Deformation Quantization
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Preface

This document is a bachelor thesis written at Leiden University under the supervision of Dr. S.P. Patil and Dr. B. Mesland. The writing of this thesis serves to fulfil the requirements of graduation for the author of this document. The thesis came into being after the subject of deformation quantization was introduced to the author by Dr. S.P. Patil. The theory of deformation quantization is a mathematically rigorous approach to solve the problem of quantization in physics. Quantization is the process in which a classical system is transitioned into a quantum mechanical system. There have been multiple candidates for solving this problem, the most important ones being deformation quantization and geometric quantization. The theory of geometric quantization has already been excellently presented in for example, the accessible and self-contained textbook "Quantum Theory for Mathematicians" written by Brian C. Hall [Hal13]. An excellent book which I cannot recommended enough. However, more or less self-contained introductions to deformation quantization at the beginning graduate level are still missing from the literature. It is therefore the goal of this thesis to give an accessible introduction into the matter. Since the goal is entirely pedagogical, it will hopefully be of some use for future undergraduate students that aim to bring their own, more original contribution to this field of study.

Roughly speaking, we can say that this text consists out of 3 parts. The first part, comprising chapter 2 and 3, introduces the Moyal-product and motivates it as the outcome of the quest for quantization in the case of Euclidean phase-space. The second part, comprising chapter 4, motivates the theory of symplectic geometry and Poisson manifolds, as the natural mathematical framework of Hamiltonian mechanics. The third and last part, comprising chapter 5, combines the previous two parts in order to motivate the definition of the star-product and studies its properties further. We will in particular give a rough overview of the formality theorem which was proven by Maxim Kontsevich in 1997 [Kos66]. This celebrated result confirms the existence of a star product on any Poisson manifold and consequently gives the deformation approach its raison d'être. In this thesis we will not pursue the problem of convergence of the star-product, since it would require us to delve even deeper into new mathematics, in an already quite substantial introduction.

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1 Introduction

Deformation quantization is a mathematical formalization and generalization of the procedure of quantizing a classical system or classical field theory into, respectively, a quantum mechanical system or quantum field theory. In the second half of the 20th century different mathematical treatments have been employed to solve the problem of quantization. The most important and widespread ones thus far have been geometric quantization and deformation quantization, as already stated in the preface. For geometric quantization the foundations were laid by A.A. Kirillov [Kir62] in the '60s, on which in the same decade B. Kostant [Kos66] and J.M. Souriau [Sou66] built their own pioneering work concerning geometric quantization. In the case of deformation quantization, it were M. Flato, A. Lichnerowicz and D. Sternheimer who in 1976 introduced this approach [F^+76] and who, along with Bayen and Fronsdal, developed it further in 1977 [BFF^+77]. In this theory a formal deformation of the commutative algebraic structure of $C^\infty(M)$ on a given smooth manifold M is introduced, the so-called star-product. In physics terminology we would say that the star-product deforms the commutative space of classical observables $C^\infty(M)$ on a given classical system M, into a non-commutative algebra that describes the quantum behaviour as a first order deviation from the classical behaviour. It was Flato who originally came up with this concept and promoted it within other fields of (mathematical) physics as well.

After the influential papers that introduced the star product and proved some interesting results concerning it, there were a whole range of important open problems. The biggest being the existence and the classification of the star products on the relevant manifolds. The first proof of existence was given in 1983 by DeWilde and Lecomte [DWL83]. They proved the existence of the star product on symplectic manifolds. Their proof relies on the fact that any symplectic manifold can locally be identified with $\mathbb{R}^{2n}$ via the Darboux theorem. Via this chart and the so called Moyal star product on $\mathbb{R}^{2n}$, a star product was locally defined on the symplectic manifold. Thereafter, these local products were all glued together to form a star product on the whole symplectic manifold using cohomological arguments. The problem of classification, was initiated by Flato [BFF^+77], by defining the relevant definition of equivalence for star products on a given smooth manifold M. Many classification theorem were subsequently given by many independent authors using very different approaches. And they all came to the collective conclusion that the obstruction of equivalence was due to the second de Rham cohomology. When the problem of classification was dealt with, the next logical step was to generalize the existence results. It was Maxim Kontsevich who in 1997 gave a substantial generalization [Kon03]. In his celebrated paper he proved the formality theorem and showed how the existence of star products on Poisson manifolds follows from this result. Several alternative proves of his results have since been made. But there are still open questions of great importance left. One of these concerns the convergence of the star product. This is an active field of research and many interesting theorems have been formulated and proven thus far. But further research into the matter is still needed for a major breakthrough.

2 A Short Introduction to Quantum Mechanics

Since this thesis is aimed towards graduated bachelor students of mathematics it is necessary to first give an elementary overview of the principles of quantum mechanics. At the end of this section we will also discuss some desirable properties of quantization maps, that we will need in the following chapters.

2.1 The Principles of Quantum Mechanics

Following [Hal13], we will give a list of the main principles of quantum mechanics. For the mathematician that is used to precise axioms, from which exclusively deductively other truths are proven, these principles will most likely feel rather vague. However, these main principles are overarching concepts of quantum mechanics that will guide the mathematician effectively, through the unknown realms of this field.

Axiom 1 (The wave function) *The state of a quantum system, is represented by a unit vector ψ in an appropriate Hilbert space $\boldsymbol{H}$. If ψ_1 and ψ_2 are two unit vectors in $\boldsymbol{H}$ with $\psi_2 = c\psi_1$ for some $c \in \mathbb{C}$, then ψ_1 and ψ_2 represent the same physical state.*

It is important to note that in the quantum mechanics literature the inner product of the chosen Hilbert space is taken to be linear in the second entry, not the first, contrary to some of the mathematics literature. For a particle moving in 1 dimension the quantum Hilbert space is most frequently taken to be $L^2(\mathbb{R}) := L^2(\mathbb{R}, \mathbb{C})$, the Hilbert space of L^2 functions from $\mathbb{R}$ to $\mathbb{C}$. The interpretation of a unit vector of this space, is that its absolute square is the probability distribution of the position of the particle.

Axiom 2 (Quantum operators) *For each classical observable, which can be represented by a real-valued function f on a classical phase space, there is an associated self-adjoint operator A_f on the Hilbert space $\boldsymbol{H}$.*

Axiom 3 (Born's rule) *Given a quantum system described by a unit vector $\psi \in \boldsymbol{H}$ and furthermore an observable f, the probability distribution for the outcome of a measurement of the observable f on the quantum system satisfies,*

$$\mathbb{E}[f^m] = \langle \psi, A_f^m \psi \rangle := \langle A_f^m \rangle_\psi \quad \forall m \in \mathbb{N}. \tag{1}$$

From the requirement in axiom 2 for operators to be self-adjoint it now follows that these expectation values $\mathbb{E}[f^m]$ are in fact real, as one would obviously desire. We also note these expectation values are the same for all unit vectors that only differ by scalar multiplication of a complex number. These states are therefore experimentally identical and thus clarify the second part of axiom 1.

Given this third axiom, it is not very surprising that the position operator, as laid out in axiom 2, in the case of a particle moving in 1 dimension that we discussed, is defined by $X\psi(x) = x\psi(x)$. And following the de Broglie hypothesis, the momentum operator is defined by $P\psi(x) = -i\hbar \frac{d}{dx}\psi(x)$. ([Hal13] has an excellent motivation/derivation of this definition in paragraph 3.3). Note that since these quantum operators are only defined on suitable dense subspaces of $L^2(\mathbb{R})$ they are unbounded. On a suitable dense subspace of $L^2(\mathbb{R})$, we can now easily calculate the commutator of the position and momentum operators to be $[X, P] = i\hbar I$. This peculiar identity is called the canonical commutation relation, and it is this anti-commutativity of operators that leads to the uncertainty principle.

Axiom 4 (Collapse of the wave function) *Given a quantum system in a particular state, described by a unit vector $\psi \in \boldsymbol{H}$ on which a measurement f is performed which results in a $\lambda \in \mathbb{R}$, the state of the quantum system will immediately after the measurement, be described by a unit vector $\psi' \in \boldsymbol{H}$ that satisfies,*

$$A_f \psi' = \lambda \psi'. \tag{2}$$

Now suppose that we have an observable f such that A_f has an orthonormal basis of eigenvectors, $\{e_n\}_{n\geq 1}$, each with unique eigenvalue λ_n. If the given quantum system is in state $\psi \in \mathbf{H}$ we can now write $\psi = \sum_{n\geq 1} a_n e_n$ for suitable $a_n \in \mathbb{C}$.

Axiom 4 now states that if we carry out a measurement of the observable f on our system and find an outcome λ_j the system will collapse to state e_j. Because Axiom 3 now gives,

$$\mathbb{E}[f] = \langle \psi, A_f \psi \rangle = \left\langle \sum_{m\geq 1} a_m e_m, A_f \sum_{n\geq 1} a_n e_n \right\rangle = \left\langle \sum_{m\geq 1} a_m e_m, \sum_{n\geq 1} a_n A_f e_n \right\rangle \tag{3}$$

$$= \left\langle \sum_{m\geq 1} a_m e_m, \sum_{n\geq 1} a_n \lambda_n e_n \right\rangle = \sum_{m,n\geq 1} \overline{a}_m a_n \lambda_n \langle e_m, e_n \rangle \tag{4}$$

$$= \sum_{m,n\geq 1} \overline{a}_m a_n \lambda_n \delta_{m,n} = \sum_{n\geq 1} |a_n|^2 \lambda_n. \tag{5}$$

We furthermore find that the probability of measuring λ_j in the first place was $|a_j|^2$.

Axiom 5 (Time-evolution of the wave function) *Given a quantum system over time t described by unit vectors $\psi(t) \in \boldsymbol{H}$, the Schrödinger equation holds,*

$$\frac{d\psi}{dt} = \frac{1}{i\hbar} \hat{H} \psi. \tag{6}$$

Where $\hat{H}$ is the operator associated to the classical Hamiltonian H, as set out in Axiom 2.

By combining this axiom with the second and third one we find that the time-evolution of the expectation value of an observable f (that is not explicitly a function of time) on a system in state $\psi(t)$ is given by,

$$\frac{d}{dt}\mathbb{E}[f] = \frac{d}{dt}\langle \psi, A_f \psi \rangle = \left\langle \frac{d}{dt}\psi, A_f \psi \right\rangle + \left\langle \psi, A_f \frac{d}{dt}\psi \right\rangle = \left\langle \frac{1}{i\hbar}\hat{H}\psi, A_f \psi \right\rangle + \left\langle \psi, A_f \frac{1}{i\hbar}\hat{H}\psi \right\rangle \tag{7}$$

$$= -\frac{1}{i\hbar}\left\langle \hat{H}\psi, A_f \psi \right\rangle + \frac{1}{i\hbar}\left\langle \psi, A_f \hat{H}\psi \right\rangle = -\frac{1}{i\hbar}\left\langle \psi, \hat{H} A_f \psi \right\rangle + \frac{1}{i\hbar}\left\langle \psi, A_f \hat{H}\psi \right\rangle \tag{8}$$

$$= \frac{1}{i\hbar}\left\langle \psi, A_f \hat{H} - \hat{H} A_f \psi \right\rangle = \frac{1}{i\hbar}\left\langle \psi, [A_f, \hat{H}]\psi \right\rangle = \frac{1}{i\hbar}\langle [A_f, \hat{H}] \rangle_\psi. \tag{9}$$

This is a very important expression. It not only immediately tells us that observables f are conserved if and only if A_f commutes with $\hat{H}$. But we can furthermore note the resemblance of the equation to the time-evolution of classical observables f, which we know from classical mechanics to be,

$$\frac{df}{dt} = \{f, H\} \tag{10}$$

where $\{\ ,\ \} : C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ denotes the well-known (standard) Poisson bracket on phase space $\mathbb{R}^{2n}$, defined by

$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j} \right) \quad \forall f, g \in C^\infty(\mathbb{R}^{2n}). \tag{11}$$

This crucial observation suggests that the mapping in axiom 2, which is frequently called the quantization map and is denoted by Q, has to satisfy the identity,

$$Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)], \tag{12}$$

for all classical observables f, g. Or at least needs to satisfy this identity in the classical limit, when $\hbar \to 0$ so that no uncertainty exists. Now we have sufficiently clarified axioms 1,3,4 and 5, it is time to address the second axiom. In the following section we will therefore pursue the construction of an explicit linear map from the space of classical observables into the space of self-adjoint operators of our underlying Hilbert space $\mathbf{H}$. In the section thereafter we will see that the property $Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)]$ can never be satisfied for all classical observables f, g, which will be the exact moment the theory of deformation quantization comes into play.

3 Weyl Quantization

In this section we will introduce many different quantization schemes and study their properties. We will subsequently find that only one of these quantizations schemes, the Weyl quantization, has the most desirable properties. Afterwards we will rigorously define and study the Weyl quantization.

3.1 Quantization Schemes

For the construction of a quantization map Q we will initially restrict ourselves to a particle moving on the real line. In the previous chapter we have seen that the underlying Hilbert space in this case, is to be taken $L^2(\mathbb{R})$ and we have furthermore seen the definitions of the position and momentum operator X, P on $L^2(\mathbb{R})$. Together with axiom 3 it is clear that for all $n \geq 1$ we require $Q(x^n) = X^n$ and $Q(p^n) = P^n$. The next logical step is to consider monomials in x, p. A naive first attempt would be to require that $Q(x^m p^n) = X^m P^n$ for all $m, n \geq 1$. But this is naive for multiple reasons. The first most obvious one is that the operators $X^m P^n$, are not self-adjoint in general, since X and P do not commute $(X^m P^n)^* = (P^*)^n (X^*)^m = P^n X^m$. The second reason why the guess $Q(x^m p^n) = X^m P^n$ is naive is because of its unnatural nature. The space of classical observables, $C^\infty(\mathbb{R}^n)$ is a commutative algebra over $\mathbb{R}$, whilst the codomain of the quantization map is the space of self-adjoint operators on $L^2(\mathbb{R}, \mathbb{C})$, which is certainly non-commutative. Incidentally, it is this last observation that gives us the solution to the problem of self-adjointness. We just need to symmetrize the monomials $X^m P^n$. This can be done by either symmetrizing in the powers of X and P, resulting in the so called symmetrized pseudodifferential operator quantization, or symmetrizing in every individual X and P, resulting in the Weyl quantization. Where the last quantization scheme feels much more natural because it is fully symmetrized. Two other quantization schemes that are frequently used in the context of quantum field theory are the so called Wick-ordered and anti-Wick-ordered quantizations. We now give an overview of these 4 quantization schemes:

Symmetrized pseudodifferential operator quantization	$Q(x^m p^n) = \frac{1}{2}(X^m P^n + P^n X^m)$
Weyl quantization	$Q(x^m p^n) = \frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} \sigma(X, ..., X, P, ..., P)$
Wick-ordered quantization *with parameter* $\alpha \in \mathbb{R}_{>0}$	$Q((x + i\alpha p)^m (x - i\alpha p)^n) = (X - i\alpha P)^n (X + i\alpha P)^m$
Anti-Wick-ordered quantization *with parameter* $\alpha \in \mathbb{R}_{>0}$	$Q((x + i\alpha p)^m (x - i\alpha p)^n) = (X + i\alpha P)^m (X - i\alpha P)^n$

for all $m, n \geq 1$. We note that because of the linearity of quantization maps, $Q(f(x,p))$ is now determined for all polynomials $f \in \mathbb{R}[x,p]$. All these operators are unbounded on $L^2(\mathbb{R})$, but are well-defined on for example the algebra of smooth functions from $\mathbb{R}$ to $\mathbb{C}$ with compact support, $C_c^\infty(\mathbb{R}, \mathbb{C})$.

3.2 Weyl Quantization on $\mathbb{R}^{2n}$

In the previous section, when considering the stark difference in algebraic structure between the domain and codomain of a quantization map, we gave an argument of symmetry to argue that the Weyl quantization is the most natural of quantization schemes. This was however done in the context of one particle moving on the real line and we only explicitly defined Weyl quantization for real polynomials in the position and momentum observables x, p. Additionally, our discussion thus far has not been of the most mathematically rigorous kind. But so is the very nature of discussions within physics. So our objective in this section is to rigorously define the Weyl quantization and to generalize it to higher dimensions and a much broader class of classical observables. We will pursue the generalization to higher dimensions first.

In classical mechanics the phase space of a system consisting out of one particle moving in $\mathbb{R}^n$ is given by $\mathbb{R}^{2n}$ where the elements of $\mathbb{R}^{2n}$ are written as $(\boldsymbol{x}, \boldsymbol{p}) = (x_1, .., x_n, p_1, ..., p_n)$, representing the position and momentum of the particle. The classical observables x_i and p_i are obviously defined as $x_i(x_1, .., x_n, p_1, ..., p_n) = x_i$ and $p_i(x_1, .., x_n, p_1, ..., p_n) = p_i$ for all $i \in \{1, 2, ..., n\}$ and are smooth real-valued functions as projections on $\mathbb{R}$. Corresponding to these classical observables we define the quantum operators on $L^2(\mathbb{R}^n)$ by $X_i\psi = x_j\psi$ and $P_i\psi = -i\hbar\frac{d}{dx_i}\psi$ for all $i \in \{1, 2, ..., n\}$. Because the only pair of operators that do not commute with each other are X_j and P_j for all $j \in \{1, 2, ..., n\}$ we only have to symmetrize in each of these n-pairs giving us the prescription for the Weyl quantizations of monomials in $x_1, ..., x_n, p_1, ..., p_n$ to be,

$$Q(x_1^{k_1}...x_n^{k_n}p_1^{l_1}...p_n^{l_n}) \tag{13}$$

$$=Q((x_1^{k_1}p_1^{l_1})...(x_n^{k_n}p_n^{l_n})) = Q(x_1^{k_1}p_1^{l_1})...Q(x_n^{k_n}p_n^{l_n}) \tag{14}$$

$$= \left(\frac{1}{(k_1+l_1)!}\sum_{\sigma_1\in S_{k_1+l_1}} \sigma_1(X_1, ..., X_1, P_1, ..., P_1)\right) ... \left(\frac{1}{(k_n+l_n)!}\sum_{\sigma_n\in S_{k_n+l_n}} \sigma_n(X_n, ..., X_n, P_n, ..., P_n)\right) \tag{15}$$

If we symmetrize in all the pairs of $x_1, ..., x_n, p_1, ..., p_n$ we of course get the same quantization, but the expression can be useful in calculations.

$$Q(x_1^{k_1}...x_n^{k_n}p_1^{l_1}..p_n^{l_n}) = \frac{1}{(k_1+...+k_n+l_1+...+l_n)!}\sum_{\sigma\in S_{k_1+...+k_n+l_1+...+l_n}} \sigma(X_1, ..., X_1, ..., P_n, ..., P_n) \tag{16}$$

From the linearity of Q we now have explicitly defined the image of any real polynomial in $x_1, ..., x_n$, $p_1, ..., p_n$. In order to generalize for other smooth functions of $x_1, ..., x_n, p_1, ..., p_n$ we could think of the direct utilisation of Taylor series or the utilisation of Taylor series to define Weyl quantization for exponentials in $x_1, ..., x_n, p_1, ..., p_n$ only to subsequently use it to define a Fourier transform. Since the last mathematical tool has nicer properties and is more natural in the context of quantum mechanics than the former, we proceed with the second route. Because we are going to Fourier transform in $(\boldsymbol{x}, \boldsymbol{p}) = (x_1, .., x_n, p_1, ..., p_n)$, we need the following lemma.

Lemma 3.2.1 *For all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ the equality $Q((\boldsymbol{a}\cdot\boldsymbol{x} + \boldsymbol{b}\cdot\boldsymbol{p})^m) = (\boldsymbol{a}\cdot\boldsymbol{X} + \boldsymbol{b}\cdot\boldsymbol{P})^m$ holds.*

Proof. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$. The following series of equalities now proves the statement,

$$Q((\boldsymbol{a}\cdot\boldsymbol{x} + \boldsymbol{b}\cdot\boldsymbol{p})^m) = Q((a_1x_1 + ... + a_nx_n + b_1p_1 + ... + b_np_n)^m) \tag{17}$$

$$= Q\left(\sum_{k_1+...+k_n+l_1+..+l_n=m} \binom{m}{k_1, ..., k_n, l_1, ..., l_n} \prod_{t=1}^{n} (a_tx_t)^{k_t}(b_tp_t)^{l_t}\right) \tag{18}$$

$$= \sum_{k_1+...+l_n=m} \binom{m}{k_1, ..., l_n} \prod_{s=1}^{n} a_s^{k_s}b_s^{l_s} Q\left(\prod_{t=1}^{n} x_t^{k_t}p_t^{l_t}\right) \tag{19}$$

$$= \sum_{k_1+...+l_n=m} \binom{m}{k_1, ..., l_n} \prod_{s=1}^{n} a_s^{k_s}b_s^{l_s} \frac{1}{m!}\sum_{\sigma\in S_m} \sigma(X_1, ..., X_1, ..., P_n, ..., P_n) \tag{20}$$

$$= \frac{1}{m!}\sum_{\sigma\in S_m}\sum_{k_1+...+l_n=m} \binom{m}{k_1, ..., l_n} \prod_{s=1}^{n} a_s^{k_s}b_s^{l_s} \sigma(X_1, ..., X_1, ..., P_n, ..., P_n) \tag{21}$$

$$= \frac{1}{m!}\sum_{\sigma\in S_m} \sigma(a_1X_1 + ... + b_nP_n, ..., a_1X_1 + ... + b_nP_n) \tag{22}$$

$$= \frac{1}{m!}\sum_{\sigma\in S_m} (a_1X_1 + ... + a_nX_n + b_1P_1 + ... + b_nP_n)^m \tag{23}$$

$$= (\boldsymbol{a}\cdot\boldsymbol{X} + \boldsymbol{b}\cdot\boldsymbol{P})^m \tag{24}$$

It would be desirable if the following equalities would hold.

$$Q(e^{i(\boldsymbol{a}\cdot\boldsymbol{x}+\boldsymbol{b}\cdot\boldsymbol{p})}) = Q\left(\sum_{m\geq 0}\frac{i^m}{m!}(\boldsymbol{a}\cdot\boldsymbol{x} + \boldsymbol{b}\cdot\boldsymbol{p})^m\right) = \sum_{m\geq 0}\frac{i^m}{m!}Q((\boldsymbol{a}\cdot\boldsymbol{x} + \boldsymbol{b}\cdot\boldsymbol{p})^m) \tag{25}$$

$$= \sum_{m\geq 0}\frac{i^m}{m!}(\boldsymbol{a}\cdot\boldsymbol{X} + \boldsymbol{b}\cdot\boldsymbol{P})^m = e^{i(\boldsymbol{a}\cdot\boldsymbol{X}+\boldsymbol{b}\cdot\boldsymbol{P})} \tag{26}$$

For any sufficiently nice classical observable f we would then have a well defined Fourier transform $\hat{f}$ and the identity,

$$f(\boldsymbol{x}, \boldsymbol{p}) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \hat{f}(\boldsymbol{a}, \boldsymbol{b}) e^{i(\boldsymbol{a}\cdot\boldsymbol{x}+\boldsymbol{b}\cdot\boldsymbol{p})}\, d\boldsymbol{a} d\boldsymbol{b}. \tag{27}$$

Which together with the previous identity would imply the following definition for the Weyl quantization,

$$Q(f) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \hat{f}(\boldsymbol{a}, \boldsymbol{b}) e^{i(\boldsymbol{a}\cdot\boldsymbol{X}+\boldsymbol{b}\cdot\boldsymbol{P})}\, d\boldsymbol{a} d\boldsymbol{b}. \tag{28}$$

There are many different convergence results regarding this particular Fourier transform. We will review one that concerns square integrable classical observables. In order to formulate this theorem we need to introduce the concept of Hilbert-Schmidt operators.

Definition 3.2.2 *Let $\boldsymbol{H}$ be a Hilbert space, $(e_n)_{n\geq 1}$ an orthonormal basis of H and A a bounded operator on $\boldsymbol{H}$. Then A is called a Hilbert-Schmidt operator whenever* $\boldsymbol{trace}(A^*A) := \sum_{n\geq 1}\langle e_n, (A^*A)e_n\rangle < \infty$.

We remark that the sum is well defined because every term is non-negative $\langle e_n, (A^*A)e_n\rangle = \langle Ae_n, Ae_n\rangle = \|Ae_n\|^2 \geq 0$, making it absolutely convergent in this case. We furthermore remark that given a second orthonormal basis $(e'_n)_{n\geq 1}$ of H, and the unitary operator U on $\mathbf{H}$ which satisfies $e'_n = Ue_n$ for all $n \geq 1$, we have $\sum_{n\geq 1}\langle e'_n, A^*Ae'_n\rangle = \sum_{n\geq 1}\langle Ue_n, A^*AUe_n\rangle = \sum_{n\geq 1}\langle e_n, U^*A^*AUe_n\rangle = \mathbf{trace}(U^*A^*AU) = \mathbf{trace}(A^*AUU^*) = \mathbf{trace}(A^*A)$, where we have used the cyclic property of the trace. So the given definition is indeed well-defined. Given two Hilbert-Schmidt operators A and B on $\mathbf{H}$ we can prove in much the same way, that $\mathbf{trace}(A^*B) := \sum_{n\geq 1}\langle e_n, (A^*B)e_n\rangle$ is well-defined, finite and absolutely convergent. One can furthermore prove that the space of Hilbert-Schmidt operators forms a vector space and even a Hilbert space under the inner product $\langle\ ,\ \rangle_{\mathrm{HS}}$ defined by $\langle A, B\rangle_{\mathrm{HS}} = \mathbf{trace}(A^*B)$. This Hilbert space is denoted by $\mathrm{HS}(\mathbf{H})$. One can then prove that Weyl quantization, when restricted to $L^2(\mathbb{R}^{2n})$, become a well-defined map onto $\mathrm{HS}(L^2(\mathbb{R}^n))$ (for details see [Hal13] chapter 13). We are now in the position to formulate the following definition and theorem.

Definition 3.2.3. *The Weyl Quantization is the map $Q : L^2(\mathbb{R}^{2n}) \to HS(L^2(\mathbb{R}^n))$ defined by*

$$f \mapsto (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \hat{f}(\boldsymbol{a}, \boldsymbol{b}) e^{i(\boldsymbol{a}\cdot\boldsymbol{X}+\boldsymbol{b}\cdot\boldsymbol{P})}\, d\boldsymbol{a} d\boldsymbol{b} \tag{29}$$

Theorem 3.2.4. (Wigner–Weyl transform) (Theorem 13.8. in [Hal13]) *The Weyl Quantization is a constant multiple of a unitary map of $L^2(\mathbb{R}^{2n})$ onto $HS(L^2(\mathbb{R}^n))$. The inverse map is called the Wigner map. For all $f \in L^2(\mathbb{R}^{2n})$ we have $Q(\overline{f}) = Q(f)^*$. So in particular it follows that $Q(f)$ is self-adjoint for square-integrable classical observables, as desired.*

As argued in the previous chapter we would like the Weyl quantization to satisfy equation (12), $Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)]$, for all classical variables $f, g \in L^2(\mathbb{R}^{2n})$. After some rather laborious but elementary calculations, one finds that the choice $f = x^2p$ and $g = xp^2$ already fails to satisfy this equation. We are now left with a problem. After a long journey we have found a natural candidate for a quantization map. But the equality $\frac{1}{i\hbar}[Q(f), Q(g)] = Q(\{f, g\})$ fails even for the simple classical variables $f = x^2p$ and $g = xp^2$. One could now be inclined to think, that this is somehow a shortcoming of the definition of the Weyl quantization. But H.J. Groenewold was the first to point out that this is in fact not the case, by proving that all the quantization maps have this problem to the same or even worse degree. This result is the so called, "No-Go" theorem of Groenewold, and was proven by him in his doctoral dissertation in 1946 [Gro46]. We will state the theorem in the next section.

3.3 Groenewold's Theorem

In order to prove Groenewold's No-Go theorem an assumption is made on the codomain of quantization maps Q. It is assumed that for all polynomials f, $Q(f)$ is contained in the algebra generated by the operators X and P. $Q(f)$ becomes consequently a differential operator with polynomial coefficients. For the proof of a more general statement see [Got99]. Before we formulate the theorem we introduce the following notation. For all $k \geq 0$, let $\mathcal{P}_{\leq k}$ denote the space of all polynomials of degree at most k.

Theorem 3.3.1 (Groenewold's theorem) *Let $\mathcal{D}(\mathbb{R}^n)$ denote the space of differential operators on $\mathbb{R}^n$ with polynomial coefficients. There does not exist a linear map $Q : \mathcal{P}_{\leq 4} \to \mathcal{D}(\mathbb{R}^n)$ satisfying the following 3 properties,*

(i) $Q(1) = I$

(ii) $Q(x_j) = X_j$ *and* $Q(p_j) = P_j$

(iii) For all $f, g \in \mathcal{P}_{\leq 3}$*, we have* $Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)]$

The proof is rather long and uninteresting, so we will not go over it. For a rigorous proof, section 13.4 in [Hal13] can be consulted. This theorem is not as problematic as one might think. On page 5 under equation (12) we already noted that equation (12), $Q(\{f, g\}) = \frac{1}{i\hbar}[Q(f), Q(g)]$ does not always have to hold. It only has to hold in the classical limit. It is therefore natural to think about (formal) power series in $\hbar$. In chapter 5 we will define the star product to make this idea rigorous.

3.4 The Moyal Product

In theorem 3.2.4. (Wigner-Weyl transform) we have seen that, given square integrable functions $f, g \in L^2(\mathbb{R}^{2n})$, their images under the Weyl quantization map are Hilbert-Schmidt operators. Because the product of Hilbert-Schmidt operators is again Hilbert-Schmidt, and the Weyl Quantization map is a bijection, there exists a unique function in $L^2(\mathbb{R}^{2n})$, denoted by $f \star g$, such that $Q(f \star g) = Q(f)Q(g)$. One can then formulate the following theorem.

Proposition 3.4.1 *The map* $\star : L^2(\mathbb{R}^{2n}) \times L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$ *defined by* $(f, g) \mapsto f \star g$ *is called the Moyal product. It is continuous and satisfies the equality* $Q(f \star g) = Q(f)Q(g)$ *for all* $f, g \in L^2(\mathbb{R}^{2n})$.

A proof of this proposition can either be found in Groenewold's doctoral dissertation [Gro46] or proposition 13.10 in [Hal13]. Unlike the name suggests, it was actually Groenewold who introduced this product and stated some of its important properties. In particular he found the following expression for the star-product,

$$(f \star g)(\boldsymbol{x}, \boldsymbol{p}) = f(\boldsymbol{x}, \boldsymbol{p}) \exp\left(i\frac{\hbar}{2}\left(\overleftarrow{\partial_{\mathbf{x}}}\overrightarrow{\partial_{\mathbf{p}}} - \overleftarrow{\partial_{\mathbf{p}}}\overrightarrow{\partial_{\mathbf{x}}}\right)\right) g(\boldsymbol{x}, \boldsymbol{p}). \tag{30}$$

If we think of the right hand side as a formal power series in $\hbar$, we may define the product for all classical observables, since we do not have to worry about convergence.

Definition 3.4.2. *We define the Moyal product on* $\mathbb{R}^{2n}$ *as the* $\mathbb{R}[\![\hbar]\!]$*-bilinear map* $\star : C^\infty(\mathbb{R}^{2n})[\![\hbar]\!] \times C^\infty(\mathbb{R}^{2n})[\![\hbar]\!] \to C^\infty(\mathbb{R}^{2n})[\![\hbar]\!]$ *satisfying,*

$$(f \star g)(\boldsymbol{x}, \boldsymbol{p}) = f(\boldsymbol{x}, \boldsymbol{p}) \exp\left(i\frac{\hbar}{2}\left(\overleftarrow{\partial_{\mathbf{x}}}\overrightarrow{\partial_{\mathbf{p}}} - \overleftarrow{\partial_{\mathbf{p}}}\overrightarrow{\partial_{\mathbf{x}}}\right)\right) g(\boldsymbol{x}, \boldsymbol{p}), \quad \forall f, g \in C^\infty(\mathbb{R}^{2n}). \tag{31}$$

Proposition 3.4.3. *The Moyal product satisfies the following important properties:*

(i) For all $f \in C^\infty(\mathbb{R}^{2n})$ *we have* $f \star 1 = 1 \star f = f$.

(ii) For all $f, g, h \in C^\infty(\mathbb{R}^{2n})$ *we have* $f \star (g \star h) = (f \star g) \star h$.

(iii) For all $f, g \in C^\infty(\mathbb{R}^{2n})$ *we have* $f \star g := fg + \sum_{n \geq 1} \hbar^n B_n(f, g)$*, where for all* $n \geq 1$ *the* $B_n : C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ *are bilinear maps that additionally are differential operators on* $\mathbb{R}^{2n}$ *in each argument. So for any* $n \geq 1$ *and smooth chart on* $\mathbb{R}^{2n}$ *we can represent* B_n *locally as,*

$$B_n(f, g) = \sum_{k_1, \ldots, l_{2n}} A_n^{k_1, \ldots, k_{2n}, l_1, \ldots, l_{2n}} \partial_1^{k_1} \ldots \partial_{2n}^{k_{2n}} f \, \partial_1^{l_1} \ldots \partial_{2n}^{l_{2n}} g,$$

where the $A_n^{k_1, \ldots, k_{2n}, l_1, \ldots, l_{2n}}$ *are smooth functions on* $\mathbb{R}^{2n}$ *for which only finitely many are non-zero.*

(iv) For all $f, g \in C^\infty(\mathbb{R}^{2n})$ *we have* $[f, g] := f \star g - g \star f = i\hbar\{f, g\} + \mathcal{O}(\hbar^3)$.

Proof. The first three properties will be proven in a more general case in the examples following definition 5.0.1. (the star product). The last property follows from the fact that for all $f, g \in C^\infty(\mathbb{R}^{2n})$ we have,

$$f \star g - g \star f = fg + i\frac{\hbar}{2}(\partial_{\boldsymbol{x}} f \partial_{\boldsymbol{p}} g - \partial_{\boldsymbol{p}} f \partial_{\boldsymbol{x}} g) - \frac{\hbar^2}{2}(\partial_{\boldsymbol{x}}^2 f \partial_{\boldsymbol{p}}^2 g - 2\partial_{\boldsymbol{x}} \partial_{\boldsymbol{p}} f \partial_{\boldsymbol{x}} \partial_{\boldsymbol{p}} g + \partial_{\boldsymbol{p}}^2 f \partial_{\boldsymbol{x}}^2 g) + \mathcal{O}(\hbar^3) \tag{32}$$

$$- gf - i\frac{\hbar}{2}(\partial_{\boldsymbol{x}} g \partial_{\boldsymbol{p}} f - \partial_{\boldsymbol{p}} g \partial_{\boldsymbol{x}} f) + \frac{\hbar^2}{2}(\partial_{\boldsymbol{x}}^2 g \partial_{\boldsymbol{p}}^2 f - 2\partial_{\boldsymbol{x}} \partial_{\boldsymbol{p}} g \partial_{\boldsymbol{x}} \partial_{\boldsymbol{p}} f + \partial_{\boldsymbol{p}}^2 g \partial_{\boldsymbol{x}}^2 f) - \mathcal{O}(\hbar^3) \tag{33}$$

$$= i\hbar(\partial_{\boldsymbol{x}} f \partial_{\boldsymbol{p}} g - \partial_{\boldsymbol{p}} f \partial_{\boldsymbol{x}} g) + \mathcal{O}(\hbar^3) = i\hbar\{f, g\} + \mathcal{O}(\hbar^3). \tag{34}$$

We now see that for the position operators $x_i \in C^\infty(\mathbb{R}^{2n})$, the momentum operators $p_j \in C^\infty(\mathbb{R}^{2n})$, the Hamiltonian $H \in C^\infty(\mathbb{R}^{2n})$ and any classical variable $f \in C^\infty(\mathbb{R}^{2n})$ we have,

$$[x_i, p_j] = x_i \star p_j - p_j \star x_i = i\hbar\{x_i, p_j\} = i\hbar\delta_{ij} \text{ and,} \tag{35}$$

$$[f, H] = f \star H - H \star f = i\hbar\{f, H\} + \mathcal{O}(\hbar^3). \tag{36}$$

So we recover both the canonical commutation relation and equation (12) in the classical limit $\hbar \to 0$. This has been achieved by only a one-parameter deformation on the unital associative algebra of classical observables $C^\infty(M)$. While previously we would require the construction of a quantization map into a Hilbert space. In this sense, this approach called deformation quantization, is the quantization scheme which is closest to classical mechanics. In (mathematical) physics, classical systems are often times modelled by either symplectic or Poisson manifolds. It is therefore most logical to review the basics in the field of differential geometry, in order to continue afterwards with the generalization of a formal deformation and in order to study it under the right conditions.

4 Differential Geometry

In the previous section we argued that deformation quantization is the quantization scheme that comes closest to the formalism of classical mechanics. The formalism of classical mechanics is however much richer than our primitive picture consisting out of: particles moving in Euclidean phase space and smooth real functions on this phase space, representing classical observables. Instead of restricting ourselves to a n-dimensional Euclidean space, wherein our particles are moving, we would like to expand to a more general manifold. Additionally we would want to convey the rich structure that the Hamiltonian of the system entails. Fortunately for us, mathematical abstractions of such descriptions have been defined an studied for quite a while now, they are symplectic and Poisson manifolds. In this chapter we will give the necessary background in order to define both structures, and many more relevant concepts. For an introduction to this topic see John M. Lee's "Introduction to Smooth Manifolds" [Lee13].

4.1 Tensors

Let V be a finite-dimensional real vector space. For k a positive integer, we define a covariant k-tensor on V to be an element of the k-fold tensor product $T^k(V^*) := V^* \otimes ... \otimes V^*$ and define a contravariant k-tensor on V to be an element of the k-fold tensor product $T^k(V) = V \otimes ... \otimes V$. These rather abstract objects can be interpreted more concretely by the following definition and theorem.

Let $V_1, ..., V_k$ and W be vector spaces. A map $F : V_1 \times ... \times V_k \to W$ is called multilinear when it is linear in each of its arguments. It is clear that the set of these multilinear maps $F : V_1 \times ... \times V_k \to W$ forms a vector space under pointwise addition and scalar multiplication, that we will denote by $L(V_1 \times .... \times V_k; W)$.

Proposition 4.1.1. *If $V_1, ..., V_k$ are finite-dimensional vector spaces the map $\phi : V_1^* \otimes ... \otimes V_k^* \to L(V_1, .., V_k; \mathbb{R})$ defined by $(\omega^1, ..., \omega^k) \mapsto ((v_1, ..., v_k) \mapsto \omega^1(v_1) \cdot ... \cdot \omega^k(v_k))$ is a canonical isomorphism, $V_1^* \otimes ... \otimes V_k^* \cong L(V_1, .., V_k; \mathbb{R})$. By taking the dual of the $V_1, .., V_k$'s and using the fact that the double dual of a finite vector space is canonically isomorphic to itself we find the canonical isomorphism $V_1 \otimes ... \otimes V_k \cong L(V_1^*, .., V_k^*; \mathbb{R})$ given by $(v_1, ..., v_k) \mapsto ((\omega^1, ..., \omega^k) \mapsto \omega^1(v_1) \cdot ... \cdot \omega^k(v_k))$.*

The proof of this proposition is omitted since it is an elementary result. From now on we will almost exclusively interpret covariant k-tensors as real-valued multilinear functions on V^k, $\omega : V^k \to \mathbb{R}$. And likewise interpret contravariant k-tensors as real-valued multilinear functions on $(V^*)^k$, $\omega : (V^*)^k \to \mathbb{R}$. A covariant k-tensor ω on a finite-dimensional vector space V that satisfies for all $v_1, ..., v_k \in V$ and all $i, j \in 1, ..., k$ with $i < j$ the property $\omega(v_1, ..., v_i, ..., v_j, ..., v_k) = -\omega(v_1, ..., v_j, ..., v_i, ..., v_k)$ is said to be alternating. In exactly the same way we call a contravariant k-tensor on a finite-dimensional vector space V, v, to be alternating when for all $\omega^1, ..., \omega^k \in V^*$ and all $i, j \in 1, ..., k$ with $i < j$ the property $v(\omega^1, ..., \omega^i, ..., \omega^j, ..., \omega^k) = -v(\omega^1, ..., \omega^j, ..., \omega^i, ..., \omega^k)$ holds. Alternating covariant k-tensors are also called k-covectors or exterior forms and they form a subspace of $T^k(V^*)$ denoted by $\bigwedge^k(V^*)$. Alternating contravariant k-tensors are also called k-vectors and they form a subspace of $T^k(V)$ denoted by $\bigwedge^k(V)$.

Let V be a finite-dimensional vector space. For each $v \in V$ we define the linear map $i_v : \bigwedge^k(V^*) \to \bigwedge^{k-1}(V^*)$, called the interior multiplication by v, by $\omega \mapsto ((w_1, ..., w_{k-1}) \mapsto \omega(v, w_1, ..., w_{k-1}))$. The notation $i_v(w) = v \llcorner w$ is also used in the literature.

4.2 Tensor Fields

Now we have defined (alternating) (co)/(contra)variant k-tensors as elements of vector spaces we want to define smooth (alternating) (co)/(contra)variant tensor fields on smooth manifolds. For this, we need to smoothly glue together, all the copies of these vector spaces, each being pinned to a specific point on the smooth manifold. In differential geometry the framework of vector bundles is used for these types of constructions. We will therefore proceed with giving the definition of smooth vector bundles.

Definition 4.2.1. *Let M be a smooth manifold. A smooth vector bundle of rank k over M is a smooth manifold E together with a surjective smooth map $\pi : E \to M$ satisfying the following conditions:*

(1) For each $p \in M$, the fiber $E_p := \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.

(2) For each $p \in M$, there exists a neighbourhood U of p in M and a diffeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$, called a local trivialization of E over U, satisfying the following conditions:

(i) $\pi_U \circ \Phi = \pi$, where $\pi_U : U \times \mathbb{R}^k \to U$ is the projection on U;

(ii) for each $q \in U$ the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

We will now introduce the following widely used notation, $T^kT^*M := \amalg_{p\in M} T^k(T_p^*M)$, $\bigwedge^k T^*M := \amalg_{p\in M} \bigwedge^k(T_p^*M)$, $T^kTM := \amalg_{p\in M} T^k(T_pM)$ and $\bigwedge^k TM := \amalg_{p\in M} \bigwedge^k(T_pM)$. The next proposition states that these sets have the natural structure of a vector bundle.

Proposition 4.2.2. *Let M be a smooth n-manifold. Both T^kT^*M, $\bigwedge^k T^*M$, T^kTM and $\bigwedge^k TM$ can be endowed with a natural topology and smooth structure making them into smooth vector bundles of rank n^k, $\binom{n}{k}$, n^k and $\binom{n}{k}$ over M respectively.*

Proof. We will only prove that T^kT^*M is a smooth vector bundle of rank n^k because the proof for the other sets is very similar in nature. We define the map $\pi : T^kT^*M \mapsto M$ by $(p, v) \mapsto p$ where $p \in M$ and $v \in T_p^*M$. We begin with endowing T^kT^*M with a manifold structure by defining the maps that will become our smooth carts. Given a smooth chart $(U, (x^1, ..., x^n))$ for M we define the map $\tilde{\phi} : \pi^{-1}U \to \mathbb{R}^{n+n^k}$ by,

$$(v_{i_1,...,i_k} dx^{i_1}|_p \otimes ... \otimes dx^{i_k}|_p) \mapsto (x^1(p), ..., x^n(p), v_{1,...,1}, ..., v_{n,...n}), \tag{37}$$

where $i_1, .., i_k \in \{1, ..., n\}$. $\tilde{\phi}$ is now clearly bijective since the inverse is given by,

$$(x^1, ..., x^n, v_{1,...,1}, ..., v_{n,...n}) \mapsto (v_{i_1,...,i_k} dx^{i_1}|_{\phi^{-1}x} \otimes ... \otimes dx^{i_k}|_{\phi^{-1}x}). \tag{38}$$

Now given two smooth charts (U, ϕ) and (V, ψ) on M, set $(\pi^{-1}U, \tilde{\phi})$ and $(\pi^{-1}V, \tilde{\psi})$ to be the corresponding charts on T^kT^*M. The sets $\tilde{\phi}(\pi^{-1}U \cap \pi^{-1}V) = \phi(U \cap V) \times \mathbb{R}^{n^k}$ and $\tilde{\psi}(\pi^{-1}U \cap \pi^{-1}V) = \psi(U \cap V) \times \mathbb{R}^{n^k}$ are then open in $\mathbb{R}^{n+n^k}$ and the transition map $\tilde{\psi} \circ \tilde{\phi}^{-1} : \phi(U \cap V) \times \mathbb{R}^{n^k} \to \psi(U \cap V) \times \mathbb{R}^{n^k}$ is given by,

$$(x^1, ..., x^n, v_{1,...,1}, ..., v_{n,...,n}) \mapsto \left(\tilde{x}^1(x), ..., \tilde{x}^n(x), v_{i_1,...,i_k} \frac{\partial \tilde{x}^1}{\partial x^{i_1}}(x) ... \frac{\partial \tilde{x}^1}{\partial x^{i_k}}(x), ..., v_{i_1,...,i_k} \frac{\partial \tilde{x}^n}{\partial x^{i_1}}(x) ... \frac{\partial \tilde{x}^n}{\partial x^{i_k}}(x)\right), \tag{39}$$

which is clearly smooth as its components are all smooth. Note that we used the chain rule in this step:

$$v_{i_1,...,i_k} dx^{i_1}|_p \otimes .... \otimes dx^{i_k}|_p = v_{i_1,...,i_k} \left(\frac{\partial \tilde{x}^{j_1}}{\partial x^{i_1}}(x) d\tilde{x}^{j_1}|_p\right) \otimes .... \otimes \left(\frac{\partial \tilde{x}^{j_k}}{\partial x^{i_1}}(x) d\tilde{x}^{j_k}|_p\right) \tag{40}$$

$$= v_{i_1,...,i_k} \frac{\partial \tilde{x}^{j_1}}{\partial x^{i_1}}(x) \frac{\partial \tilde{x}^{j_k}}{\partial x^{i_1}}(x)(d\tilde{x}^{j_1}|_p \otimes .... \otimes d\tilde{x}^{j_k}|_p). \tag{41}$$

By choosing a countable cover $\{U_i\}_{i\geq 1}$ of M by smooth coordinate domains we obtain a countable cover of T^kT^*M by smooth coordinate domains $\{\pi^{-1}U_i\}_{i\geq 1}$. Now let $(p, v), (q, w)$ be distinct points in T^kT^*M. If $p = q$, $(p, v), (q, w)$ lie in the same fiber of π and therefore lie in a common smooth chart. If p and q are distinct, there exist disjoint smooth coordinate domains U, V in M with $p \in U$ and $q \in V$ and therefore $\pi^{-1}U \ni (p, v)$ disjoint with $\pi^{-1}V \ni (q, w)$. From the smooth manifold chart lemma (page 21, lemma 1.35 in [Lee13]) it now follows that T^kT^*M is a smooth $(n + n^k)$-manifold.

We will now prove that T^kT^*M is a smooth bundle. We note that π is obviously a surjective map and also smooth since the local representations w.r.t. the described charts (U, ϕ) and $(\pi^{-1}U, \tilde{\phi})$ are of the form $\pi(p, v) = p$. For each $p \in M$ we have $E_p = T_p^*M$. For any smooth chart (U, ϕ) we defined the map $\tilde{\phi}$. We saw that this map was bijective and looking at the components of itself and its inverse we conclude that it is a diffeomorphism onto $\phi(U) \times \mathbb{R}^{n^k}$. It is clear that $\pi_U \circ \tilde{\phi} = \pi$. Looking at the expression of $\tilde{\phi}$ it is also more than clear that for each $q \in U$, the restriction of $\tilde{\phi}$ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^{n^k} \cong \mathbb{R}^{n^k}$. We conclude that T^kT^*M is a smooth vector bundle of rank n^k over M. □

Now that the set of (alternating) (co)/(contra)variant k-tensors over M are set up in the framework of vector bundles, we can define smooth fields of them very neatly. Namely, a smooth (alternating) (co)/(contra)variant k-tensor field on M is a smooth section of the smooth bundle of (alternating) (co)/(contra)- variant k-tensors over M. A smooth section of $\bigwedge^k T^*M$ is also called a differential k-form or just k-form. And a smooth section of $\bigwedge^k TM$ is also called a k-vector field or multivector field of degree k. The 4 sets, each containing all of the smooth fields of one of these 4 types, have the natural structure of a real vector space under pointwise addition and scalar multiplication. We denote these vector spaces by $\Gamma(T^kT^*M)$, $\Gamma(\bigwedge^k T^*M) = \Omega^k(M)$, $\Gamma(T^kTM)$ and $\Gamma(\bigwedge^k TM) = \mathfrak{X}^k(M)$.

Since these definitions can feel a little bit abstract, we give as an example, a more concrete description of differential k-forms. Differential k-forms are smooth maps $\omega : M \to \bigwedge^k T^*M$ such that for all $p \in M$ we have $\omega(p) \in \bigwedge^k (T_p^*M)$. It is clear that any differential k-form ω can be written locally as,

$$\omega = \sum_{1 \le i_1 < ... < i_k \le n} \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{42}$$

where the coefficients $\omega_{i_1,...,i_k}$'s and the differentials dx^i's are all smooth functions of the local coordinates $x^1, .., x^n$.

4.3 Pullbacks

The following definition is a very important concept.

Definition 4.3.1. *Let M and N be smooth manifolds and let $F : M \to N$ be a smooth map between them. For any point $p \in M$ and any k-tensor $\alpha \in T^k(T^*_{F(p)}N)$, we define a tensor $dF_p^*(\alpha) \in T^k(T_p^*M)$, called the pointwise pullback of α by F at p, by $dF_p^*(\alpha)(v_1, ..., v_k) = \alpha(dF_p(v_1), ..., dF_p(v_k))$ for any $v_1, ..., v_k \in T_pM$. Given a smooth covariant k-tensor field A on N, we define the (rough) k-tensor field F^*A on M, called the pullback of A by F, by $(F^*A)_p(v_1, ..., v_k) = A_{F(p)}(dF_p(v_1), ..., dF_p(v_k))$ for all $v_1, ..., v_k \in T_pM$. One can easily prove that this map is indeed a covariant k-tensor field on M and furthermore that it is smooth.*

4.4 Lie Derivatives and Lie Algebras

Given a smooth manifold M, we have seen that the set of smooth vector fields $\mathfrak{X}(M) := \Gamma(TM)$ has a vector space structure. An interesting question that now arises is if we can make this into an algebra or similar structure. For this we need a multiplication. A naive approach would be to define the multiplication pointwise as $(X \cdot Y)_p = (XY)_p := X_pY_p$ for all $p \in M$. Which fails because XY will in general not be a vector field. The solution is to define the multiplication $X \cdot Y = XY - YX$. We will now give the concrete definition and properties of this object.

Definition 4.3.1. *Let M be a smooth manifold. We define the Lie bracket of vector fields to be the map $[\,,\,] : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by $[X, Y] = XY - YX$.*

One can easily prove that this Lie bracket of vector fields is bilinear, skew-symmetric and satisfies the so called Jacobi identity $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in \mathfrak{X}(M)$, which we can think of as a weaker analogue of associativity. These properties motivate the definition of the following important structure, the Lie algebra.

Definition 4.3.2. *A Lie Algebra is a vector space $\mathfrak{g}$ over a field k together with a binary operation $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying for all $X, Y, Z \in \mathfrak{g}$ and $a, b \in k$, the following properties:*

(i) $[X, aY + bZ] = a[X, Y] + b[X, Z]$ and $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *(bilinearity)*

(ii) $[X, Y] = -[Y, X]$ *(skew-symmetry)*

(iii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ (*Jacobi identity*)

Equivalently we could define a Lie Algebra as a vector space $\mathfrak{g}$ *over a field* k *together with a linear map* $\bigwedge_k^2 \mathfrak{g} \to \mathfrak{g}$*, which satisfies the Jacobi identity. Given two Lie algebras over the same field,* $(\mathfrak{g}, []_g), (\mathfrak{h}, []_h)$*, a Lie algebra homomorphism* $\phi : \mathfrak{g} \to \mathfrak{h}$ *is defined to be a linear map which respects the Lie bracket structure, i.e.* $\phi([a,b]_g) = [\phi(a), \phi(b)]_h$ *for all* $a.b \in \mathfrak{g}$*.*

examples

1. Any vector space V with trivial Lie bracket satisfies the 3 properties trivially and is therefore a Lie algebra. These Lie algebras are called Abelian Lie algebras.

2. Let k be a field and A an associative k-algebra (a vector space with a bilinear operation that is associative). Then the bracket defined by $[a,b] = ab - ba$ for all $a, b \in A$ makes A into a Lie algebra. Bilinearity and skew-symmetry are clear, the Jacobi identity follows from the associativity of A, $[a,[b,c]]+[b,[c,a]]+[c,[a,b]] = a(bc-cb)-(bc-cb)a+b(ca-ac)-(ca-ac)b+c(ab-ba)-(ab-ba)c = abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac = 0$ for all $a, b, c \in A$. We note that the Lie bracket of vector fields is just the particular case $A = \mathfrak{X}(M)$, for M a smooth manifold.

3. Given a vector space V over a field k, it is well known that the vector space of endomorphisms of V, $\mathrm{End}(V)$, forms a unital associative k-algebra under the composition of endomorphisms. If we set $A = \mathrm{End}(V)$ in the previous example we get the important $\mathrm{End}(V)$ Lie algebra. Given a Lie algebra L and a vector space V, both over a common field k, we can now define a Lie algebra representation as a Lie algebra homomorphism $\phi : L \to \mathrm{End}(V)$. An important example of this is the so called adjoint representation $\mathrm{ad} : L \to \mathrm{End}(L)$ of a Lie algebra L on itself, which is given by $a \mapsto (b \mapsto [a,b] := ab - ba)$. Since the Lie bracket $[\,,\,]$ on L is bilinear it follows that ad is well defined and linear. Because for all $a, b, c \in L$ we have $[\mathrm{ad}(a), \mathrm{ad}(b)]'(c) = (\mathrm{ad}(a) \circ \mathrm{ad}(b) - \mathrm{ad}(b) \circ \mathrm{ad}(a))(c) = [a,[b,c]] - [b,[a,c]] = [[a,b],c] = (\mathrm{ad}[a,b])(c)$ it follows that ad is indeed a Lie homomorphism (in the second last equality we have used skew-symmetry and the Jacobi identity).

Definition *Given a Lie algebra* L *the lower central series is defined recursively by* $L_0 = L$ *and* $L_k = [L, L_{k-1}]$ *for all* $k \geq 1$*. A Lie algebra* L *is called nilpotent if there exists a* $n \geq 0$ *such that* $L_n = 0$*.*

Theorem (Engel's theorem) *Given a nilpotent algebra* L *every endomorphism of* L *in the image of the adjoint representation is nilpotent. The converse is in general only true when* L *is finite-dimensional.*

The relevancy of this definition and theorem lies in the fact that for nilpotent Lie algebras L, one can now introduce well-defined invertible operators in $\mathrm{End}(L)$ by $e^{[a,\cdot]} := \sum_{n\geq 0} \frac{\mathrm{ad}(a)^n}{n!}$, for all $a \in L$.

We will now proceed with another topic related to definition 4.3.1. The Lie bracket of vector fields has the following interesting connection with Lie derivatives.

Definition 4.3.3. *Let* M *be a smooth manifold, let* $X, Y \in \mathfrak{X}(M)$ *be smooth vector fields on* M *and let* θ *denote the flow of* X *on* M*. We define the Lie derivative of* Y *with respect to* X *by,*

$$(\mathcal{L}_X Y)_p = \frac{d}{dt}\bigg|_{t=0} d(\theta_{-t})_{\theta_t(p)}(Y_{\theta_t(p)}) \tag{43}$$

More generally, the Lie derivative of an arbitrary smooth tensor field A *on* M *along a smooth vector field* V *on* M*, can be defined.*

Theorem 4.3.4. *Let* M *be a smooth manifold and let* $X, Y \in \mathfrak{X}(M)$*. Then* $\mathcal{L}_X Y = [X, Y]$*.*

4.5 Exterior Derivatives

In elementary analysis the Stokes theorem gave us an easily computable condition for a differentiable vector field in $\mathbb{R}^3$ to be the gradient of a differentiable function. In order to generalize this result for higher dimensions and more generally for any smooth manifold M we need to define the counterpart of the curl for smooth vector fields on smooth manifolds. For this we introduce the exterior derivative $d : \Omega^k(M) \to \Omega^{k+1}(M)$, which as we will see, is an absolutely remarkable object because of its rich properties. We start by defining the exterior derivative on $\mathbb{R}^n$. The exterior derivative there $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, is defined as,

$$\sum_{1\leq i_1<...<i_k\leq n} \omega_{i_1,...,i_k} dx^{i_1} \wedge ... \wedge dx^{i_k} \mapsto \sum_{1\leq i_1<...<i_k\leq n} \left(\sum_{i=1}^{n} \frac{\partial \omega_{i_1,...,i_k}}{\partial x^i} dx^i \right) \wedge dx^{i_1} \wedge ... \wedge dx^{i_k} \tag{44}$$

In the case of $k = 0$, where $\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ we see that,

$$d(f) = \frac{\partial f}{\partial x^i} dx^i \tag{45}$$

Which is precisely the gradient of a smooth function. For 1-forms we get,

$$d\left(\sum_{j=1}^{n} \omega_j dx^j\right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \frac{\partial \omega_j}{\partial x^i} dx^i\right) \wedge dx^j = \sum_{i,j=1}^{n} \frac{\partial \omega_j}{\partial x^i} (dx^i \wedge dx^j) \tag{46}$$

$$= \sum_{1\leq i<j\leq n} \frac{\partial \omega_j}{\partial x^i} (dx^i \wedge dx^j) + \sum_{1\leq j<i\leq n} \frac{\partial \omega_j}{\partial x^i} (dx^i \wedge dx^j) \tag{47}$$

$$= \sum_{1\leq i<j\leq n} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) (dx^i \wedge dx^j) \tag{48}$$

In which we recognize the n-dimensional generalization of the curl.

Now we have defined the exterior derivative on $\mathbb{R}^n$ we can use the local charts on our given smooth manifold M to pull it back into an exterior derivative on M, $d(\omega) = \phi^* d(\phi^{-1*}\omega)$. We can summarize this in the following theorem (theorem 14.24 in [Lee13]).

Theorem 4.5.1. *Suppose M is a smooth manifold with or without boundary. There are unique operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for all k, called exterior differentiation, satisfying the following four properties:*

(i) *d is linear over $\mathbb{R}$.*

(ii) *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.*

(iii) *$d^2 := d \circ d = 0$.*

(iv) *For $f \in \Omega^0(M) = C^\infty(M)$, $d(f)$ is the differential of f, df.*

Definition 4.5.2. *A differential k-form $\omega \in \Omega^k$ is called closed when $d\omega = 0$.*

4.6 Symplectic Manifolds

Let V denote a finite real vector space. Given any 2-covector $\alpha \in V^* \otimes V^*$ we call it nondegenerate if the linear map $\hat{\alpha} : V \to V^*$ defined by $\hat{\alpha}(v) = v \llcorner \omega$ is invertible. Accordingly, a 2-form ω on a smooth manifold M is called nondegenerate if ω_p is nondegenerate for every $p \in M$.

Definition 4.6.1. *A symplectic form on a smooth manifold M is defined to be a closed nondegenerate 2-form. A smooth manifold M along with a specific choice of symplectic form ω is called a symplectic manifold and is most commonly denoted by (M, ω). Given a symplectic manifold (M, ω) and any smooth function $f \in C^\infty(M)$, we define the Hamiltonian vector field of f to be the smooth vector field X_f defined by $X_f = \hat{\omega}^{-1}(df)$, where $\hat{\omega} : TM \to T^*M$ is the bundle isomorphism given by $v \mapsto v \llcorner \omega$.*

4.7 Poisson Manifolds

It is well known that in mathematical physics the mathematical description of the physical concept of (phase)space is modelled as a smooth manifold. In Hamiltonian mechanics, the laws of motion give rise to the Poisson bracket, which we already came across in the first chapter. It is therefore very natural to define a more general structure that still has the same important properties as that particular Poisson bracket, in order to model more general physical systems.

Definition 4.7.1. *A Poisson manifold is a smooth manifold M together with a Poisson bracket $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, which is a Lie bracket on $C^\infty(M)$ which additionally is a derivation in each argument, i.e. $\{fg, h\} = f\{g, h\} + g\{f, h\}$ for all $f, g, h \in C^\infty(M)$. The pair $(C^\infty(M), \{\ ,\ \})$ is called a Poisson algebra.*

examples

1. In the first chapter we defined the standard Poisson bracket $\{,\} : C^\infty(\mathbb{R}^{2n}) \times C^\infty(\mathbb{R}^{2n}) \to C^\infty(\mathbb{R}^{2n})$ by,
$$\{f, g\} = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right) \quad \forall f, g \in C^\infty(\mathbb{R}^{2n}). \tag{49}$$
Proving that this binary operator is indeed a Poisson bracket is a trivial exercise.

2. Any symplectic manifold (M, ω) induces a corresponding Poisson bracket defined by the equality $\{f, g\} = \omega(X_f, X_g) = df(X_g) = X_g f$ for all $f, g \in C^\infty(M)$. We will now prove that this is indeed a Poisson bracket.
   - (i) (Bilinearity): Writing $\{f, g\} = \omega(\hat{\omega}^{-1}(df), \hat{\omega}^{-1}(dg))$ we see that the bilinearity of $\{\ ,\ \}$ follows from the linearity of the differential, the linearity of $\hat{\omega}^{-1}$ and the bilinearity of ω.
   - (ii) (Skew-symmetry): ω is by definition alternating.
   - (iii) (Jacobi-identity): For all $f, g, h \in C^\infty(M)$ we have $\{f, \{g, h\}\} = X_{\{g,h\}} f = -[X_g, X_h] f = -X_g X_h f + X_h X_g f = -X_g\{f, h\} + X_h\{f, g\} = -\{\{f, h\}, g\} + \{\{f, g\}, h\} = -\{g, \{h, f\}\} - \{h, \{f, g\}\}$, where we have
   - (iv) (Leibniz's rule): From the equalities $\{f, g\} = df(X_g)$ and $\{f, g\} = -\{g, f\} = dg(X_f)$ and the properties of the differential it follows that this Lie bracket is a derivation in each argument.

We are now going to introduce an alternative formulation of the Poisson bracket and prove that the two defintions are in fact equivalent. Before we proceed we need to introduce the following definitions.

Definition 4.7.2.*Let M be a smooth manifold. The Schouten-Nijenhuis bracket on M,*
$[\ ,\ ] : \mathfrak{X}^k(M) \otimes \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M)$ is a linear map defined on pure products by,
$[X_1, ..., X_k, Y_1, ..., Y_l] = \sum_{i=1}^{k} \sum_{j=1}^{l} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge ... \wedge \widehat{X_i} \wedge ... \wedge X_k \wedge Y_1 \wedge ... \wedge \widehat{Y_j} \wedge ... \wedge Y_l.$

Definition 4.7.3. *A Poisson bivector on a smooth manifold M is a bivector field $\pi \in \mathfrak{X}^2(M)$ such that $[\pi, \pi] = 0$. Such a Poisson bivector induces a binary operator $\{\ ,\ \}_\pi : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ given by $\{f, g\}_\pi = \pi(df, dg)$. In local coordinates $\pi = \pi^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}$ we have $\{f, g\} = \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$.*

Proposition 4.7.4. *Let M be a smooth manifold M. Given a Poisson bivector π on M the induced binary operator $\{\ ,\ \}_\pi$ is a Poisson bracket on M. Conversely, given a Poisson bracket $\{\ ,\ \}$ on M there exists a Poisson bivector π on M such that $\{\ ,\ \}_\pi = \{\ ,\ \}$.*

Proof. Let π be a Poisson bivector on M. We need to prove that $\{\ ,\ \}_\pi$ is a Poisson bracket.

- (i) (Bilinearity): From the linearity of the differential and bilinearity of π the bilinearity of $\{\ ,\ \}_\pi$ follows.
- (ii) (Skew-symmetry): π is by definition alternating.

(iii) (Jacobi-identity): For all $f, g, h \in C^\infty(M)$ we have,

$$\{f,\{g,h\}\}+\{g,\{h,f\}\}+\{h,\{f,g\}\} \tag{50}$$
$$=\pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}\left(\pi^{kl}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^l}\right)+\pi^{ij}\frac{\partial g}{\partial x^i}\frac{\partial}{\partial x^j}\left(\pi^{kl}\frac{\partial h}{\partial x^k}\frac{\partial f}{\partial x^l}\right)+\pi^{ij}\frac{\partial h}{\partial x^i}\frac{\partial}{\partial x^j}\left(\pi^{kl}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^l}\right) \tag{51}$$
$$=\pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \pi^{kl}}{\partial x^j}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j\partial x^k}\frac{\partial h}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^j\partial x^l} \tag{52}$$
$$+\pi^{ij}\frac{\partial g}{\partial x^i}\frac{\partial \pi^{kl}}{\partial x^j}\frac{\partial h}{\partial x^k}\frac{\partial f}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial g}{\partial x^i}\frac{\partial h}{\partial x^j\partial x^k}\frac{\partial f}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial g}{\partial x^i}\frac{\partial h}{\partial x^k}\frac{\partial f}{\partial x^j\partial x^l} \tag{53}$$
$$+\pi^{ij}\frac{\partial h}{\partial x^i}\frac{\partial \pi^{kl}}{\partial x^j}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial h}{\partial x^i}\frac{\partial f}{\partial x^j\partial x^k}\frac{\partial g}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial h}{\partial x^i}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^j\partial x^l} \tag{54}$$
$$=\pi^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \pi^{kl}}{\partial x^j}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j\partial x^k}\frac{\partial h}{\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^j\partial x^l} \tag{55}$$
$$+\pi^{ij}\frac{\partial g}{\partial x^i}\frac{\partial \pi^{kl}}{\partial x^j}\frac{\partial h}{\partial x^k}\frac{\partial f}{\partial x^l}-\pi^{ij}\pi^{kl}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^j\partial x^l}+\pi^{ij}\pi^{kl}\frac{\partial g}{\partial x^i}\frac{\partial h}{\partial x^k}\frac{\partial f}{\partial x^j\partial x^l} \tag{56}$$
$$+\pi^{ij}\frac{\partial h}{\partial x^i}\frac{\partial \pi^{kl}}{\partial x^j}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^l}-\pi^{ij}\pi^{kl}\frac{\partial g}{\partial x^i}\frac{\partial h}{\partial x^k}\frac{\partial f}{\partial x^j\partial x^l}-\pi^{ij}\pi^{kl}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j\partial x^k}\frac{\partial h}{\partial x^l} \tag{57}$$
$$=\pi^{ij}\frac{\partial \pi^{kl}}{\partial x^j}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^l}+\pi^{kj}\frac{\partial \pi^{li}}{\partial x^j}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^l}+\pi^{lj}\frac{\partial \pi^{ik}}{\partial x^j}\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^l} \tag{58}$$
$$=\left(\pi^{ij}\frac{\partial \pi^{kl}}{\partial x^j}+\pi^{kj}\frac{\partial \pi^{li}}{\partial x^j}+\pi^{lj}\frac{\partial \pi^{ik}}{\partial x^j}\right)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^k}\frac{\partial h}{\partial x^l}. \tag{59}$$

From we immediately see that this sum is zero for all $f, g, h \in C^\infty(M)$ if and only if,

$$0=\pi^{ij}\frac{\partial \pi^{kl}}{\partial x^j}+\pi^{kj}\frac{\partial \pi^{li}}{\partial x^j}+\pi^{lj}\frac{\partial \pi^{ik}}{\partial x^j}=[\pi,\pi]. \tag{60}$$

We therefore conclude that the Jacobi-identity is satisfied if and only if $[\pi, \pi] = 0$, which holds by definition of the Poisson bivector.

(iv) (Leibniz's rule): From the properties of the differential and bilinearity of π it follows that this Lie bracket is a derivation in each argument.

Now let $\{\ ,\ \}$ be a Poisson bracket on M. Given a smooth chart $(U, (x^i))$ on M we note that $\{x^k, x^l\} = \pi^{ij}\frac{\partial f}{\partial x^k}\frac{\partial g}{\partial x^l} = \pi^{ij}\delta^k_i\delta^l_j = \pi^{kl}$. We furthermore note that $\pi^{kl} = \{x^k, x^l\} = -\{x^l, x^k\} = -\pi^{lk}$. So on $U \subseteq M$ we can define the smooth bivector field $\pi \in \mathfrak{X}^2(M)$ with components $\pi^{ij} = \{x^i, x^j\}$. One can then prove that $\{f, g\} = \pi(df, dg)$ for all $f, g \in C^\infty(M)$. Because the Poisson bracket $\{\ ,\ \}$ satisfies the Jacobi-identity we already have seen that the identity $[\pi, \pi] = 0$ holds. We conclude that π is a Poisson bivector. This concludes the proof.

5 The Star Product

In the third chapter we came very close to defining general star products, but we were prevented by our lack of knowledge in the field of differential geometry. Now we have discussed the relevant definitions and theorems in the previous chapter, we can finally introduce the star product. The motivation for the definition of the star product can easily be seen to come from the properties of the Moyal product, see proposition 3.4.3.

Definition 5.0.1. ([F+76]) *Given a smooth m-manifold M, we define a star product $\star$ on M to be a unitary associative $\mathbb{R}[\![\hbar]\!]$-bilinear map on $C^\infty(M)[\![\hbar]\!]$, which for all $f, g \in C^\infty(M)$ is of the form,*

$$(f, g) \mapsto f \star g := fg + \sum_{n\geq 1} \hbar^n B_n(f, g), \tag{61}$$

where the B_n denote bidifferential operators on $C^\infty(M)$ of globally bounded order. More concretely, given any smooth manifold M, a star-product on M, $\star : C^\infty(M)[\![\hbar]\!] \times C^\infty(M)[\![\hbar]\!] \to C^\infty(M)[\![\hbar]\!]$ is a $\mathbb{R}[\![\hbar]\!]$-bilinear map on $C^\infty(M)[\![\hbar]\!]$ defined by the following properties (on $C^\infty(M)$ and therefore uniquely on $C^\infty(M)[\![\hbar]\!]$ since the star-product is $\mathbb{R}[\![\hbar]\!]$-bilinear):

(i) For all $f \in C^\infty(M)$ we have $f \star 1 = 1 \star f = f$.

(ii) For all $f, g, h \in C^\infty(M)$ we have $f \star (g \star h) = (f \star g) \star h$.

(iii) For all $f, g \in C^\infty(M)$ we have $f \star g := fg + \sum_{n\geq 1} \hbar^n B_n(f, g)$, where for all $n \geq 1$ the $B_n : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ are bilinear maps that additionally are differential operators on M in each argument. So for any $n \geq 1$ and smooth charts on M we can represent B_n locally as,

$$B_n(f, g) = \sum_{k_1,\dots,l_m} A_n^{k_1,\dots,k_m,l_1,\dots,l_m} \partial_1^{k_1} \dots \partial_m^{k_m} f \, \partial_1^{l_1} \dots \partial_m^{l_m} g, \tag{62}$$

where the $A_n^{k_1,\dots,k_m,l_1,\dots,l_m}$ are smooth functions on M for which only finitely many are non-zero.

Examples

1. We recall the Moyal product which was defined on the symplectic manifold $\mathbb{R}^{2n}$ by,

$$(f \star g)(\boldsymbol{x}, \boldsymbol{p}) = f(\boldsymbol{x}, \boldsymbol{p}) \exp\left(i\frac{\hbar}{2} \left(\overleftarrow{\partial_{\mathbf{x}}}\overrightarrow{\partial_{\mathbf{p}}} - \overleftarrow{\partial_{\mathbf{p}}}\overrightarrow{\partial_{\mathbf{x}}} \right) \right) g(\boldsymbol{x}, \boldsymbol{p}). \tag{63}$$

   The fact that this is a star product on $\mathbb{R}^{2n}$, follows directly from proposition 3.4.3.

2. More generally one can define on $\mathbb{R}^{2n}$ the star product,

$$(f \star g)(x) = \exp\left(i\frac{\hbar}{2} \alpha^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) f(x) g(y)|_{y=x} \tag{64}$$

   where α denotes a constant alternating (skew-symmetric) tensor field on $\mathbb{R}^{2n}$, with components α^{ij} with respect to the canonical chart. We will now prove that this is indeed a star product. It is first of all clear that this operator is well-defined on $C^\infty(\mathbb{R}^{2n})$ and $\mathbb{R}[\![\hbar]\!]$-linear.
   (i) For all $f \in C^\infty(\mathbb{R}^{2n})$ we have $(f \star 1)(x) = \exp\left(i\frac{\hbar}{2} \alpha^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) f(x)|_{y=x} = 1 \cdot f(x) = f(x)$ and $(1 \star f)(x) = \exp\left(i\frac{\hbar}{2} \alpha^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) f(y)|_{y=x} = 1 \cdot f(y)|_{y=x} = f(x)$.
   (ii) For all $f, g, h \in C^\infty(\mathbb{R}^{2n})$ we have $((f \star g) \star h) = \exp\left(i\frac{\hbar}{2} \alpha^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial z^j} \right) (f \star g)(x) h(z)|_{z=x} =$
   $\exp\left(i\frac{\hbar}{2} \alpha^{ij} (\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^j}) \frac{\partial}{\partial z^j} \right) \exp\left(i\frac{\hbar}{2} \alpha^{kl} \frac{\partial}{\partial x^k} \frac{\partial}{\partial y^l} \right) f(x) g(y) h(z)|_{z=y=x} =$
   $\exp\left(i\frac{\hbar}{2} \alpha^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial z^j} + \alpha^{kl} \frac{\partial}{\partial y^k} \frac{\partial}{\partial z^l} + \alpha^{mn} \frac{\partial}{\partial x^m} \frac{\partial}{\partial y^n} \right) f(x) g(y) h(z)|_{z=y=x} =$
   $\exp\left(i\frac{\hbar}{2} \alpha^{ij} (\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^j}) \frac{\partial}{\partial z^j} \right) \exp\left(i\frac{\hbar}{2} \alpha^{kl} \frac{\partial}{\partial y^k} \frac{\partial}{\partial z^l} \right) f(x) g(y) h(z)|_{z=y=x} =$
   $\exp\left(i\frac{\hbar}{2} \alpha^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) f(x) (g \star h)(y)|_{y=x} = (f \star (g \star h))(x)$
   (iii) The third and last property is clearly satisfied because of the properties of derivations.
   This concludes the proof.

Before we continue any further, it is relevant to remark that the previous definition relies on the more elementary definition of a formal deformation of an algebra, which was given by Gerstenhaber in 1963 [Ger63].

Definition 5.0.2. *Let A be a unital associative algebra over a commutative ring R. A formal deformation of the algebra A, is a $R[\![\hbar]\!]$-bilinear map $\star : A[\![\hbar]\!] \times A[\![\hbar]\!] \to A[\![\hbar]\!]$ such that,*

$$a \star b = ab + \sum_{n\geq 1} \hbar^n B_n(a,b), \quad \forall a, b \in A, \tag{65}$$

where the $B_n : A \times A \to A$ are R-bilinear maps and the product is furthermore associative.

5.1 Classification of Star Products

We will now generalize proposition 3.4.3.(iv) for general star products.

Proposition 5.1.1. *For any star product $\star$ on a smooth manifold M, the map $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ defined by $\{f, g\} \mapsto B_1(f, g) - B_1(g, f)$ is a Poisson bracket, making M into a Poisson manifold.*

Proof. We simply need to check all the requirements. (i) $\{\ ,\ \}$ is bilinear as a linear combination of the bilinear map B_1. (ii) Skew-symmetry is trivially satisfied. (iii) For all $f, g, h \in C^\infty(M)$ we have $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = B_1(f, B_1(g, h) - B_1(h, g)) - B_1(B_1(g, h) - B_1(h, g), f) + B_1(g, B_1(h, f) - B_1(f, h)) - B_1(B_1(h, f) - B_1(f, h), g) + B_1(h, B_1(f, g) - B_1(g, f)) - B_1(B_1(f, g) - B_1(g, f), h) = B_1(f, B_1(g, h)) - B_1(f, B_1(h, g)) - B_1(B_1(g, h), f) + B_1(B_1(h, g), f) + B_1(g, B_1(h, f)) - B_1(g, B_1(f, h)) - B_1(B_1(h, f), g) + B_1(B_1(f, h), g) + B_1(h, B_1(f, g)) - B_1(h, B_1(g, f)) - B_1(B_1(f, g), h) + B_1(B_1(g, f), h) = 0$, which proves the Jacobi identity. (iv) The Leibniz rule is satisfied since B_1 is a differential operator in each of its arguments. We conclude that $\{\ ,\ \}$ is indeed a Poisson bracket.

This observation is all fine and well, but it is precisely the converse that we are interested in. Given a Poisson manifold $(M, \{\ ,\ \})$ we want to construct a star product $\star$ on M such that its induced Poisson bracket is equal to the original one, $(f \star g - g \star f)/\hbar = \{f, g\} \bmod \hbar$ for all $f, g \in C^\infty(M)$. Before this problem can be tackled, we need to define two equivalence relations.

Definition 5.1.2. *Given a smooth manifold M and two-star products $\star$, $\star'$ on M, we say that they are equivalent if and only if there exists a power series $T = id + \sum_{n\geq 1} \hbar^n T_n$, with the T_n being linear operators on $C^\infty(M)$, such that $T(f \star g) = T(f) \star' T(g)$ holds for all $f, g \in C^\infty(M)$.*

Proposition 5.1.3. *Let $\star$ be a star product on a smooth manifold M. The induced Poisson bracket depends only on the equivalence class $[\star]$. And is therefore a well-defined property of such a class.*

Proof. Let $\star, \star'$ be equivalent star products on M. Then for all $f, g \in C^\infty(M)$ we have the consecutive equalities $fg + \hbar(B_1(f, g) + T_1(fg)) + \mathcal{O}(\hbar^2) = T(fg + \hbar B_1(f, g) + \mathcal{O}(\hbar^2)) = T(f \star g) = T(f) \star' T(g) = (f + \hbar T_1(f) + \mathcal{O}(\hbar^2)) \star' (g + \hbar T_1(g) + \mathcal{O}(\hbar^2)) = fg + \hbar(fT_1(g) + gT_1(f) + B_1'(f, g)) + \mathcal{O}(\hbar^2)$, and therefore $B_1(f, g) + T_1(fg) = fT_1(g) + gT_1(f) + B_1'(f, g)$. From which for all $f, g \in C^\infty(M)$ follows, $\{f, g\} = B_1(f, g) - B_1(g, f) = fT_1(g) + gT_1(f) + B_1'(f, g) - T_1(fg) - gT_1(f) - fT_1(g) - B_1'(g, f) + T_1(gf) = B_1'(f, g) - B_1'(g, f) = \{f, g\}'$. This completes the proof.

Definition 5.1.4. *Given a smooth manifold M and a collection $(\pi_i)_{i\geq 0}$ of Poisson bivector fields on M, we consider the formal sum in $\hbar$ of the form $\pi_\hbar := \sum_{i\geq 0} \pi_i \hbar^i \in \mathfrak{X}^2(M)[\![\hbar]\!]$, called a formal Poisson structure. Any such formal Poisson structure induces a Lie bracket $\{\ ,\ \} : C^\infty(M)[\![\hbar]\!] \times C^\infty(M)[\![\hbar]\!] \to C^\infty(M)[\![\hbar]\!]$ defined by, $\{f, g\}_\hbar = \sum_{n\geq 0} \hbar^n \sum_{i,j,k,=0 \text{ and } i+j+k=n}^{n} \pi_i(df_j, dg_k)$ for all $f = \sum_{j\geq 0} \hbar^j f_j \in C^\infty(M)[\![\hbar]\!]$ and $g = \sum_{k\geq 0} \hbar^k g_k \in C^\infty(M)[\![\hbar]\!]$.*

Definition 5.1.5. *Given two formal Poisson structures $\pi_\hbar$ and $\pi'_\hbar$ on a smooth manifold M, we say that they are equivalent, if and only if there exists a formal smooth vector field $X = \sum_{m\geq 0}^{\infty} \hbar^m X_m$ on M, such that $\pi'_\hbar := \sum_{n\geq 0} \hbar^n \sum_{i,j,k,=0 \text{ and } i+j+k=n}^{n} (\mathcal{L}_{X_i})^j \pi_k$.*

The most important corollary of Kontsevich's paper can now be formulated as follows.

Theorem 5.1.6. *([Kon03]) Let M be a smooth manifold. There exists a bijection between on the one hand, equivalence classes of formal Poisson structures $[\pi_\hbar]$, with $\pi_0 = 0$, and on the other hand, the equivalence classes of star products $[\star]$. This bijection is furthermore natural with respect to diffeomorphisms.*

In particular this means that star products always exist on Poisson manifolds. Because given a Poisson manifold (M, π). We can write $\star$ for the star product on M, corresponding to the formal Poisson structure $\pi_\hbar := \hbar\pi$. And then it follows from proposition 5.1.1. that the induced Poisson bracket is equal to the original Poisson structure. As stated, this theorem is a consequence of a more general one, called the formality theorem. In order to state this theorem we need to define a whole range of graded algebraic structures. We therefore begin with a short overview of some basic graded structures. And continue thereafter with some

5.2 Graded Algebraic Structures

Before we proceed to give any important definition, it is probably wise to give our definition of a ring R and an R-algebra, since many conventions have been used. A ring R will be defined as a set R along with two binary operations on R, addition $+$ and multiplication $\cdot$, such that the following properties hold: R is an abelian group w.r.t. the addition; R is a monoid w.r.t. multiplication and the multiplication is left and right distributive w.r.t. the addition. Given a commutative ring R we define an R-algebra A to be a R-module A together with a R-bilinear map on A, which is written multiplicatively. If this multiplication is furthermore associative, we say that A is an associative R-algebra. And when this multiplication has an identity element, we say that A is a unital R-algebra.

Definition 5.2.1. *Let G be a commutative monoid. A G-graded ring is a ring R, along with a direct sum decomposition indexed by G, $R = \bigoplus_{g\in G} R_g$, such that $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. Given a $r \in R_d$ for some $d \in G$, we say that r is a homogeneous element of degree d. A morphism between two G-graded rings $R = \bigoplus_{g\in G} R_g$ and $S = \bigoplus_{g\in G} S_g$, $f : R \to S$, is a ring homomorphism which respects the graded structure, i.e. $f(R_g) \subseteq S_g$ for all $g \in G$.*

Definition 5.2.2. *Let G be a commutative monoid and let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. A G-graded R-module is a R-module M, along with a direct sum decomposition indexed by G, $M = \bigoplus_{g\in G} M_g$, such that $R_g M_h \subseteq M_{g+h}$ for all $g, h \in G$. Given a $m \in M_d$ for some $d \in G$, we say that m is a homogeneous element of degree d. A morphism between two G-graded R-modules $M = \bigoplus_{g\in G} M_g$ and $N = \bigoplus_{g\in G} N_g$, $f : M \to N$, is a R-module homomorphism which respects the graded structure, i.e. $f(M_g) \subseteq N_{g+d}$ for all $g \in G$, where $d \in G$. We say that f is of degree d.*

Definition 5.2.3. *Let G be a commutative monoid. A G-graded R-algebra is an R-algebra A, along with a direct sum decomposition indexed by G, $A = \bigoplus_{g\in G} A_g$, such that $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$. Given an $a \in A_d$ for some $d \in G$, we say that a is a homogeneous element of degree d. A morphism between two G-graded R-algebras $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$, $f : A \to B$, is an algebra homomorphism which respects the graded structure, i.e. $f(A_g) \subseteq B_g$ for all $g \in G$.*

Definition 5.2.4. *Let R be a commutative ring, and let M be a R-module. We employ the following notation: $T^0(M) = R$, $T^n(M) := M^{\otimes_R n}$ for all $n \geq 1$ and $T(M) := \bigoplus_{n\geq 0} T^n(M)$. By extending the canonical isomorphisms $T^k(M) \otimes_R T^l(M) \to T^{k+l}(M)$ linearly onto $T(M)$, $T(M)$ is made into a $\mathbb{N}$-graded unitary associative algebra, called the tensor algebra of M over R. In a similar fashion we can define the reduced tensor algebra of M over R as the $\mathbb{N}_{\geq 1}$-graded associative algebra $\overline{T}(M) := \bigoplus_{n\geq 1} T^n(M)$.*

Definition 5.2.5. *Let R be a commutative ring, and let M be a R-module. By extending the canonical homomorphisms $\bigwedge^k(M) \wedge_R \bigwedge^l(M) \to \bigwedge^{k+l}(M)$ linearly onto $\bigwedge(M) := \bigoplus_{n\geq 0} \bigwedge^n(M)$, $\bigwedge(M)$ is made into a $\mathbb{N}$-graded unitary associative algebra, called the exterior algebra of M over R. By instead extending these homomorphisms onto $\overline{\bigwedge}(M) := \bigoplus_{n\geq 1} \bigwedge^n(M)$, $\overline{\bigwedge}(M)$ is made into a $\mathbb{N}_{n\geq 1}$-graded associative algebra, called the reduced exterior algebra of M over R.*

5.3 DGLA's

In the following sections we will, as already announced, introduce many important definitions in order to formulate the formality theorem. Many of these definitions can be found in [Esp14] and [Kel03].

Definition 5.3.1. *A graded Lie algebra (GLA) is a vector space* $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i$ *together with a linear operator* $[\,,\,] : \mathfrak{g}\otimes\mathfrak{g} \to \mathfrak{g}$ *satisfying the following conditions for all* $a \in \mathfrak{g}^\alpha$, $b \in \mathfrak{g}^\beta$ *and* $c \in \mathfrak{g}^\gamma$*:*

(i) $[a, b] \in \mathfrak{g}^{\alpha+\beta}$, *(homogeneity)*

(ii) $[a, b] = -(-1)^{\alpha\beta}[b, a]$, *(graded skew-symmetry)*

(iii) $(-1)^{\alpha\gamma}[a, [b, c]] + (-1)^{\alpha\beta}[b, [c, a]] + (-1)^{\beta\gamma}[c, [a, b]] = 0$. *(graded Jacobi identity)*

More concisely, one can define a GLA as a $\mathbb{Z}$*-graded* k*-algebra* $\mathfrak{g}$ *satisfying graded skew-symmetry and the graded Jacobi identity, where* k *denotes a field as always. A morphism of GLA's is then simply a morphism of* $\mathbb{Z}$*-graded* k*-algebras. Or stated more explicit, given GLA's* $(\mathfrak{g}, [])$ *and* $(\mathfrak{h}, []')$*, a GLA-morphism of degree* k*,* $\phi : \mathfrak{g} \to \mathfrak{h}$*, is a linear map such that* $\phi(\mathfrak{g}^i) \subseteq \mathfrak{h}^{i+k}$ *for all* $i \in \mathbb{Z}$ *and satisfies* $\phi([a, b]) = [\phi(a), \phi(b)]'$ *for all* $a, b \in \mathfrak{g}$*.*

Definition 5.3.2. *A differential graded Lie Algebra (DGLA) is a GLA* $(\mathfrak{g}, [\,,\,])$ *together with a differential* $d : \mathfrak{g} \to \mathfrak{g}$*, i.e. a linear operator of degree 1,* $d(\mathfrak{g}^i) \subseteq \mathfrak{g}^{i+1}$ *for all* $i \in \mathbb{Z}$*, which satisfies the Leibniz rule* $d[a, b] = [da, b] + (-1)^\alpha[a, db]$ *for all* $a \in \mathfrak{g}^\alpha$*,* $b \in \mathfrak{g}^\beta$ *and squares to zero* $d \circ d = 0$*.* DGLA morphisms are naturally defined as GLA morphisms that furthermore respect the differential structure. More explicitly stated, given DGLA's $(\mathfrak{g}, [], d)$ and $(\mathfrak{h}, []', d')$ a DGLA morphism is a GLA morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ which furthermore satisfies the property $\phi(d(a)) = d'(\phi(a))$ for all $a \in \mathfrak{g}$.

Examples

(i) Any Lie algebra can be considered a GLA when concentrated in degree 0.
Conversely, for any GLA $\mathfrak{g}$, $\mathfrak{g}^0$ is a Lie algebra.

(ii) For any GLA $\mathfrak{g}$, $\mathfrak{g}^{\text{even}} := \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^{2i}$ is a Lie algebra.

(iii) Any GLA can trivially be made into a DGLA by choosing the trivial zero differential.

It is clear that any DGLA $(\mathfrak{g}, [], d)$ has a natural associated cohomology complex. The chain groups are, as always, defined by $H^i(\mathfrak{g}) := \ker(d : \mathfrak{g}^i \to \mathfrak{g}^{i+1})/\mathrm{im}(d : \mathfrak{g}^{i-1} \to \mathfrak{g}^i)$ for all $i \in \mathbb{Z}$ and the differentials between them are just the ones canonically induced by d. Additionally, the set $H(\mathfrak{g}) := \oplus_{\in\mathbb{Z}} H^i(\mathfrak{g})$ has the natural structure of a GLA, as we will prove in the following proposition.

Proposition 5.3.3. *Let* $(\mathfrak{g}, [])$ *be a DGLA, then* $H(\mathfrak{g})$ *becomes a GLA when endowed with the operator* $[\,,\,]' : H(\mathfrak{g}) \otimes H(\mathfrak{g}) \to H(\mathfrak{g})$ *defined by* $[\overline{a}, \overline{b}]' \mapsto \overline{[a, b]}$*.*

Proof. We first need to check whether the operation $[\,,\,]'$ is well defined. Let $\overline{a}, \overline{c} \in H(\mathfrak{g})^i$ and $\overline{b}, \overline{d} \in H(\mathfrak{g})^j$ with $\overline{a} = \overline{c}$ and $\overline{b} = \overline{d}$. So there exist $e \in H(\mathfrak{g})^{i-1}$ and $f \in H(\mathfrak{g})^{j-1}$ such that $d(e) = a - c$ and $d(f) = b - d$. We thus find $[a, b] = [d(e) + c, d(f) + d] = [d(e), d(f)] + [d(e), d] + [c, d(f)] + [c, d] = d[e, d(f)] + (d[e, d] + (-1)^i \underline{[e, d(d)]}) + (-1)^i(d[c, f] - [d(c), f]) + [c, d] = d[e, d(f)] + d[e, d] + (-1)^i d[c, f] + [c, d]$ and therefore $[\overline{a}, \overline{b}]' = \overline{[a, b]} = \overline{[c, d]} = [\overline{c}, \overline{d}]'$. For the more general case when $a, b, c, d \in H(\mathfrak{g})$ and $\overline{a} = \overline{c}$ and $\overline{b} = \overline{d}$, we write each element out like $a = \sum_{i\in\mathbb{Z}} \lambda_i a_i$ for finitely many non-zero λ_i's in the underlying field and $a_i \in H^i(\mathfrak{g})$, from which can easily be worked out that the equality $[\overline{a}, \overline{b}]' = [\overline{c}, \overline{d}]'$ still holds. This proves the independence of representatives and therefore we conclude that the operator $[\,,\,]'$ is indeed well defined. Apart from proving that this bracket is well-defined, it is actually quite clear that $H(\mathfrak{g})$ satisfies the 3 properties. We conclude that ($H(\mathfrak{g})$,[,]') is a GLA.

Because any morphism $\phi : \mathfrak{g} \to \mathfrak{h}$ of DGLA's commutes with the corresponding brackets and differentials it is obvious that it induces a morphism $H(\phi) : H(\mathfrak{g}) \to H(\mathfrak{h})$ between cohomologies.

Definition 5.3.4. *When there exists a morphism* $\phi : \mathfrak{g} \to \mathfrak{h}$ *of DGLA's, that induces an isomorphism between cohomologies,* $H(\phi) : H(\mathfrak{g}) \to H(\mathfrak{h})$ *, we say that the DGLA's* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are quasi-isomorphic. The reason for this terminology is because given a quasi-isomorphism* $\phi : \mathfrak{g} \to \mathfrak{h}$*, an inverse may not* *exist. In the particular case of a DGLA $\mathfrak{g}$ being quasi-isomorphic to its own cohomology $H(\mathfrak{g})$ (regarded as DGLA with zero differential) we call the DGLA $\mathfrak{g}$ formal.*

In the previous definition we claimed that quasi-isomorphisms in general do not have an inverse. There however does exist a broader category that has more suitable properties. In particular DGLA's will be replaced with L_∞-algebras and quasi-isomorphisms by L_∞-morphisms. In order to define these new structures we need to introduce the graded coalgebra structure.

Definition 5.3.5. *A graded coalgebra (GCA) over a field $\mathbb{K}$ is a vector space $\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{h}^i$ over K endowed with a comultiplication, i.e. a graded linear map $\Delta : \mathfrak{h} \to \mathfrak{h} \boxtimes \mathfrak{h}$ (we use the $\boxtimes$ symbol to discern this external tensor product from the internal tensor product that we will later define on $\mathfrak{h}$), so a linear map such that $\Delta(\mathfrak{h}^i) \subseteq \bigoplus_{j+k=i} \mathfrak{h}^j \boxtimes \mathfrak{h}^k$ for all $i \in \mathbb{Z}$, which additionally satisfies the coassociativity condition $(\Delta \boxtimes id) \circ \Delta = (id \boxtimes \Delta) \circ \Delta$. It is said to a GCA with counit if there exists a linear map $\varepsilon : \mathfrak{h} \to \mathbb{K}$ such that $\varepsilon(\mathfrak{h}^i) = 0$ for any $i > 0$ and $(\varepsilon \boxtimes id) \circ \Delta = (id \boxtimes \varepsilon) \circ \Delta = id$ holds. The CGA $\mathfrak{h}$ is said to be cocommutative if $T \circ \Delta = \Delta$, where $T : \mathfrak{h} \boxtimes \mathfrak{h} \to \mathfrak{h} \boxtimes \mathfrak{h}$ is the twisting map, defined on a product of homogeneous elements of degree respectively d_x and d_y by $T(x \boxtimes y) = (-1)^{d_x d_y} y \boxtimes x$ and extended linearly.*

Definition 5.3.6. *Given two GCA's $(\mathfrak{h}, \Delta)$ and $(\mathfrak{h}', \Delta')$, a graded coalgebra morphism of degree k, $F : \mathfrak{h} \to \mathfrak{h}'$, is a graded linear map which moreover commutes with the comultiplications, i.e. a linear map $\phi : \mathfrak{h} \to \mathfrak{h}'$ such that $\phi(\mathfrak{h}^i) \subseteq \mathfrak{h}'^{i+k}$ for all $i \in \mathbb{Z}$ and we furthermore have the equality $(F \boxtimes F) \circ \Delta = \Delta' \circ F$. In the context of GCA's with counit, say ε is the counit in $\mathfrak{h}$ and ε' is the counit in $\mathfrak{h}'$, we would furthermore demand the equality $\varepsilon' \circ F = \varepsilon$.*

As a first example of a graded coalgebra we will endow tensor algebras over vector spaces, $T(V)$, with the following coalgebra structure Δ_T, which is defined on the homogeneous elements by,

$$\Delta_T(v_1 \otimes ... \otimes v_n) = 1 \boxtimes (v_1 \otimes ... \otimes v_n) + \sum_{j=1}^{j=n-1} (v_1 \otimes ... \otimes v_j) \boxtimes (v_{j+1} \otimes ... \otimes v_n) + (v_1 \otimes ... \otimes v_n) \boxtimes 1 \quad (66)$$

In this case, we also have the counit ε_T, the canonical projection $\varepsilon_T : T(V) \to V^{\otimes 0} = \mathbb{K}$. Similarly the reduced tensor algebra $\overline{T}(V)$ is a GCA without counit.
In $T(V)$ we define the ideal I_S to be generated by $v \otimes w - T(v \otimes w)$, and the ideal $I_\wedge$ to be generated by $v \otimes w + T(v \otimes w)$, where v, w are the homogeneous elements in $T(V)$. These ideals give rise to the following important unital associative algebras, the symmetric algebra $S(V) := T(V)/I_S$ and the exterior algebra $S(V) := T(V)/I_\wedge$. By swapping $T(V)$, from the beginning, with the reduced tensor algebra, $\overline{T}(V)$, we construct likewise the reduced symmetric algebra, $\overline{S}(V)$, and the reduced exterior algebra $\overline{\bigwedge}(V)$. Since for all homogeneous elements $v, w \in T(V)$, $T(v \boxtimes w)$ is of the same degree as $v \boxtimes w$, it is clear that the comultiplication on $T(V)$ and $\overline{T}(V)$ induce a comultiplications on all these newly defined algebras. Or more concretely, $S(V)$ and $\bigwedge(V)$ become GCA's with counit and $\overline{S}(V)$ and $\overline{\bigwedge}(V)$ become GCA's without counit.

By viewing the comultiplication on GCA's as the dual of the bracket of DGLA's, it is now natural to define the dual of the differential in the context of GCA's.

Definition 5.3.7 *Let $(\mathfrak{h} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{h}^i, \Delta)$ be a GCA. We define a coderivation of degree k on $\mathfrak{h}$ to be a graded linear map of degree k, $\delta : \mathfrak{h} \to \mathfrak{h}$, satisfying the property that,*

$$\Delta \circ \delta = (\delta \boxtimes id) \circ \Delta + ((-1)^{ki} id \boxtimes \delta) \circ \Delta. \quad (67)$$

A differential Q on the coalgebra $\mathfrak{h}$ is then defined to be a coderivation on $\mathfrak{h}$ of degree one, which squares to zero. A GCA endowed with a differential is called a differential graded coalgebra, DGCA. A morphism between DGCA's $(\mathfrak{h}, \Delta, Q)$ and $(\mathfrak{h}', \Delta', Q')$, $F : \mathfrak{h} \to \mathfrak{h}'$, is simply a graded coalgebra morphism such that it commutes with the differentials, i.e. $F \circ Q = Q' \circ F$.

We can now define the main object of our discussion.

Definition 5.3.8. *A L_∞-algebra is a graded vector space $\mathfrak{g}$ over a field $\mathbb{K}$ along with a differential Q on the GCA $\overline{S}(\mathfrak{g}[1])$. A L_∞-morphism $F : (\mathfrak{g}, Q) \to (\mathfrak{g}', Q')$ is a DGCA morphism $F : \overline{S}(\mathfrak{g}[1]) \to \overline{S}(\mathfrak{g}'[1])$.*

5.4 The DGLA $\mathcal{V}$

Let M be a smooth manifold. We consider the graded vector space $\mathcal{V}' := \oplus_{k\geq 0}\Gamma(\bigwedge^k TM)$, where we remind ourselves of the convention $\bigwedge^0 TM := C^\infty(M)$. We are now interested in the following generalisation of the Lie bracket of vector fields.

Definition 5.4.1. *The Schouten-Nijenhuis bracket* $[\ ,\ ] : \mathcal{V}' \otimes \mathcal{V}' \to \mathcal{V}'$ *is a linear map defined on homogeneous elements by,* $[X_1, ..., X_k, Y_1, ..., Y_l] = \sum_{i=1}^k \sum_{j=1}^l (-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge ... \wedge \widehat{X_i} \wedge ... \wedge X_k \wedge Y_1 \wedge ... \wedge \widehat{Y_j} \wedge ... \wedge Y_l$.

Proposition 5.4.2. *The Schouten-Nijenhuis bracket* $[,] : \mathcal{V}' \otimes \mathcal{V}' \to \mathcal{V}'$ *satisfies the following properties*

(i) $[X, Y] = -(-1)^{(x+1)(y+1)}[Y, X]$

(ii) $[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(x+1)y} Y \wedge [X, Z]$

(iii) $[X, [Y, Z]] = [[X, Y], Z] + (-1)^{(x+1)(y+1)}[Y, [X, Z]]$

for all $X \in \Gamma(\bigwedge^x TM)$, $Y \in \Gamma(\bigwedge^y TM)$ *and* $Z \in \Gamma(\bigwedge^z TM)$. *Making* $\mathcal{V} := \mathcal{V}'[1]$ *clearly into a GLA and therefore a DGLA, when the trivial differential has been chosen.*

Proof. Let $X_1 \wedge ... \wedge X_x = X \in \Gamma(\bigwedge^x TM)$, $Y_1 \wedge ... \wedge Y_y = Y \in \Gamma(\bigwedge^y TM)$ and $Z_1 \wedge ... \wedge Z_z = Z \in \Gamma(\bigwedge^z TM)$.

$$[X, Y] = \sum_{i=1}^{x}\sum_{j=1}^{y}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge ... \wedge \widehat{X_i} \wedge ... \wedge X_x \wedge Y_1 \wedge ... \wedge \widehat{Y_j} \wedge ... \wedge Y_y \tag{68}$$

$$= -\sum_{j=1}^{y}\sum_{i=1}^{x}(-1)^{i+j}(-1)^{(x+y-1)(y-1)}[Y_j, X_i] \wedge Y_1 \wedge ... \wedge \widehat{Y_j} \wedge ... \wedge Y_y \wedge X_1 \wedge ... \wedge \widehat{X_i} \wedge ... \wedge X_x \tag{69}$$

$$= -(-1)^{(x+1)(y+1)}\sum_{j=1}^{y}\sum_{i=1}^{x}(-1)^{i+j}[Y_j, X_i] \wedge Y_1 \wedge ... \wedge \widehat{Y_j} \wedge ... \wedge Y_y \wedge X_1 \wedge ... \wedge \widehat{X_i} \wedge ... \wedge X_x \tag{70}$$

$$= -(-1)^{(x+1)(y+1)}\sum_{i=1}^{y}\sum_{j=1}^{x}(-1)^{i+j}[Y_i, X_j] \wedge Y_1 \wedge ... \wedge \widehat{Y_i} \wedge ... \wedge Y_y \wedge X_1 \wedge ... \wedge \widehat{X_j} \wedge ... \wedge X_x \tag{71}$$

$$= -(-1)^{(x+1)(y+1)}[Y, X] \tag{72}$$

Where in the second equality we have used the fact that $\text{sgn}((1 \cdots x+y-2)^{(y-1)}) = \left((-1)^{(x+y-1)}\right)^{(y-1)} = (-1)^{(x+y-1)(y-1)}$ and in the third equality we have used that $(x+y-1)(y-1) = (x-1)(y-1)+y(y-1) = (x+1)(y+1) \mod 2$.

$$[X, Y \wedge Z] = \sum_{i=1}^{x}\sum_{j=1}^{y}(-1)^{i+j}[X_i, Y_j] \wedge X_1 \wedge ... \wedge \widehat{X_i} \wedge ... \wedge X_x \wedge Y_1 \wedge ... \wedge \widehat{Y_j} \wedge ... \wedge Y_y \wedge Z_1 \wedge ... \wedge Z_z \tag{73}$$

$$+ \sum_{i=1}^{x}\sum_{j=1}^{z}(-1)^{i+j+y}[X_i, Z_j] \wedge X_1 \wedge ... \wedge \widehat{X_i} \wedge ... \wedge X_x \wedge Y_1 \wedge ... \wedge Y_y \wedge Z_1 \wedge ... \wedge \widehat{Z_j} \wedge ... \wedge Z_z \tag{74}$$

$$= [X, Y] \wedge Z + (-1)^y(-1)^{(y+x-1)y} Y \wedge [X, Z] = [X, Y] \wedge Z + (-1)^{(x+1)y} Y \wedge [X, Z] \tag{75}$$

Where in the second equality we have used the fact that $\text{sgn}((1 \cdots x + y)^y) = \left((-1)^{(x+y-1)}\right)^y = (-1)^{(x+y-1)y}$ and in the third equality we have used $(x+y-1)y+y = xy+y^2 = xy+y = (x+1)y \mod 2$. For the third property direct computation is very involved. There are multiple ways to get around this. A simple one is the introduction of a short system of notation followed by direct computation. Another one is by consecutive induction along the degrees x, y, z, where the second property is used to lower a k-multivector field into a $k-1$-multivector field. Since there are no interesting steps in these elementary proofs, they have been omitted.

Note that the DGLA $\mathcal{V}$ represents the formal Poisson structure.

5.5 The DGLA $\mathcal{D}$

The second DGLA that we encounter in the Kontsevich formality theorem is a subalgebra of the Hochschild DGLA, which we will now introduce. To any unital associative algebra A over a field $\mathbb{K}$, we may associate the complex $C := \sum_{i=-1}^{\infty} C^i$, where $C^i := \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(i+1)}, A)$. Given a $(m+1)$-linear operator $\phi \in C^m$ and a $(n+1)$-linear operator $\psi \in C^n$ we introduce a family of compositions $(\circ_i)_{i=0}^m : C^m \times C^n \to C^{m+n}$ defined by $(\phi \circ_i \psi)(f_0, ..., f_{m+n}) = \phi(f_0, ..., f_{i-1}, \psi(f_i, ..., f_{i+n}), f_{i+n+1}, ..., f_{m+n})$. We now define the composition $\circ : C^m \times C^n \to C^{m+n}$ by $\phi \circ \psi = \sum_{i=0}^m (-1)^{ni} \phi \circ_i \psi$. We can now formulate the following important proposition.

Proposition 5.5.1. *The graded vector space C together with the Gerstenhaber bracket $[,]_G : C \otimes C \to C$ defined on homogeneous elements by $[\phi, \psi]_G = \phi \circ \psi - (-1)^{mn} \psi \circ \phi$ is a GLA, called the Hochschild GLA. We will later define a non-trivial differential on it.*

Proof. It is clear that the Gerstenhaber bracket is bilinear and homogeneous by construction. Skew-symmetry follows from the following consecutive equalities, $[\phi, \psi]_G = \phi \circ \psi - (-1)^{mn} \psi \circ \phi = -(-1)^{mn}(\psi \circ \phi - (-1)^{mn} \phi \circ \psi) = -(-1)^{mn}[\psi, \phi]_G$ for all $\phi \in C^m$ and $\psi \in C^n$. The last property is the most involved as always. So let $\chi \in C^l$, $\phi \in C^m$ and $\psi \in C^n$. It is best to first calculate the three relevant terms $[[\phi, \psi]_G, \chi]_G$, $[\phi, [\psi, \chi]_G]_G$ and $[\psi, [\phi, \chi]_G]_G$. For the first term we have,

$$[[\phi, \psi]_G, \chi]_G = (\phi \circ \psi - (-1)^{mn} \psi \circ \phi) \circ \chi - (-1)^{l(m+n)} \chi \circ (\phi \circ \psi - (-1)^{mn} \psi \circ \phi) = \tag{76}$$

$$= \sum_{i=0}^{m} \sum_{j=0}^{m+n} (-1)^{ni+jl} (\phi \circ_i \psi) \circ_j \chi - \sum_{i=0}^{n} \sum_{j=0}^{m+n} (-1)^{m(i+n)+jl} (\psi \circ_i \phi) \circ_j \chi \tag{77}$$

$$- \sum_{i=0}^{m} \sum_{j=0}^{l} (-1)^{(m+n)(j+l)+ni} \chi \circ_j (\phi \circ_i \psi) + \sum_{i=0}^{n} \sum_{j=0}^{l} (-1)^{(m+n)(j+l)+m(i+n)} \chi \circ_j (\psi \circ_i \phi) \tag{78}$$

These compositions can all be further calculated. For the composition in the first term we for example have

$$(\phi \circ_i \psi) \circ_j \chi = \begin{cases} (\phi \circ_j \chi) \circ_i \psi & j < i \\ \phi \circ_i (\psi \circ_{j-i} \chi) & i \le j \le i+n \\ (\phi \circ_{j-n} \chi) \circ_i \psi & i+n < j \end{cases} . \tag{79}$$

By completing this procedure for all 4 compositions and all 3 terms we find after laborious calculations, that the Jacobi identity is indeed satisfied.

We are now interested in associative multiplications on our unital associative algebra A, $\mathfrak{m}$. We note that these are precisely the elements of $C^1 := \mathrm{Hom}_{\mathbb{K}}(A \otimes A, A)$ that satisfy the associative property: for all $f, g, h \in A$ we have $(fg)h = f(gh) \Leftrightarrow \mathfrak{m}(\mathfrak{m}(f, g), h) - \mathfrak{m}(\mathfrak{m}f, \mathfrak{m}(g, h)) = 0$. Because for all $f, g, h \in A$ we have $[\mathfrak{m}, \mathfrak{m}]_G(f, g, h) = 2(\mathfrak{m}(\mathfrak{m}(f, g), h) - \mathfrak{m}(\mathfrak{m}f, \mathfrak{m}(g, h)))$ we conclude that the associative multiplications are precisely the $\mathfrak{m} \in C^1$ for which the Gerstenhaber bracket with itself vanishes. Given a GLA $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$ and an element $a \in \mathfrak{g}^k$ of degree k, $\mathrm{ad}_\phi := [\phi, \cdot]$ is a derivation of degree k. The Jacobi identity can be rewritten as $\mathrm{ad}_\phi[\psi, \xi] = [\mathrm{ad}_\phi \psi, \xi] + (-1)^{km}[\psi, \mathrm{ad}_\phi \xi]$ for all $\psi \in \mathfrak{g}^m$ and $\xi \in \mathfrak{g}^n$. It is therefore natural to introduce the so called Hochschild differential $d_{\mathfrak{m}} : C^i \to C^{i+1}$ defined by $\psi \mapsto [\mathfrak{m}, \psi]_G$. We furthermore note that $(d_{\mathfrak{m}} \circ d_{\mathfrak{m}})\psi = [\mathfrak{m}, [\mathfrak{m}, \psi]_G]_G = [[\mathfrak{m}, \mathfrak{m}]_G, \psi]_G - [\mathfrak{m}, [\mathfrak{m}, \psi]_G]_G = -[\mathfrak{m}, [\mathfrak{m}, \psi]_G]_G$ and so $d_{\mathfrak{m}}^2 = 0$.

Proposition 5.5.2. *The GLA C together with the differential $d_{\mathfrak{m}}$ is a DGLA.*

The explicit expression for the action on a $\psi \in C^n$ is given by,

$$(d_{\mathfrak{m}}\psi)(f_0, ..., f_{n+1}) = -\sum_{i=0}^{n}(-1)^i\psi(f_0, ..., f_{i-1}, f_i f_{i+1}, .., f_{n+1}) + f_0\psi(f_1, ..., f_{n+1}) - (-1)^n\psi(f_0, .., f_n)f_{n+1} \tag{80}$$

In the case of $A = C^\infty(M)$, we are only interested in a specific subalgebra of C. Namely, the one that consists out of smooth differential operators that vanish on constant functions. One can prove that this subalgebra $\mathcal{D}$ inherits the DGLA structure of C. Note that this DGLA represents the equivalence classes of star products.

5.6 Kontsevich Formality Theorem

We can now finally formulate the formality theorem.

Definition 5.6.1. *An L_∞-quasi isomorphism is an L_∞-morphism whose first components is a quasi-isomorphism.*

Theorem 5.6.2. ([Kon03]). *There exists a natural L_∞-quasi isomorphism $K : \mathcal{V} \to \mathcal{D}$.*

We already discussed in the previous sections how theorem 5.1.6. is a consequence of this formality theorem. Under theorem 5.1.6. we furthermore discussed how this proves that any Poisson manifold can be endowed with a star product. Which is precisely the important generalization on the existence of star products that we announced in the introduction.

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