# Rational points on a family of del Pezzo surfaces of degree one
Nijgh, Wim

**Citation**

Nijgh, W. (2022). *Rational points on a family of del Pezzo surfaces of degree one*.

# Rational points on a family of del Pezzo surfaces of degree one

W. Nijgh

Master thesis

February 2022

Supervisor: prof. dr. R.M. van Luijk

Leiden University
Mathematical Institute

# Contents

# Acknowledgements

First and foremost, I would like to thank my supervisor Ronald van Luijk. Your enthusiasm is contagious and I learned so much from you this year. You were always available and you always took all the time to answer my questions. For that, I am very thankful and I am looking forward in working together a lot more in the coming years.

I would also like to thank Rosa Winter for her help in the start of this thesis and making me enthusiastic for this topic. Your dissertation was very interesting and helpful. This thesis would not exist, if it wasn't for your research.

Thanks to all my fellow students and colleagues at the MI. Especially, the people who were working with me in the TA-room; Silke, Jelle, Tijn, Jolien, Naomi, and many more. We had a lot of fun, and although I will not share an office with you anymore, I will definitely come by plenty of times in the future. Also a special thanks to Koen and Pascal for the study sessions we had during the lockdown. It really helped me to keep motivated.

Last, but not least, I want to thank my parents for supporting me in all those years. You never doubt me and you are always there for me.

# Introduction

It has been known for a very long time that there are infinitely many Pythagorean triples $(a, b, c)$ with $a, b, c$ rational numbers. All these triples are points on the affine cone $\mathcal{Q} \subset \mathbb{A}^3_{\mathbb{Q}}$ given by the equation $x^2 + y^2 = z^2$. This gives us infinitely many so called $\mathbb{Q}$-rational points. Moreover, one can show that these points are not contained in the union of curves on this cone that are defined by finitely many equation of the form $f_i = 0$ for some polynomials $f_i \in \mathbb{Q}[x, y, z]$. This means that these points are in some sense everywhere on this cone, or as we would say 'Zariski dense' in $\mathcal{Q}$.

Now suppose that $k$ is some arbitrary field. Analogously to the $\mathbb{Q}$-points on the cone $\mathcal{Q}$, we can talk about the Zariski density of $k$-rational points on surfaces (or more generally varieties). We will make these statements more precise in subsection 1.1.4. In this thesis we will look at the Zariski density of the $k$-rational points on del Pezzo surfaces. Del Pezzo surfaces are in some sense the easiest example of algebraic surfaces. More background on these facts will be given in chapter 2.

It has been conjectured that if a del Pezzo surface has a $k$-rational point and $k$ is a number field, that the $k$-points lie Zariski dense. This conjecture has been proven for almost all del Pezzo surfaces, but there is a large family of surfaces of which the question remains wide open, namely those of degree one. There are some partial results on these surfaces, which will be given in section 2.3, but for most del Pezzo surfaces of degree one we do not know yet if the above conjecture holds. Our main result of this thesis, Theorem 3.1, proves the Zariski density of the $k$-rational points for a certain family of del Pezzo surfaces of degree one.

The family of surfaces which we will look at can be described in the affine space $\mathbb{A}^3_k$ with coordinates $x, y, t$ by an equation of the form

$$y^2 = x^3 + a(f)x + b(f),$$

where $a, b \in k[u]$ are polynomials with $\deg(a) \leq 1$ and $\deg(b) \leq 2$ and $f \in k[t]$ is a polynomial with $\deg(f) = 3$. We prove that if $k$ is of characteristic zero, then, assuming some relatively mild conditions, the $k$-rational points on a surface of the described form lie Zariski dense. Moreover, we will also show that in the case that $k$ is finitely generated over $\mathbb{Q}$, these conditions need to be fulfilled in order for the $k$-points to lie Zariski dense.

Our result is a generalization of the result given in [DW21]. In particular, our research builds on top of theirs and contains their result as a special case, namely the case that $a = 0$, $\deg(b) = 2$ and $f = t^3$. Most of their ideas will come back in our proof of the main theorem, but we had to interpret their work more geometrically in order to make the arguments work more generally.

The set-up of our proof is a bit more general. We tried to assume as little as possible, in order to try and figure out how much can be done in characteristic which is not zero. Only in the last part of the proof, we really need to assume that we work over a field of characteristic zero. In this way, one can also distil partial results in the cases outside characteristic zero.

It was our aim to assume as little background as possible. We assume that the reader has some background in algebraic geometry. In particular, we assume that the reader is familiar with the definition of schemes and morphisms of schemes. We will also assume that the reader is familiar with intersection theory on surfaces on a level that is treated in for example Chapter V.1 of [Har77]. Apart from this, no further background is needed.

In Chapter 1, we will recall some more basic concepts. Topics that are treated include base change, varieties, rational points, weighted projective spaces, elliptic curves and elliptic surfaces. More advanced readers can skip this first chapter almost entirely, except from maybe the most important definitions, which are the ones that are numbered.

Chapter 2 is an introduction on del Pezzo surfaces and especially those of degree one. In this chapter we will try to embed our research in the greater scheme of del Pezzo surfaces.

Finally, Chapter 3 is about our main result, Theorem 3.1. We will state this theorem and the rest of the chapter is dedicated to prove this result. This chapter can be read by itself.

# Chapter 1

# Background

In this chapter we give some more background on the concepts which will be used in this thesis and in particular in the proof of the main theorem. We start the first section by recalling some more elementary material on base change, varieties and rational points. We will also define what we mean by the Zariski density of rational points. In the second section, we will define weighted projective spaces and show that they are projective. In the final section, we recall the definition of an elliptic curve and give a short introduction in the theory of elliptic surfaces.

## 1.1 Varieties and rational points

This section is mostly based on Chapter 2 of [Poo17]. We will discuss varieties over any arbitrary field $k$ and define scheme-valued points. We start by recalling some definitions and common notation for schemes over a base scheme.

### 1.1.1 Base schemes and base change

Let $S$ be a scheme. A *scheme $X$ over $S$*, notation $X/S$, is a scheme $X$ together with a morphism $X \to S$. The latter morphism is often called the *structure morphism.* If $S = \operatorname{Spec} R$, then alternatively $X$ is called a *scheme over $R$*, notation $X/R$. An *$S$-morphism* between two schemes $X$ and $Y$ over $S$ is a morphism of schemes $X \to Y$ such that the obvious triangle commutes. We will denote the set of $S$-morphisms between two schemes by $\operatorname{Hom}_S(X, Y)$. In particular, for every scheme $S$ there is a unique morphism $S \to \operatorname{Spec} \mathbb{Z}$, so every scheme $X$ can be viewed as a scheme over $\mathbb{Z}$.

Let $X$ be a scheme over $S$ and $S' \to S$ be a morphism of schemes. The *base change* of $X$ with respect to $S' \to S$ is the scheme $X \times_S S'$, denoted by $X_{S'}$. If $S = \operatorname{Spec} R$ and $R \to R'$ is a ring homomorphism, then the *base change* of $X$ with respect to $R \to R'$ is the scheme $X \times_{\operatorname{Spec} R} \operatorname{Spec} R'$, which we also denote by $X_{R'}$. If $f\colon X \to Y$ is a morphism of schemes over $X$, then the *base change* of $f$ with respect to $S' \to S$ is the induced morphism $f_{S'}\colon X_{S'} \to Y_{S'}$ from the pullback diagram

$$\begin{tikzcd}
X_{S'} \arrow[r, "f_{S'}"] \arrow[d] & Y_{S'} \arrow[r] \arrow[d] & S' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r] & S
\end{tikzcd}$$

Analogously, we define the base change of $f$ with respect to $R \to R'$ and denote it by $f_{R'}\colon X_{R'} \to Y_{R'}$.

We will use the above notation in particular in the following setting.

**Example 1.1.** Let $k$ be a field and let $l/k$ be a field extension. Let $X$ be a scheme over $k$. Then the base change of $X$ with respect to $l/k$ is the scheme $X_l = X \times_{\operatorname{Spec} k} \operatorname{Spec} l$. Moreover, for every morphism $f\colon X \to Y$ of schemes over $k$, we get an induced morphism $f_l\colon X_l \to Y_l$ via this fiber-product.

**Notation 1.2.** If $k$ is a field, then we denote by $\overline{k}$ an algebraic closure of $k$. We denote by $k^{\mathrm{s}}$ a separable closure of $k$ contained in $\overline{k}$. Let $X$ be a $k$-scheme. We denote by $\overline{X}$ the base change $X_{\overline{k}}$ and by $X^{\mathrm{s}}$ the base change $X_{k^{\mathrm{s}}}$.

A property $\mathcal{P}$ of schemes over a base is said to be *preserved under base change* if whenever $X/S$ has $\mathcal{P}$, every base change $X_{S'}/S'$ has $\mathcal{P}$. Similarly, a property $\mathcal{P}$ of morphisms of schemes over $S$ is said to be *preserved under base change* if whenever $f\colon X \to Y$ has $\mathcal{P}$, every base change $f_{S'}\colon X_{S'} \to Y_{S'}$ has $\mathcal{P}$. A lot of properties of morphisms are preserved under base change, see for example [Stacks, Tag 02WE].

Let $X$ be a scheme over a field $k$. A property $\mathcal{P}$ *holds geometrically* if it holds over the base change to the algebraic closure $\overline{k}$, in other words if $\mathcal{P}$ holds for $\overline{X}$. Properties that are preserved under base change will always hold geometrically if they hold over $k$. The following example shows some properties that are not stable under base change.

**Example 1.3.**

(i) Let $X$ denote the scheme $\operatorname{Spec}(\mathbb{R}[x]/(x^2+1))$ over $\mathbb{R}$. Note that this is the scheme $\operatorname{Spec}\mathbb{C}$ viewed as a scheme over $\operatorname{Spec}\mathbb{R}$. Then $X$ is connected, but it is not geometrically connected, because we have the equality

$$\overline{X} = \operatorname{Spec}(\mathbb{C}[x]/(x+i)(x-i)) \cong \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}.$$

In particular, $X$ is not geometrically irreducible and hence not geometrically integral.

(ii) Let $X$ be denote the scheme $\operatorname{Spec}(\mathbb{Q}[x,y]/(x^2-2y^2))$ over $\mathbb{Q}$. Then $X$ is integral, but it is not geometrically irreducible, since

$$\overline{X} = \operatorname{Spec}(\overline{\mathbb{Q}}[x,y]/(x-\sqrt{2}y)(x+\sqrt{2}y)).$$

(iii) Let $X$ denote the scheme $\operatorname{Spec}(\mathbb{F}_p(t)[x,y]/(y^p-tx^p))$ over $\mathbb{F}_p$. Then $X$ is integral, but not geometrically reduced since

$$\overline{X} = \operatorname{Spec}(\overline{\mathbb{F}_p(t)}[x,y]/(y-t^{1/p}x)^p).$$

Let $f\colon X \to S$ be a morphism of schemes. Let $s \in S$ be a point and denote by $\kappa(s)$ the residue field $\mathcal{O}_{S,s}/\mathfrak{m}_{S,s}$ of the point $s$. Then there is a canonical morphism $\operatorname{Spec}(\kappa(s)) \to S$ defined on the unique point by $\operatorname{Spec}(\kappa(s)) \mapsto s$ and on sheaves by the quotient map $\mathcal{O}_{S,s} \to \kappa(s)$. The *scheme theoretic fiber* or *fiber* of $f$ above $s$ is the scheme $X_s = X \times_S \operatorname{Spec}(\kappa(s))$ induced by the canonical morphism. Note that the fiber $X_s$ is naturally a scheme over the field $\kappa(s)$.

### 1.1.2 Varieties and linear systems

In this subsection, we let $k$ be a field. We will use the following definition of a variety.

**Definition 1.4.** A *$k$-variety* $X$ is a separated scheme of finite type over $k$. We say that $X$ is a *curve* if it is a variety of pure (by which we mean that every connected component is of) dimension one and a *surface* if it is of pure dimension two. A variety $X$ is *nice* if it is projective, geometrically integral and smooth over $k$.

Note that our definition of a variety deviates from [Stacks] and [Har77], because we do not assume our varieties to be integral. The advantage of this definition is that the category of varieties is now closed under taking pullbacks and so in particular under base change. Moreover, every closed and open subset of a $k$-variety will again be a $k$-variety.

Example 1.3 shows that if we did assume that our varieties are integral and not geometrically integral, that the base change of a variety does not need to be a variety. One of the drawbacks of our definition is that our varieties can have multiple irreducible components. Moreover, a variety does not have to be connected. Also reduced varieties come with their own set of problems, but we leave this discussion for now.

Let $D$ be a divisor on a nice variety $X$ over $k$. Denote by $\mathcal{L}(D)$ the Riemann-Roch space of $D$. Recall that $\mathcal{L}(D) = \{f \in \kappa(X)^* : \operatorname{div}(f) + D \geq 0\} \cup \{0\}$, which is a finite dimensional $k$-vector space. We denote its dimension by $\ell(D)$.

The *complete linear system* $|D|$ associated to $D$ is the set of effective divisors of $X$ that are linearly equivalent to $D$. The map $f \mapsto \operatorname{div}(f) + D$ induces a bijection $(\mathcal{L}(D) - \{0\})/k^* \to |D|$. If two divisors $D_1$ and $D_2$ are linearly equivalent, we get an equality $|D_1| = |D_2|$.

A *base point* of a complete linear system is a point $P \in X$ which is in the support of each divisor in $L$. We call a complete linear system *base point free* if it has no base points.

If $\ell(D) = d > 0$, choosing a basis $(f_1, \ldots, f_d)$ of $\mathcal{L}(D)$ defines a rational map $X \dashrightarrow \mathbb{P}_k^{d-1}$ by $P \mapsto (f_1(P) : \cdots : f_d(P))$. It may be possible to extend the domain of this rational map by choosing other representatives, but it will not be a morphism in all cases. By choosing another bases, we also will get another rational map. This rational map can be defined independently of the basis by using linear systems, see for example [Har77], Section II.7. Therefore, we will call such a rational map, the *map determined by the linear system* $|D|$.

The rational map associated to the linear system $|D|$ will be defined at a point $P \in X$ if and only if $P$ is not a base point of‘$|D|$. In particular, the associated rational map is a morphism if and only if the complete linear system is base point free.

We call a divisor $D$ *very ample* if the map determined by the complete linear system $|D|$ induces a closed embedding. A divisor $D$ is called *ample* if there exists a positive integer $n$ such that $nD$ is very ample.

### 1.1.3 Scheme-valued points

Let $X$ be an $S$-scheme. If $T$ is an $S$-scheme, then *the set of $T$-points on $X$* is defined as $X(T) := \mathrm{Hom}_S(T, X)$. In the case that $T = \operatorname{Spec} R$, we will denote this also as $X(R) := \mathrm{Hom}_S(\operatorname{Spec} R, X)$. In the special case that $S = \operatorname{Spec} k$ and $T = \operatorname{Spec} l$ for some field extension $l/k$, an element of $X(l)$ is called an *$l$-rational point* or *$l$-point.* Note that we forget the $S$ in the notation but the set of $T$-points on $X$ really depends on the structure morphism $X \to S$.

One can check that the above definition defines a functor

$$h_X \colon \mathbf{Schemes}_S^{\mathrm{opp}} \to \mathbf{Sets};\ T \to X(T)$$

from the opposite category of schemes over $S$ to sets. This functor is called *the functor of points.* Given a morphism $g \colon T \to T'$ of schemes over $S$, the map $h_X(f) \colon X(T') \to X(T)$ is given by the composition $\varphi \mapsto \varphi \circ g$. In a similar way, one can check that a morphism $f \colon X \to Y$ induces a natural transformation $h_f \colon h_X \to h_Y$.

**Remark 1.5.** Much more can be said about this functor of points. As a result of the Yoneda Lemma it follows that a scheme $X$ over $S$ is defined (up to unique isomorphism) by its functor $h_X$. In fact, it is already defined by the restriction of $h_X$ to affine schemes. Therefore, it is common to identify a scheme $X$ with its functor $h_X$. We will not explicitly use this in the remainder of the thesis, but it is good to keep this in mind.

**Lemma 1.6.** Let $X$ be a scheme over $S$. The functor of points has the following properties:

(i) If $U \subset X$ is an open subscheme, then we have $U(T) \subset X(T)$ for any $S$-scheme $T$.

(ii) Suppose $S = \operatorname{Spec} k$ and $l/k$ is some field extension. If $\{X_i\}$ is an open covering for $X$, then we have the equality $\bigcup X_i(l) = X(l)$.

(iii) If $S' \to S$ is a morphism of schemes and $T$ is an $S'$-scheme, then there is an equality $X_{S'}(T) = X(T)$, where on the right we view $T$ as an $S$-scheme via the composition $T \to S' \to S$.

(iv) Suppose $X$ is separated over $S$. If $T' \to T$ is a scheme-theoretically dominant $S$-morphism, meaning that its image is dense and the corresponding map on sheaves is injective, then the induced map $X(T) \to X(T')$ is injective.

**Proof.** Statement (i) is clear, because the inclusion $\iota\colon U \to X$ induces an injective map $U(T) \to X(T)$ by composing an element of $U(T)$ with $\iota$. For (ii), observe that for any $k$-morphism $\operatorname{Spec} l \to X$, the image is exactly one point, and hence is in some $X_i$. For (iii) and (iv) see Proposition 2.3.15 and 2.3.21 of [Poo17]. ☐

We give some examples with varieties to make the results of Lemma 1.6 more concrete.

**Example 1.7.**

(i) Let $X$ denote the $n$-dimensional affine plane $\mathbb{A}^n_{\mathbb{R}} = \operatorname{Spec}(\mathbb{R}[x_1, \dots, x_n])$. Then we have a bijection $X(\mathbb{R}) \cong \operatorname{Hom}_{\mathbb{R}}(\mathbb{R}[x_1, \dots, x_n], \mathbb{R}) \cong \mathbb{R}^n$. Moreover, by part (iii) of Lemma 1.6, we also have that $X(\mathbb{C}) = \overline{X}(\mathbb{C}) \cong \mathbb{C}^n$. Part (iv) of Lemma 1.6 gives us an inclusion $X(\mathbb{R}) \hookrightarrow X(\mathbb{C})$. It this case, this exactly corresponds to the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{C}^n$.

(ii) Every projective space $\mathbb{P}^n_k$ is covered by the affine patches

$$U_i = \operatorname{Spec} k\left[\frac{X_0}{X_i}, \dots, \frac{X_{i-1}}{X_i}, \frac{X_{i+1}}{X_i}, \dots, \frac{X_n}{X_i}\right] \cong \mathbb{A}^n_k.$$

Part (ii) of Lemma 1.6 gives us that $\mathbb{P}^n_k(l) = \bigcup_{i=0}^{n} U_i(l)$ for each field extension $l/k$.

(iii) Let $X$ be a quasi-projective $k$-variety $X$, i.e. a $k$-variety $X$ such that there is an embedding $\iota\colon X \hookrightarrow \mathbb{P}^n_k$ that is locally closed. By part (i) of Lemma 1.6, we get an inclusion $X(l) \hookrightarrow \mathbb{P}^n_k(l)$ for each field extension $l/k$. Moreover, every quasi-projective scheme $X$ is separated and so by part (iv) we can identify $X(l)$ with a subset of $\mathbb{P}^n_k(\bar{l})$.

(iv) Let $l/k$ be a field extension and let $X$ be a quasi-projective $l$-variety. By part (iii) of Lemma 1.6, we have that $\mathbb{P}^n_k(l) = \mathbb{P}^n_l(l)$. Then following the previous example, we can identify $X(l)$ with a subset of $\mathbb{P}^n_k(\bar{l})$.

### 1.1.4 Zariski density of rational points on varieties

Note that in the section before, we defined rational points as morphisms. At this point, it is therefore not at all clear, what is meant for these points to lie dense in the Zariski topology of a variety. To give a hint of what is meant, we start this section with a (hopefully) motivational example.

**Example 1.8.** Let $X$ denote the affine line over $\mathbb{Q}$, in other words $X$ is the $\mathbb{Q}$-scheme $\mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec}(\mathbb{Q}[x])$. Then the following sets are in bijection:

(i) the set of closed points of $X$;

(ii) the set of maximal ideals of $\mathbb{Q}[x]$;

(iii) the set of monic irreducible polynomials of $\mathbb{Q}[x]$;

(iv) the set of $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-orbits in $X(\overline{\mathbb{Q}}) = \overline{\mathbb{Q}}$.

We can check this as follows. A point of $X$ is closed if and only if it is a maximal ideal. Every maximal ideal of $\mathbb{Q}[x]$ is of the form $(f)$ for a unique monic irreducible polynomial. Every monic irreducible polynomial defines an algebraic extension $\mathbb{Q}[x]/(f) \subset \overline{\mathbb{Q}}$ of $\mathbb{Q}$. The set of roots of this extension gives us a unique Galois-orbit in $\overline{\mathbb{Q}}$. This shows that all the above sets are indeed in bijection with each other. Moreover, as a consequence we have that the $\mathbb{Q}$-rational points $X(\mathbb{Q})$ are points with Galois-orbit of size 1, which correspond to the points $x \in X$ with residue field $\mathbb{Q}$.

The above result generalizes to arbitrary varieties in the following way. In the remainder of this section we let $k$ be a field. Identify the set of field automorphisms $\operatorname{Aut}_k(\overline{k})$ over $k$ of $\overline{k}$ with the Galois group $G_k := \operatorname{Gal}(k^s/k)$. Each $g \in G_k$, defines an automorphism $\overline{k} \to \overline{k}$ that fixes the ground field $k$. This morphism induces a morphism $g^* \colon \operatorname{Spec} \overline{k} \to \operatorname{Spec} \overline{k}$ over $k$.

Now let $X$ be a scheme over $k$. We get the following action of the group $G_k$ on the set $X(\overline{k})$ of $\overline{k}$-points of $X$. For each $g \in G_k$ and each $\overline{k}$-point $p \in X(\overline{k})$, the composition $p \circ g^* \in X(\overline{k})$ gives again a $\overline{k}$-point of $X$. This gives a well-defined right action on $X(\overline{k})$.

We get the following result for the orbits of $X(\overline{k})$ under $G_k$ in the case that $X$ is a variety.

**Proposition 1.9.** Let $X$ be a $k$-variety. Then the map

$$\begin{aligned} \{\text{Galois orbits in } X(\overline{k})\} &\to \{\text{closed points in } X\} \\ \text{orbit of } (f \colon \operatorname{Spec} \overline{k} \to X) &\mapsto f(\operatorname{Spec} \overline{k}) \end{aligned}$$

is a bijection.

**Proof.** Proposition 2.4.6 of [Poo17]. □

Recall from part (iv) of Lemma 1.6 that we have an inclusion $X(k) \subset X(\overline{k})$ and so the above proposition gives us the following corollary.

**Corollary 1.10.** The $k$-points $X(k) \subset X(\overline{k})$ are points with Galois orbit 1 and correspond under this bijection with closed points $x \in X$ with residue field $k$.

This bijection motivates why we often identify a $k$-rational point of $X$ with a point in $X$. This leads to the following definition of what it means for $k$-rational points to lie Zariski dense.

**Definition 1.11.** Let $X$ be a $k$-variety. We call $X(k)$ *Zariski dense in* $X$, or say that the *rational points lie Zariski dense in* $X$, if under the identification of Proposition 1.9 we have $\overline{X(k)} = X$.

Before we continue our discussion, we take a small side step and state some properties of $k$-varieties of which the rational points lie Zariski dense. We will not use this in the remainder of the thesis, but it is nice result about the geometry of varieties in which the $k$-rational points lie Zariski dense.

**Proposition 1.12.** Let $X$ be a $k$-variety such that $X(k)$ is Zariski dense in $X$.

1. If $X$ is irreducible, then $X$ is geometrically irreducible.
2. If $X$ is reduced, then $X$ is geometrically reduced.
3. If $X$ is integral, then $X$ is geometrically integral.

**Proof.** Proposition 2.3.26 of [Poo17]. □

Proposition 1.12 is often used to show that the $k$-points lie not Zariski dense.

Another nice result, which we will use in this thesis, is due to Châtelet.

**Proposition 1.13** (Châtelet)**.** Let $X$ be a $k$-variety of dimension $n$ such that $\overline{X} \cong \mathbb{P}^n_{\overline{k}}$. Then the following are equivalent:

(i) $X \cong \mathbb{P}^n_k$.

(ii) $X$ is birational to $\mathbb{P}^n_k$.

(iii) $X(k) \neq \emptyset$.

**Proof.** For a complete proof, we refer to Proposition 4.5.10 of [Poo17] □

## 1.2 Weighted projective space

In this thesis we will encounter weighted projective spaces. In this section we will define these spaces as a scheme and show that they can be embedded in projective space. We will also give an explicit embedding for the weighted projective space $\mathbb{P}_k(2,1,1)$ over a field $k$. This section is based on [Stacks, Tag 00JL, Tag 00JM and Tag 01M3].

### 1.2.1 Graded rings and the homogeneous spectrum

A *graded ring* $S$ is a commutative ring endowed with a direct sum decomposition $S = \bigoplus_{d \geq 0} S_d$ of the underlying abelian group such that $S_d \cdot S_e \subset S_{d+e}$ for all $d, e \geq 0$. The *irrelevant ideal* is the ideal $S_+ = \bigoplus_{d>0} S_d$. An element $f \in S$ is called *homogeneous* if $f \in S_d$ for some $d$. This $d$ is called the degree of $f$, denoted $\deg(f)$.

A *graded module* is an $S$-module $M$ endowed with a direct sum decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ of the underlying abelian group such that $S_d \cdot M_e \subset M_{d+e}$ for all $d \geq 0$ and $e \in \mathbb{Z}$. A *homogeneous ideal of* $S$ is an ideal $I \subset S$ which is also a graded submodule of $S$.

**Example 1.14.** Let $R$ be a ring. Then every polynomial ring $R[x_1, \ldots, x_n]$ is a graded ring with the grading induced by $\deg(x_i) = a_i$ for some integers $a_i \geq 0$. Note that the choice of different $a_i$ give different, possibly non-isomorphic, graded rings.

Let $S$ be a graded ring. We define $\operatorname{Proj} S$ to be the set of homogeneous ideals $\mathfrak{p}$ of $S$ such that $S_+ \not\subset \mathfrak{p}$. The set $\operatorname{Proj} S$ is a subset of $\operatorname{Spec} S$ and we endow it with the induced topology. The topological space $\operatorname{Proj} S$ is called the *homogeneous spectrum of the graded ring* $S$. Moreover, by [Stacks, Tag 01MB] there is a construction of a sheaf on $\operatorname{Proj} S$ making it into a scheme. For the construction of this sheaf, see [Stacks, Tag 01M3].

For some homogeneous polynomial $f \in S_e$, we define the set

$$D_+(f) := \{\mathfrak{p} \in \operatorname{Proj} S \mid f \notin \mathfrak{p}\}.$$

For a homogeneous ideal $I \subset S$, we also define

$$V_+(I) = \{\mathfrak{p} \in \operatorname{Proj} S \mid I \subset \mathfrak{p}\}.$$

The most important properties of these sets are summarized in the following lemma.

**Lemma 1.15.** Let $S$ be graded ring, let $I \subset S$ be a homogeneous ideal and let $f \in S_d$ for some $d > 0$. Then

(i) the set $D_+(f)$ is an affine open subset and there is a natural isomorphism $D_+(f) \cong \operatorname{Spec} S_{(f)}$, where $S_{(f)}$ denotes the degree zero part of the localization with respect to $f$;

(ii) the sets $D_+(f)$ form a basis for the topology of $\operatorname{Proj} S$;

(iii) for $g, h \in S$ homogeneous, we have $D_+(gh) = D_+(g) \cap D_+(h)$;

(iv) $V_+(I)$ is closed;

(v) $V_+(I) = \emptyset$ if and only if $S_+ \subset \sqrt{I}$;

(vi) any closed subset $T \subset \operatorname{Proj} S$ is of the form $V_+(J)$ for some homogeneous $J \subset S$.

**Proof.** [Stacks, Tag 00JP and Tag 01MB] □

**Lemma 1.16.** Let $S$ be a graded ring. Then $\operatorname{Proj} S$ is quasi-compact, meaning that every open cover of $\operatorname{Proj} S$ has a finite subcover, if and only if there are homogeneous elements $f_1, \ldots, f_n \in S_+$ such that $S_+ \subset \sqrt{(f_1, \ldots, f_n)}$. Moreover, for such $f_1, \ldots, f_n \in S_+$ we have that $\operatorname{Proj} S = \bigcup\limits_{i=1}^{n} D_+(f_i)$.

**Proof.** [Stacks, Tag 01MD] □

**Definition 1.17.** Let $R$ be a ring. A *weighted projective space over $R$* is a scheme of the form $\operatorname{Proj}(R[X_0, \ldots, X_n])$, where each $X_i$ has degree $a_i > 0$. We denote it by $\mathbb{P}_R(a_0, \ldots, a_n)$.

Note that if we take $a_i = 1$ for all $i$, we just get *$n$-dimensional projective space over $R$*, denoted as usual by $\mathbb{P}^n_R$. Every weighted projective space over $R$ comes with a canonical morphism to $\operatorname{Spec} R$ which is separated. Every weighted projective space $\mathbb{P}_k(a_0, \ldots, a_n)$ is a $k$-variety. This $k$-variety is not necessarily smooth, which we will show in the next example.

**Example 1.18.** Let $k$ be a field and let $S := k[X, Y, Z]$ be a graded ring with $\deg(X) = 2$ and $\deg(Y) = \deg(Z) = 1$. Let $W$ be the weighted projective space $W := \mathbb{P}_k(2,1,1) = \operatorname{Proj} S$ and let $x = (1:0:0) \in W$. The local ring at the point $x$ is given by

$$\mathcal{O}_{W,x} := \left\{ \frac{f}{g} : d \geq 0,\ f, g \in k[X,Y,Z]_d,\ g(1,0,0) \neq 0 \right\}.$$

The dimension of this ring is 2.

Note that because $g(1,0,0) \neq 0$, it follows that there are no fractions such that $\deg(g)$ is odd as some power of $X$ has to occur in $g$. We deduce that the maximal ideal $\mathfrak{m}_{W,x}$ of this local ring is generated by the elements $\frac{Y^2}{X}, \frac{YZ}{X}, \frac{Z^2}{X}$. In particular, the equivalence classes of these elements generate the $k$-vector space $\mathfrak{m}_{W,x}/\mathfrak{m}_{W,x}^2$. We can check that the equivalence classes of $\frac{Y^2}{X}$, $\frac{YZ}{X}$ and $\frac{Z^2}{X}$ are $k$-linearly independent and so the dimension $\dim \mathfrak{m}_{W,x}/\mathfrak{m}_{W,x}^2$ is 3. We deduce that this ring is not regular.

From the above result it follows that the point $(1:0:0)$ on $W$ is a singular point and from [Stacks, Tag 056S] it follows that $W$ is not a smooth variety. This result could alternatively, and maybe more easily, be deduced by using the isomorphism which we will see in Example 1.24.

Similarly as with projective space, weighted projective spaces can be covered by a standard set of affine opens.

**Lemma 1.19.** The set $(D(X_i))_{0 \leq i \leq n}$ is an open cover for $\mathbb{P}_R(a_0, \dots, a_n)$. In the case that $a_i = 1$, we have

$$D(X_i) \cong \operatorname{Spec}(R[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]) = \mathbb{A}_R^n,$$

where $x_j = X_j/X_i^{a_j}$.

**Proof.** Because $S_+ = (X_0, \dots, X_n)$, the first part follows from Lemma 1.16. The second part follows from the isomorphism of part (i) of Lemma 1.15. □

### 1.2.2 Embedding in projective space

Let $S$ be a graded ring and $d \geq 1$ be an integer. We define the *d-th Veronese subring* to be the subring $S^{(d)}$ given by $S^{(d)} := \bigoplus_{e \geq 0} S_{de}$. We give this ring the grading $(S^{(d)})_e = S_{de}$. We have the following results for $S^{(d)}$ and $\operatorname{Proj} S^{(d)}$. We will also give the complete proof of the Lemma 1.20, which will hopefully be motivational in understanding example 1.24.

**Lemma 1.20.** Let $S$ be a graded ring. If $\operatorname{Proj} S$ is quasi-compact, then $\operatorname{Proj} S \cong \operatorname{Proj} S^{(d)}$ for all $d \geq 1$.

**Proof.** Let $d \geq 1$ be given. By Lemma 1.16, we can choose some homogeneous elements $f_1, \ldots, f_n \in S_+$ such that both $S_+ \subset \sqrt{f_1 S + \cdots + f_n S}$ and

$$\operatorname{Proj} S = \bigcup_{i=1}^{n} D_+(f_i)$$

holds. Note that we may assume that all the $f_i$ are non-nilpotent elements. Denote by $I$ the ideal of $S$ given by $I = f_1 S + \cdots + f_n S$ and by $J$ the ideal of $S^{(d)}$ given by $J = f_1^d S^{(d)} + \cdots + f_n^d S^{(d)}$. Denote by $d_i$ the degree $\deg(f_i)$.

For each homogeneous $g \in S_+^{(d)}$ we have that $g \in \sqrt{I}$. This means that there is an equality $g^m = \sum_{i=1}^n a_i f_i$ for some $m \geq 1$ and $a_i \in S$. Moreover, we may assume that all the $a_i$ are homogeneous with $\deg(a_i) = m \cdot \deg(g) - d_i$.

Note that $g^{mnd} = (\sum_{i=1}^n a_i f_i)^{nd}$. We deduce that each term of $g^{mnd}$ is divisible by some term $f_i^d$. Hence, we get an equality $g^{mnd} = \sum\limits_{i=1}^{n} b_i f_i^d$ for some $b_i \in S$. By the assumption that $a_i$ is homogeneous, we have that $b_i$ must be homogeneous with $\deg(b_i) = mnd \cdot \deg(g) - d \cdot d_i$. Hence, $d \mid \deg(b_i)$ and so $b_i \in S^{(d)}$. It follows that $g^{mnd} \in J$. We deduce that $g \in \sqrt{J}$ and so $S_+^{(d)} \subset \sqrt{J}$ holds.

By Lemma 1.16 it now follows that

$$\operatorname{Proj} S^{(d)} = \bigcup_{i \in I} D_+(f_i^d),$$

where $D_+(f_i^d)$ is viewed as a subset of $\operatorname{Proj} S^{(d)}$. By part (i) of Lemma 1.15, we have that both $D_+(f_i) \cong \operatorname{Spec} S_{(f_i)}$ and $D_+(f_i^d) \cong \operatorname{Spec} S^{(d)}_{(f_i^d)}$. We also have the equalities

$$\begin{aligned}
S^{(d)}_{(f_i^d)} &= \left\{ \frac{h}{f_i^{de}} \mid e \geq 0,\ h \in (S^{(d)})_{d_i \cdot e} \right\} \text{ and} \\
S_{(f_i)} &= \left\{ \frac{g}{f_i^m} \mid m \geq 0,\ g \in S_{d_i \cdot m} \right\} \\
&= \left\{ \frac{g f_i^{de - m \deg(f_i)}}{f_i^{de}} \mid m \geq 0,\ e \geq \frac{m \cdot d_i}{d},\ g \in S_{d_i \cdot m} \right\}.
\end{aligned}$$

From this description it is clear that these two sets are in bijection. This bijection gives us an isomorphism of schemes $D_+(f_i) \cong D_+(f_i^d)$ for each $f_i$. These isomorphisms are compatible with the gluing of the affine opens and we may conclude that $\operatorname{Proj} S \cong \operatorname{Proj} S^{(d)}$. □

**Lemma 1.21.** Let $S$ be a graded ring which is finitely generated as an $S_0$-algebra. For some sufficiently divisible $d$, the graded ring $S^{(d)}$ is generated by $S_d$ as an $S_0$-algebra.

**Proof.** [Stacks, Tag 0EGH] ☐

With the above lemmas, we can construct an embedding of a weighted projective space in a projective space, which leads to the following proposition.

**Proposition 1.22.** Let $S$ be a graded ring. If $\operatorname{Proj} S$ is quasi-compact, then it can be embedded in a projective space $\mathbb{P}^n_{S_0}$ over $S_0$ for some integer $n \geq 0$.

**Proof.** By Lemma 1.20 and Lemma 1.21 we may assume that $\operatorname{Proj} S$ is generated by $S_1$. Now let $\{f_0, \ldots, f_n\}$ be a set of generators with $f_i \in S_1$. Then the map $\varphi\colon S_0[X_0, \ldots, X_n] \to S$ defined by $X_i \mapsto f_i$ is a surjective morphism of graded rings and hence $S$ is isomorphic to $S_0[X_0, \ldots, X_n]/\ker\varphi$. This isomorphism induces a isomorphism of $\operatorname{Proj} S$ with the corresponding closed subscheme of $\mathbb{P}^n_{S_0}$. ☐

**Corollary 1.23.** Let $k$ be a field. A weighted projective space $\mathbb{P}_k(a_0, \ldots, a_n)$ over $k$ can be embedded in projective space over $k$.

In the following example, we will give an embedding of the weighted projective space $\mathbb{P}_k(2,1,1)$ in $\mathbb{P}^3_k$. This embedding will come back in the following chapters.

**Example 1.24.** Let $k$ be a field. The weighted projective space $\mathbb{P}_k(2,1,1)$ is isomorphic to the closed subscheme $\mathcal{Q} \subset \mathbb{P}^3_k$ with coordinates $p, q, r, s$ defined by

$$\mathcal{Q} := \operatorname{Proj}\left(k[p,q,r,s]/(r^2 - qs)\right).$$

This isomorphism is constructed as follows. Write $S = k[X, Z, W]$ for the graded ring with $\deg(X) = 2$ and $\deg(Z) = \deg(W) = 1$. Then we can identify the weighted projective space as $\mathbb{P}_k(2,1,1) = \operatorname{Proj} S$. Lemma 1.20 gives us that $\operatorname{Proj} S \cong \operatorname{Proj} S^{(2)}$.

Observe that the elements $X, Z^2, ZW, W^2 \in S^{(2)}_1$ generate $S^{(2)}$ as a $k$-algebra, because every element of $S_{2d}$ for $d \geq 1$ is a $k$-linear sum of products of $d$ of these elements. The map $k[p,q,r,s] \to S^{(2)}$ defined by $p \mapsto X$, $q \mapsto Z^2$, $r \mapsto ZW$ and $s \mapsto W^2$ is surjective with kernel $(r^2 - qs)$. This map induces an isomorphism between the schemes Proj $S^{(2)}$ and $\mathcal{Q}$. This shows that $\mathbb{P}_k(2,1,1)$ is isomorphic to $\mathcal{Q}$.

We can also make this isomorphism explicit by looking what happens on the affine pieces of $\mathbb{P}_k(2,1,1)$. Observe that we will closely follow the proof of Lemma 1.20. Recall from Lemma 1.19 that $\mathbb{P}_k(2,1,1)$ is covered by the affine opens $D_+(X)$, $D_+(Z)$ and $D_+(W)$. Part (i) of Lemma 1.15 gives us that

$$D_+(X) \cong \operatorname{Spec}(k[y_1, y_2, y_3]/(y_2^2 - y_1 y_3)),$$

where $y_1 := Z^2/X$, $y_2 := ZW/X$ and $y_3 := W^2/X$. By identifying $y_1 = q/p$, $y_2 = r/p$ and $y_3 = s/p$ on the affine open $D_+(p) \subset \mathbb{P}_k^3$ we get an injective morphism $D_+(X) \hookrightarrow \mathbb{P}_k^3$. One can check that this extends to a well-defined isomorphism with image $\mathcal{Q}$. This isomorphism is defined locally on the affine opens $D_+(Z) = \operatorname{Spec}(k[X/Z^2, W/Z])$ and $D_+(W) = \operatorname{Spec}(k[X/W^2, Z/W])$ as $(a,b) \mapsto (a:1:b:b^2)$ and $(a,b) \mapsto (a:b^2:b:1)$ respectively.

## 1.3 Elliptic surfaces

In the proof of the main theorem, we will encounter an elliptic surface. Also, the blowup of a del Pezzo surface of degree 1 in the base point of the anticanonical system will be an elliptic surface. Therefore, we will shortly discuss some results on elliptic surfaces. Before we talk about elliptic surfaces, we will first talk about two other algebraic structures, namely elliptic curves and fibered surfaces.

In this section we let $k$ be an arbitrary field and $l/k$ some field extension.

### 1.3.1 Elliptic curves and Weierstrass equations

We recall the definition of an elliptic curve and its most important properties. For details we refer to sections III.1 to III.3 of [Sil09].

An *elliptic curve $E$ over $k$* is a pair $(E, O)$ such that $E$ is a nice curve over $k$ of genus one and $O \in E(k)$ is a $k$-rational point. An important property of elliptic curves is that the set of $l$-points $E(l)$ forms an abelian group with $O$ as neutral element. This group structure comes from the bijection $E(l) \mapsto \operatorname{Pic}^0(E_l)$ given by $P \mapsto [P] - [O]$.

Let $(E, O)$ be an elliptic curve. Then this curve can be embedded in $\mathbb{P}_k^2$ by the map determined by linear system of $|3O|$. The image of this embedding is given on an affine piece by the zero set of an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \text{ with } a_i \in k. \tag{1.1}$$

The point $O$ corresponds under this embedding to the unique point at infinity.

Equation (1.1) is called a *Weierstrass equation.* Every Weierstrass equation has a discriminant $\Delta$ which is a value in $k$ depending on the coefficients $a_i$. Given a Weierstrass equation with coefficients $a_i$ in a field $k$, it defines a curve of arithmetic genus 1 in $\mathbb{P}^2_k$, which is a smooth curve if and only if this discriminant $\Delta$ is not equal to zero.

In the case that $\mathrm{char}(k) \neq 2$, by completing the square on the left-hand side, we can take $a_1 = a_3 = 0$. Similarly, in the case that $\mathrm{char}(k) \neq 3$, by completing the cube on the other side, we can assume $a_2 = 0$. A Weierstrass equation such that $a_1 = a_2 = a_3 = 0$ is called a *short Weierstrass equation.*

A projective curve in $\mathbb{P}^2_k$ which is on an affine part given by a Weierstrass equation as in (1.1) is called a *Weierstrass curve.* There will be one unique point that is not on this affine. This point is also called the *point at infinity.* Note that we did not assume that a Weierstrass curve is smooth.

From now on, let $E$ denote a Weierstrass curve and let $O \in E(k)$ denote the point at infinity. Let $E_{\mathrm{ns}}$ denote the smooth part of $E$. If $E$ is not smooth, then there is an equality $E_{\mathrm{ns}} = E - \{P\}$ for some unique singular point $P \in E$. We will define an abelian group structure on the set of $l$-rational points $E_{\mathrm{ns}}(l)$ of $E_{\mathrm{ns}}$.

This group structure is very geometric and can be described as follows. Given a line in $\mathbb{P}^2_k(l)$, by Bézout's theorem, it will intersect the curve $E$ in three, possibly non-distinct, points $P, Q, R \in E(l)$. If none of these points is a singular point, then these points will add up to the point $O$. Moreover, $P, Q, R \in E_{\mathrm{ns}}(l)$ are collinear if and only if $P + Q + R = O$.

In the remainder of this subsection we will give explicit formulas for this group structure of $E$. The unique point at infinity $O \in E_{\mathrm{ns}}(l)$ is the neutral element of this group structure. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in $(E_{\mathrm{ns}} - \{O\})(l)$. We define $-P_1 := (x_1, -y_1 - a_1x_1 - a_3)$ and we define $P_1 + P_2$ to be $O$ if $P_2 = -P_1$. Otherwise, $P_1 + P_2$ is defined as follows. Let $y = \lambda x + \nu$ be an equation for the unique line through the points $P_1$ and $P_2$, or if $P_1 = P_2$, for the tangent line to $E$ at this point. Then we define

$$P_1 + P_2 = \left(\lambda^2 + a_1\lambda - a_2 - x_1 - x_2, -(\lambda + a_1)x_3 - \nu - a_3\right),$$

where $x_3$ denotes the $x$-coordinate of this new point $P_1 + P_2$.

One can check that this is a well-defined group structure which can be characterized as follows. In the smooth case, this group structure equals the group law given by the bijection with the degree zero part of the Picard group of $E$. In the singular case, if the singular point is a cusp, then $E_{\mathrm{ns}}(l)$ is isomorphic to the additive group $l^+$. Or else, if the singular point is a node, then $E_{\mathrm{ns}}(l)$ is isomorphic to multiplicative group $l^*$ or some twist of $l^*$, see Exercise 3.5 of [Sil09].

**Remark 1.25.** A point $P \in E(l)$ on the elliptic curve $E$ has order two if and only if it is a non-singular point $P = (x_P, y_P) \in (E_{\mathrm{ns}} - \mathcal{O})(l)$ on the affine part of the curve such that we have the equality $y_P = -y_P - a_1 x_P - a_3$. Moreover, if $P \in (E - O)(l)$ is a point such that this equality holds then it is either the singular point or a point of order two.

There is a special automorphism on each Weierstrass curve, which is given as follows.

**Definition 1.26.** Recall that $E$ denotes a Weierstrass curve in $\mathbb{P}^2_k$ given by the local equation (1.1). We define the automorphism $[-1]_E \colon E \to E$ by $O \mapsto O$ and

$$(x, y) \mapsto (x, -y - a_1 x - a_3) \text{ for all } P = (x, y) \in (E - O)(l).$$

Note that we only defined it on $l$-points, but one can show that this assignment really defines an automorphism $E \to E$. In the case that $E$ is smooth, the above defined morphism sends each point to its inverse. In the case $E$ has a singular point, the singular point gets mapped to itself, and on the smooth points it still maps each point to its inverse. Together with Remark 1.25 we get the following result.

**Lemma 1.27.** Let $P \in E(l)$ be an $l$-rational point on the Weierstrass curve $E$. Then the following are equivalent.

(i) $P$ is a fixed point of the map $[-1]_E \colon E \to E$;

(ii) $P$ is singular or $P$ is a smooth point of order at most two.

(iii) $P = O$ or $\dfrac{df}{dy}(P) = 0$, where $f \in k[x, y]$ is given by

$$f = y^2 + a_1 xy + a_3 y - x^3 - a_2 x^2 - a_4 x - a_6.$$

**Proof.** Statement (i) is equivalent to (ii) by the previous discussion. We prove that (ii) is equivalent to (iii). Note that $\frac{df}{dy} = 2y + a_1x + a_3$.

(ii)⇒(iii) For $P = O$ there is nothing to prove. For the points of order two, the equality holds by the first part of Remark 1.25. And for a singular point the partial derivative has to vanish.

(iii)⇒(ii) Again for $P = O$ there is nothing to prove, so suppose that $\frac{df}{dy}(P) = 0$ for $P = (x_P, y_P)$. Then we get the equality $2y_P + a_1x_P + a_3 = 0$. We conclude by the second part of Remark 1.25 that (ii) holds. □

### 1.3.2 Fibered surfaces

An elliptic surface is an example of a surface with a fibration. In the literature, there is no clear definition of what a fibration should be. In this thesis, if we talk about a fibration, we mean the following.

**Definition 1.28.** A *fibration* is a surjective morphism $f\colon X \to C$ where $X$ is a surface over $k$ and $C$ denotes a geometrically integral curve over $k$. If $X \to C$ is a fibration, we call $X$ a fibered surface.

We can see such a fibered surface $X$ as a family of curves parametrized by the curve $C$. Moreover, every fiber of such a morphism will be a curve on $X$. Because we assume in the definition of a fibration that $C$ is a geometrically integral curve, the curve $C$ has a unique generic point. The *generic fiber* of a fibration $f\colon X \to C$ is the scheme-theoretic fiber of the generic point of $C$. This generic fiber is a curve that is defined over the function field of $C$.

Let $f\colon Y \to Z$ be a morphism of varieties over $k$. Recall that a *section of* $f$ is a morphism $\sigma\colon Z \to Y$ such that $f \circ \sigma = \mathrm{id}_Z$. We denote the set of sections of $Y \to Z$ by $Y(Z)$. Note that this notation agrees with the notation of the functor of points if we view $Y$ as scheme over $Z$. An *$l$-section of* $f$ is a section of the base change $f_l\colon Y_l \to Z_l$. If $f\colon X \to C$ is a fibration and $\sigma\colon C \to X$ is a section, with abuse of language, we will also call the image $\sigma(C)$ a section.

**Example 1.29.** Set $C = \operatorname{Spec} k[t]$. Define

$$X := \operatorname{Spec}\big(k[t][x,y]/(y^2 - x^3 - ax)\big) \text{ for some element } a \in k[t].$$

Then the map $k[t] \to k[t][x,y]/(y^2 - x^3 - ax)$ induces a morphism $X \to C$ which makes $X$ a fibered surface. The generic fiber of this morphism is given by the curve $\operatorname{Spec}\big(k(t)[x,y]/(y^2 - x^3 - ax)\big)$. The map $k[t][x,y]/(y^2-x^3-ax) \to k[t]$ defined by $x \mapsto 0$, $y \mapsto 0$ and $t \mapsto t$ gives a section $\sigma\colon C \to X$ of $f$. This map is given on points by the assignment $t' \mapsto (t',(0,0))$. In other words, on each fiber $t = t'$, we get the point $(0,0)$.

Let $f\colon X \to C$ be a fibration and $\sigma\colon C \to X$ a section of $f$. Let $\kappa(C)$ denote the function field of the curve $C$. Let $\eta$ denote the generic point of $C$ and let $X_\eta$ denote the generic fiber of $f$. Note that $\kappa(\eta) = \kappa(C)$ and let $\operatorname{Spec}\kappa(C) \to C$ be the canonical morphism corresponding to the point $\eta$.

Each section $\sigma\colon C \to X$ of the fibration $f$ gives a unique $\kappa(C)$-point $\sigma^*$ on the generic fiber $X_\eta$ of $f$ induced by the following pullback diagram.

$$\begin{tikzcd}
X_\eta \arrow[r] \arrow[d] & X \arrow[d, "f"'] \\
\operatorname{Spec}\kappa(C) \arrow[r] \arrow[u, dotted, bend left, "\sigma^*"] & C \arrow[u, bend right, "\sigma"']
\end{tikzcd} \tag{1.2}$$

Moreover, if $f$ is proper and $C$ is non-singular, then the following lemma shows that there is a bijective correspondence between the $\kappa(C)$-points on the generic fiber of $f$ and the sections $C \to X$.

**Lemma 1.30.** The map $X(C) \to X_\eta(\kappa(C))$ defined by $\sigma \mapsto \sigma^*$ is injective. Moreover, if $f$ is proper and $C$ is non-singular, then this map is a bijection.

**Proof.** Recall that $C$ is a geometrically integral curve, so in particular it is reduced. Moreover, the morphism $f\colon X \to C$ is locally of finite type. It follows from [Stacks, Tag 0BX8] that there is a bijection between the set of $C$-rational maps $C \dashrightarrow X$ and the set of tuples $(x,\phi)$, where $x \in X$ lies over the generic point $\eta$ of $C$ and $\phi\colon \mathcal{O}_{X,x} \to \kappa(C)$ is a local ring map. This bijection is given by $\sigma \mapsto (\sigma(\eta), \sigma_\eta^\#)$.

Observe that for each morphism, the tuple $(\sigma(\eta), \sigma_\eta^\#)$ exactly defines a $\kappa(C)$-point $\operatorname{Spec}\kappa(C) \to X_\eta$ which defines a map that fits in diagram (1.2). In other words, this exactly defines the map $\sigma^*\colon C \dashrightarrow X$. We deduce that the map $X(C) \to X_\eta(\kappa(C))$; $\sigma \mapsto \sigma^*$ is injective.

To prove surjectivity, assume that $f\colon C \to X$ is proper and $C$ is non-singular. Because we assumed that $f$ is proper and $C$ is non-singular, the result of [Stacks, Tag 0BX7] shows that we can extend every rational map to a $C$-morphism $C \to X$. We conclude that the set of $C$-rational maps $C \dashrightarrow X$

equals the set of $C$-morphisms $C \to X$. Hence, if $f$ is proper and $C$ is non-singular, the map $X(C) \to X_\eta(\kappa(C))$ is a bijection. $\square$

### 1.3.3 Elliptic surfaces and their sections

In this subsection we will define elliptic surfaces and state some results about their sections following the results in chapter 3 of [Sil94]. An elliptic surface will roughly speaking be a fibered surface that is a one-parameter family of elliptic curves. We will use the following definition of an elliptic surface following the definition of [Sil94].

**Definition 1.31.** Let $C$ be a nice curve over $k$. An *elliptic surface* $\mathcal{E}$ over $C$ consist of a projective surface $\mathcal{E}$ over $k$ with a morphism $\pi\colon \mathcal{E} \to C$ and a section $\sigma_0\colon C \to \mathcal{E}$ such that for all but finitely many fibers of $t \in C(\overline{k})$ the fiber $\mathcal{E}_t := \pi^{-1}(t)$ is a nice curve of genus 1.

Some authors will also assume that an elliptic surface has to be non-singular, which we did not assume. Most of the time we will say that $\pi\colon \mathcal{E} \to C$ is an elliptic surface, by which we mean the triple $(\mathcal{E}, \pi, \sigma_0)$. In other words, the section is in this case implicitly given.

Let $\pi\colon \mathcal{E} \to C$ be an elliptic surface with section $\sigma_0\colon C \to \mathcal{E}$. From the definition of an elliptic surface it follows that the map $\mathcal{E} \to C$ is an example of a fibration. Moreover, the generic fiber of $\pi$ will be a nice curve of genus 1 over the function field $\kappa(C)$ of $C$. By the fact that the section $\sigma_0$ corresponds to a $\kappa(C)$-rational point on this curve, we can make this generic fiber into an elliptic curve by choosing this point.

Let $t \in C(l)$ be given and suppose $\mathcal{E}_t$ is non-singular. The section $\sigma_0\colon C \to \mathcal{E}$ gives a point $O_t = \sigma(t) \in \mathcal{E}_t(l)$ on this fiber $\mathcal{E}_t$, which gives $\mathcal{E}_t$ the structure of an elliptic curve over $l$. This non-singular fibers are called the *good fibers.* The fibers which are singular are called the *bad fibers* or the *singular fibers.*

**Example 1.32.** Recall example 1.29 and let $X \to C$ be the fibered surface given in this example. Assume that char $k \neq 2, 3$. We can embed the surface $X$ in $\mathbb{P}^1_k \times \mathbb{P}^2_k$ via the morphism $(t, (x, y)) \mapsto ((t : 1), (x : y : 1))$. Let $X^{\mathrm{c}}$ denote the closure of $X$ in $\mathbb{P}^1_k \times \mathbb{P}^2_k$. The projection map $\pi_1\colon X^{\mathrm{c}} \to \mathbb{P}^1_k$ gives a fibration.

Define the map $\sigma_0\colon \mathbb{P}^1_k \to X^{\mathrm{c}}$ by $(t : s) \mapsto ((t : s), (0 : 1 : 0))$. Note that this map is a section. A fiber above a point $(t' : 1) \in \mathbb{P}^1_k(l)$ is a smooth curve of genus 1 if and only if $a(t') \neq 0$. In the case that $a \neq 0$, we have that $a(t') = 0$

for only finitely many $t \in k$. Hence, in the case that $a \neq 0$, the surface $X^{\mathrm{c}}$ with the maps $\pi_1$ and $\sigma_0$ will be an elliptic surface.

Let $E$ denote the generic fiber of $\pi\colon \mathcal{E} \to C$. By Lemma 1.30 and part (iii) of Lemma 1.6, there is a bijection between the set of $l$-sections $\mathcal{E}_l(C_l)$ and the set of $\kappa(C_l)$-points on $E$. As we have noted, the curve $E(\kappa(C_l))$ has a group structure, so it makes sense to ask what the group structure on $\mathcal{E}_l(C_l)$ is under this bijection.

Let two sections $\sigma_1, \sigma_2 \in \mathcal{E}_l(C_l)$ be given. For almost all points $t \in C_l(\bar{l})$, the fiber $\mathcal{E}_t$ will be an elliptic curve over $\bar{l}$. Hence, we can define rational maps $\sigma_1 + \sigma_2$ and $-\sigma_1$ pointwise on an open subset of $C_l$ by

$$(\sigma_1 + \sigma_2)(t) = \sigma_1(t) + \sigma_2(t) \text{ and } (-\sigma_1)(t) = -\sigma_1(t)$$

respectively. The fact that these indeed define rational maps is proven in Proposition III.3.10(a) of [Sil94]. Since $C$ is a smooth curve and $\mathcal{E}$ is projective, it follows from Proposition II.2.1 of [Sil09] that these rational maps are morphisms.

The following proposition gives us that these assignments indeed define the group structure that corresponds with the group structure of the generic fiber.

**Proposition 1.33.** Let $\mathcal{E} \to C$ be an elliptic surface over $k$. The operations

$$\begin{aligned} \mathcal{E}(C) \times \mathcal{E}(C) \to \mathcal{E}(C);&\ \ (\sigma_1, \sigma_2) \mapsto \sigma_1 + \sigma_2 \ \text{ and,} \\ \mathcal{E}(C) \to \mathcal{E}(C);&\qquad\quad \sigma \mapsto -\sigma \end{aligned}$$

make $\mathcal{E}(C)$ into an abelian group with $\sigma_0$ as a zero-section. Moreover, the map of Lemma 1.30 is in this case a group isomorphism.

**Proof.** Proposition III.3.10(b)+(c) of [Sil09]. □

Let $X \to C$ be a fibration with $X$ a regular surface. We call $X$ *minimal* over $C$ if for each fibered surface $Y \to C$ with $Y$ a regular surface and birational map $\varphi\colon Y \dashrightarrow X$ commuting with the maps to $C$, the map $\varphi$ extends to a morphism. A *minimal elliptic surface* is a regular elliptic surface which is minimal as a fibered surface. The next theorem shows that each elliptic surface is birational to a minimal elliptic surface.

**Theorem 1.34.** Let $\mathcal{E} \to C$ be an elliptic surface. Then there exist a unique minimal elliptic surface $\mathcal{E}^{\min} \to C$ and a birational map $\varphi\colon \mathcal{E} \dashrightarrow \mathcal{E}^{\min}$ which commutes with the maps to $C$.

**Proof.** We give a sketch of the proof, because we do not have the tools at our disposal needed to give a rigorous proof. Observe that if a point is singular on $\mathcal{E}$, then it will be singular on the fiber. By assumption almost every fiber of $\mathcal{E} \to C$ is a smooth genus one curve, so there are only a finite number of fibers on which we need to resolve the singularities of this surface to obtain a regular surface. This resolving can be done by blowing up, see for example the discussion of Remark V.3.8.1 of [Har77]. After resolving the singularities, this surface will still be an elliptic surface over $C$. The result then follows from Theorem III.8.4 of [Sil94]. □

Let $\mathcal{E} \to C$ be a elliptic surface, then we call the unique minimal elliptic surface of Theorem 1.34 a *minimal model* for $\mathcal{E} \to C$. This minimal model is unique up to a unique isomorphism. Moreover, the generic fibers of the fibrations $\mathcal{E} \to C$ and $\mathcal{E}^{\min} \to C$ are isomorphic.

The (singular) fibers of a minimal elliptic surface over a perfect field are classified by the work of Kodaira and Néron. The smooth fibers, which are smooth genus one curves, are denoted $\mathrm{I}_0$ by Kodaira. For the singular fibers each irreducible component will be a curve which is birationally equivalent over its algebraic closure to the projective line. If these fibers are geometrically irreducible, then the fiber is a rational curve with a node or a cusp, denoted $\mathrm{I}_1$ and II respectively.

The classification of the reducible fibers is a bit more complex: there are two infinite families, namely $\mathrm{I}_n$ for $n \geq 1$, and $\mathrm{I}_n^*$ for $n \geq 0$ and five other types, namely III, IV, $\mathrm{II}^*$, $\mathrm{III}^*$ and $\mathrm{IV}^*$. We refer to section 3 and 4 of [SS10] or chapter IV of [Sil94] for more background on this classification. Most importantly for us, on the smooth locus of these special fibers there still will be an abelian group structure that is compatible with the group action of the sections.

# Chapter 2

# Del Pezzo surfaces

In this chapter we again let $k$ denote a field. A *Fano variety* is a nice $k$-variety with ample anti-canonical divisor. A *del Pezzo surface* is a Fano variety of dimension two. The *degree* of a del Pezzo surface is the self intersection number of the (anti)canonical divisor. By Proposition 9.2.23 of [Poo17] we have that a surface $X$ over $k$ is a del Pezzo surface if and only if $X_l$ is a del Pezzo surface for some field extension $l/k$.

Let $X$ be a del Pezzo surface over $k$ and let $K_X$ denote the canonical divisor of $X$. Set $d := \deg X = K_X^2$. By Theorem 24.5 of [Man86], $-K_X$ is very ample when $d \geq 3$. Moreover, in this case the complete linear system $|-K_X|$ embeds $X$ as a degree $d$ surface in $\mathbb{P}_k^d$. If $d = 2$, Proposition III.3.5.2 of [Kol99] gives us that $X$ embeds as a degree four surface in the weighted projective space $\mathbb{P}_k(2,1,1,1)$. If $d = 1$, Proposition III.3.5.1 of [Kol99] gives us that $X$ embeds as a degree six surface in the weighted projective space $\mathbb{P}_k(2,3,1,1)$. The latter case will be treated extensively in the second section of this chapter. First, we will state some more general results on del Pezzo surfaces.

## 2.1 General results on del Pezzo surfaces

In this section we will state the classification of del Pezzo surfaces over separably closed fields and their exceptional curves. We will also give some results on (uni)rationality of del Pezzo surfaces. For more background, we refer to section 9.4 of [Poo17]. Because almost all del Pezzo surfaces are geometrically blowups of projective space, we start this section by shortly discussing the definition of a blowup.

### 2.1.1 Blowup

Let $X$ be a scheme. An *effective Cartier divisor* on a scheme $X$ is a closed subscheme $D \subset X$, such that there exist a cover $\{U_i\}$ of $X$ with $U_i \cong \operatorname{Spec} R_i$ and $U_i \cap D \cong \operatorname{Spec} R_i/(f_i)$ for some non-zero divisor $f_i \in R_i$.

Let $Z$ be a closed subscheme of $X$ and let $\mathcal{C}$ be the full subcategory of schemes $Y$ over $X$ such that the inverse image of $Z$ under $Y \to X$ is an effective Cartier divisor. A *blowup* of $X$ along $Z$ is a terminal object in $\mathcal{C}$. In other words, a blowup is a morphism $\pi\colon X' \to X$ with the property that $\pi^{-1}(Z)$ is an effective Cartier divisor and that for each morphism $f\colon Y \to X$ with $f^{-1}(Z)$ an effective Cartier divisor, there exists a unique morphism $g\colon Y \to X'$ with $f = \pi \circ g$. The inverse image of $Z$ under this blowup is called an *exceptional divisor.*

Blowups are unique up to a unique isomorphism. In [Stacks, Tag 01OF] a construction of a blowup is given and this shows that a blowup always exists. We sometimes use the notation $\operatorname{Bl}_Z X$ to denote a scheme that is a blowup $\operatorname{Bl}_Z X \to X$ of the scheme $X$ along $Z$.

**Remark 2.1.** [Stacks, Tag 01OF] actually defines this construction as the blowup. We prefer this categorical definition, because this really defines a blowup up to isomorphism, and hence, leaves room to give blowups in another, sometimes easier or more natural way.

The following example is an elementary example of a blowup. Moreover, this example gives a local construction of a blowup in a closed point on a surface.

**Example 2.2.** Let $\mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ be the affine plane over some field $k$. Note that $(0, 0) \in \mathbb{A}^2_k(k)$ defines a closed point of $\mathbb{A}^2_k$. A blowup of $X$ along $(0, 0)$ is given by the subvariety $B$ of $\mathbb{A}^2_k \times \mathbb{P}^1_k$ defined by the equation $xt = ys$, where $s, t$ denote coordinates of $\mathbb{P}^1_k$ with the projection map $B \to \mathbb{A}^2_k$. In particular, if we write $R = k[x, y]$, then we have $B = \operatorname{Proj}(R[s, t]/(xt - ys))$ where $\deg(s) = \deg(t) = 1$, which exactly corresponds with the construction of [Stacks, Tag 01OF]. This shows that it is indeed a blowup.

Let $\pi\colon X' \to X$ be a blowup along $Z \subset X$. Then it follows from [Stacks, Tag 02OS] that the restriction $X' - \varphi^{-1}(X - Z) \to X - Z$ is an isomorphism. Moreover, the exceptional divisor $E := \varphi^{-1}(Z)$ is an effective Cartier divisor.

If $Z' \subset X$ is a closed subscheme such that $Z \cap Z' = \emptyset$, then it follows from [Stacks, Tag 080A] that the blowup $X'' \to X$ along $Z \cup Z'$ can be given as a blowup $\pi\colon X' \to X$ along $Z$ and then precomposing with the blowup $X'' \to X'$ along $\pi^{-1}(Z')$.

If $Y$ is a closed subscheme of $X$, then $Z \cap Y$ will be a closed subscheme of $Y$. By Lemma 22.2.6 of [Vak17], the closure of the pullback of $Y$ along $\pi$ will be a closed subscheme $Y'$ of $X'$ such that $\pi|'_Y\colon Y' \to Y$ is a blowup of $Y$ along $Z \cap Y$. We call this closed subscheme $Y'$ on $X'$ the *strict transform of* $Y$.

In the case that $X$ is a surface and $Z$ is a closed point, then [Har77] calls a blowup in this case a *monoidal transformation.* Almost all blowups we will encounter, are blowups in a finite number of points. In particular, they can be obtained by the compositions of monoidal transformations. Note that Example 2.2 is an example of a monoidal transformation. As a result of Proposition 3.1 of [Har77], the monoidal transformation of a nice surface, will again be a nice surface.

There is a theorem for these monoidal transformation, that gives a construction the other way around.

**Theorem 2.3** (Castelnuovo's contraction theorem)**.** Let $X$ be a nice surface over $k$ and let $C$ be a nice curve on $X$. Suppose that $C^2 = -1$ and $C \cong \mathbb{P}^1_k$. Then there exists a nice surface $Y$ with a closed point $y \in Y$ and a morphism $\pi\colon X \to Y$ with $\pi^{-1}(y) = C$, such that $\pi$ is a blowup of $Y$ along $y$.

**Proof.** For a proof that works over any field, we refer to Theorem 9.3.3 of [Poo17]. □

The blowup $\pi\colon X \to Y$ in the above theorem is also called the *blowdown* of $X$ along the curve $C$. In particular, if a curve $C$ on a nice surface satisfies the properties that $C^2 = -1$ and $C \cong \mathbb{P}^1_k$, then it is an exceptional divisor, and we will call it in this case an *exceptional curve.*

Let $\pi\colon X' \to X$ be a monoidal transformation of $X$ in a point $P$. The strict transform of a curve $C$ under this monoidal transformation is given as follows. If $P$ is not a point on $C$, then it is given by $\pi^{-1}(C)$ which is then isomorphic to $C$. If $P$ is a point in $C$, then it is given by the closure of $\pi^{-1}(C - P)$ in $X'$.

These monoidal transformations can be used to give a *normalization* of curves. If we have a singular irreducible curve $C$ on a surface $X$ over an algebraically closed field, then by Proposition V.3.8 of [Har77] there is a finite sequence of monoidal transformations $X_n \to \cdots \to X_1 \to X$ such that the strict transform of the curve $C$ on $X_n$ is non-singular.

### 2.1.2 Del Pezzo surfaces over separably closed fields

Let $n \leq 8$ and $P_1, \ldots, P_n \in \mathbb{P}^2_k(\overline{k})$. Then we say that the points $P_i$ are in general position if they are distinct, no three of them lie on a line, no six of them lie on a conic and no eight of them lie on a singular cubic, with one of these eight points at the singularity.

**Theorem 2.4** (Classification of del Pezzo surfaces over separably closed fields)**.** Let $X$ be a surface over $k$. Then $X$ is a del Pezzo surface if and only if either $X^{\mathrm{s}} \cong \mathbb{P}^1_{k^{\mathrm{s}}} \times \mathbb{P}^1_{k^{\mathrm{s}}}$, or $X^{\mathrm{s}}$ is a blowup of $\mathbb{P}^2_{k^{\mathrm{s}}}$ in at most eight points in $\mathbb{P}^2_{k^{\mathrm{s}}}(k^{\mathrm{s}})$ which are in general position. Moreover, if $X^{\mathrm{s}} \cong \mathbb{P}^1_{k^{\mathrm{s}}} \times \mathbb{P}^1_{k^{\mathrm{s}}}$, then $\deg X = 8$, and if $X^{\mathrm{s}}$ is a blowup of $\mathbb{P}^2_{k^{\mathrm{s}}}$ in $d$ points, then $\deg X = 9 - d$.

**Proof.** For the algebraically closed case, see Theorem 24.4 of [Man86]. The generalization to separably closed fields follows as a corollary of Proposition 5 and Proposition 7 of [Coo88]. For those who would rather have a complete proof, we refer to Theorem 9.4.4 of [Poo17]. □

The following result gives us the exceptional curves on del Pezzo surfaces.

**Proposition 2.5** (Exceptional curves on a del Pezzo surface)**.** Let $n$ be a non-negative integer such that $n \leq 8$. Let $X \to \mathbb{P}^2_k$ be the blowup of some $k$-rational points $x_1, \ldots, x_n \in \mathbb{P}^2_k(k)$ and assume these points are in general position. Then the exceptional curves of $X$ are exactly the fibers above the points $x_i$, together with the strict transforms of the following curves in $\mathbb{P}^2_k$:

(i) lines through two of the points $x_i$;

(ii) conics through five of the points $x_i$;

(iii) a cubic passing through seven of the $x_i$, such that one of the $x_i$ is a double point on the cubic;

(iv) a quartic passing through eight of the $x_i$, such that three of the $x_i$ are double points on the quartic;

(v) a quintic passing through eight of the $x_i$, such that six of the $x_i$ are double points on the quintic;

(vi) a sextic passing through eight of the $x_i$, such that one of the $x_i$ is a triple point and the other seven points $x_i$ are double points on this sextic.

**Proof.** Theorem 26.2(ii) of [Man86]. □

Note that only in the case $\deg X = 1$, we need all curves (i)-(vi) of Proposition 2.5. In the case of $\deg X = 2$ we already only need (i)-(iii). In the case of $\deg X = 3, 4$, we need (i) and (ii) and in all higher cases we only need (i). This already shows that the geometry of del Pezzo surfaces of lower degree become harder as we will also discover in the next subsection.

As a result of Theorem 2.4 and Proposition 2.5, the exceptional curves on a del Pezzo surface over $k$ are all defined over the separable closure $k^{\text{sep}}$. We can count these exceptional curves on a del Pezzo surface $X$, because each curve from the proposition is uniquely determined by the given points. The following table gives us the amount of exceptional curves on a del Pezzo surface $X$ in the case that it is a blowup in $9 - \deg X$ points in general position.

| $\deg X$ | 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 |
|---|---|---|---|---|---|---|---|---|---|
| # exceptional curves | 0 | 1 | 3 | 6 | 10 | 16 | 27 | 56 | 240 |

Figure 2.1: The number of exceptional curves on a del Pezzo surface $X$

### 2.1.3 (Uni)rationality of del Pezzo surfaces

In this subsection we will give results on (uni)rationality of del Pezzo surfaces over an arbitrary field $k$. A variety over $k$ is called *rational* if for some integer $n$, it is birational to $\mathbb{P}^n_k$. By Theorem 2.4 every del Pezzo surface over a separably closed field is rational. A variety $X$ is called unirational if there exists a dominant rational map $\mathbb{P}^n_k \dashrightarrow X$ for some $n \geq 0$. Note that rationality implies unirationality. In the case that $k$ is an infinite field, for any (uni)rational variety $X$ over $k$, the set of $k$-rational points $X(k)$ lies Zariski dense in $X$.

From now on, we let $X$ denote a del Pezzo surface over $k$. Suppose $\deg X \geq 5$. Then it is known that if $X$ contains a $k$-rational point, it is rational. Moreover, if $\deg X$ is 5 or 7, then $X$ always contains a $k$-rational point and hence is rational. By Proposition 1.13 every del Pezzo surface of degree 9 with a $k$-rational point is isomorphic to $\mathbb{P}^2_k$. In the case that $\deg X = 8$ and $X^s \not\cong \mathbb{P}^1_{k^s} \times \mathbb{P}^1_{k^s}$, then $X$ is isomorphic to the blowup of $\mathbb{P}^2_k$ in a $k$-rational point, and hence will be rational as well. A proof of these results and a more extensive treatment is given in section 9.4 of [Poo17].

The cases on del Pezzo surface of low degree are getting much harder. In the case that $\deg X \leq 4$, rationality will no longer be implied by the existence of a $k$-rational point and there are examples of del Pezzo surfaces that are not rational. While rationality may no longer be true for del Pezzo surface of low degree, unirationality remains true if $X$ has degree three or four and a $k$-rational point.

**Theorem 2.6.** Let $X$ be a del Pezzo surface over $k$ with $\deg X \geq 3$. If $X$ contains a $k$-rational point, then $X$ is unirational.

**Proof.** Theorem 29.4 of [Man86]. For degree three and four, this proof assumes the existence the existence of a $k$-rational point which does not lie on an exceptional curve. Theorem 30.1 shows that this is indeed the case if the field $k$ consists of at least 23 elements for degree four or at least 35 elements for degree three.

[Kol02] proved the case for degree three over general fields. Proposition 5.19 of [Pie12] seems to be the first mention of the case of degree four over general fields. □

It is not known if every del Pezzo surface over $k$ with $\deg X = 2$ and a $k$-rational point is $k$-unirational, but if this point satisfies an extra condition, then the del Pezzo surface will be $k$-unirational.

**Theorem 2.7.** Let $X$ be a del Pezzo surface over $k$ with $\deg X = 2$. If $X$ contains a $k$-rational point that is not contained in the ramification locus of the anticanonical map, nor in the intersection of four exceptional curves, then $X$ is unirational.

**Proof.** [STV14]. □

For del Pezzo surfaces of degree one even less is known. As we will see in the next section, every del Pezzo surface over $k$ of degree one contains a $k$-rational point, so one could conjecture that every such surface is unirational. But until recently, there were no known results for del Pezzo surfaces of degree one that cannot be blown down over $k$ to a del Pezzo surface of higher degree.

Kollár and Mella proved it in [KM16] for a very special type of del Pezzo surfaces of degree one, namely those that admit a certain conic bundle structure. Outside of this result, and the del Pezzo surfaces of degree one that can be blown down over $k$ to a del Pezzo surface of higher degree, we do not have one single example of a del Pezzo surface of degree one that is unirational. This leaves us with most del Pezzo surfaces of which we do not know if they are unirational. Moreover, no such surface has been found that is not unirational.

## 2.2 Degree one

The result of the main theorem applies to del Pezzo surfaces of degree one. In this section we give an explicit construction of an embedding of a del Pezzo surface of degree one over $k$ in the weighted projective space $\mathbb{P}_k(2,3,1,1)$. Furthermore, we will see that the linear system of the anticanonical divisor has a unique base point and so every del Pezzo surface of degree one over $k$ has a $k$-rational point. Moreover, we will see that the blowup in this point gives us an elliptic surface over $\mathbb{P}^1_k$.

In this entire section, $S$ will denote a del Pezzo surface of degree one over $k$ and $K_S$ denotes a fixed canonical divisor of $S$.

### 2.2.1 The anticanonical model

The *anticanonical ring* of $S$ is the graded ring

$$R(S,-K_S) := \bigoplus_{n\geq 0} \mathcal{L}(-nK_S).$$

Note that this indeed is a well-defined graded ring. The *anticanonical model* of $S$ is the scheme $\operatorname{Proj} R(S,-K_S)$. Moreover, this anticanonical model is well-defined, because $-K_S$ is ample and in particular, the ring $R(S,-K_S)$ is non-zero. Moreover, by Lemma 3.2.5.2 of [Kol99] we have that

$$\ell(-nK_S) = \frac{n(n+1)}{2} + 1. \tag{2.1}$$

**Proposition 2.8.** Let $S$ be a del Pezzo surface of degree one. Then $S$ is isomorphic to its anticanonical model. Moreover, the anticanonical model induces an embedding of $S$ in the weighted projective space $\mathbb{P}_k(2,3,1,1)$ as a surface of degree six. Conversely, every smooth surface of degree six in $\mathbb{P}_k(2,3,1,1)$ is a del Pezzo surface of degree one.

**Proof.** Proposition III.3.5.1 of [Kol99]. □

We make the embedding of Proposition 2.8 explicit by following the construction of [CO99], page 1199-1201. Details that are omitted, can be found there.

By the formula of equation (2.1), we have $\ell(-K_S) = 2$. So $\mathcal{L}(-K_S)$ is generated by two elements. We pick two linearly independent elements $z, w \in \mathcal{L}(-K_S)$. One can check that for all $n \geq 1$, we have that $z^n, z^{n-1}w, \ldots, w^n \in \mathcal{L}(-nK_S)$ are linearly independent.

Again, by equation (2.1), we have $\ell(-2K_S) = 4$. So choosing one more element $x \in \mathcal{L}(-2K_S)$ that is linearly independent of the elements $z^2, zw, w^2$, gives us a basis $(x, z^2, zw, w^2)$ of $\mathcal{L}(-2K_S)$. This already gives us six elements $xz, xw, z^3, z^2w, zw^2$ and $w^3$ of $\mathcal{L}(-3K_S)$, which again can be showed to be linearly independent. Furthermore, we have that $\ell(-3K_S) = 7$. Picking an element $y \in \mathcal{L}(-3K_S)$ that is linearly independent of those elements gives us a basis $(y, xz, xw, z^3, z^2w, zw^2, w^3)$ of $\mathcal{L}(-3K_S)$.

Now we note that $\ell(-6K_S) = 22$ and that we have 23 elements in $\mathcal{L}(-6K_S)$, namely

$$y^2, yxz, yxw, yz^3, yz^2w, yzw^2, yw^3, x^3, x^2z^2, x^2zw, xw^2,$$
$$z^6, z^5w, z^4w^2, z^3w^3, z^2w^4, zw^5, w^6$$

It follows that there must be a relation between those elements. By scaling the elements $x$ and $y$, this relation can be given by the zero set of a polynomial

$$f := y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$$

with $a_i \in k[z, w]$ homogeneous polynomials of degree $\deg(a_i) = i$.

The relation $f = 0$ defines something that looks like a Weierstrass equation. Similarly as in the case of Weierstrass curves, we have that if the characteristic of $k$ is not equal to two or three, the surface $S$ can be given by an equation such that $a_1$, $a_2$ and $a_3$ are all zero.

One can check that $x, y, z, w$ generate the anticanonical ring of $X$ and that there are no other relations then $f = 0$. It follows that

$$X \cong \operatorname{Proj} S = \operatorname{Proj} k[x, y, z, w]/(f) \hookrightarrow \mathbb{P}_k(2, 3, 1, 1).$$

This gives an embedding of the del Pezzo surface $S$ of degree one in the weighted projective space $\mathbb{P}_k(2, 3, 1, 1)$.

From the equation of $S$ it is immediately clear that $S$ contains a $k$-rational point, namely the point $(1 : 1 : 0 : 0)$. As we will see, this point is also the unique base point of the linear system $|-K_S|$. We will denote this point by $\mathcal{O}$.

There is a special family of curves on the del Pezzo surface $S$, that will play an important role in the remaining part of this thesis.

**Remark 2.9.** Let $S$ be a del Pezzo surface of degree one embedded in the weighted projective space $\mathbb{P}_k(2, 3, 1, 1)$ with coordinates $X$, $Y$, $Z$ and $W$.

Let $(z_0 : w_0) \in \mathbb{P}^1_k(k)$ be given and let $\mathcal{F}$ be the curve on $S$ given by $w_0 Z = z_0 W$. Note that every point on $\mathcal{F}$ is either the point $\mathcal{O}$ or can be described as a point with coordinates $(x_P : y_P : z_0 : w_0)$. There is an embedding $\mathcal{F} \to \mathbb{P}^2_k$ to a Weierstrass curve given by the equation $G(x, y, z_0, w_0) = 0$ such that $\mathcal{O}$ corresponds to the point at infinity. This embedding can be given on $\mathcal{F} - \mathcal{O}$ by the assignment $(x_P : y_P : z_0 : w_0) \mapsto (x_P, y_P)$. This gives us that $\mathcal{F}$ can be identified with the Weierstrass curve in $\mathbb{P}^2_k$ given by the equation $G(x, y, z_0, w_0) = 0$. Hence, the curve $\mathcal{F}$ has an induced group structure on the smooth points.

**Lemma 2.10.** Let $\mathcal{F}$ be the curve on $S$ given by $w_0 Z = z_0 W$. Then $\mathcal{F}$ is linearly equivalent to the anticanonical divisor $-K_S$.

**Proof.** Give $\mathbb{P}_k(2, 3, 1, 1)$ coordinates $X, Y, Z, W$ and let $S$ be a del Pezzo surface of degree one in $\mathbb{P}_k(2, 3, 1, 1)$. Recall the equality $\mathcal{L}(-K_S) = \langle z, w \rangle$. Without loss of generality, we may assume that $w_0 \neq 0$ and that under the embedding $z$ is given by the rational function $\frac{w_0 Z - z_0 W}{W}$ and $w$ is given by the constant function 1. Hence, it follows that $\operatorname{div}(z) = \mathcal{F} - \{W = 0\} \cap S$ and $\operatorname{div}(w) = 0$. We deduce that $\mathcal{F}$ is linearly equivalent to $-K_S$. □

### 2.2.2 Some linear systems

Recall that $S$ denotes a del Pezzo surface of degree one. In this section we identify $S$ as a subset of $\mathbb{P}_k(2, 3, 1, 1)$ with coordinates $X$, $Y$, $Z$ and $W$, defined by the polynomial $G$ given by

$$G := Y^2 + a_1 XY + a_3 Y - X^3 - a_2 X^2 - a_4 X - a_6, \tag{2.2}$$

with $a_i \in k[Z, W]$ homogeneous of degree $\deg(a_i) = i$.

The divisor $-3K_S$ is very ample and so the linear system $|-3K_S|$ determines a closed embedding of $S$ in the projective space $\mathbb{P}^6_k$, given explicitly by

$$(X : Y : Z : W) \mapsto (Y : XZ : XW : Z^3 : Z^2 W : ZW^2 : W^3).$$

This embedding of $S$ in $\mathbb{P}^6_k$ factors through the anticanonical embedding of $S$ in $\mathbb{P}_k(2, 3, 1, 1)$.

The linear system $|-2K_S|$ determines a morphism $S \to \mathbb{P}^3_k$ which is defined as

$$(X : Y : Z : W) \mapsto (X : Z^2 : ZW : W^2).$$

The image of this morphism is the cone $\mathcal{Q} := \operatorname{Proj}(k[p, q, r, s]/(r^2 - qs)) \subset \mathbb{P}^3_k$.

Moreover, this morphism is flat and finite of degree two. Outside the ramified points the morphism is étale. In particular, it is a double cover of this cone $\mathcal{Q}$. We will discuss the ramification points of this morphism in the next subsection.

The morphism determined by $|-2K_S|$ factors via the rational projection map $\mathbb{P}_k(2,3,1,1) \dashrightarrow \mathbb{P}_k(2,1,1)$ defined by $(X : Y : Z : W) \mapsto (X : Z : W)$ and the isomorphism $\mathbb{P}_k(2,1,1) \to \mathcal{Q}$ which we have encountered in example 1.24. This rational map $\mathbb{P}_k(2,3,1,1) \dashrightarrow \mathbb{P}_k(2,1,1)$ is a morphism outside the point $(0 : 1 : 0 : 0)$ and restricts to a morphism $\tau\colon S \to \mathbb{P}_k(2,1,1)$. In particular, this morphism $\tau$ is flat and finite of degree two as well and it gives a double cover of $\mathbb{P}_k(2,1,1)$.

The linear system $|-K_S|$ determines a rational map $S \dashrightarrow \mathbb{P}^1_k$ which is explicitly given by projecting onto the last two coordinates. This map is defined everywhere except in the point $\mathcal{O} = (1 : 1 : 0 : 0)$, which is a base point for this linear system. Blowing up $S$ in this base point gives a surface $\mathcal{E} := \mathrm{Bl}_{\mathcal{O}} S$ with a morphism $\pi\colon \mathcal{E} \to S$. In the next proposition we will show that the rational map $S \dashrightarrow \mathbb{P}^1_k$ extends to a morphism $\mathcal{E} \to \mathbb{P}^1_k$ which gives $\mathcal{E}$ the structure of an elliptic surface.

**Proposition 2.11.** Let $S$ be a del Pezzo surface of degree one and $\mathcal{O}$ the base point of the linear system $|-K_S|$. The surface $\mathcal{E} := \mathrm{Bl}_{\mathcal{O}} S$ can be given the structure of an elliptic surface such that

(i) the fibration $\mathcal{E} \to \mathbb{P}^1_k$ is a morphism such that this morphism extends the rational map determined by the linear system $|-K_S|$; and

(ii) the image of the zero section is the exceptional divisor of the blowup $\mathcal{E} \to S$.

**Proof.** We will use without proof that the surface $\mathcal{E}$ can be identified as the following surface in $\mathbb{P}_k(2,3,1,1) \times \mathbb{P}^1_k$. Using the coordinates $X, Y, Z, W$ for $\mathbb{P}_k(2,3,1,1)$ and $u, v$ for $\mathbb{P}^1_k$, we can define the surface $\mathcal{E}$ as the zero set of the Weierstrass equation (2.2) and the equation $uW = vZ$. The projection map to $\mathbb{P}_k(2,3,1,1)$ gives the blowup $\pi\colon \mathcal{E} \to S$. The projection map $\mathcal{E} \to \mathbb{P}^1_k$ gives a fibration.

A direct verification shows that this fibration $\mathcal{E} \to \mathbb{P}^1_k$ extends the rational map given by the composition of $\mathcal{E} \to S$ and the rational map $S \dashrightarrow \mathbb{P}^1_k$ determined by $|-K_X|$. Moreover, the section $\mathbb{P}^1_k \to \mathcal{E}$ defined by $(u : v) \mapsto (\mathcal{O}, (u : v))$, gives exactly a section with as image the exceptional divisor of the blowup.

For every point $t = (u_0 : v_0) \in \mathbb{P}^1_k(\overline{k})$, we get a fiber $\mathcal{E}_t$ of $\mathcal{E} \to \mathbb{P}^1_k$, that is defined by the equation $v_0u = u_0v$. Similarly as in Remark 2.9, this curve $\mathcal{E}_t$ will be isomorphic to a Weierstrass curve in $\mathbb{P}^2_{\overline{k}}$ which is defined on an affine piece by the Weierstrass equation $G(x, y, u_0, v_0) = 0$. If the determinant will be zero for every fiber, then $S$ will be singular, which contradicts our assumption. Hence, almost every fiber of $\mathcal{E} \to \mathbb{P}^1_k$ will be a smooth curve of genus one. We deduce that $\mathcal{E} \to \mathbb{P}^1_k$ will be an elliptic surface. □

**Remark 2.12.** Let $(z_0 : w_0) \in \mathbb{P}^1_k(k)$ be given and let $\mathcal{F}$ be the curve on $S$ given by $w_0Z - z_0W$. Note that the strict transform of $\mathcal{F}$ on $\mathcal{E}$ is exactly the fiber $\mathcal{E}_{(z_0:w_0)}$ of the morphism $\mathcal{E} \to \mathbb{P}^1_k$ above the point $(z_0 : w_0)$. Moreover, these curves are isomorphic and this isomorphism induces a group isomorphism on the sets of non-singular rational points.

The above discussion can be summarized in the following commutative diagram.

$$\begin{tikzcd}
\mathcal{F} \arrow[r, hook] \arrow[d] \arrow[rr, hook, bend left] & \mathcal{E} \arrow[r, "\pi"] \arrow[d] & S \arrow[r, hook] \arrow[dl, dashed, "|-K_S|"] \arrow[rrd, "|-2K_S|"] \arrow[rr, bend left, "\tau"] & \mathbb{P}_k(2,3,1,1) \arrow[r, dashed] & \mathbb{P}_k(2,1,1) \arrow[d, "\cong"'] & \\
\operatorname{Spec} k \arrow[r, hook] & \mathbb{P}^1_k & & & \mathcal{Q} \arrow[r, hook] & \mathbb{P}^3_k
\end{tikzcd} \tag{2.3}$$

## 2.2.3 Ramification of the map determined by $|-2K_S|$

In this section, we again let the del Pezzo surface $S$ of degree one be given by a Weierstrass equation of the form (2.2). Recall from the last section that the linear system $|-2K_S|$ determines a morphism $S \to \mathbb{P}^3_k$ with image the cone $\mathcal{Q}$. Moreover, recall that this morphism splits via the maps

$$S \xrightarrow{\tau} \mathbb{P}_k(2,1,1) \xrightarrow{\sim} \mathcal{Q} \hookrightarrow \mathbb{P}^3_k.$$

We will show that the inverse image of a point is exactly the orbit of an automorphism of $S$ of order two, which is defined as follows.

**Definition 2.13.** The automorphism $[-1]\colon S \to S$ is defined by

$$[-1]\colon (X : Y : Z : W) \mapsto (X : -Y - a_1(Z,W)X - a_3(Z,W) : Z : W). \tag{2.4}$$

Let $l/k$ be a field extension. Given an $l$-rational point $P \in S(l)$, we will denote the image $[-1](P)$ by $-P$.

Note that the map $[-1]\colon S \to S$ is indeed a well-defined automorphism, which is its own inverse.

**Remark 2.14.** Let $(z_0 : w_0) \in \mathbb{P}^1_k(k)$ be given and let $\mathcal{F}$ be the curve on $S$ given by $w_0Z - z_0W$. Then by Remark 2.9 this curve can be identified with a Weierstrass curve. Note that the restriction of the map $[-1]\colon S \to S$ to $\mathcal{F}$ exactly gives the map $[-1]_{\mathcal{F}}\colon \mathcal{F} \to \mathcal{F}$ of Definition 1.26.

**Lemma 2.15.** *Let $l/k$ be a field extension. The inverse image of an $l$-rational point $Q \in \mathcal{Q}(l)$ under the morphism determined by the linear system $|-2K_S|$ is exactly the set $\{P, -P\}$ in $S(\bar{l})$ for some point $P$ in the inverse image of $Q$.*

**Proof.** The inverse image of a point $Q \in \mathcal{Q}(l)$ is exactly the same as the inverse image under $\tau$ of the corresponding point in $\mathbb{P}_k(2,1,1)$ under the above isomorphism. This means that we can restrict ourselves to proving this for the morphism $\tau$.

Given a point $Q = (X_Q : Z_Q : W_Q) \in \mathbb{P}_k(2,1,1)(\bar{k})$ its inverse image under $\tau$ is given by $P_1 = (X_Q : Y_1 : Z_Q : W_Q)$ and $P_2 = (X_Q : Y_2 : Z_Q : W_Q)$ where $Y_1$ and $Y_2$ denote the, possibly same, two zeroes of the quadratic polynomial in $\bar{l}[Y]$ given by $G(X_Q, Y, Z_Q, W_Q)$. In the case that $Z_Q = W_Q = 0$, we just get the point $P_1 = P_2 = \mathcal{O}$.

If $Z_Q$ and $W_Q$ are not both zero, these points $P_i$ lie on the curve $\mathcal{F}$ in $S$ given by $W_QZ - Z_QW$. Note that the map $[-1]\colon S \to S$ only changes the $Y$ coordinate. Moreover, on the affine curve $\mathcal{F}-\mathcal{O}$ we only have at most two points with $X = X_Q$, which are exactly the points $P_1$ and $P_2$. The morphism $[-1]$ exactly sends these points to each other, which gives us the desired result. □

The above observation gives us a nice characterization of the ramification points of the morphism determined by the linear system $|-2K_S|$.

**Lemma 2.16.** *Let $l/k$ be a field extension. Let $P \in S(l)$ be an $l$-rational point. Let $\mathcal{F}$ be the curve defined by $W_PZ - Z_PW$. Then the following are equivalent:*

(i) *the map determined by the linear system $|-2K_S|$ is ramified at the point $P$;*

(ii) *the map $\tau\colon S \to \mathbb{P}_k(2,1,1)$ is ramified at the point $P$;*

(iii) *the point $P$ is a fixed point of the automorphism $[-1]\colon S \to S$;*

(iv) on $\mathcal{F}$ the point $P$ is either a singular point or a smooth point of order at most two in the group $\mathcal{F}_{\mathrm{ns}}(l)$;

(v) we have either have that $P = \mathcal{O}$ or that $\dfrac{\partial G}{\partial Y}(P) = 0$, where $G$ denotes the defining equation of $S$ given by (2.2).

**Proof.** First note that statements (i) and (ii) are equivalent. Statements (i) and (iii) are equivalent by Lemma 2.15. Statements (iii) and (iv) are equivalent by combining Remark 2.14 and Lemma 1.27. Statement (iv) implies statement (v) by Lemma 1.27. So it is enough to prove that statement (v) implies statement (ii).

(v)⇒(ii) Clearly $\tau$ is ramified at $\mathcal{O}$. Else if $\dfrac{dG}{dY}(P) = 0$, then the inverse image of $\tau(P) = (X_P : Z_P : W_P)$ is given by the points $(X_P : Y' : Z_P : W_P)$ such that $Y'$ denotes a zero point of the polynomial in $Y$ given by $G(X_P, Y, Z_P, W_P)$. But the latter is a polynomial in $Y$ of degree two which has $Y_P$ as a double zero point. Hence, $\tau$ is ramified at $P$. □

**Proposition 2.17.** Let $(z_0 : w_0) \in \mathbb{P}^1_k(k)$ be given and let $\mathcal{F}$ be the curve on $S$ defined by $w_0 Z - z_0 W$. If $\mathrm{char}(k) = 2$ and $\mathcal{F}$ contains a singular point that is a cusp, then the map determined by the linear system $|-2K_S|$ is ramified on all of $\mathcal{F}$. In all other cases, the set of ramified points of this map contained in $\mathcal{F}$ is at most four.

**Proof.** Note that $\mathcal{O}$ is a ramified point of the map determined by the linear system $|-2K_S|$. Every other point on $\mathcal{F}$ lies on the affine $\mathbb{A}^3_k$ and hence is a point that is singular on $\mathcal{F}$ or is smooth of order two by Lemma 2.16.

First suppose that $\mathcal{F}$ is smooth. In this case, we can identify it as an elliptic curve with $\mathcal{O}$ as the neutral element. If $\mathrm{char}(k) \neq 2$, Corollary 6.4(b) of [Sil09] gives that there are four points in the kernel of the multiplication-by-two map $[2]\colon \mathcal{F} \to \mathcal{F}$, defined by $P \mapsto 2P$. The kernel of this map are exactly the point $\mathcal{O}$ and three points of order two. So in this case by the above characterization, there are exactly four points that are ramified. In the case that $\mathrm{char}(k) = 2$, part (c) of the same corollary gives that there are at most two points in the kernel of this map $\mathcal{F}$, which means that there are at most two points, on which the map determined by the linear system $|-2K_S|$ is ramified.

Now suppose $\mathcal{F}$ is not smooth. Then it still is a geometrically irreducible Weierstrass curve, and so the unique singular point $Q \in \mathcal{F}$ will either be a cusp or a node. If $Q$ is a node, then $\mathcal{F}_{\mathrm{ns}}(\overline{k})$ is isomorphic to $\overline{k}^*$. Now if $\mathrm{char}(k) \neq 2$, the map $\overline{k}^* \to \overline{k}^*;\ x \mapsto x^2$ has kernel $\{\pm 1\}$. It follows that we have only one unique point of order two. Hence, there are in this case three points which are ramified, namely the points $\mathcal{O}$, $Q$ and this point of order two. In the case that $\mathrm{char}(k) = 2$, the map $\overline{k}^* \to \overline{k}^*;\ x \mapsto x^2$ is the Frobenius automorphism and so there are no points of order two. In this case, there are only two points, namely $\mathcal{O}$ and $Q$, on which the map determined by the linear system $|-2K_S|$ is ramified.

If the singular point $Q$ is a cusp then the group structure of $\mathcal{F}_{\mathrm{ns}}(\overline{k})$ is isomorphic to $\overline{k}^+$. In the case that $\mathrm{char}(k) \neq 2$, then this contains no points of order two, because the morphism $\overline{k}^+ \to \overline{k}^+;\ x \mapsto 2x$ is an isomorphism. So in this case there are also only two ramification points, namely $\mathcal{O}$ and $Q$. Finally, if $\mathrm{char}(k) = 2$, then every non-trivial point of $\mathcal{F}_{\mathrm{ns}}(\overline{k})$ will be a point of order two. In this case, every point of $\mathcal{F}$ will be a ramified point of the map determined by the linear system $|-2K_S|$. ☐

## 2.3 Zariski density

If $k$ is an infinite field, then unirationality of a variety implies the Zariski density of the $k$-rational points. It follows from Theorem 2.6 that for any del Pezzo surface $X$ of degree three or higher over an infinite field, the $k$-rational points $X(k)$ lie Zariski dense in $X$ if and only if $X(k) \neq \emptyset$. In degree two, we have found a partial solution due to Theorem 2.7.

As we have noted in section 2.1.3, the question of unirationality of del Pezzo surfaces of degree one is still open. While unirationality for degree one does not seem in reach at the moment, there are some partial results on the Zariski density of the $k$-rational points of a del Pezzo surface. We will discuss them below. In particular, we will discuss the work of Desjardins and Winter.

### 2.3.1 Partial results in degree one

In the last fifteen years, some partial results on the Zariski density of the $k$-rational points on del Pezzo surfaces of degree one have been proven. All these results are proven for del Pezzo surfaces given by an equation of the form $Y^2 = X^3 + a_4X + a_6$ with $a_4, a_6 \in k[Z, W]$ homogeneous of degree four and six respectively.

First of all, there are some results of Ulas. In [Ula07], he has proven the Zariski density of $\mathbb{Q}$-rational points in the following cases:

(i) $a_4(Z, 1)$ has degree at most 3 and $a_6 = 0$ (Theorem 2.2);

(ii) $a_4(Z, 1)$ has degree 4 and is not even and $a_6 = 0$ (Theorem 2.1(1));

(iii) $a_4 = 0$ and $a_6(Z, 1)$ is monic of degree 6 and not even (Theorem 3.1).

Moreover, he has proven that the result still holds in the even cases of (ii) and (iii) under the extra condition that there exists a fiber of $\mathcal{E} \to \mathbb{P}^1_k$ with infinitely many $\mathbb{Q}$-rational points (Theorems 2.1(2) and 3.2). In [Ula09] he proved the Zariski density for the rational points for the surfaces with $a_4 = 0$ and $a_6(Z, 1) = Z^5 + aZ^3 + bZ^2 + cZ + d$ for some $a, b, c, d \in \mathbb{Z}$ under the condition that the set of $\mathbb{Q}$-rational points on the elliptic curve defined by the Weierstrass equation $y^2 = x^3 + 135(2a - 15)x - 1350(5a + 2b - 26)$ is infinite.

Another result of Ulas is a joint work with Togbé. In [UT10] they proved the Zariski density for the surfaces where $a_4(Z, 1)$ and $a_6(Z, 1)$ have degree 4 and are even, again under the condition that there exists a fiber of $\mathcal{E} \to \mathbb{P}^1_k$ with infinitely many $\mathbb{Q}$-rational points. Some of these results are generalized by Salgado and van Luijk to arbitrary fields in [SL14].

Another result is that of Várilly-Alvarado. In [Vár11] he has proven the density of the set of $\mathbb{Q}$-rational points for surfaces with $a_4 = 0$ and $a_6 = aZ^6 + bW^6$ for some $a, b \in \mathbb{Z}$ with $3 \cdot \frac{a}{b}$ not a square or $\gcd(a, b) = 1$ and $9 \nmid ab$ under some extra technical condition.

### 2.3.2 Desjardins and Winter

The most recent result and in some sense the strongest result about the Zariski density of a family of del Pezzo surfaces of degree one is that of [DW21]. Their result not only gives a sufficient condition for the $k$-rational points to lie Zariski dense, but one which is also necessary in the case that $k$ is a number field, or more general, of finite type over $\mathbb{Q}$. The statement of the theorem is as follows.

**Theorem 2.18** (Desjardins and Winter)**.** Let $k$ be a field of characteristic 0, and let $a, b, c \in k$ with $a, c$ non-zero. Let $S$ be the del Pezzo surface given by

$$Y^2 = X^3 + aZ^6 + bZ^3W^3 + cW^6$$

in the weighted projective space $\mathbb{P}_k(2, 3, 1, 1)$ with coordinates $X$, $Y$, $Z$ and $W$. Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the base point of the linear system $|-K_S|$. If $S$ contains a rational point with non-zero $Z, W$-coordinates, such that the corresponding point on $\mathcal{E}$ is non-torsion on its fiber, then $S(k)$ is dense in $S$ with respect to the Zariski topology. If $k$ is of finite type over $\mathbb{Q}$, the converse holds as well.

The proof of this theorem makes use of a family of 'multisections', by which we mean a family of curves on $\mathcal{E}$ that intersect each fiber of $\mathcal{E} \to \mathbb{P}^1_k$ the same number of times, counted with multiplicity. This family of curves is parametrized by the fiber which is mentioned in the theorem. With the help of these curves, the Zariski density of the $k$-rational points is proven.

The proof distinguishes three cases. In two cases the density is proven by showing that there is a $k$-section of infinite order on (a base change of) $\mathcal{E}$. In the more difficult case, they construct a new elliptic surface which parametrizes a family of smooth curves which are the normalizations of the multisections mentioned earlier. They then show that this constructed elliptic surface maps dominantly to $S$ and finish the proof by showing that the $k$-rational points of this new elliptic surface lie Zariski dense.

In the next chapter we will generalize the result of Theorem 2.18 to a bigger family of del Pezzo surfaces of degree one which contains the family of this theorem as a subset. Theorem 2.18 assumes the existence of another $k$-rational point that is non-torsion on its fiber on $\mathcal{E}$. This implies that this fiber contains infinitely many $k$-rational points. In our proof, only this latter assumption is needed. Moreover, we can avoid using the elliptic fibration $\mathcal{E} \to \mathbb{P}^1_k$ almost completely.

Except from these points, we will still use a lot of ideas of [DW21]. Especially, we will create a fibered surface, consisting of a family of curves, defined similarly as in [DW21]. In order to make the arguments work more generally, we had to reinterpret some computations more geometrically, as some of the computations would have been too large to do in more generality.

# Chapter 3

# Zariski density on a family of surfaces

The goal of this chapter will be to prove the following main theorem of this thesis and give some examples of del Pezzo surfaces to which we can apply it.

**Theorem 3.1.** Let $k$ be a field of characteristic 0. Let $f := f_3t^3+f_2t^2+f_1t+f_0$ with $f_i \in k$ and $f_3 \neq 0$ be a polynomial in $k[t]$. Let $a_1, a_2, a_3, a_4, a_6 \in k[u]$ be polynomials such that $3\deg(a_i) \leq i$. Suppose the surface $S$ in $\mathbb{P}_k(2,3,1,1)$ given by the equation

$$Y^2 + a_1XYW + a_3(f(Z/W))YW^3 = X^3 + a_2X^2W^2 + a_4(f(Z/W))XW^4 + a_6(f(Z/W))W^6 \tag{3.1}$$

is smooth. Let $t_0 \in k$ be an element. Define the curve $\mathcal{F}$ on $S$ by $Z - t_0W = 0$. If $f - f(t_0)$ is separable, $3t_0 \neq \dfrac{-f_2}{f_3}$ and $|\mathcal{F}(k)| = \infty$, then $S(k)$ is dense in $S$ with respect to the Zariski topology. Moreover, if $k$ is of finite type over $\mathbb{Q}$ and $S(k)$ lies Zariski dense in $S$, then there exists a $t_0 \in k$ satisfying these properties.

Note that by Proposition 2.8, the surface $S$ of Theorem 3.1 is a del Pezzo surface of degree 1, because $S$ is a smooth surface of degree six in the weighted projective space $\mathbb{P}_k(2,3,1,1)$. In particular, Theorem 3.1 is a result on the Zariski density of the $k$-rational points of del Pezzo surfaces of degree one. To give the reader a general idea of what is going to happen in this chapter, we start by giving a sketch of the proof.

**Sketch of the proof of Theorem 3.1:**

In order to prove the theorem, we will construct a surface $\mathcal{C}$ which maps dominantly to $S$. Because dense sets map to dense sets under dominant maps, it follows that in order to prove the Zariski density of the $k$-rational points on $S$ it is enough to prove the Zariski density of the $k$-points on $\mathcal{C}$.

This surface $\mathcal{C}$ parametrizes a family of curves, which are all curves on $S$. To be more specific, there will be a fibration $\mathcal{C} \to \mathcal{F}_0$ where $\mathcal{F}_0$ denotes an open part of the curve $\mathcal{F}$ such that each fiber is a curve on $S$. We will define these curves in section 3.1. In the remainder of this section we will prove some properties of these curves. The surface $\mathcal{C}$ will be defined afterward in subsection 3.2.1.

In order to prove the Zariski density of the $k$-points on $\mathcal{C}$, we will first show that there are infinitely many sections of the fibration $\mathcal{C} \to \mathcal{F}_0$. The generic fiber of the fibration $\mathcal{C} \to \mathcal{F}_0$ will be of geometric genus 0 or 1, so this gives us two cases which we should distinguish. The genus 0 case will be relatively easy.

In the genus 1 case, we need to do a bit more work. Therefore we will construct another fibered surface $\mathscr{D} \to \mathbb{P}^1_k$, that is related to this surface $\mathcal{C}$. This work is done in the remainder of section 3.2. In the case that the generic fiber has geometric genus 1, we will show in subsection 3.3.1 that $\mathscr{D}$ is an elliptic surface. On this elliptic surface there will be a section of infinite order and from this section we can construct infinitely many sections of $\mathcal{C} \to \mathcal{F}_0$.

In both cases, the existence of these sections will imply the existence of infinitely many distinct curves on $\mathcal{C}$, which have infinitely many $k$-rational points. This will imply the Zariski density of the $k$-rational points on $\mathcal{C}$ from which we will conclude the first part of the statement.

For the second part of the statement of the theorem, we will use a generalization of Merel's Theorem, [Mer96]. This theorem states that the torsion points on elliptic curves over some finitely generated field extension of $\mathbb{Q}$ are bounded, and this bound only depends on the field of definition.

**Assumptions in this chapter:**

We suppose that $k$ is a field. We define the polynomials $a_i \in k[u]$ and $f \in k[t]$ as in Theorem 3.1 and assume $S$ is a smooth surface in the weighted projective space $\mathbb{P}_k(2,3,1,1)$ given by equation 3.1. Recall that by Proposition 2.8, the surface $S$ is a del Pezzo surface of degree 1. We denote by $\mathcal{O} = (1:1:0:0)$ the base point of the canonical system $|-K_S|$.

In some sections we will make further assumptions, for example on the characteristic of $k$, but for now we will try to prove things in more generality.

## 3.1 A family of curves

The goal of this section is to introduce a family of curves and give some properties which hold for them. We start by introducing some notation which we will use in this section.

Identify the affine piece $W \neq 0$ of $\mathbb{P}_k(2,3,1,1)$ with the affine space $\mathbb{A}^3_k$. Give $\mathbb{A}^3_k$ coordinates $x = \frac{X}{W^2}$, $y = \frac{Y}{W^3}$ and $t = \frac{Z}{W}$. Define the polynomial $g \in k[x, y, u]$ by

$$g := y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6. \tag{3.2}$$

Note that the variety $S \cap \mathbb{A}^3_k$ is given in $\mathbb{A}^3_k$ by the zero set of the polynomial $g(x, y, f(t))$. Furthermore, define the affine surface $S'$ in $\mathbb{A}^3_k$ as the zero set of the polynomial $g(x, y, t)$.

The polynomial $f$ induces an endomorphism of $\mathbb{A}^3_k$ given by

$$\varphi\colon \mathbb{A}^3_k \to \mathbb{A}^3_k;\ (x_R, y_R, t_R) \mapsto (x_R, y_R, f(t_R)). \tag{3.3}$$

This morphism $\varphi$ is a finite morphism of degree three which is ramified at the points $R = (x_R, y_R, t_R)$ with $f - f(t_R)$ inseparable. Moreover, $\varphi$ is a flat morphism and so outside these ramified points the morphism $\varphi$ is étale. The image of the surface $S \cap \mathbb{A}^3_k$ under $\varphi$ is the surface $S'$.

**Definition 3.2.** Let $l/k$ be a field extension and let $R \in (S \cap \mathbb{A}^3_k)(l)$ be an $l$-rational point on the affine piece with coordinates $R = (x_R, y_R, t_R)$ such that $\varphi(R)$ is smooth over $l$ on $(S')_l$. We define the surface $H_R$ in $\mathbb{A}^3_l$ to be the pullback of the tangent plane at the point $\varphi(R)$ to $(S')_l$ under the map $\varphi_l$. Moreover, we define the curve $C_R$ on $S_l$ to be the closure of $S_l \cap H_R$ in $\mathbb{P}_l(2,3,1,1)$. Or equivalently, $C_R$ is given by the intersection $S_l \cap H^{\mathrm{c}}_R$ in $\mathbb{P}_l(2,3,1,1)$, where $H^{\mathrm{c}}_R$ denotes the closure of $H_R$ in $\mathbb{P}_l(2,3,1,1)$.

**Remark 3.3.** Note that the plane $H_R$ in $\mathbb{A}^3_l$ of Definition 3.2 is explicitly given by the zero set of the polynomial

$$\frac{\partial g}{\partial x}(\varphi(R))(x - x_R) + \frac{\partial g}{\partial y}(\varphi(R))(y - y_R) + \frac{\partial g}{\partial u}(\varphi(R))(f(t) - f(t_R)). \tag{3.4}$$

Therefore, the closure $H_R^{\mathrm{c}}$ of $H_R$ in $\mathbb{P}_l(2,3,1,1)$ is given by the equation

$$\frac{\partial g}{\partial x}(\varphi(R))(XW - x_R W^3) + \frac{\partial g}{\partial y}(\varphi(R))(Y - y_R W^3) + \frac{\partial g}{\partial u}(\varphi(R))(f(Z,W) - f(t_R))W^3. \tag{3.5}$$

At least if the characteristic is not three, the surface $S'$ will be smooth almost everywhere, because the map $\varphi$ will be étale almost everywhere and we assumed that $S$ is a smooth surface. Hence, the curve $C_R$ of Definition 3.2 is defined for almost every point $R \in S \cap \mathbb{A}_k^3(l)$. In particular, $C_R$ will be defined for all points $R = (x_R, y_R, t_R)$ with $f - f(t_R)$ separable.

**Additional assumptions in the remainder of this section:**

Recall that on the affine piece $\mathbb{A}_k^3$ of $\mathbb{P}_k(2,3,1,1)$ where $W \neq 0$, the surface $S$ is given by the equation $g(x, y, f(t))$ where $g$ is defined as in (3.2). Moreover, recall that we have a morphism $\varphi\colon \mathbb{A}_k^3 \to \mathbb{A}_k^3$, defined as in (3.3), which maps the surface $S$ to the surface $S'$ given by $g(x,y,t) = 0$.

We assume that $l/k$ is a field extension and let $R \in (S \cap \mathbb{A}_k^3)(l)$ be an $l$-point with affine coordinates $R = (x_R, y_R, t_R)$ such that $\varphi(R)$ is smooth on $(S')_l$. Moreover, we define the surface $H_R$ on $\mathbb{A}_l^3$ and the curve $C_R$ on $S_l$ as in Definition 3.2. We will often identify the point $R$ with the corresponding point on the surface $S$ or the corresponding point on the curve $C_R$.

### 3.1.1 An isomorphic curve in $\mathbb{P}_k(2,1,1)$

Recall from section 2.2.2 that the rational map $\mathrm{pr}_Y : \mathbb{P}_k(2,3,1,1) \dashrightarrow \mathbb{P}_k(2,1,1)$ defined by $(X : Y : Z : W) \mapsto (X : Z : W)$, restricts to a morphism on $S$. We will again denote this morphism by $\tau\colon S \to \mathbb{P}_k(2,1,1)$. Recall that this morphism $\tau$ is a double cover of $\mathbb{P}_k(2,1,1)$. We will show that in the case that $\tau$ is unramified at $R$, that the curve $C_R$ is isomorphic with its image in the weighted projective space $\mathbb{P}_l(2,1,1)$.

**Lemma 3.4.** If $\tau$ is unramified at $R$, then the restriction of $\tau_l\colon S_l \to \mathbb{P}_l(2,1,1)$ to $C_R$ is an isomorphism onto its image. Moreover, this image is defined by the polynomial $h \in k[X,Z,W]$ with $h := W^6 g(X/W^2, \mathcal{Y}_R(X,Z,W)/W^3, f(Z/W))$ and

$$\mathcal{Y}_R(X,Z,W) = y_R W^3 - \frac{\frac{\partial g}{\partial x}(\varphi(R))(XW - x_R W^3) + \frac{\partial g}{\partial t}(\varphi(R))(f(Z/W) - f(t_0))W^3}{\frac{\partial g}{\partial y}(\varphi(R))}. \tag{3.6}$$

**Proof.** By the assumption that $\tau$ is unramified at the point $R$, it follows from Lemma 2.16 that $\dfrac{\partial g}{\partial y}(\varphi(R)) \neq 0$. Therefore, we can define the morphism

$$\mathbb{P}_l(2,1,1) \to H_R^{\mathrm{c}}; \quad (X : Z : W) \mapsto (X : \mathcal{Y}_R(X,Z,W) : Z : W).$$

Because the defining equation (3.5) of $H_R^{\mathrm{c}}$ is linear in $Y$, this gives us that the restriction of the map $\mathrm{pr}_Y$ to $H_R^{\mathrm{c}}$ is an inverse of this map. Hence, the latter map is an isomorphism and so the restriction $\tau_l|_{C_R}$ will also be an isomorphism onto its image. The equation of this image is given by evaluating at $\mathcal{Y}_R$ in the defining equation of $S$, which is exactly the given polynomial $h$. □

Recall that $R$ is an $l$-rational point on $S$. The automorphism $[-1]\colon S \to S$ of Definition 2.13 gives us another $l$-rational point

$$-R := [-1](R) = (x_R, -y_R - a_1 x_R - a_3(f(t_R)), t_R), \tag{3.7}$$

which also lies on $S \cap \mathbb{A}_k^3(l)$. One can check that the point $\varphi(-R)$ will be a smooth point on $(S')_l$ as well. Therefore, the curve $C_{-R}$ of Definition 3.2 is well-defined. We will show that this curve $C_{-R}$ maps to the same curve in $\mathbb{P}_k(2,1,1)$ as $C_R$.

**Lemma 3.5.** The images of the curves $C_R$ and $C_{-R}$ under the morphism $\tau_l\colon S_l \to \mathbb{P}_l(2,1,1)$ are the same.

**Proof.** Give $\mathbb{P}_k(2,1,1)$ coordinates $X$, $Z$ and $W$. Identify the affine plane $\mathbb{A}_k^2$ as the subset of $\mathbb{P}_k(2,1,1)$ where $W \neq 0$. Define $\mathrm{pr}_y\colon \mathbb{A}_k^3 \to \mathbb{A}_k^2$ by

$$(x,y,t) \mapsto (x,t).$$

Note that $\mathrm{pr}_y$ is a local description of the rational map $\mathrm{pr}_Y$. Restricting $\mathrm{pr}_y$ to the surface $S \cap \mathbb{A}_k^3$ gives a local description of the morphism $\tau\colon S \to \mathbb{P}_k(2,1,1)$.

Define two automorphisms $\chi, \chi'\colon \mathbb{A}_k^3 \to \mathbb{A}_k^3$ by

$$\chi\colon (x,y,t) \mapsto (x, -y - a_1 x - a_3(f(t)), t),$$
$$\chi'\colon (x,y,t) \mapsto (x, -y - a_1 x - a_3(t), t).$$

Note that both $\chi^2 = \mathrm{id}$ and ${\chi'}^2 = \mathrm{id}$. Moreover, these automorphisms restrict to automorphisms $\chi|_{S\cap\mathbb{A}_k^3}\colon S \cap \mathbb{A}_k^3 \to S \cap \mathbb{A}_k^3$ and $\chi'|_{S'}\colon S' \to S'$. Also note that $\chi|_{S\cap\mathbb{A}_k^3}$ corresponds with the map $[-1]\colon S \to S$.

Recall the morphism $\varphi\colon \mathbb{A}^3_k \to \mathbb{A}^3_k$ from (3.3) and note that $\chi' \circ \varphi = \varphi \circ \chi$. Also define a morphism $\varphi'\colon \mathbb{A}^2_k \to \mathbb{A}^2_k$ by $(x,t) \mapsto (x, f(t))$. The above defined morphisms give us the following commutative diagram.

$$\begin{tikzcd}
\mathbb{A}^3_k \arrow[r, "\chi"] \arrow[d, "\varphi"] \arrow[rr, bend left, "\mathrm{pr}_y"] & \mathbb{A}^3_k \arrow[r, "\mathrm{pr}_y"] \arrow[d, "\varphi"] & \mathbb{A}^2_k \arrow[d, "\varphi'"] \\
\mathbb{A}^3_k \arrow[r, "\chi'"] \arrow[rr, bend right, "\mathrm{pr}_y"'] & \mathbb{A}^3_k \arrow[r, "\mathrm{pr}_y"] & \mathbb{A}^2_k
\end{tikzcd} \tag{3.8}$$

Now recall from Definition 3.2 that the curve $C_R$ is defined by the closure of the pullback of the tangent plane $T$ to $(S')_l$ at $\varphi(R)$ intersected with $S$. Similarly, the curve $C_{-R}$ is defined by the tangent plane $T'$ to $(S')_l$ at $\varphi(-R)$.

The morphism $\chi'$ is a linear transformation with

$$\chi'(\varphi(R)) = \varphi(\chi(R)) = \varphi(-R).$$

This means that the tangent plane $T$ of $\varphi(R)$ gets mapped isomorphically to the tangent plane $T'$ under $\chi'$. It follows that the pullbacks $H_R = \varphi^{-1}(T)$ and $H_{-R} = \varphi^{-1}(T')$ are isomorphic under $\chi$. Hence, we get the equality

$$\chi(C_R \cap \mathbb{A}^3_k) = \chi(H_R \cap (S \cap \mathbb{A}^3_k)) = \chi(H_R) \cap \chi(S \cap \mathbb{A}^3_k) = H_{-R} \cap (S \cap \mathbb{A}^3_k) = C_{-R}.$$

Moreover, we have the equality $\mathrm{pr}_y = \mathrm{pr}_y \circ \chi$ and it follows that $H_R$ and $H_{-R}$ have the same image under $\mathrm{pr}_y$. Because the map $\mathrm{pr}_y$ is a local representation of $\tau$, we deduce that the curves $C_R$ and $C_{-R}$ have the same image under the morphism $\tau\colon S \to \mathbb{P}_k(2,1,1)$. □

Combining the above results, we get the following corollary, which is the main result to take away of this section.

**Corollary 3.6.** Suppose that the point $R$ is not a ramified point of the map determined by the linear system $|-2K_S|$. Then the curves $C_R$ and $C_{-R}$ are isomorphic and they map isomorphically to the same curve in $\mathbb{P}_l(2,1,1)$ as given in Lemma 3.4.

**Proof.** Combine Lemma 3.4 and 3.5 with Lemma 2.16. □

### 3.1.2 General properties

In this subsection we prove some general properties of the curve $C_R$.

**Lemma 3.7.** The scheme-theoretical fiber $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ above $\varphi(R)$ of the morphism $\varphi|_S \colon S \to \mathbb{A}^3_k$ is contained in the curve $C_R$. Moreover, this fiber contains at most three $\bar{l}$-points, with equality if and only if $f(t) - f(t_R)$ is separable. In the latter case, these are either three $l$-points or it consists of a unique $l$-point and the other two points are defined over some quadratic extension of $l$ and correspond with the same scheme-theoretic point.

**Proof.** Recall that $R \in (S \cap \mathbb{A}^3_k)(l)$ is an $l$-rational point. The scheme-theoretical fiber $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ of the morphism $\varphi$ is an $l$-variety that is defined by the equations $x = x_R$, $y = y_R$ and $f(t) = f(t_R)$. These equations define a variety of dimension zero, and it follows that $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ is in bijection with the Galois orbits of the set

$$\{(x_R, y_R, t) \in (S \cap \mathbb{A}^3_k)(\bar{l}) : f(t) = f(t_R)\}.$$

All points of this set are $\bar{l}$-rational points on $C_R$ and this gives an inclusion of $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ in $C_R$. Moreover, it holds that $|(S \cap \mathbb{A}^3_k)_{\varphi(R)}(\bar{l})| = |f^{-1}(f(t_R))|$. Because $f$ is a polynomial of degree 3, we have that $|f^{-1}(f(t_R))| \leq 3$ with equality if and only if $f - f(t_R)$ is separable.

In the latter case, there are two options. Either the polynomial $f - f(t_R)$ splits into linear factors and then the set $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ has three points defined over $l$. Or the polynomial $f$ has a linear factor given by $t - t_R$ and a irreducible quadratic factor. These correspond with the point $R$ which is defined over $l$ and one other point with a residue field which is some quadratic extension $l'$ of $l$. □

**Lemma 3.8.** If $f - f(t_R)$ is separable, then all points $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ are singular points of $C_R$.

**Proof.** By assumption, the morphism $\varphi \colon S \cap \mathbb{A}^3_k \to S'$ is unramified at $R$. Restricting $\varphi$ to the curve $C_R \cap \mathbb{A}^3_k$ still gives a finite morphism unto its image, which is étale at $R$. The image $\varphi(C_R \cap \mathbb{A}^3_k)$ of this map is the tangent plane to $S'$ at $\varphi(R)$ intersected with $S'$. Therefore, the point $\varphi(R)$ is a singular point on $\varphi(C_R \cap \mathbb{A}^3_k)$. Because $\varphi|_{C_R \cap \mathbb{A}^3_k}$ is étale at $R$, we deduce that all points in $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ are singular points of the curve $C_R$. □

**Lemma 3.9.** The curve $C_R$ is linearly equivalent to $-3K_{S_l}$.

**Proof.** The defining equation of $C_R$ on $S_l$ is given by the zero set of (3.5), which is a homogeneous polynomial of degree 3. The rational function with numerator the defining polynomial of $C_R$ and denominator $W^3$, is a rational function in $\mathcal{L}(-3K_S)$. We deduce that $C_R = \operatorname{div}(f) + 3\mathcal{F}$ where $\mathcal{F}$ denotes the curve on $S$ defined by $W = 0$. By Lemma 2.10, we have that $\mathcal{F} \sim -K_{S_l}$ and hence $C_R$ is linearly equivalent to $-3K_{S_l}$. □

**Lemma 3.10.** Suppose that the map determined by the linear system $|-2K_S|$ is unramified at the point $R$. Let $(z_0 : w_0) \in \mathbb{P}^1_l(l)$ be given and define the curve $\mathcal{F}$ on $S$ by $w_0 Z = z_0 W$. The curve $C_R$ does not contain the curve $\mathcal{F}$ and intersects $\mathcal{F}$ in three points, counted with multiplicity.

**Proof.** If $(z_0 : w_0) = (1 : 0)$ the curve $\mathcal{F}$ is defined by the equation $W = 0$. By equation (3.5), we can deduce that the curve $\mathcal{F}$ is not contained in $C_R$.

In all other cases, the curve $\mathcal{F}$ can be given by $t = t'$ on the affine piece $W \neq 0$ where $t' = \dfrac{z_0}{w_0}$. We can use the isomorphism of Corollary 3.6. The defining equation of the image on the affine piece is given by $g(x, \mathcal{Y}_R(x,t,1), f(t)) = 0$. This is polynomial in $x$ and $t$ which has a constant coefficient at $x^3$. Substituting $t = t'$, this still gives a non-zero polynomial, and hence this gives finitely many solutions. We deduce that $C_R$ does not contain $\mathcal{F}$.

Now we calculate the intersection number. Recall that $C_R \sim -3K_{S_l}$ by Lemma 3.9. Similarly, we can deduce that $\mathcal{F} \sim -K_{S_l}$. It follows that

$$C_R \cdot \mathcal{F} = (-3K_{S_l}) \cdot (-K_{S_l}) = 3 \cdot {K_{S_l}}^2 = 3.$$

Because $C_R$ does not contain any fiber, it intersects every fiber in three points, counted with multiplicity. □

**Lemma 3.11.** Suppose that the map determined by the linear system $|-2K_S|$ is unramified at the point $R$. Let $\mathcal{F}$ be the curve on $S_l$ given by $Z = t_R W$. Then the intersection $C_R \cap \mathcal{F}$ equals the set $\{R, -2R\}$, where the point $-2R$ is given by the group operation in the group $\mathcal{F}_{\text{ns}}(l)$, where $\mathcal{F}_{\text{ns}}$ denotes the locus of the non-singular points.

**Proof.** Recall that by Remark 2.9 the curve $\mathcal{F}$ can be identified with a Weierstrass curve. This identification can be done on the hyperplane defined by $t - t_R$ in $\mathbb{A}^3_k$. Note that the equality $H_R^{\text{c}} \cap \mathcal{F} = C_R \cap \mathcal{F}$ holds, where $H_R$ denotes the surface from the Definition 3.2 of $C_R$. This line goes through the point $R$. Because $R$ is a singular point of $C_R$, this implies that the intersection multiplicity of this line with $\mathcal{F}$ is at least 2.

Let $Q$ denote the third point of intersection on this line. By the assumption that the point $R$ is an unramified point of the map determined by $|-2K_S|$, it is not a singular point on $\mathcal{F}$. If $Q$ is not equal to $R$ then it has intersection multiplicity one, and so cannot be a singular point on $\mathcal{F}$ either. Hence, by the group structure of the Weierstrass equation we have that $2R + Q = \mathcal{O}$. We deduce that $Q$ equals the point $-2R$. □

**Corollary 3.12.** The point $R$ is the only point in the intersection $C_R \cap \mathcal{F}$ if and only if it has order three in the group $\mathcal{F}_{\text{ns}}(l)$.

In the following lemmas, we give results on the genus of $C_R$. In the proof the adjunction formula will be used.

**Lemma 3.13.** Let $C$ be an integral curve on a nice surface $X$. Denote by $p_a(C)$ the arithmetic genus of $C$ and by $K_X$ the canonical divisor of $X$. Then we have the equality

$$2p_a(C) - 2 = C \cdot (C + K_X). \tag{3.9}$$

**Proof.** A proof of this fact for nice curves can be found in Proposition V.1.5 of [Har77], which can be generalized to integral curves, e.g. Exercise V.1.3 of [Har77]. □

**Lemma 3.14.** If $C_R$ is geometrically integral, then its arithmetic genus $p_a(C_R)$ is 4.

**Proof.** Assume that $C_R$ is geometrically integral. From the adjunction formula (3.9) it follows that

$$p_a(C_R) = \frac{1}{2}C_R \cdot (C_R + K_S) + 1 = -\frac{3}{2}K_S \cdot (-2K_S) + 1 = 3{K_S}^2 + 1 = 4.$$

□

**Lemma 3.15.** Suppose that the map determined by the linear system $|-2K_S|$ is unramified at the point $R$. Suppose that the curve $C_R$ is geometrically integral and that $f - f(t_R)$ is separable. Then the geometric genus is at most 1.

**Proof.** If $f - f(t_R)$ is separable, it follows from Remark 3.7 that there are exactly three points in $(S \cap \mathbb{A}^3_k)_{\varphi(R)}$ defined over $l$ or one point defined over $l$ and one point defined over some quadratic extension $l'$ of $l$. According to Lemma 3.8 these points must be singular on the curve $C_R$. Lemma 3.14 gives that the arithmetic genus is four, so we deduce from Corollary V.3.7 of [Har77] that the geometric genus of $C_R$ is at most 1. ☐

### 3.1.3 The curve obtained from the generic point of a fiber

Let $t_0 \in k$ and let $\mathcal{F} \subset S$ be the curve on $S$ given by the equation $Z = t_0 W$. Let $\eta$ be the generic point of $\mathcal{F}$. We can identify this point with the $\kappa(\eta)$-point $(\tilde{x}, \tilde{y}, t_0) \in S(\kappa(\eta))$, where

$$\kappa(\eta) = k(\tilde{x}, \tilde{y}) = \mathrm{Frac}(k[x, y]/g(x, y, f(t_0)))$$

denotes the residue field of $\eta$. Define the curve $C_\eta$ as in Definition 3.2. Note that this curve is defined over $\kappa(\eta)$ and that it is a curve on the base change $S_{\kappa(\eta)}$. We have the following result on this curve $C_\eta$.

**Proposition 3.16.** Suppose that the map determined by the linear system $|-2K_S|$ is unramified at $\eta$. If $f - f(t_0)$ is separable, then the curve $C_\eta$ is geometrically integral of geometric genus 0 or 1.

**Proof.** To derive a contradiction, suppose that the curve $C_\eta$ is not geometrically integral. The curve $\overline{C_\eta}$ has degree 3 over $\overline{\mathbb{P}^1_{\kappa(\eta)}}$ and so if it is not geometrically integral, it would contain a geometrically integral curve, say $C'$, which has degree 1 over $\overline{\mathbb{P}^1_{\kappa(\eta)}}$. In this case, $C'$ intersects every curve on $\overline{S_{\kappa(\eta)}}$ of the form $w_0 Z - z_0 W$, where we take $(w_0 : z_0) \in \mathbb{P}^1_{\kappa(\eta)}(\overline{\kappa(\eta)})$, in exactly one point. By Lemma 3.11 either $\eta$ or $-2\eta$ would be a point on this curve.

By our assumption and from Lemma 2.16, it follows that $\frac{\partial g}{\partial y}(\eta) \neq 0$. Because $\mathcal{O}$ is not on the curve $C_\eta$, it follows that $\mathcal{O}$ is not a point on the curve $C'$. Because $C'$ intersects every fiber of the morphism $\overline{S_{\kappa(\eta)}} - \mathcal{O} \to \overline{\mathbb{P}^1_{\kappa(\eta)}}$ in exactly one point, it defines a section $\overline{\mathbb{P}^1_{\kappa(\eta)}} \to \overline{S_{\kappa(\eta)}} - \mathcal{O}$. This gives us that $C' \cong \overline{\mathbb{P}^1_{\kappa(\eta)}}$.

Because the closure of every fiber of this morphism is in $|-K_S|$ and $C'$ intersects every fiber once, it follows that $C' \cdot K_{\overline{S_{\kappa(\eta)}}} = -1$. By the adjunction formula (3.9), we find that

$$C'^2 = 2(p_a(C') - 1) - C' \cdot K_{\overline{S_{\kappa(\eta)}}} = -2 + 1 = -1.$$

This means that $C'$ is an exceptional curve.

Note that $\kappa(\eta)$ is transcendental over $k$ and that the exceptional curves of $S$ are already defined over $k^{\text{sep}}$. Moreover, there are only finitely many exceptional curves on $S$ and any exceptional curve does not contain the curve $\mathcal{F}$. We deduce that both points $\eta$ and $-2\eta$ are not on a exceptional curve on the surface $S_{\kappa(\eta)}$. Hence, we derived a contradiction and we conclude that $C_\eta$ is geometrically integral. ☐

By Proposition 2.17, the point $\eta$ will in almost all cases be an unramified point of the map determined by the linear system $| - 2K_S|$. If follows that Proposition 3.16 is almost always true. In particular, it is true if $\text{char}(k) \neq 2$.

## 3.2 Some fibered surfaces

In this section we define some fibered surfaces. These surfaces will be related to the curves $C_R$ which we have encountered in section 3.1. The first surface $\mathcal{C}$ will be the family of curves $C_R$ parametrized over an open subset of the curve $\mathcal{F} \subset S$ given by $Z = t_0 W$ for some $t_0 \in k$. As mentioned before, this surface will play an important role in the proof of Theorem 3.1.

After defining the surface $\mathcal{C}$, the remaining goal of this section is constructing a fibration $\mathscr{D} \to \mathbb{P}^1_k$, of which a model can be given in $\mathbb{P}^1_k \times \mathbb{P}^2_k$. The fibered surface $\mathscr{D}$ will be closely related to $\mathcal{C}$ and will be needed to define an infinite family of sections $\mathcal{F}_0 \to \mathcal{C}$ later on.

We define this surface in the following way. We first construct a fibered surface $\mathcal{D} \to A$ over some curve $A$, with the property that $\mathcal{C}$ is a fiber product of $\mathcal{D}$ with $\mathcal{F}_0$ over $A$. From this surface we obtain a surface $\tilde{\mathcal{D}}$ by blowing up $\mathcal{D}$ along a closed subscheme. We show that the resulting surface can be identified as a surface in $\mathbb{P}^1_k \times \mathbb{P}^2_k$. The closure of this surface will exactly be the surface $\mathscr{D}$.

The reason why we have to go through all this trouble, is the fact that obtaining equations for a minimal regular model for $\mathcal{C} \to \mathcal{F}$ is rather difficult, if not impossible. It involves too many equations with too many transcendentals. The construction of this section gives a way to work around this.

**Assumptions of this section:**

Recall the assumptions sketched at the beginning of this chapter. In particular, recall that $S \subset \mathbb{P}_k(2, 3, 1, 1)$ is a del Pezzo surface defined by equation (3.1), in which we used a polynomial $f \in k[t]$ with $\deg(f) = 3$.

Let $t_0 \in k$ be an element such that $f - f(t_0)$ is separable. Let $\mathcal{F} \subset S$ be the curve on $S$ given by the equation $Z = t_0 W$. Let $\eta$ be the generic point of $\mathcal{F}$. Let

$$\kappa(\eta) = k(\tilde{x}, \tilde{y}) = \mathrm{Frac}(k[x, y]/g(x, y, f(t_0)))$$

denote the residue field of $\eta$. Note that this is also the function field of the fiber $\mathcal{F}$.

Denote by $\mathcal{F}_0$ the open subvariety of $\mathcal{F}$ where the map determined by the linear system $|-2K_S|$ is unramified. We assume this set to be non-empty. By Proposition 2.17 in almost all cases, and in particular if char $k \neq 2$, this subvariety $\mathcal{F}_0$ will indeed be non-empty.

Observe that $\mathcal{F}_0$ is a smooth curve, because the singular points of $\mathcal{F}$ are ramification points of the map determined by $|-2K_S|$. Moreover, it is contained in the affine $S \cap \mathbb{A}^3_k$. This means that for every field extension $l/k$ and $l$-rational point $R \in \mathcal{F}_0(l)$, the point $\varphi(R)$ is smooth in $(S')_l$ by the fact that the map $\varphi\colon \mathbb{A}^3_k \to \mathbb{A}^3_k$ is étale at $R$. It follows that we can define the curve $C_R$ of Definition 3.2 for every point $R \in \mathcal{F}_0(l)$.

### 3.2.1 Construction of the surface $\mathcal{C}$

Let $\mathbb{A}^3_k \times \mathbb{P}_k(2, 3, 1, 1)$ be given with coordinates $x, y, t$ and $X, Y, Z, W$. Observe that we can identify $\mathcal{F}_0 \times S$ as a subvariety of $\mathbb{A}^3_k \times \mathbb{P}_k(2, 3, 1, 1)$.

**Definition 3.17.** Define the surface $\mathcal{C}$ as the variety over $k$ given by the intersection of $\mathcal{F}_0 \times S$ with the zero set of the equation

$$\frac{\partial g}{\partial x}(x, y, f(t_0))(XW - \tilde{x}W^3) + \frac{\partial g}{\partial y}(x, y, f(t_0))(Y - \tilde{y}W^3) + \frac{\partial g}{\partial t}(x, y, f(t_0))(f(Z/W) - f(t_0))W^3. \tag{3.10}$$

Note that equation (3.10) is almost the same as equation (3.5). Moreover, there is a projection morphism $\mathcal{C} \to \mathcal{F}_0$ onto the first coordinate, which makes this surface $\mathcal{C}$ a fibered surface over $\mathcal{F}_0$. The fibers of this morphism are exactly the curves $C_R$, which we encountered in section 3.1. The fiber above the generic point $\eta \in \mathcal{F}_0$ under $\mathcal{C} \to \mathcal{F}_0$ is the curve $C_\eta$ of section 3.1.3.

For each field extension $l/k$, we get the following description of the $l$-points of $\mathcal{C}$, namely

$$\mathcal{C}(l) := \{(R, P) : R \in \mathcal{F}_0(l), P \in C_R(l)\} \subset (\mathcal{F}_0 \times S)(l). \tag{3.11}$$

Let us motivate why we defined this surface $\mathcal{C}$. There is a projection morphism $\mathcal{C} \to S$ onto the second coordinate. We will show in Lemma 3.18 that this morphism is dominant. Hence, this morphism will map dense sets to dense sets. It follows that if we want to prove the Zariski density of the rational points of $S$, it is enough to prove the Zariski density of rational points in $\mathcal{C}$.

**Lemma 3.18.** The surface $\mathcal{C}$ maps dominantly to $S$ under the projection morphism $\mathcal{C} \to S$.

**Proof.** For each $R \in \mathcal{F}_0(\overline{k})$ the curve $\{R\} \times C_R \subset \overline{\mathcal{C}}$ gets mapped to $C_R$ on $\overline{S}$. Each $C_R$ intersects $\overline{\mathcal{F}}$ in only three points counted with multiplicity by Lemma 3.10. Because $R$ is the only double point of $C_R$ on $C_R \cap \overline{\mathcal{F}}$, every $C_R$ will be mapped to a different curve. We deduce that the image of $\overline{\mathcal{C}} \to \overline{S}$ must contain infinitely many different curves $C_R$ and therefore the image of $\mathcal{C} \to S$ must be dense in $S$. ☐

**Corollary 3.19.** If $\mathcal{C}(k)$ lies Zariski dense in $\mathcal{C}$, then $S(k)$ lies Zariski dense in $S$.

### 3.2.2 Construction of the surface $\mathcal{D}$

Let $p_x \colon \mathcal{F}_0 \to \mathbb{A}^1_k$ be the morphism defined by $(x, y, t_0) \mapsto x$. Let $A$ denote the image of $\mathcal{F}_0$ in $\mathbb{A}^1_k$. Recall that the set $\mathcal{F}_0$ is the subset of $\mathcal{F}$, that is unramified for the map determined by the linear system $|-2K_S|$.

Given an element $x_0 \in A(l)$ for some field extension $l/k$, the fiber above $x_0$ under $p_x$ contains two points, which are given by a set $\{R, -R\}$, where the point $R$ denotes some $\bar{l}$-rational point $R = (x_0, y_0, t_0)$ in $\mathcal{F}_0(\bar{l})$. The points $R$ and $-R$ in the inverse image of $x_0$ are not necessarily $l$-rational points. If they are not, they will be $l'$-rational points, for some field extension $l'/l$ of degree 2.

Let $l/k$ be a field extension and let $x_0 \in A(l)$. Set $R = (x_0, y_0, t_0) \in \mathcal{F}_0(\bar{l})$ for some point $R$ in the inverse image of the point $x_0$ under the map $\mathcal{F}_0 \to A$. Recall from Lemma 3.5 that the image of the curves $C_R$ and $C_{-R}$ under the rational map $\mathbb{P}_{\bar{l}}(2,3,1,1) \dashrightarrow \mathbb{P}_{\bar{l}}(2,1,1)$ agree. Hence, the image of this curve is independent of the choice of $R$ and only depends on the $x$-coordinate of $R$. It follows that we get a well-defined curve that is defined over $l$.

**Definition 3.20.** Let $x_0 \in A(l)$. Then we define the curve $D_{x_0} \subset \mathbb{P}_l(2,1,1)$ to be the image of the curve $C_R$ for some $R \in \mathcal{F}(\bar{l})$ with $x(R) = x_0$.

We have the following result for these curves.

**Lemma 3.21.** Let $x_0 \in A(l)$ and $D_{x_0}$ be the curve as defined in Definition 3.20. Let $C_R$ denote a fiber of the morphism $\mathcal{C} \to \mathcal{F}_0$ above $R \in \mathcal{F}_0(\bar{l})$ of $\mathcal{C} \to \mathcal{F}_0$ such that $p_x(R) = x_0$. Then the following statements hold:

(i) there is an isomorphism $C_R \cong \overline{D_{x_0}}$ given by Lemma 3.4;

(ii) the curve $D_{x_0}$ has the same arithmetic genus and geometric genus as $C_R$;

(iii) under the above isomorphism, the scheme-theoretic fiber $(S \cap \mathbb{A}_k^3)_{\varphi(R)}$ of $\varphi(R)$ under $\phi$, containing three singular $\bar{l}$-points of $C_R$, descends to a set consisting of three singular $\bar{l}$-points of $D_{x_0}$. Moreover, this set is defined over $l$ by the equations $X = x_0W^2$ and $f(Z/W)W^3 = f(t_0)W^3$.

**Proof.** Statement (i) is just a rephrasing of Lemma 3.4.

By Proposition III.9.3 of [Har77] we have the equality $p_a(C_R) = p_a(D_{x_0})$ for the arithmetic genus. Recall that the morphism $\mathcal{F}_0 \to A$ is étale, and in particular unramified. This means that $R$ is already defined over $l^s$ and so $C_R$ as well. Theorem 2.5.1 of [Poo17] gives us an equality $g(C_R) = g(D_{x_0})$ for the geometric genus as well. This proves statement (ii).

For statement (iii), note that non-singular points map to non-singular points under the base change $\overline{D_{x_0}} \to D_{x_0}$. Because both the arithmetic genus as the geometric genus stays the same, it follows that the subscheme $(S \cap \mathbb{A}_k^3)_{\varphi(R)}$ maps to a set of singular points of $D_{x_0}$.

Now write $R = (x_0, y_0, t_0) \in (S \cap \mathbb{A}_k^3)(\bar{l})$. Then $(S \cap \mathbb{A}_k^3)_{\varphi(R)}$ can be identified as a subscheme of $\mathbb{P}_l(2,3,1,1)$ given by the equations $X = x_0W^2$, $Y = y_0W^3$ and $f(Z/W)W^3 = f(t_0)W^3$. The image of $(S \cap \mathbb{A}_k^3)_{\varphi(R)}$ under the map $\tau$ of Lemma 3.4 is given by the equations $X = x_0W^2$ and $f(Z/W)W^3 = f(t_0)W^3$. Because by assumption $x_0 \in l$ and $t_0 \in k \subset l$, this set is defined over $l$. □

Recall the surface $\mathcal{C}$ that we defined in section 3.2.1. Also recall that this surface comes with morphisms $\mathcal{C} \to \mathcal{F}_0$ and $\mathcal{C} \to S$, which we can compose with the morphisms $p_x \colon \mathcal{F}_0 \to A$ and $\tau \colon S \to \mathbb{P}_k(2,1,1)$ respectively. These compositions induce a morphism $\mathcal{C} \to A \times \mathbb{P}(2,1,1)$ which gives rise to the following surface.

**Definition 3.22.** The surface $\mathcal{D}$ is defined as the image of $\mathcal{C}$ in $A \times \mathbb{P}(2,1,1)$ under the morphism that is induced by the diagram

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[d, "p_x"] & \mathcal{C} \arrow[l] \arrow[r] \arrow[d, dotted] & S \arrow[d, "\tau"] \\
A & A \times \mathbb{P}_k(2,1,1) \arrow[l] \arrow[r] & \mathbb{P}_k(2,1,1)
\end{tikzcd} \tag{3.12}$$

By Lemma 3.4, we deduce that this surface $\mathcal{D}$ is exactly the family of curves $D_x$ over $A$. Let $\tilde{x}$ denote the generic point of $A$, then the curve $D_{\tilde{x}}$ will be the generic fiber of the surface $\mathcal{D}$. For each field extension $l/k$ the $l$-points of $\mathcal{D}$ are

$$\mathcal{D}(l) = \{(x, P) : x \in A(l), P \in D_x(l)\} \subset (A \times \mathbb{P}(2,1,1))(l). \tag{3.13}$$

The relation between the surfaces $\mathcal{C}$ and $\mathcal{D}$ is given in the following lemma.

**Lemma 3.23.** The following diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{F}_0 \arrow[r] & A
\end{tikzcd} \tag{3.14}$$

is a pullback diagram, i.e. $\mathcal{C} \cong \mathcal{F}_0 \times_A \mathcal{D}$

**Proof.** Diagram (3.14) is commutative, so it induces a map $\mathcal{C} \to \mathcal{F}_0 \times_A \mathcal{D}$. Moreover, by our construction this map is explicitly given by the assignment $(R, P) \mapsto (R, (x(R), \mathrm{pr}_Y(P)))$. An inverse is given by the assignment

$$(R, (x, P)) \mapsto (R, (X(P) : (\mathcal{Y}_R(X, Z, W))(P) : Z(P) : W(P))),$$

where $\mathcal{Y}_R$ is defined as in Lemma 3.4. The fact that this assignment indeed gives a well-defined morphism with image $\mathcal{C}$ follows from the following two facts. For each point $(R, (x, P)) \in \mathcal{F}_0 \times_A \mathcal{D}$, we must have that $x(R) = x$. And secondly, for each point $R \in \mathcal{F}_0(l)$, we have by Lemma 2.16 that $\frac{\partial g}{\partial y}(\varphi(R)) \neq 0$, so indeed $\mathcal{Y}_R$ is well-defined. □

### 3.2.3 Construction of the surface $\tilde{\mathcal{D}}$

By Lemma 3.21 each fiber of the surface $\mathcal{D}$ of Definition 3.22 is a singular curve of which we know some of the singularities. Our next objective is to define a fibered surface of which the fibers are exactly blowups of the curves of Definition 3.20 which resolves these singularities. In this section, we will denote the coordinates of $\mathbb{P}_k(2,1,1)$ by $X, W, Z$ respectively.

**Definition 3.24.** Let $x \in A(l)$ be given. Recall that $D_x$ is the curve of Definition 3.20 and is a curve in $\mathbb{P}_l(2,1,1)$. We define $B_x$ to be the closed subscheme of the curve $D_x$ given by $X = xW^2$ and $f(Z/W)W^3 = f(t_0)W^3$. We define the curve $\tilde{D}_x$ to be the curve obtained from blowing up $D_x$ in this closed subscheme $B_x$.

Recall that we defined a blowup in section 2.1.1 via a universal property. This means that the curve in Definition 3.24 is uniquely defined up to a unique isomorphism. So actually speaking we should talk about 'a' curve $D_x$, but the latter makes it reasonable to talk about 'the' curve $\tilde{D}_x$ instead.

The following lemma gives us results on these family of curves $D_x$.

**Lemma 3.25.** Let $x \in A(l)$ be given. The arithmetic genus of the curve $\tilde{D}_x$ is one and exactly resolves the singularities of $D_x$ that are contained in $B_x$. Moreover, the multiplicity of each singular point is exactly 2.

**Proof.** By Proposition III.9.3 of [Har77], the equality $p_a(\tilde{D}_x) = p_a(\overline{\tilde{D}_x})$ holds. So without loss of generality we may assume that $l = \bar{l}$. Recall from Lemma 3.21 that for each fiber $D_x$ of $\mathcal{D} \to A$, the subscheme $B_x$ is a singular subscheme of $D_x$. Moreover, there are exactly three points in $\overline{B_x}$ which are all singular on $\overline{D_x}$.

One can show that the blowup of $D_x$ can be obtained as the strict transform of $D_x$ under a composition of monoidal transformations of the weighted projective space $\mathbb{P}_l(2,1,1)$. By Corollary V.3.7 of [Har77], we deduce that the arithmetic genus of $\tilde{D}_x$ will be

$$p_a(\tilde{D}_x) = p_a(D_x) - \frac{r_0(r_0-1) + r_1(r_1-1) + r_2(r_2-1)}{2},$$

where $r_i$ denote the multiplicities of the points. Because each of the three points is singular, we have $r_i > 1$. Moreover, this sum has to be non-negative and this is the case if and only if $r_i = 2$ for all $i$. We deduce the equality $p_a(\tilde{D}_x) = 1$ and that $\tilde{D}_x$ resolves the singularities contained in $B_x$. □

We will now define a fibered surface that parametrizes this family of curves of Definition 3.24.

**Definition 3.26.** Recall that $\mathcal{D}$ is the surface of Definition 3.22 and that it is a subvariety of $A \times \mathbb{P}_k(2,1,1)$. Let $B$ denote the subscheme defined by the equations $X = xW^2$ and $f(Z/W)W^3 = f(t_0)W^3$. The surface $\tilde{\mathcal{D}}$ is the surface obtained by blowing up $\mathcal{D}$ in the subscheme $B$.

As with the case of the curves $\tilde{D}_x$, we have defined the surface $\tilde{\mathcal{D}}$ uniquely up to a unique isomorphism. The goal in the remainder of this subsection is to show that this blowup $\tilde{\mathcal{D}}$ can be obtained as a fibered surface in $A \times \mathbb{P}^2_k$. This discussion will be quite technical and readers who are willing to accept this statement, can skip this discussion and continue reading in section 3.2.4.

Recall the cone $\mathcal{Q} = \operatorname{Proj}\big(k[p,q,r,s]/(r^2 - qs)\big) \subset \mathbb{P}^3_k$ of Example 1.24. To obtain the desired construction of $\tilde{\mathcal{D}}$ as a surface in $A \times \mathbb{P}^2_k$, we first give a construction of how to obtain the projective plane $\mathbb{P}^2_k$ from the cone $\mathcal{Q}$ by blowing up some points and blowing down some $-1$-curves.

Let $t_1, t_2 \in \overline{k}$ be the other zeroes of the polynomial $f - f(t_0)$. Because we assumed that $f - f(t_0)$ is separable that $t_0, t_1, t_2$ are all different elements of $\overline{k}$. Set $K = k(t_1) = k(t_2)$ and note that $K/k$ is a field extension of at most two. For $i = 0,1,2$, define the $K$-points $Q_i = (0 : {t_i}^2 : t_i : 1)$ on the surface $\mathcal{Q}$. The point $Q_0$ is already defined over $k$ and the points $Q_1$ and $Q_2$ are defined over $K$ and if $K \neq k$ holds, then the points are Galois conjugates.

Note that $\mathcal{Q}$ has a unique singular point $O := (1 : 0 : 0 : 0)$. We have three distinct lines through this singular point and the points $Q_0$, $Q_1$ and $Q_2$ respectively. We will denote these distinct lines by $L_0$, $L_1$ and $L_2$ respectively. The line $L_0$ is defined over $k$ so it is isomorphic to $\mathbb{P}^1_k$. Furthermore, the lines $L_1$ and $L_2$ are isomorphic to $\mathbb{P}^1_K$ and in the case that $k \neq K$ then they are each other's Galois conjugates.

Define the curve $C$ on $\mathcal{Q}$ to be the intersection of the hyperplane $x = 0$ with the cone $\mathcal{Q}$. This hyperplane contains the points $Q_0$, $Q_1$ and $Q_2$. Because the curve $C$ is a conic and contains the $k$-rational point $Q_0$, it will be isomorphic to $\mathbb{P}^1_k$.

The next step is to blow up $\mathcal{Q}$ in the singular point $O$. This blowup gives a nice surface $\tilde{\mathcal{Q}} := \operatorname{Bl}_O \mathcal{Q}$ with a morphism $\pi_1 \colon \tilde{\mathcal{Q}} \to \mathcal{Q}$. Let $E$ denote the exceptional curve of this blowup. The lines $L_i$ have different slopes at the singular point $O$. Therefore, their strict transforms, which we will denote by $L'_i$, will not intersect each other in any point. Because $L'_i \sim L'_j$, we have that ${L'_i}^2 = 0$ on $\tilde{\mathcal{Q}}$.

The curve $C$ does not contain the point $O$, so the strict transform $C'$ will just be $\pi_1^{-1}(C)$. We get that $C' \cdot L'_i = 1$, because $C'$ and $L'_i$ are meeting transversely.

We also have that the strict transform $C'$ has self intersection number ${C'}^2 = 2$, because $C$ is a conic.

Let $Q_i'$ denote the points on $\tilde{\mathcal{Q}}$ that are the strict transforms of the points $Q_i$. We will now blowup in the points $Q_i'$ simultaneously. The points $Q_1$ and $Q_2$ are defined over $k$ or they are each other's Galois conjugates and so this blowup will be defined over $k$. From this blowup, we obtain a surface $\tilde{\mathcal{S}} := \mathrm{Bl}_{Q_0',Q_1',Q_2'} \tilde{\mathcal{Q}}$ with a morphism $\pi_2 \colon \tilde{\mathcal{S}} \to \tilde{\mathcal{Q}}$. Note that, as a result of [Stacks, Tag 080A], we have that $\tilde{\mathcal{S}} \cong \mathrm{Bl}_{O,Q_0,Q_1,Q_2} \mathcal{Q}$.

We denote the exceptional curves above $Q_i$ by $F_i$ for $i = 0, 1, 2$. Furthermore, we denote the strict transforms of $L_i'$ on $\tilde{\mathcal{S}}$ by $E_i$ for $i = 0, 1, 2$. By Proposition V.3.2 and V.3.6 of [Har77], it follows that $E_i^2 = L_i'^2 - 1 = -1$ and that $E_i \cdot E_j = 0$ for $i \neq j$. Denote the strict transform of $C'$ by $E_3$. By the same propositions, the curve $E_3$ has self intersection $E_3^2 = {C'}^2 - 3 = -1$ as well. Moreover, for $i \neq 3$ we get that

$$E_i \cdot E_3 = (\pi_2^*(L_i') - F_i) \cdot (\pi_2^*(C') - (F_0 + F_1 + F_2)) = L_i' \cdot C' - F_i^2 = 1 - 1 = 0.$$

Hence, we have found four disjoint $-1$-curves on the surface $\tilde{\mathcal{S}}$. By blowing down these curves $E_i$ simultaneously, we obtain a surface $\mathcal{S}$ with a blowup morphism $\pi_3 \colon \tilde{\mathcal{S}} \to \mathcal{S}$. Note that the latter morphism is indeed defined over $k$. We will show in Lemma 3.27 that $\mathcal{S} \cong \mathbb{P}^2_k$.

The above construction can be summarized in the following diagram.

$$\begin{tikzcd}
 & & \tilde{\mathcal{S}} \arrow[dl, "\pi_2"'] \arrow[dd, "\pi_3"] \\
 & \tilde{\mathcal{Q}} \arrow[dl, "\pi_1"'] & \\
\mathcal{Q} \arrow[rr, dotted, leftrightarrow] & & \mathcal{S}
\end{tikzcd} \tag{3.15}$$

In the following lemma, the notion of a weak del Pezzo surface is used. A *weak del Pezzo surface* is a nice surface $X$ such that $K_X$ is big and nef, meaning that $K_X^2 \geq 1$ and $K_X \cdot C \geq 0$ for each curve $C \subset X$. For more background on these surfaces, we refer to [Dol12], p395.

**Lemma 3.27.** The surface $\mathcal{S}$ constructed as above is isomorphic to $\mathbb{P}^2_k$.

**Proof.** First we show that $\tilde{\mathcal{Q}}$ is a weak del Pezzo surface. Write $\tilde{E}$ for the exceptional curve of the blowup $\pi_1 \colon \tilde{\mathcal{Q}} \to \mathcal{Q}$. By Theorem III.7.11 of [Har77] and following Example II.8.20.3 of [Har77], we find that the dualizing sheaf $\omega_{\mathcal{Q}}^{\circ}$ on $\mathcal{Q}$ is isomorphic to the invertible sheaf $\mathcal{O}_{\mathcal{Q}}(-2)$.

We can define a canonical divisor $K_{\mathcal{Q}}$ on the cone $\mathcal{Q}$ to be a divisor with support in $\mathcal{Q}-\{O\}$ corresponding to $\omega_{\mathcal{Q}}^{\circ}$. Then we have that $K_{\mathcal{Q}} \sim -2H$, where $H$ is the intersection of $\mathcal{Q}$ with a hyperplane in $\mathbb{P}_k^3$ which does not contain the point $O$. Using the fact that the singularity of the cone is an ordinary double point, it follows from Proposition 8.1.10 of [Dol12] that the equality $K_{\tilde{\mathcal{Q}}} = \pi_1^*(K_{\mathcal{Q}})$ holds.

We have that $K_{\tilde{\mathcal{Q}}}^2 = \pi_1^*(-2H)^2 = 4\deg(\mathcal{Q}) = 8$, which shows that $K_{\tilde{\mathcal{Q}}}$ is big. We also get that $-K_{\tilde{\mathcal{Q}}} \cdot \tilde{E} = 2\pi_1^*(H) \cdot \tilde{E} = 0$. Furthermore, for every other curve $F$ on $\tilde{\mathcal{Q}}$ that is not this exceptional curve, we have $-K_{\tilde{\mathcal{Q}}} \cdot F \geq 0$, because the intersection number is given by the sum of the intersection multiplicities under the image of $\pi_1$ on the cone $\mathcal{Q}$. Hence, $-K_{\tilde{\mathcal{Q}}}$ is also nef and so we deduce that $\tilde{\mathcal{Q}}$ is a weak del Pezzo surface.

Because $K_{\tilde{\mathcal{Q}}}$ is big and nef with intersection number 8, it follows from Proposition 8.2.25 of [Dol12], that there is only one $-2$-curve on the surface $\tilde{\mathcal{Q}}$. This is exactly the curve denoted by $\tilde{E}$. This means that the points $Q_i$ do not lie on a $-2$-curve. Because this holds, it follows from Proposition 8.1.23(b) of [Dol12], that the blowup $\tilde{\mathcal{S}}$ is a weak del Pezzo surface as well. Moreover, by Proposition V.3.3 of [Har77] the equality $K_{\tilde{\mathcal{S}}}^2 = K_{\tilde{\mathcal{Q}}}^2 - 3 = 5$ holds.

The surface $\mathcal{S}$ is obtained by blowing down the surface $\tilde{\mathcal{S}}$ in four disjoint exceptional curves. From Proposition 8.1.23(a) of [Dol12] it follows that $\mathcal{S}$ is a weak del Pezzo surface. Again by Proposition V.3.3 of [Har77], we get for the canonical divisor that $K_{\mathcal{S}}^2 = K_{\tilde{\mathcal{S}}}^2 + 4 = 9$. It follows from Theorem 8.1.15 of [Dol12] that $\overline{\mathcal{S}} \cong \mathbb{P}_{\overline{k}}^2$. Moreover, $\mathcal{S}$ contains a $k$-rational point $P_0$, coming from blowing down the curve $E_0$. By Proposition 1.13, we conclude that $\mathcal{S} \cong \mathbb{P}_k^2$. □

Now we show how the above construction gives us a model of $\tilde{\mathcal{D}}$ in $A \times \mathbb{P}_k^2$. Recall that the isomorphism $\mathbb{P}_k(2,1,1) \to \mathcal{Q}$ from Example 1.24, which we will denote by $\psi_1$, is explicitly given by

$$\psi_1 \colon (X : Z : W) \mapsto (X : Z^2 : ZW : W^2).$$

Let $x \in A(l)$ be given and define the automorphism

$$\begin{aligned} \psi_2 \colon & \mathbb{P}_l(2,1,1) \to \mathbb{P}_l(2,1,1); \\ & (X : Z : W) \mapsto (X - xW^2 : Z : W). \end{aligned}$$

Under the isomorphism $(\psi_1)_l \circ \psi_2$, the curve $D_x$ embeds in $\mathcal{Q}_l$ such that the image of $B_x$ is given in $\mathcal{Q}_l$ by the equations $p = 0$ and $f(r/s)s^3 = f(t_0)s^3$. This set corresponds under the map of Proposition 1.9 with the Galois orbits of the points $Q'_i$.

We will show in the next proposition that the construction of blowing up and down the cone $\mathcal{Q}$, gives a model for the curve $\tilde{D}_x$ in $\mathbb{P}^2_l$.

**Proposition 3.28.** Let $x \in A(l)$ be given. From the above blowup and blowdown construction of $\mathcal{Q}$, we obtain a blowup $\tilde{D}_x \to D_x$. In particular, $\tilde{D}_x$ can be identified as a curve in $\mathbb{P}^2_l$.

**Proof.** The point $O$ does not lie on $D_x$. Therefore, we deduce that the strict transform of $((\psi_1)_l \circ \psi_2)(D_x)$ in $\mathcal{Q}_l$ under the blowup $(\pi_1)_l$ will be isomorphic to $D_x$. Denote this strict transform by $D'_x$. The blowup $(\pi_2)_l \colon \tilde{\mathcal{S}}_l \to \tilde{\mathcal{Q}}_l$ resolves the singularities corresponding to $B_x$ of the curve $D'_x$. Moreover, the strict transform of $D'_x$ on $\tilde{\mathcal{S}}_l$ is exactly a curve which satisfies Definition 3.24 and so we denote this curve by $\tilde{D}_x$.

To make notation more easy in this proof, we drop the subscript $l$ for the curves on the surfaces in the remainder of the proof. E.g. with abuse of notation, we will just write $L'_i$ instead of $(L'_i)_l$, but one should really use the base change!

Because the curve $D_x$ intersects $L_i$ in three points counted with multiplicity, it follows that $D'_x \cdot L'_i = 3$ for the strict transforms of these curves under $\pi_1$. Recall that the points $Q_i$ have multiplicity two by Lemma 3.14. We use Proposition V.3.2 and V.3.6 of [Har77] to deduce the equality

$$\tilde{D}_x \cdot E_i = (\pi_2^*(D'_x) - 2E_i) \cdot (\pi_2^*(L'_i) - E_i) = D'_x \cdot L'_i + 2E_i^2 = 3 - 2 = 1$$

for $0 \leq i \leq 2$. In particular, the curve $\tilde{D}_x$ intersects $E_i$ for $0 \leq i \leq 2$ in only one point.

Observe that the curve $C \sim 2 \cdot L_i$ on $\mathcal{Q}$, so $C' \sim 2 \cdot (L'_i + E)$. From Proposition V.3.2 and V.3.6 of [Har77] we again deduce that the intersection number of the strict transforms of these curves under $\pi_1$, is given by

$$D'_x \cdot C' = 2D'_x \cdot (L'_i + E) = 2 \cdot (3 + 0) = 6.$$

Furthermore, the points of intersection are exactly the strict transforms $Q'_i$ of the points $Q_i$, which are singular on $D_x$. We deduce that these curves meet transversally.

Again we use Proposition V.3.2 and V.3.6 of [Har77], to deduce that

$$\begin{aligned}\tilde{D}_x \cdot E_3 &= (\pi_2^*(D'_x) - 2(E_0 + E_1 + E_2)) \cdot (\pi_2^*(C') - (E_0 + E_1 + E_2)) \\ &= D'_x \cdot C' + 2(E_0^2 + E_1^2 + E_2^2) = 6 + 2 \cdot (-3) = 0.\end{aligned}$$

Hence, the curves $\tilde{D}_x$ and $E_3$ do not meet.

The morphism $\pi_3$ blows down these four exceptional curves $E_i$. Denote the restriction of $(\pi_3)_l$ to $\tilde{D}_x$ by $\pi'_3$. Then, because $\tilde{D}_x$ intersects each exceptional curve $E_i$ at most once, this map $\pi'_3$ will be an isomorphism which gives an isomorphic curve $\tilde{D}_x$ on $\mathcal{S}_l \cong \mathbb{P}_l^2$. The maps in the other direction will now give the blowup

$$\left(\psi_2^{-1} \circ \psi_1^{-1} \circ \pi_1 \circ \pi_2\right)_l \circ (\pi'_3)^{-1} \colon \tilde{D}_x \to D_x.$$

□

The above discussion extends to the surface $\tilde{\mathcal{D}}$ in the following way.

**Proposition 3.29.** The surface $\tilde{\mathcal{D}}$ can be identified as a surface in $A \times \mathbb{P}_k^2$ such that the induced morphism $\tilde{\mathcal{D}} \to A$ is a fibration. Moreover, for each $x \in A(l)$, the fiber of this morphism is the curve of Definition 3.24.

**Proof.** The universal property of the product gives us an isomorphism

$$\mathrm{id}_A \times \psi_1 \colon A \times \mathbb{P}_k(2,1,1) \to A \times \mathcal{Q}.$$

Moreover, we can precompose this with the automorphism

$$\begin{aligned}\Psi_2 \colon A \times \mathbb{P}_k(2,1,1) &\to A \times \mathbb{P}_k(2,1,1); \\ (x, (X : Z : W)) &\mapsto (x, (X - xW^2 : Z : W)).\end{aligned}$$

Restricting this morphism to $\mathcal{D}$ gives us an embedding of $\mathcal{D}$ in $A \times \mathcal{Q}$ such that on each fiber the points in the set $B_x$ are mapped to the points $Q_i$. Let $\mathcal{D}'$ denote the image of $\mathcal{D}$ in $A \times \mathcal{Q}$ under the isomorphism

$$(\mathrm{id}_A \times \psi_1) \circ \Psi_2 \colon A \times \mathbb{P}_k(2,1,1) \to A \times \mathcal{Q}.$$

This isomorphism maps the closed subscheme $B$ to the subscheme $B' \subset A \times \mathcal{Q}$ defined by $p = 0$ and $f(r/s)s^3 = f(t_0)s^3$. The points of this scheme correspond exactly to the Galois orbits of the set

$$\left\{(x, Q_i) : x \in A(\overline{k}) \text{ and } i = 0, 1, 2\right\}.$$

Recall the blowup and down construction given in diagram (3.15). By [Stacks, Tag 0805] blowups commute with flat base change. This gives us that the morphism $A \times \tilde{\mathcal{S}} \to A \times \mathcal{Q}$ is exactly the blowup of $\mathcal{Q}$ along $B' \cup (A \times \{O\})$. Composing with the isomorphism $(\mathrm{id}_A \times \psi_1) \circ \Psi_2$, we get a blowup construction

$$\begin{tikzcd}
A \times \tilde{\mathcal{S}} \arrow[d, "\mathrm{id}_A \times \pi_2"] & & \\
A \times \tilde{\mathcal{Q}} \arrow[d, "\mathrm{id}_A \times \pi_1"] & & \\
A \times \mathcal{Q} \arrow[r, "\sim", "\mathrm{id}_A \times \psi_1^{-1}"'] & A \times \mathbb{P}_k(2,1,1) \arrow[r, "\sim", "\Psi_2^{-1}"'] & A \times \mathbb{P}_k(2,1,1)
\end{tikzcd} \tag{3.16}$$

of $A \times \mathbb{P}_k(2,1,1)$ along some closed subscheme.

The surface $\mathcal{D}'$ does not meet the subscheme $A \times \{O\}$. Therefore, the strict transform of $\mathcal{D}'$ under this blowup, gives a blowup

$$\Psi_2^{-1} \circ \mathrm{id}_A \times (\psi_1^{-1} \circ \pi_1 \circ \pi_2)|_{\tilde{\mathcal{D}}} \colon \tilde{\mathcal{D}} \to \mathcal{D}$$

of $\mathcal{D}$ along $B$.

Again by the univerisal property of the product, we have a map

$$\mathrm{id}_A \times \pi_3 \colon A \times \tilde{\mathcal{S}} \to A \times \mathcal{S}.$$

The restriction of this map to $\tilde{\mathcal{D}}'$ is an isomorphism on each fiber above $A$ by Proposition 3.28, and hence is itself an isomorphism. Moreover, from the result of Lemma 3.27 it follows that $A \times \mathcal{S} \cong A \times \mathbb{P}^2_k$. Hence, this gives us a model of $\tilde{\mathcal{D}}$ in $A \times \mathbb{P}^2_k$.

For each $x \in A(l)$, the fiber of the projection morphism $\tilde{\mathcal{D}} \to A$ gives us by construction the curve $\tilde{D}_x$ of Definition 3.24. □

### 3.2.4 Construction of the surface $\mathscr{D}$

Embed $A$ into $\mathbb{P}^1_k$ by $x \mapsto (x : 1)$. This induces an embedding of $A \times \mathbb{P}^2_k$ in $\mathbb{P}^1_k \times \mathbb{P}^2_k$. The proof of Proposition 3.29 gives us an explicit description of $\tilde{\mathcal{D}}$ in $A \times \mathbb{P}^2_k$. Hence, this gives us an embedding of $\tilde{\mathcal{D}}$ in $\mathbb{P}^1_k \times \mathbb{P}^2_k$.

**Definition 3.30.** The surface $\mathscr{D}$ is defined as the closure of the image of $\tilde{\mathcal{D}}$ in $\mathbb{P}^1_k \times \mathbb{P}^2_k$ as described above.

Note that $\mathscr{D} \to \mathbb{P}^1_k$ is a fibration. Moreover, for each $x \in A(l) \subset \mathbb{P}^1_k(l)$, we get a fiber of $\tilde{\mathcal{D}}$ which is given by the curve of Definition 3.24. In particular, if $\tilde{x}$ denotes the generic point of $\mathbb{P}^1_k$, then the generic fiber of this fibration is the curve $\tilde{D}_{\tilde{x}}$. This curve can be identified as the generic fiber of $\tilde{\mathcal{D}}$ as well.

Diagram 3.17 gives us an overview of all the surfaces and morphisms we have encountered.

$$\begin{tikzcd}
\mathcal{C} \arrow[d, hook] \arrow[dr, "\sim"] \arrow[ddr] \arrow[drr, bend left] & & \tilde{\mathcal{D}} \arrow[r, hook] \arrow[d] & \mathscr{D} \arrow[r, hook] \arrow[ddr] & \mathbb{P}^1_k \times \mathbb{P}^2_k \arrow[dd] \\
\mathcal{F}_0 \times S \arrow[d] \arrow[dr] & \mathcal{F}_0 \times_A \mathcal{D} \arrow[r] \arrow[d] & \mathcal{D} \arrow[r, hook] \arrow[d] & A \times \mathbb{P}_k(2,1,1) \arrow[dl] & \\
S & \mathcal{F}_0 \arrow[r] & A \arrow[rr, hook] & & \mathbb{P}^1_k
\end{tikzcd} \tag{3.17}$$

**Additional assumption:** From now on, we assume that char $k \neq 2, 3$!

In the remainder of this subsection we will give an explicit model for this surface $\mathscr{D}$ in the case that characteristic of $k$ is not two or three. The construction of this model will follow from the embedding of the proof of Proposition 3.29 and uses the isomorphism $\mathbb{P}_k(2,1,1) \cong \mathcal{Q}$. Explicit calculations will be done with MAGMA. The code can be find in the Appendix.

Recall that the del Pezzo surface $S$ is given by equation (3.1). Because we assumed that char $k \neq 2, 3$, by completing the square and competing the cube, we may assume that $a_1, a_2, a_3 = 0$, as noted in section 2.2.1. Moreover, we can choose these transformations such that $t_0$ and the polynomial $f$ will not change.

We can also apply the linear transformation $\mathbb{P}_k(2,3,1,1) \to \mathbb{P}_k(2,3,1,1)$ given by

$$(X : Y : Z : W) \mapsto (X : Y : Z - t_0 W : W).$$

This gives another embedding of our del Pezzo surface $S$ in the weighted projective space which sends the curve $\mathcal{F}$, which we have defined at the beginning of this section, to the curve defined by the equation $Z = 0$.

We recall some notation from section 3.1. Identify $\mathbb{A}^3_k$ with the standard affine of the weighted projective space $\mathbb{P}_k(2,3,1,1)$ where $W \neq 0$ and denote its coordinates by $x = \dfrac{X}{W^2}$, $y = \dfrac{Y}{W^3}$ and $t = \dfrac{Z}{W}$. The equation of the surface $S$

under the above isomorphism can then be given on this affine $\mathbb{A}^3_k$ by the solution set of

$$y^2 = x^3 + (b_1 f'(t) + b_0)x + (c_2 f'(t)^2 + c_1 f'(t) + c_0), \tag{3.18}$$

where $f' = f'_3 t^3 + f'_2 t^2 + f'_1 t + f'_0$ and $b_1, b_0, c_2, c_1, c_0, f'_3, f'_2, f'_1, f'_0 \in k$. The relation of the $f$ we started with and $f'$ is given by $f'(t) = f(t + t_0)$. The curve $\mathcal{F}$ is under this above isomorphism given on $\mathbb{A}^3_k$ by the equation $t = 0$. Write $f' - f'_0 = f'_3 t(t - t'_1)(t - t'_2)$, where $t'_1, t'_2 \in \overline{k}$ denote the other zeroes of the polynomial $f' - f'_0$.

Recall that $\eta$ denotes the generic point of $\mathcal{F}$ and that $C_\eta$ denotes the curve defined as in Definition 3.2. Also recall that $\tilde{x}$ denotes the generic point of $A$ and that $D_{\tilde{x}}$ denotes the curve of Definition 3.20. By Lemma 3.4 the curve $C_\eta$ gets mapped isomorphically to its image under $\tau_{\kappa(\eta)} \colon S_{\kappa(\eta)} \to \mathbb{P}_{\kappa(\eta)}(2,1,1)$. Moreover, by definition of $D_{\tilde{x}}$ we get an equality $\tau_{\kappa(\eta)}(C_\eta) = (D_{\tilde{x}})_{\kappa(\eta)}$.

Recall that the generic point $\eta$ can be identified with a $\kappa(\eta)$-rational point on $\mathcal{F}$. This point is explicitly given by $(\tilde{x}, \tilde{y}, 0) \in (S \cap \mathbb{A}^3_k)(\kappa(\eta))$. The curve $C_\eta$ is then defined in $\mathbb{P}_{\kappa(\eta)}(2,3,1,1)$ by equation (3.18) and the zero set of the polynomial given in equation (3.5), where we substitute $x_R = \tilde{x}$, $y_R = \tilde{y}$ and $t_R = 0$. The projection onto $\mathbb{P}_{\kappa(\eta)}(2,1,1)$ now gives us an explicit equation of $D_{\tilde{x}}$, which is given explicitly by the equation of Lemma 3.4.

If we now restrict the automorphism of $\mathbb{P}_{k(\tilde{x})}(2,1,1)$ given by

$$(X : Z : W) \mapsto (X - \tilde{x}W^2 : Z : W)$$

to the curve $D_{\tilde{x}}$, the three singular points of $D_{\tilde{x}}$ get mapped to the points $(0 : 0 : 1)$, $(0 : t'_1 : 1)$ and $(0 : t'_2 : 1)$. Let us denote by $D'$ the image of $D_{\tilde{x}}$ under the above isomorphism intersected with the affine where the last coordinate is not zero. Giving this affine $\mathbb{A}^2_{k(\tilde{x})}$ coordinates $x$ and $t$, we can find with MAGMA that $D'$ is given by the solution set of the polynomial given in (3.19), which can be found on the next page.

On this affine, we can give an explicit construction of the blowup of $D'$ in the points corresponding to $(0 : 0 : 1)$, $(0 : t_1 : 1)$ and $(0 : t_2 : 1)$. It is given by the strict transform $\tilde{D}'$ under the blowup

$$\{((x,t),(u:v)) \in \mathbb{A}^2_{k(\tilde{x})} \times \mathbb{P}^1_{k(x)} : xu = (f'(t) - f'_0)v\} \to \mathbb{A}^2_{k(\tilde{x})}.$$

Observe that there is an embedding

$$\tilde{D}'|_{v \neq 0} \to \mathbb{A}^2_{k(\tilde{x})};\ ((x,t),(u:v)) \mapsto \left(\frac{u}{v}, t\right).$$

*Defining polynomial of $D'$:*

$$
\begin{multline}
\left(\tilde{x}^3 + \left(b_0 + b_1 f_0'\right)\tilde{x} + \left(c_0 + c_1 f_0' + c_2 {f_0'}^2\right)\right) x^3 \\
+ \bigg(\frac{3}{4}\tilde{x}^4 + \left(3/2 b_0 + 3/2 b_1 f_0'\right)\tilde{x}^2 + \\
\left(3c_0 + 3c_1 f_0' + 3c_2 {f_0'}^2\right)\tilde{x} - \frac{1}{4}b_0^2 - \frac{1}{2}b_0 b_1 f_0' - \frac{1}{4}b_1^2 {f_0'}^2\bigg) x^2 \\
+ \bigg(-\frac{1}{2}b_1 f_3'\tilde{x}^3 + \left(-3/2 c_1 f_3' - 3c_2 f_3' f_0'\right)\tilde{x}^2 + \left(\frac{1}{2}b_0 b_1 f_3' + \frac{1}{2}b_1^2 f_3' f_0'\right)\tilde{x} \\
+ \left(c_0 b_1 f_3' - \frac{1}{2}c_1 b_0 f_3' + \frac{1}{2}c_1 b_1 f_3' f_0' - c_2 b_0 f_3' f_0'\right)\bigg) x t^3 \\
+ \bigg(-\frac{1}{2}b_1 f_2'\tilde{x}^3 + \left(-3/2 c_1 f_2' - 3c_2 f_2' f_0'\right)\tilde{x}^2 + \left(\frac{1}{2}b_0 b_1 f_2' + \frac{1}{2}b_1^2 f_2' f_0'\right)\tilde{x} \\
+ \left(c_0 b_1 f_2' - \frac{1}{2}c_1 b_0 f_2' + \frac{1}{2}c_1 b_1 f_2' f_0' - c_2 b_0 f_2' f_0'\right)\bigg) x t^2 \\
+ \bigg(-\frac{1}{2}b_1 f_1'\tilde{x}^3 + \left(-3/2 c_1 f_1' - 3c_2 f_1' f_0'\right)\tilde{x}^2 + \left(\frac{1}{2}b_0 b_1 f_1' + \frac{1}{2}b_1^2 f_1' f_0'\right)\tilde{x} \\
+ \left(c_0 b_1 f_1' - \frac{1}{2}c_1 b_0 f_1' + \frac{1}{2}c_1 b_1 f_1' f_0' - c_2 b_0 f_1' f_0'\right)\bigg) x t \\
+ \left(c_2 {f_3'}^2\tilde{x}^3 - \frac{1}{4}b_1^2 {f_3'}^2\tilde{x}^2 + \left(-\frac{1}{2}c_1 b_1 {f_3'}^2 + c_2 b_0 {f_3'}^2\right)\tilde{x} + c_0 c_2 {f_3'}^2 - \frac{1}{4}c_1^2 {f_3'}^2\right) t^6 \\
+ \left(2c_2 f_3' f_2'\tilde{x}^3 - \frac{1}{2}b_1^2 f_3' f_2'\tilde{x}^2 + \left(-c_1 b_1 f_3' f_2' + 2c_2 b_0 f_3' f_2'\right)\tilde{x} + 2c_0 c_2 f_3' f_2' - \frac{1}{2}c_1^2 f_3' f_2'\right) t^5 \\
+ \bigg(\left(2c_2 f_3' f_1' + c_2 {f_2'}^2\right)\tilde{x}^3 + \left(-\frac{1}{2}b_1^2 f_3' f_1' - \frac{1}{4}b_1^2 {f_2'}^2\right)\tilde{x}^2 \\
+ \left(-c_1 b_1 f_3' f_1' - \frac{1}{2}c_1 b_1 {f_2'}^2 + 2c_2 b_0 f_3' f_1' + c_2 b_0 {f_2'}^2\right)\tilde{x} \\
+ \left(2c_0 c_2 f_3' f_1' + c_0 c_2 {f_2'}^2 - \frac{1}{2}c_1^2 f_3' f_1' - \frac{1}{4}c_1^2 {f_2'}^2\right)\bigg) t^4 \\
+ \left(2c_2 f_2' f_1'\tilde{x}^3 - \frac{1}{2}b_1^2 f_2' f_1'\tilde{x}^2 + \left(-c_1 b_1 f_2' f_1' + 2c_2 b_0 f_2' f_1'\right)\tilde{x} + 2c_0 c_2 f_2' f_1' - \frac{1}{2}c_1^2 f_2' f_1'\right) t^3 \\
+ \left(c_2 {f_1'}^2\tilde{x}^3 - \frac{1}{4}b_1^2 {f_1'}^2\tilde{x}^2 + \left(-\frac{1}{2}c_1 b_1 {f_1'}^2 + c_2 b_0 {f_1'}^2\right)\tilde{x} + c_0 c_2 {f_1'}^2 - \frac{1}{4}c_1^2 {f_1'}^2\right) t^2
\tag{3.19}
\end{multline}
$$

Let us give this affine plane $\mathbb{A}^2_{k(\tilde{x})}$ coordinates $v', t'$. Again with MAGMA, we deduce that the image in $\mathbb{A}^2_{k(\tilde{x})}$ is given by the zero set of the polynomial given in (3.20).

$$\begin{aligned}
&\left(\tilde{x}^3 + (b_0 + b_1 f_0')\tilde{x} + c_0 + c_1 f_0' + c_2 {f_0'}^2\right)\left(f_3' {t'}^3 + f_2' {t'}^2 + f_1' t'\right) \\
&\quad + \left(c_2 \tilde{x}^3 - \frac{1}{4} b_1^2 \tilde{x}^2 + \left(-\frac{1}{2} c_1 b_1 + c_2 b_0\right)\tilde{x} + c_0 c_2 - \frac{1}{4} c_1^2\right) {v'}^3 \\
&+ \left(-\frac{1}{2} b_1 \tilde{x}^3 - \left(\frac{3}{2} c_1 + 3 c_2 f_0'\right)\tilde{x}^2 + \frac{1}{2} b_1 (b_0 + b_1 f_0')\tilde{x}\right. \\
&\qquad\qquad \left. + c_0 b_1 - \frac{1}{2} c_1 b_0 + \frac{1}{2} c_1 b_1 f_0' - c_2 b_0 f_0'\right) {v'}^2 \\
&+ \left(\frac{3}{4}\tilde{x}^4 + \frac{3}{2}(b_0 + b_1 f_0')\tilde{x}^2 + 3(c_0 + c_1 f_0' + c_2 {f_0'}^2)\tilde{x}\right. \\
&\qquad\qquad \left. - \frac{1}{4} b_0^2 - \frac{1}{2} b_0 b_1 f_0' - \frac{1}{4} b_1^2 {f_0'}^2\right) v'.
\end{aligned} \tag{3.20}$$

An explicit equation for the surface $\mathscr{D}$ in $\mathbb{P}^1_k \times \mathbb{P}^2_k$ can now be constructed as follows. It is given by viewing the polynomial in (3.20) as a polynomial in $k[\tilde{x}][v', t']$ and homogenizing it both with respect to $\tilde{x}$ as with respect to $v', t'$. Let us give $\mathbb{P}^1_k \times \mathbb{P}^2_k$ coordinates $S_0, S_1$ and $T_0, T_1, T_2$. Identify $\tilde{x} = \frac{S_0}{S_1}$, $v' = \frac{T_0}{T_2}$ and $t' = \frac{T_1}{T_2}$. The surface $\mathscr{D}$ is given by the solution set of (3.21).

$$\begin{aligned}
&\left(S_0^3 S_1 + (b_0 + b_1 f_0') S_0 S_1^3 + (c_0 + c_1 f_0' + c_2 {f_0'}^2) S_1^4\right)\left(f(T_1/T_2) - f_0\right) {T_2}^3 \\
&+ \left(c_2 S_0^3 S_1 - \frac{1}{4} b_1^2 S_0^2 S_1^2 + \left(-\frac{1}{2} c_1 b_1 + c_2 b_0\right) S_0 S_1^3 + \left(c_0 c_2 - \frac{1}{4} c_1^2\right) S_1^4\right) {T_0}^3 \\
&+ \left(-\frac{1}{2} b_1 S_0^3 S_1 - \left(\frac{3}{2} c_1 + 3 c_2 f_0'\right) S_0^2 S_1^2 + \frac{1}{2} b_1 (b_0 + b_1 f_0') S_0 S_1^3\right. \\
&\qquad\qquad \left. + \left(c_0 b_1 - \frac{1}{2} c_1 b_0 + \frac{1}{2} c_1 b_1 f_0' - c_2 b_0 f_0'\right) S_1^4\right) T_0^2 T_2 \\
&+ \left(\frac{3}{4} S_0^4 + \frac{3}{2}(b_0 + b_1 f_0') S_0^2 S_1^2 + 3(c_0 + c_1 f_0' + c_2 {f_0'}^2) S_0 S_1^3\right. \\
&\qquad\qquad \left. - \left(\frac{1}{4} b_0^2 + \frac{1}{2} b_0 b_1 f_0' + \frac{1}{4} b_1^2 {f_0'}^2\right) S_1^4\right) T_0 T_2^2.
\end{aligned} \tag{3.21}$$

We deduce that the surface $\mathscr{D}$ is a family of cubic curves in $\mathbb{P}^2_k$.

## 3.3 Zariski density

In this section we will give a proof of Theorem 3.1, showing that under mild conditions the $k$-rational points of the surface $S$ lie Zariski dense in $S$. To prove the Zariski density of the $k$-rational points in $S$, it is enough by Corollary 3.19 to prove that $\mathcal{C}(k)$ is Zariski dense in $\mathcal{C}$, where $\mathcal{C}$ is the fibered surface defined as in Definition 3.17. By Proposition 3.16 the generic fiber of $\mathcal{C} \to \mathcal{F}_0$, has geometric genus zero or one. Before we go to the proof of Theorem 3.1, we first take a closer look at the case that the geometric genus of $C_\eta$ equals one.

**Assumptions of this section:**

We assume that we are in the same setting as sketched in the beginning of section 3.2. Moreover, we assume that the characteristic of $k$ is zero. In this case $\mathcal{F}_0$ is certainly non-empty. We define the surfaces $\mathcal{C}$, $\mathcal{D}$, $\tilde{\mathcal{D}}$ and $\mathscr{D}$ as in section 3.2. Recall that these surfaces are all fibered surfaces. Note that $\tilde{\mathcal{D}}$ can be identified as a subset of $\mathscr{D}$. Again denote by $\tilde{x}$ the generic point of $A \subset \mathbb{P}^1_k$ and recall that $\eta$ denotes the generic point of $\mathcal{F}_0$. Also recall that the generic fibers of the morphisms $\mathcal{C} \to \mathcal{F}_0$, $\mathcal{D} \to A$ and $\mathscr{D} \to \mathbb{P}^1_k$ are given by $C_\eta$, $D_{\tilde{x}}$ and $\tilde{D}_{\tilde{x}}$ respectively.

### 3.3.1 A section of infinite order

In this subsection we suppose that $C_\eta$ has geometric genus one. We will show that in this case the surface $\mathscr{D}$ can be given the structure of an elliptic surface which has under a mild condition a section of infinite order. Before we can show this, we first need to construct the zero section and this section of infinite order. Therefore, we take a closer look at some points of $C_\eta$.

Observe that the only singular points of $C_\eta$ are in the genus one case given by the subscheme $(S \cap \mathbb{A}^3_k)_{\varphi(\eta)}$, because the arithmetic genus is four and this set consists of three points. Recall that this subscheme consists of three $\overline{\kappa(\eta)}$-points.

First of all $\eta$ is a $\kappa(\eta)$-point of this subscheme. In the remainder of this section we will denote this point $\eta$ by $\eta_0$. The other two points, which we will denote by $\eta_1$ and $\eta_2$, are either defined over $\kappa(\eta)$ or over some quadratic extension of $\kappa(\eta)$. In the latter case, the points $\eta_1$ and $\eta_2$ are in the same Galois orbit of $C_\eta(\overline{\kappa(\eta)})$ of $\mathrm{Gal}(\kappa(\eta)^s/\kappa(\eta))$.

The points $\eta_0$, $\eta_1$ and $\eta_2$ will all define the same curve $\overline{C_\eta}$ over the algebraic closure. Therefore, Lemma 3.11 gives us three other points $-2\eta_0$, $-2\eta_1$

and $-2\eta_2$ in the group $\mathcal{F}_{\mathrm{ns}}(\overline{\kappa(\eta)})$, which can all be identified as $\overline{\kappa(\eta)}$-points on the curve $C_\eta$. Note that the point $-2\eta_0$ is a $\kappa(\eta)$-point, and the points $-2\eta_1$ and $-2\eta_2$ are defined over the same field as the points $\eta_1$ and $\eta_2$.

Recall that we assumed that the characteristic is zero, so it is unequal to three. This means that $\mathcal{F}_{\mathrm{ns}}(k)$ does not have exponent three. Because $\eta_0$ is a point defined over some transcendental extension of $k$, this means that $\eta_0$ does not have order three. The same argument holds for $\eta_1$ and $\eta_2$ and so in particular we deduce that $\eta_i \neq -2\eta_i$.

By Lemma 3.21, the curve $D_{\tilde{x}}$ has geometric genus one as well. So in this case, the blowup $\tilde{D}_{\tilde{x}}$ is a non-singular curve of genus one. The duplication formula given in III.2.3(d) of [Sil09], gives that the $x$-coordinate of $-2\eta_0$ only depends on $\tilde{x}$ and not on $\tilde{y}$. It follows that the point $-2\eta_0$ maps under the map $C_\eta \to D_{\tilde{x}}$ to a $\kappa(\tilde{x})$-point of $D_{\tilde{x}}$.

The strict transform of this $\kappa(\tilde{x})$-point under $\tilde{D}_{\tilde{x}} \to D_{\tilde{x}}$ gives a $\kappa(\tilde{x})$-point $Q_0$ on $\tilde{D}_{\tilde{x}}$. This means that we can give $\tilde{D}_{\tilde{x}}$ the structure of an elliptic curve. Moreover, we will show in the next lemma that the surface $\mathscr{D}$ is an elliptic surface with a zero section coming from the point $Q_0$.

**Lemma 3.31.** Suppose $C_\eta$ has geometric genus one. Then $\mathscr{D} \to \mathbb{P}^1_k$ is an elliptic surface with zero section $\sigma_0\colon \mathbb{P}^1_k \to \mathscr{D}$ corresponding under the bijection of Lemma 1.30 to the $\kappa(\tilde{x})$-point $Q_0$.

**Proof.** The surface $\mathscr{D}$ is projective, so in particular it is proper over $k$. It follows from [Stacks, Tag 01W6] that the fibration $\mathscr{D} \to \mathbb{P}^1_k$ is a proper morphism. By Lemma 1.30, we get that the point $Q_0$ gives a section $\sigma_0\colon \mathbb{P}^1_k \to \mathscr{D}$ of this fibration. Moreover, almost all fibers of this fibration will be smooth. These fibers are cubic curves by equation (3.21). It follows that almost all fibers of $\mathscr{D} \to \mathbb{P}^1_k$ will be non-singular curves of genus one. Hence, $\mathscr{D} \to \mathbb{P}^1_k$ will be an elliptic surface with section $\sigma_0\colon \mathbb{P}^1_k \to \mathscr{D}$. □

Recall that we have the two other $\overline{\kappa(\eta)}$-points $-2\eta_1$ and $-2\eta_2$ on $C_\eta$. These points give two $\overline{\kappa(\tilde{x})}$-rational points on $D_{\tilde{x}}$. Again by the duplication formula, we can deduce that these points will be either two $\kappa(\tilde{x})$-rational points or they are defined over some quadratic extension and are each others Galois conjugates.

The strict transforms of these $\overline{\kappa(\tilde{x})}$-points under $\tilde{D}_{\tilde{x}} \to D_{\tilde{x}}$ give two points $Q_1$ and $Q_2$. Moreover, because $(\tilde{D}_{\tilde{x}}, Q_0)$ is an elliptic curve, we get a new $\overline{\kappa(\tilde{x})}$-point $Q$ given by the addition $Q := Q_1 + Q_2$. This point is stable under Galois, and hence is a $\kappa(\tilde{x})$-point on $\widetilde{D}_{\tilde{x}}$. By Lemma 1.30, this point $Q$ will correspond to a section $\sigma\colon \mathbb{P}^1_k \to \mathscr{D}$ of $\mathscr{D} \to \mathbb{P}^1_k$.

Now let $K$ denote the residue field of $Q_1$. Then the points $Q_1$ and $Q_2$ will define two sections $\sigma_1, \sigma_2\colon \mathbb{P}^1_K \to \mathscr{D}_K$ of the base change $\mathscr{D}_K \to \mathbb{P}^1_K$. Moreover, the base change $\sigma_K\colon \mathbb{P}^1_K \to \mathscr{D}_K$ is then given by the sum $\sigma_K = \sigma_1 + \sigma_2$.

Our next goal is to show that under mild conditions, this section $\sigma$ will be of infinite order. To show this, we will use the model of $\mathscr{D}$ described in section 3.2.4. Recall that for this model we assumed that $a_1 = a_2 = a_3 = 0$. Following the construction of 3.2.4, we can find explicit descriptions for the sections $\sigma_i$ for $i = 0, 1, 2$.

**Lemma 3.32.** Recall that the surface $\mathscr{D} \to \mathbb{P}^1_k$ can be given by the zero set of (3.21). The section $\sigma_0\colon \mathbb{P}^1_k \to \mathscr{D}$ and the sections $\sigma_1, \sigma_2\colon \mathbb{P}^1_K \to \mathscr{D}_K$ are given by the description

$$(S_0 : S_1) \mapsto ((S_0 : S_1), (0 : t_i : 1)).$$

Moreover, for each smooth fiber we have that

$$\sigma(S_0 : S_1) = \sigma_1(S_0 : S_1) + \sigma_2(S_0 : S_1) = ((S_0 : S_1), (0 : t_1' : 1) + (0 : t_2' : 1)),$$

where the + denotes the pointwise addition on this fiber.

**Proof.** We shortly recall the setting of 3.2.4. Because the characteristic is not two or three, we assumed that $a_1 = a_2 = a_3 = 0$. Moreover, we have applied the linear transformation $(X : Y : Z : W) \mapsto (X : Y : Z - t_0 W : W)$ on $\mathbb{P}_k(2,3,1,1)$. This embeds $S$ in $\mathbb{P}_k(2,3,1,1)$ such that the curve defined by $Z = t_0 W$ is mapped to the curve $Z = 0$. Recall that we defined the polynomial $f'$ which is given by $f'(t) = f(t + t_0)$. We denote the three zeroes of $f'(t)$ by $t_i'$, where $t_0' = 0$.

Now the map $S \to \mathbb{P}_{k(\tilde{x})}(2,1,1)$ given by $(X : Y : Z : W) \mapsto (X : Z : W)$ sends the point $-2\eta_i$ to the point given by $(h(\tilde{x}) : t_i' : 1)$ where $h$ denotes the duplication formula give in III.2.3(d) of [Sil09]. Recall from section 3.2.4 that the blowup on the affine part of $\mathbb{P}_{k(\tilde{x})}(2,1,1)$ where the last coordinate is non-zero is given by

$$\{((x,t),(u:v)) \in \mathbb{A}^2_{k(\tilde{x})} \times \mathbb{P}^1_{k(x)} : xu = (f'(t) - f_0')v\} \to \mathbb{A}^2_{k(\tilde{x})}.$$

The strict transform of the point $(h(\tilde{x}) : t_i' : 1) \in \mathbb{P}_{k(\tilde{x})}(2,1,1)$ is given by $((h(\tilde{x}), t_i'), (0 : 1))$. We have embedded the strict transform of the curve $D_{\tilde{x}}$ in $\mathbb{P}_k^2$ by the map $((x, t), (u : v)) \mapsto (u : tv : v)$. The point $((h(\tilde{x}), t_i'), (0 : 1))$ gets mapped under this embedding to the point $(0 : t_i' : 1)$.

From the above construction, we deduce that the sections $\sigma_i$ for $i = 0, 1, 2$ are on each point defined by

$$\sigma_i(S_0 : S_1) = ((S_0 : S_1), (0 : t_i' : 1)).$$

The result for $\sigma$ follows by definition. □

With these explicit descriptions, we can show that $\sigma$ is a section of infinite order. We will show this on a singular fiber. Observe that the result of Lemma 3.32 only says something on the smooth fibers. Recall from section 1.3 that in the case of a minimal regular model this extends to the singular fibers. We will use this result in the next lemma.

**Lemma 3.33.** Suppose that $-f_2 \neq 3f_3t_0$ and that $C_\eta$ has geometric genus one. Then the section $\sigma\colon \mathbb{P}_k^1 \to \mathscr{D}$ has infinite order in $\mathscr{D}(\mathbb{P}_k^1)$.

**Proof.** Again we use that $\mathscr{D}$ can be given in $\mathbb{P}_k^1 \times \mathbb{P}_k^2$ by the zero set of (3.21). Write $f' = f_3't^3 + f_2't^2 + f_1't + f_0'$. Observe that $f_2' = f_2 + 3f_3t_0$. This gives us that the assumption $-f_2 \neq 3f_3t_0$ implies that $f_2' \neq 0$.

We will check that $\sigma$ has infinite order, by checking it on the fiber above $(1 : 0)$ of $\mathscr{D} \to \mathbb{P}_k^1$. Setting $S_0 = 1$ and $S_1 = 0$ in equation (3.21) gives us that this fiber is given by the equation $\frac{3}{4}T_0T_2^2 = 0$. Hence, we get a double line given by $T_2 = 0$ and another line given by $T_0 = 0$. We will denote the latter line by $L$.

By Theorem 1.34 the surface $\mathscr{D}$ has a minimal regular model. A calculation with MAGMA gives us that on this double line $T_2 = 0$, there will be three singular points of the surface. Blowing up these points will give us locally a regular model for the elliptic surface $\mathscr{D}$ such that this fiber is of type $\mathrm{I}_0^*$.

The above observation implies that this fiber has additive reduction. Moreover, the line $L$ will correspond to the zero component on this model, because the point $\sigma_0(1 : 0) = ((1 : 0), (0 : 0 : 1))$ denotes the zero of the additive group on the smooth part of this fiber. In particular, the group structure on this line is already given on this singular surface $\mathscr{D}$.

This line $L$ contains one singular point of the fiber above $(1 : 0)$ of the morphism $\mathscr{D} \to \mathbb{P}_k^1$, namely $P := ((1 : 0), (0 : 1 : 0))$. We deduce that there is a group isomorphism $(L - \{P\})(K) \xrightarrow{\sim} K$ given by $((1:0),(0:t':1)) \mapsto t'$. It follows that we can identify the addition on $L - \{P\}$ with the normal addition in the second component of the second coordinate.

Now observe that the other points $\sigma_1(1:0)$ and $\sigma_2(1:0)$ are also on the line $L$. It follows that

$$\sigma(1:0) = (\sigma_1 + \sigma_2)(1:0) = ((1:0),(0:t_1+t_2:1)) = ((1:0),(0:-f_2':1)).$$

Recall that by assumption we have $f_2' \neq 0$. It follows that for each integer $m$ we have $m \cdot \sigma(1:0) = ((1:0),(0:-mf_2':1))$. Because $k$ has characteristic zero, we find that $-mf_2' \neq 0$. We conclude that this section $\sigma$ has infinite order on $\mathscr{D}$ if $-f_2 \neq 3f_3t_0$. □

### 3.3.2 Proof of Theorem 3.1

Recall that $\mathcal{C}$ is the fibered surface defined as in section 3.2.1 and that $C_\eta$ is the generic fiber of the fibration of $\mathcal{C} \to \mathcal{F}_0$. We will first give two last partial results on the surface $\mathcal{C}$. These two results show that $\mathcal{C} \to \mathcal{F}_0$ has infinitely many sections. These sections will be used to show the Zariski density of the $k$-rational points of $\mathcal{C}$.

**Lemma 3.34.** Suppose that $C_\eta$ has geometric genus zero. Then there are infinitely many distinct sections of $\mathcal{C} \to \mathcal{F}_0$.

**Proof.** Suppose that $C_\eta$ has geometric genus zero. In this case, we have that the normalization $\tilde{C}_\eta$ is isomorphic to $\overline{\mathbb{P}^1_{\kappa(\eta)}}$ over the algebraic closure of $\kappa(\eta)$. The point $-2\eta$ of Lemma 3.11 is a $\kappa(\eta)$-rational point of $C_\eta$. Recall that this point is not equal to $\eta$ by our assumption that $\mathrm{char}(k) = 0 \neq 3$.

Moreover, this point $-2\eta$ has intersection multiplicity 1 on $\mathcal{F} \cap C_\eta$, which means that $-2\eta$ is a smooth point of $C_\eta$. Hence, its strict transform on the normalization $\tilde{C}_\eta$ gives a $\kappa(\eta)$-rational point. It now follows from Proposition 1.13 that $\tilde{C}_\eta$ is isomorphic to $\mathbb{P}^1_{\kappa(\eta)}$. We deduce that $|C_\eta(\kappa(\eta))| = \infty$.

The surface $\mathcal{C}$ can be identified as a closed subvariety in $\mathbb{P}^6_{\mathcal{F}_0}$ in the following way. We have an inclusion of $S \hookrightarrow \mathbb{P}^6_k$ which is the map determined by the linear system $|-3K_S|$. This induces an embedding

$$\mathcal{C} \subset \mathcal{F}_0 \times S \hookrightarrow \mathcal{F}_0 \times \mathbb{P}^6_k \cong \mathbb{P}^6_{\mathcal{F}_0}.$$

Hence, $\mathcal{C} \to \mathcal{F}_0$ is a projective morphism.

From the above observation, it follows that the morphism $\mathcal{C} \to \mathcal{F}_0$ is proper. Recall that $\mathcal{F}_0$ is a smooth curve. By Lemma 1.30, every point in $C_\eta(\kappa(\eta))$ gives a section $\mathcal{F}_0 \to \mathcal{C}$. Hence, we have found infinitely many sections $\mathcal{F}_0 \to \mathcal{C}$. □

**Proposition 3.35.** Suppose that $-f_2 \neq 3f_3t_0$ and that $C_\eta$ has geometric genus one. Then there are infinitely many distinct sections of $\mathcal{C} \to \mathcal{F}_0$.

**Proof.** Suppose that $C_\eta$ has geometric genus one. We can apply Lemma 3.33. This tells us that each multiple of the section $\sigma\colon \mathbb{P}^1_k \to \mathscr{D}$ will give us a different section on $\mathscr{D}$ which is defined over $k$. If we restrict these sections to $A$ then the image is in $\tilde{\mathcal{D}}$. Now composing them with the map $\tilde{\mathcal{D}} \to \mathcal{D}$, we get infinitely many different sections $A \to \mathcal{D}$. Under the universal property, all these sections pull back to sections $\mathcal{F}_0 \to \mathcal{C}$ via the pullback diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathcal{F}_0 \arrow[r] \arrow[u, dotted, bend left] & A \arrow[u, bend right]
\end{tikzcd}$$

These are all different sections of $\mathcal{C} \to \mathcal{F}_0$, because $\mathcal{F}_0 \to A$ is surjective. □

Now we are ready to tie everything together and give a proof of the main theorem.

**Proof of Theorem 3.1.** Suppose that we are in the setting of the theorem. Define the surface $\mathcal{C}$ as in section 3.2.1. By Lemma 3.15 we have two cases which we should distinguish, namely when the generic fiber $C_\eta$ of $\mathcal{C} \to \mathcal{F}_0$ has genus 0 and when $C_\eta$ has genus 1. By Lemma 3.34 and Proposition 3.35 we have in both cases that there are infinitely many sections $\mathcal{F}_0 \to \mathcal{C}$.

Because we assumed that $|\mathcal{F}(k)| = \infty$, and $\mathcal{F}_0$ is an open subset of $\mathcal{F}$, it follows that $|\mathcal{F}_0(k)| = \infty$. Hence, every section will contain infinitely many $k$-points on $\mathcal{C}$. Because the image of a section is defined by some curve over $k$ on $\mathcal{C}$, we conclude that in both cases $\mathcal{C}(k)$ lies Zariski dense in $\mathcal{C}$. By Corollary 3.19 we deduce that $S(k)$ lies Zariski dense in $S$.

Conversely, suppose that every curve $\mathcal{F}$ on $S$ of the form of the theorem does not contain infinitely many $k$-rational points. Recall from Proposition 2.11 that we can obtain an elliptic surface $\mathcal{E} \to \mathbb{P}^1_k$ by blowing up the base point of the linear system $|-K_S|$. By our assumption $\mathcal{E}(k)$ would be contained in the union of the following set:

(i) the fiber above $(1:0)$ of $\mathcal{E} \to \mathbb{P}^1_k$;

(ii) the fibers above $(t_1 : 1)$ with $t_1 = -\frac{f_2}{3f_3}$ or where $f - f(t_1)$ is inseparable;

(iii) a finite set of points for each fiber $(t_1 : 1)$ with $t_1 \in k$.

There are at most three curves where $f - f(t_1)$ is inseparable. So we deduce that this set consists of at most five fibers on $\mathcal{E}$. Moreover, all the fibers of (iii) must be smooth fibers of $\mathcal{E} \to \mathbb{P}^1_k$, because else they would contain infinitely many points.

Now suppose that $k$ is finitely generated over $\mathbb{Q}$. Then by a generalization of the theorem of Merel, [Mer96], see for example the appendix of [DW21] or footnote 1 of [CT12], the order of the torsion points of an elliptic curve over $k$ are bounded and this bound only depends on $k$. It follows that the torsion points can be given by some division polynomials which are defined as in Exercise 3.7 of [Sil09]. These polynomials define a finite set of curves on $\mathcal{E}$.

We deduce that the set $\mathcal{E}(k)$ of $k$-points of $\mathcal{E}$ is contained in a finite union of curves defined by the division polynomials and the fibers of (i) and (ii). It follows that $\mathcal{E}(k)$ does not lie Zariski dense in $\mathcal{E}$. Because $S$ is birational to $\mathcal{E}$, this implies that $S(k)$ does not lie Zariski dense in $S$. □

### 3.3.3 Application of Theorem 3.1: some examples

We conclude this thesis by giving some concrete examples where we apply this theorem. First of all, we want to mention that all three examples that are discussed in section 5 of [DW21], Theorem 3.1 can be applied to. These are the special cases where $f = t^3$ and $a_i = 0$ for all $i \leq 4$. Here we give some other examples, which cannot be proved by the theorem of [DW21].

In the first example we give a del Pezzo surface of the form of equation (3.1), with $f = t^3$ and $a_4 \neq 0$.

**Example 3.36.** Let $S$ be the surface in $\mathbb{P}_{\mathbb{Q}}(2, 3, 1, 1)$ given by the equation

$$Y^2 = X^3 + (Z^3W - W^4)X + Z^6 + W^6.$$

We can check that this surface is non-singular, which means that it is a del Pezzo surface of degree one. This surface can be given on the affine where $W \neq 0$ by the equation $y^2 = x^3 + (f - 1)x + f^2 - 1$ where $f = t^3$.

Let $\mathcal{F}$ be the curve defined by $t = 1$. We have $-f_2 = 0 \neq 3 = 3f_3t_0$ and $t^3 - 1$ is separable. Moreover, $\mathcal{F}$ is isomorphic to the singular curve with Weierstrass equation $y^2 = x^3$, hence $|\mathcal{F}(\mathbb{Q})| = \infty$. We conclude that the curve $\mathcal{F}$ satisfies the properties of Theorem 3.1, and it follows from the theorem that $S(\mathbb{Q})$ is dense in $S$.

In the next example, we give a family of del Pezzo surfaces for which the assumption of the theorem is satisfied.

**Example 3.37.** Let $S$ be the surface in $\mathbb{P}_{\mathbb{Q}}(2,3,1,1)$ given on the affine $\mathbb{A}^3_{\mathbb{Q}}$ by the equation $y^2 = x^3 + afx + bf^2 + cf + 2$ with $f = t^3 - 2t + 1$ and $a, b, c \in \mathbb{Q}$ such that $S$ is non-singular. Define the curve $\mathcal{F}$ on $S$ by $t = 1$. This curve is isomorphic to the curve given by the Weierstrass equation $y^2 = x^3 + 2$. This is an elliptic curve, because the determinant is non-zero.

We can check for example with Magma that the point $(-1, 1)$ is a point on this curve with infinite order in $\mathcal{F}(\mathbb{Q})$, hence $|\mathcal{F}(\mathbb{Q})| = \infty$. Observe that $t^3 - 2t$ is separable and that $-f_2 = 0 \neq 3 = 3f_3t_0$ holds. We deduce that the curve $\mathcal{F}$ satisfies the properties of the theorem. Again we conclude that $S(\mathbb{Q})$ is dense in $S$.

# Bibliography

[CO99] P. Cragnolini and Paolo A. Oliverio. “Lines on del Pezzo surfaces with $K_S^2 = 1$ in characteristic $\neq 2$”. In: *Communications in Algebra* 27.3 (1999), pp. 1197–1206. DOI: `10.1080/00927879908826489`.

[Coo88] Kevin R. Coombes. “Every rational surface is separably split.” In: *Commentarii mathematici Helvetici* 63.2 (1988), pp. 305–311. URL: `http://eudml.org/doc/140125`.

[CT12] Anna Cadoret and Akio Tamagawa. “Uniform boundedness of $p$-primary torsion of abelian schemes”. In: *Inventiones mathematicae* 188 (Apr. 2012), pp. 83–125. DOI: `10.1007/s00222-011-0343-6`.

[Dol12] Igor V. Dolgachev. *Classical Algebraic Geometry: A Modern View*. Cambridge University Press, 2012.

[DW21] Julie Desjardins and Rosa Winter. *Density of rational points on a family of del Pezzo surfaces of degree one*. 2021. arXiv: `2102.05563 [math.AG]`.

[Har77] Robin C. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer New York, 1977.

[KM16] János Kollár and Massimiliano Mella. *Quadratic families of elliptic curves and unirationality of degree 1 conic bundles*. 2016. arXiv: `1412.3673 [math.AG]`.

[Kol02] János Kollár. “Unirationality of cubic hypersurfaces”. In: *Journal of the Institute of Mathematics of Jussieu* 1.3 (2002), pp. 467–476. DOI: `10.1017/S1474748002000117`.

[Kol99] János Kollar. *Rational Curves on Algebraic Varieties*. Springer Berlin Heidelberg, 1999.

[Man86] Yuri I. Manin. *Cubic Forms*. North-Holland Publishing Co., Amsterdam, 1986.

[Mer96] L. Bornes Merel. “Pour la torsion des courbes elliptiques sur les corps de nombres”. In: *Inventiones mathematicae* 124 (Feb. 1996), pp. 437–449. DOI: `10.1007/s002220050059`.

[Pie12] Marta Pieropan. *On the unirationality of del Pezzo surface over an arbitrary field.* Master thesis, available at `http://www.algant.eu/documents/theses/pieropan.pdf`, 2012.

[Poo17] Bjorn Poonen. *Rational Points on Varieties.* Providence: American Mathematical Society (AMS), 2017.

[Sil09] Joseph H. Silverman. *The Arithmetic of Elliptic Curves.* Graduate Texts in Mathematics. Springer New York, 2009.

[Sil94] Joseph H. Silverman. *Advanced Topics in the Arithmetic of Elliptic Curves.* Springer-Verlag, 1994.

[SL14] Cecilia Salgado and Ronald van Luijk. *Density of rational points on del Pezzo surfaces of degree one.* 2014. arXiv: `1212.2364 [math.AG]`.

[SS10] Matthias Schuett and Tetsuji Shioda. *Elliptic Surfaces.* 2010. arXiv: `0907.0298 [math.AG]`.

[Stacks] The Stacks Project Authors. *Stacks Project.* `https://stacks.math.columbia.edu`.

[STV14] Cecília Salgado, Damiano Testa, and Anthony Várilly-Alvarado. "On the unirationality of del Pezzo surfaces of degree 2". In: *Journal of the London Mathematical Society* 90.1 (Apr. 2014), pp. 121–139. DOI: `10.1112/jlms/jdu014`.

[Ula07] Maciej Ulas. "Rational points on certain elliptic surfaces". In: *Acta Arithmetica* 129.2 (2007), pp. 167–185. DOI: `10.4064/aa129-2-3`.

[Ula09] Maciej Ulas. *Rational points on certain del Pezzo surfaces of degree one.* 2009. arXiv: `0901.2658 [math.NT]`.

[UT10] Maciej Ulas and Alain Togbé. "On the diophantine equation $z^2 = f(x)^2 \pm f(y)^2$". In: *Bulletin of the Australian Mathematical Society* 82 (Oct. 2010). DOI: `10.1017/S0004972710000377`.

[Vak17] Ravi Vakil. *The Rising Sea: Foundations Of Algebraic Geometry.* published at `http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`, version of Nov. 18, 2017.

[Vár11] Anthony Várilly-Alvarado. "Density of rational points on isotrivial rational elliptic surfaces". In: *Algebra & Number Theory* 5.5 (Dec. 2011), pp. 659–690. DOI: `10.2140/ant.2011.5.659`.

# Appendix: Magma code

```
# Calculating a model of the elliptic surface C,
# where C is the surface obtained from the curves CR
# with R points on the fiber t=0.

# Defining field of definition
Q:=Rationals();
k<c0,c1,c2,b0,b1,f3,f2,f1,f0>:=FunctionField(Q,9);
kx<x0>:=FunctionField(k);

# Defining f and functionfield of eta
R<z>:=PolynomialRing(kx);
f:=f3*z^3+f2*z^2+f1*z+f0;
K<y0>:=quo<R|-z^2+x0^3+
    (b0+b1*f0)*x0+c0+f0*c1+f0^2*c2>;

# Defining polynomials for dP-surface S and surface
# S accent
Rs<x,y,t>:=PolynomialRing(K,3);
A3t:=AffineSpace(Rs);
A3u<xp,yp,u>:=AffineSpace(K,3);
G:=-yp^2+xp^3+(b0+b1*u)*xp+c0+c1*u+c2*u^2;
ft:=Evaluate(f,t);
F:=Evaluate(G,[x,y,ft]);

# Surface S accent
T:=Scheme(A3u,G);
# Surface S
S:=Scheme(A3t,F);

```

```
# Defining CR as pullback of tangent space
P:=S![x0,y0,0];
pi:=map<S->A3u | [x,y,Evaluate(ft,[x,y,t])] >;
U:=TangentSpace(T,pi(P));
CR:=Curve(Pullback(pi,U));

# Defining D
Ceq1:=DefiningEquations(CR)[1];
yy:=y-Ceq1/MonomialCoefficient(Ceq1,y);
pD:=Evaluate(F,[x,yy,t]);
Rx<xx,tx>:=PolynomialRing(kx,2);
D:=ClearDenominators(Evaluate(pD,[xx,0,tx]));

# Translating singular points to plane x=0
DD:=Evaluate(D,[xx+x0,tx]);

# Equation for D accent
DD;

# Blowup D accent
RR<tt,vv>:=PolynomialRing(kx,2);
ftt:=Evaluate(f,tt);
BD:=ClearDenominators(RR!(vv^3*Evaluate(DD,
      [(ftt-f0)/vv,tt])/(ftt-f0)^2));

# Equation for the blowup D accent
BD;

# Defining the surface D on three affine patches to
# calculate what happens at the fiber above  s:=1/x0=0
# Here we make s(=1/x0) into variable to obtain surface
# D, where D1 is the standard affine patch, D2 and
# D3 are the affine patches obtained by taking
# coordinates t/v,1/v and 1/t,v/t respectively.

RRR<sss,ttt,vvv>:=PolynomialRing(k,3);
D1:=RRR!(sss^4*(&+[
      Evaluate(MonomialCoefficient(BD,m),
      1/sss)*Evaluate(m,[ttt,vvv]) :
```

```
    m in Monomials(BD)]));
D2:=RRR!(sss^4*vvv^3*(&+[
    Evaluate(MonomialCoefficient(BD,m),
    1/sss)*Evaluate(m,[ttt/vvv,1/vvv]) :
    m in Monomials(BD)]));
D3:=RRR!(sss^4*ttt^3*(&+[
    Evaluate(MonomialCoefficient(BD,m),
    1/sss)*Evaluate(m,[1/ttt,vvv/ttt]) :
    m in Monomials(BD)]));

# Calculating singular points on the special fiber
# at s=1/x0=0.
A3R:=AffineSpace(RRR);
SD1:=Scheme(A3R,D1);
SD2:=Scheme(A3R,D2);
SD3:=Scheme(A3R,D3);
D1S:=SingularSubscheme(SD1);
D2S:=SingularSubscheme(SD2);
D3S:=SingularSubscheme(SD3);
S0:=Scheme(A3R,sss);
Dimension(D1S meet S0);
Dimension(D2S meet S0);
Dimension(D3S meet S0);
Degree(D2S meet S0);
IrreducibleComponents(D2S meet S0);
Degree(D3S meet S0);
IrreducibleComponents(D3S meet S0);
```