we've found solutions for out-of-phase oscillons (for negative k), as well as in-phase oscillons (positive k). In one dimension the localized solution of (229) is our well known sech-function, $a(\rho) = \sqrt{\frac{8\Lambda}{3}}\,\mathrm{sech}(\rho)$. In higher dimensions the solutions can easily be found using numerical methods. The profiles in real space for different ϵ are given in figure 17 and 18.

If there is a mass difference Δ^2 between the fields the previous simplification is not possible. This can easily be seen by the difference in asymptotic behavior of the fields. In particular $b(\rho) \propto e^{-\sqrt{(m^2-\Delta^2)}\rho}$ while $b(\rho) \propto e^{-m\rho}$ as $\rho \to \infty$. The full system must thus be solved numerically in this case. In our numerical analysis we'll use the solutions for $m_1 = m_2$ as initial conditions and investigate the influence of a mass difference Δ^2 on our simulation of the equations of motion.

5. *Numerical stability analysis*

We checked the stability of the oscillon solutions by evolving the full system of equations of motion numerically. As initial conditions we chose $\phi(x,0) = k\chi(x,0) = \epsilon a(\epsilon x)$, $\dot{\phi}(x,0) = \dot{\chi}(x,0) = 0$ for in-phase oscillons and $\phi(x,0) = -k\chi(x,0) = \epsilon a(\epsilon x)$, $\dot{\phi}(x,0) = \dot{\chi}(x,0) = 0$ for out-of-phase oscillons. Here $a(\epsilon x)$ are solutions of the profile equations for $m_1 = m_2$ and $k = \sqrt{1 - \frac{1}{\Lambda}}$. We also investigated the influence of a mass mismatch by introducing a mass difference through $m_1^2 - m_2^2 =$

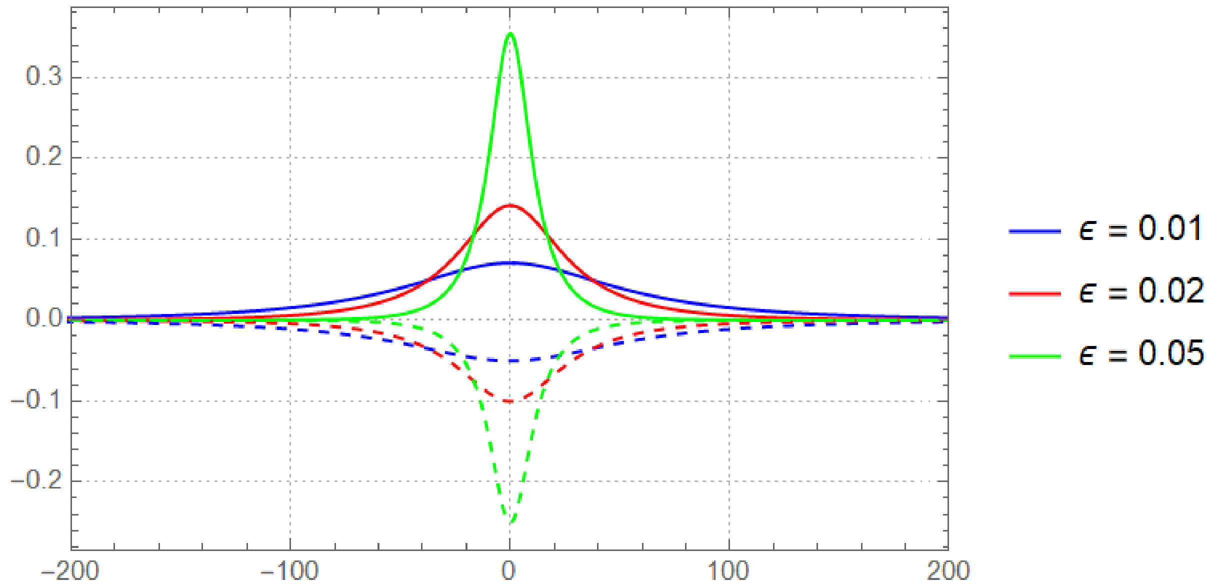


FIG. 18: In this model we have out-of-phase solutions of the profile equations as well. This is probably a general feature of multi-field oscillons with symmetric interactions.

$\Delta^2\epsilon^2$, with the same initial conditions. In our investigations we set $\epsilon = 0.01$. The oscillon region was again defined as the region in space that contained 95% of the total energy. The energy in this region was tracked for both fields individually. The main conclusions of these simulations can be summarized as follows:

- No stable oscillons were found to exist in this model in three dimensions. The contrary is true for oscillons in one dimension. Oscillons seemed to be exceedingly stable in this case and exist for a large range of values for Δ^2.

- Adding a mass difference Δ^2 to the system makes it possible for one field to deposit its energy content into the other field. The fields interact until an equilibrium state is reached in three dimensions. The equilibrium state is reached earlier if the absolute mass difference $|\Delta^2|$ is bigger. In one dimension an equilibrium state never seems to be reached and interactions persist. This field "pumping" becomes less violent as we increase the absolute value of Δ^2 until the fields eventually decouple.

- As was the case for oscillons studied in the previous chapter, the decay of the configuration happens via dispersion. In three dimensions, the lifetime of the oscillon seems to decrease as we increase the mass difference Δ^2.

Although we were not able to perform a full analytical stability analysis for the oscillon solutions in this model, we will discuss some general ideas and give qualitative interpretations of the numerical results in the following section.

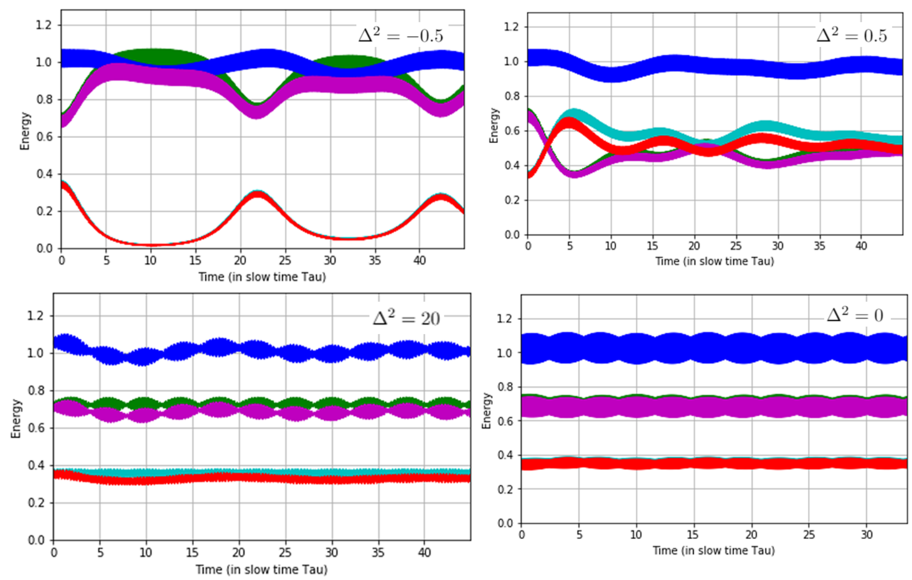


FIG. 19: The energy evolution of the localized solutions in one dimension for different values of the mass mismatch Δ^2. We see here the energy in the field χ (light blue), the energy in the field ϕ (green), the localized energy in χ (red), the localized energy in ϕ (purple) and the sum the localized energies (dark blue). Adding a mass mismatch to the system introduces interactions between the fields. As the Δ^2 increases the fields eventually decouple. Oscillons are very stable in one dimension as the energy remains localized for many periods of oscillation.

B. Qualitative discussion and considerations for asymmetric couplings

I would first like to address the fact that no stable oscillons were found in three dimension, while the opposite was found in one dimension. This distinction is interesting since it hints at a possible extension of the V-K criterion. It can be shown that in one dimension the solutions of the profile equations never fail the V-K criterion (at least for $m_1 = m_2$), while this is not the case for the solutions in three dimensions. This is ultimately tied to the "trivial" height-width relations of these sech-like oscillons. A similar conclusion was reached in [11] where the V-K criterion also seemed to apply in a more general two-field model. A thorough mathematical derivation of the criterion is still lacking however, but an extension to more general models might be possible.

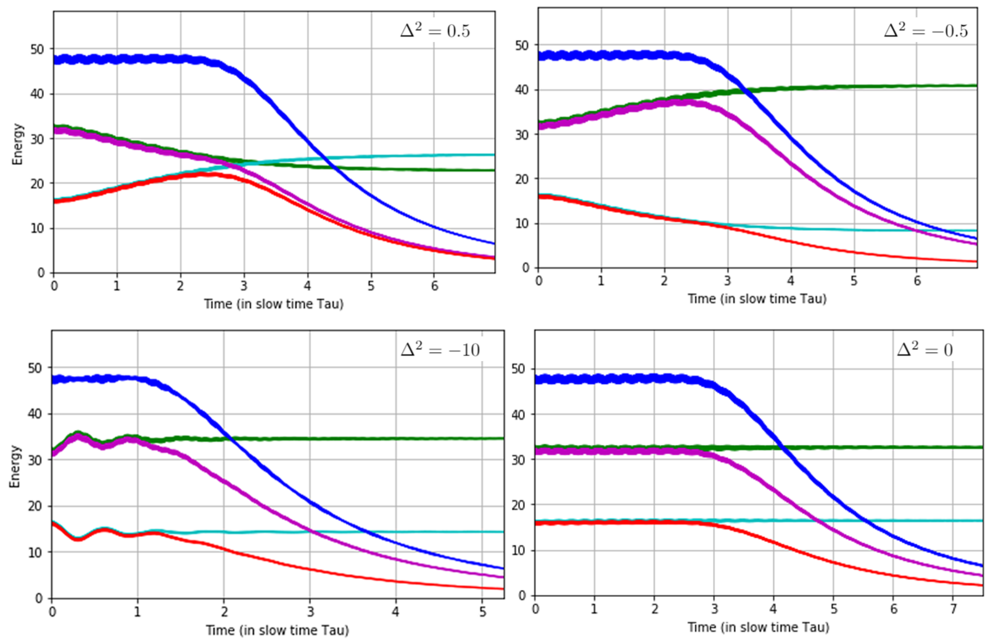


FIG. 20: The energy evolution of the localized solutions in three dimensions for different values of the mass mismatch Δ^2. We see here the energy in the field χ (light blue), the energy in the field ϕ (green), the localized energy in χ (red), the localized energy in ϕ (purple) and the sum the localized energies (dark blue). Adding a mass mismatch to the system introduces interactions between the fields until an equilibrium is reached. As the Δ^2 increases the fields eventually decouple. We weren't able to find stable oscillons in three dimensions as the configuration eventually disperses its energy away. If the mass mismatch is larger dispersion starts earlier.

The effect of introducing a mass mismatch through Δ^2 might be best understood using intuition from Feynman perturbation theory. In reality the depositing of energy into the other field is a non-perturbative effect since we use large coupling constants, but we can gain some intuition from particle physics. The pumping of the fields happens through the interaction $2\phi \to 2\chi$ and vice versa. Conservation of momentum then requires

$$m_1^2 + k_\phi^2 = m_2^2 + k_\chi^2 \tag{230}$$

And we conclude that $\Delta^2\epsilon^2 = k_\chi^2 - k_\phi^2$. Since an oscillon only contains k-modes up until $O(\epsilon)$ the interaction strength should decrease as $\Delta^2 \to O(1)$. This is approximately what we observed in our numerical stability analysis, where the fields start to decouple as we increase $|\Delta^2|$. Extending this intuition to asymmetric vertices, we predict that in asymmetric potentials the pumping mechanism becomes increasingly important. This is simply due to the fact that the cross-section in one

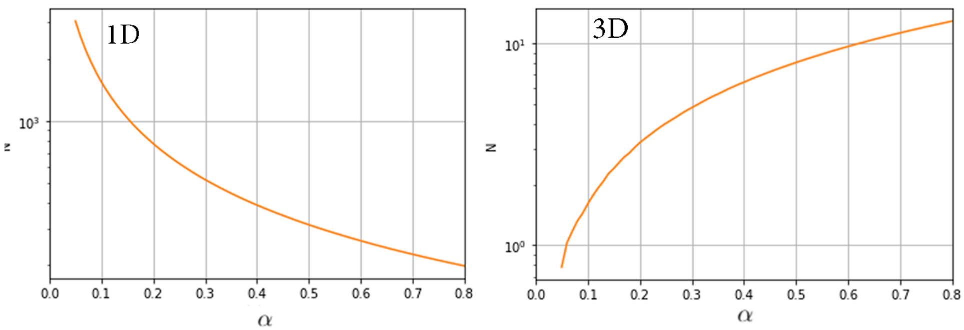


FIG. 21: The V-K criterion visualized for the model under consideration. We've defined $N = \int d^d\rho(1+k^2)a(\rho)^2$ and considered the profiles for the case $m_1 = m_2$. α is equivalent to ϵ in this model. We see that the solutions fail the criterion in three dimensions, while the opposite is true in one dimension. This agrees with our numerical analysis and might indicate that a general extension of the V-K criterion exists.

direction of the vertex is larger due to the larger phase space available to the particles. The fact that the decay of the oscillon via dispersion happens earlier for larger mismatches is ultimately tied to the fact that $\sigma(t) \propto 1/m^2$, as shown in (14). A larger mass mismatch just means a larger total mass and thus more rapid dispersion.

An interesting characteristic of oscillons in symmetric potentials is the fact that all solutions branch off into two distinct ways: any solution will have an in-phase configuration and an out-of-phase configuration. This is just a general consequence of the fact that in symmetric potentials the profile equations obtain an overall minus sign when sending $b \to -b$ or $a \to -a$. It is not expected that this is true for asymmetric potentials, where such a symmetry doesn't exist.

In this chapter we've discussed oscillons in a more general system containing two scalar fields in a symmetric potential. We've seen how a mass difference between the fields will in general alter the profile equations. The solutions were shown to be unstable in three dimensions although they were exceedingly stable in one dimension. Initial investigations into asymmetric potentials with cubic interactions also resulted in unstable oscillons. A full analysis of oscillons in more general potentials will be left for future work.

VI. CONCLUSION AND FUTURE DIRECTIONS

In this thesis we've looked at oscillons in the context of scalar field theories. I've started by reviewing oscillons in single-field models focusing on their analytical construction and linear stability analysis. We've seen how oscillons can emerge during preheating and argued that the general nature of oscillon solutions demands a better understanding of oscillons in multi-field theories. We were able to fully analyse oscillons in a specific model containing an "exchange" symmetry. We saw that out-of-phase oscillons also existed in this model and it is plausible that this is a general characteristic for multi-field oscillons in symmetric potentials.

The stability of these oscillons was analysed by extending the Vakhitov-Kolokolov criterion. The theoretical predictions of this criterion seemed to agree with our numerical simulations. Although the criterion only applies to a specific subsection of possible initial conditions, it was argued that it is this section that is the most important in assessing stability. The reason that stable oscillons were found in this model was due to the large sextic term in the potential. The localized zero-mode solutions had a non-trivial width/height relation which stabilizes the solutions. It could be that a large nonlinear term can stabilize solutions in more general models as well, as it was hinted that the V-K criterion might be extended.

An area that we still understand poorly is the exact mechanisms that come into play as the oscillon collapses. In our simulations we saw decay in the form of dispersion, but other types of decay might also exist (e.g. collapse instabilities). We expect that the mass difference that can be present in multi-field theories will influence decay significantly, as well as whether the scalar potential is symmetric or asymmetric.

It is clear that still a lot needs to be done in the area of multi-component oscillons to understand them fully. In future work we will focus on oscillons in general multi-field models, mapping the influence of mismatches between the masses of the fields on stability. We will also investigate the effects of symmetric versus asymmetric couplings. Another interesting approach to this analysis would be to perform a full lattice simulation and see if the multi-field oscillons emerge naturally. It is still an open question whether the multi-component oscillons found in this thesis are more or less attractive in the space of possible field configurations than the isolated oscillons, and performing these types of simulations might grant us some insight into this question.

VII. ACKNOWLEDGEMENTS

I want to thank Evangelos Sfakianakis, Ana Achúcarro and Alessandra Silvestri for supervising this project. It was very exciting to take my first steps in real research, especially since the territory of multi-component oscillons is relatively uncharted. I'm looking forward to continuing this work in the future!

[1] S. Coleman, “Q-balls,” Nuclear Physics B **262**, 744 (1985).

[2] Fuminobu Takahashi S. Kasuya, M. Kawasaki, “I-balls,” Physics Letters B **559**, 99–106 (2003), arXiv:0209358 [hep-ph].

[3] V.G. Makhankov I.L. Bogolyubsky, “Lifetime of pulsating oscillons in certain classical models,” Pisma Zh.Eksp.Teor.Fiz. **24**, 15–18 (1976).

[4] D. Holm R. Camassa, “An integrable shallow water equation with peaked solitons,” Physics Review Letters **71**, 1661 (1993), arXiv:9305002 [nlin.PS].

[5] Mariya A. Lizunova Vakhid A. Gani, Alexander E. Kudryavtsev, “Kink interactions in the (1+1)-dimensional ψ^6-model,” Physics Review D **89**, 125009 (2014), arXiv:1402.5903 [hep-th].

[6] Marcelo Gleiser, “Pseudo-Stable Bubbles,” Phys. Rev. D **49**, 2978 (1994), arXiv:9308279 [hep-ph].

[7] David Sicilia Marcelo Gleiser, “A General Theory of Oscillon Dynamics,” Physics Review D **80**, 125037 (2009), arXiv:0910.5922 [hep-th].

[8] Marcelo Gleiser, “d-dimensional Oscillating Scalar Field Lumps and the Dimensionality of Space,” Physics Letter B **600**, 126–132 (2004), arXiv:0408221 [hep-th].

[9] N. Graham, “An Electroweak Oscillon,” Phys. Rev. Lett. **98** (2007), 10.1103/PhysRevLett.98.101801, arXiv:0610267 [hep-th].

[10] Mustafa A. Amin and David Shirokoff, “Flat-top oscillons in an expanding universe,” Phys. Rev. D **81**, 085045 (2010), arXiv:1002.3380 [astro-ph.CO].

[11] Evangelos I. Sfakianakis, “Analysis of Oscillons in the SU(2) Gauged Higgs Model,” (2012), arXiv:1210.7568 [hep-ph].

[12] A. H. Guth N. Iqbal R. R. Rosales N. Stamatopoulos E. Farhi, N. Graham, “Emergence of Oscillons in an Expanding Background,” Physical Review D **77** (2008), 10.1103/PhysRevD.77.085019, arXiv:0712.3034 [hep-th].

[13] A. A. Kolokolov N. G. Vakhitov, “Stationary solutions of the wave equation in a medium with nonlinearity saturation,” Radiophysics and Quantum Electronics **16**, 783–789 (1973).

[14] Alim Sukhtayev Gregory Berkolaiko, Andrew Comech, “Vakhitov-Kolokolov and energy vanishing conditions for linear instability of solitary waves in models of classical self-interacting spinor fields,” (2013), arXiv:1306.5150 [math-ph].

[15] H.-R. Mueller E.J. Copeland, M. Gleiser, “Oscillons: Resonant Configurations During Bubble Collapse,” Physics Review D **52**, 1920 (1995), arXiv:9503217 [hep-ph].

[16] Igor I. Tkachev Edward W. Kolb, “Non-linear axion dynamics and formation of cosmological pseudo-solitons,” Phys. Rev. D **49**, 5040 (1994), arXiv:9311037 [astro-ph].

[17] Hal Finkel Mustafa A. Amin, Richard Easther, “Inflaton Fragmentation and Oscillon Formation in Three Dimensions,” Journal of Cosmology and Astroparticle Physics **2010** (2010), 10.1088/1475-7516/2010/12/001, arXiv:1009.2505 [astro-ph.CO].

[18] Mustafa A. Amin, “Inflaton fragmentation: Emergence of pseudo-stable inflaton lumps (oscillons) after inflation,” (2010), 10.1103/PhysRevD.80.125037, arXiv:1006.3075 [astro-ph.CO].

[19] John McDonald, “Inflaton Condensate Fragmentation in Hybrid Inflation Models,” Physical Review D **66** (2002), 10.1103/PhysRevD.66.043525, arXiv:0105235 [hep-ph].

[20] Jerome Martin Karsten Jedamzik, Martin Lemoine, “Generation of gravitational waves during early structure formation between cosmic inflation and reheating,” Journal of Cosmology and Astroparticle Physics **2010**, 021–021 (2010), arXiv:1002.3278 [astro-ph.CO].

[21] Marcelo Gleiser, Noah Graham, and Nikitas Stamatopoulos, “Generation of Coherent Structures After Cosmic Inflation,” Phys. Rev. D **83**, 096010 (2011), arXiv:1103.1911 [hep-th].

[22] N. Graham, “Numerical Simulation of an Electroweak Oscillon,” Physical Review D **76**, 085017 (2007), arXiv:0706.4125 [hep-th].

[23] Mustafa A. Amin, “K-oscillons: Oscillons with noncanonical kinetic terms,” Phys. Rev. D **87**, 123505 (2013), arXiv:1303.1102 [astro-ph.CO].

[24] Z. K. Yankauskas, “Radial Field distributions in a self-focusing light-beam ,” Izvestiya VUZ. Radiofizika **9**, 412–415 (1966).

[25] MD Kruskal H Segur, “Nonexistence of Small-Amplitude Breather Solutions in phi4 Theory,” Physical Review Letters **58**, 747–750 (1987).

[26] Philippe Grandclément István Rácz Gyula Fodor, Péter Forgács, “Oscillons and Quasi-breathers in the ψ^4 Klein-Gordon model,” Physics Review D **74**, 124003 (2006), arXiv:0609023v1 [hep-th].

[27] Zalán Horváth Márk Mezei Gyula Fodor, Péter Forgács, “Radiation of scalar oscillons in 2 and 3 dimensions,” Physics Letters B **674**, 319–324 (2009), arXiv:0903.0953 [hep-th].

[28] Zalán Horváth Márk Mezei Gyula Fodor, Péter Forgács, “Computation of the radiation amplitude of oscillons,” Physics Review D **79**, 065002 (2009), arXiv:0812.1919 [hep-th].

[29] Mark P. Hertzberg, “Quantum Radiation of Oscillons,” Physics Review D **82**, 045022 (2010), arXiv:1003.3459 [hep-th].

[30] Lev A. Kofman, “The Origin of Matter in the Universe: Reheating after Inflation,” (1996), arXiv:9605155 [astro-ph].

[31] James B. Gilmore Richard Easther, Raphael Flauger, “Delayed Reheating and the Breakdown of Coherent Oscillations,” Journal of Cosmology and Astroparticle Physics **2011** (2011), 10.1088/1475-7516/2011/04/027, arXiv:1003.3011 [astro-ph.CO].

[32] Jerome Martin Karsten Jedamzik, Martin Lemoine, “Collapse of Small-Scale Density Perturbations during Preheating in Single Field Inflation,” Journal of Cosmology and Astroparticle Physics **2010** (2010), 10.1088/1475-7516/2010/09/034, arXiv:1002.3039 [astro-ph.CO].

[33] Michele Liguori Sabino Matarrese Maria Chiara Guzzetti, Nicola Bartolo, “Gravitational waves from inflation,” La Rivista Del Nuovo Cimento (2016), 10.1393/ncr/i2016-10127-1, arXiv:1605.01615 [astro-ph.CO].

[34] Stefano Orani Stefan Antusch, Francesco Cefala, “Gravitational waves from oscillons after inflation,” Physical Review Letters **118**, 011303 (2017), arXiv:1607.01314 [astro-ph.CO].

[35] Richard Easther Hal Finkel Zong-Gang Mou Paul M. Saffin Shuang-Yong Zhou, Edmund J. Copeland, “Gravitational Waves from Oscillon Preheating,” Journal of High Energy Physics (2013), 10.1007/JHEP10(2013)026, arXiv:1304.6094 [astro-ph.CO].

[36] Fabrizio Rompineve Jan Olle, Oriol Pujolas, “Oscillons and Dark Matter,” Journal of Cosmology and Astroparticle Physics (2020), 10.1088/1475-7516/2020/02/006, arXiv:1906.06352 [hep-ph].

[37] M. Yu. Khlopov, “Primordial Black Holes,” Research in Astronomy and Astrophysics **10** (20107), 10.1088/1674-4527/10/6/001, arXiv:0801.0116 [astro-ph].

[38] John Joseph M. Carrasco, Renata Kallosh, Andrei Linde, and Diederik Roest, “Hyperbolic geometry of cosmological attractors,” Phys. Rev. D **92**, 041301 (2015), arXiv:1504.05557 [hep-th].

[39] Tomasz Krajewski, Krzysztof Turzyński, and Michał Wieczorek, “On preheating in α-attractor models of inflation,” Eur. Phys. J. C **79**, 654 (2019), arXiv:1801.01786 [astro-ph.CO].

[40] Oksana Iarygina, Evangelos I. Sfakianakis, Dong-Gang Wang, and Ana Achucarro, “Universality and scaling in multi-field α-attractor preheating,” JCAP **06**, 027 (2019), arXiv:1810.02804 [astro-ph.CO].

[41] Oksana Iarygina, Evangelos I. Sfakianakis, Dong-Gang Wang, and Ana Achcarro, “Multi-field inflation and preheating in asymmetric α-attractors,” (2020), arXiv:2005.00528 [astro-ph.CO].