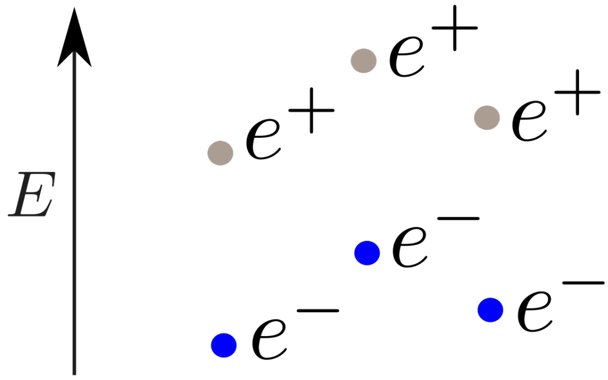
Back reaction of Schwinger pair production



THESIS

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Back reaction of Schwinger pair production

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Abstract

We compute the quantum back reaction of Schwinger pair production in the context of spinor QED. Using path integral and Schwinger's proper time methods, we employ a WKB approximation to find the current-current correlation function and relate it to the current expectation value in a manner analogous to fluctuation-dissipation theory. The same expression for the current expectation value in terms of the electric field is derived using point-splitting to cure UV divergences. We use this current value to compute the back reaction on the electric field via Maxwell's equation, obtaining a non-linear complex differential equation. A stable numerical solution is found, whose real part is the electric field. The field, initially just below the Schwinger limit, undergoes a sharp drop followed by exponential-like decay. Further analysis on the behavior of the current, production rate, and the energy density is performed using numerical integration.

Contents

Chapter 1

Introduction

One of the remarkable quantitative QFT predictions is the Schwinger production of charged particle-antiparticle fermionic pairs in a strong electric field. Schwinger calculated the production rate [1] by using what is now called the Schwinger proper time method. He used this method in the context of QED to find the imaginary part of the effective interaction Lagrangian $\mathcal{L}$, which translates directly to the production rate Γ: the probability per unit time and unit volume that an electron-positron pair is created by a constant electric field E. He found:

$$\Gamma = \frac{\alpha E^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-\frac{n\pi m^2}{eE}} \tag{1.1}$$

where $\alpha \approx \frac{1}{137}$ is the fine structure constant, m is the electron mass, and e is the electron charge. To first order, the production is suppressed by a factor of $\exp\left(-\pi m^2/eE\right)$. This means that pair production begins to be physically relevant when the field is close to the Schwinger limit value $eE \sim \pi m^2$. Thinking about this in terms of quantum tunneling is useful to get a qualitative picture. An electron can be modeled to inhabit a potential well that binds it to the positron. This binding potential is deformed by the electric potential eEx, as seen by the solid line in figure 1.1 taken from [2]. One can estimate the tunneling probability Γ_{tunnel} by considering the binding energy to be $V_0 \sim 2m$. Then one finds:

$$\Gamma_{tunnel} = e^{-\frac{4\sqrt{2mV_0}V_0}{3eE}} \sim e^{-c\frac{m^2}{eE}} \tag{1.2}$$

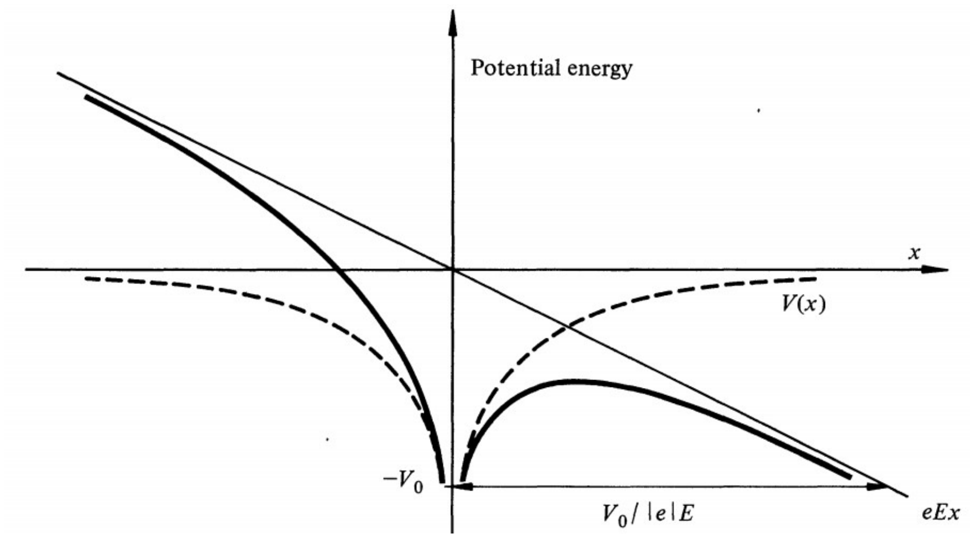

Figure 1.1: *Potential energy of an electron bound to a positron via a binding potential $V(x)$ (dashed line) in the presence of an electric potential eEx. [2]*

where c is some constant.* The tunneling picture shows a good qualitative picture for pair production and predicts the Schwinger limit.

Schwinger's production rate was obtained in a constant field approximation, but we know that the electron-positron pair will back-react on the field, causing it to decay. This is because creating massive particles will absorb energy from the field, and because the charged pairs will screen the field. Calculating the production rate while taking into account back-reaction, and predicting a self-consistent behavior of the field as a function of time, is a well known problem in theoretical physics that I address in this thesis.

Previous studies use WKB approximations in order to take back-reaction into account. Notably, the study by Mottola et.al [3] uses a WKB approximation that allows the study of the large momentum behavior when eE/ω^2 is small, where E is the electric field and ω is the energy of the produced fermions. Curiously, their study predicts oscillations of the field and current, with energy being transferred periodically between the two. On first glance, we expect their results to hold up for the initial conditions when E is near the critical value of $eE \approx \pi m^2$, but below the critical regime, and especially when $E \sim 0$, the energy of any produced particles will be small, and their approximation is no longer adequate. Moreover, after half a period of oscil-

*In this text, the speed of light and Plank's constant $c = \hbar = 1$.

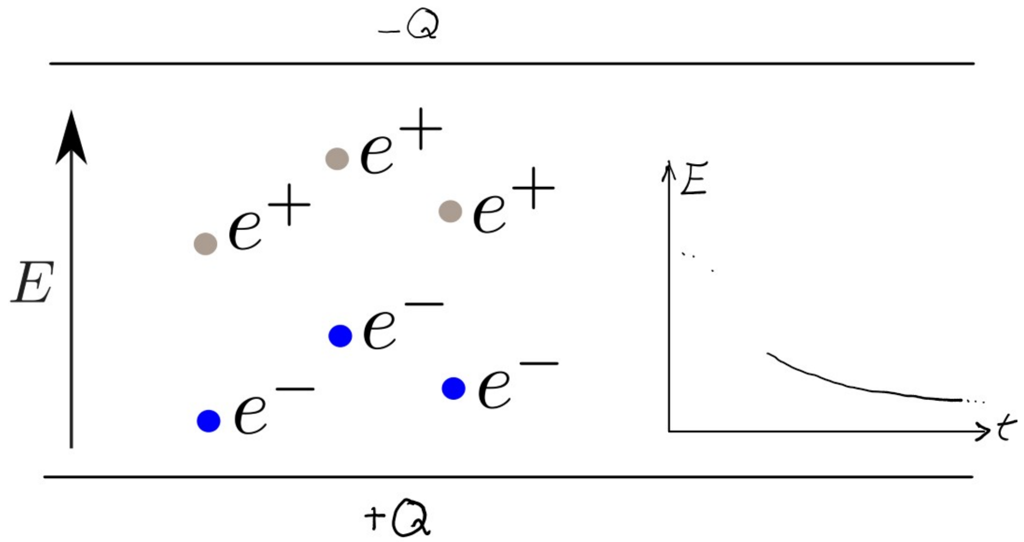

Figure 1.2: *System dynamics can be intuitively understood as a capacitor discharging by polarizing the vacuum.*

lation, when most of the energy of the system again resides in the electric field, the energy of fermions is small and indeed goes to zero while E remains finite.

Another reason we expect this to be the wrong description is that the produced fermions are only on-shell at spatial infinity. This can be understood in the tunneling picture presented earlier, where electrons can only tunnel to infinity when the field is small. Therefore, if and when the electric field goes to zero, giving all of its energy to the fermions, fermions of opposite charges are infinitely far apart and will not be attracted back to the origin to produce an oscillating field. A good way to gain more intuition is to think of a capacitor with its plates infinitely far apart. The electric field between the plates produces fermionic pairs that form a current. This current discharges the capacitor, with produced fermions annihilating opposing charges on the plates. The pair production rate falls exponentially as the electric field decays, and if we wait a very long time, the capacitor will discharge completely, leaving a vanishing electric field and zero current. For this reason, we expect the oscillation of the system to be overdamped. A visualization of this process can be seen in figure 1.2†. In this study, we use the WKB approximation to calculate the back-reaction on the electric field and probe the dynamics of the system. We compare our results to the intuition we have

†This figure and cover page figure from [4] established, especially for the late time behavior, and assess the consistency of our approximations with the results.

Chapter 2

Theoretical Framework

Schwinger's original treatment of the pair production problem involved the theory of Quantum Electrodynamics (QED), where we have Dirac electrons and positrons of mass m and charge $\pm e$, represented by the spinor ψ, interacting with a quantized electromagnetic field A_μ. We will use the same theory here.

2.1 QED

Greek indices count spacetime coordinates: $\mu \in \{0, 1, 2, 3\}$, while Latin indices run over spatial coordinates: $i \in \{1, 2, 3\}$. We can write the Dirac Lagrangian density: $\mathcal{L}_{Dirac} = \bar{\psi}(i\not{\partial} - m)\psi$, Maxwell's Lagrangian density: $\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor, and combine them with an interaction term $-j^\nu A_\nu = -e\bar{\psi}\not{A}\psi$ to get the QED Lagrangian and action:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - e\not{A} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} - j^\nu A_\nu \tag{2.1}$$

$$S = \int d^4x[\mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} - j^\nu A_\nu] = S_{Dirac} + S_{Maxwell} + S_I \tag{2.2}$$

where $S_I = -\int d^4x j^\nu A_\nu$ is the interaction action. The equations of motion read:

$$(i\not{\partial} - e\not{A} - m)\psi(x) = 0 \tag{2.3}$$

$$\partial_\mu F^{\mu\nu} = j^\nu = e\bar{\psi}\gamma^\nu\psi \tag{2.4}$$

The problem as originally formulated by Schwinger [1] considers a non-dynamic external field A_μ which generates an electric current whose expectation value can be obtained by the path integral:

$$\langle\psi(A)|\, j_\mu(x)\,|\psi(A)\rangle = \frac{i\delta}{\delta A^\mu(x)} \ln \int D[\bar{\psi},\psi] e^{iS_{Dirac}+iS_I} \tag{2.5}$$

$$\langle j_\mu(x)\rangle = \frac{i\delta}{\delta A^\mu(x)} \ln \mathcal{Z}_I[A] \tag{2.6}$$

As a first step we will do the same, but then substitute the current expectation value into the classical Maxwell's equation, restoring the dynamics of the field to compute the back-reaction. We will later discuss the validity of this approach within our approximation limits and check for self consistency.

2.2 Effective Lagrangian and proper time

Our aim is to compute (2.6). Note that this is not what Schwinger computed. If we define the Dirac operator: $\not{D} = \not{\partial} + ieA(x)$, we know [5] that the path integral can be evaluated as:

$$\mathcal{Z}_I[A] = \int D[\bar{\psi},\psi] e^{i\int dx^4 \bar{\psi}(i\not{D}-m)\psi} = N \det[i\not{D} - m] \tag{2.7}$$

where N is a normalization factor I will drop later.

$$\ln \mathcal{Z}_I[A] = \ln\det[i\not{D} - m] + \ln N \tag{2.8}$$

We can then use the property: $\ln\det[i\not{D} - m] = \mathrm{Tr}\,\mathrm{tr}\ln[i\not{D} - m]$ where Tr specifies tracing over Dirac gamma matrices and $\mathrm{tr}\big[\hat{O}\big] = \int dx^4 \langle x|\,\hat{O}\,|x\rangle$. We therefore have:

$$\ln \mathcal{Z}_I[A] = \int d^4x\, \mathrm{Tr}\,\langle x|\ln(i\not{D} - m)\,|x\rangle + \ln N \tag{2.9}$$

We can construct an effective interaction Lagrangian defined as $\mathcal{Z}_I[A] = e^{i\int dx^4 \mathcal{L}_I}$.

$$\mathcal{L}_I = -i\,\mathrm{Tr}\,\langle x|\ln(i\not{D} - m)\,|x\rangle \tag{2.10}$$

where we dropped the normalization factor because we can shift the Lagrangian by any number without affecting the physics. We can use the charge conjugation matrix: $C\gamma_\mu C^{-1} = -\gamma_\mu^T$ and the fact that the $\mathrm{Tr}\ln\hat{O}$ is invariant under transposition of $\hat{O}$ and under charge conjugation to write:

$$\mathcal{L}_I = -i\,\mathrm{Tr}\,\langle x|\ln(i\not{D} - m)\,|x\rangle = -i\,\mathrm{Tr}\,\langle x|\ln(-i\not{D} - m)\,|x\rangle \tag{2.11}$$

$$\mathcal{L}_I = -\frac{i}{2}\,\mathrm{Tr}\,\langle x|\ln\Big(\not{D}^2 + m^2\Big)\,|x\rangle \tag{2.12}$$

where in the last step we averaged the two expressions above. The logarithm is difficult to evaluate, but we can take a derivative with respect to m^2 to reach an easier expression:

$$\frac{d}{dm^2}\mathcal{L}_I = -\frac{i}{2}\operatorname{Tr}\langle x|\frac{1}{\not{D}^2+m^2}|x\rangle = \frac{i}{2}\operatorname{Tr}\langle x|\frac{1}{-\not{D}^2-m^2+i\epsilon}|x\rangle \tag{2.13}$$

where in the last step we use the Feynman prescription. We can now invoke the mathematical identity which allows Schwinger's proper time method:

$$\frac{1}{-\not{D}^2-m^2+i\epsilon} = -i\int_0^\infty ds e^{is(-\not{D}^2-m^2+i\epsilon)} \tag{2.14}$$

where s has dimensions of energy^{-2}, and is called the Schwinger proper time. Integrating back we have:

$$\mathcal{L}_I = \frac{i}{2}\operatorname{Tr}\langle x| -i\int dm^2\int_0^\infty ds e^{is(-\not{D}^2-m^2+i\epsilon)}|x\rangle \tag{2.15}$$

$$= \frac{i}{2}\operatorname{Tr}\langle x|\int_0^\infty \frac{ds}{s}e^{is(-\not{D}^2-m^2+i\epsilon)}|x\rangle \tag{2.16}$$

and therefore, with $\hat{p}_\mu = i\partial_\mu$ we can write:

$$\mathcal{L}_I = \frac{i}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2}\operatorname{Tr}\langle x|e^{-i\hat{H}s}|x\rangle \tag{2.17}$$

$$\hat{H} = -\not{D}^2 = -\Pi^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} \tag{2.18}$$

$$\Pi_\mu(\hat{x}) = \hat{p}_\mu - eA_\mu(\hat{x}) \tag{2.19}$$

The Feynman prescription $m^2 \to m^2 + i\epsilon$ will be restored when needed.

2.3 Current expectation value

We can now write the current expectation value from (2.6) in terms of $\mathcal{L}_I$:

$$j_\nu(y) = \frac{i\delta}{\delta A^\nu(y)}\ln \mathcal{Z}_I[A] = \frac{-\delta}{\delta A^\nu(y)}\int dx^4\mathcal{L}_I \tag{2.20}$$

$$= \frac{-i}{2}\int_0^\infty \frac{ds}{s}e^{-ism^2}\int dx^4\operatorname{Tr}\langle x|\frac{\delta}{\delta A^\nu(y)}e^{-i\hat{H}s}|x\rangle \tag{2.21}$$

The derivative is elaborated upon in appendix A and can be calculated as follows:

$$\frac{\delta}{\delta A^\nu(y)}e^{-i\hat{H}s} = i\int_0^s d\alpha e^{i\not{\Pi}^2\alpha}(\frac{\delta}{\delta A^\nu(y)}\not{\Pi}^2)e^{i\not{\Pi}^2(s-\alpha)} \tag{2.22}$$

$$\frac{\delta\hat{\not{\Pi}}^2}{\delta A^\nu(y)} = -e(\not{\Pi}\gamma_\nu\delta(\hat{x}-y) + \delta(\hat{x}-y)\gamma_\nu\not{\Pi}) \tag{2.23}$$

Note that the operator trace ($\int_x \langle x| \hat{O} |x\rangle$) does not allow us to cycle operators in this case: $\hat{p}$ is not a bounded operator and so trace cyclicity isn't a well-defined operation. As an example, think about the trace of the $[\hat{x}, \hat{p}]$ commutator: cyclicity makes it zero whereas we know that the commutator is $-i\hbar$. To simplify the expressions we turn the delta operator into a ketbra by inserting the identity $\delta(\hat{x} - y') = \int_x |x\rangle\langle x| \delta(\hat{x} - y') = |y'\rangle\langle y'|$. For the current we then find:

$$\begin{aligned} j_\nu(y) =& \frac{-e}{2} \int_0^\infty \frac{ds}{s} e^{-ism^2} \int dx^4 \,\mathrm{Tr} \\ & \langle x| \int_0^s d\alpha e^{-i\hat{H}\alpha} [\not{\Pi}\gamma_\nu |y\rangle\langle y| + |y\rangle\langle y| \gamma_\nu \not{\Pi}] e^{-i\hat{H}(s-\alpha)} |x\rangle \end{aligned} \tag{2.24}$$

$$= \frac{-e}{2} \int_0^\infty \frac{ds}{s} e^{-ism^2} \int_0^s d\alpha \,\mathrm{Tr} \langle y| e^{-i\hat{H}s} \not{\Pi}\gamma_\nu + \gamma_\nu \not{\Pi} e^{-i\hat{H}s} |y\rangle \tag{2.25}$$

$$= -e \int_0^\infty ds e^{-ism^2} \,\mathrm{Tr} \langle y| \gamma_\nu \not{\Pi} e^{-i\hat{H}s} |y\rangle \tag{2.26}$$

where in the last step we use the cyclicity of the trace over Dirac γ matrices and the fact that $\not{\Pi}$ commutes with the Hamiltonian.

2.4 WKB and evolving operators

We will need to know how to evolve the operator $\Pi(s)$ in Schwinger's proper time s with the Hamiltonian $\hat{H}(s)$. This we can find from the Heisenberg equation of motion:

$$\frac{d\hat{\Pi}_\mu}{ds} = i[\hat{H}, \hat{\Pi}_\mu] = 2eF_{\mu\nu}\hat{\Pi}^\nu + \frac{e}{2}\partial_\mu \sigma_{\alpha\beta} F^{\alpha\beta} - ie\partial_\nu F^{\nu\mu} \tag{2.27}$$

To move forward, we use a WKB approximation to neglect the last two terms. We consider the change of the electromagnetic field with time to be much less than the evolution of the fermionic state, this implies $\dot{E}/(E\omega) \ll 1$. We can thus consider $F_{\alpha\beta}$ to be time-independent and approximate its commutator with $\hat{p}$ to vanish. With these simplifications we have:

$$\frac{d\hat{\Pi}_\mu}{ds} = i[\hat{H}, \hat{\Pi}_\mu] = 2eF_{\mu\nu}\hat{\Pi}^\nu \tag{2.28}$$

$$\hat{\Pi}_\mu(s) = e^{iHs}\hat{\Pi}_\mu(0)e^{-iHs} = [e^{2es\mathbf{F}}]_{\mu\nu}\hat{\Pi}^\nu(0) \tag{2.29}$$

We can also find (using matrix notation):

$$\frac{d\mathbf{x}}{ds} = i[\hat{H}, \mathbf{x}] = 2\mathbf{\Pi} \tag{2.30}$$

This gives:

$$\mathbf{x}(s) = \mathbf{x}(0) + 2e^{es\mathbf{F}}\frac{\sinh(es\mathbf{F})}{e\mathbf{F}}.\mathbf{\Pi}(0) = \mathbf{x}(0) + \mathbf{N^{-1}}.\mathbf{\Pi}(0) \tag{2.31}$$

$$\mathbf{\Pi}(0) = e^{-es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}.[\mathbf{x}(s) - \mathbf{x}(0)] = \mathbf{N}.[\mathbf{x}(s) - \mathbf{x}(0)] \tag{2.32}$$

$$\mathbf{\Pi}(s) = e^{es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}.[\mathbf{x}(s) - \mathbf{x}(0)] = \mathbf{M}.[\mathbf{x}(s) - \mathbf{x}(0)] \tag{2.33}$$

Notice that one should be careful taking the limit $s \to 0$ in the above expressions. Equations (2.31) and (2.33) are sufficient to show that in the case of a constant field $F_{\alpha\beta}$ (see Schwinger's paper [1] or section 2-5-4 of Itzykson and Zuber [2]):

$$\begin{aligned}\langle y| e^{-i\hat{H}s} |x\rangle &= \frac{-i}{16\pi^2 s^2}\exp\Big[ie\int_x^y dz^\alpha\Big(A_\alpha(z) + \frac{1}{2}F_{\alpha\beta}(z^\beta - y^\beta)\Big)\Big] \\ &\times \exp\Big[i(\mathbf{y}-\mathbf{x})\frac{e\mathbf{F}}{4}\coth es\mathbf{F}(\mathbf{y}-\mathbf{x}) - i\frac{es}{2}\sigma_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\operatorname{tr}\ln[\frac{\sinh(es\mathbf{F})}{es\mathbf{F}}]\Big]\end{aligned} \tag{2.34}$$

and therefore

$$\langle x| e^{-i\hat{H}s} |x\rangle = \frac{-i}{16\pi^2 s^2}\exp\Big[-i\frac{es}{2}\sigma_{\alpha\beta}F^{\alpha\beta} - \frac{1}{2}\operatorname{tr}\ln[\frac{\sinh(es\mathbf{F})}{es\mathbf{F}}]\Big] \tag{2.35}$$

where $\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$. Since we wish to describe a uniform electric field in the z direction, we can take $F_{03} = E$. Then:

$$\exp\Big[-\frac{1}{2}\operatorname{tr}\ln[\frac{\sinh(es\mathbf{F})}{es\mathbf{F}}]\Big] = \exp\Big[\frac{1}{2}\ln\det\Big[\frac{es\mathbf{F}}{\sinh(es\mathbf{F})}\Big]\Big] \tag{2.36}$$

$$= \exp\Big[\frac{1}{2}\ln[\frac{esF_{03}}{\sinh(esF_{03})}\frac{esF_{30}}{\sinh(esF_{30})}]\Big] = \frac{esE}{\sinh esE} \tag{2.37}$$

A more elaborate calculation of matrix elements is presented in appendix B.1. Noting that $(\gamma^0\gamma^3)^2 = 1$, Taylor expanding the exponent we find:

$$e^{-i\frac{es}{2}\sigma_{\alpha\beta}F^{\alpha\beta}} = \gamma^0\gamma^3\sinh(esE) + \cosh(esE) \tag{2.38}$$

Combining the above we find:

$$\langle x| e^{-i\hat{H}s} |x\rangle = \frac{-ieE}{16\pi^2 s}[\gamma^0\gamma^3 + \coth(esE)] \tag{2.39}$$

Using this, the effective interaction Lagrangian in the WKB approximation reads:

$$\mathcal{L}_I = \frac{i}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2}\,\mathrm{Tr}\,\langle x|\, e^{-i\hat{H}s}\,|x\rangle \tag{2.40}$$

$$= \frac{eE}{8\pi^2}\int_0^\infty \frac{ds}{s^2} e^{-ism^2}\coth esE \tag{2.41}$$

This expression is divergent for $s \to 0$. The pole at $s = 0$ can be removed by minimal subtraction, a common treatment found in Schwinger's paper [1] whereby the minimal number of terms in a Taylor expansion are subtracted to do away with singular terms.

$$\mathcal{L}_I \to \frac{eE}{8\pi^2}\int_0^\infty \frac{ds}{s^2} e^{-ism^2}[\coth esE - \frac{1}{esE} - \frac{esE}{3}] \tag{2.42}$$

This Lagrangian can be shown to have an imaginary part, first computed by Schwinger, that translates to the production rate. To find it, note that the integrand of $\mathrm{Im}[\mathcal{L}_I]$ is even in s since $\mathrm{Im}[e^{-ism^2}]$ is odd. We can extend the integration from $[0,\infty)$ to $(-\infty,\infty)$ and divide by 2, then analytically continue s to the complex plane, and close the line integral from below so that the contribution of the half-circle is zero since $e^{-ism^2} \to 0$ when $s \to -\infty$. The contour is shown in figure 2.1 and picks up poles of $\coth esE$ along the negative imaginary axis. The poles are at $esE = -in\pi$ for $n \in \{1,2,3,...\}$, and the residue of $\coth esE$ is $1/eE$. We find:

$$\mathrm{Im}[\mathcal{L}_I] = \frac{eE}{8\pi^2}\int_0^\infty \frac{ds}{s^2}\,\mathrm{Im}[e^{-ism^2}](\coth esE - \frac{1}{esE} - \frac{esE}{3}) \tag{2.43}$$

$$= \frac{eE}{16\pi^2}\,\mathrm{Im}[\int_{-\infty}^\infty \frac{ds}{s^2} e^{-ism^2}(\coth esE - \frac{1}{esE} - \frac{esE}{3})] \tag{2.44}$$

$$= \mathrm{Im}[\frac{eE}{16\pi^2}(-2\pi i)\sum_{n=1}^\infty \frac{eE}{-\pi^2 n^2} e^{\frac{-n\pi m^2}{eE}}] \tag{2.45}$$

$$= \frac{e^2E^2}{8\pi^3}\sum_{n=1}^\infty \frac{1}{n^2} e^{\frac{n\pi m^2}{eE}} \tag{2.46}$$

We can also find the production rate Γ, which is equal to twice the imaginary part of the Lagrangian [1]*. Using the fine structure constant $\alpha = \frac{e^2}{4\pi} \approx 1/137$ we can write:

$$\Gamma = 2\,\mathrm{Im}[\mathcal{L}_I] = \frac{\alpha E^2}{\pi^2}\sum_{n=1}^\infty \frac{1}{n^2} e^{-\frac{n\pi m^2}{eE}} = \frac{\alpha E^2}{\pi^2}\,\mathrm{Li}_2[e^{-\frac{\pi m^2}{eE}}] \tag{2.47}$$

where Li_2 is the second-order polylogarithm function. As for the real part of $\mathcal{L}_I$, we show in appendix C that it is zero.

*Unlike Schwinger's paper, Itzykson and Zuber [2] give the correct factor of $\frac{\alpha}{\pi^2}$.

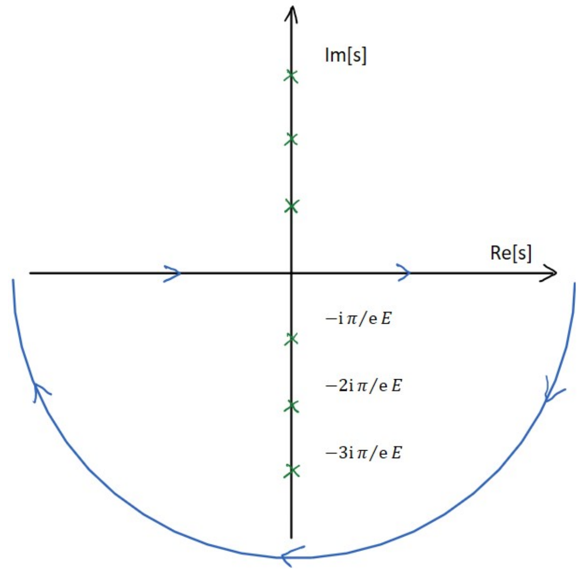

Figure 2.1: *Contour integration for the imaginary part of the Lagrangian and the current*

2.5 A naively vanishing current

As explained in the introduction, the production of electron-positron pairs generate an electric current. Formally, we can find the expectation value of this current from (2.20) and (2.26). We can use (2.33) to obtain:

$$\begin{aligned}\langle y|\,\hat{\Pi}_\nu e^{-i\hat{H}s}\,|y\rangle =& \langle y|\,e^{-i\hat{H}s}\hat{\Pi}_\nu(s)\,|y\rangle = \langle y|\,e^{-i\hat{H}s}M_{\nu\sigma}[\hat{x}^\sigma(s)-\hat{x}^\sigma(0)]\,|y\rangle \\ =& M_{\nu\sigma}[\langle y|\,\hat{x}^\sigma(0)e^{-i\hat{H}s}\,|y\rangle - \langle y|\,e^{-i\hat{H}s}\hat{x}^\sigma(0)\,|y\rangle] \\ =& M_{\nu\sigma}(y-y)^\sigma\,\langle y|\,e^{-i\hat{H}s}\,|y\rangle = 0\end{aligned} \tag{2.48}$$

and so the current as written in (2.26) vanishes if we use the WKB approximation this naive way. This is not a surprise since Maxwell's equation in the gauge $A_3 \approx Et$ reads:

$$j_3 = \ddot{A}_3 \tag{2.49}$$

and since our WKB approximation used $\dot{E} = 0$, the current vanishes. We know that the current should not vanish, but this tells us that it is very small. A simple way to show what we expect the current to look like is the following:

If we go back to expression (2.20) for the current, we can heuristically write:

$$j_\nu(y) = -\frac{\delta}{\delta A^\nu(y)} \int d^4x \mathcal{L}_I \sim -\frac{d}{dA^\nu(y)} \mathcal{L}_I \tag{2.50}$$

$$j_3 \sim \frac{-d}{tdE} \mathcal{L}_I \tag{2.51}$$

This shows that we can expect the current to be imaginary like the effective interaction Lagrangian. In the next sections, we try to improve upon the WKB approximation to reach the non-vanishing expression for the current.

Chapter 3

Current from the response tensor

3.1 Current as path integral over the gauge field

We know the current must not vanish and we would like to reach an expression of the current in terms of the electric field, then use Maxwell's equation to connect the electric field via the current to its derivative at each time. Much like we did in the previous section, this method we call "bootstrapping", since we are relating the current at each time instance to the field dynamics, and in a way, tying the field to itself. However, we saw that the way we used the WKB approximation resulted in a vanishing current. We can think in terms of Ohm's law, where the current is the conductivity times the electric field (not its derivative). We see that somehow, our approximation killed the conductivity. This motivates us to try and calculate the conductivity directly. Formally, we can write a path integral equation that resembles Ohm's law:

$$j_\mu(x) = \int \mathcal{D}A_\nu(x') \frac{\delta j_\mu(x)}{\delta A_\nu(x')} = \int \mathcal{D}A^\nu(x') K_{\mu\nu}(x, x') \tag{3.1}$$

where $K_{\mu\nu}(x, x') \equiv \frac{\delta j_\mu(x)}{\delta A_\nu(x')}$ is what is known in response theory literature as the response tensor. To get a better intuition for this quantity, we can recall expression (2.6) for the current and write:

$$
\begin{aligned}
K_{\mu\nu}(x,x') =& \frac{i\delta^2 \ln \mathcal{Z}_I}{\delta A^\mu(x)\delta A^\nu(x')} \\
= \mathcal{Z}_I^{-1}\frac{i\delta^2}{\delta^2 A^\mu(x)\delta A^\nu(x')}\mathcal{Z}_I[A] - i\bigg(&\mathcal{Z}_I^{-1}\frac{\delta}{\delta A^\nu(x')}\mathcal{Z}_I\bigg)\bigg(\mathcal{Z}_I^{-1}\frac{\delta}{\delta A^\mu(x)}\mathcal{Z}_I\bigg) \\
=& -i\langle j_\mu(x) j_\nu(x')\rangle + i\langle j_\mu(x)\rangle\langle j_\nu(x')\rangle
\end{aligned}
\tag{3.2}
$$

Equation (3.2) expresses the fluctuation-dissipation theorem; in the last line of (3.2), we see the structure of quantum thermal fluctuations of j. On the other hand, $K_{\mu\nu}$ expresses conductance, the dissipation of energy among intrinsic excitations of the system [6]. Our aim now is to evaluate $K_{\mu\nu}$, which encompasses various aspects of electromagnetic linear response: Off-diagonal spatial elements $K_{i\neq j}$ measure Hall conductivity, K_{00} measure density response, and K_{ii} measure linear conductivity. Our gauge means that we are interested in K_{33}, the conductivity relating j_3 to A_3. It is crucial here to note that we are computing $K_{\mu\nu}$ out of equilibrium in a non-linear setting where $E \neq 0$ and $A_3 \neq 0$. In contrast to linear response, we will get an expression for K_{33} containing E.

3.2 The response tensor

The functional derivatives can be calculated as:

$$
\begin{aligned}
K_{\mu\nu}(y',y) =& \frac{\delta}{\delta A^\mu(y')} j_\nu(y) \\
=& \frac{-i}{2}\int_0^\infty \frac{ds}{s} e^{-ism^2} \int d^4x \, \mathrm{Tr}\, \langle x| \frac{\delta}{\delta A^\mu(y')}\frac{\delta}{\delta A^\nu(y)} e^{-i\hat{H}s} |x\rangle
\end{aligned}
\tag{3.3}
$$

Notice the symmetry of exchanging (μ, y') and (ν, y). The derivative is calculated as in (2.22) and we find:

$$
\begin{aligned}
\frac{\delta}{\delta A^\mu(y')}\frac{\delta}{\delta A^\nu(y)} e^{-i\hat{H}s} =& 2ie^2 g_{\mu\nu}\delta(\hat{x}-y)\delta(\hat{x}-y') s e^{-i\hat{H}s} \\
& - \int_0^s d\alpha \int_0^\alpha d\beta e^{-i\hat{H}\beta} \frac{\delta \hat{\Pi}^2}{\delta A^\mu(y')} e^{-i\hat{H}(\alpha-\beta)} \frac{\delta \hat{\Pi}^2}{\delta A^\nu(y)} e^{-i\hat{H}(s-\alpha)} \\
& - \int_0^s d\alpha \int_0^\alpha d\beta e^{-i\hat{H}\beta} \frac{\delta \hat{\Pi}^2}{\delta A^\nu(y)} e^{-i\hat{H}(\alpha-\beta)} \frac{\delta \hat{\Pi}^2}{\delta A^\mu(y')} e^{-i\hat{H}(s-\alpha)}
\end{aligned}
\tag{3.4}
$$

We have:

$$\frac{\delta \hat{\not{\Pi}}^2}{\delta A^\nu(y)} = -e(\not{\Pi}\gamma_\nu \delta(\hat{x}-y) + \delta(\hat{x}-y)\gamma_\nu \not{\Pi}) \tag{3.5}$$

As before, we can turn deltas into ketbras. For last term in (3.4) we get:

$$\begin{aligned}\int d^4x \operatorname{Tr} \langle x| e^{-i\hat{H}\beta} \frac{\delta \hat{\not{\Pi}}^2}{\delta A^\nu(y)} e^{-i\hat{H}(\alpha-\beta)} \frac{\delta \hat{\not{\Pi}}^2}{\delta A^\mu(y')} e^{-i\hat{H}(s-\alpha)} |x\rangle = \\ e^2 \operatorname{Tr}\{ \langle y| e^{i\hat{H}(\alpha-\beta-s)} \not{\Pi}\gamma_\nu |y'\rangle\langle y'| e^{i\hat{H}(\beta-\alpha)} \not{\Pi}\gamma_\mu |y\rangle \\ + \langle y| \not{\Pi}\gamma_\mu e^{i\hat{H}(\alpha-\beta-s)} |y'\rangle\langle y'| \not{\Pi}\gamma_\nu e^{i\hat{H}(\beta-\alpha)} |y\rangle \\ + \langle y| \not{\Pi}\gamma_\mu e^{i\hat{H}(\alpha-\beta-s)} \not{\Pi}\gamma_\nu |y'\rangle\langle y'| e^{i\hat{H}(\beta-\alpha)} |y\rangle \\ + \langle y| e^{i\hat{H}(\alpha-\beta-s)} |y'\rangle\langle y'| \not{\Pi}\gamma_\nu e^{i\hat{H}(\beta-\alpha)} \not{\Pi}\gamma_\mu |y\rangle\}\end{aligned} \tag{3.6}$$

We still have to integrate over α and β. We notice that the integrand is a function of $\alpha-\beta$ and use this fact to change the variables of the area integration. Using $u=\alpha-\beta, v=\alpha+\beta$, and performing the integral over v, we finally get after one last change of variable ($\theta = s-u$):

$$\begin{aligned}\int_0^s d\alpha \int_0^\alpha d\beta \int_x \operatorname{Tr} \langle x| e^{-i\hat{H}\beta} \frac{\partial \hat{\not{\Pi}}^2}{\partial A^\nu(y)} e^{-i\hat{H}(\alpha-\beta)} \frac{\partial \hat{\not{\Pi}}^2}{\partial A^\mu(y')} e^{-i\hat{H}(s-\alpha)} |x\rangle \\ = e^2 \int_0^s d\theta\theta \operatorname{Tr}\{ \langle y| e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu |y'\rangle\langle y'| e^{-i\hat{H}(s-\theta)} \not{\Pi}\gamma_\mu |y\rangle \\ + \langle y| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} |y'\rangle\langle y'| \not{\Pi}\gamma_\nu e^{-i\hat{H}(s-\theta)} |y\rangle \\ + \langle y| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu |y'\rangle\langle y'| e^{-i\hat{H}(s-\theta)} |y\rangle \\ + \langle y| e^{-i\hat{H}\theta} |y'\rangle\langle y'| \not{\Pi}\gamma_\nu e^{-i\hat{H}(s-\theta)} \not{\Pi}\gamma_\mu |y\rangle\}\end{aligned} \tag{3.7}$$

Now recall that to this term we are adding another with $(\mu, y') \to (\nu, y)$ exchanged. This, combined with the possibility of changing the integration variable $\theta \to s-\theta$ for some terms, shows that the 8 terms cancel in pairs, leaving only 4 terms and a factor of s, so we have:

$$
\begin{aligned}
&\int_0^s d\alpha \int_0^\alpha d\beta \int_x \mathrm{Tr} \left\langle x \right| e^{-i\hat{H}\beta} \frac{\partial \hat{\not{\Pi}}^2}{\partial A^\nu(y)} e^{-i\hat{H}(\alpha-\beta)} \frac{\partial \hat{\not{\Pi}}^2}{\partial A^\mu(y')} e^{-i\hat{H}(s-\alpha)} \left| x \right\rangle + \\
&\int_0^s d\alpha \int_0^\alpha d\beta \int_x \mathrm{Tr} \left\langle x \right| e^{-i\hat{H}\beta} \frac{\partial \hat{\not{\Pi}}^2}{\partial A^\mu(y')} e^{-i\hat{H}(\alpha-\beta)} \frac{\partial \hat{\not{\Pi}}^2}{\partial A^\nu(y)} e^{-i\hat{H}(s-\alpha)} \left| x \right\rangle \\
&\qquad = e^2 s \int_0^s d\theta \, \mathrm{Tr}\{ \left\langle y \right| e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu \left| y' \right\rangle\!\left\langle y' \right| e^{-i\hat{H}(s-\theta)} \not{\Pi}\gamma_\mu \left| y \right\rangle \\
&\qquad\qquad + \left\langle y \right| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} \left| y' \right\rangle\!\left\langle y' \right| \not{\Pi}\gamma_\nu e^{-i\hat{H}(s-\theta)} \left| y \right\rangle \\
&\qquad\qquad + \left\langle y \right| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu \left| y' \right\rangle\!\left\langle y' \right| e^{-i\hat{H}(s-\theta)} \left| y \right\rangle \\
&\qquad\qquad + \left\langle y' \right| \not{\Pi}\gamma_\nu e^{-i\hat{H}\theta} \not{\Pi}\gamma_\mu \left| y \right\rangle\!\left\langle y \right| e^{-i\hat{H}(s-\theta)} \left| y' \right\rangle \}
\end{aligned}
\tag{3.8}
$$

We can see $(\mu, y') \to (\nu, y)$ symmetry in the first two terms on the right-hand side accompanied by changing the integration variable $\theta \to s - \theta$. In the last two terms, this symmetry is manifest. Combining it all we obtain:

$$
\begin{aligned}
K_{\mu\nu}(y', y) = e^2 g_{\mu\nu} \delta(y - y') &\int_0^\infty ds e^{-ism^2} \, \mathrm{Tr} \left\langle y \right| e^{-i\hat{H}s} \left| y \right\rangle \\
+ \frac{ie^2}{2} &\int_0^\infty ds e^{-ism^2} H_{\mu\nu}(y', y; s)
\end{aligned}
\tag{3.9}
$$

where we define the key quantity:

$$
\begin{aligned}
H_{\mu\nu}(y', y; s) \equiv \int_0^s d\theta \, \mathrm{Tr}\{ &\left\langle y \right| e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu \left| y' \right\rangle\!\left\langle y' \right| e^{-i\hat{H}(s-\theta)} \not{\Pi}\gamma_\mu \left| y \right\rangle \\
+ &\left\langle y \right| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} \left| y' \right\rangle\!\left\langle y' \right| \not{\Pi}\gamma_\nu e^{-i\hat{H}(s-\theta)} \left| y \right\rangle \\
+ &\left\langle y \right| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu \left| y' \right\rangle\!\left\langle y' \right| e^{-i\hat{H}(s-\theta)} \left| y \right\rangle \\
+ &\left\langle y' \right| \not{\Pi}\gamma_\nu e^{-i\hat{H}\theta} \not{\Pi}\gamma_\mu \left| y \right\rangle\!\left\langle y \right| e^{-i\hat{H}(s-\theta)} \left| y' \right\rangle \}
\end{aligned}
\tag{3.10}
$$

We can gain some intuition for the response tensor by relating it to the current-current correlation function $\langle j_\mu(x) j_\nu(x') \rangle$ as seen in equation (3.2). Using Maxwell's equation, one can relate the current-current correlation to $\langle A_\mu(x) A_\nu(x') \rangle$ (see section 67 in [7]). The correlation, to first order in perturbation, has the two Feynman diagrams seen in figure 3.1. The first term in (3.9), which has a spacetime delta function, is an ultra-local term that corresponds to the first diagram with one-point interaction. The second term corresponds to the second diagram with two vertices.

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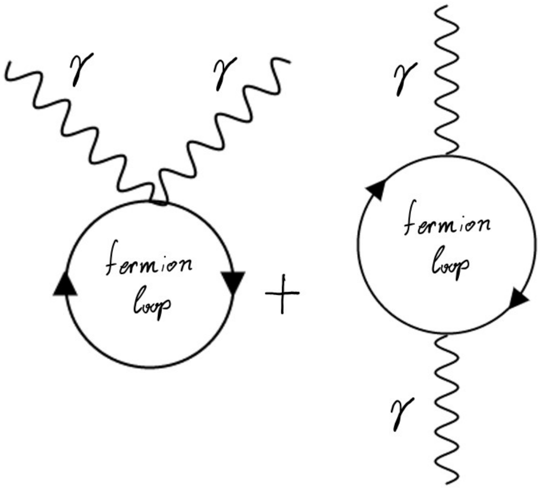

Figure 3.1: *Two Feynman diagrams for current-current correlation. The two photons connect to two currents at different spacetime points.*

Computing the second term is not an easy task as it involves long expressions with large matrices, traces of up to 8 γ matrices, renormalization of UV divergences, and contour integration of a lengthy integrand. A big part of this project was spent on these calculations and they are found in appendix B. Using the WKB approximation we show that the second term becomes negligibly small. As for the first term, we can use (2.39) to simplify it and obtain:

$$K_{\mu\nu}(y', y) = -\frac{ie^3 E}{4\pi^2} g_{\mu\nu}\delta(y' - y) \int_0^\infty \frac{ds}{s} e^{-ism^2} \coth(esE) \tag{3.11}$$

To perform this integral, we need to cure the UV divergence at $s = 0$. We do so by employing minimal subtraction:

$$K_{\mu\nu}(y', y) = \frac{-ie^3 E}{4\pi^2} g_{\mu\nu}\delta(y' - y) \int_0^\infty \frac{ds}{s} e^{-ism^2} (\coth(esE) - \frac{1}{esE}) \tag{3.12}$$

For the imaginary part of $K_{\mu\nu}(y', y)$, we can find:

$$\mathrm{Im}[K_{\mu\nu}(y', y)] = g_{\mu\nu}\delta(y' - y)\,\mathrm{Im}[\frac{-ie^3 E}{8\pi^2} \int_{-\infty}^\infty \frac{ds}{s} e^{-ism^2} (\coth(esE) - \frac{1}{esE})] \tag{3.13}$$

$$= g_{\mu\nu}\delta(y' - y)\,\mathrm{Im}[\frac{-ie^3 E}{8\pi^2} (-2\pi i) \sum_{n=1}^\infty \frac{1}{-in\pi} e^{-n\pi m^2/eE}] \tag{3.14}$$

$$= g_{\mu\nu}\delta(y' - y) \frac{e^3 E}{4\pi^2} \ln[1 - e^{-\pi m^2/eE}] \tag{3.15}$$

where the contour integral is performed in a manner analogous to the previous one for the imaginary part of the effective interaction Lagrangian. The contour is shown in figure 2.1. In the last step we use: $\ln[1 - e^{-\pi m^2/eE}] = -\sum_{n=1}^{\infty} \frac{1}{n} e^{-n\pi m^2/eE}$. Finding the real part of $K_{\mu\nu}$ is not an easy task, especially since the integrand becomes odd in s, and the integration limits cannot be extended to cover the real axis. In appendix C, we take the derivative with respect to m^2 to get an even integrand. We then perform a contour integration and show that the real part vanishes.

We can take the expression we found for the response tensor and substitute it back in (3.1) to try and find the current:

$$j_\mu(x) = \int \mathcal{D}A_\nu(x') \frac{\delta j_\mu(x)}{\delta A_\nu(x')} = \int \mathcal{D}A_3(x') \frac{\delta j_\mu(x)}{\delta A_3(x')} \tag{3.16}$$

$$j_3(x) \approx Mt' \int dE K_{33}(x' - x) \tag{3.17}$$

$$\mathrm{Im}[j_3] \sim Mt'\delta(x - x') \int dE \frac{e^3 E}{4\pi^2} \ln[1 - e^{-\pi m^2/eE}] \tag{3.18}$$

where M is some appropriate measure for the path integral. We also use the approximation that we are only interested in gauge field configurations where $A_3(t) = Et$ as a first approximation. The expression we end up with has a delta function and an undefined measure, which makes it ambiguous and difficult to use.

Chapter 4

Response current as a linear approximation

Since using (3.1) is difficult, we can think of brutally replacing $\int \mathcal{D}A_\mu$ with a multiplication by A_μ. This could hold for a constant integrand. Looking at our integrand we see that it is a function of E, which our WKB approximation deems constant, so replacing an integral with multiplication is not unrealistic, and should give us some physical intuition. Along the same line of thought, we can get inspiration from linear response theory to draft an approximation when the electric field is below the Schwinger limit $eE \lesssim \pi m^2$. Formally, we can think of the current $j_\mu(x)$ as a functional of $A^\nu(x')$; the current is a response to a gauge field taking the role of a generalized force. For small fields, the response current should be small and can be approximated linearly. Therefore, we assume that a linear approximation in $A^\nu(x')$ is satisfactory to describe the value of j_μ. We therefore take the first order term of the usual Taylor expansion* for the functional $j_\mu[A(x)]$:

$$j_\mu[A(x)] \approx j_\mu[A(x) - \Delta A(x)] + \int dx' \frac{\delta j_\mu(x)}{\delta A^\nu(x')} A^\nu(x') \tag{4.1}$$

This equation gives us the current at spacetime point x, when the field is $A(x)$ in terms of the current when the field was $A(x) - \Delta A(x)$. We define $\Delta A(x)$ to identify $A(x) - \Delta A(x)$ with the field's initial value, $A(0)$. In our problem, the initial value of the current when $A(0) = 0$ vanishes, and

*The form of linear approximation we use is given a brief explanation in appendix D therefore $j_\mu[A(x) - \Delta A(x)] = j_\mu[A(0)] = 0$, so we can write:

$$j_\mu(x) \approx \int dx' \frac{\delta j_\mu(x)}{\delta A^\nu(x')} A^\nu(x') = \int dx' K_{\mu\nu}(x, x') A^\nu(x') \tag{4.2}$$

Unlike mainstream linear response theory [6], $K_{\mu\nu}(x, x')$ here is evaluated for $A_\mu \neq 0$, which will allow us to bootstrap the fields at t via the current and Maxwell's equation to $\ddot{A}(t)$. This linear approximation may not work when $eE \gtrsim \pi m^2$, so we make sure that initially E is below the Schwinger limit. The approximation should work better as $E \to 0$, which is an interesting regime for us since we wish to know whether the field oscillates or decays exponentially. In any case, this is the leading term of the Taylor expansion and the assumption is that it dominates the dynamics below the Schwinger limit. A further study can inspect higher terms in the expansion and analyse their effect on the dynamics even above the Schwinger limit.

4.1 Complex differential equation, real electric field

If we indeed use equation (4.2), knowing (3.15), we will have an imaginary current. This is not a surprise as explained after we derived an imaginary effective interaction Lagrangian. The imaginary current reads:

$$j_0 = j_1 = j_2 = 0 \tag{4.3}$$

$$j_3 = \frac{-ie^3 E}{4\pi^2} A_3 \int_0^\infty \frac{ds}{s} e^{-ism^2} (\coth(esE) - \frac{1}{esE}) \tag{4.4}$$

$$j_3 = i\frac{e^3 E}{4\pi^2} A_3 \ln\Big(1 - e^{-\pi m^2/eE}\Big) \tag{4.5}$$

We can now use Maxwell's equation to reach an ordinary differential equation for A. Keeping in mind that in the end we are looking for a real electric field, we will interpret the real part of the solution to be the physical field.

$$\ddot{A}_3 = i\frac{e^3 E}{4\pi^2} A_3 \ln\Big(1 - e^{-\pi m^2/eE}\Big) \tag{4.6}$$

This differential equation is non-linear and finding an analytic solution is too difficult. What we can do instead, is to use numerical methods to obtain an intuition for the system dynamics. If we solve (4.6) numerically with initial conditions $A_3(0) = 0$, $eE(0) = m^2$, then take the real part of A_3, E, and $\ddot{A}_3 = j_3$, we obtain the physical fields and the current.

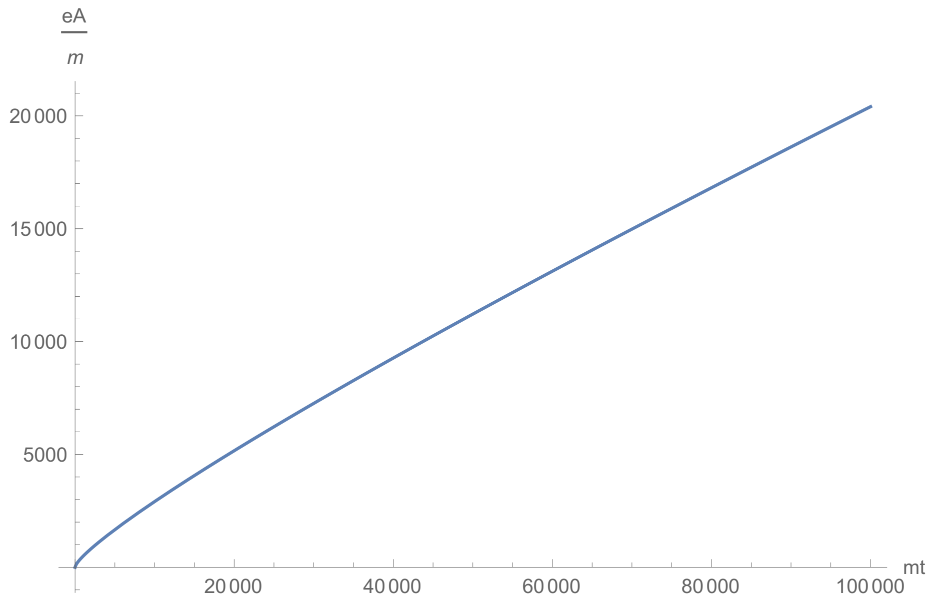

Figure 4.1: *Normalized real gauge field eA/m vs mt for initial conditions $A_3(0) = 0$, $eE(0) = m^2$.*

4.2 Numerical results

We take real quantities $A \equiv \mathrm{Re}[A_3]$, $E \equiv \mathrm{Re}[\dot{A}_3]$, $j \equiv \mathrm{Re}[j_3]$ to be the physical quantities. To plot them we normalize using m: time is measured in units of $1/m$, so we plot mt on the horizontal axis. On the vertical axis we plot eA/m, eE/m^2, $-ej/m^3$, and Γ/m^4, where Γ is the production rate found in equation (2.47). Plots are shown in figures 4.1, 4.2, 4.3, 4.4, and 4.5.

The plots tell a story consistent with the intuition we established in the introduction: the slope of E initially is zero, which is consistent with the initial conditions we wish to impose; no current and a large electric field. At first, the field is at $\frac{1}{\pi}$ of the Schwinger limit, so it creates pairs of low particle number density and generates a small current.

There is a short period when the electric field is almost constant. We will discuss this initial plateau in a bit. Soon after, as charged particles emerge they absorb energy from the field and screen it, which amounts to a back-reaction that sharply decreases the electric field. This happens during the first 100 time units. By then, the current peaks, and starts to drop as the production rate (plotted in 4.5) undergoes a sharp decline to $\sim 1/10$ of its original value. This means that the back-reaction starts to decrease, halting

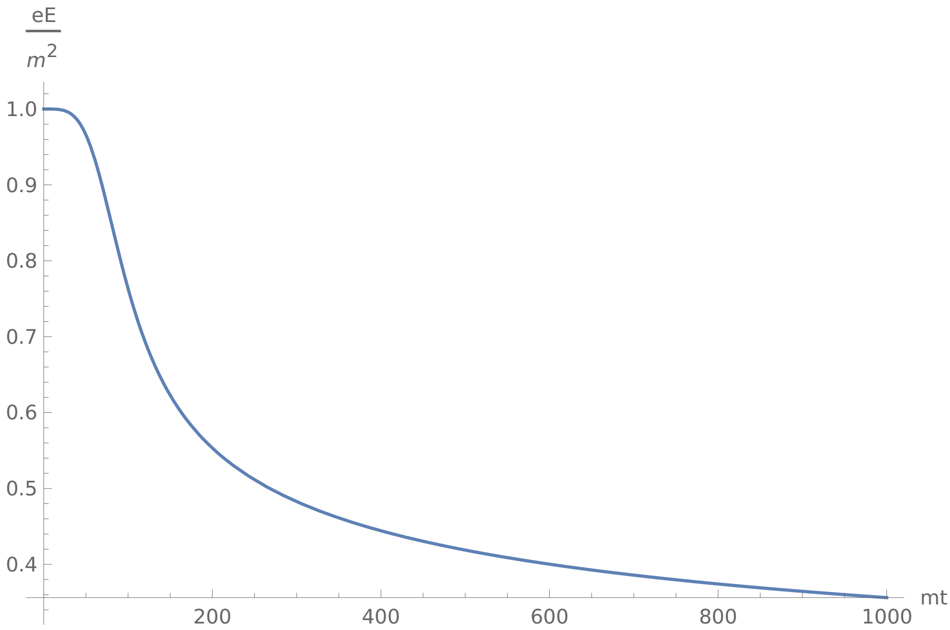

Figure 4.2: *Normalized real electric field eE/m^2 vs mt for initial conditions $A_3(0) = 0$, $eE(0) = m^2$.*

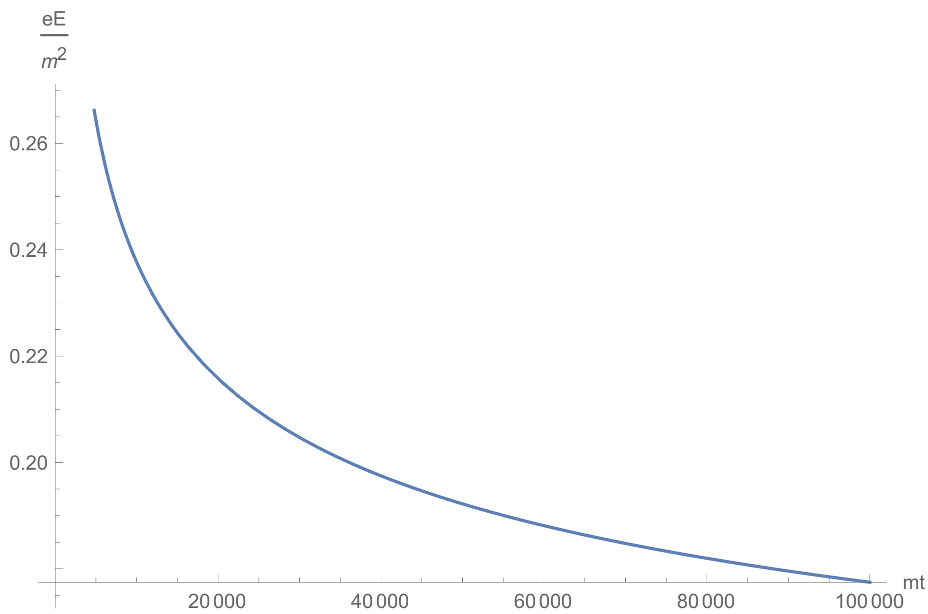

Figure 4.3: *Normalized real electric field eE/m^2 vs mt for initial conditions $A_3(0) = 0$, $eE(0) = m^2$, showing E after a long time.*

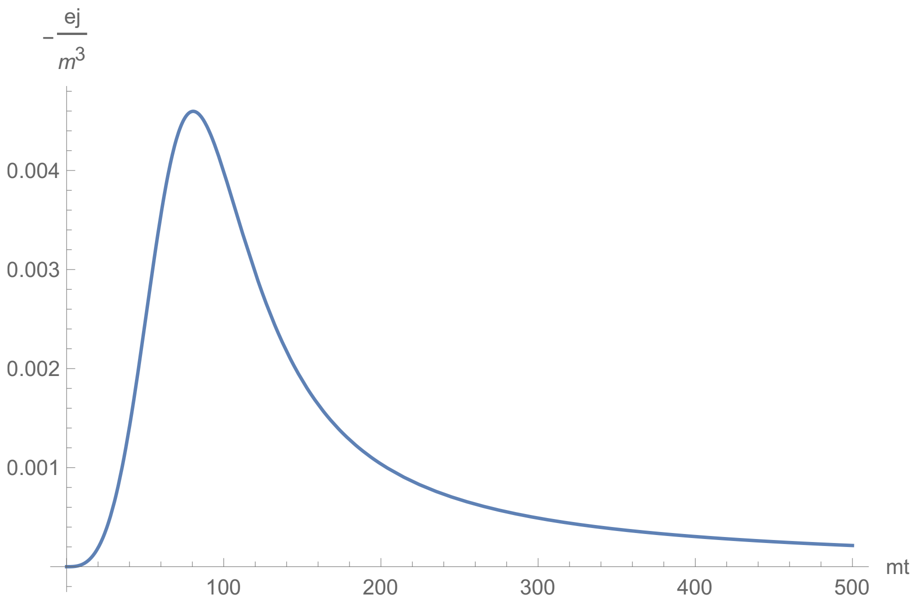

Figure 4.4: *Normalized real current* $-ej/m^3$ *vs* mt *for initial conditions* $A_3(0) = 0$, $eE(0) = m^2$.

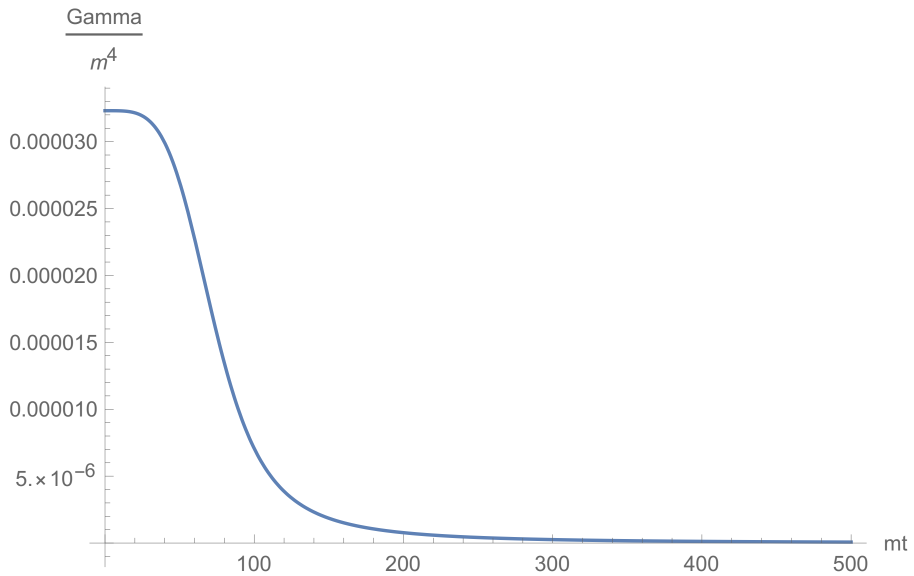

Figure 4.5: *Normalized production rate* Γ/m^4 *vs time in units of mass (*mt*).*

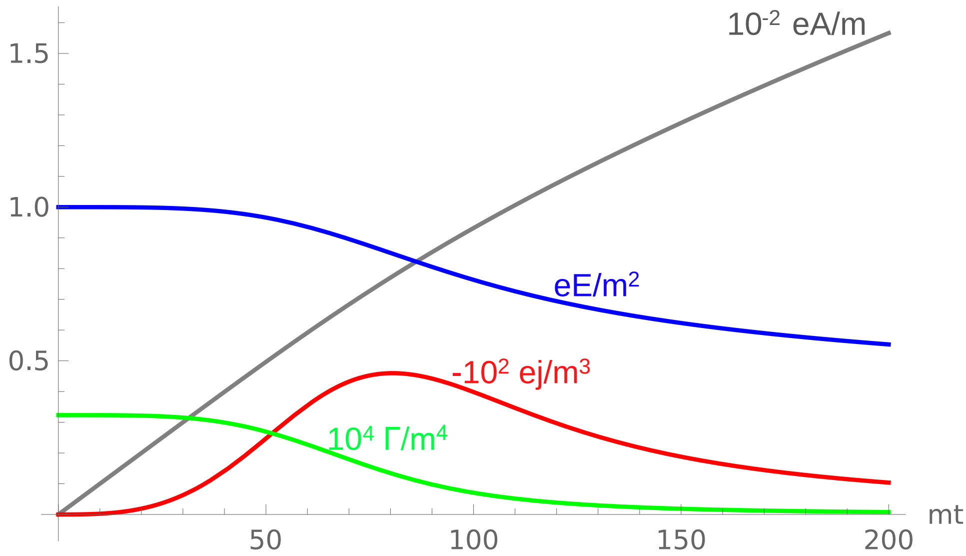

Figure 4.6: *$10^{-2}eA/m$ in grey, eE/m^2 in blue, $-10^2ej/m^3$ in red, $10^4\Gamma/m^4$ in green, all vs mt.*

the sharp drop of the electric field. At 200 time units, the field is about half its initial value and starts a slow exponential-like decay to 0, reaching $eE \approx 0.1m^2$ after a very long time when $mt = 10^7$. The current also starts an exponential-like decay to 0.

We see in figure 4.5 that the production rate starts already at a very small value, producing particles at low density. The rate quickly drops to infinitesimal values and becomes approximately zero around 400 time units. After this time, the electric field decreases very slowly due to the little pair production and the back-reaction of the existing pairs as they further absorb energy from the field.

In figure 4.6 we compare previous plots for the gauge field, electric field, electric current, and production rate. In order to fit them all into one figure with clarity, we rescale and plot: $10^{-2}eA/m$, eE/m^2, $-10^2ej/m^3$, $10^4\Gamma/m^4$.

4.3 Sensitivity to initial conditions

What is the reason for the initial plateau we see in the plot for the real part of E? We expect the very early behavior to be sensitive to the initial conditions

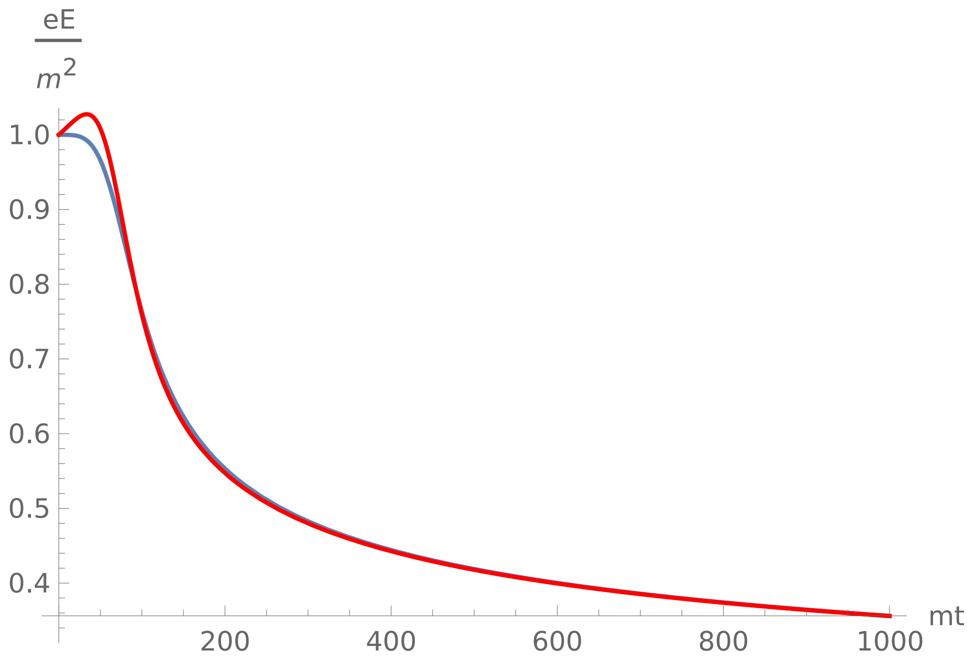

Figure 4.7: *Normalized real electric field eE/m^2 vs mt showing sensitivity to initial conditions in early behavior. In red: $eA_3(0) = 10im^2$. In blue: $A_3(0) = 0$.*

we impose on the system. In equation (4.6), we have a complex gauge field, A_3. Since $\mathrm{Im}[A_3]$ it is not a physical field, insisting that $\mathrm{Im}[A_3(0)] = 0$ is an auxiliary initial condition. One sees from equation (4.6) that $\mathrm{Im}[A_3]$ on the right-hand side feeds $\mathrm{Re}[\ddot{A}_3]$ on the left-hand side. This means that the initial behavior of $\mathrm{Re}[\dot{A}_3]$ should be sensitive to $\mathrm{Im}[A_3(0)]$. Indeed, the condition $eA_3(0)/m^2 = 10i$ results in an initial spike in $E = \mathrm{Re}[\dot{A}_3]$ at first, as seen in the red line in figure 4.7. This sensitivity of early behavior to initial conditions persists until $mt \approx 100$. In 4.7 we see a peak in E very early, and then the predicted late time behavior is the same as we saw earlier in figure 4.2 for $A_3(0) = 0$, plotted here in blue. For physical analysis we can focus on times after $mt \sim 100$ and know that our system has settled into dynamics unaffected by $A_3(0)$.

4.4 Energy of the system

Initially, all energy is stored in the electric field. The energy density $\mathcal{E}$ when $eE = m^2$ amounts to:

$$\mathcal{E}_E = \frac{1}{2}E^2 = \frac{1}{2}\frac{m^4}{e^2} = \frac{m^4}{8\pi\alpha} \tag{4.7}$$

The field then loses energy and decreases. After a long time, say $mt = 100000$, the field is $E \approx 0.18m^2/e$. This means that the field lost 0.96 of its energy, almost all of it. Energy is then stored in the mass of particles and their kinetic energy, and some energy is radiated away. We can find out how much energy density ends up stored in the form of particle mass, $\mathcal{E}_m$, by calculating the number density of pairs n as an integral of the production rate over a very long time. Each pair has mass $2m$, therefore:

$$\mathcal{E}_m = 2mn = 2m \int_0^\infty \Gamma dt = 2m^4 \int d(mt) \frac{\Gamma}{m^4} \tag{4.8}$$

where the last expression is convenient for numerical integration. We integrate Γ/m^4 from $t = 0$ until $mt = 100000$, and check that the answer, $\sim 2.7 \times 10^{-3}$, does not change much if we go to higher times. This means that the energy density stored in the form of fermion mass is $\mathcal{E}_m \sim 5 \times 10^{-3} m^4$. This is much less than the initial energy stored in the electric field $\mathcal{E}_E \sim 5m^4$, which means that almost all energy is transferred to the first produced pair and stored as kinetic energy, or lost in the form of radiation.

This trend remains for a smaller initial field. Production rate becomes exponentially small and a much smaller number of pairs is available to absorb the field's energy. For example, if we start with $eE(0) = 0.2m^2$, when $mt = 100000$ the field becomes $eE \approx 0.18m^2$. This means

$$\Delta\mathcal{E}_E = \frac{m^4}{8\pi\alpha}[(0.18)^2 - (0.2)^2] \approx -0.04m^4 \tag{4.9}$$

On the other hand, pair production is very low and

$$\mathcal{E}_m = 2m^4 \int d(mt) \frac{\Gamma}{m^4} = 6 \times 10^{-7} m^4 \tag{4.10}$$

much less than the available energy lost by the electric field. This means that a larger percentage of energy is stored as kinetic energy or radiation.

4.5 Error analysis

So far, we have used two approximations: The first is $eE < \pi m^2$, used to justify our linear approximation. As seen in figure 4.8, this approximation is self-consistent and gets better with time as we approach small E. This is fortunate because the regime we wish to inspect and compare to previous studies [3] is in fact the small field regime. We see that our results disagree with the oscillation behaviour predicted by Mottola et. al. The second approximation we did is a WKB approximation, where we take the small parameter to be $\dot{E}/E\omega$. The physical meaning of this approximation is that the electric field changes much slower than the evolution of the fermionic state, so the fermions do not "feel" the change of the field. The fermion energy $\omega = \sqrt{k^2 + m^2}$ is at least equal to m. With this in mind, our small parameter for the WKB approximation is at most $\dot{E}/Em$. We see in figure 4.8 that the absolute value of this parameter is never more than 5×10^{-3}, and gets smaller at later times, so our results are consistent with the WKB approximation.

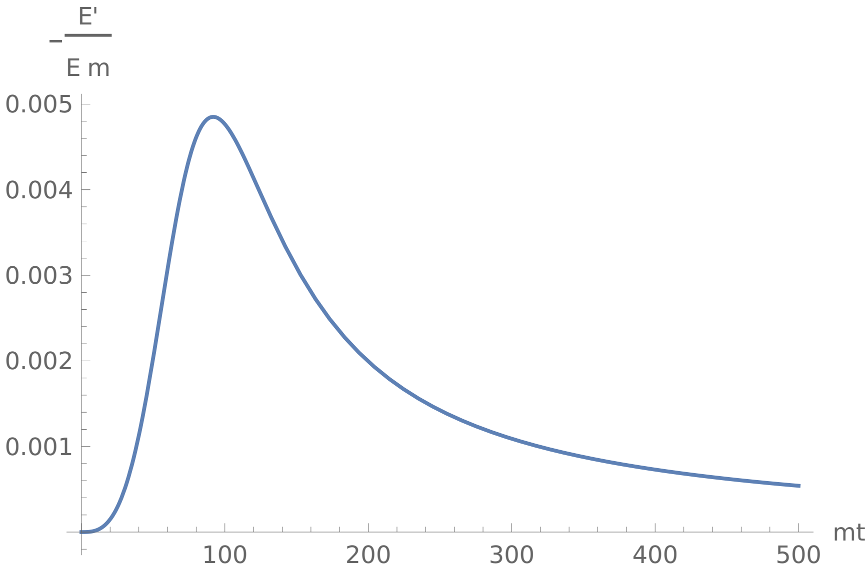

Figure 4.8: *WKB error ($-\dot{E}/Em$) vs mt with initial conditions $eE = m^2$ and $A(0) = 0$.*

Chapter 5

Current from point splitting and beyond WKB

5.1 Point splitting at small proper time

As we saw, the problem with calculating the current expectation value is that it gives zero in the WKB approximation. It does so because the current becomes proportional to $y - y$ as seen in eq. (2.26). From eq. (2.33), it is clear that this amounts to taking the limit $x(s) \to x(0)$, zooming in on the UV behavior, and taking the limit $s \to 0$. One can see throughout expressions (2.33), (2.39), leading to (2.48) that there is a subtlety in the limit $x(s) = x(0)$ since for $s = 0$ the expressions diverge. This divergence requires a proper regularization. However, we can gain some intuition for what happens near $s = 0$ if we simplify the mathematics by approximating $M_{\mu\nu}$, which is calculated in appendix B.1, about $s = 0$:

$$\mathbf{\Pi}(s) = e^{es\mathbf{F}} \frac{e\mathbf{F}}{2\sinh(es\mathbf{F})}.[\mathbf{x}(s) - \mathbf{x}(0)] = \mathbf{M}.[\mathbf{x}(s) - \mathbf{x}(0)] \tag{5.1}$$

$$M_\mu{}^\nu(\mathbf{F}) = \frac{1}{2s} \begin{pmatrix} esE \coth esE & 0 & 0 & -esE \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -esE & 0 & 0 & esE \coth esE \end{pmatrix} \tag{5.2}$$

$$\approx \frac{1}{2s} \begin{pmatrix} 1 & 0 & 0 & -esE \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -esE & 0 & 0 & 1 \end{pmatrix} \tag{5.3}$$

where in the last step, $esE \ll 1$. Note that s is the proper time because it parameterizes the fermion path. The use of a small s approximation can be interpreted, after Fourier transformation, to corresponds to an approximation for large fermion momentum. This approximation is found in the work of Mottola et. al [3] and is justified within the WKB approximation we did, in which the small quantity is $\dot{E}/(E\omega)$.

As for the current, we can write starting from (2.26) and using (2.48):

$$j_\nu(y) = -e\int_0^\infty dse^{-ism^2}\,\mathrm{Tr}\,\langle y|\,\gamma_\nu \not{\Pi} e^{-i\hat{H}s}\,|y\rangle \tag{5.4}$$

$$= -e\int_0^\infty dse^{-ism^2}\,\mathrm{Tr}\{\gamma_\nu\gamma_\mu M^{\mu\sigma}(y-y)_\sigma\,\langle y|\,e^{-i\hat{H}s}\,|y\rangle\} \tag{5.5}$$

The current is still proportional to $y - y$, but we can introduce point splitting in the time direction:

$$j(y) \sim (y-y) \to t - t' \tag{5.6}$$

$$j(y') - j(y) \sim (t'-t) - (t-t') = 2(t'-t) \tag{5.7}$$

$$dj \sim 2dt \tag{5.8}$$

So in the current expression, we will replace $y - y$ with $2dt$ and j with dj:

$$dj_\nu(y) \approx -e\int_0^\infty dse^{-ism^2}\,\mathrm{Tr}\{\gamma_\nu\gamma_\mu\,\langle y|\,e^{-i\hat{H}s}\,|y\rangle\}M^{\mu 0}2dt \tag{5.9}$$

$$\frac{dj_3}{dt} = -2e\int_0^\infty dse^{-ism^2}\,\mathrm{Tr}\{\gamma_3\gamma_\mu\frac{-ieE}{16\pi^2 s}[-\gamma_0\gamma_3 + \coth esE]\}M^{\mu 0} \tag{5.10}$$

$$= \frac{ie^2E}{2\pi^2}\int_0^\infty \frac{ds}{s}e^{-ism^2}[M^{00} - M^{30}\coth esE] \tag{5.11}$$

$$\frac{dj_3}{dt} \approx \frac{-ie^3E^2}{4\pi^2}\int_0^\infty \frac{ds}{s}e^{-ism^2}(\coth esE - \frac{1}{eEs}) \tag{5.12}$$

where in the last step, we substitute the matrix elements of $\mathbf{M}$ from the $s \to 0$ expression in (5.3): $M^{00} = 1/2s$ and $M^{30} = eE/2$. The result is the current we got by using response and the formally loose and unrigorous linear approximation, and we even get minimal subtraction for free! This becomes clear if we keep in mind that $A_3 \approx Et$ and write from (4.4):

$$j_3 = \frac{-ie^3E}{4\pi^2}A_3\int_0^\infty \frac{ds}{s}e^{-ism^2}(\coth(esE) - \frac{1}{esE}) \tag{5.13}$$

$$dj_3 \approx \frac{-ie^3E}{4\pi^2}dA_3\int_0^\infty \frac{ds}{s}e^{-ism^2}(\coth(esE) - \frac{1}{esE}) \tag{5.14}$$

$$= \frac{-ie^3E^2}{4\pi^2}dt\int_0^\infty \frac{ds}{s}e^{-ism^2}(\coth(esE) - \frac{1}{esE}) \tag{5.15}$$

so we expect the dynamics to be the same. The method presented in this section forms another educated guess that supports the results we obtained earlier, but must be taken with a grain of salt until matched with experimental data that exhibit our predicted system dynamics.

5.2 The current beyond WKB

Since the current expectation value was calculated to be zero in the naive usage of WKB, we try to find the current without making the approximation. If we go back to computing commutators of operators with the Hamiltonian, we have:

$$\frac{d\hat{\Pi}_\mu}{ds} = i[\hat{H}, \hat{\Pi}_\mu] = 2eF_{\mu\nu}\hat{\Pi}^\nu + \Delta_\mu \tag{5.16}$$

$$\Delta_\mu \equiv \frac{e}{2}\partial_\mu \sigma_{\alpha\beta}F^{\alpha\beta} - ie\partial_\nu F^{\nu\mu} \tag{5.17}$$

Instead of neglecting Δ_μ, we now keep it in our equations. We can solve the differential equation (5.16) for $\mathbf{\Pi}(s)$ to obtain:

$$\mathbf{\Pi}(s) = e^{2es\mathbf{F}}.\mathbf{\Pi_0} - \frac{\mathbf{\Delta}}{2e\mathbf{F}} \tag{5.18}$$

where $\mathbf{\Pi_0}$ is some s-independent operator. One can check that this is indeed the solution by substituting it into (5.16). We can use this solution to integrate the differential equation for the position operator:

$$\frac{d\mathbf{x}}{ds} = i[\hat{H}, \mathbf{x}] = 2\mathbf{\Pi} \tag{5.19}$$

$$\mathbf{x}(s) - \mathbf{x}(0) = -\frac{\mathbf{\Delta}s}{e\mathbf{F}} + 2e^{es\mathbf{F}}\frac{\sinh es\mathbf{F}}{e\mathbf{F}}\mathbf{\Pi_0} \tag{5.20}$$

we can use these two solutions to write $\mathbf{\Pi}(s)$ in terms of $\mathbf{x}(s) - \mathbf{x}(0)$:

$$\mathbf{\Pi}(s) = -\frac{\mathbf{\Delta}}{2e\mathbf{F}} + e^{es\mathbf{F}}\frac{e\mathbf{F}}{2\sinh es\mathbf{F}}[\mathbf{x}(s) - \mathbf{x}(0) + \frac{\mathbf{\Delta}s}{e\mathbf{F}}] \tag{5.21}$$

$$= \mathbf{M}.[\mathbf{x}(s) - \mathbf{x}(0)] - (1 - 2s\mathbf{M})\frac{\mathbf{\Delta}}{2e\mathbf{F}} \tag{5.22}$$

$$= \mathbf{M}.[\mathbf{x}(s) - \mathbf{x}(0)] - \mathbf{P}.\mathbf{\Delta} \tag{5.23}$$

The difference is a new term proportional to Δ, with

$$\mathbf{P} = (1 - e^{es\mathbf{F}}\frac{es\mathbf{F}}{\sinh(es\mathbf{F})})/2e\mathbf{F} \tag{5.24}$$

$$P_{\mu}{}^{\nu} = \frac{-s}{2}\begin{pmatrix} 1 & 0 & 0 & \frac{1-esE\coth esE}{esE} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1-esE\coth esE}{esE} & 0 & 0 & 1 \end{pmatrix} \tag{5.25}$$

where the last expression is in our chosen gauge $F_{03} = E$. We can also find:

$$\Delta_{\mu} = ie(-\gamma_0\gamma_3 g_{\mu 0} + g_{\mu 3})\dot{E} \tag{5.26}$$

As for equation (2.39), it gets modified as:

$$\langle x| e^{-i\hat{H}s} |x\rangle = \frac{-ieE}{16\pi^2 s}[\gamma^0\gamma^3 + \coth(esE)]e^{-\int ds(\mathbf{P}.\mathbf{\Delta})^2} \tag{5.27}$$

and does not change to first order in Δ_{μ}. Recalling our original equation for the current:

$$\langle j_{\nu}(y)\rangle = -e\int_0^{\infty} dse^{-ism^2}\,\mathrm{Tr}\,\langle y|\,\gamma_{\nu}\not{\Pi}e^{-i\hat{H}s}\,|y\rangle \tag{5.28}$$

Substituting $\mathbf{\Pi}$ from (5.23) and knowing that the first term vanishes, we can use equation (5.27) to compute:

$$\langle j_3(y)\rangle = e\int_0^{\infty} dse^{-ism^2}\,\mathrm{Tr}\,\gamma_3\gamma_{\mu}\,\langle y|\,e^{-i\hat{H}s}P^{\mu\nu}\Delta_{\nu}\,|y\rangle \tag{5.29}$$

$$= \frac{e^3E\dot{E}}{4\pi^2}\int_0^{\infty}\frac{ds}{s}e^{-ism^2}[(P^{00} + P^{33})\coth(esE) - P^{03} - P^{30}]e^{-\int ds(\mathbf{P}.\mathbf{\Delta})^2} \tag{5.30}$$

$$\langle j_3(y)\rangle = 0 \tag{5.31}$$

where in the last step we simply substituted the matrix elements of $\mathbf{P}$. We conclude that there is not much to be gained by going beyond the WKB approximation in this manner, and that a WKB approximation should be sufficient for a good qualitative description of the system.

Chapter 6

Conclusion

We sat out to solve the problem of back reaction on a large electric field due to fermionic Schwinger pair production in the context of spinor QED. Our strategy involved a WKB approximation where we first consider the electric field E to be an external constant field. We then seek an expression for the current at time t in terms of $E(t)$, then restore field dynamics, using Maxwell's equation which in our chosen gauge reads: $j = \dot{E}$. This relates the electric field to $\dot{E}$ and produces a differential equation which should give a good qualitative description for the dynamics of the system.

Although we know the current must not vanish, the most formal usage of the WKB approximation resulted in a vanishing current, which forced us to resort to less rigorous methods. We presented two ways to find a non-vanishing electric current; one employs the fluctuation-dissipation theorem, computing the response tensor in combination with a linear approximation of the electric current functional $j^\nu[A_\mu]$, and the other uses point splitting combined with the approximation of small Schwinger proper time. The two methods result in essentially identical expressions for the current, which gives some confidence in our approach as an educated guess. However, the crudeness of these methods leaves room for suspicion in the results that follow. The hope is that experimental data can soon falsify or confirm our results.

The electric current expression and Maxwell's equation lead to a non-linear complex differential equation for E. After finding a stable numerical solution, we interpret the real part of E to be the physical electric field. System dynamics show a predicted behavior; Starting with an electric field $E = m^2/e$ just below the Schwinger limit, the field undergoes a sharp drop due to the back reaction of pair production, followed by a slow exponential-like decay soon after as pair production diminishes drastically. Along with the gauge field A and the electric field E, we also obtain and plot the electric current and production rate. We then analyse the error and conclude that our results are consistent with the WKB approximation we used. A basic examination of the energy of the system indicates that most of the energy after pair production is not stored in the electric field nor in the mass of the produced pairs, but either stored as kinetic energy of the fermions or as radiation (or both). This last point should be taken with skepticism until further investigation.

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Appendix A

Functional derivative of exponential

We need to find:

$$\delta e^{-i\hat{H}s} = \delta \sum_{n=0}^{\infty} \frac{1}{n!}(is\not{\Pi}^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n-1} (is\not{\Pi}^2)^m \delta(is\not{\Pi}^2)(is\not{\Pi}^2)^{n-1-m} \tag{A.1}$$

$$= \sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \frac{1}{n!}(is\not{\Pi}^2)^m \delta(is\not{\Pi}^2)(is\not{\Pi}^2)^{n-1-m} \tag{A.2}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{(n+m+1)!}(is\not{\Pi}^2)^m \delta(is\not{\Pi}^2)(is\not{\Pi}^2)^{n} \tag{A.3}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{m!n!}{(n+m+1)!}\frac{1}{m!}(is\not{\Pi}^2)^m \delta(is\not{\Pi}^2)\frac{1}{n!}(is\not{\Pi}^2)^{n} \tag{A.4}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int_0^1 d\alpha \alpha^m (1-\alpha)^n \frac{1}{m!}(is\not{\Pi}^2)^m \delta(is\not{\Pi}^2)\frac{1}{n!}(is\not{\Pi}^2)^{n} \tag{A.5}$$

$$= is \int_0^1 d\alpha e^{i\not{\Pi}^2 s\alpha}(\delta\not{\Pi}^2)e^{i\not{\Pi}^2 s(1-\alpha)} \tag{A.6}$$

$$= i \int_0^s d\alpha e^{i\not{\Pi}^2 \alpha}(\delta\not{\Pi}^2)e^{i\not{\Pi}^2 (s-\alpha)} \tag{A.7}$$

Appendix B

Calculating the second term of the response tensor

Recall:

$$K_{\mu\nu}(y',y) = \frac{-ie^3E}{4\pi^2} g_{\mu\nu}\delta(y'-y)\int_0^\infty \frac{ds}{s} e^{-ism^2}\coth(esE) + \frac{ie^2}{2}\int_0^\infty dse^{-ism^2} H_{\mu\nu}(y',y;s) \tag{B.1}$$

where:

$$\begin{aligned} H_{\mu\nu}(y',y;s) \equiv \int_0^s d\theta \, \mathrm{Tr}\{ &\langle y| e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu \left|y'\middle\rangle\middle\langle y'\right| e^{-i\hat{H}(s-\theta)} \not{\Pi}\gamma_\mu |y\rangle \\ &+ \langle y| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} \left|y'\middle\rangle\middle\langle y'\right| \not{\Pi}\gamma_\nu e^{-i\hat{H}(s-\theta)} |y\rangle \\ &+ \langle y| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta} \not{\Pi}\gamma_\nu \left|y'\middle\rangle\middle\langle y'\right| e^{-i\hat{H}(s-\theta)} |y\rangle \\ &+ \left\langle y'\right| \not{\Pi}\gamma_\nu e^{-i\hat{H}\theta} \not{\Pi}\gamma_\mu |y\rangle\langle y| e^{-i\hat{H}(s-\theta)} \left|y'\right\rangle \} \end{aligned} \tag{B.2}$$

$H_{\mu\nu}$ can be computed using equations (2.33) and (2.34) in the same way as the term for the current. Simplifying the second two terms involves commuting Π and x, and gives:

$$\langle y| \not{\Pi}\gamma_\mu e^{-i\hat{H}\theta}\not{\Pi} \left|y'\right\rangle = \gamma_\rho M_\theta^{\rho\gamma} Y_\gamma \gamma_\mu \left\langle y\middle|y';\theta\right\rangle Y_\delta M_\theta^{\delta\sigma}\gamma_\sigma + i\gamma_\rho\gamma_\mu \left\langle y\middle|y';\theta\right\rangle M_\theta^{\rho\sigma}\gamma_\sigma \tag{B.3}$$

Where $Y = y' - y$, M_θ is M as a function of θ instead of s in (2.33), and $|y;\theta\rangle = e^{-i\hat{H}\theta}|y\rangle$. We can put this back to find $H_{\mu\nu}$, noting that we can use the cyclicity of the trace in the last term, and changing integration variables only in the last term to make its trace match that of other terms and factor it out. We find:

$$\begin{aligned} &H_{\mu\nu}(Y = y' - y; s) = \\ &\int_0^s d\theta[(-M^{\delta\rho}_{s-\theta}M^{\gamma\sigma}_{\theta} - M^{\rho\gamma}_{\theta}M^{\sigma\delta}_{s-\theta} + M^{\rho\gamma}_{\theta}M^{\delta\sigma}_{\theta} + M^{\gamma\rho}_{s-\theta}M^{\sigma\delta}_{s-\theta})Y_\gamma Y_\delta \\ &+ i(M^{\rho\sigma}_{\theta} + M^{\sigma\rho}_{s-\theta})]\,\mathrm{Tr}\{\gamma_\rho\gamma_\mu\left\langle y\middle|y';\theta\right\rangle\gamma_\sigma\gamma_\nu\left\langle y'\middle|y;s-\theta\right\rangle\} \end{aligned} \tag{B.4}$$

where we used $M^{\mu\nu} = N^{\nu\mu}$ to turn all matrices into $M^{\mu\nu}$, which is defined:

$$\mathbf{M}_s = \frac{1}{s}e^{es\mathbf{F}}\frac{es\mathbf{F}}{2\sinh(es\mathbf{F})} \tag{B.5}$$

To make expressions shorter we define:

$$\begin{aligned} D^{\rho\sigma} = (-M^{\delta\rho}_{s-\theta}M^{\gamma\sigma}_{\theta} - M^{\rho\gamma}_{\theta}M^{\sigma\delta}_{s-\theta} + M^{\rho\gamma}_{\theta}M^{\delta\sigma}_{\theta} + M^{\gamma\rho}_{s-\theta}M^{\sigma\delta}_{s-\theta})Y_\gamma Y_\delta \\ +i(M^{\rho\sigma}_{\theta} + M^{\sigma\rho}_{s-\theta}) \end{aligned} \tag{B.6}$$

$$H_{\mu\nu}(Y = y' - y; s) = \int_0^s d\theta D^{\rho\sigma}\,\mathrm{Tr}\{\gamma_\rho\gamma_\mu\left\langle y\middle|y';\theta\right\rangle\gamma_\sigma\gamma_\nu\left\langle y'\middle|y;s-\theta\right\rangle\} \tag{B.7}$$

Now we need to find the trace. Recalling (2.34), (2.36), and (2.38), we obtain:

$$\begin{aligned} &\mathrm{Tr}\{\gamma_\rho\gamma_\mu\left\langle y\middle|y';\theta\right\rangle\gamma_\sigma\gamma_\nu\left\langle y'\middle|y;s-\theta\right\rangle\} = \\ &\frac{e^2E^2\theta(s-\theta)}{-64\pi^4s^4}e^{i\mathbf{Y}\frac{e\mathbf{F}}{4}[\coth e\theta\mathbf{F} + \coth e(s-\theta)\mathbf{F}]\mathbf{Y}} \\ &[\mathrm{Tr}\{\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\}\coth e\theta E\coth e(s-\theta)E \\ &- \mathrm{Tr}\{\gamma_\rho\gamma_\mu\gamma_0\gamma_3\gamma_\sigma\gamma_\nu\}\coth e(s-\theta)E \\ &- \mathrm{Tr}\{\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\gamma_0\gamma_3\}\coth e\theta E + \mathrm{Tr}\{\gamma_\rho\gamma_\mu\gamma_0\gamma_3\gamma_\sigma\gamma_\nu\gamma_0\gamma_3\}] \end{aligned} \tag{B.8}$$

In our chosen gauge, $\nu = 3$, which simplifies the process of finding the traces. To make the equations even shorter we define $\phi = s - \theta$. We find the functions:

$$\begin{aligned} H_{11} = e^2E^2\int_0^s d\theta\tau(\theta)\{[D^{11} - D^{22}][\coth e\theta E\coth e\phi E + 1] \\ + [D^{00} - D^{33}][\coth e\theta E\coth e\phi E - 1] - [D^{30} - D^{03}][\coth e\phi E - \coth e\theta E]\} \end{aligned} \tag{B.9}$$

$$H_{12} = e^2E^2\int_0^s d\theta\tau(\theta)[D^{12} + D^{21}][\coth e\theta E\coth e\phi E + 1] \tag{B.10}$$

$$\begin{aligned}H_{13} = e^2E^2\int_0^s d\theta\tau(\theta)\{[D^{13}+D^{31}]\coth e\theta E\coth e\phi E \\ -[D^{01}-D^{10}]\coth e\phi E + [D^{01}+D^{10}]\coth e\theta E + [D^{13}-D^{31}]\}\end{aligned} \tag{B.11}$$

$$\begin{aligned}H_{33} = e^2E^2\int_0^s d\theta\tau(\theta)\{[D^{00}-D^{11}-D^{22}+D^{33}]\coth e\theta E\coth e\phi E \\ +[D^{03}+D^{30}](\coth e\phi E + \coth e\theta E) + \mathrm{tr}(D)\}\end{aligned} \tag{B.12}$$

where

$$\tau(\theta) = -\frac{\theta\phi}{16\pi^4 s^4}\exp\left\{i\mathbf{Y}\frac{e\mathbf{F}}{4}[\coth e\theta\mathbf{F} + \coth e\phi\mathbf{F}]\mathbf{Y}\right\} \tag{B.13}$$

B.1 Calculating the matrices

In the last expression, we see that we need to find the matrices D. We note the following: for any function $k^{\mu\nu}$ of $\mathbf{F}$ we can use a Taylor expansion. Taking $F^{30} = -F^{03} = E$ gives:

$$(e^{es\mathbf{F}})_\mu{}^\nu = \begin{pmatrix} \cosh esE & 0 & 0 & -\sinh esE \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh esE & 0 & 0 & \cosh esE \end{pmatrix} \tag{B.14}$$

In order to find $\mathbf{M} = \frac{1}{s}e^{es\mathbf{F}}\frac{es\mathbf{F}}{2\sinh(es\mathbf{F})}$, we need to find the matrix $\mathbf{T} = \frac{es\mathbf{F}}{\sinh es\mathbf{F}}$ which is defined by the equation:

$$\mathbf{T}.\sinh es\mathbf{F} = es\mathbf{F} \tag{B.15}$$

$$\mathbf{T}_\mu{}^\nu . \begin{pmatrix} 0 & 0 & 0 & -\sinh esE \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sinh esE & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -esE \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -esE & 0 & 0 & 0 \end{pmatrix} \tag{B.16}$$

It is clear that the elements of indices $\{1,2\}$ are ill-defined. We can find them if we start with an electromagnetic tensor that contains a magnetic field in the z direction, then after finding T we take the limit of the magnetic field to zero, we find that the right matrix to fill up the middle square of indices $\{1,2\}$ is the identity matrix. For insight to the problem, we will keep B explicit in the equations that follow, and take the limit of zero magnetic field at the end. We find:

$$\mathbf{F}_\mu{}^\nu = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ -E & 0 & 0 & 0 \end{pmatrix} \tag{B.17}$$

$$\mathbf{M}_{\mu}{}^{\nu} = \frac{1}{2s}\begin{pmatrix} esE\coth esE & 0 & 0 & -esE \\ 0 & esB\cot esB & esB & 0 \\ 0 & -esB & esB\cot esB & 0 \\ -esE & 0 & 0 & esE\coth esE \end{pmatrix} \tag{B.18}$$

Taking the limit $B \to 0$, we find that the $\{1,2\}$ diagonal elements of $\mathbf{T}$ are 1, and $\mathbf{M}$ with zero magnetic field becomes:

$$M_{\mu}{}^{\nu}(\mathbf{F}) = \frac{1}{2s}\begin{pmatrix} esE\coth esE & 0 & 0 & -esE \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -esE & 0 & 0 & esE\coth esE \end{pmatrix} \tag{B.19}$$

With the shorthand notation $y' - y = Y$, in the same way one finds the exponent in τ:

$$\tau = -\frac{\theta\phi}{16\pi^4 s^4}\exp\Big\{i\mathbf{Y}\frac{e\mathbf{F}}{4}[\coth e\theta\mathbf{F} + \coth e\phi\mathbf{F}]\mathbf{Y}\Big\} = -\frac{\theta\phi}{16\pi^4 s^4}$$
$$\exp\bigg\{\frac{i}{4}\bigg[eE(Y_0^2 - Y_3^2)(\coth[eE\phi] + \coth[eE\theta]) - (Y_1^2 + Y_2^2)(\frac{1}{\theta} + \frac{1}{\phi})\bigg]\bigg\} \tag{B.20}$$

Recall:

$$H_{33} = e^2E^2\int_0^s d\theta\tau(\theta)\{[D^{00} + D^{33}](\coth e\theta E \coth e\phi E + 1) +$$
$$[D^{03} + D^{30}](\coth e\phi E + \coth e\theta E) - [D^{11} + D^{22}](\coth e\theta E \coth e\phi E - 1)\} \tag{B.21}$$

We find:

$$D^{00} + D^{33} = \frac{e^2E^2}{-8}(Y_0^2 + Y_3^2)\frac{3 + \cosh[2eEs] - 2\cosh[2eE\phi] - 2\cosh[2eE\theta]}{\sinh^2(eE\phi)\sinh^2(eE\theta)} \tag{B.22}$$

$$D^{03} + D^{30} = e^2E^2Y^0Y^3(-2 + \frac{1}{2}(\coth eE\phi - \coth eE\theta)^2) \tag{B.23}$$

$$D^{11} + D^{22} = \frac{-is}{\theta\phi} + \frac{(\phi-\theta)^2}{4\phi^2\theta^2}(Y_1^2 + Y_2^2) \tag{B.24}$$

The second term of the response, to calculate j_3 in our gauge, is:

$$C_{33} = \frac{ie^2}{2}\int_0^\infty ds e^{-ism^2} H_{33} \tag{B.25}$$

B.2 rescaling and contour integration

We can simplify this by rescaling the variables $\theta \to s\theta$ and then $eEs \to s$. We find:

$$C_{33} = \frac{-ie^5E^3}{32\pi^4}\int_0^\infty \frac{ds}{s}e^{-ism^2/eE}\int_0^1 d\theta\theta\phi$$
$$e^{\frac{i}{4}eE[(Y_0^2-Y_3^2)(\coth[s\phi]+\coth[s\theta])-(Y_1^2+Y_2^2)/(\theta\phi s)]}\{$$
$$D_1(\coth s\theta\coth s\phi+1)+D_2(\coth s\phi+\coth s\theta)-D_3(\coth s\theta\coth s\phi-1)\} \tag{B.26}$$

with

$$D_1 = -\frac{eE}{8}(Y_0^2+Y_3^2)\frac{3+\cosh(2s)-2\cosh(2s\phi)-2\cosh(2s\theta)}{\sinh^2(s\phi)\sinh^2(s\theta)} \tag{B.27}$$

$$D_2 = eEY^0Y^3(-2+\frac{1}{2}(\coth s\phi-\coth s\theta)^2) \tag{B.28}$$

$$D_3 = \frac{-i}{\theta\phi s}+eE\frac{(\phi-\theta)^2}{4\phi^2\theta^2s^2}(Y_1^2+Y_2^2) \tag{B.29}$$

We can perform the s integral using a contour. Let us first delay talking about the pole at $s=0$.

Since $\mathrm{Re}[e^{is}]$ is even and $\mathrm{Re}[ie^{is}]$ is odd, by taking the real part of the expression the integrand becomes even in s except for the second term involving D_2, which becomes odd. This term is odd in Y_3, and when we integrate C_{33} over y_3' to find the current, it drops. For this reason, we drop it from now, and the entire integrand becomes even in s, so we can extend the integration to $\pm\infty$. We can then close the contour by taking an infinite loop in the lower imaginary plane. This picks up poles of the coth function on the negative imaginary axis, and the contribution from the infinite line integral at infinite negative imaginary values for s vanishes due to the exponent. If we are only interested in the real part of C_{33}. We have:

$$C_{33} = \frac{-e^5E^3}{64\pi^4}\mathrm{Re}[i\int_{-\infty}^\infty \frac{ds}{s}e^{-ism^2/eE}$$
$$\int_0^1 d\theta\phi e^{\frac{i}{4}eE[(Y_0^2-Y_3^2)(\coth[s\phi]+\coth[s\theta])-(Y_1^2+Y_2^2)/(\theta\phi s)]} \tag{B.30}$$
$$[D_1(\coth s\theta\coth s\phi+1)-D_3(\coth s\theta\coth s\phi-1)]$$

There are two types of poles: $s=\frac{-in\pi}{\theta}$ and $s=\frac{-in\pi}{\phi}$. For the second, we can change variables $\theta\to\phi$ in the integral, and since the integrand is symmetric under this transformation, we end up with a duplication of poles of the first type, so we use only the first poles. As for the exponent, at the poles we can use:

$$\lim_{\epsilon\to 0}\frac{1}{\sqrt{2\pi\epsilon}}e^{-\frac{x^2}{2\epsilon}} = \delta(x) \tag{B.31}$$

to find:

$$e^{\frac{i}{4}eE[(Y_0^2-Y_3^2)(\coth[s\phi]+\coth[s\theta])-(Y_1^2+Y_2^2)/(\theta\phi s)]} \to \frac{4\pi}{eE\coth(s\theta)}\delta(Y_0)\delta(Y_3) \tag{B.32}$$

and:

$$\begin{aligned}C_{33} = \frac{-e^5E^3}{64\pi^4}\,\mathrm{Re}[i(-2\pi i)\sum_{n=1}^{\infty}\frac{e^{-n\pi m^2/eE}}{-in\pi}\int_0^1 d\theta\theta^2\phi\frac{4\pi}{eE}\delta(Y_0)\delta(Y_3)\\ \frac{i}{\theta}\cot(n\pi\phi/\theta)\,\mathrm{Res}\left\{\frac{1}{\coth(s\theta)}(D_1-D_3)\right\}\end{aligned} \tag{B.33}$$

$D_1/\coth(s\theta)$ has a finite residue, but the delta functions demand $Y_0 = Y_3 = 0$ and so $D_1 = 0$. As for D_3, other than the pole at $s = 0$ it has a finite residue, but $\mathrm{Res}\left\{\frac{1}{\coth(s\theta)}\right\} = 0$, so this makes the whole expression zero except for when $s = 0$.

As for when $s = 0$, the exponent becomes a four-dimensional delta, so the only term that remains is the one in D_3 which is $\sim 1/s$. A calculation shows that the leading term of the integrand becomes $\sim 1/s^2$ which has a vanishing residue, so the entire expression is zero. In detail:

$$C_{33} \to \frac{-e^3E}{2\pi^2}\int_0^1 d\theta\phi^2\theta^2\,\mathrm{Res}\,\frac{D_1-D_3}{s}\delta^{(4)}(Y) \tag{B.34}$$

$$= -\frac{ie^3E}{12\pi^2}\delta^{(4)}(Y)\,\mathrm{Res}\,\frac{1}{s^2} = 0 \tag{B.35}$$

Appendix C

Real part of response and effective interaction Lagrangian

The Integral we wish to evaluate is:

$$I = \mathrm{Re}[-i\int_0^\infty \frac{ds}{s} e^{-ism^2}(\coth(esE) - \frac{1}{esE})] \tag{C.1}$$

By taking a derivative with respect to m^2 we can make the integrand even in s, then we integrate back.

$$\frac{d}{dm^2} I = \mathrm{Re}[-i\int_0^\infty \frac{ds}{s}(-is) e^{-ism^2}(\coth(esE) - \frac{1}{esE})] \tag{C.2}$$

$$= -\frac{1}{2}\mathrm{Re}[\int_{-\infty}^\infty ds e^{-ism^2}(\coth(esE) - \frac{1}{esE})] \tag{C.3}$$

We were able to do the last step because $\mathrm{Re}[e^{-ism^2}]$ is even in s. Now the poles are along the imaginary axis. Closing the contour from below as we did in 2.1, we find:

$$\frac{d}{dm^2} I = -\frac{1}{2}\mathrm{Re}[-2\pi i \sum_{n=1}^\infty \frac{1}{eE} e^{-n\pi m^2/eE}] \tag{C.4}$$

$$I = \mathrm{Re}[i \sum_{n=1}^\infty \frac{\pi}{eE}\int dm^2 e^{-n\pi m^2/eE}] \tag{C.5}$$

$$= -\mathrm{Re}[i \sum_{n=1}^\infty \frac{1}{n} e^{-n\pi m^2/eE}] = 0 \tag{C.6}$$

As for the effective interaction Lagrangian, we can employ the same method to show that it vanishes. The integral we need to evaluate is:

$$I_2 = \mathrm{Re}[\int_0^\infty \frac{ds}{s^2} e^{-ism^2}[\coth esE - \frac{1}{esE} - \frac{esE}{3}] \tag{C.7}$$

$$\frac{d}{dm^2} I_2 = \mathrm{Re}[-i \int_0^\infty \frac{ds}{s} e^{-ism^2}(\coth(esE) - \frac{1}{esE})] = I \tag{C.8}$$

$$\frac{d}{dm^2} I_2 = -\mathrm{Re}[i \sum_{n=1}^{\infty} \frac{1}{n} e^{-n\pi m^2/eE}] \tag{C.9}$$

$$I_2 = \mathrm{Re}[i \sum_{n=1}^{\infty} \frac{eE}{n^2\pi} e^{-n\pi m^2/eE}] = 0 \tag{C.10}$$

where we remained faithful to *minimal* subtraction by removing $-\frac{esE}{3}$ since it is no longer needed to remove the divergence, and in any case, once the residue is evaluated, it has zero residue.

Appendix D

A short justification for the form of linear approximation

To intuitively justify (4.1) we treat the analogy of the usual form of linear approximation for a function $f(x)$ is:

$$f(x + \Delta x) \approx f(x) + \Delta x f'(x) \tag{D.1}$$

by replacing $\Delta x \to -\Delta x$ we reach the form presented in (4.1):

$$f(x - \Delta x) \approx f(x) - \Delta x f'(x) \tag{D.2}$$

$$f(x) \approx f(x - \Delta x) + \Delta x f'(x) \tag{D.3}$$

We then choose the point $x - \Delta x = 0$. This form allows us to approximate $f(x)$ using the slope at x instead of the slope at 0:

$$f(x) \approx f(0) + \Delta x f'(x) \tag{D.4}$$

This is significant since the usual treatment of linear response [6] evaluates the response tensor $K_{\mu\nu}$, which plays the role of the slope, at $A = 0$. Instead, we evaluate it at x when $A(x) \neq 0$.

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