Vector bundles and the Berry phase

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Abstract

A mathematically rigorous description is given of Berry's geometric phase in quantum mechanics for a wave function in a system described by a finite-dimensional Hilbert space $\mathcal{H}$ together with a Hamiltonian $\hat{H}$ smoothly dependent on a parameter.
Under the assumption of the adiabatic approximation, traversing a smooth path in the parameter space results in the accumulation of a geometric phase that can locally be computed as the line integral of local 1-forms, emerging naturally from the eigensections on the trivial vector bundle over the parameter space. By Stokes' theorem, the integral may be calculated as a surface integral of Berry's 2-form, and a global description of this 2-form is given. When instead of the trivial bundle an eigenbundle of $\hat{H}$ is considered, solutions of Schrödinger's equation emerge naturally as horizontal lifts with respect to the canonical connection on this bundle. It also becomes clear that Berry's phase is a geometric quantity.
Finally, a generalization of this geometric phase due to Ahranov and Anandan is described in terms of the tautological line bundle with base manifold the complex projective space $\mathbb{P}\mathcal{H}$. This generalization is exactly valid, even if the Hamiltonian varies non-adiabatically.

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Chapter 1

Introduction

In our understanding of the physical world, quantum mechanics plays a crucial role. It allows physicists to describe a particle as a *wave function* $|\psi\rangle$ that, from a mathematical perspective, is a vector in a complex Hilbert space $\mathcal{H}$. In experiments, such a wave function can not be measured directly. In fact, if $|\psi\rangle$ and $|\psi'\rangle$ are two wave functions such that there is a *phase angle* $\theta \in \mathbb{R}$ with $|\psi'\rangle = e^{i\theta}|\psi\rangle$, no direct measurement can differentiate between the two. When an experimenter brings the two into contact, however, it is possible to measure their phase difference θ due to interference.

The equation of motion for wave functions is known as Schrödinger's equation,

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\psi;t\rangle = \hat{H}(t)|\psi;t\rangle,$$

where $\hat{H}(t)$ at any time is a Hermitian operator on $\mathcal{H}$ known as the *Hamiltonian* of the system.

It was first discovered by Berry [3] in 1984 that when a wave function evolves due to a time-dependent Hamiltonian and after some time returns to its initial state, in certain cases a phase difference is accumulated that is purely geometric in nature. Not long thereafter, Simon [18] noted that this phase change may be described mathematically by a connection in a fiber bundle. In the subsequent years, multiple generalizations were proposed. Of these, the geometric phase Ahranov and Anandan [1] described in 1987 is among the more interesting.

Though these results have been confirmed experimentally – an overview of these experiments may be found in [23] – from the perspective of a mathematician these papers are lacking in rigor. In this thesis, an attempt is made to give a mathematically compelling description of the mechanisms behind this geometric phase.

As in the setup described by Berry, the assumption is made that the t-dependence of the Hamiltonian is due to a slowly varying parameter. We require the parameter dependence to be 'smooth', or 'differentiable', though the parameter space is not necessarily Euclidean. A natural choice is to model it by a *smooth manifold M*: a topological space that is locally 'similar' to Euclidean space and thus inherits a notion of smoothness and differentiability. It will be convenient not to consider a map from this space directly into $\mathcal{H}$, but rather to 'attach' a copy of $\mathcal{H}$ to each point $R \in M$ and study the action of $\hat{H}(R)$ on this copy. The mathematical structure that allows us to do this, is called the *trivial vector bundle* with base space M and typical fiber $\mathcal{H}$. We consider *local eigensections* of this bundle: smooth assignments of an eigenvalue of $\hat{H}(R)$ for each parameter R in an open subset of M. A *connection* in a vector bundle is an extra structure that for any path in the base space tells us how to 'connect' the vector spaces above this path. It will be shown that as a consequence of the adiabatic approximation, the geometric phase can be calculated locally by a path integral on M in terms of the trivial connection and eigensection on this bundle. The geometric phase will also be written in terms of a surface integral of a globally defined 2-form, called *Berry's 2-from*.

Then, like Simon did, we will consider an *eigenbundle* of $\hat{H}$, modeling a smooth assignment of an eigenspace of $\hat{H}(R)$ 'attached' to each $R \in M$. It will be shown that solutions of Schrödinger's equation emerge naturally from the canonical connection on this bundle.

Finally, the generalization due to Ahranov and Anandan will be described that gives us a way to calculate the geometric phase exactly, without requiring any approximation. For this, we will consider a vector bundle known as the *tautological line bundle*. The base manifold of this vector bundle is the complex projective space $\mathbb{P}\mathcal{H}$, consisting of wave functions except that the phase is 'forgotten', elements of $\mathbb{P}\mathcal{H}$ are sometimes referred to as *physical states*. In this context, the geometric phase is also described by a line integral in the base space.

In order make this mathematically rigorous, in Chapter 2 an outline is given of some mathematical preliminaries in the theory of smooth manifolds. Section 2.1 gives the definition of a smooth manifold, and Section 2.2 discusses tangent vectors and vector fields. Then Section 2.3 defines cotangent vectors and 1-forms before delving into the theory of alternating multilinear

maps in order to define general k-forms on manifolds.
Chapter 3 develops the theory of vector bundles, with Section 3.1 giving the definition of a smooth fiber bundle and the special case of a vector bundle. Section 3.2 explains connections in vector bundles, giving rise to the covariant derivative on sections of the bundle, and Section 3.3 uses this concept to define the horizontal lift of a curve in the base space of a vector bundle. Section 3.4 then discusses the explicitly trivial vector bundle and the tautological line bundle, two vector bundles of central importance in this thesis.
This mathematical formalism is put to use in Chapter 4 where in Section 4.1 the adiabatic theorem is introduced, Section 4.2 explores the implications of this theorem for the geometric phase described by Berry and Simon, and Section 4.3 describes the geometric phase due to Ahranov and Anandan.
Finally, in Chapter 5 some directions for further research are given.

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Chapter 2

Mathematical preliminaries

This chapter is meant to familiarize the reader with some concepts in differential geometry that are extensively used in the rest of the text. In that light, I have decided to exclude the proofs. This section is mainly based on [14] and I recommend the reader who is interested in the proofs or in a more exhaustive discussion on smooth manifolds to consult it. For the discussion on vector-valued forms, [20] was also consulted.

2.1 Smooth manifolds

A smooth manifold M of dimension n is a space that locally resembles the Euclidean space $\mathbb{R}^n$, and admits a cover of coordinate patches. Where two patches overlap, there is a smoothness criterion for their transition function. This allows us to generalize many concepts of multivariate calculus to spaces that only locally resemble Euclidean space.

The definition of a manifold requires a few basic concepts from topology. A reader who is unfamiliar with these definitions might, for example, consult [16].

Definition 2.1.1. Let $n \in \mathbb{Z}_{\geq 0}$. If M is a topological space that satisfies the following conditions:

(a) M is Hausdorff and second-countable;

(b) There exists an ***atlas*** $\mathcal{A}$ of M, consisting of pairs (U, ϕ) with $U \subseteq M$

an open subset and $\phi\colon U \to \mathbb{R}^n$ a continuous, open and injective map called a ***chart***, such that $\{U \subseteq M : (U,\phi) \in \mathcal{A}\}$ is an open cover of M;

(c) The atlas $\mathcal{A}$ is smooth: for $(U,\phi),(V,\psi) \in \mathcal{A}$, the ***transition map from ϕ to ψ*** written as $(\psi \circ \phi^{-1})\colon \phi[U \cap V] \to \psi[U \cap V]$ is smooth (i.e. infinitely differentiable);

(d) The atlas $\mathcal{A}$ is maximal: if (U,ϕ) is a chart and $\mathcal{A} \cup \{(U,\phi)\}$ is a smooth atlas of M, then $(U,\phi) \in \mathcal{A}$;

then $(M,\mathcal{A})$ is called a ***smooth manifold of dimension*** n.

Often a manifold is written as M and the maximal smooth atlas $\mathcal{A}$ is implied.

A chart $\phi\colon U \to \mathbb{R}^n$ is a homeomorphism of U and the image of ϕ in $\mathbb{R}^n$. It gives rise to a ***local coordinate system*** $(x^1,\ldots,x^n)$, where the $x^i\colon U \to \mathbb{R}$ are the ***coordinate functions*** defined such that $\phi(p) = \big(x^1(p),\ldots,x^n(p)\big)$ for $p \in U$. The condition that $\{U \subseteq M : (U,\phi) \in \mathcal{A}\}$ is an open cover of M guarantees that any $p \in M$ is contained in a local coordinate system.

In order to see that the transition map from ϕ to ψ is well-defined, first note that ϕ is open, and therefore the image $\phi[U \cap V]$ is open in $\mathbb{R}^n$. By injectivity, for each $x \in \phi[U \cap V]$ there is a unique $p_x \in U \cap V$ with $\phi(p_x) = x$. With this notation, $(\psi \circ \phi^{-1})\colon x \mapsto \psi(p_x)$ is a well-defined map between open subsets of $\mathbb{R}^n$. We call a function $f\colon \mathbb{R}^n \to \mathbb{R}^m$ ***smooth*** if it is infinitely differentiable: for any $k \in \mathbb{Z}_{\geq 0}$ the k^{th} order partial derivatives must exist and be continuous.

In practice it is unwieldy to provide a maximal smooth atlas. Luckily, any smooth atlas $\mathcal{A}$ on a second-countable Hausdorff space M defines a unique maximal smooth atlas consisting of all charts compatible with $\mathcal{A}$.

Example 2.1.2. Any finite-dimensional real vector space V with $\dim V = n$ is a smooth manifold of dimension n. Let $(|e_i\rangle)$ be a basis of V, and let $\phi\colon V \to \mathbb{R}^n$ be the chart defined by $|v\rangle \mapsto (v^i)$ for $v^i \in \mathbb{R}$ such that $|v\rangle = |e_i\rangle v^i$, written in the Einstein summation convention. The atlas $\mathcal{A} := \{(V,\phi)\}$ is trivially smooth, and so induces a manifold structure on V.

Example 2.1.3. For any finite-dimensional real vector space V with $\dim V = n \geq 1$, the projective space $\mathbb{P}V$, defined to be the set of one-dimensional subspaces of V, is a smooth manifold of dimension $n-1$. Let $(|e_i\rangle)$ be a basis of V, and for $v^i \in \mathbb{R}$ not all zero, we set

$$(v^1 : v^2 : \ldots : v^n) := \operatorname{span}\big(|e_i\rangle v^i\big) \in \mathbb{P}V. \tag{2.1}$$

Note that $(v^1 : \ldots : v^n) = (\lambda v^1 : \ldots : \lambda v^n)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. Now define open sets $U_j \subseteq \mathbb{P}$ for $j \in \{1, \ldots, n\}$ by $U_j = \{\operatorname{span}(|e_i\rangle v^i) : v^j \neq 0\}$, and define $\phi_j \colon U_j \to \mathbb{R}^{n-1}$ by

$$\phi_j \colon (v^1 : \ldots : v^j : \ldots : v^n) \mapsto \frac{1}{v^j} \cdot (v^1, \ldots, v^{j-1}, v^{j+1}, \ldots, v^n).$$

With inverse given by $(x^i) \mapsto (x^1 : \ldots : x^{j-1} : 1 : x^{j+1} : \ldots : x^{n-1})$. This map is well-defined, and it is easy to check that the transition maps are smooth.

Note that both examples generalize to the case of complex vector spaces, since $\mathbb{C}$ itself is a smooth 2-dimensional manifold.

Now that we have a structure that is in some sense 'smooth', we are able to define what smooth functions and maps are.

Definition 2.1.4. Let $(M, \mathcal{A})$ be a smooth manifold. A map $f \colon M \to \mathbb{R}$ is called a ***smooth function on*** M if for every $p \in M$ there is a chart $(U, \phi) \in \mathcal{A}$ such that U is an open neighborhood of p and $(f \circ \phi^{-1}) \colon \phi[U] \to \mathbb{R}$ is smooth.

We denote the set of all smooth functions on a smooth manifold M by $C^\infty(M)$. If V is a finite-dimensional real vector space with basis $(|e_i\rangle)$, a map $f \colon M \to V$ is called smooth if there are $f^i \in C^\infty(M)$ such that $f = |e_i\rangle f^i$. The set of smooth V-valued functions is denoted by $C^\infty(M, V)$. Both sets have a natural real vector space structure with operations pointwise scalar multiplications and addition.

A type of smooth functions that will prove to be useful later in this thesis are the so-called *bump functions*. These are identically zero outside of a specified open set.

Definition 2.1.5. Let M be a smooth manifold and let $f \in C^\infty(M)$ be a smooth function, the losure of the set $\{p \in M : f(p) = 0\}$ is called the ***support of*** f, written as

$$\operatorname{supp} f := \overline{\{p \in M : f(p) = 0\}}.$$

Lemma 2.1.6. *If $U \subseteq M$ is an open subset of a smooth manifold, and $A \subseteq U$ is closed, there exists a smooth function $f \in C^\infty(M)$ such that* $\operatorname{supp} f \subseteq U$ *and $f|_A \equiv 1$, called a* ***bump function for A with support in U****.*

Here, '$\equiv$' should be read as 'is identically' and be taken to mean that $f|_A(p) = 1$ for each p in the domain on $f|_A$.

A similarly useful construct is the following:

Lemma 2.1.7. *If M is a smooth manifold and $\mathcal{U} = \{U_\alpha\}$ is an open cover of M, there exist smooth functions $\phi^\alpha \in C^\infty(M)$ such that $\operatorname{supp}\phi^\alpha \subseteq U_\alpha$ and for every $p \in M$, $0 \leq \phi^\alpha(p) \leq 1$, and p has an open neighborhood U_p such that $\operatorname{supp}\phi^\alpha(p) \cap U_p = \emptyset$ for all but finitely many α, and*

$$\sum_\alpha \phi^\alpha(p) = 1.$$

*The family $\{\phi^\alpha\}$ is called a **smooth partition of unity subordinate to** $\mathcal{U}$.*

Maps from one manifold to another also have a criterion for smoothness, derived from their atlases.

Definition 2.1.8. Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be smooth manifolds. A map $F\colon M \to N$ is called a ***smooth map from** M **to** N* if for every $p \in M$ there are charts $(U, \phi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$ such that U is an open neighborhood of p, $F[U]$ is contained in V, and $(\psi \circ F \circ \phi^{-1})\colon \phi[U] \to \psi[V]$ is smooth.

It may be shown that the composition of two smooth maps again smooth. A smooth map that has a smooth inverse is called a ***diffeomorphism***.

2.2 Tangent vectors

A tangent vector to a point $p \in M$ should be thought of as a direction one can travel starting at p within the manifold M. Since the smoothness criterion of a manifold is based on infinite differentiability of some maps, it is natural to define these tangent vectors in terms of differentiation. Tangent vectors in Euclidean space $\mathbb{R}^n$ are correspond to directional derivatives, and this gives us the idea to think of derivations on M as tangent vectors.

Definition 2.2.1. Let M be a smooth manifold, then $X_p\colon C^\infty(M) \to \mathbb{R}$ is called a ***derivation at** p* if it is linear and satisfies the ***product rule***: for all $f, g \in C^\infty(M)$ we require

$$X_p(fg) = f(p)X_p(g) + g(p)X_p(f).$$

The real vector space of all derivations at p, denoted T_pM, is called the ***tangent space to*** M ***at*** p, and has the same dimension as M. The disjoint union of these tangent spaces is denoted by

$$TM := \bigsqcup_{p \in M} T_pM \tag{2.2}$$

and has the structure of a smooth manifold of twice the dimension of M. It comes equipped with a natural ***projection*** $\pi_T \colon TM \to M$ that maps a tangent vector $X_p \in TM$ to the unique $p \in M$ with $X_p \in T_pM$.

Given a local coordinate system (x^i) near a point $p \in M$, and write

$$[\partial_i]_p := \left.\frac{\partial}{\partial x^i}\right|_p : f \longmapsto \frac{\partial f(x^i, \dots, x^n)}{\partial x^i}(p) \tag{2.3}$$

then $([\partial_i]_p)$ is a basis of the tangent space T_pM.

Lemma 2.2.2. *If M is a smooth manifold, X_p is a derivation at some $p \in M$ and $f \equiv c$ for some $c \in \mathbb{R}$ is a constant function on M, we have $X_p f = 0$.*

Definition 2.2.3. Let M be a smooth manifold. A smooth map $X \colon M \mapsto TM$ such that $\pi_T \circ X = \mathrm{id}_M$ is called a ***vector field on*** M, and we often write X_p for $X(p)$.

The condition $\pi_T \circ X = \mathrm{id}_M$ guarantees that X is a smooth assignment of a tangent vector at p for each $p \in M$. We write $\Gamma(TM)$ for the set of all vector fields on M, for reasons that will become apparent below, under Definition 3.1.2.

Since a vector field $X \in \Gamma(TM)$ on a smooth manifold M at each point is a derivation, we can apply X to a function $f \in C^\infty(M)$ and the result is another smooth function $Xf \in C^\infty(M)$, defined by $[Xf](p) = X_p f \in \mathbb{R}$. By the product rule, for $f, g \in C^\infty(M)$ we have $X(fg) = f(Xg) + g(Xf)$, since $[X(fg)](p) = f(p)X_p(g) + g(p)X_p(f) = [f(Xg) + g(Xf)](p)$. We call an $X \colon C^\infty(M) \to C^\infty(M)$ with this property a ***derivation***.

Differentials

We saw above that if f is a function on a manifold, tangent vectors act on f in a way reminiscent of differentiation. Using this notion, it makes sense to define the differential of a function as a smooth function on TM.

Definition 2.2.4. Let M be a smooth manifold and $f \in C^\infty(M)$ be a smooth function. We define the ***differential of*** f as the function $df \in C^\infty(TM)$ given by $df\colon X_p \mapsto X_p f \in \mathbb{R}$.

Note that the restriction $df_p := df|_{T_pM}$ is a linear function $T_pM \to \mathbb{R}$.

We list a few properties of the differentials of smooth functions:

Proposition 2.2.5. *If M is a smooth manifold, $f, g\colon M \to \mathbb{R}$ are smooth function, and $a, b \in \mathbb{R}$ are numbers, then:*

(a) $d(af + bg) = a\,df + b\,dg$;

(b) $d(fg) = f\,dg + g\,df$;

(c) If $I \subseteq \mathbb{R}$ is an open interval containing $\operatorname{im} f$*, and $h\colon J \to \mathbb{R}$ is smooth, then $d(h \circ f) = (h' \circ f)\,df$ where h' is the derivative of h.*

If $F\colon M \to N$ is a smooth map between manifolds, then similarly there is a notion of a differential. In this case, for any point $p \in M$, the map F induces a map between the tangent spaces TM and TN.

Definition 2.2.6. Let M, N be two smooth manifolds and let $F\colon M \to N$ be a smooth map between them. We define the ***differential of*** F as the map $dF\colon TM \to TN$ given by

$$dF\colon X_p \longmapsto \big(f \mapsto X_p(f \circ F)\big).$$

Note that for any $p \in M$, the restriction $dF_p := dF|_{T_pM}$ becomes a linear map $dF_p\colon T_pM \to T_{F(p)}N$, called the ***differential of*** F ***at*** p.

It is important to distinguish Definitions 2.2.4 and 2.2.6. Both are called differentials, but the differential of a function is again a function and the differential of a map is again a map. However, these definitions are equivalent in the sense that there is a canonical identification $\mathbb{R} \simeq T_a\mathbb{R}$ for each $a \in \mathbb{R}$, and with respect to this identification the differential of a function may also be seen as the differential of a map into the manifold $\mathbb{R}$. The precise details of this identification may be found in [14, Ch. 3].

We list a few properties of the differentials of smooth maps:

Proposition 2.2.7. *If M, N and P are smooth manifolds, $F\colon M \to N$ and $G\colon N \to P$ are smooth maps, and $p \in M$, then:*

(a) dF_p is linear;

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p$;

If we consider functions or maps from an open interval, seen as a one-dimensional manifold with coordinate t, the differential gives rise to the definition of a velocity at each $t_0 \in I$.

Definition 2.2.8. Let M be a smooth manifold, $I \subseteq \mathbb{R}$ an open interval with coordinate t, and $f \in C^\infty(I)$ a smooth function. For $t_0 \in I$, we define the ***velocity of*** f ***at*** t_0, denoted by $\dot{f}(t_0)$, as

$$\dot{f}(t_0) := df_{t_0}\left(\left.\frac{\partial}{\partial t}\right|_{t_0}\right) \in \mathbb{R}.$$

Note that $\dot{f} \in C^\infty(I)$ is again a smooth function on I. We also write

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} f := \dot{f}(t_0). \tag{2.4}$$

Given a (parametrized) smooth curve γ into a smooth manifold M, i.e. a smooth map $\gamma\colon I \to M$, the differential also provides us with a notion of the velocity of this curve.

Definition 2.2.9. Let M be a smooth manifold, $I \subseteq \mathbb{R}$ an open interval with coordinate t, and $\gamma\colon I \to M$ a smooth curve. For $t_0 \in I$, we define the ***velocity of*** γ ***at*** t_0, denoted by $\dot{\gamma}(t_0)$, as

$$\dot{\gamma}(t_0) := d\gamma_{\gamma(t_0)}\left(\left.\frac{\partial}{\partial t}\right|_{t_0}\right) \in T_{\gamma(t_0)}M,$$

meaning $\dot{\gamma}(t_0)$ acts on a smooth function $f \in C^\infty(M)$ by

$$\dot{\gamma}(t_0)f = d\gamma_{\gamma(t_0)}\left(\left.\frac{\partial}{\partial t}\right|_{t_0}\right) f = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} (f \circ \gamma).$$

We see that $\dot{\gamma}$ becomes a smooth map $\dot{\gamma}\colon I \to TM$ such that $\pi_T \circ \dot{\gamma} = \gamma$.

2.3 Differential forms

If V is a vector space, the vector space of linear maps $V \to \mathbb{R}$ is called the ***dual of*** V, and denoted by $V^* := L(V; \mathbb{R})$. Elements of the dual space are called ***covectors***. In the theory of manifolds, cotangent vectors play an important role.

Definition 2.3.1. Let M be a smooth manifold, then for each $p \in M$ we define the ***cotangent space at*** p as $T_p^*M = (T_pM)^*$.

Like we did for tangent spaces below Definition 2.2.1, we write

$$T^*M := \bigsqcup_{p \in M} T_p^*M \tag{2.5}$$

for the smooth manifold consisting of all cotangent spaces over M, again of twice the dimension of M. The ***projection*** $\pi_{T^*}\colon T^*M \to M$ now is the one that maps a cotangent vector $\omega \in T^*M$ to the unique $p \in M$ with $\omega \in T_p^*M$.

Definition 2.3.2. Let M be a smooth manifold. A smooth map $\omega\colon M \to T^*M$ such that $\pi_{T^*} \circ \omega = \mathrm{id}_M$ is called a ***covector field on*** M or ***1-form on*** M, and we often write ω_p for $\omega(p)$.

We denote by $\Omega^1(M)$ the set of all 1-forms on M. As with Definition 2.2.3, the condition $\pi_{T^*} \circ \omega = \mathrm{id}_M$ guarantees that ω_p is a covector at p for each $p \in M$. Therefore, ω_p becomes a linear map of T_pM into $\mathbb{R}$, and a 1-form may be seen as a smooth map $\omega\colon TM \to \mathbb{R}$ such that $\omega|_{T_pM}$ is linear for each $p \in M$. The converse is also true: any such smooth map ω may be seen as a 1-form.

Example 2.3.3. For $f \in C^\infty(M)$ a smooth function, we saw in Definition 2.2.4 that $df\colon TM \to \mathbb{R}$ is a smooth map that is linear when restricted to any tangent space, thus for $p \in M$ we have $df_p \in T_p^*M$, and df is indeed a 1-form.

Let (x^i) be local coordinates on a open subset $U \subseteq M$ and consider the local coordinate functions $x^i\colon U \to \mathbb{R}$ and corresponding differentials dx^i. Then we may write

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{2.6}$$

In particular, the basis $(dx^i|_p)$ of T_p^*M is the dual basis of $(\frac{\partial}{\partial x^i}|_p)$, the standard basis of TM.

Integration of 1-forms

Definition 2.3.4. Let M, N be two smooth manifolds, and $F\colon M \to N$ a smooth map between them. If $\omega \in \Omega^1(N)$ is a 1-form on N, we define the 1-form $F^*\omega \in \Omega^1(M)$ called the ***pullback of*** ω ***along*** F pointwise by

$$[F^*\omega]_p\colon X_p \longmapsto \omega([dF]_p X_p). \tag{2.7}$$

If we have some 1-form ω on an open subset I of the real numbers containing $[a, b] \subseteq \mathbb{R}$, we may write $\omega_t = f(t)dt$ for some $f\colon \in C^\infty(I)$. We then define

$$\int_{[a,b]} \omega := \int_a^b f(t)\mathrm{d}t \tag{2.8}$$

where the right-hand side is just the Riemann integral of f.

Definition 2.3.5. Let M be a smooth manifold, $\gamma\colon I \to M$ a smooth curve, and let ω be a covector field over M, then we define the ***line integral of*** ω ***over*** γ on a closed interval $[a, b] \subseteq I$ to be

$$\int_\gamma \omega := \int_{[a,b]} \gamma^*\omega = \int_a^b \big(\omega_{\gamma(t)}\gamma'(t)\big)\mathrm{d}t.$$

The result is independent of the parametrization of γ: if $\gamma'\colon [c, d] \to M$ is another smooth curve such that there exists an increasing diffeomorphism $\phi\colon [a, b] \to [c, d]$ with $\gamma = \gamma' \circ \phi$, for any 1-form ω we have $\int_\gamma \omega = \int_{\gamma'} \omega$. Another important result on line integrals is the following:

Theorem 2.3.6 (Fundamental theorem for line integrals)**.** *If M is a smooth manifold, $f \in C^\infty(M)$ is a smooth function and $\gamma\colon [a, b] \to M$ is a smooth curve in M, then*

$$\int_\gamma df = f(\gamma(b)) - f(\gamma(a)).$$

This follows from the fundamental theorem of calculus. A reader interested in more details may find them in [14, Theorem 11.39].

Tensor products of multilinear maps

A comprehensive discussion of tensor products is outside the scope of this thesis. Instead, we focus on the tensor product of covectors. We write $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$.

Definition 2.3.7. Let $V_1, \dots, V_k$ and W be $\mathbb{F}$-vector spaces, then a map $F\colon V_1 \times \cdots \times V_k \to W$ is called ***multilinear*** if for each $i \in \{1, \dots, k\}$ and each $v_1 \in V_1, \dots, v_k \in V_k$, $v_i' \in V_i$ and $\lambda, \lambda' \in \mathbb{F}$ we have

$$F(v_1, \dots, \lambda v_i + \lambda' v_i', \dots, v_k) = \lambda F(v_1, \dots, v_k) + \lambda' F(v_1, \dots, v_i', \dots, v_k).$$

We denote the vector space of multilinear maps $V_1 \times \cdots \times V_k \to W$ by $L(V_1, \dots, V_k; W)$.

A multilinear map with two arguments $F\colon V_1 \times V_2 \to W$ is often called ***bilinear***.

Definition 2.3.8. Let $V_1, \dots, V_k$ and $W_1, \dots, W_l$ be $\mathbb{F}$-vector spaces, and $F \in L(V_1, \dots, V_k; \mathbb{F})$, $G \in L(W_1, \dots, W_l; \mathbb{F})$ be two multilinear functions. We define their ***tensor product*** as the multilinear function $F \otimes G$, the element of $L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{F})$ given by

$$F \otimes G\colon v_1, \dots, v_k, w_1, \dots, w_l \longmapsto F(v_1, \dots, v_k) G(w_1, \dots, w_l).$$

This product is obviously associative.

We introduce the shorthand notation $T^k(V^*) := L(V, \dots, V; \mathbb{F})$ where the right-hand side has k copies of V. This notation emphasizes that $T^k(V^*)$ may be seen as a generalization of $V^* = L(V; \mathbb{F})$. Elements of $T^k(V^*)$ are called ***covariant*** k***-tensor on*** V.

Definition 2.3.9. Let V, W be $\mathbb{F}$-vector spaces. A multilinear map $F\colon V \times \cdots \times V \to W$ is called ***symmetric*** if its value is invariant under permuting any pair of argument, so if for indices $i < j$ we have

$$F(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = F(v_1, \dots, v_j, \dots, v_i, \dots, v_k),$$

and is called ***alternating*** or ***antisymmetric*** if such a permutation introduces a minus sign, meaning we have

$$F(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -F(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

The vector space $T^k(V^*)$ admits projection maps into a subspace of symmetric tensors $\Sigma^k(V^*)$ and a subspace of alternating tensors $\Lambda^k(V^*)$.

Definition 2.3.10. Let V be a $\mathbb{F}$-vector space. We define the ***symmetrization*** by

$$\begin{aligned} \mathrm{Sym}\colon T^k(V^*) &\longrightarrow \Sigma^k(V^*) \\ \alpha &\longmapsto \left(v_1, \dots, v_k \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} \alpha\left(v_{\sigma(1)}, \dots, v_{\sigma(k)}\right) \right) \end{aligned}$$

and the ***alternation*** by

$$\begin{aligned}\text{Alt}\colon T^k(V^*) &\longrightarrow \Lambda^k(V^*)\\ \alpha &\longmapsto \left(v_1,\dots,v_k \mapsto \frac{1}{k!}\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)\alpha\left(v_{\sigma(1)},\dots,v_{\sigma(k)}\right)\right).\end{aligned}$$

The symmetrization and alternation maps are clearly linear, and map into the correct subspaces. If α is symmetric, for any $\sigma \in S_k$ we have

$$\alpha\left(v_{\sigma(1)},\dots,v_{\sigma(k)}\right) = \alpha\left(v_1,\dots,v_k\right) \tag{2.9}$$

and $\operatorname{Sym}\alpha = \alpha$, thus $\operatorname{Sym}^2 = \operatorname{Sym}$ and $\operatorname{im}\operatorname{Sym} = \Sigma^k(V^*)$. Similarly $\operatorname{Alt}^2 = \operatorname{Alt}$ and $\operatorname{im}\operatorname{Alt} = \Lambda^k(V^*)$.

Alternating covariant tensors will be used to generalize the concept of 1-forms, and therefore are of special significance to us. Foreshadowing this, we call them ***exterior forms***.

Note that the tensor product of two exterior forms is not necessarily alternating. However, there is a natural analogue called the wedge product that does guarantee the result is again an exterior form.

Definition 2.3.11. Let V be a $\mathbb{F}$-vector space, and let $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^l(V^*)$ be exterior forms. We define their ***wedge product*** as the exterior form $\omega \wedge \eta \in \Lambda^{k+l}(V)$ given by

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!}\operatorname{Alt}(\omega\otimes\eta).$$

The wedge product is bilinear, associative and anticommutative.

Notice that $\Lambda^0(V^*)$ is by definition taken to consist of the real numbers. Thus, for $\lambda \in \mathbb{R}$ a scalar and ω a exterior k-form, we have

$$\lambda\wedge\omega = \operatorname{Alt}(\lambda\otimes\omega) = \operatorname{Alt}(\lambda\omega) = \lambda\operatorname{Alt}\omega = \lambda\omega. \tag{2.10}$$

Differential forms on manifolds

For $k \in \mathbb{Z}_{\geq 0}$ and $p \in M$ a point on a smooth manifold, we denote

$$T^{\diamond k}M := \bigsqcup_{p\in M}(T_pM)^k \tag{2.11}$$

where the right-hand side denotes the Cartesian product of k copies of T_pM. $T^{\diamond k}M$ thus consists of k-tuples of tangent vectors at the same point.

Definition 2.3.12. Let M be a smooth manifold. A smooth map $\omega\colon T^{\diamond k}M \to \mathbb{R}$ such that $\omega|_{(T_pM)^k}$ is an exterior form for all $p \in M$ is called a ***differential*** ***k-form on M***, and we often write ω_p for $\omega|_{(T_pM)^k}$.

The set of all k-forms on M is denoted by $\Omega^k(M)$. The wedge product of two differential forms is defined pointwise by $(\omega \wedge \eta)_p := \omega_p \wedge \eta_p$.

Note that 0-forms are just smooth functions, and a 1-form in the definition above is exactly the same as in Definition 2.3.2, as per the remark directly below. As we have seen earlier, the differential of a smooth function $f \in C^\infty(M)$ is the 1-form df. It turns out we can generalize this notion and to each k-form ω assign a $(k+1)$-form $d\omega$ that behaves like a derivative of ω.

Theorem 2.3.13 (Exterior differentiation)**.** *Let M be a smooth manifold, then there are operators $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$, called* ***exterior derivatives****, uniquely defined by the following properties:*

(a) Each d is $\mathbb{R}$-linear.

(b) For each $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(c) $d \circ d = 0$.

(d) For $f \in \Omega^0(M) = C^\infty(M)$, df is given by $X \mapsto Xf$.

From *(d)*, it is clear that the differential operator in Definition 2.2.4 is an exterior derivative.

Below, we will extensively use maps that send tangent vectors of a manifold (multi)linearly and alternatingly into a vector space. This is entirely analogous to Definition 2.3.12, and therefore these maps are called vector-valued forms.

Definition 2.3.14. Let M be a smooth manifold and V a vector space. A smooth map $\omega\colon T^{\diamond k}M \to V$ such that $\omega|_{(T_pM)^k}$ is multilinear and alternating for all $p \in M$ is called a ***V-valued differential k-form on M***, and we often write ω_p for $\omega|_{(T_pM)^k}$.

For M and V as above, we denote the set of all V-valued k-forms on M by $\Omega^k(M,V)$.

If $\omega \in \Omega^k(M,V)$ for V finite-dimensional, and we choose a basis $\{|e_i\rangle\}$ of V, there are unique real k-forms $\{\omega^i\}$ such that $\omega = |e_i\rangle\omega^i$. This decomposition may be used to make sense of notions like the wedge products and exterior derivatives for vector-valued forms.

If V, W and Z are finite-dimensional vector spaces, a bilinear map $\mu\colon V \times W \to Z$ gives rise to a wedge product $\wedge_\mu$ of V-valued and W-valued forms, that results in a Z-valued form.

Definition 2.3.15. Let V, W and Z be finite-dimensional vector spaces such that V has basis $(|v_i\rangle)$, W has basis $(|w_j\rangle)$, and $\mu\colon V \times W \to Z$ is a bilinear map. Let $\alpha = |v_i\rangle\alpha^i \in \Omega^k(M,V)$ and $\beta = |w_j\rangle\beta^j \in \Omega^l(M,W)$ for certain $\alpha^i \in \Omega^k(M)$, $\beta^j \in \Omega^l(M)$ be two vector-valued forms, then their wedge product is defined by

$$\alpha \wedge_\mu \beta := (\alpha^i \wedge \beta^j)\mu(v_i, w_j) \in \Omega^{k+l}(Z).$$

Scalar multiplication gives natural bilinear maps $V \times \mathbb{R} \to V$ and $\mathbb{R} \times V \to V$, and the wedge product with respect to these maps is just denoted by $\wedge$. If $\wedge_\mu$ is a wedge product as above, and we have forms $\alpha = |v_i\rangle\alpha^i \in \Omega^k(M,V)$, $\beta = |w_j\rangle\beta^j \in \Omega^l(M,W)$, and $\omega \in \Omega^m(M)$, we have

$$\alpha \wedge \omega = (\alpha^i \wedge \omega)v_i = \big((-1)^{k+m}\omega \wedge \alpha^i\big)v_i = (-1)^{k+m}\omega \wedge \alpha. \tag{2.12}$$

and

$$(\alpha \wedge \omega) \wedge_\mu \beta = (\alpha^i \wedge \omega \wedge \beta^j)\mu(v_i, w_j) = \alpha \wedge_\mu (\omega \wedge \beta). \tag{2.13}$$

If α, β are complex-valued forms, the bilinear multiplication map $\cdot\colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is implied, and we just write $\alpha \wedge \beta$ for $\alpha \wedge_\cdot \beta$.

Vector-valued k-forms inherit a notion of exterior differentiation from the k-forms: for $\alpha = |v_i\rangle\alpha^i \in \Omega^k(M,V)$ we set $d\alpha := |v_i\rangle(d\alpha^i) \in \Omega^{k+1}(M,V)$.

Proposition 2.3.16. *If V, W and Z are finite-dimensional vector spaces and $\mu\colon V \times W \to Z$ is a bilinear map, for $\alpha \in \Omega^k(M,V)$ and $\beta \in \Omega^l(M,W)$, we have*

$$d(\alpha \wedge_\mu \beta) = d\alpha \wedge_\mu \beta + (-1)^k \alpha \wedge_\mu d\beta.$$

A proof may be found in [20, Prop. 21.3].

Chapter 3

Vector bundles

This chapter contains the central mathematical concepts required in the study of geometric phases in Chapter 4. The literature on vector bundle connections in quite scattered, and while the approach in this chapter is based on [13], some concepts had to be proven in a more general context, and I also utilized [9, 12, 19, 20].

3.1 Definition

Definition 3.1.1. Let $\pi\colon E \to M$ be a smooth map between smooth manifolds. If there exists a smooth manifold F, called the ***model fiber***, and for each $p \in M$ the spaces E_p and F are diffeomorphic, and the base space M admits an open cover $\{U_\alpha\}$ such that there are diffeomorphisms $\phi^\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times F$, each called a ***local trivialization***, such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\;\phi^\alpha\;} & U_\alpha \times F \\ & \searrow^{\pi} \quad \swarrow_{\pi_{U_\alpha}} & \\ & M & \end{array}$$

commutes, we call (E, M, π) a ***smooth fiber bundle***.

Definition 3.1.2. Let $\pi\colon E \to M$ be a smooth fiber bundle, then a map $\sigma\colon M \to E$ is called a ***smooth section of*** E if $\pi \circ \sigma = \mathrm{id}_M$.

The set of smooth sections is denoted by $\Gamma(E)$. Not every fiber bundle admits a global section, thus we may need to consider section only locally.

Definition 3.1.3. Let $\pi\colon E \to M$ be a smooth fiber bundle and $U \subseteq M$ an open subset, then a map $\sigma\colon U \to E$ is called a ***smooth local section of*** E ***on*** U if $\pi \circ \sigma = \mathrm{id}_U$.

The set of smooth local sections on U is denoted by $\Gamma_U(E)$.

Definition 3.1.4. Let $\pi\colon E \to M$ be a smooth fiber bundle with typical fiber a real (complex) vector space V with $k = \dim V$. The bundle is called a ***smooth real (complex) vector bundle of rank*** k if for each $p \in M$, the fiber $E_p = \pi^{-1}(p)$ has the structure of a k-dimensional real (complex) vector space, and for each local trivialization ϕ^α and each $p \in U_\alpha \subseteq M$, the restriction $\phi^\alpha|_{\{p\}}\colon E_p \to \{p\} \times V$ is a linear isomorphism.

In the Dirac bra–ket notation, a section $\sigma \in \Gamma(E)$ of a vector bundle will be written $|\sigma\rangle$, and this section evaluated at a point $p \in M$ as $|\sigma\rangle_p$.

Example 3.1.5. We have already seen two important examples of vector bundles: the tangent bundle (below Definition 2.2.1) and cotangent bundle (below Definition 2.3.1) of a smooth manifold. Smooth sections of these bundles are exactly the vector field (Definition 2.2.3) and 1-forms (Definition 2.3.2).

Note that each vector bundle admits a global section: for example, the zero-section $|0\rangle\colon p \mapsto (p, 0)$ is smooth.

For calculations in vector spaces, it is often useful to chose a basis. The concept of a basis may locally be extended to vector bundles.

Definition 3.1.6. Let $\pi\colon E \to M$ be a smooth vector bundle, and let $U \subseteq M$ be an open set and $|e_j\rangle \in \Gamma_U(E)$ local sections on U. If for every $p \in U$ the collection $(|e_j\rangle_p)$ is a basis for E_p, the collection $(|e_j\rangle)$ is called a ***local basis of sections of*** E.

Since any vector bundle is locally trivial, it is always possible to find a local basis of sections near a point $p \in M$. This is clear from the observation that for local trivialization ϕ^α, a choice of basis on the typical fiber V gives rise to a local basis of sections on U_α. If a vector bundle admits a global basis of sections, it is called ***trivial***.

An interesting fact on local trivializations on a compact manifold is the following:

Lemma 3.1.7. *If M is a compact smooth manifold and $\pi\colon E \to M$ is a smooth vector bundle, M admits a finite cover $\mathcal{U}$ such that each $U \in \mathcal{U}$ is connected and admits a local basis of sections of E.*

Proof. This follows directly from the fact that any open cover of a compact space admits a finite subcover. Let $\{U_\alpha\}$ be a cover of M like in Definition 3.1.1. Write C_α for the set of connected components of U_α and take $\tilde{\mathcal{U}} := \bigcup_\alpha C_\alpha$. It is obvious that this set still covers M. If we take $\mathcal{U}$ to be a finite subcover of $\tilde{\mathcal{U}}$, clearly each element admits a local trivialization and is connected. □

Given a smooth map $F\colon M \to N$ of a manifold into the base space of a vector bundle, we may pull back the vector bundle structure along F. This results in a vector bundle with base space M where the fiber above a point $p \in M$ is taken to be the fiber above $F(p)$.

Definition 3.1.8. Let M, N be smooth manifolds and let $\pi\colon E \to N$ be a vector bundle. For a smooth map $F\colon M \to N$, the bundle

$$F^*\pi\colon F^*E \to M$$

with $F^*E := \{(p, v) \in M \times E : \pi(v) = F(p)\}$ and $F^*\pi$ the obvious projection on the first coordinate is called the ***pullback bundle of π along F***.

Definition 3.1.9. Let M, N be smooth manifolds, let $\pi\colon E \to N$ be a vector bundle and $\sigma \in \Gamma(E)$ a smooth section. For a smooth map $F\colon M \to N$, the section

$$\begin{aligned} F^*\sigma\colon N &\longrightarrow F^*E \\ p &\longmapsto \big(p, \sigma(F(p))\big) \end{aligned}$$

is called the ***pullback section of σ along F***.

An important additional structure on a vector space is a choice of inner product. We want to generalize this to vector bundles, making a smooth choice of inner products on the fibers.

Definition 3.1.10. Let $\pi\colon E \to M$ be a real (complex) vector bundle. A ***Riemannian (Hermitian) metric on*** E is an assignment of inner products $\langle \_ | \_ \rangle_p\colon E_p \times E_p \to \mathbb{F}$ to each vector space E_p such that for smooth sections $|\sigma_1\rangle, |\sigma_2\rangle$ the inner product $\langle \sigma_1 | \sigma_2 \rangle\colon M \to \mathbb{F}$ defined pointwise by $\langle \sigma_1 | \sigma_2 \rangle(p) := \langle \sigma_1(p) | \sigma_2(p) \rangle_p$ is smooth.

Here we follow the convention from quantum mechanics that the inner product on a complex vector space is linear in the *second* coordinate.

We will simply refer to a real vector bundle endowed with a Riemannian metric as a ***Riemannian vector bundle***, and to a complex vector bundle endowed with a Hermitian metric as a ***Hermitian vector bundle***.

3.2 Connections

Definition 3.2.1. Let $\pi\colon E \to M$ be a vector bundle. A map

$$\nabla\colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E),$$

denoted by $\nabla_X|\sigma\rangle := \nabla(X, |\sigma\rangle)$ is called a ***connection in*** E if it satisfies the following properties:

(a) ∇ is linear over $C^\infty(M)$ in the first coordinate, i.e. for all $X_1, X_2 \in \Gamma(TM)$, $|\sigma\rangle \in \Gamma(E)$, $f, g \in C^\infty(M)$ we have

$$\nabla_{fX_1+gX_2}|\sigma\rangle = f\nabla_{X_1}|\sigma\rangle + g\nabla_{X_2}|\sigma\rangle;$$

(b) ∇ is linear over $\mathbb{R}$ in the second coordinate, i.e. for all $X \in \Gamma(TM)$, $|\sigma_1\rangle, |\sigma_2\rangle \in \Gamma(E)$, $a, b \in \mathbb{R}$ we have

$$\nabla_X(a|\sigma_1\rangle + b|\sigma_2\rangle) = a\nabla_X|\sigma_1\rangle + b\nabla_X|\sigma_2\rangle;$$

(c) ∇ satisfies the following product rule: for all $X \in \Gamma(TM)$, $|\sigma\rangle \in \Gamma(E)$, $f \in C^\infty(M)$ we have

$$\nabla_X(f|\sigma\rangle) = f\nabla_X|\sigma\rangle + (Xf)|\sigma\rangle.$$

A connection on the tangent bundle $TM \to M$ is an important object in the field of differential geometry, and is called a ***linear connection on*** M.

For fixed X and $|\sigma\rangle$, the section $\nabla_X|\sigma\rangle$ is referred to as the ***covariant derivative of*** $|\sigma\rangle$ ***in the direction of*** X and is sometimes denoted by $|\nabla_X\sigma\rangle$.

Keeping the section $|\sigma\rangle$ fixed, we may view a connection ∇ as a linear map from $\Gamma(TM)$ to $\Gamma(E)$, denoted by $\nabla|\sigma\rangle\colon X \mapsto \nabla_X|\sigma\rangle$. Since $\Gamma(E)$ is a vector space, and by the $C^\infty(M)$-linearity of the connection, $\nabla|\sigma\rangle$ is just a $\Gamma(E)$-valued 1-form.

Lemma 3.2.2. *If ∇ is a connection in a vector bundle $\pi\colon E \to M$, for any $X \in \Gamma(TM)$, $|\sigma\rangle \in \Gamma(E)$ and $p \in M$, the value of $|\nabla_X\sigma\rangle_p$ depends* only *on the value of X at p and the value of $|\sigma\rangle$ near p, in the sense that if there are $X' \in \Gamma(TM)$ and $|\sigma'\rangle \in \Gamma(E)$, and $U \subseteq M$ is an open neighborhood of p satisfying $X_p = X'_p$ and $|\sigma\rangle|_U = |\sigma'\rangle|_U$, then $|\nabla_X\sigma\rangle_p = |\nabla_{X'}\sigma'\rangle_p$.*

For a proof, the reader might consult [13, Lemma 4.1 and 4.2]. As a result of this lemma, we may unambiguously write $\nabla_{X_p}|\sigma\rangle$ for $|\nabla_X\sigma\rangle_p$, and it makes sense to write $|\nabla_X\sigma\rangle$ even if $|\sigma\rangle$ and X are only local sections defined the same domain in M.

Lemma 3.2.3. *If $\pi\colon E \to M$ is a vector bundle with $\dim M = m$, $\dim E = m+n$, and $U \subseteq M$ is an open neighborhood that admits local coordinates (x^i) and a local basis of sections $(|e_j\rangle)$ of E, there exist functions $\Gamma^k_{ij} \in C^\infty(U)$ for $1 \le i \le m$ and $1 \le j,k \le n$ called the* ***Christoffel symbols of*** *∇ with respect to the local coordinates and the basis of sections, such that*

$$|\nabla_{\partial_i}e_j\rangle = \Gamma^k_{ij}|e_k\rangle.$$

For any local vector field $X \in \Gamma_U(TM)$ and section $|\sigma\rangle \in \Gamma_U(E)$ expressed in local coordinates by $X = X^i\partial_i$ and $|\sigma\rangle = \sigma^j|e_j\rangle$, we have

$$|\nabla_X\sigma\rangle = (X\sigma^k + X^i\sigma^j\Gamma^k_{ij})|e_k\rangle.$$

Proof. By Lemma 3.2.2 ∇ acts unambiguously on local sections. Since at any point $p \in U$, $(|e_k\rangle_p)$ is a basis of E_p, we may select $\Gamma^k_{ij}(p)$ to be the unique scalars that satisfy

$$|\nabla_{\partial_i}e_j\rangle_p = \Gamma^k_{ij}(p)|e_k\rangle_p.$$

For the second part, by the properties of ∇ we have

$$|\nabla_X\sigma\rangle = \sigma^j|\nabla_X e_j\rangle + (X\sigma^j)|e_j\rangle = X^i\sigma^j|\nabla_{\partial_i}e_j\rangle + (X\sigma^j)|e_j\rangle.$$

Substituting the definition of Γ^k_{ij} and changing the summation index j in $(X\sigma^j)|e_j\rangle$ to k gives the proposed equation. □

Lemma 3.2.4. *If M, N are smooth manifolds, ∇ is a connection in a vector bundle $\pi\colon E \to N$, and $F\colon M \to N$ is a smooth map, there is a unique connection $F^*\nabla$ on the pullback bundle $F^*\pi\colon F^*E \to M$ satisfying*

$$[F^*\nabla]_X(F^*\sigma) = F^*(\nabla_{(dF)X}\sigma), \tag{3.1}$$

for all $X \in \Gamma(TM)$ and $\sigma \in \Gamma(E)$. This connection is called the ***pullback connection of*** ∇ ***along*** *F.*

Proof. Suppose $F^*\nabla$ is such a connection. For each $p \in M$, let $U \subseteq M$ be an open neighborhood of p such that U admits local coordinates (x^i) and $F[U]$ is contained in a coordinate patch (y^l) that also admits a local basis of sections $(|e_j\rangle)$ of E. Clearly, $(F^*|e_j\rangle)$ forms a local basis of sections of F^*E on U, and we calculate the Christoffel symbols of $F^*\nabla$ with respect to this at p. Let $Y_i^l(p) \in \mathbb{R}$ such that $dF_p\partial_i = Y_i^l\partial_l \in T_pN$, then

$$\begin{aligned}\big[[F^*\nabla]_{\partial_i}(F^*|e_j\rangle)\big](p) &= \big[F^*(\nabla_{dF\partial_i}|e_j\rangle)\big](p)\\ &= \big(p, \big[\nabla_{dF\partial_i}|e_j\rangle\big](F(p))\big)\\ &= \big(p, Y_i^l(p)\Gamma_{lj}^k(F(p))|e_k\rangle\big)\\ &= Y_i^l(p)\Gamma_{lj}^k(F(p))(F^*|e_k\rangle).\end{aligned}$$

Thus $F^*\nabla$ has Christoffel symbols $[F^*\Gamma]_{ij}^k(p) = Y_i^l(p)\Gamma_{lj}^k(F(p))$, and the second part of Lemma 3.2.3 gives the existence and uniqueness of $F^*\nabla$. □

If a vector bundle has extra structure, like a metric as in Definition 3.1.10, we are interested in connections that respect this structure.

Definition 3.2.5. Let $\pi\colon E \to M$ be a Riemannian (Hermitian) vector bundle. A connection ∇ is called a ***Riemannian (Hermitian) connection*** if for any $|\sigma_1\rangle, |\sigma_2\rangle \in \Gamma(E)$ and $X \in \Gamma(TM)$ we have

$$X\langle\sigma_1|\sigma_2\rangle = \langle\nabla_X\sigma_1|\sigma_2\rangle + \langle\sigma_1|\nabla_X\sigma_2\rangle.$$

3.3 Horizontal lifts

Now that we have developed the methods of covariant derivatives, we apply them to lift a curve $\gamma\colon I \to M$ *horizontally* into $\tilde{\gamma}\colon I \to E$: requiring that $\tilde{\gamma}$ projects down to γ, and is locally 'flat' with respect to the connection on E.

Definition 3.3.1. Let $\pi\colon E \to M$ be a vector bundle, and let $\gamma\colon I \to M$ be a smooth curve. A smooth map $\zeta\colon I \to E$ is called a ***section of*** E ***along*** γ or a ***lift of*** γ ***into*** E if we have $\pi \circ \zeta = \gamma$.

We denote the space of section of E along γ by $\Gamma_\gamma(E)$.

Example 3.3.2. If M is a manifold and $\gamma\colon I \to M$ is a smooth curve, the velocity $\dot{\gamma}$ we saw in Definition 2.2.9 defines a section of TM along γ.

Note that if γ is self-intersecting, i.e. there are distinct $t, t' \in I$ with $\gamma(t) = \gamma(t')$, we allow $\zeta(t) \neq \zeta(t')$, and in such cases it is not possible to extend ζ to a (local) section of E.

Definition 3.3.3. Let $\pi\colon E \to M$ be a vector bundle, and let $\gamma\colon I \to M$ be a smooth curve. A section $\zeta \in \Gamma_\gamma(E)$ along γ is called ***extendible*** if there exists an open neighborhood U of $\operatorname{im}\gamma$ and a local section $|\sigma\rangle \in \Gamma_U(E)$, such that $\zeta = \sigma \circ \gamma$, and $|\sigma\rangle$ is called an ***extension*** of ζ.

Lemma 3.3.4. *If $\pi\colon E \to M$ is a vector bundle with connection ∇, then for each curve $\gamma\colon I \to M$ the connection defines a unique operator $D_t\colon \Gamma_\gamma(E) \to \Gamma_\gamma(E)$ called the* ***covariant time derivative*** *satisfying the following properties:*

(a) D_t is linear over $\mathbb{R}$, i.e. for all $\zeta_1, \zeta_2 \in \Gamma_\gamma(E)$, $a, b \in \mathbb{R}$ we have

$$D_t(a\zeta_1 + b\zeta_2) = aD_t\zeta_1 + bD_t\zeta_2;$$

(b) D_t satisfies the following product rule: for all $\zeta \in \Gamma_\gamma(E)$, $f \in C^\infty(I)$ we have

$$D_t(f\zeta) = \dot{f}\zeta + fD_t\zeta;$$

(c) D_t is compatible with ∇ in the sense that for all $\zeta \in \Gamma_\gamma(E)$, if ζ has an extension $|\sigma\rangle$, at every $t_0 \in I$ we have

$$[D_t\zeta](t_0) = \nabla_{\dot{\gamma}(t_0)}|\sigma\rangle.$$

The proof is adapted from that of [13, Lem. 4.9].

Proof. Suppose D_t is such an operator. For any $t_0 \in I$, consider an open neighborhood $J \subseteq I$ of t_0 such that an open subset $U \subseteq M$ containing $\gamma[J]$ exists that admits a local basis of sections $(|e_j\rangle)$.

First we show that if $\zeta, \zeta' \in \Gamma_\gamma(E)$ suffice $\zeta|_J = \zeta'|_J$ we have $[D_t\zeta](t_0) = [D_t\zeta'](t_0)$. By linearity, it suffices to proof that $[D_t(\zeta - \zeta')](t_0) = 0$. By

Lemma 2.1.6, there exists a bump function with support in J such that $f(t_0) = 1$. Then

$$0 \equiv D_t 0 = D_t f(\zeta - \zeta') = \dot{f}(\zeta - \zeta') + f D_t(\zeta - \zeta') = f D_t(\zeta - \zeta')$$

which proofs our claim. $[D_t\zeta](t_0)$ thus only depends on the behavior of ζ near t_0.

Now for $t \in J$ we can write $\zeta(t) = |e_j\rangle_{\gamma(t)}\zeta^j(t)$ and $\dot{\zeta}(t) = |e_j\rangle_{\gamma(t)}\dot{\zeta}^j(t)$, and we have

$$\begin{aligned}[D_t\zeta](t_0) &= \dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)[D_t|e_j\rangle_{\gamma(t)}](t_0) \\ &= \dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)\nabla_{\dot{\gamma}(t_0)}|e_j\rangle.\end{aligned}$$

This shows the value of $D_t\zeta$ at t_0 is uniquely determined by the choice of ζ and the connection ∇.

We can use the expression above as a definition to construct the value of $D_t\zeta$ at any point in I. We need to proof this is indeed a covariant time derivative by checking the properties.

(a) If $\zeta_1, \zeta_2 \in \Gamma_\gamma(E)$ and $a, b \in \mathbb{R}$, for any $t_0 \in I$ construct a local frame as above, then $a\zeta_1(t) + b\zeta_2(t) = \left(a\zeta_1^i(t) + b\zeta_2^i(t)\right)|e_j\rangle_{\gamma(t)}$, and the required linearity follows from the construction and the linearity of the velocity.

(b) If $\zeta \in \Gamma_\gamma(E)$, $f \in C^\infty(I)$ and again we construct a local frame, we have $[f\zeta](t) = |e_j\rangle_{\gamma(t)} f(t)\zeta^j(t)$. Now we have

$$\begin{aligned}[D_t f\zeta](t_0) &= [\dot{f}\zeta^j + f\dot{\zeta}^j](t_0)|e_j\rangle_{\gamma(t_0)} + [f\zeta^j](t_0)\nabla_{\dot{\gamma}(t_0)}|e_j\rangle \\ &= [\dot{f}\zeta^j](t_0)|e_j\rangle_{\gamma(t_0)} + f(t_0)\left(\dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)\nabla_{\dot{\gamma}(t_0)}|e_j\rangle\right) \\ &= [\dot{f}\zeta + fD_t\zeta](t_0).\end{aligned}$$

(c) If $\zeta \in \Gamma_\gamma(E)$ has an extension $\sigma \in \Gamma_V(E)$ and we construct a local frame on an open $U \subseteq V$, and $\sigma^j \in C^\infty(U)$ are such that $\sigma = |e_j\rangle\sigma^j$, we have

$$\begin{aligned}\nabla_{\dot{\gamma}(t_0)}|\sigma\rangle &= [\dot{\gamma}(t_0)\sigma^j]|e_j\rangle_{\gamma(t_0)} + \sigma^j(\gamma(t_0))\nabla_{\dot{\gamma}(t_0)}|e_j\rangle \\ &= \dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)\nabla_{\dot{\gamma}(t_0)}|e_j\rangle \\ &= [D_t\zeta](t_0)\end{aligned}$$

since $\zeta^j = \sigma^j \circ \gamma$, and $\dot{\gamma}(t_0)\sigma^j = \dot{\zeta}^j(t_0)$ per Definition 2.2.9.

□

Definition 3.3.5. Let $\pi\colon E \to M$ be a vector bundle with connection ∇, and let $\gamma\colon I \to M$ be a smooth curve. A section $\zeta \in \Gamma_\gamma(E)$ along γ is called ***horizontal*** with respect to ∇ if we have $D_t\gamma \equiv 0$.

The central theorem for parallel transport in vector bundles is the following:

Theorem 3.3.6. *If $\pi\colon E \to M$ is a vector bundle with connection ∇ and $\gamma\colon I \to M$ is a smooth curve, for $t_0 \in I$ and $|v\rangle_p \in E_p$ with $p = \gamma(t_0) \in M$ there exists a unique horizontal section of E along γ with value $|v\rangle_p$ at t_0, denoted by $\tilde{\gamma}_{|v\rangle_p} \in \Gamma_\gamma(E)$ and called **the horizontal lift of γ with value $|v\rangle_p$ at** t_0.*

The proof is adapted from that of [13, Thm. 4.11].

Proof. Suppose $\tilde{\gamma}_{|v\rangle_p}$ is such a lift and take $m := \dim M$, $n := \dim E - m$. Let $J \subseteq I$ be an open interval containing t_0 such that an open neighborhood $U \subseteq M$ of $\gamma[J]$ exists that admits local coordinates (x^i) and a local basis of sections $(|e_j\rangle)$ of E. Ease notation by setting $\zeta := \tilde{\gamma}_{|v\rangle_p}|_J$ and by an abuse of notation $\gamma := \gamma|_J$, then from Lemma 3.2.3 we have

$$\nabla_{\dot{\gamma}}|e_l\rangle = (\dot{\gamma}\delta_l^k + \dot{\gamma}^i\delta_l^j\Gamma_{ij}^k)|e_k\rangle = \dot{\gamma}^i\Gamma_{il}^k|e_k\rangle$$

since we can write $|e_l\rangle = \delta_l^j|e_j\rangle$ for $\delta_l^j \in C^\infty(U)$ the constant function $\delta_l^j \equiv 0$ for $l \neq j$ and $\delta_l^j \equiv 1$ for $l = j$. Because $\dot{\gamma}$ is a derivation, the term $\dot{\gamma}\delta_l^j$ at each time is a derivation acting on a constant function and thus vanishes. From the proof of Lemma 3.3.4 it follows that we can write

$$0 = \dot{\zeta}^j|e_j\rangle + \zeta^j\nabla_{\dot{\gamma}}|e_j\rangle = \dot{\zeta}^j|e_j\rangle + \dot{\gamma}^i\Gamma_{ij}^k|e_k\rangle = \left(\dot{\zeta}^k + \zeta^j\dot{\gamma}^i\Gamma_{ij}^k\right)|e_k\rangle,$$

and this results in the system of n ordinary differential equations (ODE's) on I given by

$$\dot{\zeta}^k(t) = -\dot{\gamma}^i(t)\Gamma_{ij}^k(\gamma(t)) \cdot \zeta^j(t).$$

The existence and uniqueness of a solution to such a system of linear first-order ODE's is well known. The interested reader might consult [7, Theorem 3.4].

Now that we know how to construct local solutions, we 'glue' these together into a global solution. Since by an argument similar to that of Lemma 3.1.7 we may cover M in a finite number of coordinate patches, we may take

$(J_z)_{z\in Z}$ to be a collection of connected open subsets of I satisfying the conditions above, indexed by $Z = \{z \in \mathbb{Z} : z^- \leq z \leq z^+\}$ for some boundaries $z^-, z^+ \in \mathbb{Z}$, such that $I = \bigcup_{z\in Z} J_z$ and for $z^- \leq z < z^+$ there exists an element $t_z \in J_z \cap J_{z+1}$. A global solution is constructed inductively: if a local solution $\zeta_z \colon I_z \to E$ is found, the values $\zeta_z(t_z)$ and $\zeta_z(t_{z+1})$ may be used as initial values to find a solution in I_{z-1} and I_{z+1} as described above, and the uniqueness guarantees the solutions agree on the overlap. $\square$

3.4 Examples

3.4.1 Explicitly trivial vector bundle

Definition

For M a manifold and V a real vector space, the explicitly trivial vector bundle over M with typical fiber V has total space $E = M \times V$ and projection $\pi\colon E \to M$ given by projection on the first coordinate. Note that this bundle is trivial in the sense that it has a global basis of sections: for any basis $\{|\tilde{e}_j\rangle\}$ of V the sections $|e_j\rangle\colon p \mapsto (p, |\tilde{e}_j\rangle)$ clearly suffice.

Connection

Firstly we construct a connection on E. Note that there is a canonical bijection between $\Gamma(E)$, the smooth sections of E, and $C^\infty(M, V)$, the smooth functions of M into V. This identification is given by

$$|\sigma\rangle \mapsto \big(f_\sigma\colon p \mapsto \pi_V(\sigma(p))\big), \tag{3.2}$$

where $\pi_V\colon E \to V$ is the projection on the second coordinate. That this is indeed a bijection is seen by constructing the inverse identification

$$f \mapsto \big(|\sigma_f\rangle\colon p \mapsto (p, f(p))\big). \tag{3.3}$$

Now the exterior derivative of the vector valued function f_σ gives rise to the *trivial connection* on E. For each vector field $X \in \Gamma(TM)$ and smooth section $|\sigma\rangle \in \Gamma(E)$, this connection is given by

$$|\nabla_X \sigma\rangle\colon p \mapsto \big(p, [df_\sigma]_p X_p\big). \tag{3.4}$$

Since $[df_\sigma]_p\colon T_pM \to V$ this is indeed a section of E.

This indeed satisfies the conditions in Definition 3.2.1: the linearity conditions are obvious, and the product rule follows from Proposition 2.2.5.

Covariant time derivative

We now construct the covariant time derivative associated with this connection. For a smooth curve $\gamma\colon I \to M$ with lift $\zeta \in \Gamma_\gamma(E)$, the proof of

Lemma 3.3.4 gave a way to calculate $[D_t\zeta](t_0)$ for any $t_0 \in I$:

$$\begin{aligned}[D_t\zeta](t_0) &= \dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)\nabla_{\dot{\gamma}(t_0)}|e_j\rangle \\ &= \Big[\dot{f_\zeta}^j(t_0)|e_j\rangle + \zeta^j(t_0)|e_i\rangle(\dot{\gamma}(t_0)\delta^i_j)\Big]_{\gamma(t_0)} \\ &= \Big[\dot{f_\zeta}^j(t_0)|e_j\rangle\Big]_{\gamma(t_0)} \\ &= \big(\gamma(t_0), \dot{f_\zeta}(t_0)\big). \end{aligned} \tag{3.5}$$

The horizontal lifts must satisfy $D_t\zeta \equiv 0$, thus $\dot{f_\zeta} = 0$ along the curve and are therefore given by $t \mapsto (\gamma(t), |v\rangle)$ for some constant $|v\rangle \in V$. This justifies calling the connection trivial: it just 'connects' elements in different fibers if they represent the same vector in V.

Riemannian or Hermitian connection

Now consider the special case that V is a real or complex inner product space. Let $|\sigma_1\rangle, |\sigma_2\rangle \in \Gamma(E)$ be sections of E, and define $f^\flat_{\sigma_1} \colon M \to V^*$ by $f^\flat_{\sigma_1}(p) \colon |v\rangle \mapsto \langle f_\sigma(p)|v\rangle$. Now $f^\flat_{\sigma_1}$ and f_{σ_2} are respectively V^*-valued and V-valued 0-forms, and since there is a natural bilinear inner product map $\mathrm{ip} \colon V^* \times V \to \mathbb{F}$ given by $\mathrm{ip}(\langle v|, |w\rangle) = \langle v|w\rangle$, we may write

$$\langle\sigma_1|\sigma_2\rangle = f^\flat_{\sigma_1} \wedge_{\mathrm{ip}} f_{\sigma_2}. \tag{3.6}$$

For any vector field $X \in \Gamma(TM)$, this leads to the equality

$$X\langle\sigma_1|\sigma_2\rangle = d(f^\flat_{\sigma_1} \wedge_{\mathrm{ip}} f_{\sigma_2})X = \big([df^\flat_{\sigma_1}] \wedge_{\mathrm{ip}} f_{\sigma_2} + f^\flat_{\sigma_1} \wedge_{\mathrm{ip}} [df_{\sigma_2}]\big)X. \tag{3.7}$$

On the other hand, we have

$$\langle\sigma_1|\nabla_X\sigma_2\rangle_p = \mathrm{ip}\big(f^\flat_{\sigma_1}(p), [df_{\sigma_2}]_pX\big) = \big(f^\flat_{\sigma_1} \wedge_{\mathrm{ip}} [df_{\sigma_2}]\big)_pX \tag{3.8}$$

and

$$\langle\nabla_X\sigma_1|\sigma_2\rangle_p = \overline{\mathrm{ip}\big(f^\flat_{\sigma_2}(p), [df_{\sigma_1}]_pX\big)} = \big([df^\flat_{\sigma_1}] \wedge_{\mathrm{ip}} f_{\sigma_2}\big)_pX, \tag{3.9}$$

since we have $X_p(f^\flat_{\sigma_1}) = (X_pf_{\sigma_1})^\flat$.

This shows that the trivial connection of the trivial bundle $E = M \times V$ with V a real or complex inner product space is Riemannian in the former case and Hermitian in the latter.

3.4.2 Tautological line bundle

Definition

Let V be a real (complex) inner product space and

$$\mathbb{P}V = \{W \subseteq V \mid W \text{ is a 1-dimensional subspace of } V\}$$

be the projective vector space. The ***tautological line bundle*** $\mathcal{O}_V$ may be thought of as the vector bundle over $\mathbb{P}V$ where the fiber of an element $W \in \mathbb{P}V$ is $W \subseteq V$: the bundle is tautological. Specifically, we have

$$\mathcal{O}_V = \{(W, |w\rangle) \in \mathbb{P}V \times V \mid |w\rangle \in W\}$$

together with the obvious projection $\pi_{\mathcal{O}} \colon \mathcal{O}_V \to \mathbb{P}V$ defined by $(W, |w\rangle) \mapsto W$.

Connection

The tautological line bundle has a canonical connection it inherits from the explicitly trivial vector bundle $\mathbb{P}V \times V$. In order to construct it, we define the projection operator $p_{\mathcal{O}} \colon \mathbb{P}V \times V \to \mathbb{P}V \times V$ by $p_{\mathcal{O}}(W, |v\rangle) = (W, p_W(|v\rangle))$ where p_W is just projection onto the subspace W. We clearly have $\operatorname{im} p_W = \mathcal{O}_V \subseteq \mathbb{P}V \times V$

Now, if $|\sigma\rangle$ is a smooth section of $\mathbb{P}V \times V$, then $p_{\mathcal{O}} \circ |\sigma\rangle$ is a smooth section of $\mathcal{O}_V$, and since each section of $\mathcal{O}_V$ can be seen as a section of $\mathbb{P}V \times V$, this leads to the definition for a connection $\nabla^{\mathcal{O}}$ on the tautological line bundle:

$$\nabla^{\mathcal{O}}_X |\sigma\rangle := p_{\mathcal{O}} \circ |\nabla_X \sigma\rangle. \tag{3.10}$$

This satisfies the conditions in Definition 3.2.1: since ∇ is a connection the linearity conditions follow directly from the linearity of $P_{\mathcal{O}}$ on each fiber, and the product rule follows from linearity and the fact that $P_{\mathcal{O}} \circ |\sigma\rangle = |\sigma\rangle$ for $|\sigma\rangle \in \Gamma(\mathcal{O}_V)$.

Covariant time derivative

We again consider a smooth curve $\gamma \colon I \to \mathbb{P}V$ with lift $\zeta \in \Gamma_\gamma(\mathcal{O}_V)$. Let $t_0 \in I$ and let $U \subseteq \mathbb{P}V$ be an open neighborhood of $\gamma(t_0)$ that admits a local basis of sections $(|e_j\rangle)$, and extend this basis to a basis of sections of $\mathbb{P}V \times V$, by an abuse of notation also denoted by $(|e_j\rangle)$. We use the proof of Lemma 3.3.4 to calculate

$$\begin{aligned}
[D_t^{\mathcal{O}}\zeta](t_0) &= \dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)\nabla^{\mathcal{O}}_{\dot{\gamma}(t_0)}|e_j\rangle \\
&= \dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)p_{\mathcal{O}}\big(\nabla_{\dot{\gamma}(t_0)}|e_j\rangle\big) \\
&= p_{\mathcal{O}}\big(\dot{\zeta}^j(t_0)|e_j\rangle_{\gamma(t_0)} + \zeta^j(t_0)\nabla_{\dot{\gamma}(t_0)}|e_j\rangle\big) \\
&= p_{\mathcal{O}}\big([D_t\zeta](t_0)\big) \\
&= \Big(\gamma(t_0), p_{\gamma(t_0)}\big(\dot{f}_\zeta(t_0)\big)\Big). \qquad (3.11)
\end{aligned}$$

Chapter 4

Geometric phases

After quantum mechanics was first developed, it lasted until 1984 before it was realized that the phase transition of a system described by a time-dependent Hamiltonian is a meaningful quantity. Berry [3] found this phase difference to be a measurable effect. His paper sparked interest in the research of *geometric phases*, and in the subsequent years many interpretations and generalizations were published of which we will discuss the most important.
This chapter constitutes the main result of the thesis: the mathematical machinery developed above is put to use to formalize the concepts in [3], [18] and [1]. Zwanzinger et al. [23] provide an overview of Berry's phase and some generalizations, and I would recommend the interested reader to consult it. Apart from the aforementioned original papers for Section 4.2, [10, 17] were also consulted, and for Section 4.3 [4, 6, 15].

4.1 The adiabatic theorem

An important result in quantum mechanics is the *adiabatic theorem*, proposed in the early days of quantum mechanics by Ehrenfest [8] in 1917 and first proven by Born and Fock [5] in 1928 and in a more general context by Kato [11] in 1950.

Consider a system in quantum mechanics that is described during time $[0, T]$ by a time-dependent Hamiltonian operator $\hat{H}(t)$ on a complex Hilbert space $\mathcal{H}$, for a smooth assignment $t \mapsto \hat{H}(t)$. Suppose that at any time

$t \in [0, T]$, $\hat{H}(t)$ has an eigenvalue $E_n(t)$ such that $t \mapsto E_n(t)$ is smooth. Further assume an $\varepsilon > 0$ exists such that the spectrum of $\hat{H}(t)$ is discrete within a distance ε from $E_n(t)$ at any time $t \in [0, T]$. For $\tau \in \mathbb{R}_{>0}$, let $|\psi_\tau; \_\rangle$ be a solution to the slowed-down Schrödinger equation

$$i\hbar \cdot \frac{t}{\tau} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t |\psi_\tau; t\rangle = \big(\hat{H}(t) - E_n(t)\big)|\psi_\tau; t\rangle \tag{4.1}$$

with initial condition $\hat{H}(0)|\psi_\tau; 0\rangle = E_n(0)|\psi_\tau; 0\rangle$.

Theorem 4.1.1 (Adiabatic theorem)**.** *If we have a system as described above, the limit*

$$|\psi; t\rangle := \lim_{\tau\to\infty} |\psi_\tau; t\rangle \tag{4.2}$$

exists and for any time $t \in [0, T]$ we have $\hat{H}(t)|\psi; t\rangle = E_n(t)|\psi; t\rangle$.

Working with this theorem is awkward, and physicists tend to use it to justify the *adiabatic approximation*: if the time-dependence of the Hamiltonian is slow enough, we may assume up to a very small error that a solution of Schrödinger's equation is an instantaneous eigenvector of the Hamiltonian operator at some time, stays an eigenvector of the operator.

4.2 Berry–Simon phase

The basic setup described by Berry [3] is a finite-dimensional Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geq 1$ together with a Hermitian operator $\hat{H}(R)\colon \mathcal{H} \to \mathcal{H}$ called the Hamiltonian that smoothly depends on a parameter $R \in M$ with M a compact smooth manifold called the *parameter space*. Here it is assumed that the Hamiltonian for any parameter has discrete and distinct eigenvalues. We will vary the parameters as represented by a smooth curve $\rho\colon I \to M$ with I an open interval containing $[0, T]$, and for each time $t \in I$ consider the Hamiltonian $\hat{H}(\rho(t))$. In this setup, a wave function

$$\begin{aligned} |\psi; \_\rangle\colon I &\longrightarrow \mathcal{H} \\ t &\mapsto |\psi; t\rangle \end{aligned} \tag{4.3}$$

is said to solve the Schrödinger equation if at any $t_0 \in I$ we have

$$i\hbar \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} |\psi; \_\rangle = \hat{H}(\rho(t_0))|\psi; t_0\rangle. \tag{4.4}$$

4.2.1 Berry's phase

Formalism of vector bundles

A natural way to describe this system is as a trivial vector bundle over M with typical vector the Hilbert space $\mathcal{H}$, and considering $\hat{H}(R)$ to be a Hermitian operator on the Hilbert space above R for each $R \in M$. This vector bundle is defined by $E = M \times \mathcal{H}$ together with the obvious projection $\pi\colon E \to M$. We endow this bundle with the trivial connection ∇ defined in Section 3.4.1.

For any parameter R, there exists an open neighborhood of $U \subseteq M$ of R on which smooth sections $|n_U\rangle \in \Gamma_U(E)$ may be defined that have unit norm everywhere and suffice

$$\hat{H}|n_U\rangle = E_n|n_U\rangle, \tag{4.5}$$

for a function $E_n \in C^\infty(U)$ called an ***eigenfunction***. These sections are called ***local eigensections***, since at any parameter $R \in U$ this is just the eigenvalue equation well-known in quantum mechanics. Since the eigenvalues of $\hat{H}$ are discrete and non-degenerate, it is never ambiguous which eigenvalue is the n^{th} and E_n is globally well-defined.

Instead of studying solutions of Equation (4.4) directly, we consider the induced section $\psi \in \Gamma_\rho(E)$ of E along ρ, defined by

$$\psi\colon t \longmapsto \big(\rho(t), |\psi; t\rangle\big). \tag{4.6}$$

In this context, Schrödinger's equation becomes

$$i\hbar\, D_t\psi = \hat{H}\psi. \tag{4.7}$$

We assume the curve ρ in parameter space is traversed *adiabatically*, i.e. slowly enough that the adiabatic approximation is valid. Thus a wave function that is at some time $t_0 \in [0, T]$ an eigenvalue of $\hat{H}(t_0)$, say the n^{th}, *stays* an eigenvalue. We claim that without loss of generality, we may assume $E_n \equiv 0$. This may look like a restriction, but it is not. For a general Hamiltonian $\hat{H}(R)$, we may instead consider the Hamiltonian $\hat{H}'(R) = \hat{H}(R) - E_n(R)$ for which the n^{th} eigenvalue is indeed identically zero, moreover the solutions of $\hat{H}'(R)$ and $\hat{H}(R)$ will only differ by an easily computable dynamic phase: if $|\psi, \_\rangle$ solves Equation (4.4), then

$$|\psi', t\rangle := e^{-i\hbar^{-1}\int_0^t E_n(\rho(t))\mathrm{d}t}|\psi, t\rangle \tag{4.8}$$

solves

$$i\hbar \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} |\psi'; \_\rangle = \big(\hat{H}(\rho(t_0)) - E_n(\rho(t))\big)|\psi'; t_0\rangle = \hat{H}'(\rho(t_0))|\psi'; t_0\rangle. \tag{4.9}$$

In the language of local eigensections, let U be an open neighborhood of $\rho(t_0)$ that admits local eigensections and let $J \subseteq I$ be an open interval such that $\rho[J] \subseteq U$. Set $\nu \in \Gamma_{\rho|_J}(E)$ to be the restriction of $|n_U\rangle$ to ρ, i.e. the section along $\rho|_J$ defined by $\nu(t) = |n_U\rangle_{\rho(t)}$ for $t \in J$. If ψ is a solution of Equation (4.7) that at time t_0 is in the n^{th} eigenspace, so $\hat{H}(\rho(t_0))\psi(t_0) = 0$, then due to the adiabatic approximation there is a smooth function $\gamma \in C^\infty(J)$ such that

$$\psi(t) = e^{i\gamma(t)}\nu(t), \tag{4.10}$$

for $t \in J$. Substituting this into Equation (4.7) gives

$$\begin{aligned} i\hbar\, D_t\psi &= \hat{H}\psi \\ D_t(e^{i\gamma}\nu) &= \hat{H}(e^{i\gamma}\nu) \\ i\dot{\gamma}e^{i\gamma}\nu + e^{i\gamma}\, D_t\nu &= 0 \\ \dot{\gamma}\nu &= i\, D_t\nu \\ \dot{\gamma} &= i\,\nu^\flat\, D_t\nu, \end{aligned} \tag{4.11}$$

where we denote by $\nu^\flat \in \Gamma_{\rho|J}(E^*)$ the section of the dual bundle $E^* := M \times \mathcal{H}^*$ defined by $\nu^\flat := \langle\nu|\_\rangle$. We similarly define $\langle n_U| \in \Gamma_U(E^*)$ by $\langle n_U| := \langle n_U|\_\rangle$. Since ν is by definition extendible on U, we get

$$\dot{\gamma}(t_0) = i\langle n_U|\nabla_{\dot{\rho}(t_0)}|n_U\rangle. \tag{4.12}$$

We saw in Section 3.4.1 that ∇ is a Hermitian connection, and hence we have for any $X \in \Gamma_U(TM)$

$$\langle m_U|\nabla_X n_U\rangle + \langle\nabla_X m_U|n_U\rangle = X\langle m_U|n_U\rangle \equiv 0, \tag{4.13}$$

since $\langle m_U|n_U\rangle$ is constant by the orthonormality of the basis. In particular, we have

$$\langle n_U|\nabla_X n_U\rangle = -\langle\nabla_X n_U|n_U\rangle = -\overline{\langle n_U|\nabla_X n_U\rangle} \tag{4.14}$$

which is true only if $\langle n_U|\nabla_X n_U\rangle$ is purely imaginary. We may conclude that the complex number $\langle n_U|\nabla_{\dot{\rho}(t_0)}|n_U\rangle$ is purely imaginary, and multiplying by i guarantees the result of Equation (4.12) is real.

The calculation of the total geometric phase is more complicated than simply using Equation (4.11) and integrating, since that equation is only locally valid. Instead, by an argument similar to Lemma 3.1.7 – this will be made precise below – we may select a partition

$$0 = T_0 < T_1 < \cdots < T_k = T \tag{4.15}$$

such that for $0 \leq i < n$ the closed interval $[T_i, T_{i+1}]$ is contained in an interval $J_i \subseteq I$ such that $U_i \subseteq M$ is an open subset with $\rho[J_i] \subseteq U_i$ that admits a local eigensection $|n_{U_i}\rangle \in \Gamma_{U_i}(E)$ and associated local section $\nu_i \in \Gamma_{\rho_i}(E)$ where $\rho_i := \rho|_{J_i}$. We may than integrate Equation (4.11) locally, and add the results together. Still, there is a problem left: we selected the $|n_{U_i}\rangle$ arbitrarily. For any other choice $|n'_{U_i}\rangle \in \Gamma_{U_i}(E)$, since they differ by a unit phase, we may construct smooth functions $\alpha_i \in C^\infty(J_i)$ such that $\nu'_i = e^{i\alpha}\nu_i$, and calculate

$$\begin{aligned} i(\nu'_i)^\flat D_t \nu'_i &= i(e^{i\alpha}\nu_i)^\flat D_t(e^{i\alpha}\nu_i) \\ &= ie^{-i\alpha}\nu_i^\flat\big(i\dot{\alpha}\, e^{i\alpha}\nu_i + e^{i\alpha} D_t\nu_i\big) \\ &= -\dot{\alpha}\nu_i^\flat\nu_i + i\nu_i^\flat D_t\nu_i \\ &= i\nu_i^\flat D_t\nu_i - \dot{\alpha}. \end{aligned} \tag{4.16}$$

Integrating this gives

$$\int_{T_i}^{T_{i+1}} i(\nu'_i)^\flat D_t\nu'_i \mathrm{d}t = i\big(\alpha_i(T_{i+1}) - \alpha_i(T_i)\big) + \int_{T_i}^{T_{i+1}} i\nu_i^\flat D_t\nu_i \mathrm{d}t, \tag{4.17}$$

and the first term on the right-hand side is in general non-zero. This means the geometric phase of a general path ρ is ill-defined, but if $\rho(0) = \rho(T)$ this problem can be overcome. We may without loss of generality impose that for $0 \leq i < k$ we have $\nu_i(T_{i+1}) = \nu_{i+1}(T_{i+1})$ and $\nu_0(T_0) = \nu_{k-1}(T_k)$. Now we define the total phase to be

$$\gamma(T) = i\sum_{i=0}^{k-1}\int_{T_i}^{T_{i+1}} \nu_i^\flat D_t\nu_i \,\mathrm{d}t \tag{4.18}$$

To check that this is gauge invariant, let (ν'_i) be another choice of local sections, then repeatedly applying Equation (4.17) gives

$$\gamma'(T) = \gamma(T) + i\big(\alpha_0(p_{T_0}) - \alpha_0(p_{T_1}) + \alpha_1(p_{T_1}) - \cdots - \alpha_{n-1}(p_{T_k})\big) \tag{4.19}$$

and since it must be so that all differences $\alpha_i(T_{i+1}) - \alpha_{i+1}(T_{i+1})$ and $\alpha_0(T_0) - \alpha_{n-1}(T_k)$ are a multiple of 2π in order to satisfy the boundary conditions imposed on the ν_i and ν'_i, we see $\gamma(T)$ is well-defined up to addition of a multiple of 2π.

Berry's 2-form

We will show that Berry's phase is conveniently described as the surface integral of a 2-form on M. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of M such that on each U_α a local basis of eigensections $(|m_\alpha\rangle)$ for $1 \leq m \leq \dim \mathcal{H}$ exists on U_α. Then Equation (4.12) suggests we may define 1-forms on U_α by

$$A_\alpha := i\langle n_\alpha|\nabla|n_\alpha\rangle. \tag{4.20}$$

We saw in Section 3.2 that $\nabla|n_\alpha\rangle$ is a $\Gamma_{U_\alpha}(E)$-valued 1-form, and letting $\langle n_\alpha|$ act on such a section results in a complex functions. Due to Equation (4.14) this is in fact a real function on U_α. Restricting Equation (4.20) to a single tangent space is clearly linear, hence A_α is indeed a real 1-form as we claimed. Moreover, if ρ is a circuit entirely contained in one open set U_α, we clearly have

$$\gamma_n(T) = \oint_\rho A_\alpha. \tag{4.21}$$

If ρ is contractible to a point within U_α, the generalization of Stokes' theorem [14, Theorem 16.11] allows us to write $\gamma_n(T)$ in terms the local 2-form $V_\alpha := dA_\alpha$, called a ***local Berry's 2-form***. If C is an oriented smooth surface in M with boundary $\partial C = \rho$, then

$$\oint_\rho A_\alpha = \int_C V_\alpha, \tag{4.22}$$

which allows us to describe $\gamma_n(T)$ in terms of gauge-invariant quantities. To see this, first note that the 1-form A_α depends on a *choice of gauge*: if we consider another basis of sections $(|m'_\alpha\rangle)$ with $|n'_\alpha\rangle = e^{i\theta}|n_\alpha\rangle$ for some $\theta \in C^\infty(U_\alpha)$, we get

$$\begin{aligned} A'_\alpha(X_R) &= i\langle n'_\alpha|\nabla_{X_R}|n'_\alpha\rangle \\ &= ie^{-i\theta}\langle n_\alpha|\nabla_{X_R}\left[e^{i\theta}|n_\alpha\rangle\right] \\ &= ie^{-i\theta}\langle n_\alpha|\left[e^{i\theta(R)}\nabla_{X_R}|n_\alpha\rangle + \left(X_R e^{i\theta}\right)|n_\alpha\rangle\right] \\ &= i\langle n_\alpha|\nabla_{X_R}|n_\alpha\rangle + ie^{-i\theta}\langle n_\alpha|n_\alpha\rangle d\left(e^{i\theta}\right)X_R \\ &= \left[i\langle n_\alpha|\nabla|n_\alpha\rangle + ie^{-i\theta}d\left(e^{i\theta}\right)\right]X_R \\ &= \left[A_\alpha - d\theta\right]X_R, \end{aligned} \tag{4.23}$$

where $d\left(e^{i\theta}\right) = ie^{i\theta}d\theta$ follows from Proposition 2.2.5. However, we have

$$V'_\alpha = dA'_\alpha = dA_\alpha - d(d\theta) = dA_\alpha = V_\alpha, \tag{4.24}$$

since $d \circ d = 0$ by Theorem 2.3.13, and V_α is indeed independent of this choice of gauge.

It is possible to give an explicit formula to calculate V_α from $\hat{H}$. In order to do this, we remember that (local) sections of E give rise to (local) functions from the base space to $\mathcal{H}$. Let $f_{m_\alpha} \in C^\infty(U_\alpha, \mathcal{H})$ and $f^\flat_{m_\alpha} \in C^\infty(U_\alpha, \mathcal{H}^*)$ be the functions associated with the local sections $|m_\alpha\rangle$ and $\langle m_\alpha|$, respectively. The parameter-dependent Hamiltonian, restricted to U_α, may be seen as a smooth function $\hat{H}_\alpha := \hat{H}|_{U_\alpha} \in C^\infty\big(U_\alpha, \mathrm{Herm}(\mathcal{H})\big)$ into the vector space of Hermitian operators on $\mathcal{H}$. Remembering that these smooth functions may be seen as vector-valued 0-forms on U_α, the obvious bilinear maps $\mathrm{ip}\colon \mathcal{H}^* \times \mathcal{H} \to \mathbb{C}$, $\mathrm{ev}\colon \mathrm{Herm}(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$ and $\mathrm{ev}^*\colon \mathcal{H}^* \times \mathrm{Herm}(\mathcal{H}) \to \mathcal{H}^*$ give rise to wedge products, by Definition 2.3.15.

In terms of these vector-valued forms, we may write Equation (4.20) as

$$A_\alpha = i(f^\flat_{n_\alpha} \wedge_{\mathrm{ip}} df_{n_\alpha}). \tag{4.25}$$

This leads to the following expression for V_α:

$$V_\alpha = dA_\alpha = d\big(i(f^\flat_{n_\alpha} \wedge_{\mathrm{ip}} df_{n_\alpha})\big) = i(df^\flat_{n_\alpha} \wedge_{\mathrm{ip}} df_{n_\alpha}). \tag{4.26}$$

Since we assumed the eigensections $(|m_\alpha\rangle)$ form a basis, we get the completeness condition

$$\sum_m |m_\alpha\rangle\langle m_\alpha| \equiv \mathrm{id}_{\mathcal{H}}\,. \tag{4.27}$$

This suggests we may write

$$V_\alpha = \sum_{m \neq n} i\big((df^\flat_{n_\alpha} \wedge_{\mathrm{ip}} f_{m_\alpha}) \wedge (f^\flat_{m_\alpha} \wedge_{\mathrm{ip}} df_{n_\alpha})\big), \tag{4.28}$$

where the exclusion of n in the sum is justified by the anticommutativity of the wedge product and fact that $(df^\flat_{n_\alpha} \wedge_{\mathrm{ip}} f_{n_\alpha}) = -(f^\flat_{n_\alpha} \wedge_{\mathrm{ip}} df_{n_\alpha})$ due to Equation (4.14).

In terms of vector-valued forms, Equation (4.5) has the form

$$\hat{H}_\alpha \wedge_{\mathrm{ev}} f_{n_\alpha} \equiv 0. \tag{4.29}$$

Taking the exterior derivative, we get the result

$$d\hat{H}_\alpha \wedge_{\mathrm{ev}} f_{n_\alpha} + \hat{H} \wedge_{\mathrm{ev}} df_{n_\alpha} \equiv 0, \tag{4.30}$$

which allows us to calculate

$$\begin{aligned} f^{\flat}_{m_\alpha} \wedge_{\text{ip}} (d\hat{H}_\alpha \wedge_{\text{ev}} f_{n_\alpha}) &= -f^{\flat}_{m_\alpha} \wedge_{\text{ip}} (\hat{H}_\alpha \wedge_{\text{ev}} df_{n_\alpha}) \\ &= -(f^{\flat}_{m_\alpha} \wedge_{\text{ev}^*} \hat{H}_\alpha) \wedge_{\text{ip}} df_{n_\alpha} \\ &= -(f^{\flat}_{m_\alpha} \wedge E_{m_\alpha}) \wedge_{\text{ip}} df_{n_\alpha} \\ &= -E_{m_\alpha}(f^{\flat}_{m_\alpha} \wedge_{\text{ip}} df_{n_\alpha}). \end{aligned} \tag{4.31}$$

Since the eigenvalues of $\hat{H}$ are everywhere non-degenerate and $E_n \equiv 0$, we have E_{m_α} everywhere non-zero, hence we may write

$$f^{\flat}_{m_\alpha} \wedge_{\text{ip}} df_{n_\alpha} = -\frac{f^{\flat}_{m_\alpha} \wedge_{\text{ip}} (d\hat{H}_\alpha \wedge_{\text{ev}} f_{n_\alpha})}{E_{m_\alpha}}. \tag{4.32}$$

By Equation (4.13), we get

$$df^{\flat}_{n_\alpha} \wedge_{\text{ip}} f_{m_\alpha} = -\frac{f^{\flat}_{n_\alpha} \wedge_{\text{ip}} (d\hat{H}_\alpha \wedge_{\text{ev}} f_{m_\alpha})}{E_{m_\alpha}}. \tag{4.33}$$

Substituting these results into Equation (4.26), the formula for the local Berry's 2-form becomes

$$V_\alpha = i \sum_{m \neq n} \left(\frac{\left(f^{\flat}_{n_\alpha} \wedge_{\text{ip}} (d\hat{H}_\alpha \wedge_{\text{ev}} f_{m_\alpha})\right) \wedge \left(f^{\flat}_{m_\alpha} \wedge_{\text{ip}} (d\hat{H}_\alpha \wedge_{\text{ev}} f_{n_\alpha})\right)}{E^2_{m_\alpha}} \right), \tag{4.34}$$

which pointwise may be written as

$$[V_\alpha]_R = i \sum_{m \neq n} \left(\frac{\langle n; R | d\hat{H}_R | m; R \rangle \wedge \langle m; R | d\hat{H}_R | n; R \rangle}{E_m(R)^2} \right) \tag{4.35}$$

in the more familiar language of quantum mechanics, where we take $|m; R\rangle := f_{m_\alpha}(R) \in \mathcal{H}$. Though this equation appears to depend profoundly on the details of $\hat{H}$, it will become clear below that Berry's phase actually is an effect only of the geometry of the eigensections of $\pi \colon E \to M$.

Now that we have a description of the local Berry's 2-form on each U_α in the cover $\mathcal{U}$, we will give a description of the globally defined Berry's 2-form. Note that for $U_\alpha, U_\beta \in \mathcal{U}$, if we write $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the sections $|n_\alpha\rangle|_{U_{\alpha\beta}}$ and $|n_\beta\rangle|_{U_{\alpha\beta}}$ span the same eigenspace for each parameter, and thus differ only by a gauge transform. However, we know from Equation (4.24) that the local Berry's 2-forms are gauge invariant, and so $[V_\alpha]_R = [V_\beta]_R$ must hold for $R \in M$.

In order to 'glue' the local 2-forms together, let $\{\phi^\alpha\}$ be a smooth partition of unity subordinate to $\mathcal{U}$ as in Lemma 2.1.7. Since ϕ^α is a function on M with support in U_α, we may consider $\phi^\alpha V_\alpha$ to be a 2-form on M that is identically zero on $M \setminus U_\alpha$. We define ***Berry's 2-form on*** M as the 2-form $V \in \Omega^2(M)$ given by

$$V := \phi^\alpha V_\alpha. \tag{4.36}$$

Any $R \in M$ has an open neighborhood which intersects the support of only finitely many ϕ^α, say $\phi^1, \dots, \phi^k$. Then by gauge invariance, we have $[V_1]_R = \cdots = [V_k]_R$ and

$$V_R = \phi^i(R)[V_i]_R = (\phi^1(R) + \cdots + \phi^k(R))[V_1]_R = [V_1]_R. \tag{4.37}$$

This definition thus makes sense and is independent of our choice of a partition of unity, and it is easy to see that for *any* contractible circuit $\rho\colon [0,T] \to M$, if C is an oriented smooth surface in M with boundary $\partial C = \rho$, we have

$$\gamma_n(T) = \int_C V, \tag{4.38}$$

and this is in agreement with Equation (4.18) up to addition of a multiple of 2π.

4.2.2 Simon connection

Eigenbundle

It was first noted by Simon [18] that the natural language of Berry's phase is the mathematics of fiber bundles. He did not describe a trivial vector bundle, like we did in Section 4.2.1, but rather the eigenbundle which we will define below.

We again assume ρ is traversed slowly enough to justify the adiabatic approximation. If $|\psi; t_0\rangle$ at some time $t_0 \in [0,T]$ is an eigenvector with eigenvalue 0, $|\psi; t\rangle$ will cling to the n^{th} eigenspace, and we actually only need to consider the complex line bundle $\pi_{\mathcal{E}}\colon E_{\mathcal{E}_n} \to M$, called the ***$n^{th}$ eigenbundle***, Such that $\pi_{\mathcal{E}}^{-1}(R)$ is the eigenspace of $\hat{H}(R)$ with eigenvalue 0 over each parameter $R \in M$. Since we assumed the eigenvalues of the Hamiltonian are non-degenerate, the typical fiber is isomorphic to the complex line $\mathbb{C}$.

A way to construct this bundle is as the pullback of the tautological line bundle seen in Section 3.4.2. Recall this is the subbundle $\mathcal{O}_{\mathcal{H}} \subseteq \mathbb{P}\mathcal{H} \times \mathcal{H}$ over $\mathbb{P}\mathcal{H}$ defined such that $\pi_{\mathcal{O}}^{-1}(W) = W$ for each one-dimensional subspace $W \subseteq \mathcal{H}$. Let $\mathcal{E}_n \colon M \to \mathbb{P}\mathcal{H}$ be the map that for each parameter $R \in M$ selects the eigenspace $\mathcal{E}_n(R)$ of $\hat{H}(R)$ of the eigenvalue 0, which is sooth per assumption. This is per assumption a one-dimensional subspace of $\mathcal{H}$ and so an element of $\mathbb{P}\mathcal{H}$. Taking the pullback of the tautological bundle then guarantees that the fiber over each point in parameter space is indeed the corresponding eigenspace.

Now the identification with the pullback bundle of $\mathcal{O}_{\mathcal{H}}$ along $\mathcal{E}_n$ is made by setting $E_{\mathcal{E}_n} := \mathcal{E}_n^* \mathcal{O}_{\mathcal{H}}$ and $\pi_{\mathcal{E}} := \mathcal{E}_n^* \pi_{\mathcal{O}}$. This bundle inherits a pullback connection from $\mathcal{O}_{\mathcal{H}}$, defined by

$$[\mathcal{E}_n^* \nabla^{\mathcal{O}}]_X (\mathcal{E}_n^* \sigma) = \mathcal{E}_n^* (\nabla^{\mathcal{O}}_{(d\mathcal{E}_n)X} \sigma), \tag{4.39}$$

for $\sigma \in \Gamma(\mathcal{O}_{\mathcal{H}})$. This connection is denoted by $\nabla^{\mathrm{S}} := \mathcal{E}_n^* \nabla^{\mathcal{O}}$ and called the ***Simon connection***.

The n^{th} eigenbundle is in general not trivial, and does not admit a global basis of sections. Lemma 3.1.7 does guarantee we can cover M with a finite number of open sets U_α that each admit a local basis of sections, and since $E_{\mathcal{E}_n}$ is a vector bundle of rank 1, this means that a non-zero local section exists on each U_α. In the bundle E, this gives rise to a normalized n^{th} eigensection on U_α, and therefore justifies the existence of the partition in Equation (4.15).

Claim. *Define the map* $p_{\mathcal{E}_n} \colon E \to E_{\mathcal{E}_n}$ *by*

$$p_{\mathcal{E}_n}(R, |v\rangle) = \big(R, (\mathcal{E}_n(R), p_{\mathcal{E}_n(R)} |v\rangle)\big).$$

The Simon connection is given by

$$\nabla^S = p_{\mathcal{E}_n} \circ \nabla.$$

Proof. By the uniqueness of the pullback connection, we only need to show that for any $\sigma \in \Gamma(\mathcal{O}_{\mathcal{H}})$, $X \in \Gamma(TM)$ and $R \in M$ we have

$$p_{\mathcal{E}_n}\big([\nabla_X(\mathcal{E}_n^* \sigma)](R)\big) = \big(\mathcal{E}_n^* \big[\nabla^{\mathcal{O}}_{(d\mathcal{E}_n)X} \sigma\big]\big)(R).$$

The right-hand side may be written as

$$\begin{aligned}
\big[\mathcal{E}_n^*\big(\nabla^{\mathcal{O}}_{(d\mathcal{E}_n)X}\sigma\big)\big](R) &= \big[\mathcal{E}_n^*\big(p_{\mathcal{O}} \circ \nabla_{(d\mathcal{E}_n)X}\sigma\big)\big](R) \\
&= \Big(R, \big[p_{\mathcal{O}} \circ \nabla_{(d\mathcal{E}_n)X}\sigma\big]\big(\mathcal{E}_n(R)\big)\Big) \\
&= \Big(R, p_{\mathcal{O}}\Big(\big[\nabla_{(d\mathcal{E}_n)X}\sigma\big]\big(\mathcal{E}_n(R)\big)\Big)\Big) \\
&= \Big(R, p_{\mathcal{O}}\big(\mathcal{E}_n(R), [df_\sigma]_{\mathcal{E}_n(R)}(d\mathcal{E}_n)X_R\big)\Big) \\
&= \Big(R, p_{\mathcal{O}}\big(\mathcal{E}_n(R), d(f_\sigma \circ \mathcal{E}_n)_R X_R\big)\Big) \\
&= \Big(R, \Big(\mathcal{E}_n(R), p_{\mathcal{E}_n(R)}\big(d(f_\sigma \circ \mathcal{E}_n)_R X_R\big)\Big)\Big).
\end{aligned}$$

Meanwhile, the left-hand side gives

$$\begin{aligned}
p_{\mathcal{E}_n}\big([\nabla_X(\mathcal{E}_n^*\sigma)](R)\big) &= p_{\mathcal{E}_n}\big(R, [df_{\mathcal{E}_n^*\sigma}]_R X_R\big) \\
&= \Big(R, \big(\mathcal{E}_n(R), p_{\mathcal{E}_n(R)}([df_{\mathcal{E}_n^*\sigma}]_R X_R)\big)\Big).
\end{aligned}$$

Since from Definition 3.1.9 it is clear we have $f_{\mathcal{E}_n^*\sigma} = f_\sigma \circ \mathcal{E}_n$ these are indeed the same. □

Above, we identify a section $\sigma \in \Gamma(E_{\mathcal{E}_n})$ with the section $(i \circ \sigma) \in \Gamma(E)$ where $i\colon E_{\mathcal{E}_n} \hookrightarrow E$ is the obvious inclusion map, and by a slight abuse of notation we denote both by σ.

Claim. *The covariant time derivative with respect to the Simon connection is given by*

$$D_t^{\mathsf{S}} = p_{\mathcal{E}_n} \circ D_t.$$

Proof. Consider any smooth curve $\rho\colon I \to M$, $\zeta \in \Gamma_\rho(E_{\mathcal{E}_n})$, and $t_0 \in I$. Find an open neighborhood $U \subseteq M$ that admits a local basis of sections $(|e_j\rangle)$ of $E_{\mathcal{E}_n}$, and by an abuse of notation extend this to a basis of sections of E, then like in the proof of Lemma 3.3.4:

$$\begin{aligned}
[D_t^{\mathsf{S}}\zeta](t_0) &= \dot\zeta^j(t_0)|e_j\rangle_{\rho(t_0)} + \zeta^j(t_0)\nabla^{\mathsf{S}}_{\dot\rho(t_0)}|e_j\rangle \\
&= \dot\zeta^j(t_0)|e_j\rangle_{\rho(t_0)} + \zeta^j(t_0)p_{\mathcal{E}_n}\big(\nabla_{\dot\rho(t_0)}|e_j\rangle\big) \\
&= p_{\mathcal{E}_n}\big(\dot\zeta^j(t_0)|e_j\rangle_{\rho(t_0)} + \zeta^j(t_0)\nabla_{\dot\rho(t_0)}|e_j\rangle\big) \\
&= p_{\mathcal{E}_n}\big([D_t\zeta](t_0)\big).
\end{aligned}$$

□

Berry's phase and the Simon connection

If ρ is a smooth curve in parameter space M, and $|\psi;t\rangle$ is a solution of Equation (4.4), consider the lift $\psi \in \Gamma_\rho(E_{\mathcal{E}_n})$ of ρ given by

$$\psi\colon t \mapsto \big(\rho(t), |\psi;t\rangle\big), \tag{4.40}$$

then we have

$$D_t^{\mathrm{S}}\psi\colon t \longmapsto \left(\rho(t), |\psi;t\rangle\!\langle\psi;t| \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_t |\psi;t\rangle\right). \tag{4.41}$$

We see that the condition $D_t^{\mathrm{S}}\psi \equiv 0$ corresponds to Schrödinger's equation (4.7), since $\hat{H}\phi \equiv 0$. The solutions of Schrödinger's equation for a path ρ in parameter space are thus exactly the lifts of ρ into $E_{\mathcal{E}_n}$ that are horizontal with respect to the Simon connection.

We may also use this connection to calculate Berry's phase. This should convince the reader that this phase may be seen as an effect of the geometry of the line bundle, since Simon's connection is canonical in $E_{\mathcal{E}_n}$.

For any time $t_0 \in [0,T]$, consider a neighborhood U of $\psi(t_0)$ that admits a local section $|n_U\rangle \in \Gamma_U(E_{\mathcal{E}_n})$ everywhere of unit norm. Let $J \subseteq I$ be an open interval such that $\rho[J] \subseteq U$, then since $\psi(t)$ and $|n\rangle_{\rho(t)}$ at any time $t \in J$ differ only by a complex phase, we may write $\psi(t) = e^{i\gamma(t)}|n\rangle_{\rho(t)}$ for some function $\gamma \in C^\infty(J)$. This is exactly Berry's geometric phase. We have

$$\begin{aligned}
[D_t^{\mathrm{S}}\psi](t_0) &= \left[D_t^{\mathrm{S}}\left(e^{i\gamma(t)}|n\rangle_{\rho(t)}\right)\right](t_0) \\
&= \left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} e^{i\gamma(t)}\right)|n\rangle_{\rho(t_0)} + e^{i\gamma(t_0)}[D_t^{\mathrm{S}}|n\rangle_{\rho(t)}](t_0) \\
&= e^{i\gamma(t_0)}\big(i\dot{\gamma}(t_0)|n\rangle_{\rho(t_0)} + [D_t^{\mathrm{S}}|n\rangle_{\rho(t)}](t_0)\big).
\end{aligned} \tag{4.42}$$

The parallel transport law $[D_t^{\mathrm{S}}\psi](t_0) = 0$ then reads

$$\begin{aligned}
i\dot{\gamma}(t_0)|n\rangle_{\rho(t_0)} &= -[D_t^{\mathrm{S}}|n\rangle_{\rho(t)}](t_0) \\
i\dot{\gamma}(t_0)|n;\rho(t_0)\rangle &= -|n;\rho(t_0)\rangle\!\langle n;\rho(t_0)| \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} |n;\rho(t)\rangle \\
\dot{\gamma}(t_0) &= i\langle n;\rho(t_0)| \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} |n;\rho(t)\rangle,
\end{aligned} \tag{4.43}$$

and Equation (4.11) is recovered.

4.3 Aranov–Anandan phase

Aharonov and Anandan [1] realized that the description of the geometric phase given by Berry is independent of the exact Hamiltonian chosen, but depends only on the curve $t \mapsto \pi(|\psi;t\rangle)$ of physical states. In fact the geometric phase is a property of the image of $\pi(|\psi;\_\rangle)$ in projective Hilbert space independent of parametrization. It therefore makes sense to study Berry's phase independent of the parameter space for the Hamiltonian, but rather as a direct feature of $\mathbb{P}\mathcal{H}$. This is not just a reformulation of the Berry's phase but a generalization as it allows us to drop the adiabatic approximation and the assumption that $|\psi;t\rangle$ is an eigenvector of $\hat{H}(\rho(t))$, and derive the geometric phase of any cyclic evolution.

Consider a finite-dimensional Hilbert space $\mathcal{H}$ and let $t \mapsto \hat{H}(t)$ be a smooth assignment of Hermitian operators on $\mathcal{H}$ for $t \in I$ with I an open subset of $\mathbb{R}$ containing a closed interval $[0,T]$. If $t \mapsto |\psi;t\rangle$ is a normalized solution of Schrödinger's equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi;\_\rangle = \hat{H}(t)|\psi;t\rangle \tag{4.44}$$

we claim it accumulates both a geometric phase γ and a dynamic phase given by

$$\theta(t) = -\frac{1}{\hbar}\int_0^t \langle\psi;t|\hat{H}(t)|\psi;t\rangle \mathrm{d}t. \tag{4.45}$$

To demonstrate this, we consider the tautological line bundle $\pi_{\mathcal{O}}\colon \mathcal{O}_{\mathcal{H}} \to \mathbb{P}\mathcal{H}$ already seen in Section 3.4.2. We view the evolution of $|\psi;t\rangle$ as a path in $\mathcal{O}_{\mathcal{H}}$ given by $\psi\colon t \mapsto (\mathrm{span}_{\mathbb{C}}|\psi;t\rangle, |\psi;t\rangle)$. For any $t \in [0,T]$, we write $p_t := \pi(\psi(t)) \in \mathbb{P}\mathcal{H}$ to ease notation.

Now consider a fixed $t_0 \in [0,T]$, an let $U \subseteq \mathbb{P}\mathcal{H}$ be an open neighborhood of p_{t_0} such that there exists a section $|\sigma\rangle \in \Gamma_U(\mathcal{O}_{\mathcal{H}})$ that everywhere has norm 1. Since the fiber over each point is a complex line, this is equivalent with the existence of a basis of sections on U. Now take $J \subseteq I$ to be an open interval containing t_0 such that $p_t \in U$ hold for all $t \in J$. Write $\varsigma\colon t \mapsto |\sigma\rangle_{p_t}$ for the section of $\mathcal{O}_{\mathcal{H}}$ along the curve defined by p_t in base space, for $t \in J$. For such t, the unit vectors $\psi(t)$ and $\varsigma(t) = |\sigma\rangle_{p_t}$ differ only by a unit scalar. We denote this complex number by $e^{i\beta(t)} \in \mathbb{C}$ for a smooth function $\beta \in C^\infty(J)$ such that we have

$$e^{i\beta}\varsigma = \psi. \tag{4.46}$$

Now from the fact that $|\psi;t\rangle$ solves Equation (4.44), we get

$$\begin{aligned} i\hbar[D_t\psi](t_0) &= \hat{H}(t_0)\psi(t_0) \\ i\hbar\big[D_t\big(e^{i\beta}\varsigma\big)\big](t_0) &= \hat{H}(t_0)e^{i\beta(t_0)}\varsigma(t_0) \\ i\hbar\left(\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} e^{i\beta}\right)\varsigma(t_0) + i\hbar e^{i\beta}[D_t\varsigma](t_0) &= \hat{H}(t_0)e^{i\beta(t_0)}\varsigma(t_0) \\ -\hbar\dot{\beta}(t_0)\varsigma(t_0) + i\hbar[D_t\varsigma](t_0) &= \hat{H}(t_0)\varsigma(t_0) \\ i[D_t\varsigma](t_0) - \frac{1}{\hbar}\hat{H}(t_0)\varsigma(t_0) &= \dot{\beta}(t_0)\varsigma(t_0) \\ i\langle\sigma|\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}|\sigma\rangle_{p_t} - \frac{1}{\hbar}\langle\sigma|\hat{H}(t_0)|\sigma\rangle_{p_{t_0}} &= \dot{\beta}(t_0). \end{aligned} \tag{4.47}$$

The total phase is, as claimed, the sum of a dynamic phase θ and a geometric phase γ. From Equation (4.45) we have

$$\dot{\gamma}(t_0) = \dot{\beta}(t_0) - \dot{\theta}(t_0) = i\langle\sigma|\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0}|\sigma\rangle_{p_t}. \tag{4.48}$$

This equation is only locally valid and depends on an arbitrary choice of phase for the section σ, however in exactly the same way we did for Berry's phase in deriving Equation (4.18) we can construct a global solution. Since $\mathbb{P}\mathcal{H}$ is compact we may by Lemma 3.1.7 select a partition

$$0 = T_0 < T_1 < \cdots < T_k = T \tag{4.49}$$

such that there are open intervals J_i and open subsets $U_i \subseteq M$ that admit a local section $|\sigma_i\rangle$ of unit norm, and the inclusions $[T_i, T_{i+1}] \subseteq J_j$ and $\rho[J_i] \subseteq U_i$ hold. If $p_0 = p_T$, we may without loss of generality impose that for $0 \leq i < n$ we have $\sigma_i(p_{T+1}) = \sigma_{i+1}(p_{T+1})$ and $\sigma_0(p_{T_0}) = \sigma_{n-1}(p_{T_n})$. The total phase than is

$$\gamma(T) = i\sum_{i=0}^{k-1}\int_{T_i}^{T_{i+1}}\langle\sigma_i|\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t}|\sigma_i\rangle_{p_t}\mathrm{d}t \tag{4.50}$$

which again is is well-defined up to addition of a multiple of 2π.

So while the dynamic phase θ depends on the details of $\hat{H}(t)$, we see that the geometric phase depends only on the path $t \mapsto p_t = \pi\big(\psi(t)\big)$ that the physical state of $|\psi;t\rangle$ traces out in projective Hilbert space.

Chapter 5

Outlook

In this thesis, we have seen a mathematically rigorous description of the result of [1, 3, 18] for finite-dimensional systems. However, we made extensive use of the adiabatic approximation to justify that a solution of Schrödinger's equation stays an eigenvector. An obvious way this could be formalized is by direct usage of the Adiabatic theorem 4.1.1. Since the solutions of Equation 4.1 will not in general be instantaneous eigenvalues of $\hat{H}$, a description in terms of the n^{th} eigenbundle is not possible.

Though the scope of this thesis is limited to finite-dimensional vector bundles, the most interesting quantum systems are described by an infinite-dimensional Hilbert space. The formalism above should be generalized, for example by considering *Banach manifolds* [22] which are defined by replacing $\mathbb{R}^n$ in Definition 2.1.1 by any Banach space B. It would be interesting to see under what conditions a notion of horizontal lifting as in Theorem 3.3.6 exists when considering vector bundles with typical fiber an infinite-dimensional Hilbert space $\mathcal{H}$.

Another description of the geometric phases can be given in terms of principal G-bundles, a good description of which can be found in [19]. Specifically, we may consider *principal* $\mathrm{U}(1)$*-bundles* instead of complex line bundles. These consist of a copy of the circle group above each point, and thus also convey phase information. In these bundles, one may define an Ehresmann connection which is not an operator like in Definition 3.2.1, but rather an explicit choice of horizontal tangent vectors on the bundle. This also leads to a horizontal lifting that turns out to be equivalent with Theorem 3.3.6. The equivalent of a tautological line bundle in this paradigm is the Hopf fibration, and may be described by the obvious map $S(\mathcal{H}) \to \mathbb{P}\mathcal{H}$ form the unit vectors to the projective space. This may be easier to generalize to the infinite-dimensional case then the description in terms of vector bundles.

Another interesting generalization may be to replace the parameter space in Section 4.2 by a (Banach) manifold of (bounded) Hermitian operators with discrete and non-degenerate spectrum. In the k-dimensional case, the Hermitian matrices form a k^2-dimensional real vector space $\mathrm{Herm}(\mathcal{H})$. They must all have discrete spectrum, and the requirement that the spectrum of a matrix $A \in \mathrm{Herm}(\mathcal{H})$ is non-degenerate is equivalent with the requirement that the discriminant of the characteristic polynomial of A is non-zero, written as

$$\mathrm{Herm}'(\mathcal{H}) := \{A \in \mathrm{Herm}(\mathcal{H}) : \Delta(\det(A - \lambda I_n)) \neq 0\}.$$

This setup is discussed by Arnold [2, §6], who states that the set of Hermitian matrices with degenerate eigenvalues is of codimension 3. This means that $\mathrm{Herm}'(\mathcal{H})$ is simply connected, and Berry's phase for *any* circuit can be expressed as the integral of a 2-form over a surface in $\mathrm{Herm}'(\mathcal{H})$. It would be interesting to see an equivalent of this in the infinite-dimensional case.

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