Locality of quantum-related measurement statistics

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Abstract

Quantum mechanics either violates realism (non-determinism) or locality. This follows from Bell's theorem by showing the measurement statistics of the EPR-experiment—specifically, $\mathbb{E}[\sigma_\alpha \otimes \sigma_\beta] = -\alpha \cdot \beta$—can never be fully reproduced by any local hidden variable model (LHV-model). Research has been done in the last couple of decades into the degree to which LHV-models can reproduce measurement statistics approximating the EPR's statistics. Specifically of interest is finding a critical amplitude p_c for which amplitudes $0 \leq p \leq p_c$ can be reproduced by LHV-models with $\mathbb{E}[A(\alpha)B(\beta)] = -p\alpha \cdot \beta$ and for $p > p_c$ it cannot. Our analysis and the literature show: a density matrix separability criterion on the two-electron singlet state yields $p_c \geq \frac{1}{3}$; constructs by Kaszlikowski and Żukowski find $p_c \leq \frac{3}{4}$; and finally $p_c = \frac{1}{K_G(d+1)}$ with d the dimension of the measurement settings in S^d and K_G the Grothendieck constant. Further, open question arise and new approaches to the locality of quantum states are given.

Contents

Chapter 1

Introduction

1.1 Intro

Quantum mechanics (QM) is a special kind of physics. Its probabilistic nature is in stark contrast to classical physics and its corresponding determinism. In classical physics, measurement outcomes are precisely defined through intrinsic properties of objects. Sometimes these outcomes can be calculated by proven formulas, but even if not, these intrinsic properties assure equal measurement outcomes for same objects under same conditions.

This is, however, not the case with quantum mechanics. Its mathematical framework allows for properties of objects, such as electrons, to yield different outcomes by repeating exactly the same measurement under the same conditions. Also, it seemingly allows to violate the concept of *locality*: information cannot travel faster than the speed of light.

This seemingly probabilistic nature and violation of locality of quantum mechanics has not always been appreciated. Physicists such as Einstein, Podolsky and Rosen suggested that QM's mathematical description was flawed. Their belief: quantum mechanics can and should be formulated as a local hidden variable theory that instills non-determinism and locality. [1]

However, in 1968 [2] John Stewart Bell showed by a short, but powerful proof that the quantum formalism could not be formulated as such a model. A specific quantum experiment, the EPR-experiment, was analyzed and for this shown its measurement statistics could never fit a local hidden variable theory and thus always either violates determinism or locality. And, experiments have shown the formalism to calculate real-world experiments correctly.

In my research, I will analyze, with the help of literature, to answer what variations of the EPR-experiment *can* be reproduced by a LHV-model. Doing so, I give an overview of different methods to answer this question and state open problems still concerning.

This thesis is divided into three chapters: the first gives an introduction to all the necessary mathematical tools of quantum mechanics, the second the mathematical analysis and the third chapter the conclusion and discussions.

1.2   Quantum formalism

In this section, we outline the basic construct of the mathematical formalism of quantum mechanics and corresponding properties.

Hilbert spaces form the framework of quantum mechanics.[*] We define a Hilbert space $\mathcal{H}$ and its dual $\mathcal{H}'$ and denote elements of the former as $|\cdot\rangle$, called the *ket*, and elements of the latter as $\langle\cdot|$, the *bra*. This is called the *braket* notation.[†] There exists a natural set isomorphism between $\mathcal{H}$ and $\mathcal{H}'$

$$J : \mathcal{H} \xrightarrow{\sim} \mathcal{H}'$$
$$\alpha|x\rangle \mapsto \bar{\alpha}\langle x|. \tag{1.1}$$

which shows that for every *ket* in $\mathcal{H}$ there is a corresponding *bra* in $\mathcal{H}'$ and vice versa and constants change to their complex conjugate.[‡]

The inner product corresponding to the Hilbert space is denoted as

$$(f|x\rangle, |y\rangle) = \langle y| f |x\rangle \tag{1.2}$$

with $|x\rangle, |y\rangle \in \mathcal{H}$ and for all continuous functions $f : \mathcal{H} \to \mathcal{H}$.[§] By definition, inner products are linear and continuous in both arguments.

In quantum mechanics, the state of an object, with all its properties, is described by an element from a particular $\mathcal{H}$. Therefore, this set is often called the *state space* and its elements *state vectors* or *pure states*.[¶]

Physics is the science of measuring: this formalism also provides a framework for measurements on observables.[||] We define quantum operators as linear and continuous

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[*]A *Hilbert space* is an inner product space complete in respect to its metric.
[†]For a Hilbert space $\mathcal{H}$, we define its dual as the set $\mathcal{H}' = \{f : V \to \mathbb{C} \mid f \text{ continuous and linear}\}$ where f is called a *(continuous) linear form.*
[‡]The existence of this isomorphism follows from the the Riesz–Fréchet representation theorem. [3]
[§]All this notation is the default through all literature.
[¶]In this thesis all the state are normalized such that $\langle\psi|\psi\rangle = 1$.
[||]Observables are physical quantities that can be measured—velocity, momentum, position, for example.

functions

$$\sigma : \mathcal{H} \to \mathcal{H} \tag{1.3}$$

that are self-adjoint, or Hermitian. Then, a state vector $|v_\lambda\rangle \in \mathcal{H}$ for which holds

$$\sigma |v_\lambda\rangle = \lambda |v_\lambda\rangle \tag{1.4}$$

is called an *eigenstate* of σ with *eigenvalue** $\lambda \in \mathbb{C}$. Note, the set of eigenstates for any operator forms an orthonormal basis for $\mathcal{H}$.

Using this, we define how observables are measured by quantum operators:

– If state $|\psi\rangle \in \mathcal{H}$ is an eigenstate of operator σ, then the measurement outcome is its eigenvalue λ.

– If $|\psi\rangle$ is not an eigenstate, then it can be written as a complex sum of eigenstates:

$$|\psi\rangle = \sum_\lambda z_\lambda |v_\lambda\rangle \tag{1.5}$$

with $z_\lambda \in \mathbb{C}$ and $\sum_\lambda |z_\lambda|^2 = 1$. Then, the measurement outcome can take all the eigenvalues λ, each with probability $|z_\lambda|^2$.

Surprisingly, this means a state that is not an eigenvector has no pre-known measurement outcome, but can actually take different measurement values—this is called non-determinism.

Due to this non-determinism, we define the *expected value* for a quantum operator σ on state $|\psi\rangle$ as

$$
\begin{aligned}
\mathbb{E}_{|\psi\rangle}[\sigma] &\equiv \langle\psi|\,\sigma\,|\psi\rangle \\
&= \left[\sum_\lambda \overline{z_\lambda}\langle v_\lambda|\right] \sigma \left[\sum_\lambda z_\lambda |v_\lambda\rangle\right] \\
&= \left[\sum_\lambda \overline{z_\lambda}\langle v_\lambda|\right] \left[\sum_\lambda \sigma(z_\lambda |v_\lambda\rangle)\right] \\
&= \left[\sum_\lambda \overline{z_\lambda}\langle v_\lambda|\right] \left[\sum_\lambda z_\lambda \lambda |v_\lambda\rangle\right] \\
&= \sum_\lambda |z_\lambda|^2 \lambda
\end{aligned}
\tag{1.6}
$$

by exploiting the linearity of σ and the orthonormality of the eigenstates.

---

*In this thesis, operators can often be written as matrices—therefore the word *eigenvalue* is used.

In the scope of this thesis, we can write *kets* and *bras* as (vector) arrays.* Using this, we define the computational useful *density matrix* $\rho_{|\psi\rangle}$ for a state $|\psi\rangle$:

$$
\begin{aligned}
\rho_{|\psi\rangle} &\equiv |\psi\rangle\langle\psi| \\
&\cong \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \end{pmatrix} \\
&= \begin{pmatrix} |a|^2 & a\bar{b} \\ b\bar{a} & |b|^2 \end{pmatrix}.
\end{aligned}
\tag{1.7}
$$

We call density matrices of this form *pure state* density matrices. Following, we have

$$
\mathbb{E}_{|\psi\rangle}[\sigma] = \mathrm{Tr}[\sigma\rho_{|\psi\rangle}].
\tag{1.8}
$$

Often times, not all observables of an object are considered simultaneously but only a single property is. For single properties the Hilbert space formalism still holds and a space is defined for specifically that property.

In this thesis, we will only focus on the observable *electron spin* that electrons have.

1.2.1　Electron spin

The *electron spin* is an intrinsic property of a sub-atomic particle, the electron. On a large scale, there is no exact replica of this property, but a very close analogy can be made.

Large objects can have a *spin angular momentum*: the object spinning about an axis located through its center of mass (figure 1.1). Letting α be the vector describing the direction of this axis taking values in S^2, we see an object has three options: rotating in a pre-defined direction we call clockwise (value 1), moving oppositely called counterclockwise (value -1), or rotating not all in that direction (value 0).†

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*We use the symbol $\cong$ when changing between *braket* and *array* notation.
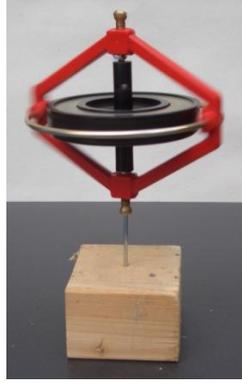†S^2 is the surface of a three-dimensional sphere.

Figure 1.1: *A gyroscope with spin angular momentum. The object rotates about its center of mass axis.[4]*

As the electron is a (near) point-like particle, it cannot have a spin angular momentum as other objects have. However, measurements have shown electrons do carry a similar property called *electron spin*. In short, we can act as if electrons carry spin angular momentum and base our mathematics on it. There is single exception: the spin value can never be 0—it is always either ± 1 in every direction $\alpha \in S^2$, corresponding to a 'clockwise' and 'counterclockwise' rotation.[*]

Returning to quantum formalism, we introduce the Hilbert space $\mathcal{H}$ that compactly denotes electron spin. We define

$$\mathcal{H} = \left\{ |\psi\rangle \equiv \begin{pmatrix} a \\ b \end{pmatrix} \cong a|0\rangle + b|1\rangle \mid \forall a, b \in \mathbb{C}, \ |a|^2 + |b|^2 = 1 \right\} \tag{1.9}$$

with the corresponding dual space

$$\mathcal{H}' = \left\{ \langle\psi| \equiv \begin{pmatrix} \overline{a} & \overline{b} \end{pmatrix} \cong \overline{a}\langle 0| + \overline{b}\langle 1| \mid \forall a, b \in \mathbb{C}, \ |a|^2 + |b|^2 = 1 \right\}^{\dagger} \tag{1.10}$$

and the inner product

$$\langle \cdot || \cdot \rangle : \mathcal{H}' \times \mathcal{H} \to \mathbb{C}$$
$$\langle \psi_1 || \psi_2 \rangle \mapsto \begin{pmatrix} \overline{a_1} & \overline{b_1} \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \overline{a_1}a_2 + \overline{b_1}b_2 \tag{1.11}$$

where we denote $|\psi_1\rangle \cong \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$ and $|\psi_2\rangle \cong \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$.[‡]

---

[*]The analogy between *spin angular momentum* and *electron spin* is so strong that throughout the literature electron spin is indeed described as a rotation.

[‡]Due to the isomorphism between $\mathcal{H}$ and $\mathcal{H}'$ (1.1), this defines the inner product on $\mathcal{H} \times \mathcal{H}$ also.

Electron spin measurement

For the electron spin, we define the quantum observable that, given a direction $\alpha \in S^2$, outputs the direction ± 1—we call this the spin measurement. Let σ_α be this measurement. Then it is defined as

$$\sigma_\alpha \equiv \alpha \cdot \sigma \equiv \alpha \cdot \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix} \tag{1.12}$$

$$\equiv \alpha_x \sigma_x + \alpha_y \sigma_y + \alpha_z \sigma_z$$

with

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{1.13}$$

$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

the well-known Pauli matrices.

By this definition, we find that the spin measurements in the positive standard directions (x, y, z) are the corresponding Pauli matrices.

To give an example, we perform a spin measurement in the positive z-direction on the general state $|\psi\rangle = a|0\rangle + b|1\rangle \in \mathcal{H}$. As $|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are eigenstates with eigenvalue 1 resp. -1, this means—according to calculation rules (1.5)—that observing 1 as measurement outcome occurs with probability $|a|^2$ and -1 with probability $|b|^2$.

Two-particle formalism

The formalism of the electron spin can also be extended when talking about two electrons instead of one using tensor products. The state vectors are then of the form

$$|\psi_{AB}\rangle = \sum_{i,j \in \{0,1\}} z_{i,j} |i\rangle \otimes |j\rangle \tag{1.14}$$

with $z_{i,j} \in \mathbb{C}$ and $\sum_i |z_i|^2 = 1$ and corresponding co-vector

$$\langle \psi_{AB}| = \sum_{i,j \in \{0,1\}} \overline{z_{i,j}} \langle i| \otimes \langle j|. \tag{1.15}$$

A special class of measurements in the multi-particle formalism are local measurements. Here, the particles are all measured individually by a respective quantum operator. In literature, for the two-particle situation, this often described as *Alice* measuring particle A by quantum operator σ_A and Bob particle B by σ_B.

The two-particle local measurements are of the form

$$\sigma_A \otimes \sigma_B : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \tag{1.16}$$

with the property

$$\begin{aligned}(\sigma_A \otimes \sigma_B)\left(|\psi_A\rangle \otimes |\psi_B\rangle\right) &= (\sigma_A|\psi_A\rangle) \otimes (\sigma_B|\psi_B\rangle) \\ &= (\lambda_A|\psi_A\rangle) \otimes (\lambda_B|\psi_B\rangle) \\ &= (\lambda_A\lambda_B)(|\psi_A\rangle \otimes |\psi_B\rangle)\end{aligned} \tag{1.17}$$

where the second equality holds in case $|\psi_A\rangle$ and $|\psi_B\rangle$ are eigenstates for σ_A resp. σ_B. This means the outcome of such a measurement is defined as the product of the two separate local measurements. It follows that $\sigma_A \otimes \mathbb{I}_2$ is the measurement that only outputs the measurement outcome of σ_A on the first electron and analogously for $\mathbb{I}_2 \otimes \sigma_B$ on the second electron. For vectors and arrays, the tensor product works at the Kronecker product.

Non-determinism

One peculiarity of quantum physics we have already encountered upon is its non-determinism. A object can have different measurement outcomes for the same observable in exactly the same situation. For example, look at the electron in the state

$$\begin{aligned}|\psi\rangle &= \frac{1}{\sqrt{5}}|0\rangle + \frac{2}{\sqrt{5}}|1\rangle \\ &\cong \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix} \\ &= \frac{3}{\sqrt{10}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\right] - \frac{1}{\sqrt{10}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right].\end{aligned} \tag{1.18}$$

Note that in the last expression, the state is written as a sum of the eigenstates of σ_x with respective eigenvalues 1 and -1. Therefore, the probability of σ_x measuring 1 on this state is $\left(\frac{3}{\sqrt{10}}\right)^2 = \frac{9}{10}$ and similarly for -1 the probability is $\left(\frac{1}{\sqrt{10}}\right)^2 = \frac{1}{10}$.

1.3   EPR-experiment

In this thesis, the EPR-experiment plays a central role. In this, there are two electrons and two observers, as is common, Alice and Bob. The experiment is done by Alice performing a spin measurement with angle α on the first electron and Bob on the second with angle β. The electrons are in the so-called singlet state, i.e. the two-particle quantum state

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}} \left[|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle \right] \\
&\cong \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right].
\end{aligned}
\tag{1.19}
$$

The corresponding density matrix of this state is

$$
\begin{aligned}
\rho_{|\psi\rangle} &= |\psi\rangle\langle\psi| \\
&= \frac{1}{2} \left(|0\rangle \otimes |1\rangle \, \langle 0| \otimes \langle 1| - |0\rangle \otimes |1\rangle \, \langle 1| \otimes \langle 0| - |1\rangle \otimes |0\rangle\langle 0| \otimes \langle 1| \right. \\
&\hspace{6cm} \left. + |1\rangle \otimes |0\rangle\langle 1| \otimes \langle 0| \right) \\
&\cong \frac{1}{2} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \right. \\
&\hspace{0.5cm} \left. - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \end{pmatrix} \right) \\
&= \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\end{aligned}
\tag{1.20}
$$

The expected values of the measurement outcomes are then

$$
\begin{aligned}
\mathbb{E}[\sigma_\alpha] &\equiv \mathbb{E}_{|\psi\rangle}[\sigma_\alpha \otimes \mathbb{I}] \\
&= \mathrm{Tr}[\sigma_\alpha \otimes \mathbb{I} \, \rho_{|\psi\rangle}] \\
&= \mathrm{Tr}\left[\begin{pmatrix} \sigma_\alpha & 0 \\ 0 & \sigma_\alpha \end{pmatrix} \rho_{|\psi\rangle} \right] \\
&= \frac{1}{4} \mathrm{Tr}\left[\begin{pmatrix} \sigma_\alpha & -\sigma_\alpha \\ -\sigma_\alpha & \sigma_\alpha \end{pmatrix} \right] \\
&= \frac{1}{2} \mathrm{Tr}[\sigma_\alpha] \\
&= 0
\end{aligned}
\tag{1.21}
$$

as the Pauli matrices have zero trace. Analogously, $\mathbb{E}[\sigma_\beta] \equiv \mathbb{E}_{|\psi\rangle}[\mathbb{I} \otimes \sigma_\beta] = 0$. For the product measurement, one finds

$$\mathbb{E}[\sigma_\alpha \otimes \sigma_\beta] \equiv \mathbb{E}_{|\psi\rangle}[\sigma_\alpha \otimes \sigma_\beta] = \text{Tr}[\sigma_\alpha \otimes \sigma_\beta \, \rho_{|\psi\rangle}]$$
$$= -\alpha \cdot \beta \tag{1.22}$$

so, when Alice and Bob both measure in the same direction, it holds that

$$\mathbb{E}[\sigma_\alpha \otimes \sigma_\alpha] = -|\alpha|^2 = -1. \tag{1.23}$$

As the possible outcomes are ± 1, this statement concludes when measuring in the same direction the product of the outcome will always be -1. Accordingly, if Alice observes measurement outcome 1, Bob observes -1 and vice versa.

1.4   Research question

As mentioned in the introduction, the EPR-experiment violates one of the following properties

  – Locality: information can not travel faster than the speed of light;

  – Realism: observable are deterministic—i.e., for a property of an object, equal measurements under equal conditions on this property yield equal outcomes. These properties are embedded as 'variables' in the objects and may or may not be directly observable (hidden).[*]

The mathematical formalism of quantum mechanics described in the previous sections indeed suggests either one of these principles is violated. The electron with spin state (1.18) has two different possible measurement (non-realism) outcomes and the expected value of (1.23) seemingly concludes two electrons interact simultaneously (non-local).

However, these violations may be solely a result of the mathematical formalism and not of the underlying physics we measure on. For example, the physics behind the state of (1.18) may be describing not one electron, but a box of ten electrons with *nine* having an electron spin in the positive *x*-direction and *one* in the negative *x*-direction—measuring this state actually means randomly receiving one of these electrons from the box and *then* measuring the electron spin deterministically. The probabilities would then fit the ones calculated in the quantum formalism. Likewise, the result of (1.23) can be explained by introducing two pairs of two inseparable electrons (a left and right one): one pair with the left having an electron spin in the positive *z*-direction and the right one

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[*]An example of a hidden variable is *mass*. It is not possible to directly observe the mass of an object—only indirect inferences are possible by measuring the gravitational *weight*, for example.

in the negative *z*-direction; and the other pair equally but the left and right interchanged. Then, the measurement could again mean randomly receiving one of these pairs and measuring the spins.

Both examples are mathematical formulations that yield the same result as the quantum formalism's, but *do* instill locality and realism. Theories that follow these properties are called *local hidden variable theories* with the corresponding mathematical constructions called *local hidden variable models* (LHV-model).*

An interesting question arises whether all results in quantum mechanics, including the EPR's, can be formulated as a LHV-model. This would deem the previously explained quantum formalism obsolete and show quantum mechanics is in fact *local* and *realistic*. However, in 1964 the physicist John Stewart Bell proved such a model does not exist, concluding that quantum mechanics is inherently non-local and/or non-realistic.[2]

In this research, I analyze to what degree LHV-models *can* explain QM. One question is central: what is the highest amplitude $p > 0$ for which a LHV-model can reproduce the quantum measurement statistics

$$E[\sigma_\alpha \otimes \sigma_\beta] = -p\alpha \cdot \beta \tag{1.24}$$

with

$$\mathbb{E}[\sigma_\alpha] = \mathbb{E}[\sigma_\beta] = 0. \tag{1.25}$$

We analyze whether there exists a critical value p_c such that LHV-models exist for all $p \leq p_c$ and definitely not for all $p > p_c$.

Chapter 2 contains the mathematical analysis of this research and Chapter 3 the conclusions and discussion.

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*All realistic theories are defined by (hidden) variables and vice versa—therefore the word realistic is substituted for *hidden variable*. *Local* refers to the *locality* principle.

2

Local hidden variable theories

In this chapter, we introduce the mathematical construct of local hidden variable theory as discussed in the previous section and show a main result in this field: Bell's theorem. Also, we analyze how, and to what degree, LHV-models can reproduce the measurement statistics found in the EPR-experiment.

We start off with the mathematical construct.

2.1 Mathematical construct

In the previous chapter, we discussed LHV-models as having the properties *locality*— information cannot travel faster than the speed of light—and realism—observables are deterministic. Fortunately, these two requirements lead to a general (and easy) mathematical construct that imbed these properties precisely: all LHV-models can be written such way. In this construct, we only need two things: measurement functions

$$
\begin{aligned}
&A, B : S^d \times \Omega \to \{\pm 1\} \\
&\quad A : (\alpha, \omega) \mapsto A(\alpha, \omega) \in \{\pm 1\} \\
&\quad B : (\beta, \omega) \mapsto B(\beta, \omega) \in \{\pm 1\}
\end{aligned}
\tag{2.1}
$$

where S^d is the surface of a $(d+1)$-dimensional sphere* and a probability space

$$
(\Omega, \mathcal{F}, \mathbb{P}_\Omega)
\tag{2.2}
$$

with the sample set Ω, the σ-algebra event set $\mathcal{F}$, and the probability function $\mathbb{P}_\Omega : \mathcal{F} \to [0,1]$. In this thesis, it is often unnecessary—or clear from context—to explicitly

---

*We define $S^d \equiv \{v \in \mathbb{R}^{d+1} \mid \|v\| = 1\}$. Note that S^2 was also previously defined.

mention $\mathcal{F}$. It can be the power set of Ω or another applicable σ-algebra. Also, $\mathbb{P}_\Omega$ is often omitted in exchange for the more useful density measure p_Ω.

The idea is that two observers, Alice and Bob, both choose a measurement setting α and β for their measurement devices and nature randomly selects an element in $\omega \in \Omega$ as the *object* they will measure. Clearly, when the measurement settings and the ω stay the same, the measurement outcome will also, so the formalism is deterministic (realistic). Also, the system is local as functions A and B do not interact with each other—A does not depend on β and B not on α. If measurements are repeated and give different outcomes this is either because the measurement settings have changed (α and β) or a different object, ω_i, has been given.

In the most general framework, we allow communication between Alice and Bob so their selection of ω is equal during their measurements. This is a justified addition, because the underlying physics may be organized in such way Alice and Bob receive the same object ω_i. This does not violate locality or realism. Also, two separate probability spaces can always be combined in one space with elements that describe both. *

An easy example is measuring the width, depth, or height of three-dimensional objects. We define four different objects Alice and Bob can measure. They cannot select an object freely, but have the equal probability $\frac{1}{4}$ of measuring a particular one. Alice and Bob always measure the same object, but may differ in what direction they measure: this is notated by Alice's options $\alpha_x, \alpha_y, \alpha_z$ and similarly $\beta_x, \beta_y, \beta_z$ for Bob. The probability space (2.2) is then defined by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ the set of the four objects and $\mathbb{P}(\omega_i) = \frac{1}{4}$ for all i. We then have Alice's measurement outcome $A(\alpha_j, \omega_i)$, defined by (2.1), as the size of object ω_i in the j-direction and analogously for Bob. This model is local as Alice's measurement does not influence Bob's and vice versa, and it is realistic as a repeated measurement on the same object yields the same result.

2.2 Bell's theorem

As mentioned in the previous chapter, Bell's theorem shows no LHV-model exists that can fully reproduce the measurement statistics of the quantum formalism—i.e.,

$$\mathbb{E}[A(\alpha)] = \mathbb{E}[B(\beta)] = 0 \text{ and}$$
$$\mathbb{E}[A(\alpha)\,B(\beta)] = -\alpha \cdot \beta. \tag{2.3}$$

---

*Most generally: $\Omega = \Omega_A \times \Omega_B = \left\{ \omega = \begin{pmatrix} \omega_A \\ \omega_B \end{pmatrix} \mid \omega_1 \in \Omega_A,\ \omega_B \in \Omega_B \right\}$ and $p_{AB}(\omega) = p_A(\omega_A)p_B(\omega_B)$.

Theorem 1 (Bell's theorem (1964) [2],[5]*)**.** *Quantum mechanics cannot be formulated as a LHV-model.*

Proof. Let two electrons be in the two-electron singlet state

$$|\psi\rangle \cong \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right] \tag{2.4}$$

with the corresponding measurements statistics ((1.21) and (1.22)). Assume there exists a LHV-model which reproduces the statistics—i.e., measurement functions A, B and a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\Omega)$ described by equations 2.1 and 2.2 and having expected values as 2.3.

Then, let $\alpha, \alpha', \beta, \beta' \in S^d$ be four random measurement settings and define

$$\begin{aligned} S &= A(\alpha)\, B(\beta) - A(\alpha)\, B(\beta') + A(\alpha')\, B(\beta) + A(\alpha')\, B(\beta') \\ &= A(\alpha)\, \left(B(\beta) - B(\beta')\right) + A(\alpha')\, \left(B(\beta) + B(\beta')\right) \end{aligned} \tag{2.5}$$

and note either

$$|\, B(\beta) - B(\beta')\,| = 0 \text{ and } |\, B(\beta) + B(\beta')\,| = 2 \tag{2.6}$$

or

$$|\, B(\beta) - B(\beta')\,| = 2 \text{ and } |\, B(\beta) + B(\beta')\,| = 0 \tag{2.7}$$

as the functions only output ± 1. Combined with the fact that $|\, A(\alpha)\,|, |\, A(\alpha')\,| = 1$, we find:

$$|S| \leq 2 \tag{2.8}$$

and the so-called *CHSH-inequality*

$$\mathbb{E}\left[|S|\right] \leq 2. \tag{2.9}$$

Now we define our four measurement settings $\alpha, \alpha', \beta, \beta'$ such that they lie in a plane and can be parameterized by a single angle. The angles we choose are $\theta_\alpha = 0$, $\theta_{\alpha'} =$

---

*While John Stewart Bell was the first to prove this theorem, a different, more common prove is given here, using the *CHSH-inequality*. [5]

$\frac{\pi}{2}$, $\theta_\beta = \frac{\pi}{4}$ and $\theta_{\beta'} = \frac{3\pi}{4}$. Then we have,

$$
\begin{aligned}
\mathbb{E}[|S|] &= \mathbb{E}\left[A\left(\theta = 0\right)\right]\mathbb{E}\left[B\left(\theta = \frac{\pi}{4}\right)\right] - \mathbb{E}\left[A\left(\theta = 0\right)\right]\mathbb{E}\left[B\left(\theta = \frac{3\pi}{4}\right)\right] \\
&\quad + \mathbb{E}\left[A\left(\theta = \frac{\pi}{2}\right)\right]\mathbb{E}\left[B\left(\theta = \frac{\pi}{4}\right)\right] + \mathbb{E}\left[A\left(\theta = \frac{\pi}{2}\right)\right]\mathbb{E}\left[B\left(\theta = \frac{3\pi}{4}\right)\right] \\
&= \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) \\
&= 2\sqrt{2}
\end{aligned}
$$
(2.10)

using the EPR-measurement statistics and the known relation $\alpha \cdot \beta = \cos\theta$.
Now note that $2\sqrt{2} > 2$, concluding the CHSH-inequality (2.9) is violated and thus the LHV-model is not well-defined. In conclusion, no LHV-model can reproduce the measurements statistics of the quantum state. □

Bell's theorem elegantly shows that no LHV-model can yield the same measurement statistics as the EPR-experiment. This means, quantum mechanics can be—as is the case with the EPR-experiment—*intrinsically* non-local and/or non-realistic.

A different question is, however, whether similar correlations can be reproduced by a LHV-model. *Specifically of interest*, is finding a $p_c \in \mathbb{R}$ such that for all positive values $p \leq p_c$ there exists a LHV-model with

$$
\mathbb{E}[A(\alpha)B(\beta)] = -p\,\alpha \cdot \beta.
$$
(2.11)

Note, from Bell's theorem (1), we know $p_c \leq \frac{1}{\sqrt{2}}$—the *CHSH-bound*.

This chapter is organized in few sections where we explore this question. The main results of this chapter is

$$
p_c(d) = \frac{1}{K_G(d+1)}
$$
(2.12)

using the real Grothendieck constant and the measurement setting dimension d.

2.3   Quantum separability

The first method we use is to introduce separable quantum density matrices.

Definition 1. *A density matrix ρ_{AB} is called* separable *if it can be written as*

$$
\begin{aligned}
\rho_{AB} &= \sum_i p_i \, \rho_{i,A} \otimes \rho_{i,B} \\
&\equiv \sum_i p_i |\psi_{i,A}\rangle\langle\psi_{i,A}| \otimes |\psi_{i,B}\rangle\langle\psi_{i,B}|
\end{aligned}
\tag{2.13}
$$

for $p_i \geq 0$, $\sum_i p_i = 1$ and $|\psi_{i,A}\rangle, |\psi_{i,B}\rangle \in \{|0\rangle, |1\rangle\}$.

Werner [6] and Terhal [7] proved separability is a sufficient requirement for a LHV-model to exist that reproduces the same expected value outcomes as the quantum model.

Definition 2. *A quantum state is called* local *if there exists a LHV-model that reproduces the state's measurement statistics—i.e.,*

$$
\begin{aligned}
E_\rho[\sigma_\alpha \otimes \sigma_\beta] &= \mathbb{E}_M[\mathrm{A}(\alpha)\,\mathrm{B}(\beta)] \\
E_\rho[\sigma_\alpha \otimes \mathbb{I}] &= \mathbb{E}_M[\mathrm{A}(\alpha)] \\
E_\rho[\mathbb{I} \otimes \sigma_\beta] &= \mathbb{E}_M[\mathrm{B}(\beta)]
\end{aligned}
\tag{2.14}
$$

for a quantum density matrix ρ and LHV-model M.

Lemma 1. *Two-electron states with separable density matrices are local.*

Sketch of proof. Write ρ_{AB} in the form of equation 2.13. From the values p_i, define a probability space with the corresponding events $\rho_{i,A} \otimes \rho_{i,B}$. These events individually can always be further modelled by a clever mathematical construction; depending on the measurements. [7] See Model 1 as an example of such an construction. □

So, finding a separable quantum density matrix implies its expected outcomes can be reproduced by a LHV-model. The next theorem gives a general result for separable 2×2 density matrices.

Theorem 2. *Let ρ be a separable two-qubit density matrix.*
If $\mathbb{E}_\rho[\sigma_\alpha] = \mathbb{E}_\rho[\sigma_\beta] = 0$ and $\mathbb{E}_\rho[\sigma_\alpha \otimes \sigma_\beta] = \gamma \, \alpha \cdot \beta$ for all $\alpha, \beta \in S^2$ and for a $\gamma \in \mathbb{R}$, then $-\frac{1}{3} \leq \gamma \leq \frac{1}{3}$.

Proof. We prove the theorem by introducing a general density matrix ρ, applying the expected value constraints, and then using a necessary and sufficient requirement for the separability for ρ.

We first note that the the set of Pauli matrices together with the identity matrix form a real basis for the single-qubit density matrices. Namely, all ρ_A can be written as:

$$\rho_A = \frac{1}{2}[\mathbb{I}_2 + \sigma \cdot \delta] \tag{2.15}$$

for a $\delta \in \mathbb{R}^3$ with $\|\delta\| \leq 1$ and $\sigma = \begin{pmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{pmatrix}$ (1.13). [8]

Using the two-qubit formalism, this means, using some abuse by notation, that $B \times B = \{\sigma_i \otimes \sigma_j \mid \sigma_{i,j} \in \{\sigma_0, \sigma_1, \sigma_2, \sigma_3\} \equiv \{\mathbb{I}, \sigma_x, \sigma_y, \sigma_z\}\}$ forms a basis for the two-qubit density matrices—i.e.,

$$\rho = \sum_{i,j=0}^{3} R_{i,j}\, \sigma_i \otimes \sigma_j \tag{2.16}$$

for all ρ and appropriate 4×4 real matrix R.

Based on this, we can find an expression for the joint correlations for certain $s, t \in \{0, 1, 2, 3\}$:

$$
\begin{aligned}
\mathbb{E}_\rho[\sigma_s \otimes \sigma_t] = \mathrm{Tr}[(\sigma_s \otimes \sigma_t)\rho] &= \sum_{i,j=0}^{3} R_{i,j}\, \mathrm{Tr}[(\sigma_s \otimes \sigma_t)(\sigma_i \otimes \sigma_j)] \\
&= \sum_{i,j=0}^{3} R_{i,j}\, \mathrm{Tr}[\sigma_s\sigma_i \otimes \sigma_t\sigma_j] \\
&= \sum_{i,j=0}^{3} R_{i,j}\, \mathrm{Tr}[\sigma_s\sigma_i]\, \mathrm{Tr}[\sigma_t\sigma_j] \\
&= \sum_{i,j=0}^{3} R_{i,j}\, 4\delta_{s,i}\delta_{t,j} \\
&= 4R_{s,t}
\end{aligned}
\tag{2.17}
$$

by linearity of the trace and tensor product properties.

Note that clearly $\mathbb{E}_\rho[\mathbb{I} \otimes \mathbb{I}] = 1$, so $R_{0,0} = \frac{1}{4}$. Also, the expected value constraints $\mathbb{E}_\rho[\sigma_\alpha] = \mathbb{E}_\rho[\sigma_\beta] = 0$ for all directions α and β are exactly met when $\mathbb{E}_\rho[\sigma_i \otimes \mathbb{I}] = \mathbb{E}_\rho[\mathbb{I} \otimes \sigma_i] = 0$ for all $1 \leq i \leq 3$, concluding $R_{i,0} = R_{0,i} = 0$ for same i.

We now rotate matrix R to a more practical one. Ben-Aryeh et al. [9] show for density matrices in the form of 2.16 with the above expected values, there always exists a vector

basis $\{v_1, v_2, v_3\}$ for $\mathbb{R}^3$ such that

$$\rho = \frac{1}{4}\mathbb{I} \otimes \mathbb{I} + \sum_{i=1}^{3} \overline{R_i}\, \sigma_{v_i} \otimes \sigma_{v_j}. \tag{2.18}$$

in our standard notation $\sigma_{v_i} = v_i \cdot \sigma$. As this set of v_i is a basis, there exists a transformation matrix M such that

$$M^\top \sigma_{\lambda_i} M = \sigma_i \tag{2.19}$$

for all i.[*] We then define

$$
\begin{aligned}
\overline{\rho} &\equiv (M \otimes M)\rho(M \otimes M)^\top \\
&= (M \otimes M)\rho(M^\top \otimes M^\top) \\
&= (M \otimes M)\left(\frac{1}{4}\mathbb{I} \otimes \mathbb{I} + \sum_{i=1}^{3} \overline{R_i}\, \sigma_{\lambda_i} \otimes \sigma_{\lambda_i}\right)(M^\top \otimes M^\top) \\
&= \frac{1}{4}\mathbb{I} \otimes \mathbb{I} + \sum_{i=1}^{3} \overline{R_i}\left(M\sigma_{\lambda_i}M^\top\right) \otimes \left(M\sigma_{\lambda_i}M^\top\right) \\
&= \frac{1}{4}\mathbb{I} \otimes \mathbb{I} + \sum_{i=1}^{3} \overline{R_i}\, \sigma_i \otimes \sigma_i.
\end{aligned}
\tag{2.20}
$$

In other words, $\overline{\rho}$ is the rotated density matrix with respect to the new basis.

Now note, that density transformations of the form

$$\rho \mapsto (M_A \otimes M_B)\rho(M_A \otimes M_B)^\dagger \tag{2.21}$$

preserve separability as

$$
\begin{aligned}
(M_A \otimes M_B)\rho(M_A \otimes M_B)^\dagger &= (M_A \otimes M_B)\left(\sum_i p_i\, \rho_A \otimes \rho_B\right)(M_A \otimes M_B)^\dagger \\
&= \sum_i p_i\left(M_A\rho_A M_A^\dagger\right) \otimes \left(M_B\rho_B M_B^\dagger\right)
\end{aligned}
\tag{2.22}
$$

is again a sum of pure state density matrices. [†]

---

[*]Operator σ_{λ_i} is a 3×3 matrix, so M also. As M is a transformation matrix, it is also by definition invertible.

[†]This can be checked by noting $\mathrm{Tr}\left[M_A\rho_A M_A^\dagger\right]^2 = 1$. [10]

Hence, ρ is separable if and only if $\overline{\rho}$ is. Using 2.17, we find for all measurement settings on $\overline{\rho}$:

$$
\begin{aligned}
\mathbb{E}_{\overline{\rho}}[\sigma_\alpha \otimes \sigma_\beta] &= \mathbb{E}_{\overline{\rho}}\left[\left(\sum_{i=1}^{3} a_i\sigma_i\right) \otimes \left(\sum_{i=1}^{3} b_j\sigma_j\right)\right] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \, \mathbb{E}_{\overline{\rho}}[\sigma_i \otimes \sigma_j] \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \, \overline{R_i}\delta_{i,j} \\
&= \sum_{i=1}^{3} a_i b_i \overline{R_i}.
\end{aligned}
\tag{2.23}
$$

Then, inserting the theorem's prerequisite

$$
\mathbb{E}_{\overline{\rho}}[\sigma_\alpha \otimes \sigma_\beta] = \gamma\, \alpha \cdot \beta
\tag{2.24}
$$

leads to

$$
\overline{R_i} = \gamma
\tag{2.25}
$$

so,

$$
\overline{\rho} = \frac{1}{4}\left[\mathbb{I} \otimes \mathbb{I} + \gamma(\sigma_1 \otimes \sigma_1 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3\right].
\tag{2.26}
$$

Changing from braket to array notation, the tensor product is applied as the Kronecker product. With this, we can explicitly calculate the tensor products of the Pauli matrices and find

$$
\begin{aligned}
4\overline{\rho} = \mathbb{I}_4 + \gamma &\left(\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \\
= &\begin{pmatrix} 1+\gamma & 0 & 0 & 0 \\ 0 & 1-\gamma & 2\gamma & 0 \\ 0 & 2\gamma & 1-\gamma & 0 \\ 0 & 0 & 0 & 1+\gamma \end{pmatrix}
\end{aligned}
\tag{2.27}
$$

Now that we have the general, explicit expression for density matrices with our measurement requirements, we need only to look when it is separable. To do this, we apply the Peres-Horodecki criterion [11]. For this, we calculate the partial transpose matrix

ρ^{PT} of ρ defined as the matrix of transposed 2-by-2 block elements of ρ and look at its eigenvalues. We find

$$4\rho^{PT} = \begin{pmatrix} 1+\gamma & 0 & 0 & 2\gamma \\ 0 & 1-\gamma & 0 & 0 \\ 0 & 0 & 1-\gamma & 0 \\ 2\gamma & 0 & 0 & 1+\gamma \end{pmatrix} \tag{2.28}$$

which has eigenvalues $\Omega_{\rho^{PT}} = \left\{ \frac{1-3\gamma}{4}, \frac{1-\gamma}{4}, \frac{1-\gamma}{4}, \frac{1+3\gamma}{4} \right\}$. According to the criterion, as the dimension of this space is 2×2, ρ is separable if and only if all the eigenvalues are non-negative which is the case if and only if $-\frac{1}{3} \leq \gamma \leq \frac{1}{3}$.

In conclusion, under the expected value restrictions we posed, only for these values of γ, $\bar{\rho}$ is separable and thus ρ. □

A slightly different reasoning of the separability of 2.27 is also given. Ben-Aryeh et al. also show for these states, the density matrix can be expressed as

$$4\bar{\rho} = \sum_{i=1}^{3} 2|\gamma| \left[\left(\frac{\mathbb{I}-\sigma_i}{2} \otimes \frac{\mathbb{I} - \text{sign}(\gamma)\sigma_i}{2} \right) \right. $$
$$\left. + \left(\frac{\mathbb{I}+\sigma_i}{2} \otimes \frac{\mathbb{I} + \text{sign}(\gamma)\sigma_i}{2} \right) \right] + \frac{\mathbb{I} \otimes \mathbb{I}}{4} 4(1 - 3|\gamma|) \tag{2.29}$$

Then, because for all $i \in \{1, 2, 3\}$,

$$\frac{\mathbb{I} \pm \sigma_i}{2} = |j_\pm\rangle\langle j_\pm| \tag{2.30}$$

is the density matrix of an eigenvector $(|j_\pm\rangle)$ of one of the Pauli matrices, and thus a pure state, 2.29 is a weighted sum of pure states.[*] The total weight is $\frac{1}{4}(12|\gamma| + 4 - 12|\gamma|) = 1$. Hence, if these weights are all positive, $\bar{\rho}$ is separable according to the definition which occurs only if $|\gamma| \leq \frac{1}{3}$.

This method of finding an exact expression for the separability sum (2.13) is useful to find *sufficient* requirements—such as $|\gamma| \leq \frac{1}{3}$—but provides no *necessary* requirements as the density matrix may be also written as a sum of other pure states. However, these sums do help out when creating an explicit LHV-model, see model 1 in section 2.4.4.

---

[*]This is also verified by the fact that $\text{Tr}\left[\left(\frac{\mathbb{I}+\sigma_i}{2} \right)^2 \right] = 1$. [10]

2.4 Werner states

Most often, however, the locality of models is done only on a specific quantum state: the Werner state.

Definition 3. *The two-electron Werner state is a quantum state described by the density matrix*

$$\rho_{W_p} = p|\psi^-\rangle\langle\psi^-| + \frac{1-p}{4}\mathbb{I}_4 \tag{2.31}$$

for $0 \le p \le 1$ and $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ the singlet state.

A Werner state has the advantage that its measurement statistics is comparable to that of the singlet state.

Lemma 2. *The Werner states ρ_{W_p} have expected outcomes*
$\mathbb{E}_{\rho_{W_p}}[\sigma_\alpha \otimes \sigma_\beta] = -p\,\boldsymbol{\alpha}\cdot\boldsymbol{\beta}$ *for all $\alpha, \beta \in S^2$.*

Proof. We calculate,

$$\begin{aligned}
\mathbb{E}[\sigma_\alpha\sigma_\beta] &= \mathrm{Tr}[\sigma_\alpha \otimes \sigma_\beta\, \rho_{W_p}] \\
&= p\,\mathrm{Tr}[\sigma_\alpha \otimes \sigma_\beta|\psi^-\rangle\langle\psi^-|] + \frac{1-p}{4}\mathrm{Tr}[\sigma_\alpha \otimes \sigma_\beta\,\mathbb{I}_4] \\
&= -p\,\alpha\cdot\beta.
\end{aligned} \tag{2.32}$$

using $\mathbb{E}[\sigma_\alpha \otimes \sigma_\beta] = -\alpha\cdot\beta$ for the *singlet state* and the fact that the spin measurements have zero trace. $\qquad\square$

2.4.1 Separability $\left(p \le \frac{1}{3}\right)$

Using the density matrix of the singlet state (1.20), we find

$$\rho_{W_p} = \frac{p}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1-p}{4}\mathbb{I}_4 \tag{2.33}$$

$$= \frac{1}{4}\begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & p+1 & -2p & 0 \\ 0 & -2p & p+1 & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}. \tag{2.34}$$

This is the density matrix (2.27) seen in the previous section. So, from Theorem 2, we find ρ_{W_p} is separable if and only if $|p| \le \frac{1}{3}$ and thus has a LHV-model with equal measurement statistics for these values by Theorem 1.

Unfortunately, Theorem 1 does not include a constructive proof with an exact model. However, inspired by the separability expression (2.29), as mentioned, we can construct a LHV-model with same correlations as this density matrix.

2.4.2 Visibility and locality $\left(p \geq \frac{3}{4}\right)$

A different analysis worth highlighting is the one done by Kaszlikowski and Żukowski [12]. They prove Bell's theorem (theorem 1) by introducing the construct of an inner product, define probability functions P_{QM} and P_{LHV}, corresponding to quantum mechanical models and LHV-models, and show

$$\langle P_{LHV}|P_{QM}\rangle < \|P_{QM}\|^2 \tag{2.35}$$

implying $P_{LHV} \neq P_{QM}$. With this construction, they prove the EPR-experiment cannot be reproduced by a LHV-model. We give a short outline of their construction and relate it to our Werner models.

In the context of the EPR-experiment, the functions P_{QM} and P_{LHV} are defined as

$$P_{QM}, P_{LHV} : \{\pm 1\}^2 \times \left(S^d\right)^2 \to [0, 1] \tag{2.36}$$

with $P_{QM}(m, m', \alpha, \beta)$ the probability of observing measurement outcomes m and m' for spin measurement settings α and β. P_{LHV} is defined analogously for a LHV-model, not further specified. The inner product is defined as an integration over all the possible measurement settings α and β and the measurement outcomes $m, m' \in \{\pm 1\}$:

$$\langle f|g\rangle \equiv \sum_{m,m'\in\{\pm 1\}} \iint d\Omega_a \, d\Omega_b \, f(m, m'; \alpha, \beta)g(m, m'; \alpha, \beta) \tag{2.37}$$

where Ω_a, Ω_b denote all the measurement settings by spherical parameters.

This inner product gives the following relations

$$\begin{aligned} \|P_{QM}\|^2 &\equiv \langle P_{QM}|P_{QM}\rangle \\ &= \frac{16}{3}\pi^2 \end{aligned} \tag{2.38}$$

and

$$\langle P_{QM}|P_{LHV}\rangle \leq 5\pi^2. \tag{2.39}$$

This shows equation (2.35) is always held, for all possible LHV-models, concluding quantum mechanics cannot be reproduced by any LHV-model.

More interesting, is their consideration of *visibility*. In real life experiments, the visibility tells us how often the apparatus register measurements correctly and not noise. This is relevant for experiments that observe signals continuously—at a set time interval, for example—rather than registering signals only after it has been produced. In case of the former, when misregistered noise is added to the data, the measurement statistics become skewed and data should be treated differently.

We let $V \in [0,1]$ denote this visibility with $V = 1$ indicating no noise is measured and $V = 0$ solely noise is measured—i.e., no correct measurements are done at all.

Taking visibility into account, Kaszlikowski and Żukowski show the following relations hold:

$$\left\| P_{QM} \right\|^2 = (2\pi)^2 \left(1 + \frac{V^2}{3} \right) \tag{2.40}$$

and

$$\langle P_{QM} | P_{LHV} \rangle \leq (2\pi)^2 \left(1 + \frac{V}{4} \right). \tag{2.41}$$

In this case, we have

$$\langle P_{LHV} | P_{QM} \rangle < \left\| P_{QM} \right\|^2 \tag{2.42}$$

if and only if $V > \frac{3}{4}$. This means, when is assured that the measurement apparatus correctly measure 75%, or more, of the data entries and the data follows the EPR's measurement statistics, it is proven this data could not have been generated by a LHV-model. For situations where this visibility is less, this conclusion cannot be made and the found statistics may be due to random effects of the noise.

Making the comparison to Werner states, this visibility directly translates to the the parameter p of these states. This is because, simply put, looking at its definition, a parameter p implies a $1 - p$ probability of measuring the density matrix $\rho = \mathbb{I} \otimes \mathbb{I}$ which works as noise. In other words, a Werner state ρ_{W_p} with parameter p is equivalent to measuring a singlet state electron pair with visibility $V = p$. Therefore, this analysis proves that for $p \geq \frac{3}{4}$, the state ρ_{W_p} is non-local. An interesting approach, albeit not a new insight as the CHSH-bound proved the same for $p \geq \frac{1}{\sqrt{2}} \approx 0.71$.

2.4.3   Grothendieck constant $\left(p \leq \frac{1}{K_G(d+1)} \right)$

We find separability gives a partial answer to what types of density matrices can be reproduced by LHV-models and also an intuition (Lemma 1) how to find an explicit model. However, there is no reason to believe that a non-separable [*] density matrix

---

[*]Also called an *entangled* density matrix

cannot be reproduced by LHV-model either.

This section gives a sharper answer to our question using the interesting mathematical Grothendieck constant. First we introduce this constant and then formulate a theorem using this number.

Definition 4. *For all $d \geq 2$, the (real) Grothendieck constant $K_G(d)$ in dimension d is the smallest contant with the property: if M is a $m \times m$ real matrix with*

$$\left| \sum_{i,j=1}^{m} M_{ij} s_i t_j \right| \leq 1 \tag{2.43}$$

for all real numbers $|s_i|, |t_j| \leq 1$, then

$$\left| \sum_{i,j=1}^{m} M_{i,j} \langle \mathbf{s}_i, \mathbf{t}_j \rangle \right| \leq K_G(d) < \infty \tag{2.44}$$

for all vectors $\mathbf{s}_i, \mathbf{t}_j$ in the unit ball of a d-dimensional Hilbert space $\mathcal{H}$, i.e. $\|\mathbf{s}_i\|, \|\mathbf{t}_j\| \leq 1$.

So, the Grothendieck constant relates the size of a matrix when multiplied by real numbers, to the size when multiplied by an inner product—if the matrix is small enough for *real numbers* in inequality 2.43, it will similarly be so for *inner products* in equation 2.44.

Only for $d = 2$, the constant is known to have an exact expression—for the other dimensions there are approximate upper and lower bound (table 2.1).

d	$K_G(d) \geq$	$K_G(d) \leq$
2	$\sqrt{2}$	$\sqrt{2}$
3	1.4359	1.4644
4	1.4841	1.5708
5	1.46112	
6	1.47017	
7	1.47583	
8	1.47972	1.6641
9	1.48608	
∞	1.67696	1.78221

Table 2.1: *Lower and upper bounds of the real Grothendieck constant for different dimensions.[*] $K_G(\infty)$, often written as K_G, is defined as $\lim_{d \to \infty} K_G(d)$.*

Toner et al. ([19],[20]) have shown the following remarkable result.[*]

Theorem 3. *A Werner state ρ_{W_p} is local if and only if $p \leq p_c = \frac{1}{K_G(d+1)}$ with d the dimension of the measurement settings.*

Before we prove this, we introduce the following lemma and make a subsequent definition.

Lemma 3. *For a LHV-model M with corresponding measurement functions* A *and* B, *there exists a LHV-model $\widetilde{M}$ with the same measurement functions and*
$\mathbb{E}_{\widetilde{M}}[A(\alpha)\,B(\beta)] = \mathbb{E}_M[A(\alpha)\,B(\beta)]$ *and* $\mathbb{E}_{\widetilde{M}}[A(\alpha)] = \mathbb{E}_{\widetilde{M}}[B(\beta)] = 0.$

Proof. Introducing a model $\widetilde{M}$, it is always possible to extend the elements $\omega \in \Omega$ with an extra random variable that changes the sign of the measurement value of both $A(\alpha)$ and $B(\beta)$ with probability $\frac{1}{2}$. Then, the measurement outcomes of the product $A(\alpha)\,B(\beta)$ remain the same and as both the positive and negative outcomes are measured with equal probability, we get $\mathbb{E}_{\widetilde{M}}[A(\alpha)] = \mathbb{E}_{\widetilde{M}}[B(\beta)] = 0$. □

In short, when constructing a LHV-model M reproducing the measurement statistics of a Werner state, we only need to ensure the correlations are equal, i.e. $\mathbb{E}_M[A(\alpha)B(\beta)] = \mathbb{E}_{\rho_{W_p}}[\sigma_\alpha \otimes \sigma_\beta] = -p\alpha \cdot \beta$ and need not look at the requirements
$\mathbb{E}_M[A(\alpha)] = \mathbb{E}_{\rho_{W_p}}[\sigma(\alpha)] = 0.$

To prove the theorem, we also need to define *Bell inequalities*.[†]

Definition 5. *For LHV-models, Bell inequalities are general statements that bound the sum of expected values of measurement values. All LHV-models satisfy these inequalities.*

An example of a Bell inequality is the aforementioned CHSH-inequality used in the proof of Bell's theorem. This inequality states all LHV-models satisfy

$$\left| \mathbb{E}\left[A(\alpha)\,B(\beta)\right] - \mathbb{E}\left[A(\alpha)\,B(\beta')\right] + \mathbb{E}\left[A(\alpha')\,B(\beta)\right] + \mathbb{E}\left[A(\alpha')\,B(\beta')\right] \right| \leq 2 \tag{2.45}$$

for all measurement settings $\alpha, \alpha', \beta, \beta' \in S^d$ with A, B the measurement functions of the model.

The Bell inequalities can also be written in the form of quantum formalism. For example, the CHSH-inequalitiy transforms to

$$\left| \mathbb{E}_\rho\left[\sigma_\alpha \otimes \sigma_\beta\right] - \mathbb{E}_\rho\left[\sigma_\alpha \otimes \sigma_{\beta'}\right] + \mathbb{E}_\rho\left[\sigma_{\alpha'} \otimes \sigma_\beta\right] + \mathbb{E}_\rho\left[\sigma_{\alpha'} \otimes \sigma_{\beta'}\right] \right| \leq 2. \tag{2.46}$$

---

[*][13] ($d = 2$, upper bounds $d = 4, 8, \infty$); [14] (lower bounds $d = 3, 4$); [15] (upper bound $d = 3$); [16] (lower bounds $d = 5, 6, 7, 8$); [17] (lower bound $d = 9$); [18] (lower bound $d = \infty$)

[*]Up until now, we solely considered the case of $d = 2$: measurement settings defined as points on the unit sphere. Following the definition of S^d, for $d = 1$ we restrict settings to a *circle* on this sphere. Higher values of d are considered in section 3.2.

[†]*Bell inequalities* are defined more technically—our definition will suffice for this thesis.

 It is clear that local quantum states need to satisfy these transformed quantum states. Furthermore, these inequalities form a sufficient requirement for locality.

Lemma 4 ([21])*. The set of all Bell inequalities transformed to apply for a quantum states form a necessary and sufficient requirement for locality of quantum states.*

A special class of Bell inequalities are the *correlation* Bell inequalities which only include expected values of *product* measurements and not the local measurements—i.e. only $\mathbb{E}[A(\alpha)\,B(\beta)]$ and not $\mathbb{E}[A(\alpha)], \mathbb{E}[B(\beta)]$ are include in the sum. Due to lemma 3, we can restrict our focus solely to *correlation* Bell inequalities as LHV-models with wanted correlations can always be made into a model with zero local expected values.

Using this definition and lemma 3, we can now prove theorem 3.

Proof of Theorem 3. ([19],[20]) We start off with giving a mathematical description of Bell inequalities. Correlation Bell inequalities in their most general form can be written as

$$\left| \sum_{i,j=1}^{m} M_{i,j}\mathbb{E}[A(\alpha_i)B(\beta_j)] \right| \leq 1 \tag{2.47}$$

where M is a $m \times m$ matrix and $\{(\alpha_i, \beta_j) \mid i,j \in \{1,\ldots,m\}\}$ a set of m random measurement settings for A and B. For example, the CHSH-inequality is described by $m = 2$, $M_{1,2} = -\frac{1}{2}$, and $M_{i,j} = \frac{1}{2}$ for all the other i, j.

We can further normalize M in such way that 2.47 holds and also

$$\max_{\{a_i, b_j\} \in \{\pm 1\}^{2m}} \left| \sum_{i,j=1}^{m} M_{i,j}a_ib_j \right| = 1. \tag{2.48}$$

Then for all real numbers $|a_i|, |b_j| \leq 1$, we have

$$\left| \sum_{i,j=1}^{m} M_{i,j}a_ib_j \right| \leq 1 \tag{2.49}$$

which allows us to apply the definition of Grothendieck constant:

$$\left| \sum_{i,j=1}^{m} M_{i,j}\,\mathbf{s}_i \cdot \mathbf{t}_j \right| \leq K_G(d+1) \tag{2.50}$$

for all $\mathbf{s}_i, \mathbf{t}_j \in \mathbb{R}^{d+1}$ with $\|\mathbf{s}_i\|, \|\mathbf{t}_j\| \leq 1$.[*]

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[*]Following, we denote this norm restriction as $\in \mathbb{R}^{d+1}_{\leq 1}$.

As the Grothendieck constant is independent of m and M_m, and as the inequality is an upper bound for the (absolute) sum, we can take the supremum of *all* possible Bell inequalities of size m while maximizing over the measurement settings and (still) find[*]

$$\sup_{M_m} \max_{\mathbf{s}_i, \mathbf{t}_j \in \mathbb{R}^{d+1}_{\leq 1}} \left| \sum_{i,j=1}^{m} M_{i,j}\, \mathbf{s}_i \cdot \mathbf{t}_j \right| \leq K_G(d+1). \tag{2.51}$$

This bound also allows us to take the limit of m to infinity. As (2.51) is increasing in m and $K_G(d+1)$ is the smallest value with the property (2.50), we find equality as

$$\lim_{m \to \infty} \sup_{M_m} \max_{\mathbf{s}_i, \mathbf{t}_j \in \mathbb{R}^{d}_{\leq 1}} \left| \sum_{i,j=1}^{m} M_{i,j}\, \mathbf{s}_i \cdot \mathbf{t}_j \right| = K_G(d+1). \tag{2.52}$$

With this, we can prove our theorem. First, let $\rho \equiv \rho_{W_p}$ be a Werner state with parameter p. According to lemma 4, ρ is local if and only if inequality (2.47)—transformed to quantum formalism—is satisfied for the previously defined α_i, β_j:

$$\left| \sum_{i,j=1}^{m} M_{i,j} \mathbb{E}[\sigma_{\alpha_i} \otimes \sigma_{\beta_j}] \right| \leq 1. \tag{2.53}$$

We need not enforce the statistics $\mathbb{E}[\sigma_\alpha] = \mathbb{E}[\sigma_\beta] = 0$ for all α, β, because lemma 2 makes these requirements obsolete.

Following, we find

$$1 \geq \left| \sum_{i,j=1}^{m} M_{i,j} \mathbb{E}[\sigma_{\alpha_i} \otimes \sigma_{\beta_j}] \right| = p \left| \sum_{i,j=1}^{m} M_{i,j} \alpha_i \cdot \beta_j \right| \tag{2.54}$$

and using (2.52), we have

$$1 \geq \lim_{m \to \infty} \sup_{M_m} \max_{\alpha_i, \beta_j \in \mathbb{R}^{d}_{\leq 1}} p \left| \sum_{i,j=1}^{m} M_{i,j} \alpha_i \cdot \beta_j \right| = p K_G(d+1) \tag{2.55}$$

from which follows

$$p \leq p_c \equiv \frac{1}{K_G(d+1)}. \tag{2.56}$$

<div align="right">□</div>

---

[*]The supremum over the measurement settings is indeed a maximum as the $d+1$-dimensional unit sphere is a compact set.

In other words, this theorem shows the critical value of p for which no possible Bell-inequality is violated—regardless of the number of measurement values and even letting it go to infinity—is fixed at a (somewhat) known expression. Consequently, LHV-models for these Werner state exist.

2.4.4 Models

After the theoretical analyses of the previous sections, here we introduce some explicit examples of LHV-models with interesting measurement statistics. Following lemma 2, we are only interested in the product expected values.

Model 1. *Let (Ω, p_Ω) define a probability space with*

$$\Omega = \left\{ \begin{pmatrix} f \\ s \\ \lambda \\ \delta \end{pmatrix} \mid f \in \{1,2,3\}, \ s \in \{\pm 1\} \right\} \text{ and } \lambda, \delta \in \left[-\tfrac{1}{2}, \tfrac{1}{2} \right] \right\} \text{ the event set and } p_\Omega \begin{pmatrix} f \\ s \\ \lambda \\ \delta \end{pmatrix} = $$

$\frac{1}{6}$ *the probability density. Define measurements*

$$A : S^2 \times \Omega \to \{\pm 1\}$$
$$\left(\alpha, \begin{pmatrix} f \\ s \\ \lambda \\ \delta \end{pmatrix} \right) \mapsto \text{sign}\left(\lambda + \frac{1}{2} s \alpha_f \right) \tag{2.57}$$

and

$$B : S^2 \times \Omega \to \{\pm 1\}$$
$$\left(\beta, \begin{pmatrix} f \\ s \\ \lambda \\ \delta \end{pmatrix} \right) \mapsto \text{sign}\left(\delta - \frac{1}{2} s \beta_f \right) \tag{2.58}$$

with $\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ and $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$.

Lemma 5. *Model 1 has expected outcomes $\mathbb{E}[A(\alpha)] = 0$ and correlation $\mathbb{E}[A(\alpha) B(\beta)] = -\frac{1}{3} \boldsymbol{\alpha} \cdot \boldsymbol{\beta}$.*

Proof.

$$\mathbb{E}[A(\alpha)] = \int_\Omega \mathbb{P}(\omega)\, A(\alpha, \omega)\, d\omega$$

$$= \sum_f \sum_s \iint_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{P}\begin{pmatrix} f \\ s \\ \lambda \\ \gamma \end{pmatrix} A\left(\alpha, \begin{pmatrix} f \\ s \\ \lambda \\ \gamma \end{pmatrix}\right) d\lambda\, d\delta$$

$$= \frac{1}{6} \sum_f \sum_s \int_{-\frac{1}{2}}^{\frac{1}{2}} \text{sign}\left(\lambda + \frac{1}{2}s\,\alpha_f\right) d\lambda \qquad (2.59)$$

$$= \frac{1}{6} \sum_f \sum_s \left[\int_{-\frac{1}{2}s\alpha_f}^{\frac{1}{2}} (1)\, d\lambda + \int_{-\frac{1}{2}}^{-\frac{1}{2}s\alpha_f} (-1)\, d\lambda \right]$$

$$= \frac{1}{6} \sum_f \sum_s s\alpha_f = \frac{1}{6} \sum_f [\alpha_f - \alpha_f] = 0$$

and similarly $\mathbb{E}[B(\beta)] = 0$. Following, we have

$$\mathbb{E}[A(\alpha)\, B(\beta)] = \int_\Omega \mathbb{P}(\omega)\, A(\alpha, \omega)\, B(\beta, \omega)\, d\omega$$

$$= \sum_f \sum_s \iint_{-\frac{1}{2}}^{\frac{1}{2}} \mathbb{P}\begin{pmatrix} f \\ s \\ \lambda \\ \gamma \end{pmatrix} A\left(\alpha, \begin{pmatrix} f \\ s \\ \lambda \\ \gamma \end{pmatrix}\right) B\left(\beta, \begin{pmatrix} f \\ s \\ \lambda \\ \gamma \end{pmatrix}\right) d\lambda\, d\delta$$

$$= \frac{1}{6} \sum_f \sum_s \iint_{-\frac{1}{2}}^{\frac{1}{2}} \text{sign}\left(\lambda + \frac{1}{2}s\,\alpha_f\right) \text{sign}\left(\gamma - \frac{1}{2}s\,\beta_f\right) d\lambda\, d\delta \qquad (2.60)$$

$$= \frac{1}{6} \sum_f \sum_s s\alpha_f\, (-s)\beta_f$$

$$= -\frac{1}{6} \sum_f \sum_s \alpha_f \beta_f = -\frac{1}{3} \sum_f \alpha_f \beta_f = -\frac{1}{3}\alpha \cdot \beta.$$

$\square$

We give a short geometrical interpretation of this model. From equation (2.29), you see the Werner state with this correlation ($\gamma = -\frac{1}{3}$) is the sum of six pure state density matrices: three different *sets* and per set two sign versions. In the LHV-model, we model that when measuring, you are in fact randomly given one of these six pure state density matrices on which you do a spin measurement. This is constructed by introducing random variables f and s with probabilities equal to the weight of the density matrices

in (2.29), namely $\frac{1}{6}$. The other random variables λ and δ and measurement functions are then defined in such a way to simulate the measurements on the individual pure states. The probability distribution of λ and δ and the parameters of the functions are found by experimenting.

Toner [19] introduces models close to the limit of $p \leq \frac{1}{K_G(d+1)}$.

Model 2 ([19]). *Let* $(\Omega, \mathbb{P})$ *define a probability space with*

$$\Omega = \left\{ \lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \end{pmatrix} \in \mathbb{R}^\infty \right\} \text{ and } p_\Omega : \lambda \sim \mathcal{N}(0, \mathbb{I}) \text{ the infinite-dimensional multivariate}$$

normal distribution. Using the given functions $\widetilde{A}, \widetilde{B} : S^2 \to l^\infty$ *in [19], we define*

$$A, B : S^2 \times \Omega \to \{\pm 1\}$$
$$A : (\alpha, \lambda) \mapsto \text{sign}\left(\widetilde{A}(\alpha) \cdot \lambda\right) \tag{2.61}$$
$$B : (\beta, \lambda) \mapsto \text{sign}\left(\widetilde{B}(\beta) \cdot \lambda\right)$$

Theorem 4 ([19]). *Model 2 has correlation* $\mathbb{E}[A(\alpha) B(\beta)] \approx -0.6595 \, \alpha \cdot \beta$.

Compare this amplitude with $-0.696 \leq -p_c = -\frac{1}{K_G(3)} \leq -0.683$.

Toner also introduces a model for measurement settings with $d = 1$. For this, we first define some functions.

Let $f, g : [-\pi, \pi) \to [-1, 1]$ with

$$f(t) = \begin{cases} 1 & \text{if } |t| \in \left[0, \frac{\pi}{4}\right] \\ 3\left(1 - \frac{2t}{\pi}\right) - 4\left(1 - \frac{2t}{\pi}\right)^3 & \text{if } |t| \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \\ -f(\pi - |t|) & |t| \in \left[\frac{\pi}{2}, \pi\right] \end{cases} \tag{2.62}$$

and

$$g(t) = \text{sign}(\cos(t)). \tag{2.63}$$

Let $(c_{2k+1})_{k \in \mathbb{N}}$ and $(d_{2k+1})_{k \in \mathbb{N}}$ denote the Fourier coefficients of f, g, i.e. such that $f(t) = \sum_{k=0}^\infty c_{2k+1} \cos(2k+1)t$ and $g(t) = \sum_{k=0}^\infty d_{2k+1} \cos(2k+1)t$.

Then define sequences

$$a_{2k+1} = (-1)^k \frac{2}{\pi(2k+1)} c_{2k+1} \text{ and}$$
$$b_{2k+1} = -\frac{1}{a_1} \sum_{d|(2k+1), d \neq 1} a_d b_{(2k+1)/d}. \tag{2.64}$$

From this we define the following model.

Model 3 ([19]). *Let (Ω, p_Ω) define a probability space with*

$$\Omega = \{2n+1 \mid n \in \mathbb{N}\} \times [-\pi, \pi) \times \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \text{ and } p_\Omega \begin{pmatrix} 2n+1 \\ t \\ \lambda \end{pmatrix} = \tfrac{1}{2\pi}|b_{2n+1}|. \text{ We define}$$

measurements

$$
\begin{aligned}
&\mathrm{A}, \mathrm{B} : [-\pi, \pi) \times \Omega \to \{\pm 1\} \\
&\mathrm{A} : (x, \omega) \mapsto -\operatorname{sign}(b_{2n+1}) g[(2n+1)(x-t)] \\
&\mathrm{B} : (y, \omega) \mapsto \operatorname{sign}\left(\frac{1}{2}f[(2n+1)(t-y)] - \lambda\right).
\end{aligned}
\tag{2.65}
$$

Theorem 5. *Model 3 has correlation* $\mathbb{E}[\mathrm{A}(x)\,\mathrm{B}(y)] = -\frac{1}{\sqrt{2}}\cos(x-y)$

Proof. By the probability space we find

$$\mathbb{P}\begin{pmatrix} \cdot \\ \cdot \\ \lambda \end{pmatrix} = \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} \frac{1}{2\pi}|b_{2n+1}|\,dt = \sum_{n=0}^{\infty} |b_{2n+1}| = 1 \tag{2.66}$$

which shows λ is uniformly random. Continuing, we have a marginal expected value

$$
\begin{aligned}
\mathbb{E}[\mathrm{B}(y, (2n+1, t, \cdot))] &= \int_{-\frac{1}{2}}^{\frac{1}{2}} B(y, (2n+1, t, \lambda))\,d\lambda \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \operatorname{sign}\left(\frac{1}{2}f[(2n+1)(t-y)] - \lambda\right)d\lambda \\
&= f[(2n+1)(t-y)]
\end{aligned}
\tag{2.67}
$$

similar to an earlier calculation (2.60). Then [19] proves the result. $\square$

Surprisingly, this model's correlations has an amplitude exactly the critical value $p_c = \frac{1}{K_G(2)} = \frac{1}{\sqrt{2}}$. The theoretical existence of a LHV-model proved by theory 3 is now made explicit.

3

Conclusion and discussion

3.1 Conclusion

In the previous sections, different methods are explored to find the locality of quantum states and approximate p_c. Specifically, the analysis of Werner states and theorem 3 give the most powerful insight to this question and allow us to focus mainly on Grothendieck constant. For $d = 1$, the constant's exact known value gives a sharp answer.

In this section, we make conclusions of our findings. In the following section we pose open questions and further research.

3.1.1 $d = 1$

For $d = 1$, figure 3.1 summarizes the findings from theorem 3. In this figure a number of graphs are shown, each representing the product correlation function for a certain Werner state ρ_{W_p} with θ the angle between the two measurement settings α and β. The solid black and red line represent the correlation of Werner states with respective parameters $p = 1$ and $p = \frac{1}{\sqrt{2}}$ with the area in between colored red.

As $p_c = \frac{1}{\sqrt{2}}$ from theorem 3, we conclude that Werner states with product correlation functions in the red area ($p > p_c$) are local and those in the white below ($p \leq p_c$) are not. For example, the Werner state with the green, dashed product correlation function can be reproduced by a LHV-model ($p \approx 0.2$), but the one with the blue, dashed correlation cannot ($p \approx 0.8$).

Note that this bound p_c is surprisingly the same as the CHSH-bound in Bell's theorem (1). The main difference between these values: the CHSH-bound works on *all* LHV-

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models, and p_c is only defined in the context of LHV-models for the quantum mechanical *Werner states*. However, as these values are equal—seemingly coincidentally and very elegantly, we can conclude that LHV-models with

$$\mathbb{E}[A(\alpha) B(\beta)] = -p \, \alpha \cdot \beta \tag{3.1}$$

exist for all $0 \leq p \leq \frac{1}{\sqrt{2}}$ and do not exist for $p \geq \frac{1}{\sqrt{2}}$. Even stronger, model 3 constructs it for $p = p_c$.
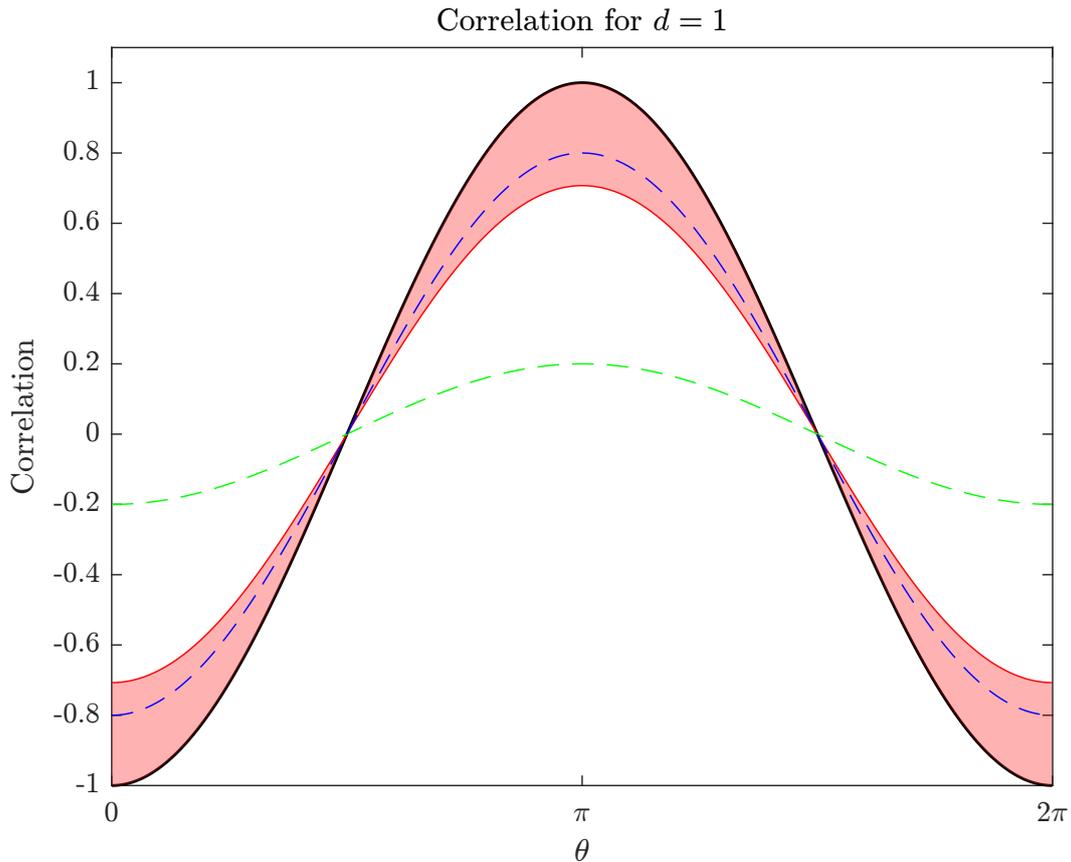

Figure 3.1: *Product correlations of Werner states with the measurement settings in S^1 with angle θ in between them. The red area represents correlations not reproducible by LHV-models (e.g. blue line) and the white area below vice versa (e.g. green line).*

3.1.2 $d = 2$

For $d = 2$, the conclusion is less sharp. In figure 3.2, you can see the results for this dimension. The same colors represent the maximum amplitude (black, $p = 1$) and the two test correlation (blue and green)—the CHSH-bound of $\frac{1}{\sqrt{2}}$ is the dashed black

line. Zooming in, this becomes better visible (figure 3.2b). The red lines, dashed and non-dashed, correspond to the most accurate up and lower bound approximations of $p_c = \frac{1}{K_G(3)}$, respectively.

Just as with $d = 1$, all the cosine correlations in the white (non-red) area correspond to amplitudes $p < p_c$. This means the Werner states with those amplitudes are local by theorem 3 and thus a LHV-model can reproduce their measurement statistics. For amplitudes above the dashed red line, so for $p > p_c$, the respective *Werner states* are not local.

Now lets look at the area between the upper bound of p_c and the CHSH-bound. Cosine correlations above the CHSH-bound can never be reproduced by any LHV-model, but below this bound and above the upper bound of p_c, only the non-locality of respective Werner states is proved by theorem 3. This means no conclusion can be made about all *LHV-models* in general. It is possible there is a LHV-model, not related to a Werner state, that has its correlation in this area.
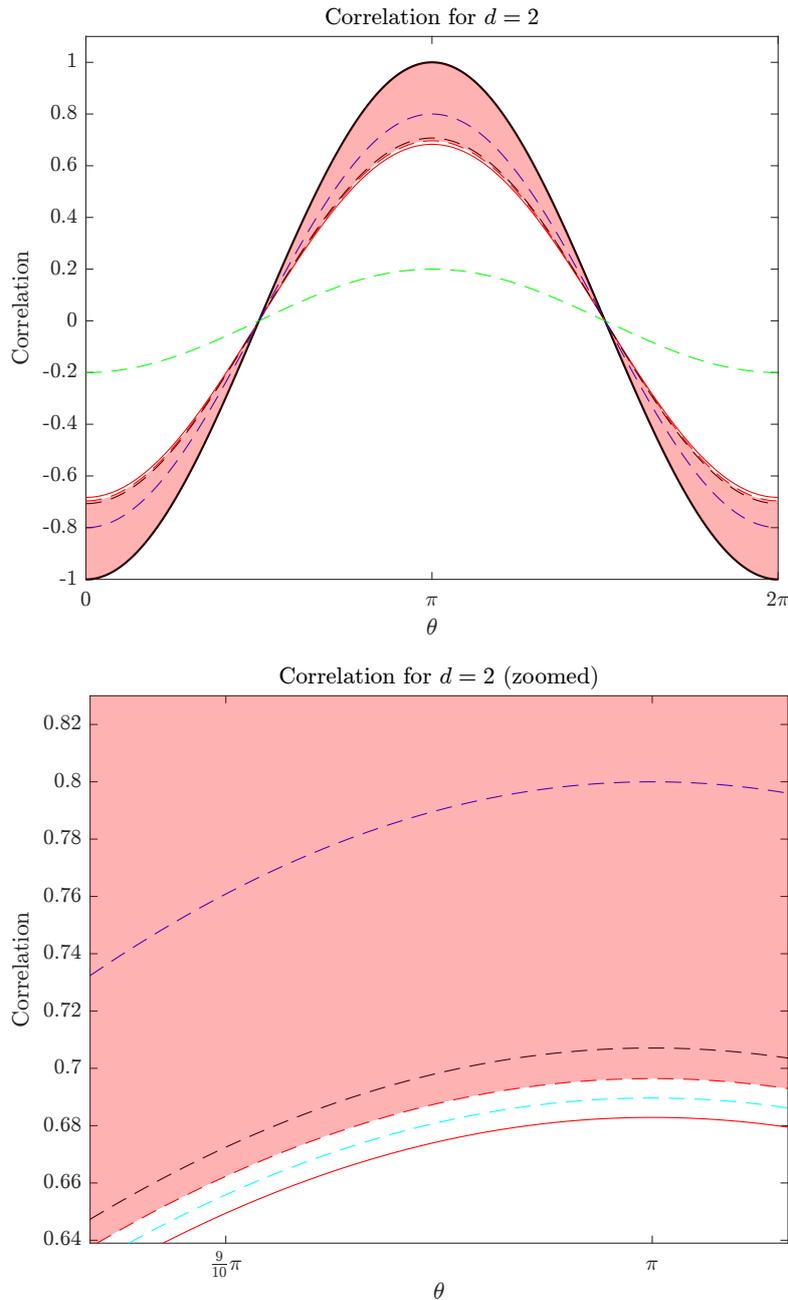
(b) Zoomed-in. The black and red lines indicate the CHSH-bound, the upper bound, and lower bound of p_c, respectively. The CHSH-bound and p_c are different and the approximation of p_c causes a gap.

Figure 3.2: *Product correlations of Werner states with the measurement settings in S^2 with angle θ in between them. The red area represents non-local Werner states (e.g. blue line). The area between the CHSH-bound (black, dashed) and the upper bound of p_c may be reproduced by a LHV-model.*

3.2   Further research

In the conclusion, we only consider the situation of $d = 1$ and $d = 2$ and look at the results of theorem 3 and the CHSH-bound. However, more options can be explored with corresponding open questions.

Non-quantum locality

All the methods in the previous sections work from *quantum-to-locality* principle: one introduces a quantum state with some favorable properties, such as Werner states, and gives an analysis when such a state is local. The advantage is that clear expressions for these quantum states exist which often simplify necessary calculations. For example, the Werner states are simply expressed by their density matrix (1.20) and their correlations are easily calculated ((1.21) and (1.22)). A disadvantage is non-trivial locality requirements such as Theorem 3.

A different approach, however, is to disregard the *quantum states* altogether. One can try to make a LHV-model that has measurement statistics similar to EPR's without constructing it from a local quantum state. The main argument for this approach is that it is more general and in some ways more natural.

To explain this, for Werner states we can conclusively say which parameters allow for local models and which not. For those that are local, there definitely exist LHV-models— albeit possibly not constructed (yet)—and these may come close to CHSH-bound. However, this does not make a conclusion for *all* possible LHV-models. There might be a LHV-model for measurement dimension $d = 2$ with correlation similar to $-\frac{1}{\sqrt{2}}\alpha \cdot \beta$. The only statement Theorem 3 makes, is that it cannot be a local *Werner* state.

The easiest example is the triangle correlation generated by a model described by Bell [2]: a simple construction of the LHV-model without basing it on a quantum state.

A disadvantage of this approach is that it is difficult to make general conclusions. Looking at *all* the possible LHV-models is difficult, if not impossible.

Further research can be done into finding classes of cosine-like correlation functions, or different classes, that LHV-models can produce.

Increasing d

We only analyzed $d = 1$ and $d = 2$ in the previous sections, because these are the dimensions spin measurements are defined for. However, we can also consider general (projective) measurements of arbitrary high dimensions. Then the critical value $p_c(d) = \frac{1}{K_G(d+1)}$ still holds for Werner states [19] where $p_c(d)$ indicates $\alpha, \beta \in S^d$.

Comparing $d = 2$ and 3, $K_G(d)$ is increasing in d, but for higher dimensions this is more difficult to say—in part because no exact values of the constant are known (figure 4). These bounds, however, do give a slight suggestion the constant increases. The following lemma proofs it is indeed non-decreasing.

Lemma 6. *Grothendieck constant $K_G(d)$ is non-decreasing in d*

Proof. Assume the Grothendieck constant decreases between two subsequent dimensions—i.e. $K_G(d + 2) < K_G(d + 1)$ for a certain d. This implies from theorem 3, $p_c(d + 1) > p_c(d)$. Then, define a Werner state ρ_{W_p} with $p_c(d + 1) > p > p_c(d)$. The second inequality shows the state is non-local for measurement settings defined in S^d and the first inequality shows the state is local for measurements in S^{d+1}. Due to this locality, a probability space and measurements functions A, B exist (2.1, 2.2) taking measurement settings in S^{d+1}.

Now note, that there exists a natural imbedding from S^d to S^{d+1}, for example, by mapping vectors

$$S^d \ni v \mapsto \begin{pmatrix} v \\ 0 \end{pmatrix} \in S^{d+1}. \tag{3.2}$$

Using this imbedding, we define d-dimensional measurement functions $\widetilde{A}, \widetilde{B}$ by $\widetilde{A}(v, \omega) \equiv A\left(\begin{pmatrix} v \\ 0 \end{pmatrix}, \omega\right)$ and B analogously. These functions combined with the probability space form a LHV-model and its correlation is easily calculated:

$$\mathbb{E}[\widetilde{A}(\alpha)\widetilde{B}(\beta)] = \mathbb{E}\left[A\left(\begin{pmatrix} \alpha \\ 0 \end{pmatrix}\right) B\left(\begin{pmatrix} \beta \\ 0 \end{pmatrix}\right)\right]$$
$$= \mathbb{E}_{\rho_{W_p}}\left[\sigma_{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}} \otimes \sigma_{\begin{pmatrix} \beta \\ 0 \end{pmatrix}}\right] \tag{3.3}$$
$$= -p \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \beta \\ 0 \end{pmatrix}$$
$$= -p\alpha \cdot \beta.$$

This is the Werner's state correlation function, so the state is local for *d*-dimensional measurement settings. This is a contradiction to the previous, concluding that $K_G(d)$ is non-decreasing in *d*.   □
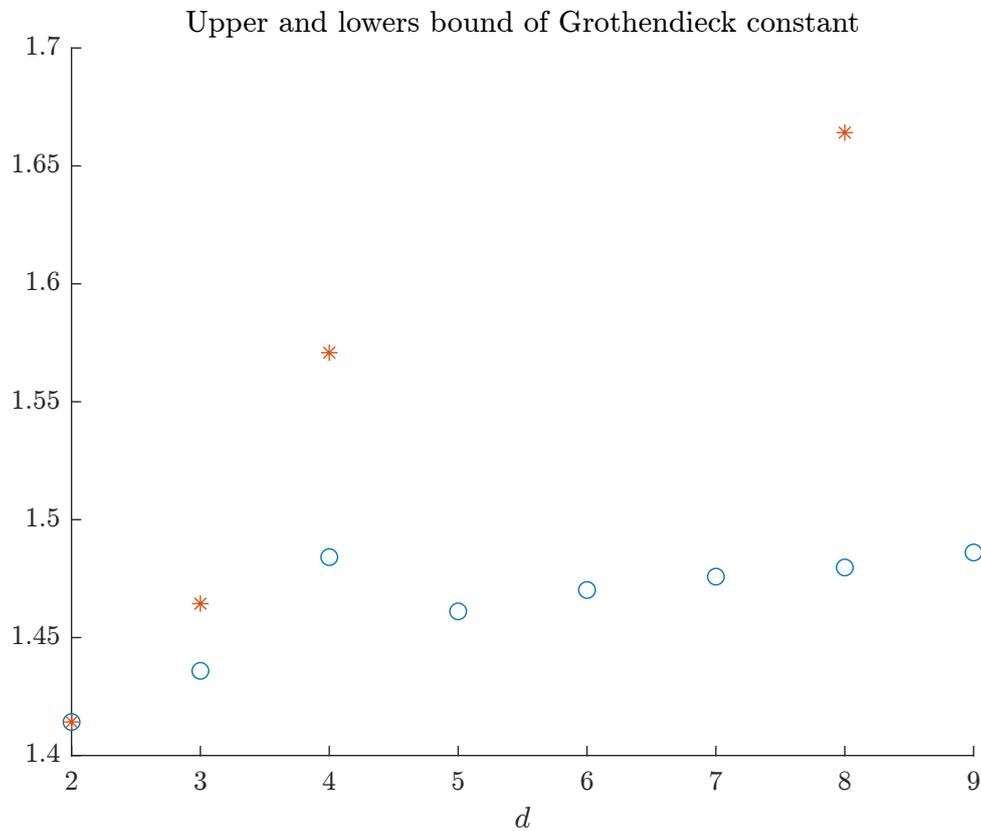

Figure 3.3: *Upper (red stars) and lower bounds (blue circles) of the Grothendieck constant for different dimensions. The non-monotone development may be due to different calculation methods. See table, 2.1.*

Different measurement space

In our analysis we only looked at measurement settings defined in S^d, the surface of a $(d+1)$–dimensional sphere. However, an open interesting question arises how the boundary value p_c changes if this measurement space is changed. For example, if we limit this set to a part of S^2, such as the upper hemisphere, can a different value of p_c be given? It seems logical to think a higher value of p_c then holds, because less measurement options makes it 'easier' for a LHV-model to reproduce the same statistics—as is the case with $p_c(1)$ and $p_c(2)$. A analysis on the so-called *Grasshopper problem* by Goulko and Kent [22] might give interesting insight into the effect of changing the measurement set.

3.3 Discussion

My research has been two-folded: a literature research into the current standing of the locality problem of *EPR-like* quantum states and my own research/mathematics into some problems often inspired from this literature.

My own research started off by trying to numerically find density matrices with expected values coming close to that of the EPR-experiment by using MATLAB. While doing literature research simultaneously, I found a basis of a more theoretical approach by analyzing general two-qubit density matrices and the realization that separable density matrices can always be reproduced by a LHV-model. After this, the Grothendieck constant and corresponding properties came to my knowledge and gave interesting results. After the following theorem (3) basically fully answered the research question, I went on to make conclusions and find more interesting research questions—some of which would have fit neatly here if time was not so brutal.

I want to thank my supervisors, Richard and Jordi, greatly for their support, guidance, and their seemingly infinite amount of time they were willing to spend on me. I hope this thesis can be of some use for people interested in the topic.

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