Homotopy and magnetic monopoles

Colling, Alex

Homotopy and magnetic monopoles

THESIS

submitted in partial fulfillment of the
requirements for the degree of

BACHELOR OF SCIENCE
in
PHYSICS & MATHEMATICS

Author :	A. Colling
Student ID :	s2564335
Supervisor Physics :	prof. dr. A. Achúcarro
Supervisor Mathematics :	dr. R. de Jong

Leiden, The Netherlands, June 28, 2022

Homotopy and magnetic monopoles

A. Colling

Huygens-Kamerlingh Onnes Laboratory, Leiden University
P.O. Box 9500, 2300 RA Leiden, The Netherlands

June 28, 2022

Abstract

This thesis studies magnetic monopoles and their relation to homotopy theory. Magnetic monopoles can appear as topological defects in certain gauge theories extending the Standard Model. The existence and stability of these defects is strongly related to homotopy theory. We classify topological defects using homotopy groups and show that monopoles are expected to exist in many theories based on this classification. We discuss how to assign a magnetic charge to defects using a topological interpretation of this charge. Lastly, we present an example where new phenomena emerge in the topological description of defects due to a nontrivial π_1-action on the homotopy groups.

Contents

Version of June 28, 2022– Created June 28, 2022 - 12:44

Chapter 1

Introduction

For a long time people have speculated about the existence of *magnetic monopoles*, particles carrying a net magnetic charge. Magnetic poles appear to always come in pairs: a north and a south pole. Cutting a magnet in half does not isolate the poles. Rather, it results in two new magnets, both with a north and south pole. In fact, one of Maxwell's equations states that the magnetic field $\mathbf{B}$ satisfies $\nabla \cdot \mathbf{B} = 0$. This implies that the magnetic flux through any closed surface is zero and that isolated magnetic charges are impossible.

Nevertheless, magnetic monopoles have been extensively studied for various reasons. Their existence would make the Maxwell equations completely symmetric under the interchange of the electric and magnetic fields. Moreover, in 1931 Dirac showed that the existence of a single magnetic monopole would imply that electric charge is quantised [9], providing an attractive explanation for the quantisation of electric charge observed in nature. Interest in monopoles was renewed in 1974 when Gerard 't Hooft [18] and Alexander Polyakov [28] showed that magnetic monopole solutions exist as *topological defects* in certain theories extending the Standard Model.

Topological defects are solutions of a system of partial differential equations of nonzero energy that are stable for topological reasons, because it is not possible to continuously deform them into a ground state. They can be classified using homotopy theory, which is a branch of mathematics that studies continuous deformations of maps between topological spaces. Defects are known to form in phase transitions in condensed matter systems. Examples include the trapping of a magnetic field in flux lines in superconductors and the formation of domain walls in ferromagnets.

This thesis focuses on topological defects in gauge theories that might have formed in phase transitions in the early universe. We are particularly interested in monopole solutions and their relation to homotopy theory. None of the treated results are new. The purpose of this thesis is to give a mathematically rigorous account of the subject. In particular, Propositions A.2 and B.4 contain proofs supporting statements in the physics literature of which we have not encountered the details.

We draw attention to the fact that quite a lot of theory from both mathematics and physics is used in this thesis. From the mathematical perspective, we apply elements of homotopy theory and differential geometry, including homotopy groups, Lie groups and connections. From the physical standpoint we use classical field theory and gauge theory to describe our models. We will do our best to guide the reader through all of these subjects.

We begin in section 2 by providing the intuition behind topological defects. Some examples are presented in the language of field theory. We discuss defects in the abelian Higgs model [27] and introduce the Georgi-Glashow model [11]. We also illustrate the formation of defects in the early universe through the Kibble mechanism [20]. In section 3 homotopy groups are defined. These are groups containing the topological information of a space relevant for the study of defects. The long exact sequence of homotopy groups is our most important tool to calculate these groups. We apply this sequence to the Hopf fibration and Lie group actions.

Subsequently we turn to gauge theory in section 4 and construct the Yang-Mills-Higgs Lagrangian [44]. Gauge fields need to be added to our models in more than one space dimension to ensure that the energy of the defects remains finite. In gauge theory, interactions are modelled using fields that have unphysical degrees of freedom in the sense that different field configurations correspond to the same physical state. This redundancy in the mathematical description is called *gauge symmetry*. Mathematically, the fields in our models are given by sections of principal fibre bundles and their associated vector bundles. Different sections of the same bundle correspond to different field configurations describing the same physical state and are related by a *gauge transformation*.

We present a slightly simplified version of this formalism without introducing associated vector bundles, as this is not required for our discussion. Moreover, we consider gauge transformations defined on strict subsets of spacetime. These transformations are called *local*. If the domain is all of spacetime the transformation is *global*, see Definition 4.4. This should not be confused with the more usual definition in the physics literature where a global transformation refers to a transformation that does not depend on the location in spacetime.

Armed with the theory of sections 3 and 4 we return to topological defects in section 5. The Higgs mechanism [17] is described as it plays an important role in the formation of defects. The defects are classified using homotopy theory. It is shown that many Grand Unified Theories predict the formation of monopoles carrying a *topological charge* in the early universe. In section 6 we study magnetic monopoles, starting with the Dirac monopole and its relation to the Hopf fibration. We show that the topological charge of monopole defects can in some cases naturally be identified with a magnetic charge. This correspondence is worked out explicitly for the ’t Hooft-Polyakov monopole.

In section 7 we consider a model where the classification using homotopy groups is more involved. This complication arises because the homotopy groups come with a basepoint that is not necessarily fixed in the defect configurations. We encounter this situation for topological spaces that have a nontrivial π_1-action on the homotopy groups. In our example we can interpret the π_1-action physically as the influence of Alice strings [37] on monopoles winding around them. Relevant homotopical constructions and background information on Lie groups, Lie group actions and connections can be found in Appendices A and B.

Chapter 2

Topological defect models

We introduce the simplest models describing different types of topological defects from the physical perspective. The kink and the vortex are discussed, as well as the abelian Higgs model and the Georgi-Glashow model. Moreover, we illustrate the Kibble mechanism using these models. The purpose of this section is to explain the intuition behind the defects and hint at the connection between defects and magnetic fields. We return to most of the models in later sections, where we analyse them in detail using the theory developed there.

2.1 Field theory

To describe topological defects we need to introduce the formalism of classical field theory. In field theory, particles and their interactions are modelled using *fields*. Examples are gravitational potentials or electric and magnetic fields. Mathematically, a field is a smooth map defined on all of *spacetime* M. We will assume there is no gravity. We can then view spacetime as $M = \mathbb{R}^4$, where the first coordinate corresponds to time and the other three to space.

We employ fundamental units ($\hbar = c = 1$) and use the metric signature $(+, -, -, -)$, see Definition 4.1. Throughout, Greek indices refer to space and time and (in three space dimensions) take values $\mu, \nu, \lambda, \rho = 0, 1, 2, 3$. Latin indices from the middle of the alphabet refer to space and take values $i, j, k = 1, 2, 3$. This subsection is based on [39].

The simplest example of a field is a real scalar field $\varphi : M \to \mathbb{R}$. A model contains a *Lagrangian density* $\mathcal{L}$, which is a scalar function constructed from φ and its derivatives. For our scalar field, the Lagrangian might look like

$$\mathcal{L} = \partial_\mu \varphi \partial^\mu \varphi - \tfrac{1}{2} m^2 \varphi^2 = \tfrac{1}{2} (\partial_0 \varphi)^2 - \sum_{i=1}^{3} \tfrac{1}{2} (\partial_i \varphi)^2 - \tfrac{1}{2} m^2 \varphi^2 \,. \tag{2.1}$$

In this equation m is a constant, $\partial_0 \varphi$ is the time derivative and $\partial_i \varphi$ the space derivatives ($i = 1, 2, 3$). The Lagrangian is also written in relativistic notation (see section 4.1), where summation over the repeated index $\mu = 0, 1, 2, 3$ is implied. The way fields evolve is dictated by the *field equations*, which are derived by requiring that the field extremises the *action* $S = \int d^4x\, \mathcal{L}$. For a Lagrangian describing a scalar field φ, the field equation reads

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \sum_{\mu=0}^{3} \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0\,. \tag{2.2}$$

The quantity

$$E = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \partial_0 \varphi} \partial_0 \varphi - \mathcal{L} \right) \tag{2.3}$$

is conserved if the Lagrangian does not depend explicitly on time and is identified with the energy (or Hamiltonian) of the field. For our Lagrangian (2.1) the energy is

$$E = \int d^3x \left(\tfrac{1}{2}(\partial_0 \varphi)^2 + \sum_{i=1}^{3} \tfrac{1}{2}(\partial_i \varphi)^2 + \tfrac{1}{2} m^2 \varphi^2 \right). \tag{2.4}$$

Moreover, using (2.2) we obtain the *Klein-Gordon equation* as equation of motion:

$$(\Box + m^2)\varphi = 0\,, \tag{2.5}$$

where $\Box = \partial_0^2 - \sum_{i=1}^3 \partial_i^2$ is the d'Alembert operator. For $m = 0$ this reduces to the wave equation. The general solution of (2.5) can be found using standard techniques from Fourier analysis and is a real superposition of solutions of the form $\varphi_k = A_k e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}}$ with (complex) amplitude A_k, frequency ω_k and Fourier wave vector $\mathbf{k}$ satisfying $\omega_k^2 = |\mathbf{k}|^2 + m^2$. If we view φ_k as the wave function of a particle with energy $E = \omega_k$ and momentum $\mathbf{p} = \mathbf{k}$ we find the relation $E^2 = p^2 + m^2$. This is the relativistic formula for the energy of a particle of mass m. For this reason we call m the mass of the field φ.

2.2 The kink

We now turn to models admitting topological defect solutions. For our first example we work in one space dimension and follow section 6 of [8]. The Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - U(\varphi) = \tfrac{1}{2}(\partial_0 \varphi)^2 - \tfrac{1}{2}(\partial_1 \varphi)^2 - U(\varphi)\,. \tag{2.6}$$

In this equation $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a real scalar field in one space dimension and U is the potential given by $U(\varphi) = \frac{\lambda}{2}(\varphi^2 - a^2)^2$, where λ and a are positive constants. We will always set the minimum of the potential to zero, which is possible in the absence of gravity. The energy reads

$$E = \int dx \left(\tfrac{1}{2}(\partial_0 \varphi)^2 + \tfrac{1}{2}(\partial_1 \varphi)^2 + U(\varphi) \right). \tag{2.7}$$

Note that the Lagrangian is invariant under the transformation $\varphi \mapsto -\varphi$. The energy is also invariant, and whenever φ is a solution of the field equation $-\varphi$ is as well. This is called a *global symmetry* of the model with symmetry group $\mathbb{Z}_2$.

The *ground states* or *vacuum states* are the solutions that minimise the energy (2.7). In this model there are two ground states where φ is constant and equal to a or $-a$. The ground states get interchanged under the transformation $\varphi \mapsto -\varphi$ and hence are not invariant under the symmetry. For this reason, if all the energy is taken out of the system and the field φ decays into a ground state we say that the symmetry is *spontaneously broken*. Spontaneous symmetry breaking plays an important role in the formation of topological defects, as is explained in section 2.5.

We will search for time-independent solutions of finite energy different from the ground states. For any finite energy configuration, it is necessary in order for the energy integral to converge that φ tends to a zero of U as x tends to plus or minus infinity. Hence, we could for example look for finite energy solutions by requiring that φ tends to a as $x \to \infty$ and to $-a$ as $x \to -\infty$. Indeed, an explicit solution of the time-independent field equation $\partial_1^2 \varphi = \frac{\partial U}{\partial \varphi}$ with these boundary conditions is

$$\varphi = a \tanh(a\sqrt{\lambda}x) \,. \tag{2.8}$$

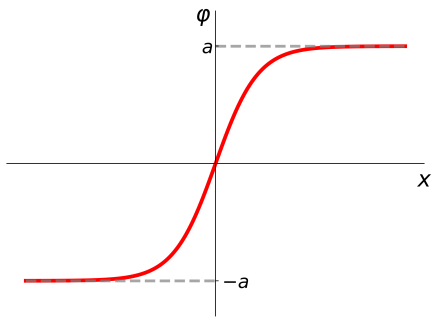


Figure 2.1: The kink solution given by equation (2.8).

This solution is called the *kink* and is shown in Figure 2.1. The energy of the kink is concentrated around $x = 0$ in a region with size of order $(a\sqrt{\lambda})^{-1}$. The total energy can be calculated from (2.7), giving $E = \frac{4}{3}a^3\sqrt{\lambda}$. It is intuitively clear that the kink solution is stable: for it to decay to a ground state we would have to "pull down" the entire right branch of the solution to the value $\varphi = -a$ or push the left branch up to a, which costs an infinite amount of energy. We can also view the kink as a solution in three space dimensions that does not depend on the y and z coordinates. In this case the kink has nonzero energy density concentrated in a plane and is called a *domain wall*. The domain wall has finite energy per unit area.

The fundamental reason for the existence of the kink is that the set of minima of the potential $\mathcal{M} = U^{-1}(0) = \{\pm a\}$ is not connected. This set $\mathcal{M}$ is called the *vacuum manifold**. Any finite energy solution φ defines a mapping φ_∞ from two points at ∞ and $-\infty$ to $\mathcal{M}$ that cannot change over time while keeping the energy finite. For the ground states this associated map φ_∞ sends both points at infinity to the same element of $\mathcal{M}$. If a solution has a different associated map (as is the case for the kink), it can never decay into a ground state. In this case the solution is stable and traps energy in a finite region of space where the field interpolates between a and $-a$. This is an example of a topological defect. The topological aspect is analysed in more detail in section 5.2.

*We show in section 5.1 that under certain assumptions $\mathcal{M}$ is always a manifold, as the name suggests.

2.3 The vortex

For the second example of a topological defect we consider a model in two space dimensions as presented in [35]. The model contains a complex scalar field $\varphi : \mathbb{R}^3 \to \mathbb{C}$ in the Lagrangian

$$\mathcal{L} = (\partial_\mu \varphi)^* \partial^\mu \varphi - U(\varphi) = (\partial_0 \varphi)^* \partial_0 \varphi - \sum_{i=1}^{2} (\partial_i \varphi)^* \partial_i \varphi - U(\varphi) \,. \tag{2.9}$$

In this equation $*$ denotes complex conjugation. The potential $U(\varphi) = \frac{1}{4}\lambda(\varphi^* \varphi - v^2)^2$ is called the *Mexican hat potential* (see Figure 2.2) with positive constants λ and v.

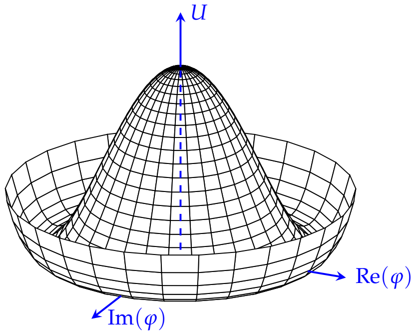


Figure 2.2: The Mexican Hat potential U. The minima of U form a circle of radius v. This figure is adapted from [46].

The energy is given by

$$E = \int d^2x \left((\partial_0 \varphi)^* \partial_0 \varphi + \sum_{i=1}^{2} (\partial_i \varphi)^* \partial_i \varphi + U(\varphi) \right) . \tag{2.10}$$

Note that this model contains a global symmetry consisting of rotations in the complex plane: the Lagrangian is invariant under the substitution $\varphi \mapsto e^{i\alpha}\varphi$ for $\alpha \in \mathbb{R}$. The symmetry group is $\mathrm{U}(1)$ (see Example B.3). A ground state is a state of zero energy where φ is constant and equal to an element of the vacuum manifold $\mathcal{M} = U^{-1}(0)$, which is a circle in the complex plane of radius v. Such a ground state is only invariant under the trivial rotation (over zero degrees) and hence breaks the symmetry. We will represent elements of $\mathcal{M}$ using arrows, i.e. the element $ve^{i\theta} \in \mathcal{M}$ corresponds to an arrow of size v making an angle θ with the positive real axis.

For a finite energy solution it is necessary that φ tends to a zero of U in every direction in space. Just like for the kink, we can attempt to create a configuration different from a ground state by assigning different elements of $\mathcal{M}$ to different directions in space. We will consider a configuration at a constant time where the field points radially outward in all directions, i.e. $\varphi(r, \theta) = ve^{i\theta}$ in polar coordinates for large values of r (see Figure 2.3). In order for the field to be continuous we then must have $\varphi = 0$ somewhere in the center of the configuration, resulting in a nonzero localised energy density.

It should be noted that this configuration does not actually have finite energy: the dependence of φ on the polar angle θ results in a contribution to the derivatives that goes as $1/r$. Since the derivatives enter squared in the energy, we obtain terms proportional to $1/r^2$ in the energy density which make the energy integral diverge logarithmically. We will sidestep this problem for now and solve it in section 2.4 by including a gauge field in the model.

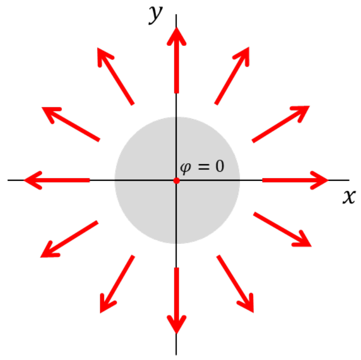


Figure 2.3: A defect configuration of φ. Far away from the origin the field takes values in $\mathcal{M}$ and points radially outward. In the center of the configuration (grey area) φ leaves the set $\mathcal{M}$ and becomes equal to zero somewhere in the middle to preserve continuity.

A solution that looks like Figure 2.3 can be found by solving the field equations (numerically) with this configuration as initial condition. Since the energy stays finite as time evolves, on a very large circle in space the field φ will at any time induce a map φ_∞ taking values in $\mathcal{M}$. For the initial configuration the map φ_∞ points radially outward in all directions. It is not possible to continuously deform this configuration while keeping the energy finite such that the map φ_∞ becomes constant. This statement is related to the topology of $\mathcal{M}$ and will be made precise later in this thesis. It follows that the solution cannot decay into a ground state (because then φ_∞ would be constant) and hence is stable. We have found another example of a topological defect.

This solution is called a *vortex*. Viewed as a solution in three space dimensions with no dependence on the third coordinate, it becomes a *cosmic string* with an energy density concentrated on a line and (if we include a gauge field) finite energy per unit length.

2.4 Defect models in gauge theories

In gauge theories it is possible to assign a finite energy to configurations like the vortex in Figure 2.3. We show how this works for a vortex in the abelian Higgs model following [35]. In addition to the complex scalar field φ the model now contains a *gauge field* A. In two space dimensions, this can be viewed in this model as a triplet of real scalar fields (A_0, A_1, A_2). The *field strength* F encodes the kinetic and gradient energy of the gauge field. It has 9 components (3 of which are independent) given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $\mu, \nu = 0, 1, 2$. The ordinary derivative $\partial_\mu \varphi$ is replaced by the *covariant derivative* $D_\mu \varphi = \partial_\mu \varphi - ieA_\mu \varphi$, where e is a real constant describing the strength of the interaction between A and φ. The Lagrangian (2.9) and energy (2.10) now read

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* D^\mu \varphi - U(\varphi) \tag{2.11}$$

$$= \sum_{i=1}^{2} \tfrac{1}{2} F_{0i}^2 - \tfrac{1}{2} F_{12}^2 + (D_0 \varphi)^* D_0 \varphi - \sum_{i=1}^{2} (D_i \varphi)^* D_i \varphi - U(\varphi)\,. \tag{2.12}$$

$$E = \int d^2x \left(\sum_{i=1}^{2} \tfrac{1}{2} F_{0i}^2 + \tfrac{1}{2} F_{12}^2 + (D_0 \varphi)^* D_0 \varphi + \sum_{i=1}^{2} (D_i \varphi)^* D_i \varphi + U(\varphi) \right). \tag{2.13}$$

We will construct a configuration for which $A_0 = 0$ (this gets rid of some of the redundancy in the mathematical description mentioned in the introduction). Just like before, in a finite energy configuration φ has to tend to an element of the vacuum manifold $\mathcal{M}$ far away from the origin in any direction. We accomplish this by requiring that in polar coordinates (r, θ) for large values of r the field φ has the form

$$\varphi(r, \theta) = v\exp(if(\theta)) , \tag{2.14}$$

where f is a smooth real-valued function. Note that a necessary condition for φ to be single-valued is that $f(2\pi) - f(0) = 2\pi n$ for some integer n. We will see later that this integer characterises the relevant topological properties of φ. The space derivatives of (2.14) are

$$\partial_j \varphi = i\partial_j f(\theta) v\exp(if(\theta)) \tag{2.15}$$

for $j = 1, 2$. The integral over these derivatives squared generally diverges. However, the quantity that enters the energy integral (2.13) is now not the ordinary derivative but the covariant derivative $D_j\varphi = \partial_j\varphi - ieA_j\varphi$. If we set our gauge field equal to $A_j = e^{-1}\partial_j f(\theta)$ at large distances, we obtain $D_j\varphi = 0$. Moreover, we find $F_{\mu\nu} = 0$ and $D_0\varphi = 0$ if we choose our initial configuration such that $\partial_0\varphi = 0$. It follows that all the terms in (2.13) are zero outside of the core of the defect and hence that the energy is finite. A vortex solution is again obtained by solving the field equations with this initial condition.

We note that the requirement for the energy to be finite forces the gauge field to be nonzero. If we view the vortex as a cosmic string in three space dimensions with an extra component A_3 of the gauge field that is equal to zero, we can identify the spatial part (A_1, A_2, A_3) of A with minus the magnetic vector potential $\mathbf{A}$ (see section 4.2). The magnetic field $\mathbf{B}$ is defined as the curl of the vector potential: $\mathbf{B} = \nabla \times \mathbf{A}$. By Stokes' theorem, the magnetic flux Ψ through the plane where the third space coordinate is zero then equals

$$\Psi = \iint \mathbf{B} \cdot d\mathbf{s} = \oint \mathbf{A} \cdot d\mathbf{l} = \frac{1}{e}\int_0^{2\pi} -\nabla f(\theta) \cdot R\hat{\boldsymbol{\theta}} d\theta \tag{2.16}$$

$$= -\frac{1}{e}\int_0^{2\pi} \partial_\theta f(\theta) d\theta = -\frac{1}{e}(f(2\pi) - f(0)) = -\frac{2\pi n}{e} , \tag{2.17}$$

where the line integral is taken over a large circle of radius R in the plane and ∂_θ denotes the derivative with respect to θ. We find that the string carries a quantised magnetic flux given by the integer n. The configuration in Figure 2.3 corresponds to the map $f(\theta) = \theta$ and hence has magnetic flux $\Psi = -\frac{2\pi}{e}$.

In three space dimensions we can have defects with energy concentrated in a point, which are called *monopoles*. The simplest model admitting monopole solutions is the Georgi-Glashow model discussed in section 6.4. This model contains a field φ whose values can be seen as vectors in $\mathbb{R}^3$. The gauge field components A_μ are also three-dimensional vectors, and the covariant derivative is $D_\mu\varphi = \partial_\mu\varphi + eA_\mu \times \varphi$. In this equation $\times$ is the cross product. The field strength now has an extra term: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + eA_\mu \times A_\nu$. The model has Lagrangian and energy

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}(D_\mu\varphi)(D^\mu\varphi) - U(\varphi) \tag{2.18}$$

$$= \sum_{i=1}^{3}\tfrac{1}{2}F_{0i}^2 - \sum_{i,j=1}^{3}\tfrac{1}{4}F_{ij}^2 + \tfrac{1}{2}(D_0\varphi)^2 - \sum_{i=1}^{3}\tfrac{1}{2}(D_i\varphi)^2 - U(\varphi)\,, \tag{2.19}$$

$$E = \int d^3x \left(\sum_{i=1}^{3}\tfrac{1}{2}F_{0i}^2 + \sum_{i,j=1}^{3}\tfrac{1}{4}F_{ij}^2 + \tfrac{1}{2}(D_0\varphi)^2 + \tfrac{1}{2}\sum_{i=1}^{3}(D_i\varphi)^2 + U(\varphi)\right), \tag{2.20}$$

where all multiplications of vectors are performed using the standard inner product on $\mathbb{R}^3$. The potential $U(\varphi) = \frac{1}{4}\lambda(\varphi^2 - v^2)^2$ is a generalisation of the Mexican hat potential. The set of minima $\mathcal{M}$ of U forms a sphere of radius v. Just like in our other models, in a finite energy configuration φ tends to an element of $\mathcal{M}$ in any direction in space and hence induces a map φ_∞ from a very large sphere in space to $\mathcal{M}$. We can again find a defect configuration by requiring that φ_∞ points radially outward (Figure 2.4), because in that case φ_∞ cannot be deformed into a constant map. This is called the *hedgehog* configuration. Near the origin the field φ leaves the set $\mathcal{M}$ and traps energy in the form of a monopole.

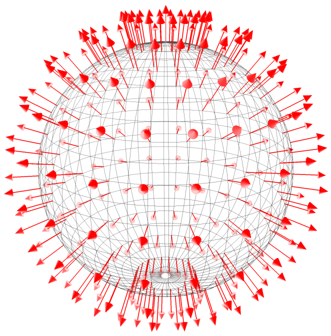


Figure 2.4: The hedgehog configuration of the field φ in the Georgi-Glashow model. The figure is taken from [45].

In order for the energy to be finite the covariant derivatives $D_\mu\varphi$ have to go to zero fast enough far away from the origin. To accomplish this, the gauge field has to be nonzero to cancel the contribution from the partial derivatives $\partial_\mu\varphi$. By identifying a part of the gauge field with the electromagnetic potential, we deduce in section 6.4 that this configuration carries a magnetic charge and hence can be called a magnetic monopole. The magnetic charge is quantised and directly related to the topological properties of φ.

We note that just like for vortices, monopoles can also exist in models with a global symmetry (i.e. without gauge fields). For such solutions the energy integral diverges linearly. In practice however, the integral is cut off by approximately the distance to the nearest anti-monopole (e.g. a solution that looks like Figure 2.4 with all the arrows pointing inward). In this thesis we focus on defects in gauge theories, because we need the gauge field to assign a magnetic field to the monopoles. See [42, Sect. 14.5] for a discussion of global monopoles.

2.5 The Kibble mechanism

The way in which topological defects form in the early universe is described by the Kibble mechanism, first formulated in 1976 by Tom Kibble. We illustrate the mechanism using the vortex described above following [33].

The idea is as follows. In the earliest moments of the universe the temperature was very high and there was a lot of energy available. The symmetry of the system was unbroken and the field φ was zero everywhere. As the universe cooled down, the system was losing energy. At some point a phase transition occurred and the field decayed to a ground state in which the symmetry is broken. We say that φ obtains an orientation (given by the chosen ground state).

Since information cannot travel faster than the speed of light, distant points cannot communicate during the transition. All orientations are equally likely because of the symmetry and hence distant points receive (in general) different orientations. It follows that the field will be uniform on domains of size at most ξ given by the distance that light can travel during the phase transition. We call ξ the correlation length. The field interpolates smoothly where the domains meet.

In some cases (depending on the topology of the vacuum manifold $\mathcal{M}$) it is possible that the field interpolates in such a way that we obtain a configuration that is topologically nontrivial. An example of this for the vortex configuration is shown in Figure 2.5. For the kink, this situation occurs if the field decays to the ground state $\varphi = a$ on one side of the universe and to $\varphi = -a$ on the other side. In order for the field to remain continuous, it is necessary for it to leave the set $\mathcal{M}$ somewhere in the space enclosed by these regions, resulting in a nonzero energy density in a finite region of space: a topological defect.

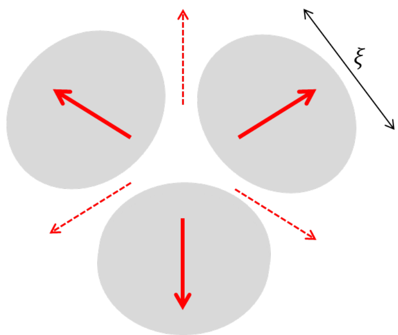


Figure 2.5: An illustration of the Kibble mechanism for the vortex. The field is uniform at distances less than the correlation length ξ (grey regions) and interpolates smoothly where these regions meet, creating the vortex configuration of Figure 2.3. Energy is trapped in the middle.

By estimating the correlation length ξ and the probability that the configuration of the field is topologically nontrivial, the number density of defects can be estimated. These calculations show that topological defects form whenever it is (topologically) possible [20]. In this case, since the defects are stable, some of them are expected to still be around today. The defects are remnants of the broken symmetry and hence could provide us with precious information about the early universe. This prospect, together with the connection to topology, forms the motivation for the study of topological defects and magnetic monopoles in the subsequent sections.

Chapter 3

Homotopy theory

The topological information of a space relevant for the study of topological defects is stored in its homotopy groups. In this section we define these groups and introduce the long exact sequence of homotopy groups following chapter 4 of [16]. We discuss some examples and we apply the long exact sequence to Lie group actions in section 3.3.

Some remarks on the notation: by a space X we mean a topological space and every map between spaces is required to be continuous. We write I^n for the n-dimensional unit cube $[0,1]^n$ and denote its boundary, the subspace of points with at least one coordinate equal to 0 or 1, by ∂I^n. For spaces A, B, X, Y with $A \subseteq X$ and $B \subseteq Y$, we write $f : (X, A) \to (Y, B)$ for a map $f : X \to Y$ satisfying $f(A) \subset B$. If A (or B) contains only one element x_0, we write (X, x_0) instead of $(X, \{x_0\})$.

3.1 Homotopy groups

Homotopy theory deals with the question when two maps can be continuously deformed into each other. We formalise what we mean by continuous deformations as follows.

Definition 3.1. Let X be a space with basepoint $x_0 \in X$. We say that two maps $f, g : (I^n, \partial I^n) \to (X, x_0)$ are *homotopic* if there exists a map $\Gamma : (I^n \times I, \partial I^n \times I) \to (X, x_0)$ satisfying $\Gamma(x, 0) = f(x)$ and $\Gamma(x, 1) = g(x)$ for all $x \in I^n$.

We call the map Γ the *homotopy* between f and g. The notion of being homotopic defines an equivalence relation, and the equivalence classes under this relation are called homotopy classes. We do not want to distinguish between homotopic maps, so we are only interested in the homotopy classes.

Definition 3.2. Let X be a space with basepoint $x_0 \in X$. We define the n-th *homotopy group* $\pi_n(X, x_0)$ to be the set of homotopy classes of maps $(I^n, \partial I^n) \to (X, x_0)$. For $n > 0$, the operation

$$(f \odot g)(s_1, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n)\,, & s_1 \in [0, \frac{1}{2}] \\ g(2s_1 - 1, s_2, \ldots, s_n)\,, & s_1 \in [\frac{1}{2}, 1] \end{cases}$$

is well-defined on homotopy classes and makes $\pi_n(X, x_0)$ into a group. The unit element is the class of the constant map sending I^n to x_0, and the inverse of $[f] \in \pi_n(X, x_0)$ is the class of the map $f^{-1}(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n)$.

For $n = 0$ we set $I^0 = \{0\}$ and $\partial I^0 = \emptyset$. The definitions above then imply that $\pi_0(X, x_0)$ can be identified with the set of path components of X. There is no natural way to define a group structure on this set. Taking $n = 1$, we recover the definition of the fundamental group.

A map $(I^n, \partial I^n) \to (X, x_0)$ is the same as a map from the quotient $I^n/\partial I^n$ (obtained by identifying all the points of ∂I^n) to X sending the point $s_0 = \partial I^n/\partial I^n$ to x_0. We have $I^n/\partial I^n \cong S^n$ for $n > 0$, where S^n is the n-dimensional unit sphere $S^n = \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. For all $n \geq 0$ we find that we can view representatives of elements of $\pi_n(X, x_0)$ as maps $(S^n, s_0) \to (X, x_0)$. From this viewpoint homotopies are maps of the form $\Gamma : (S^n \times I, \{s_0\} \times I) \to (X, x_0)$.

Just like for the fundamental group, a map $\varphi : (X, x_0) \to (Y, y_0)$ between pointed spaces induces a map $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined by $\varphi_*([f]) = [\varphi \circ f]$. It can be verified that φ_* is a group homomorphism for $n > 0$. Moreover, the induced maps satisfy the functorial properties $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$ and $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_n(X)}$. If φ is a homotopy equivalence, the induced map φ_* is an isomorphism: see Lemma A.1 in the appendix for a proof.

Lemma 3.1. *Let X be a contractible space and $x_0 \in X$. Then $\pi_n(X, x_0) = 0$ for all n.*

Proof. Since X is contractible, there exists a homotopy equivalence from X to a one point space $\{*\}$. The induced map on π_n is an isomorphism by Lemma A.1, which is only possible if $\pi_n(X, x_0) = 0$ since the homotopy groups of $\{*\}$ are trivial. □

Homotopy groups behave nicely with respect to products.

Lemma 3.2. *Let I be an index set and X_i a space with basepoint $x_i \in X_i$ for all $i \in I$. Then there are isomorphisms $\pi_n(\prod_{i\in I} X_i, (x_i)_{i\in I}) \xrightarrow{\sim} \prod_{i\in I} \pi_n(X_i, x_i)$ for all n.*

Proof. A map $(I^n, \partial I^n) \to (\prod_{i\in I} X_i, (x_i)_{i\in I})$ is the same thing as a collection of maps $(I^n, \partial I^n) \to (X_i, x_i)$ for all $i \in I$. This correspondence is well-defined on homotopy classes because a homotopy $(I^n \times I, \partial I^n \times I) \to (\prod_{i\in I} X_i, (x_i)_{i\in I})$ is also the same as a homotopy $(I^n \times I, \partial I^n \times I) \to (X_i, x_i)$ for all i. □

In general computing homotopy groups is quite involved. This becomes apparent when considering the homotopy groups $\pi_i(S^n)$ of spheres. Note that we are leaving out the basepoint in our notation for path-connected spaces: see Appendix A.1 for details. The simplest example is the fundamental group of the circle $\pi_1(S^1)$, which is isomorphic to $\mathbb{Z}$. An isomorphism can be constructed by associating a *winding number* $n \in \mathbb{Z}$ to any loop $\gamma : I \to S^1$ in the following way [23, p.224].

We view S^1 as the unit circle in $\mathbb{C}$ and consider the universal covering map $p : \mathbb{R} \to S^1$ defined by $p(t) = e^{2\pi i t}$. Given a loop $\gamma : I \to S^1$ there exists a lift $\tilde{\gamma} : I \to \mathbb{R}$ of γ satisfying $\gamma = p \circ \tilde{\gamma}$. Since $\gamma(0) = \gamma(1)$ we must have $e^{2\pi i \tilde{\gamma}(0)} = e^{2\pi i \tilde{\gamma}(1)}$, i.e. $\tilde{\gamma}(1) - \tilde{\gamma}(0) \in \mathbb{Z}$. The integer $n = \tilde{\gamma}(1) - \tilde{\gamma}(0)$ is independent of the choice of lift and representative of $[\gamma] \in \pi_1(S^1)$ and is the winding number of γ. In particular, if $\tilde{\gamma}(t) = nt$ we find the following.

Lemma 3.3. *Let $n \in \mathbb{Z}$. The winding number of the loop $\gamma : I \to S^1$ given by $\gamma(t) = e^{2\pi i n t}$ is n.*

If we view a loop as a map $f : S^1 \to S^1$, the associated winding number is given by the winding number of the loop $\gamma(t) = f(e^{2\pi i t})$. The map $\pi_1(S^1) \to \mathbb{Z}$ sending $[\gamma]$ to the winding number of γ is an isomorphism.

We can in fact construct isomorphisms $\pi_n(S^n) \xrightarrow{\sim} \mathbb{Z}$ for all $n \geq 1$.

Theorem 3.1. *For all $n \geq 1$ we have $\pi_n(S^n) \cong \mathbb{Z}$, generated by the identity map id_{S^n}.*

Proof. For $n = 1$ we indeed have $\pi_1(S^1) \cong \mathbb{Z}$, and the identity map id_{S^1} is a generator because it corresponds to the loop $\gamma(t) = e^{2\pi i t}$, which has winding number 1 by Lemma 3.3. Given a map $f : S^n \to S^n$ we define the *suspension* $Sf : S^{n+1} \to S^{n+1}$ of f to be the map that leaves the north and south poles $(0, 0, \dots, \pm 1)$ invariant and applies f on each subspace S^n where the last coordinate x_{n+1} is constant and satisfies $-1 < x_{n+1} < 1$. It can be shown that the suspension map $S : \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is well-defined on homotopy classes and induces an isomorphism $\pi_n(S^n) \cong \pi_{n+1}(S^{n+1})$ [16, Cor. 4.25]. By induction we find $\pi_n(S^n) \cong \mathbb{Z}$ for all n, and because $S\mathrm{id}_{S^n} = \mathrm{id}_{S^{n+1}}$ the identity map generates $\pi_n(S^n)$. □

The groups $\pi_i(S^n)$ with $i < n$ are also known:

Theorem 3.2. *For $i < n$ we have $\pi_i(S^n) = 0$.*

Proof. Given a map $f : S^i \to S^n$ with $i < n$, it is a nontrivial fact that it is possible to continuously deform f such that it is not surjective [16, Thm. 4.8]. We can then take a point $p \in S^n$ outside the image of f and view f as a map $S^i \to S^n \setminus \{p\}$. The space $S^n \setminus \{p\}$ is homeomorphic to $\mathbb{R}^n$ through stereographic projection and hence contractible. By Lemma 3.1 it follows that $f : S^i \to S^n \setminus \{p\}$ is homotopic to a constant map, and so $[f] = 0$ in $\pi_i(S^n)$. □

The complexity of the homotopy groups of spheres lies in the groups $\pi_i(S^n)$ for $i > n > 1$. These groups turn out to be very difficult to compute, and over the years a lot of effort has been put into their calculation. Even though a lot of patterns have been discovered, a lot of them are still unknown. Using the theory of fibre bundles in section 3.2 we will find the homotopy groups $\pi_i(S^1)$ and $\pi_3(S^2)$. Heavier machinery than discussed in this thesis is required to go further.

3.2 Fibre bundles

To introduce our main tool for the calculation of homotopy groups, the long exact sequence, we need to study fibre bundles.

Definition 3.3. A *fibre bundle* with *total space* E, *base space* B and *fibre* F is a tuple (E, p, B, F) where $p : E \to B$ is a map such that each point of B has a neighborhood U for which there is a homeomorphism $h : p^{-1}(U) \xrightarrow{\sim} U \times F$ such that the following diagram commutes.

$$\begin{array}{ccc} p^{-1}U & \xrightarrow{\quad h \quad} & U \times F \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \mathrm{proj}_1} & \\ & U & \end{array}$$

Here proj_1 is the projection on the first coordinate. The map h is called a *local trivialisation*.

The definition above can be interpreted as stating that the map p locally over B looks like a projection $U \times F \to U$. In particular, p is surjective and each fibre $p^{-1}(b)$ is homeomorphic to F. We will often write a fibre bundle as $F \to E \xrightarrow{p} B$.

Example 3.1. The simplest example of a fibre bundle is the trivial bundle $F \to B \times F \to B$, where the map p is the projection on the first coordinate: we can take $U = B$ and $h = \mathrm{id}$ in the definition. A more interesting example is a covering map $p : E \to B$ with B connected. In this case every point of B has a neighborhood U such that p restricts to a homeomorphism $p^{-1}U \xrightarrow{\sim} U \times J$, where J is some index set (if B is not connected the set J can depend on the point of B). It follows that a covering map over a connected base space is a fibre bundle with discrete fibre.

Example 3.2. Consider the complex projective line $\mathbb{C}P^1$ consisting of all lines through the origin in $\mathbb{C}^2$. By viewing S^3 as the unit sphere in $\mathbb{C}^2$, we can identify $\mathbb{C}P^1$ with the quotient space of S^3 under the equivalence relation $(z_1, z_2) \sim \lambda(z_1, z_2)$ for λ in the unit circle S^1 of $\mathbb{C}$. The map $p : S^3 \to \mathbb{C}P^1$ sending a point (z_1, z_2) to its equivalence class $[z_1 : z_2]$ then has fibres homeomorphic to S^1 and defines a fibre bundle. Indeed, for $i = 1, 2$ consider the open sets $U_i = \{[z_1 : z_2] \in \mathbb{C}P^1 \mid z_i \neq 0\}$ and the trivialisations

$$h_i : p^{-1}(U_i) \to U_i \times S^1\,, \qquad (z_1, z_2) \mapsto ([z_1 : z_2], z_i|z_i|^{-1})\,. \tag{3.1}$$

It is clear that the h_i are homeomorphisms and they determine commuting diagrams as in Definition 3.3. For $(z_1, z_2) \in S^3$, direct computations give $h_i^{-1}([z_1 : z_2], \lambda) = \lambda |z_i| z_i^{-1}(z_1, z_2)$. Observe that the base space $\mathbb{C}P^1$ is homeomorphic to the one point compactification $\mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$ through the map $[z_1 : z_2] \mapsto z_1/z_2$ (this fraction is ∞ if $z_2 = 0$). Moreover, we can construct a homeomorphism $S^2 \xrightarrow{\sim} \mathbb{C} \cup \{\infty\}$ by sending the north pole to ∞ and mapping the rest of the sphere onto $\mathbb{C}$ using the stereographic projection. It follows that we have found a fibre bundle $S^1 \to S^3 \to S^2$ consisting of only spheres, called the *Hopf bundle* or *Hopf fibration*.

The Hopf bundle is important in both physics and mathematics. It allows us to calculate some homotopy groups of spheres (see Example 3.4) and forms the topological structure of the Dirac monopole, as explained in section 6.2. There are only three more fibre bundles consisting of only spheres, which are obtained by replacing $\mathbb{C}$ by one of the other normed real division algebras $\mathbb{R}, \mathbb{H}$ or $\mathbb{O}$ in the above.

The reason why we are interested in fibre bundles is the following theorem [16, Thm. 4.41].

Theorem 3.3. (Long exact sequence of homotopy groups) *Let $F \to E \xrightarrow{p} B$ be a fibre bundle with basepoints $b_0 \in B$, $e_0 \in p^{-1}(b_0)$. Identify F with the fibre $p^{-1}(b_0)$ and let $i : (F, e_0) \to (E, e_0)$ be the inclusion. Then if B is path-connected, there is a long exact sequence*

$$\cdots \to \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{p_*} \pi_n(B, b_0) \xrightarrow{\delta} \pi_{n-1}(F, e_0) \to \cdots \to \pi_0(E, e_0) \to 0\,. \tag{3.2}$$

Note that the sequence is not really exact at the end because π_0 is not a group, but exactness still holds in the sense that the image of one map equals the kernel of the next. The *connecting homomorphism* $\delta : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ is not so simple to define because it is not induced by a map $B \to F$. Its construction is given in Appendix A.2.

Example 3.3. Let $p : E \to B$ be a covering map over a path-connected base space. We have seen in Example 3.1 that p induces a fibre bundle $J \to E \xrightarrow{p} B$ with discrete fibre J. We have $\pi_n(J, e_0) = 0$ for $n > 0$ and so the long exact sequence shows that $p_* : \pi_n(E, e_0) \xrightarrow{\sim} \pi_n(B, b_0)$ is an isomorphism for $n > 1$. In addition, if E is path-connected we find a short exact sequence

$$0 \to \pi_1(E, e_0) \xrightarrow{p_*} \pi_1(B, b_0) \xrightarrow{\delta} \pi_0(J, e_0) \to 0\,. \tag{3.3}$$

We see that $p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$ is injective. Comparing with the construction of δ in the appendix it can also be deduced that $J = \pi_0(J, e_0)$ can be identified with the set of cosets of $p_*\pi_1(E, e_0)$ in $\pi_1(B, b_0)$ via pathlifting, as we did in our definition of the winding number.

Corollary 3.1. $\pi_n(S^1) = 0$ *for* $n > 1$.

Proof. The map $p_* : \pi_n(\mathbb{R}) \to \pi_n(S^1)$ induced by the covering map $p : \mathbb{R} \to S^1$ is an isomorphism for $n > 1$, and the homotopy groups of $\mathbb{R}$ are trivial by Lemma 3.1. □

Example 3.4. Applying the long exact sequence to the Hopf bundle $S^1 \to S^3 \xrightarrow{p} S^2$ gives

$$\cdots \to \pi_n(S^1) \xrightarrow{i_*} \pi_n(S^3) \xrightarrow{p_*} \pi_n(S^2) \xrightarrow{\delta} \pi_{n-1}(S^1) \to \dots \tag{3.4}$$

For $n \geq 3$ we have $\pi_n(S^1) = \pi_{n-1}(S^1) = 0$ by Corollary 3.1 and so $\pi_n(S^3) \cong \pi_n(S^2)$. In particular, Theorem 3.1 with $n = 3$ gives

Corollary 3.2. *$\pi_3(S^2)$ is isomorphic to $\mathbb{Z}$ and is generated by the Hopf map $p : S^3 \to S^2$.*

3.3 Homotopy groups of compact Lie groups

In physics the spaces of interest are often acted upon by a compact and connected Lie group G. The homotopy groups of G can provide important information in these situations and therefore are worth studying. The necessary prior knowledge on Lie groups can be found in Appendix B.

We begin with a result on the second homotopy group of a compact connected Lie group.

Theorem 3.4. *The second homotopy group $\pi_2(G)$ of a compact connected Lie group G is trivial.*

Note that we can leave out the assumption that G is connected by Lemma A.2. The proof of this theorem is beyond the scope of this thesis and can be found in [5, Prop. V.7.5]. We can however verify the theorem explicitly for some matrix groups using the long exact sequence. Proposition 3.1 indicates the type of exact sequence we consider. The reader is advised to read Examples B.12 and B.13 in the appendix.

Proposition 3.1. *Let G be a compact connected Lie group acting from the left on a manifold $\mathcal{M}$. Let $p \in \mathcal{M}$ and let O_p and G_p denote the orbit and the stabiliser of p respectively. Then there is a long exact sequence*

$$\cdots \to \pi_n(G_p) \to \pi_n(G) \to \pi_n(O_p) \to \pi_{n-1}(G_p) \to \cdots \to \pi_0(G_p) \to 0\,. \tag{3.5}$$

Proof. By Lemma B.2 there is a fibre bundle $G_p \to G \to G/G_p$, and the quotient G/G_p is diffeomorphic to O_p by Proposition B.3. The result now follows from the exact sequence (3.2). □

Example 3.5. Consider the action of $\mathrm{SU}(n)$ on $\mathbb{C}^n$ by matrix-vector multiplication. The vector $e_1 = (1, 0, 0, \dots)$ has orbit S^{2n-1}, while the stabiliser is diffeomorphic to $\mathrm{SU}(n-1)$. We find an exact sequence

$$\cdots \to \pi_{i+1}(S^{2n-1}) \to \pi_i(\mathrm{SU}(n-1)) \to \pi_i(\mathrm{SU}(n)) \to \pi_i(S^{2n-1}) \to \dots \tag{3.6}$$

Theorem 3.2 implies that $\pi_{i+1}(S^{2n-1}) = \pi_i(S^{2n-1}) = 0$ for $i = 2$ and $n \geq 3$, so the exact sequence gives $\pi_2(\mathrm{SU}(n-1)) \cong \pi_2(\mathrm{SU}(n))$. Recall from Example B.4 that $\mathrm{SU}(2)$ is diffeomorphic to S^3, implying that $\pi_2(\mathrm{SU}(2)) = 0$. An easy inductive argument now gives $\pi_2(\mathrm{SU}(n)) = 0$ for all $n \geq 2$. Similarly, taking $i = 1$ it follows that $\mathrm{SU}(n)$ is simply connected.

Example 3.6. The arguments in Example 3.5 can be adapted to the action of $\mathrm{SO}(n)$ on $\mathbb{R}^n$ with orbit S^{n-1} and stabiliser $\mathrm{SO}(n-1)$. For the case $n = 3$ we need the fact that there is a two-sheeted covering $p : \mathrm{SU}(2) \to \mathrm{SO}(3)$, see [34, Prop 2.12A]. From Example 3.3 and the results $\pi_1(\mathrm{SU}(2)) = \pi_2(\mathrm{SU}(2)) = 0$ we then find $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$ and $\pi_2(\mathrm{SO}(3)) = 0$. Moreover, we can apply the long exact sequence (3.5) and deduce that $\pi_i(\mathrm{SO}(n-1))$ is isomorphic to $\pi_i(\mathrm{SO}(n))$ for $i = 1, 2$ and $n \geq 4$. By induction, this implies $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ and $\pi_2(\mathrm{SO}(n)) = 0$ for all $n \geq 3$. For $n = 2$ we have $\mathrm{SO}(2) \cong \mathrm{U}(1) \cong S^1$ by Example B.3. This means that $\mathrm{SO}(2)$ and $\mathrm{U}(1)$ have fundamental group $\mathbb{Z}$ and trivial higher homotopy groups (Corollary 3.1).

The order of the fundamental group of Lie groups is related to the notion of being semisimple, defined in Definition B.5.

Theorem 3.5. *Let G be a compact, connected and semisimple Lie group. Then $\pi_1(G)$ is finite.*

For the proof we again refer to [5, V.7.13]. Note that the semisimple groups $\mathrm{SU}(n \geq 2)$ and $\mathrm{SO}(n \geq 3)$ indeed have finite fundamental groups, while $\mathrm{SO}(2)$ and $\mathrm{U}(1)$ are not semisimple and have an infinite fundamental group.

Chapter 4

Gauge theory

Gauge theory is used to describe forces in relativistic quantum systems. The formalism contains a redundancy in the mathematical description characterised by a *gauge group* G. Using the gauge group we can define *gauge transformations* that act on the fields, and the physical states correspond to the orbits of this action. For this reason any physical observable is required to be invariant under gauge transformations. In this section we follow [10, 15] to construct a *gauge invariant* Lagrangian, the Yang-Mills-Higgs Lagrangian, which is the central object in the physical models discussed in this thesis. Subsequently, we show how to formulate electromagnetism in the language of gauge theory. The relevant notions from differential geometry are covered in Appendix B.

From now on repeated indices always imply summation. This summation is over the values $0, 1, 2, 3$ for Greek indices μ, ν, λ, ρ and over $1, 2, 3$ for Latin indices i, j, k from the middle of the alphabet.

4.1 Yang-Mills-Higgs Lagrangian

The terms of the Lagrangian are required to be *Lorentz invariant*, meaning that they do not depend on the choice of inertial frame. For background on Lorentz transformations and special relativity we refer the reader to chapter 12 of [13]. Lorentz invariant terms can be constructed with the metric tensor. Using the standard basis e_μ on spacetime $M = \mathbb{R}^4$ we define the corresponding coordinate vector fields $\partial_\mu = \frac{\partial}{\partial x^\mu} \in \mathfrak{X}(M)$ and covector fields $\mathrm{d}x^\mu \in \Omega^1(M)$.

Definition 4.1. The *Minkowski metric tensor* η is the symmetric nondegenerate (0,2) tensor field on M given by

$$\eta = \eta_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \mathrm{d}x^0 \otimes \mathrm{d}x^0 - \mathrm{d}x^1 \otimes \mathrm{d}x^1 - \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3 \,. \tag{4.1}$$

The metric tensor gives an isomorphism $TM \to T^*M$ between the tangent and cotangent bundles that can be used to raise and lower indices of tensor fields as explained in section 13 of [24]. For example, if $A = A_\mu \mathrm{d}x^\mu$ and $F = F_{\mu\nu} \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ are tensor fields, we have

$$A^\mu = \eta^{\mu\nu} A_\nu \quad \text{and} \quad F^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\lambda} F_{\rho\lambda} \,. \tag{4.2}$$

In this equation $\eta^{\mu\nu}$ are the components of the inverse metric, which are equal to $\eta_{\mu\nu}$ for the Minkowski metric. Scalars of the form $A_\mu A^\mu$ or $F_{\mu\nu}F^{\mu\nu}$ obtained by tensor contraction are Lorentz invariant and can appear in the Lagrangian. Note that the Lagrangians in section 2 indeed contain terms of this form.

We fix a gauge group G, which is a compact connected Lie group G of dimension n describing the gauge symmetry of the model. To simplify our discussion, we will assume in this section that G is a matrix group that is either simple or equal to $\mathrm{U}(1)$. Let $\tilde{A} \in \Omega^1(P, \mathfrak{g})$ be a gauge field on a principal G-bundle $\pi : P \to M$ (see Definition B.11). Because M is contractible, every principal bundle over M is trivial and it is always possible to choose a global section $s : M \to P$ [15, Cor. 4.2.9]. In the following we assume this has always been done, so that we can view our gauge field as a $\mathfrak{g}$-valued 1-form $A \in \Omega^1(M, \mathfrak{g})$ on spacetime given by the pullback $A = s^*\tilde{A}$. The gauge field has components $A_\mu = A(\partial_\mu) \in C^\infty(M, \mathfrak{g})$. These are smooth maps on M with values in the Lie algebra $\mathfrak{g}$.

We can use the gauge field and the Lie bracket $[\cdot,\cdot]_\mathfrak{g}$ on $\mathfrak{g}$ to define a 2-form $[A, A] \in \Omega^2(M, \mathfrak{g})$ by $[A, A](X, Y) = [A(X), A(Y)]_\mathfrak{g}$ for vector fields $X, Y \in \mathfrak{X}(M)$.

Definition 4.2. Let $A \in \Omega^1(M, \mathfrak{g})$ be a gauge field. The *field strength* $F \in \Omega^2(M, \mathfrak{g})$ is given by

$$F = \mathrm{d}A + [A, A] \,. \tag{4.3}$$

In this equation d is the exterior derivative. The components $F_{\mu\nu} = F(\partial_\mu, \partial_\nu)$ satisfy

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\mathfrak{g} \,. \tag{4.4}$$

In addition to the gauge field with corresponding field strength, our model contains a *Higgs field* $\Phi \in C^\infty(M, V)$, where V is a finite dimensional real or complex vector space forming a representation of G given by $\rho : G \to \mathrm{GL}(V)$. We refer to V as the Higgs vector space or field space. Note that Φ is not (necessarily) the same as the Higgs field in the Standard Model, but it plays an analogous role in the process of symmetry breaking described in section 5.1. To simplify notation, for $g \in G$ and $v \in V$ we will write gv instead of $\rho(g)(v)$.

Definition 4.3. The *Higgs potential* U is a smooth map $U \in C^\infty(V, \mathbb{R})$ that has minimal value $U_{\min} = 0$ and is G-invariant, i.e. $U(v) = U(gv)$ for all $g \in G$ and $v \in V$.

The Higgs potential evaluated at Φ can be integrated over all of space to find the potential energy of a configuration of the Higgs field. The energy, like all physical properties, is required to reflect the gauge symmetry described by G. For this reason, the Higgs potential needs to be G-invariant. In general, all physical observables have to be invariant under gauge transformations.

Definition 4.4. Let $W \subset M$ be open. A *gauge transformation* is a smooth map $g : W \to G$. The gauge transformation is *global* if $W = M$ and *local* otherwise.

A gauge transformation $g : W \to G$ acts on the gauge field A and Higgs field Φ, which transform as

$$A \mapsto g \cdot A \cdot g^{-1} + g \cdot \mathrm{d}g^{-1} \,, \tag{4.5}$$

$$\Phi \mapsto g\Phi \,. \tag{4.6}$$

In equation (4.5) the dot $\cdot$ denotes matrix multiplication and $\mathrm{d}g^{-1}$ is the differential of each matrix entry of $g^{-1} : W \to G$. Note that the first term is just the action of the adjoint representation of G. For the second term, $\mathrm{d}g^{-1}$ can be seen as taking values in $T_{g^{-1}}G$, which land in $T_eG = \mathfrak{g}$ after left multiplication with g. In (4.6) we suppressed the map ρ in the notation. In components, equation (4.5) reads

$$A_\mu \mapsto g \cdot A_\mu \cdot g^{-1} + g \cdot \partial_\mu g^{-1} \,. \tag{4.7}$$

It can be verified that under this transformation the field strength transforms as

$$F \mapsto g \cdot F \cdot g^{-1} \,. \tag{4.8}$$

Field configurations that can be transformed into each other using a global gauge transformation are *gauge equivalent*. The physical states can be viewed as the set of equivalence classes under the relation of being gauge equivalent. A choice of a representative of an equivalence class is called a *gauge*. Physical observables do not depend on the choice of gauge: they are *gauge invariant*. It can be shown that choosing a different global section $s : M \to P$ results in a gauge equivalent configuration of A [15, Thm. 5.4.2]. A gauge can in fact be identified with the choice of a section s. Local gauge transformations will be considered in section 6 in a situation where it is not possible to choose a global section.

The Lagrangian also needs to contain terms involving the derivatives of Φ to account for the kinetic and gradient energy of the Higgs field. The usual derivative $\partial_\mu \Phi$ transforms as $\partial_\mu \Phi \mapsto g\partial_\mu \Phi + (\partial_\mu g)\Phi$. We want to get rid of the second term, so that the derivative of Φ transforms just like Φ. For this we need a new definition of the derivative.

Definition 4.5. The *covariant derivative* is the map $D : C^\infty(M, V) \to \Omega^1(M, V)$ given for $\Phi \in C^\infty(M, V)$ and $X \in \mathfrak{X}(M)$ by

$$D_X \Phi = D(\Phi)(X) = \mathrm{d}\Phi(X) + \rho_*(A(X))\Phi \,. \tag{4.9}$$

In this equation ρ_* is the representation of $\mathfrak{g}$ induced by ρ.

We will again suppress the map ρ_* in our notation. The covariant derivative in components $D_\mu \Phi = D_{\partial_\mu} \Phi$ then reads

$$D_\mu \Phi = \partial_\mu \Phi + A_\mu \Phi \,. \tag{4.10}$$

The second term is chosen such that the covariant derivative transforms just like Φ (i.e. $D_\mu \Phi \mapsto g D_\mu \Phi$). Formula (4.9) is derived using a notion of parallel transport defined by the gauge field $\tilde{A}$. Just like for the exterior derivative, this construction can be generalised to a map $\Omega^k(M, V) \to \Omega^{k+1}(M, V)$, see section 5.12 in [15].

In order to define our Lagrangian, we choose a G-invariant inner product $\langle \cdot, \cdot \rangle_V$ on V and an Ad-invariant inner product $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ (which exist by Theorem B.2 since G is compact). Lastly, we introduce a *coupling constant* $e \in \mathbb{R}$ describing the strength of the interactions of the fields. Our assumption that G is simple or $\mathrm{U}(1)$ ensures that there is only one coupling constant (see Example B.11).

Definition 4.6. The *Yang-Mills-Higgs Lagrangian* $\mathcal{L}[A,\Phi] \in C^\infty(M,\mathbb{R})$ is given by

$$\mathcal{L}[A,\Phi] = -\tfrac{1}{4e^2}\langle F_{\mu\nu}, F^{\mu\nu}\rangle_{\mathfrak{g}} + \tfrac{1}{2}\langle D_\mu\Phi, D^\mu\Phi\rangle_V - U(\Phi)\,. \tag{4.11}$$

The prefactor $\frac{1}{2}$ of the covariant derivative term is omitted if V is a complex vector space. We choose an orthonormal basis $\{t_a\}_{a=1}^n$ of $\mathfrak{g}$ and introduce *structure constants* $f_{abc} \in \mathbb{R}$ such that $[t_a, t_b] = f_{abc}t_c$. We can expand A_μ and $F_{\mu\nu}$ in this basis as $A_\mu = eA_\mu^a t_a$ and $F_{\mu\nu} = eF^a_{\mu\nu}t_a$. The components are then related by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ef_{abc}A^b_\mu A^c_\nu\,. \tag{4.12}$$

The fields $A^1_\mu, \dots, A^n_\mu$ describe *n gauge bosons*, which are particles that act as force carriers. The field strength encodes the kinetic and gradient energy of the gauge bosons. We can re-express the Lagrangian as follows:

$$\mathcal{L}[A,\Phi] = -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \tfrac{1}{2}\langle D_\mu\Phi, D^\mu\Phi\rangle_V - U(\Phi)\,. \tag{4.13}$$

The Lagrangian is gauge invariant, Lorentz invariant and contains all the information of the model. Just like for the Klein-Gordon Lagrangian (2.1) the appearance of terms that are quadratic in the fields (e.g. a term proportional to $\langle\Phi,\Phi\rangle_V$) are interpreted by saying that the field Φ is massive. Terms of higher order in the fields correspond to interactions. The field equations are again derived by requiring the action $S = \int d^4x\, \mathcal{L}$ to be extremal. The energy of a field configuration also follows from the Lagrangian. It is given by a gauge invariant version of Equation 2.3 (see [39, Sect. 5.7]) and can be calculated using

$$E[A,\Phi] = \int d^3x \left(\tfrac{1}{2}F^a_{0i}F^a_{0i} + \tfrac{1}{4}F^a_{ij}F^a_{ij} + \tfrac{1}{2}\langle D_0\Phi, D_0\Phi\rangle_V + \tfrac{1}{2}\langle D_i\Phi, D_i\Phi\rangle_V + U(\Phi)\right). \tag{4.14}$$

Recall that there is a summation over a and the spatial indices $i, j = 1, 2, 3$ in this equation.

4.2 Electromagnetism

The gauge group of electromagnetism is $G = \mathrm{U}(1)$. We choose an inner product on $\mathfrak{u}(1)$ given by $\langle u, v\rangle_{\mathfrak{u}(1)} = -uv$ and pick the orthonormal basis $\{-i\}$. This basis can be used to view A_μ as a map $M \to \mathbb{R}$, i.e. the corresponding $\mathfrak{u}(1)$-valued gauge field is $-ieA_\mu$. The real valued gauge field A_μ describes the photon (the gauge boson of electromagnetism) and is called the *electromagnetic four potential*. The coupling constant e can be interpreted as (a dimensionless version of) the elementary electric charge. Because G is abelian, the adjoint representation is trivial and hence A_μ transforms under a gauge transformation g as

$$A_\mu \mapsto A_\mu + \frac{i}{e}g\partial_\mu g^{-1}\,. \tag{4.15}$$

If $g = \exp(ie\alpha)$ with α a smooth real valued function, the transformation becomes

$$A_\mu \mapsto A_\mu + \partial_\mu\alpha\,. \tag{4.16}$$

The electric potential A^0 and magnetic vector potential $\mathbf{A}$ are given by $A^\mu = (A^0, \mathbf{A})$ and the electric and magnetic fields can be computed using

$$\mathbf{E} = -\nabla A^0 - \partial_0 \mathbf{A}\,, \tag{4.17}$$

$$\mathbf{B} = \nabla \times \mathbf{A}\,. \tag{4.18}$$

The expression for the field strength simplifies to $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ since the bracket $[\cdot,\cdot]_{\mathfrak{u}(1)}$ is zero. Moreover, F is gauge invariant because G is abelian. The components of F can be expressed in matrix form as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{4.19}$$

In this equation $\mathbf{E} = (E_1, E_2, E_3)$ is the electric field and $\mathbf{B} = (B_1, B_2, B_3)$ is the magnetic field. We see that

$$E_i = F_{0i}\,, \tag{4.20}$$

$$B_i = -\tfrac{1}{2}\varepsilon_{ijk}F_{jk}\,, \tag{4.21}$$

where ε_{ijk} denotes the three-dimensional Levi-Civita symbol. This symbol equals zero if any index is repeated and equals the sign of the permutation $(1,2,3) \mapsto (i,j,k)$ otherwise. Because F is gauge invariant, from (4.20) and (4.21) it follows that the electromagnetic fields are gauge invariant as well.

We set the vacuum permittivity ε_0 and vacuum permeability μ_0 equal to 1. In the absence of any other fields, the Lagrangian of electromagnetism then reads

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} = \tfrac{1}{2}(|\mathbf{E}|^2 - |\mathbf{B}|^2)\,. \tag{4.22}$$

Requiring the action to be extremal leads to the field equations $\partial_\mu F^{\mu\nu} = 0$, which are equivalent to Maxwell equations 1 and 4 in vacuum. The other two Maxwell equations can be stated as $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$. This identity is automatically satisfied by the definition of F. The energy reduces to

$$E = \int d^3x\,\left(\tfrac{1}{2}F_{0i}F_{0i} + \tfrac{1}{4}F_{ij}F_{ij}\right) = \int d^3x\,\tfrac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}|^2)\,. \tag{4.23}$$

To add a complex scalar field Φ of charge $q = ne$ with $n \in \mathbb{Z}$ to the model, we choose the representation $\rho : \mathrm{U}(1) \to \mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^*$ given by $\rho(g) = g^n$. Under a gauge transformation $g = \exp(ie\alpha)$ the field then transforms as $\Phi \mapsto \exp(iq\alpha)\Phi$. We see that the phase of Φ can be gauge transformed into any function and therefore has no physical meaning. If $n \neq 0$ the covariant derivative $D\Phi$ contains interaction terms between Φ and the gauge field A.

Chapter 5

Topological defects in gauge theories

In some gauge theories the minima of the potential are not invariant under the entire gauge group G. For global symmetries we referred to this situation in section 2 by saying that the symmetry is spontaneously broken. We will use the same terminology for gauge theories and talk about spontaneous symmetry breaking of a gauge symmetry. It is however strictly speaking not possible to break a gauge symmetry: the group G represents a mathematical redundancy that is still present after the "symmetry breaking". Nevertheless, the formation of topological defects described in section 2.5 can also occur in spontaneously broken gauge theories. The Standard Model and Grand Unified Theories extending the Standard Model are based on such theories.

We first discuss the Higgs mechanism, which occurs during the breaking of a gauge symmetry and is used to explain the mass of gauge bosons in the Standard Model. This mechanism helps us to define the electromagnetic field of monopole defects in section 6. Subsequently, we classify topological defect solutions of finite energy in a general setting using homotopy groups. It is shown that monopoles are expected to exist in many Grand Unified Theories based on this classification.

5.1 Symmetry breaking and the Higgs mechanism

Gauge symmetry generally requires gauge bosons to be massless, because terms proportional to $A^a_\mu A^{a\mu}$ in the Lagrangian are not gauge invariant. However, if the gauge field interacts with a Higgs field it can acquire mass through the Higgs mechanism during symmetry breaking. We introduce this mechanism by way of an example following [35] and [15].

For the example, we consider the abelian Higgs model introduced in section 2.4. Recall that the model contains a complex scalar field $\Phi : M \to \mathbb{C}$ (which we now call the Higgs field) in the Mexican hat potential of Figure 2.2. The model has gauge group $G = \mathrm{U}(1)$, and as in section 4.2 we view the gauge field as a map $A_\mu : M \to \mathbb{R}$ by choosing the basis $\{-i\}$ of $\mathfrak{u}(1)$. The covariant derivative has components $D_\mu\Phi = \partial_\mu\Phi - ieA_\mu\Phi$ and the field strength is $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu$. The Lagrangian and energy are equal to

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\Phi)^* D^\mu\Phi - \tfrac{1}{4}\lambda(\Phi^*\Phi - v^2)^2\,, \tag{5.1}$$

$$E = \int d^3x\ \left(\tfrac{1}{2}F_{0i}F_{0i} + \tfrac{1}{4}F_{ij}F_{ij} + (D_0\Phi)^* D_0\Phi + (D_i\Phi)^* D_i\Phi + \tfrac{1}{4}\lambda(\Phi^*\Phi - v^2)^2\right). \tag{5.2}$$

We again remind the reader that there is a summation in (5.2) over the spacial indices i and j. The Higgs field transforms in the fundamental representation (corresponding to a charge $q = e$), i.e. after a gauge transformation g the transformed field $g\Phi$ is given by pointwise multiplication of g and Φ, and similarly for $A_\mu \Phi$.

We are interested in configurations of minimal energy, for which we need the following definitions. Note that our definitions apply to any theory with a Yang-Mills-Higgs Lagrangian which is invariant under a compact and connected gauge group G. The terminology is inherited from global symmetry breaking.

Definition 5.1. The *vacuum manifold* $\mathcal{M}$ is the set of all minima of the Higgs potential $U \in C^\infty(V, \mathbb{R})$, i.e. $\mathcal{M} = \{w \in V \mid U(w) = 0\}$. Elements of $\mathcal{M}$ are *vacuum vectors*.

We will make two important assumptions about the vacuum manifold $\mathcal{M}$:

1. Because the potential U is gauge invariant, we can restrict the action of G on V to $\mathcal{M}$. We will assume that G acts transitively on $\mathcal{M}$, i.e. given $w \in \mathcal{M}$ all zeros of U are of the form gw for some $g \in G$. This means that there is no accidental degeneracy or additional symmetry in the model. In particular, $\mathcal{M} \subset V$ is an orbit of the G-action on V and therefore a compact and connected embedded submanifold by Proposition B.3.

2. We assume that $\mathcal{M}$ is n-simple for $n = 1, 2$ (see Definition A.1). Proposition A.1 then implies that we can drop the basepoint condition on maps and homotopies. Recall that sufficient conditions for this to hold are that $\mathcal{M}$ is simply connected or a topological group. A situation where $\mathcal{M}$ is not n-simple is analysed in section 7.

Definition 5.2. A *vacuum state* is an equivalence class (under global gauge transformations) of a pair $(\Phi^{(v)}, A^{(v)})$ consisting of a Higgs field and a gauge field that minimise the energy. That is: the corresponding field strength $F^{(v)}$ and covariant derivative $D\Phi^{(v)}$ vanish and the value of $\Phi^{(v)}$ is in the vacuum manifold at every point of spacetime M.

Since the energy is gauge invariant, the definition above does not depend on the choice of representative $(\Phi^{(v)}, A^{(v)})$. A gauge for the vacuum state is called a *pure gauge*. There always exists a pure gauge where $A^{(v)}$ equals zero and $\Phi^{(v)}$ is constant [15, Prop. 8.1.5], called a *vacuum gauge*. Because G acts transitively on $\mathcal{M}$, it follows that the vacuum state is unique.

In the abelian Higgs model, the vacuum manifold is a circle of radius v in the complex plane and $G = \mathrm{U}(1)$ acts transitively on $\mathcal{M}$. A vacuum gauge is given by $A^{(v)} \equiv 0$ and $\Phi^{(v)} \equiv v$. The vacuum vector v is not invariant under the G-action. Hence, analogous to a global symmetry, if the system loses energy and decays into the vacuum state we say that the gauge symmetry is broken to the subgroup of G that fixes the vacuum vector v.

Definition 5.3. Let $w \in \mathcal{M}$ be a vacuum vector. The *unbroken subgroup H* is the stabiliser subgroup $G_w \subset G$ of w. If $H \subsetneq G$, we say that the gauge symmetry of the system is *spontaneously broken*.

The unbroken group H is a Lie subgroup of G and by Proposition B.3 the manifold $\mathcal{M}$ is diffeomorphic to G/H. Although the explicit embedding of H in G depends on the choice of vacuum vector, the group H is independent of this choice up to isomorphism. In the abelian Higgs model H is trivial and hence $\mathcal{M}$ is diffeomorphic to $G = \mathrm{U}(1)$. In particular, $\mathcal{M}$ is a topological group and the second assumption about the vacuum manifold is satisfied.

In quantum field theory small perturbations of the fields around the vacuum state correspond to particles. Let us consider perturbations in the vacuum gauge $(A^{(v)}, \Phi^{(v)}) \equiv (0, v)$ described by A_μ and $\Phi = v + \chi + i\theta$, where χ and θ are (small) real scalar fields. Inserting the perturbations into the Lagrangian (5.1), neglecting terms of higher than quadratic order in the fields and their derivatives, we obtain

$$\mathcal{L}^{(2)} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\chi\partial^\mu\chi + e^2v^2\left(A_\mu - \tfrac{1}{ev}\partial_\mu\theta\right)\left(A^\mu - \tfrac{1}{ev}\partial^\mu\theta\right) - \lambda v^2\chi^2\,. \tag{5.3}$$

Only the derivatives of θ appear in the Lagrangian, implying that θ is a massless field. The reason for this is that θ corresponds to a perturbation tangent to $\mathcal{M}$. Since the potential U attains its minimum on all of $\mathcal{M}$, it follows using a Taylor expansion that it is zero up to second order in θ (see [15, Thm. 8.1.15] for a proof in the general case). This means that the field θ only enters (5.3) through the covariant derivative, which up to second order only contains the derivatives of θ.

A massless field corresponding to perturbations tangent to $\mathcal{M}$ is called a *Nambu-Goldstone field* [12]. If the broken symmetry were global, there would be no gauge field and the Nambu-Goldstone field θ would contribute a term $\partial_\mu\theta\partial^\mu\theta$ to the Lagrangian (5.3). In this case the corresponding massless particle would be observable. However, for broken gauge symmetries the Nambu-Goldstone fields are not observed. In fact, the field θ can be made to disappear from the Lagrangian (5.3) using the transformation $g = \exp(-i\theta/v)$, showing that it is not physical in gauge theories. A gauge where all Nambu-Goldstone fields disappear is called a *unitary gauge*. In such a gauge we have

$$\mathcal{L}^{(2)} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\chi\partial^\mu\chi + e^2v^2A_\mu A^\mu - \lambda v^2\chi^2\,. \tag{5.4}$$

We see that the field χ describing a perturbation orthogonal to $\mathcal{M}$ and the gauge field are massive. To find their masses, we need to compare the field equations to the Klein-Gordon equation (2.5). Extremising the action leads to the equation (2.2) for χ and A_μ, which in the unitary gauge read

$$(\Box + \lambda v^2)\chi = 0\,, \tag{5.5}$$
$$\partial^\mu F_{\mu\nu} + 2e^2v^2A_\nu = 0\,. \tag{5.6}$$

From (5.5) we see that the field χ has mass $m_\chi = \sqrt{\lambda}v$. Equation (5.6) requires a bit more work: we have $\partial^\nu\partial^\mu F_{\mu\nu} = 0$ by the antisymmetry of $F_{\mu\nu}$, so differentiating (5.6) we obtain $\partial^\nu A_\nu = 0$. Writing out the term $\partial^\mu F_{\mu\nu}$ in the equation of motion and using this equality yields $(\Box + 2e^2v^2)A_\nu = 0$. Hence, the gauge field has mass $m_A = \sqrt{2}ev$. We say the gauge boson has acquired mass by "absorbing" the Nambu-Goldstone field. This is the Higgs mechanism.

The general case is covered in section 6 of [35] and in section 8 of [15] from a more mathematical perspective. We choose a vacuum gauge where $A^{(v)}$ is zero and $\Phi^{(v)}$ is constant and equal to a vacuum vector Φ_0. If the gauge group G and the unbroken group H have dimensions n and k respectively, we can find a suitable orthonormal basis $t_1, \dots, t_n$ of $\mathfrak{g}$ where $t_1, \dots, t_k$ form a basis of the Lie algebra $\mathfrak{h}$ of H. The generators of $\mathfrak{h}$ are called the *unbroken generators*. By Lemma B.1 they satisfy $t_i\Phi_0 = 0$ $(1 \leq i \leq k)$. The associated gauge bosons remain massless*.

*Note that the unbroken part of the gauge field satisfies $D_\mu\Phi = 0$. This implies $D_\mu(D_\nu\Phi) - D_\nu(D_\mu\Phi) = 0$, which reduces to $F_{\mu\nu}\Phi = 0$. This equation can be used to find the massless components in any gauge.

Each of the *broken generators* $t_{k+1}, \dots, t_n$ corresponds to a perturbation tangent to $\mathcal{M}$ and hence would give rise to a Nambu-Goldstone field. These fields get "absorbed" by the corresponding gauge bosons, which become massive. In addition, a massive scalar field appears for each perturbation orthogonal to $\mathcal{M}$ (with respect to the inner product $\mathrm{Re}\langle\cdot,\cdot\rangle_V$ on $T_{\Phi_0}V \cong V$).

The Higgs mechanism is used to explain why the gauge bosons W^+, W^- and Z^0 of the weak interaction have mass. In this model, the *electroweak* interaction is spontaneously broken into the weak interaction and electromagnetism at energies below the electroweak scale (around 246 GeV). The electroweak gauge group is $\mathrm{SU}(2) \times \mathrm{U}(1)$, which is broken down to a subgroup $\mathrm{U}(1)_{\mathrm{em}}$ describing electromagnetism. The resulting massive gauge bosons correspond to the gauge bosons of the weak interaction, while the massless gauge boson is identified with the photon. The model correctly predicts the masses and other properties of the $W^\pm$ and Z^0 bosons. The massive particle corresponding to the perturbation orthogonal to $\mathcal{M}$, the *Higgs boson*, was discovered in 2012 at CERN [6].

5.2 Topological classification of defects

In theories with spontaneous symmetry breaking, topological defects can exist depending on the homotopy groups of the vacuum manifold $\mathcal{M} = U^{-1}(0)$. Examples include the kink and the vortex described in section 2. We now turn to a general description of defects in gauge theories following [8] and [3]. The considerations are done at a fixed time, and hence the explicit time dependence of the fields is dropped. We begin by analysing the situation in three space dimensions. Consider a general theory with a compact and connected gauge group G and Lagrangian (4.13), where the Higgs potential is such that symmetry breaking occurs with unbroken group $H \subsetneq G$. We are looking for topological defect configurations for which the energy (4.14) is finite. To simplify the problem, we fix a gauge to get rid of the unphysical degrees of freedom. It is always possible to make a gauge transformation such that $A_0 = 0$ [8]. The energy then reads

$$E = \int d^3x \left(\tfrac{1}{2}(\partial_0 A_i^a)(\partial_0 A_i^a) + \tfrac{1}{4}F_{ij}^a F_{ij}^a + \tfrac{1}{2}\langle \partial_0\Phi, \partial_0\Phi\rangle_V + \tfrac{1}{2}\langle D_i\Phi, D_i\Phi\rangle_V + U(\Phi) \right). \quad (5.7)$$

We still have the freedom to make time-independent gauge transformations, since these do not change A_0 (see Equation 4.7). This freedom can be used to transform to a *radial gauge* in which the radial component $A_r = A(\frac{\partial}{\partial r})$ is zero for $r \geq 1$ [8]. Note that we have to exclude a small region around the origin because the radial component is not defined at the origin. Since all the terms in (5.7) are nonnegative, we obtain the following inequality in spherical coordinates:

$$\begin{aligned} E &\geq \int_1^\infty \int_0^\pi \int_0^{2\pi} r^2 \sin(\theta) d\phi d\theta dr \left(\tfrac{1}{2}\langle D_r\Phi, D_r\Phi\rangle_V + U(\Phi) \right) && (5.8) \\ &= \int_1^\infty \int_0^\pi \int_0^{2\pi} \sin(\theta) d\phi d\theta dr \left(\tfrac{1}{2}\langle r\partial_r\Phi, r\partial_r\Phi\rangle_V + r^2 U(\Phi) \right). && (5.9) \end{aligned}$$

In particular, if the energy is to be finite it is necessary for the radial integral to converge that the integrand tends to zero as $r \to \infty$ in any direction $\mathbf{n} \in S^2$:

$$\lim_{r\to\infty} r\partial_r \Phi(r\mathbf{n}) = 0\,, \tag{5.10}$$

$$\lim_{r\to\infty} r^2 U(\Phi(r\mathbf{n})) = 0\,. \tag{5.11}$$

To meet requirement (5.10) we will assume that $\lim_{r\to\infty}\Phi(r\mathbf{n})$ exists. From (5.11) it then follows that the limit must be an element of $\mathcal{M}$. Hence, to any finite energy configuration we can associate an asymptotic function $\Phi_\infty : S^2 \to \mathcal{M}$ given by

$$\Phi_\infty(\mathbf{n}) = \lim_{r\to\infty} \Phi(r\mathbf{n})\,. \tag{5.12}$$

We will assume that this limit is attained in any direction at a finite radius, so that Φ_∞ is equal to Φ restricted to a large sphere centered at the origin (this is generally true up to a negligible error). The map Φ_∞ is then continuous and defines an element $[\Phi_\infty] \in \pi_2(\mathcal{M})$ of the second homotopy group of the vacuum manifold (recall the assumption that $\mathcal{M}$ is 2-simple). Since time evolution is continuous and keeps the energy finite, it induces a continuous deformation of Φ_∞ and hence the homotopy class $[\Phi_\infty]$ does not change over time.

In order for this homotopy class to be physically interesting, it needs to be verified that it does not change under global gauge transformations. This is the content of the following lemma, adapted from section 6.3 in [8].

Lemma 5.1. *Let Φ and Φ' be gauge equivalent field configurations of finite energy such that the associated mappings Φ_∞ and Φ'_∞ exist. Then Φ_∞ and Φ'_∞ are homotopic.*

Proof. Let $g : M \to G$ be a gauge transformation such that $\Phi'(\mathbf{x}) = g(\mathbf{x})\Phi(\mathbf{x})$. On a very large sphere of radius r we then find $\Phi'_\infty(\mathbf{n}) = g(r\mathbf{n})\Phi_\infty(\mathbf{n})$ for all $\mathbf{n} \in S^2$. Consider the map $\Gamma_1 : S^2 \times [0,1] \to \mathcal{M}$ given by $\Gamma_1(\mathbf{n}, s) = g(sr\mathbf{n})\Phi_\infty(\mathbf{n})$. This map defines a homotopy from $g(0)\Phi_\infty$ to Φ'_∞. Moreover, since G is connected there exists a path γ from the unit element $e \in G$ to $g(0)$. This path induces the homotopy $\Gamma_2(\mathbf{n}, s) = \gamma(s)\Phi_\infty(\mathbf{n})$ from Φ_∞ to $g(0)\Phi_\infty$. It follows that Φ and Φ' are homotopic through the homotopies Γ_1 and Γ_2. □

We find that the class $[\Phi_\infty] \in \pi_2(\mathcal{M})$ is a conserved physical property of a finite energy configuration. We will refer to it as the *topological charge* of Φ. The topological charge can correspond to a physical observable, like a magnetic charge (see section 6). If the topological charge is nontrivial, the field Φ cannot decay into the vacuum state since the vacuum state corresponds to a constant map Φ and hence to the trivial homotopy class. In this case Φ describes a topological defect configuration. If Φ takes values in $\mathcal{M}$ in all of space, we can define a homotopy from Φ_∞ to a constant map by restricting Φ to smaller and smaller spheres. Hence, for defect configurations the map Φ has to leave $\mathcal{M}$ somewhere near the origin. This results in a nonzero localised energy density of the defect.

Next, we show how to construct topological defects of finite energy if $\pi_2(\mathcal{M}) \neq 0$. We will again be working in the gauge $A_0 = 0$ and find a field configuration such that each of the terms in (5.7) gives a finite contribution to the energy. Let $\Psi_\infty : S^2 \to \mathcal{M}$ be a smooth representative of a nontrivial homotopy class of $\pi_2(\mathcal{M})$. The existence of such a map follows from the Whitney approximation theorem [24, Thm. 6.26]. We can then choose a Higgs field of the form $\Phi(r\mathbf{n}) = f(r)\Psi_\infty(\mathbf{n})$, where $f : [0,\infty) \to [0,1]$ is a smooth function that is equal to zero in a neighborhood of 0 and equal to 1 for $r \geq 1$. It follows that Φ is smooth and its associated asymptotic function is Ψ_∞.

Since $\Phi(r\mathbf{n})$ lies in the vacuum manifold for $r > 1$, the partial derivatives $\partial_i \Phi(r\mathbf{n})$ are tangent to $\mathcal{M}$. Because G acts transitively on $\mathcal{M}$, at any point $r\mathbf{n}$ we can find an element $A_i(r\mathbf{n}) \in \mathfrak{g}$ such that

$$D_i \Phi(r\mathbf{n}) = \partial_i \Phi(r\mathbf{n}) + A_i(r\mathbf{n})\Phi(r\mathbf{n}) = 0\,, \tag{5.13}$$

i.e. the gauge field cancels an infinitesimal change in Φ. This follows from the fact that the map $\mathfrak{g} \to T_{f(r\mathbf{n})}\mathcal{M}$ sending X to $X\Phi(r\mathbf{n})$ is surjective: the kernel has dimension equal to $\dim H = \dim G - \dim \mathcal{M}$ by Lemma B.1, and therefore the image has dimension $\dim \mathcal{M}$. We show in Proposition B.4 using an Ehresmann connection that it is possible to choose an element $A_i(r\mathbf{n})$ satisfying (5.13) at every point with $r > 1$ in a smooth way.

The construction of Proposition B.4 gives us a gauge field such that the fourth term of the energy (5.7) is zero outside of a bounded region and hence gives a finite contribution. Moreover, since $\Phi(r\mathbf{n}) = \Phi(\mathbf{n})$ for $r \geq 1$ the derivatives $\partial_i \Phi(r\mathbf{n})$ fall off like $1/r$ as $r \to \infty$ (this follows directly from the expression for the gradient $\nabla\Phi$ in spherical coordinates). Together with equation (5.13), we find that the gauge field also goes like $1/r$ as $r \to \infty$. The field strength then goes like $1/r^2$, and its square like $1/r^4$. This means that the field strength goes to zero fast enough for the integral over the second term in (5.7) to converge. Note that this does not work in higher dimensions.

The term $U(\Phi)$ is zero for $r \geq 1$, so the only contributions to the energy that are left are the terms involving the time derivatives $\partial_0\Phi$ and $\partial_0 A_i^a$. However, these time derivatives form an independent set of data, and we can even choose our initial conditions such that $\partial_0\Phi = \partial_0 A_i^a = 0$. It follows that we have found a finite energy configuration with associated nontrivial homotopy class $[\Psi_\infty]$. A corresponding topological defect can be found by solving the field equations numerically with the configuration (Φ, A) as initial condition. We see that topological defects exist whenever $\pi_2(\mathcal{M}) \neq 0$.

The arguments in this section can be generalised to $d \leq 3$ space dimensions by considering maps from S^{d-1} into $\mathcal{M}$. The defects in $d = 3$ dimensions discussed above are monopoles. They are pointlike and have finite energy. For $d = 2$ we obtain vortices classified by $\pi_1(\mathcal{M})$. Viewed as solutions in three space dimensions that are independent of one of the space coordinates, these are cosmic strings that have their energy concentrated on a line and have finite energy per unit length. In $d = 1$ space dimension the relevant set is $\pi_0(\mathcal{M})$. The defects correspond to two-dimensional domain walls of finite energy per unit area. Note that $\mathcal{M}$ is always connected in the gauge theories considered in this thesis, so this kind of defect can only occur in models with a global symmetry.

Example 5.1. The kink covered in section 2.2 is a topological defect in $d = 1$ space dimension in a model with a global $\mathbb{Z}_2$ symmetry. It has vacuum manifold $\mathcal{M} = \{\pm a\}$ and the kink solution φ (equation 2.8) defines a mapping $\varphi_\infty : S^0 \to \mathcal{M}$ belonging to the nontrivial homotopy class of $\pi_0(\mathcal{M}) \cong \mathbb{Z}_2$ because it sends the two elements of S^0 to different path components of $\mathcal{M}$.

Example 5.2. The abelian Higgs model discussed in sections 2.4 and 5.1 has vacuum manifold $\mathcal{M} \cong S^1$, and since $\pi_1(\mathcal{M}) \cong \mathbb{Z}$ we were able to construct vortex configurations in this model. Our configurations had an asymptotic Higgs field $\Phi_\infty : S^1 \to \mathcal{M}$ of the form $\Phi_\infty(\theta) = v\exp(if(\theta))$. From Lemma 3.3 it follows that the winding number n of Φ_∞ is given by $2\pi n = f(2\pi) - f(0)$. This winding number is the topological charge of Φ, and we have seen in section 2.4 that this integer determines the magnetic flux of the defect.

Example 5.3. The Georgi-Glashow model introduced in section 2.4 has vacuum manifold $\mathcal{M} \cong S^2$. We have $\pi_2(S^2) \cong \mathbb{Z}$, and therefore this model admits monopole solutions. The hedgehog configuration of Figure 2.4 corresponds to the identity map $S^2 \to S^2 \cong \mathcal{M}$. Theorem 3.1 implies that the corresponding monopole has topological charge 1.

5.3 Monopoles in Grand Unified Theories

We have seen that topological defect solutions exist in a theory where symmetry breaking occurs in such a way that $\pi_n(\mathcal{M}) \neq 0$ for some $n \leq 2$. Moreover, such solutions are always formed in this case through the Kibble mechanism. For the electroweak symmetry breaking in the Standard Model, the vacuum manifold $\mathcal{M} \cong (\mathrm{SU}(2) \times \mathrm{U}(1))/\mathrm{U}(1)_{\mathrm{em}}$ is a 3-sphere. Since $\pi_n(S^3) = 0$ for $n \leq 2$ by Theorem 3.2, no topological defects are formed.

However, topological defects are a general prediction of Grand Unified Theories (GUTs). These are models that try to unify all three gauge interactions (electromagnetism, weak force and strong force) into a simple compact connected gauge group G at very high energies (around 10^{15} GeV), so that all interactions are described by a single coupling constant e. This symmetry group G is then broken down to the symmetry group $H = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ of the Standard Model, and even further to $\mathrm{SU}(3) \times \mathrm{U}(1)_{\mathrm{em}}$ through electroweak symmetry breaking. We will show following [3] that monopoles are generally predicted by GUTs, even if the gauge group is only semisimple.

Since the gauge boson of electromagnetism (the photon) is massless, the corresponding $\mathrm{U}(1)$ gauge group must be unbroken in nature. For this reason, the unbroken group H during the symmetry breaking of G contains a factor $\mathrm{U}(1)$, and hence by Lemma 3.2 the fundamental group $\pi_1(H)$ contains a factor $\mathbb{Z}$. Moreover, by Proposition 3.1 there is an exact sequence (recall the assumption that G acts transitively on $\mathcal{M}$)

$$\cdots \to \pi_2(G) \to \pi_2(\mathcal{M}) \to \pi_1(H) \to \pi_1(G) \to \dots \tag{5.14}$$

Theorem 3.4 gives $\pi_2(G) = 0$, so we have $\pi_2(\mathcal{M}) \cong \ker(\pi_1(H) \to \pi_1(G))$. Using the fact that G is semisimple, Theorem 3.5 implies that this kernel is nontrivial (and even infinite) because $\pi_1(G)$ is finite and $\pi_1(H)$ is not. We see that $\pi_2(\mathcal{M}) \neq 0$, i.e. monopole solutions exist in (many) GUTs. Since the solutions are stable, the monopoles are expected to still be around today.

So why have these monopoles never been observed? One possible explanation is that they simply were never formed, for example because the early universe is not described by a GUT or because the unification temperature was never reached. The other possibility is that they do exist but we have not found them (yet). This option is problematic, because calculations show that the monopoles formed in the early universe would dominate the energy density of the universe today [30]. This is known as the *monopole problem*. An elegant solution to this problem could be provided by cosmic inflation [14]. Inflation refers to a period of time shortly after the monopole formation during which the universe expanded at an exponential rate, diluting the monopole density to an acceptable level. Besides the monopole problem, cosmic inflation could also solve two other major problems in cosmology: the horizon and flatness problems.

Chapter 6

Magnetic monopoles

The monopoles constructed in section 5.2 carry a topological charge given by the homotopy class of the map Φ_∞. In this section we discuss in what sense this topological charge is magnetic. We review the Dirac monopole and its connection to homotopy theory. This leads to a topological interpretation of magnetic charge, and in section 6.3 it is shown that in some gauge theories the topological charge of monopole solutions is magnetic in this sense. Section 6.4 covers the 't Hooft-Polyakov monopole, which is an example of a topological defect with a magnetic charge.

6.1 Dirac's quantisation condition

We follow the approach by Wu and Yang [43] as treated in [31] and [7] to describe the Dirac monopole. In this subsection, we consider vector potentials $\mathbf{A} = (A^1, A^2, A^3)$ defined on open subsets W of $\mathbb{R}^3$. The signature of the three dimensional metric is $(+,+,+)$, i.e. the corresponding gauge field is given by $A = -ieA_i \mathrm{d}x^i = -ieA^i \mathrm{d}x^i \in \Omega^1(W, \mathfrak{u}(1))$.

The electric field $\mathbf{E}$ of an electric charge q positioned at the origin is defined on $\mathbb{R}^3 \setminus \{0\}$ and given by $\mathbf{E} = \frac{q}{4\pi r^2}\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector in the radial direction. Similarly, we can consider the magnetic field

$$\mathbf{B} = \frac{g}{r^2}\hat{\mathbf{r}}\,. \tag{6.1}$$

For this field the magnetic flux through a sphere around the origin is $4\pi g$. We call g, the magnetic flux divided by 4π, the *magnetic charge* contained inside the sphere. Objects carrying a nonzero magnetic charge are magnetic monopoles. There exists no vector potential $\mathbf{A}$ defined on $\mathbb{R}^3 \setminus \{0\}$ describing (6.1): if $\mathbf{B} = \nabla \times \mathbf{A}$ the magnetic flux through a sphere around the origin would be zero by Stokes' theorem instead of $4\pi g$. However, it is possible to define vector potentials locally on the upper and lower regions of space

$$W^U = \{(r,\phi,\theta) \in \mathbb{R}^3 \setminus \{0\} \mid \theta < \tfrac{3}{4}\pi\}\,, \tag{6.2}$$
$$W^L = \{(r,\phi,\theta) \in \mathbb{R}^3 \setminus \{0\} \mid \theta > \tfrac{1}{4}\pi\}\,, \tag{6.3}$$

by

$$\mathbf{A}^U = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\boldsymbol{\phi}}\,, \tag{6.4}$$

$$\mathbf{A}^L = -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{\boldsymbol{\phi}}\,. \tag{6.5}$$

$\mathbf{A}^U$ and $\mathbf{A}^L$ are defined to be zero for $\theta = 0$ and $\theta = \pi$ respectively. The magnetic field given by these potentials reduces to (6.1). The corresponding $\mathfrak{u}(1)$-valued 1-forms are

$$A^U = -ieg(1-\cos\theta)\mathrm{d}\phi\,, \tag{6.6}$$

$$A^L = ieg(1+\cos\theta)\mathrm{d}\phi\,. \tag{6.7}$$

For the two vector potentials to describe the same physics, they must be related by a gauge transformation on $W^U \cap W^L$. Indeed, for $\Omega = \exp(i2eg\phi)$ we find

$$A^U - A^L = -2ieg\mathrm{d}\phi = \Omega\mathrm{d}\Omega^{-1}\,. \tag{6.8}$$

Under this gauge transformation, a complex scalar field Φ of charge e transforms in the fundamental representation as $\Phi \mapsto \Omega\Phi$. In order for the transformed field to be single valued, we must require that $\Omega(\phi = 2\pi) = \Omega(\phi = 0)$. This leads to the *Dirac quantisation condition*

$$2eg \in \mathbb{Z}\,. \tag{6.9}$$

It follows that magnetic charge is quantised. Conversely, if a single magnetic monopole of charge g exists, we find that all electric charges must be multiples of $\frac{1}{2g}$. Hence magnetic monopoles provide an attractive explanation for the quantisation of electric charge observed in nature.

Note that the integer $n = 2eg$ is equal to the winding number of the gauge transformation $\Omega = \exp(i2eg\phi)$ restricted to a circle centered at the origin by Lemma 3.3. In general, we can interpret Ω as an object detecting the magnetic charge [31, Sect. 4.1].

Lemma 6.1. *Suppose a magnetic field is given by $A^U \in \Omega^1(W^U, \mathfrak{u}(1))$ and $A^L \in \Omega^1(W^L, \mathfrak{u}(1))$. Let Ω be a gauge transformation relating A^U and A^L on $W^U \cap W^L$, i.e.*

$$A_i^U = A_i^L + \frac{i}{e}\Omega\partial_i\Omega^{-1}\,. \tag{6.10}$$

Consider a circle S^1_R of radius R centered at the origin in the $\theta = \frac{\pi}{2}$ plane. Then the total magnetic charge g contained inside the sphere S^2_R of radius R centered at the origin is equal to $g = \frac{n}{2e}$, where n is the winding number of $\Omega|_{S^1_R}$.

Proof. Let $f : I \to \mathbb{R}$ be a continuous lift of the loop $\gamma(t) = \Omega|_{S^1_R}(\phi = 2\pi t)$ along the covering map $p : \mathbb{R} \to \mathrm{U}(1)$, i.e. $\Omega|_{S^1_R}(\phi = 2\pi t) = \exp(2\pi i f(t))$. Recall that the winding number of $\Omega|_{S^1_R}$ is given by $n = f(1) - f(0)$. Moreover, f is smooth because p is a local diffeomorphism.

Let $H_R^U, H_R^L \subset S_R^2$ denote the upper an lower hemispheres. By Stokes' theorem, the magnetic flux Ψ through S_R^2 is

$$\begin{aligned}\Psi &= \iint_{S_R^2} \mathbf{B}\cdot d\mathbf{s} = \iint_{H_R^U} (\nabla\times\mathbf{A}^U)\cdot d\mathbf{s} + \iint_{H_R^L} (\nabla\times\mathbf{A}^L)\cdot d\mathbf{s} = \oint_{S_R^1} \mathbf{A}^U\cdot d\mathbf{l} - \oint_{S_R^1} \mathbf{A}^L\cdot d\mathbf{l}\\ &= \oint_{S_R^1} \frac{i}{e}\Omega\nabla\Omega^{-1}\cdot d\mathbf{l} = \frac{i}{e}\int_0^{2\pi} (\Omega\nabla\Omega^{-1})|_{S_R^1}(\phi)\cdot(R\hat{\boldsymbol{\phi}}d\phi) = \frac{i}{e}\int_0^{2\pi}(\Omega\partial_\phi\Omega^{-1})|_{S_R^1}(\phi)d\phi\\ &= \frac{i}{e}\int_0^1 (\Omega\partial_t\Omega^{-1})|_{S_R^1}(\phi=2\pi t)dt = \frac{i}{e}\int_0^1 -2\pi i\partial_t f(t)dt = \frac{2\pi}{e}(f(1)-f(0)) = \frac{2\pi n}{e}\,.\end{aligned} \tag{6.11}$$

The magnetic charge producing this flux is $g = \frac{n}{2e}$. □

Note that the map $\Omega|_{S_R^1}$ changes continuously as we vary R, so the winding number n is constant as a function of R. By considering smaller and smaller circles, we find that the magnetic charge must be concentrated at the origin and hence the monopole is a point singularity.

6.2 Dirac monopole and the Hopf fibration

The Dirac monopole has a singularity at the location of the monopole and hence is defined on $\mathbb{R}^3\setminus\{0\}$, which is not a contractible space. Therefore, not all principal fibre bundles over $\mathbb{R}^3\setminus\{0\}$ have to be trivial, so we can not always choose a global section as we did in section 4.1. This is exactly what happens with the gauge field describing (6.1): it is given by a connection on a nontrivial principal U(1)-bundle, and after choosing local sections it takes the form of (6.6) and (6.7). Because these expressions do not depend on the radial coordinate, we can view the gauge field as being defined on S^2. Under this identification, the gauge field is the natural connection on the Hopf bundle defined in Example B.15 [25].

Proposition 6.1. *The Dirac monopole with magnetic field (6.1) and unit magnetic charge $g = \frac{1}{2e}$ is described by the natural connection $\tilde{A}$ on the Hopf bundle. Let h_1 and h_2 be the local trivialisations and $\tau : S^2 \xrightarrow{\sim} \mathbb{C}\mathrm{P}^1$ the diffeomorphism defined in Example 3.2. Then after choosing the local sections $s_U(x) = h_1^{-1}(\tau(x),1)$ and $s_L(x) = h_2^{-1}(\tau(x),1)$ we have $A^U = s_U^*\tilde{A}$ and $A^L = s_L^*\tilde{A}$, where A^L and A^S are given by (6.6) and (6.7).*

Proof. The sets $U_i = \{[z^1 : z^2] \in \mathbb{C}\mathrm{P}^1 \mid z^i \neq 0\}$ on which the trivialisations are defined correspond to $S^2\setminus\{(0,0,-1)\}$ for $i=1$ and $S^2\setminus\{(0,0,1)\}$ for $i=2$ under τ^{-1}, implying that s_U and s_L are well-defined on W^U and W^L respectively. Note that s_U and s_L are indeed local sections, as follows from the fact that the h_i are local trivialisations. Let us begin by computing the map τ. The homeomorphism $\sigma : S^2 \xrightarrow{\sim} \mathbb{C}\cup\{\infty\}$ is constructed using the stereographic projection and sends the north pole to ∞:

$$\sigma(x^1,x^2,x^3) = \begin{cases}\frac{x^1+ix^2}{1-x^3}\,, & x^3\neq 1\\ \infty\,, & x^3 = 1.\end{cases} \tag{6.12}$$

Composing with the inverse of $\mu : \mathbb{C}\mathrm{P}^1 \xrightarrow{\sim} \mathbb{C}\cup\{\infty\}$, $[z^1 : z^2]\mapsto \frac{z^1}{z^2}$ with $(z^1,z^2)\in S^3$ gives

$$\tau(x^1,x^2,x^3) = (\mu^{-1}\circ\sigma)(x^1,x^2,x^3) = \begin{cases}\left[\frac{x^1+ix^2}{\sqrt{2(1-x^3)}} : \frac{1-x^3}{\sqrt{2(1-x^3)}}\right], & x^3\neq 1\\ [1:0]\,, & x^3 = 1.\end{cases} \tag{6.13}$$

We can now write down an explicit formula for s_U and s_L using τ and the maps h_i^{-1} given by $h_i^{-1}([z^1 : z^2], \lambda) = \lambda |z^i| (z^i)^{-1}(z^1, z^2)$. Using spherical coordinates (ϕ, θ) on S^2 we obtain

$$s_U(\phi,\theta) = h_1^{-1}(\tau(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta), 1) = (\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}e^{-i\phi}) \,, \tag{6.14}$$
$$s_L(\phi,\theta) = h_2^{-1}(\tau(\cos\phi\sin\theta, \sin\phi\sin\theta, \cos\theta), 1) = (\cos\tfrac{\theta}{2}e^{i\phi}, \sin\tfrac{\theta}{2}) \,. \tag{6.15}$$

For $(z^1, z^2) \in S^3$ we view $T_{(z^1,z^2)}S^3$ as a subspace of $\mathbb{C}^2$. The differentials of s_U and s_L are then equal to

$$D_{(\phi,\theta)}s_U = (-\tfrac{1}{2}\sin\tfrac{\theta}{2}\mathrm{d}\theta, \tfrac{1}{2}\cos\tfrac{\theta}{2}e^{-i\phi}\mathrm{d}\theta - i\sin\tfrac{\theta}{2}e^{-i\phi}\mathrm{d}\phi) \,, \tag{6.16}$$
$$D_{(\phi,\theta)}s_L = (-\tfrac{1}{2}\sin\tfrac{\theta}{2}e^{i\phi}\mathrm{d}\theta + i\cos\tfrac{\theta}{2}e^{i\phi}\mathrm{d}\phi, \tfrac{1}{2}\cos\tfrac{\theta}{2}\mathrm{d}\theta) \,. \tag{6.17}$$

Here $(D_{(\phi,\theta)}s_U)(v)$ and $(D_{(\phi,\theta)}s_L)(v)$ are given in terms of $\mathrm{d}\theta|_{(\phi,\theta)}(v)$ and $\mathrm{d}\phi|_{(\phi,\theta)}(v)$ for $v \in T_{(\phi,\theta)}S^2$. Finally, we compute the pullbacks of $\tilde{A}$ (see equation (B.18)):

$$\begin{aligned}
A^U = s_U^*\tilde{A} &= \tfrac{1}{2}((\cos\tfrac{\theta}{2})(-\tfrac{1}{2}\sin\tfrac{\theta}{2}\mathrm{d}\theta) - (\cos\tfrac{\theta}{2})(-\tfrac{1}{2}\sin\tfrac{\theta}{2}\mathrm{d}\theta) \\
&+ (\sin\tfrac{\theta}{2}e^{i\phi})(\tfrac{1}{2}\cos\tfrac{\theta}{2}e^{-i\phi}\mathrm{d}\theta - i\sin\tfrac{\theta}{2}e^{-i\phi}\mathrm{d}\phi) - (\sin\tfrac{\theta}{2}e^{-i\phi})(\tfrac{1}{2}\cos\tfrac{\theta}{2}e^{i\phi}\mathrm{d}\theta + i\sin\tfrac{\theta}{2}e^{i\phi}\mathrm{d}\phi)) \\
&= -i\sin^2\tfrac{\theta}{2}\mathrm{d}\phi = -\tfrac{1}{2}i(1-\cos\theta)\mathrm{d}\phi \,, \qquad (6.18) \\
A^L = s_L^*\tilde{A} &= \tfrac{1}{2}((\cos\tfrac{\theta}{2}e^{-i\phi})(-\tfrac{1}{2}\sin\tfrac{\theta}{2}e^{i\phi}\mathrm{d}\theta + i\cos\tfrac{\theta}{2}e^{i\phi}\mathrm{d}\phi) \\
&- (\cos\tfrac{\theta}{2}e^{i\phi})(-\tfrac{1}{2}\sin\tfrac{\theta}{2}e^{-i\phi}\mathrm{d}\theta - i\cos\tfrac{\theta}{2}e^{-i\phi}\mathrm{d}\phi) + (\sin\tfrac{\theta}{2})(\tfrac{1}{2}\cos\tfrac{\theta}{2}\mathrm{d}\theta) - (\sin\tfrac{\theta}{2})(\tfrac{1}{2}\cos\tfrac{\theta}{2}\mathrm{d}\theta)) \\
&= i\cos^2\tfrac{\theta}{2}\mathrm{d}\phi = \tfrac{1}{2}i(1+\cos\theta)\mathrm{d}\phi \,. \qquad (6.19)
\end{aligned}$$

Comparing with (6.6) and (6.7) we indeed see that this gauge field describes a Dirac monopole of unit charge $g = \frac{1}{2e}$. □

We find that the Hopf bundle provides the principal bundle structure of the Dirac monopole, showing that the Hopf map $S^3 \to S^2$ has applications not just in mathematics (e.g. to calculate $\pi_3(S^2)$ as in section 3) but also in physics. An overview of other physical applications of the Hopf bundle can be found in [41].

6.3 Magnetic charge of topological defects

In section 5.3 we have seen that $\pi_2(\mathcal{M})$ is nonzero in many GUTs, and hence there exist field configurations (Φ, A) describing a monopole defect. In this section we show that in many situations there is a natural way to assign a magnetic charge to these configurations [31]. We again work at a constant time and drop the time dependence. We will assume that the gauge group G is a simply connected compact matrix group and the unbroken group $H \cong \mathrm{U}(1)$ describes electromagnetism. From the exact sequence (5.14) we then obtain isomorphisms $\pi_2(\mathcal{M}) \cong \pi_1(H) \cong \mathbb{Z}$, so we can view the topological charge $[\Phi_\infty] \in \pi_2(\mathcal{M})$ of the Higgs field Φ as an integer n. Note that the topological charge in this sense is only defined up to sign until we choose an isomorphism $\pi_2(\mathcal{M}) \xrightarrow{\sim} \mathbb{Z}$. Similarly, every loop in H has a winding number $m \in \mathbb{Z}$ defined up to sign.

We will identify the massless gauge boson corresponding to the unbroken generator of the Lie algebra $\mathfrak{h}$ of H with the photon. To calculate the electromagnetic field, we would like to transform to a unitary gauge where Φ is constant and equal to a vacuum vector $\Phi_0 \in \mathcal{M}$ and embed H into G as the stabiliser of Φ_0. However, if $n \neq 0$ this cannot be achieved globally by Lemma 5.1. We can try to transform to a unitary gauge locally far away from the core of the defect by defining gauge transformations on the sets $V^U = W^U \cap S^2_\infty$ and $V^L = W^L \cap S^2_\infty$. Here W^U and W^L are given by (6.2) and (6.3) and S^2_∞ is a very large sphere around the origin such that $\Phi = \Phi_\infty$ on this sphere. The domains can always be extended to W^U and W^L (such that the transformations are independent of the radial coordinate) to meet the requirement of Definition 4.4 that the domains are open in $\mathbb{R}^3$.

Lemma 6.2. *There exist continuous maps $g^U : V^U \to G$ and $g^L : V^L \to G$ such that the maps $g^U\Phi|_{V^U} : V^U \to \mathcal{M}$ and $g^L\Phi|_{V^L} : V^L \to \mathcal{M}$ are constant and equal to Φ_0 everywhere.*

Proof. Recall from Proposition 3.1 that $H \to G \to \mathcal{M}$ is a fibre bundle, where $\pi : G \to \mathcal{M}$ is defined by $\pi(g) = g\Phi_0$. Since the closures $\overline{V^U}$ and $\overline{V^L}$ are homeomorphic to the closed disk D^2, by the homotopy lifting property (see Definition A.2) we can lift $\Phi|_{V^U}$ and $\Phi|_{V^L}$ along π to maps $\tilde{\Phi}|_{V^U}$ and $\tilde{\Phi}|_{V^L}$ into G (by restricting the lifts on $\overline{V^U}$ and $\overline{V^L}$). The desired maps are then given by $g^U = (\tilde{\Phi}|_{V^U})^{-1}$ and $g^L = (\tilde{\Phi}|_{V^L})^{-1}$. □

We will assume that we can choose the lifts in Lemma (6.2) such that the maps g^U and g^L are smooth, making them into local gauge transformations. It is claimed in section 4.3 of [31] that this is possible, and we will construct smooth maps g^U and g^L explicitly for the 't Hooft-Polyakov monopole in section 6.4.

Let us denote the gauge field after the transformations g^U and g^L on V^U and V^L by A^U and A^L respectively. In this unitary gauge the Higgs field is constant far away from the origin, and so the first term in the covariant derivative (4.9) vanishes. In order for the energy to remain finite, the second term must also (asymptotically) vanish, i.e. $A^U_\mu\Phi_0 = A^L_\mu\Phi_0 \equiv 0$. It follows that the gauge field represents a small perturbation taking values in $\mathfrak{h}$ at large distances in this gauge and hence describes the electromagnetic field.

On $V^U \cap V^L$ the fields A^U and A^L are related by the gauge transformation $\Omega = g^U(g^L)^{-1}$. Since $g^U\Phi = g^L\Phi \equiv \Phi_0$ on $V^U \cap V^L$ we find $\Omega\Phi_0 = \Phi_0$, so Ω takes values in the unbroken subgroup H. On the equator $S^1_\infty \subset S^2_\infty$ we then find $\Omega A^L_\mu \Omega^{-1} = A^L_\mu$ (the adjoint representation of H on $\mathfrak{h}$ is trivial because $H \cong \mathrm{U}(1)$ is abelian), so the transformation rule (4.7) reads

$$A^U_\mu = A^L_\mu + \Omega\partial_\mu\Omega^{-1} . \tag{6.20}$$

This equation looks very similar to (6.10). Let us choose a generator t_1 of $\mathfrak{h}$ and identify the t_1-component of A_μ with a coupling constant e times the electromagnetic four potential, as we did in section 4.2 for $H = \mathrm{U}(1)$. Just like in Lemma 6.1 we can calculate the magnetic flux through S^2_∞ and find that the magnetic charge of the defect equals (up to sign) m times a minimal magnetic charge $g_{\min}$, where m equals the winding number of $\Omega|_{S^\infty_1} : S^\infty_1 \to H$. Hence, we only need to verify that $m \neq 0$ in order to conclude that the topological defect is indeed a magnetic monopole. This follows from the characterisation of the connecting homomorphism $\delta : \pi_2(\mathcal{M}) \to \pi_1(H)$ given in Proposition A.2.

Proposition 6.2. *The winding number m of the gauge transformation $\Omega|_{S^\infty_1}$ is equal up to sign to the topological charge n of Higgs field Φ.*

Proof. Note that $(g^U)^{-1}$ and $(g^L)^{-1}$ are lifts of Φ along $\pi : G \to \mathcal{M}$, $\pi(g) = g\Phi_0$. Recall from Proposition B.3 that π induces a diffeomorphism $G/H \xrightarrow{\sim} \mathcal{M}$. Under this diffeomorphism, the maps $(g^U)^{-1}$ and $(g^L)^{-1}$ correspond to the lifts Ψ^+ and Ψ^- in Proposition A.2. This proposition tells us that $\delta([\Phi_\infty]) = [\Omega|_{S^1_\infty}]$, where $\delta : \pi_2(\mathcal{M}) \to \pi_1(H)$ is the connecting homomorphism. Because $\pi_2(G) = \pi_1(G) = 0$ the long exact sequence (5.14) gives that δ is an isomorphism, and the result follows. □

We see that the topological charge of the monopole can be identified with the magnetic charge*. Just like for the Dirac monopole, the magnetic charge is quantised. However, GUT monopoles are nonsingular in all of space. For the Dirac monopole we found a singularity at the origin because we cannot continuously change the winding number of the gauge transformation Ω to zero as the radius of S^1_∞ decreases. In GUTs Ω can leave the U(1) subgroup and wander through the larger group G as the radius decreases, which is why we can avoid the singularity.

More generally, suppose H is any (compact and connected) gauge group. Consider a configuration described by $\mathfrak{h}$-valued gauge fields A^U on V^U and A^L on V^L. We can define the magnetic charge enclosed by S^2_∞ to be the homotopy class $[\Omega|_{S^1_\infty}] \in \pi_1(H)$, where Ω is a H-valued gauge transformation relating A^U and A^L on $V^U \cap V^L$ [31, Sect. 4.2]. Note that H has to be connected for this element to be well-defined: an example where this is not the case is worked out in section 7. We have seen in Lemma 6.1 that this definition coincides with our expectation when $H = \mathrm{U}(1)$. For any monopole defect created during the breaking of a simply connected gauge group G to H we find that the topological charge can be identified with the magnetic charge. If G is not simply connected, every monopole still has a nonzero magnetic charge: the map $\delta : \pi_2(\mathcal{M}) \to \pi_1(H)$ is still injective because $\pi_2(G) = 0$.

6.4 't Hooft-Polyakov monopole

In this section, we study monopole defects in the Georgi-Glashow model introduced in section 2.4. A monopole solution carrying a magnetic charge was found in 1974 by 't Hooft [18] and Polyakov [28]. Our approach follows [35] and [10]. The model has gauge group $G = \mathrm{SU}(2)$ and the Higgs field Φ is in the adjoint representation (i.e. the Higgs vector space is $\mathfrak{su}(2)$ and G acts on Φ as $\Phi \mapsto g\Phi g^{-1}$, see Example B.9). We choose a basis $t_a = \frac{1}{2i}\tau_a$ ($a = 1, 2, 3$) of $\mathfrak{su}(2)$, where τ_a are the Pauli matrices given by

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.21}$$

This basis is orthonormal with respect to -2 times the trace, i.e. $\langle A, B \rangle_{\mathfrak{g}} = -2\mathrm{Tr}(AB)$. The structure constants are given by the Levi-Civita symbol $f_{abc} = \varepsilon_{abc}$ (see section 4.2).

For any $g \in \mathrm{SU}(2)$, the linear map $\mathrm{Ad}(g) : \mathfrak{su}(2) \to \mathfrak{su}(2)$, $A \mapsto gAg^{-1}$ preserves the norm and can hence be seen as an orthogonal transformation, i.e. an element of O(3). Since G is connected, its image under Ad is connected as well. The image contains $\mathrm{Ad}(I) = \mathrm{Id}$ and therefore is part of the connected component SO(3) of O(3). It follows that we can view a group element g as a rotation in field space. It can be verified that $\mathrm{Ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$ is surjective and has kernel $\{\pm I\}$, and in fact this map defines the double cover of SO(3) mentioned in section 3.3.

*Note that we have disregarded the electric charge in the considerations above. It is in fact possible for a defect to carry both electric and magnetic charge: such configurations are called dyons [38].

The field Φ can be described using three real scalar fields Φ^a such that $\Phi = \Phi^a t_a$, and the gauge field is expanded in the usual way as $A_\mu = eA^a_\mu t_a$ with coupling constant e. The gauge field acts on Φ via the commutator $[A_\mu, \Phi]$. Hence, the covariant derivative expressed in components reads

$$(D_\mu \Phi)^a = \partial_\mu \Phi^a + e\varepsilon_{abc} A^b_\mu \Phi^c \,. \tag{6.22}$$

The Lagrangian is equal to

$$\mathcal{L}[A, \Phi] = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \tfrac{1}{2}(D_\mu \Phi)^a (D^\mu \Phi)^a - \tfrac{1}{4}\lambda(\Phi^a \Phi^a - v^2)^2 \,, \tag{6.23}$$

where λ and v are positive constants. Note that this Lagrangian equals (2.19) if we identify $\mathfrak{su}(2)$ with $\mathbb{R}^3$ using the chosen basis. The potential leads to symmetry breaking: the vacuum manifold $\mathcal{M}$ is a 2-sphere with radius v in field space. $\mathcal{M}$ is simply connected and therefore n-simple for all n, and G acts transitively on $\mathcal{M}$ because any two vacuum vectors are related by a rotation. We choose $\Phi_0 = vt_3$ as our vacuum vector. Then t_3 is the only generator that commutes with Φ_0 and hence is unbroken. It gives rise to an unbroken subgroup

$$H = \{\exp(xt_3) \mid x \in \mathbb{R}\} = \left\{ \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix} \end{pmatrix} \middle| \, x \in \mathbb{R} \right\} \tag{6.24}$$

that is isomorphic to U(1) and consists of elements acting as rotations around the third axis in field space. Due to the Higgs mechanism two of the gauge bosons acquire a mass $m = ev$, while the third one remains massless. In a gauge where $\Phi \equiv \Phi_0$ the massless component of the gauge field is A^3_μ and we will identify this with the electromagnetic four potential. Because SU(2) is simply connected, the exact sequence (5.14) gives $\pi_2(\mathcal{M}) \cong \pi_1(\mathrm{U}(1)) \cong \mathbb{Z}$. Hence, monopole defects exist in this model. The field equations are (see [10])

$$\left(D_\mu F^{\mu\nu}\right)^a = e\varepsilon_{abc}(D^\nu \Phi)^b \Phi^c \,, \tag{6.25}$$
$$\left(D_\mu D^\mu \Phi\right)^a = -\lambda(\Phi^b \Phi^b - v^2)\Phi^a \,. \tag{6.26}$$

We will search for monopole solutions that do not depend on time in the gauge $A_0 = 0$, implying that the Higgs field has no kinetic energy ($D_0 \Phi = 0$) and by (4.17) that there is no electric field. Recall that this still leaves us the freedom to make time-independent gauge transformations. The energy is given by

$$E[A, \Phi] = \int d^3x \, \left(\tfrac{1}{4} F^a_{ij} F^a_{ij} + \tfrac{1}{2}(D_i \Phi)^a (D_i \Phi)^a + \tfrac{1}{4}\lambda(\Phi^a \Phi^a - v^2)^2 \right) . \tag{6.27}$$

The asymptotic Higgs field Φ_∞ needs to belong to a nontrivial homotopy class of $\pi_2(\mathcal{M})$, which we can accomplish using the hedgehog configuration of Figure 2.4. We require that

$$\Phi^a(r\mathbf{n}) = n^a v \text{ as } r \to \infty \,. \tag{6.28}$$

In this equation $n^a = x^a/r$ is a unit vector pointing in the x^a-direction. We have seen in Example 5.3 that the hedgehog configuration has topological charge $n = 1$. The partial derivatives of the field have asymptotics

$$\partial_i \Phi^a(r\mathbf{n}) = \frac{1}{r}(\delta_i^a - n^a n_i)v \text{ as } r \to \infty\,, \tag{6.29}$$

where δ_i^a is the Kronecker delta function that equals 1 if $a = i$ and zero otherwise. There is no distinction between n_i and n^i. Just like in section 5.2, to make sure the energy (6.27) remains finite the second term of the covariant derivative (6.22) needs to cancel the $1/r^2$ contribution of the partial derivatives (6.29). This is achieved with the asymptotics

$$A_i^a(r\mathbf{n}) = \frac{1}{er}\varepsilon_{aij}n_j \text{ as } r \to \infty\,, \tag{6.30}$$

as can be verified using (6.22). A solution with the required asymptotics can be found by solving the field equations numerically with the Ansatz

$$\Phi^a = n^a v F(r)\,, \tag{6.31}$$

$$A_i^a = \frac{1}{er}\varepsilon_{aij}n_j G(r)\,, \tag{6.32}$$

where F and G are unknown smooth functions tending to 1 as $r \to \infty$ and to zero as $r \to 0$ fast enough so that the fields are smooth.

To find the magnetic charge, we could proceed as in the previous section by transforming locally to a unitary gauge where $\Phi \equiv \Phi_0$ and then calculating the flux through a very large sphere as in Lemma 6.1. However, a more efficient method is to define the electromagnetic field strength $\mathcal{F}_{\mu\nu}$ in such a way that it reduces to $\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$ far from the origin in the unitary gauge $\Phi \equiv \Phi_0$, where the electromagnetic field is described by A_μ^3. Inside the core of the monopole the SU(2) symmetry is unbroken and there is no unambiguous notion of a magnetic field. Therefore, there are many possible definitions of $\mathcal{F}_{\mu\nu}$ that lead to completely different magnetic fields in the core. Far away from the origin all these definitions agree. One possibility is

$$\mathcal{F}_{\mu\nu} = -\frac{2}{ev}\mathrm{Tr}(F_{\mu\nu}\cdot\Phi) = \frac{1}{v}F_{\mu\nu}^a\Phi^a\,, \tag{6.33}$$

where the dot denotes matrix multiplication. If $\Phi \equiv \Phi_0$ we have $\mathcal{F}_{\mu\nu} = F_{\mu\nu}^3$, which indeed equals $\partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$ far from the origin because the massive fields A_μ^2 and A_μ^1 are (almost) zero there (as follows from $D_\mu\Phi = 0$). Moreover, the field equation (6.25) reduces to the Maxwell equations $\partial_\mu \mathcal{F}^{\mu\nu} = 0$ in this region. Since the expression (6.33) is gauge invariant, we can compute the magnetic field at large distances in any gauge using

$$B_i = -\frac{1}{2}\varepsilon_{ijk}\mathcal{F}_{jk}\,. \tag{6.34}$$

Filling in (6.28) and (6.30) into (6.33) and (6.34) we find

$$B_i = \frac{1}{er^2}n_i\,. \tag{6.35}$$

Comparing with (6.1) we see that the monopole has magnetic charge $g = \frac{1}{e}$, which is twice the minimal charge allowed by the Dirac quantisation condition (6.9). In this model the possible magnetic charges for a topological defect are $\frac{n}{e}$ with $n \in \mathbb{Z}$ because the solution we found has topological charge $n = 1$.

We can verify the results of Lemma 6.2 and Proposition 6.2 explicitly for our example by computing the gauge transformation Ω. A straightforward calculation shows that the functions

$$g^U(\phi, \theta) = \exp(\phi t_3) \exp(-\theta t_2) \exp(-\phi t_3)\,, \tag{6.36}$$

$$g^L(\phi, \theta) = 2t_2 \exp(\phi t_3) \exp((\pi - \theta) t_2) \exp(-\phi t_3)\,. \tag{6.37}$$

take values in G and transform Φ_∞ into $\Phi_0 = vt_3$. Note that g^U and g^L are ill-defined for $\theta = \pi$ and $\theta = 0$ respectively: this is unavoidable by Lemma 5.1. The maps g^U and g^L are indeed smooth, as was assumed in section 6.3. On the equator S^1 where $\theta = \frac{\pi}{2}$ the transformation $\Omega = g^U(g^L)^{-1}$ is given by

$$\Omega|_{S^1}(\phi) = g^U(g^L)^{-1}(\theta = \tfrac{\pi}{2}) = -\exp(2\phi t_3) = \begin{pmatrix} -e^{-i\phi} & 0 \\ 0 & -e^{i\phi} \end{pmatrix}. \tag{6.38}$$

We see that $\Omega|_{S^1}$ takes values in H and even gives a homeomorphism $S^1 \xrightarrow{\sim} H$. The induced map $(\Omega|_{S^1})_* : \pi_1(S^1) \to \pi_1(H)$ therefore is an isomorphism and sends $[\mathrm{id}_{S^1}]$ to $[\Omega|_{S^1}]$. It follows that $\Omega|_{S^1}$ has winding number $m = \pm 1$, which is consistent with Proposition 6.2 since Φ_∞ has topological charge $n = 1$.

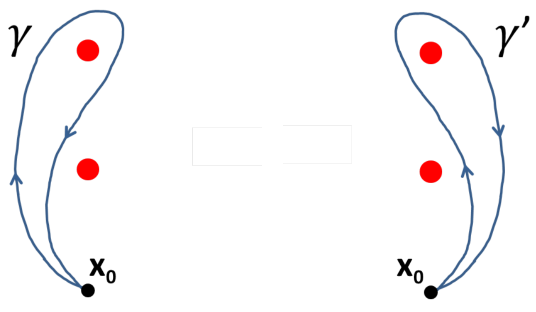
Chapter 7

Monopoles and Alice strings

Up until this point we assumed that the vacuum manifold $\mathcal{M}$ is n-simple for $n = 1, 2$, meaning that we can drop the basepoint condition on maps and homotopies. In this section we consider a model where $\mathcal{M}$ is not 2-simple and the unbroken group is not connected. Such a model necessarily satisfies $\pi_1(\mathcal{M}) \neq 0$, implying the existence of cosmic string solutions. It is shown that these strings can change the sign of both the topological and magnetic charges of monopoles in the model. The strings act as a "mirror" on the monopoles and are therefore called *Alice strings*, after the main character in the novel *Through the looking glass* by Lewis Carroll in 1871. They were first studied in 1982 by Schwarz [37]. It is worth mentioning that the material covered in Appendix A.1 may aid the reading of this section.

7.1 $[S^n, X]$ versus $\pi_n(X)$

In the previous sections we have identified the n-th homotopy group $\pi_n(X)$ of a topological space X with the set $[S^n, X]$. This set consists of the homotopy classes of maps $S^n \to X$, where the homotopies are maps $S^n \times [0,1] \to X$ without any basepoint requirement. This identification is not always justified, as we will show with an example (see [22]). Let X be the space obtained from $\mathbb{R}^2$ by taking out two points and choose a basepoint $x_0 \in X$. Consider the loops $\gamma, \gamma' : (S^1, s_0) \to (X, x_0)$ shown in Figure 7.1.



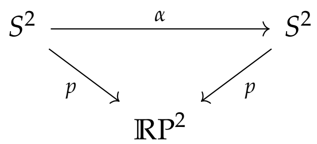
(a) (b)

Figure 7.1: Two loops $\gamma, \gamma' \in \pi_1(X, x_0)$. The red dots represent the two holes in the space X.

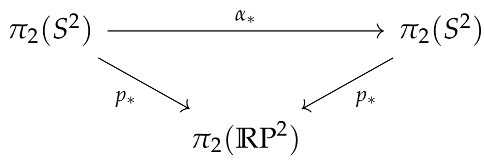
The loops γ and γ' are homotopic in $[S^1, X]$: we can simply move the basepoint x_0 around the bottom red hole and deform γ into γ'. If we require that the basepoint x_0 remains fixed, however, the maps γ and γ' are not homotopic. We see that $[\gamma] = [\gamma']$ in $[S^1, X]$, but not in $\pi_1(X, x_0)$. It is shown in Proposition A.1 that in this case the loops γ and γ' are in the same orbit of the π_1-action on $\pi_1(X, x_0)$. Indeed, if we denote by α a loop based at x_0 winding counter-clockwise around the bottom hole it can be verified that γ' is homotopic to $\alpha \odot \gamma \odot \alpha^{-1}$ in $\pi_1(X, x_0)$. We will work out the details in a similar example in section 7.3.

An example of a space that is not 2-simple (i.e. for which $[S^2, X] \neq \pi_2(X)$) is the real projective plane $X = \mathbb{RP}^2$. This is defined as the space obtained from S^2 by identifying antipodal points. Neglecting the basepoint when working with $\pi_2(\mathbb{RP}^2)$ can lead to contradictions, as we will show following [4].

Let $\alpha : S^2 \to S^2$ be the antipodal map. We argue in Example A.3 that this map is homotopic to a representative of -1 in $\pi_2(S^2) \cong \mathbb{Z}$, and so the induced map $\alpha_* : \pi_2(S^2) \to \pi_2(S^2)$ is given by multiplication by -1. Moreover, the quotient map $p : S^2 \to \mathbb{RP}^2$ is a two-sheeted covering map, and it fits into a commutative diagram

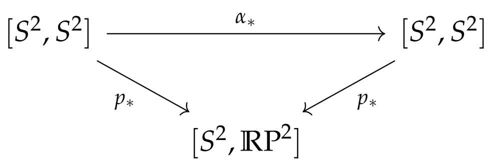


However, the induced diagram



does not commute: the map p_* is an isomorphism by Example 3.3 and so $p_*(1) \neq p_*(-1)$. This seems to contradict the functorial properties of π_2. The problem here is that π_2 is not a functor on the category of topological spaces, but on the category of *pointed* topological spaces. In particular, the map α does not preserve the basepoint of S^2.

From the fact that there is a double cover $S^2 \to \mathbb{RP}^2$ we can deduce that $\pi_2(\mathbb{RP}^2) = \mathbb{Z}$ and $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$. The nontrivial element of $\pi_1(\mathbb{RP}^2)$ acts on $\pi_2(\mathbb{RP}^2)$ by changing the sign (see Example A.3). Proposition A.1 shows that the set $[S^2, \mathbb{RP}^2]$ is the orbit space $\mathbb{Z}/\mathbb{Z}_2$ of this action, which is the set of integers modulo their sign. Since the π_1-action is nontrivial, we find that there is a difference between $[S^2, \mathbb{RP}^2]$ and $\pi_2(\mathbb{RP}^2)$. If we replace the homotopy groups $\pi_2(S^2)$ and $\pi_2(\mathbb{RP}^2)$ by $[S^2, S^2]$ and $[S^2, \mathbb{RP}^2]$ in the diagram above, we obtain



The map p_* now corresponds to the quotient map $\mathbb{Z} \to \mathbb{Z}/\mathbb{Z}_2$ and is no longer bijective. We therefore do not encounter the paradox from the previous diagram. For example, the identity map id_{S^2} and α are not homotopic in $[S^2, S^2]$, but their images under p_* are equal. We will show how the distinction between $[S^2, \mathbb{RP}^2]$ and $\pi_2(\mathbb{RP}^2)$ affects the classification of monopoles in a model with vacuum manifold $\mathcal{M} \cong \mathbb{RP}^2$.

7.2 The model

The following account is based on [29] and [32]. We consider a model with gauge group $G = \mathrm{SO}(3)$ and a Higgs field Φ taking values in the 5-dimensional representation V of G consisting of real symmetric traceless 3×3 matrices. A group element g acts on Φ by conjugation $g\Phi g^{-1}$. The model also contains a $\mathfrak{so}(3)$-valued gauge field A. We choose the basis $(t_a)_{ij} = \varepsilon_{aji}$ (with $a = 1,2,3$) of $\mathfrak{so}(3)$:

$$t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; t_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; t_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.1}$$

This basis is orthonormal with respect to $-\frac{1}{2}$ times the trace. We expand A_μ in the basis as $A_\mu = eA_\mu^a t_a$. The gauge field acts on the Higgs field via the commutator of matrices $[A_\mu, \Phi]$.

It is possible to choose a Higgs potential $U : V \to \mathbb{R}$ such that the zeros of U are of the form

$$\mathcal{M} = \left\{ v(I - 3\mathbf{n}\mathbf{n}^T) \,\middle|\, \mathbf{n} \in S^2 \right\}. \tag{7.2}$$

In this equation $\mathbf{n}^T$ is the transpose of the unit vector $\mathbf{n}$, v is a positive constant and I is the identity matrix. See [29] for the construction of a suitable potential. Consider the surjective map $p : S^2 \to \mathcal{M}$ given by $p(\mathbf{n}) = v(I - 3\mathbf{n}\mathbf{n}^T)$. This map satisfies $p(\mathbf{n}) = p(\mathbf{n}')$ if and only if $\mathbf{n} = \pm\mathbf{n}'$ and induces a homeomorphism $\mathbb{RP}^2 \xrightarrow{\sim} \mathcal{M}$. In particular, $\mathcal{M}$ is not 2-simple.

Note that for any $g \in G$ the transformed field $g\Phi g^{-1} = I - 3(g\mathbf{n})(g\mathbf{n})^T$ is again a zero of U. Moreover, G acts transitively on the vacuum manifold $\mathcal{M}$ because any unit vector $\mathbf{n}$ is obtained by a suitable rotation g. Let us choose the vacuum vector

$$\Phi_0 = I - 3\mathbf{e}_3\mathbf{e}_3^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{7.3}$$

The unbroken group H consists of all group elements h that leave the matrix $\mathbf{e}_3\mathbf{e}_3^T$ invariant. This implies $h\mathbf{e}_3 = \pm\mathbf{e}_3$. It follows that H consists of all rotations around the third axis, as well as all of these rotations composed with an element $\Omega \in G$ that sends $\mathbf{e}_3$ to $-\mathbf{e}_3$. An example of such an element is

$$\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7.4}$$

By considering the action of H on the plane $x^3 = 0$ in $\mathbb{R}^3$, we find that Ω acts as a reflection in the x^1-axis and we deduce that H is isomorphic to $\mathrm{O}(2)$. In particular, H is not connected: the component H_e connected to the identity is isomorphic to $\mathrm{SO}(2)$. The generator t_3 of $\mathfrak{g}$ commutes with Φ_0 and hence is unbroken. Since the Lie algebra $\mathfrak{h}$ of H is one-dimensional, it is generated by t_3.

The homotopy groups of $\mathcal{M}$ are $\pi_1(\mathcal{M}) = \mathbb{Z}_2$ and $\pi_2(\mathcal{M}) = \mathbb{Z}$. Therefore, both monopoles and strings exist in this model. Any monopole solution has an asymptotic Higgs field $\Phi_\infty : S^2 \to \mathcal{M}$. The map Φ_∞ does not have to satisfy any basepoint condition, so we cannot unambiguously define the topological charge as an element of $\pi_2(\mathcal{M})$. We can only assign the class $[\Phi_\infty] \in [S^2, \mathcal{M}]$ to the monopole. This leads to a different classification of monopole solutions because $[S^2, \mathcal{M}] \cong \mathbb{Z}/\mathbb{Z}_2$. Note that this classification does not affect the criterion for the existence of monopoles. The trivial homotopy class of $\pi_2(\mathcal{M})$ always forms an orbit under the π_1-action, i.e. we have $[S^2, \mathcal{M}] \neq 0$ whenever $\pi_2(\mathcal{M}) \neq 0$.

We can also attempt to assign a magnetic charge to the monopole defects by identifying the massless t_3-component A^3_μ of the gauge field with the electromagnetic four potential in the unitary gauge $\Phi = \Phi_0$. However, this definition of magnetic charge is not gauge invariant in this model because H is not connected. If we apply the gauge transformation $\Omega \in H \setminus H_e$, the Higgs field is unchanged but the t_3 component of A changes sign since $\Omega t_3 \Omega^{-1} = -t_3$. It follows that a monopole can be gauge transformed into an antimonopole, a monopole with the opposite magnetic charge. It is however possible to distinguish between a pair of monopoles and a monopole-antimonopole pair by bringing them together and observing whether they annihilate or not. The relative charge between two monopoles is still well-defined.

7.3 Alice strings

Since $\pi_1(\mathcal{M}) = \mathbb{Z}_2$ there are also cosmic strings in the model, which we have ignored until now. The strings can affect both the topological and magnetic charge of defects, as we will show in this section following [32, 42]. We treat the topological and magnetic charges separately because the results of section 6.3 do not apply in this model where $\mathcal{M}$ is not 2-simple and H is not connected.

A string configuration Φ has to induce a nontrivial map $\Phi_\infty : S^1 \to \mathcal{M}$ far away from the origin. Consider the fibre bundle $\{\pm 1\} \to S^2 \xrightarrow{p} \mathcal{M} \cong \mathbb{RP}^2$. A loop γ in $\mathcal{M}$ is nontrivial if and only if its image under the connecting homomorphism $\delta : \pi_1(\mathcal{M}) \to \pi_0(\{\pm 1\})$ is nonzero. This happens exactly when a lift $\tilde{\gamma} : I \to S^2$ of γ along the map $p : S^2 \to \mathcal{M}$ satisfies $\tilde{\gamma}(1) = -\tilde{\gamma}(0)$ (see Appendix A.2). We can therefore construct a noncontractible loop by letting $\tilde{\gamma}$ be a path in S^2 from the north to the south pole and setting $\gamma = p \circ \tilde{\gamma}$. This gives us the string configuration shown in Figure 7.2.

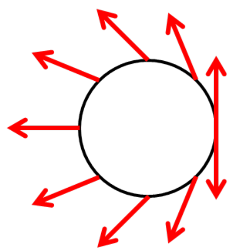


Figure 7.2: The asymptotic Higgs field $\Phi_\infty : S^1 \to \mathcal{M} \cong \mathbb{RP}^2$. The arrows represent unit vectors in S^2, which become elements of $\mathcal{M}$ after identifying arrows pointing in opposite directions. The arrows describe a path from the north to the south pole in S^2. This induces a noncontractible loop in $\mathcal{M}$.

As an explicit formula for the asymptotic Higgs field Φ_∞ we can take

$$\Phi_\infty(\theta) = \exp(\theta t_1/2)\Phi_0 \exp(-\theta t_1/2)\,. \tag{7.5}$$

Indeed, a lift of the loop $\gamma(s) = \Phi_\infty(\theta = 2\pi s)$ is $\tilde{\gamma}(s) = \exp(\pi t_1 s)\mathbf{e}_3$, which is a path from the north to the south pole. Note that the vacuum vector Φ_0 gets transformed by the gauge transformation $g(\theta) = \exp(\theta t_1/2)$ as we go around the string. The unbroken generator has to annihilate the vacuum vector and hence is also transformed by $g(\theta)$. In particular, after winding around the string once the unbroken generator becomes $g(2\pi)t_3 g(2\pi)^{-1}$. Since the path $\tilde{\gamma}(s) = g(\theta = 2\pi s)\mathbf{e}_3$ ends at $-\mathbf{e}_3$, the transformation $g(2\pi)$ has to be in the component of H disconnected from the identity. A calculation shows that $g(2\pi)$ equals the transformation Ω given by (7.4), which is indeed not in H_e.

It follows that $g(2\pi)t_3 g(2\pi)^{-1} = -t_3$, i.e. the electromagnetic field changes sign. We obtain the gauge invariant statement that a monopole that winds once around a string becomes an antimonopole. The same happens to electrically charged particles. This is why the string is called an Alice string. As mentioned earlier, this process can have observable consequences in the presence of a second monopole. Whether two monopoles annihilate or not can depend on the number of times they wind around the string while moving towards each other.

Alice strings have a similar effect on the topological charge of monopoles. The following arguments are based on [22]. Let us again consider a situation where a monopole and a string coexist at a fixed time. Let $X \subset \mathbb{R}^3$ be the region outside the cores of the defects where the Higgs field takes values in $\mathcal{M}$. The space X is topologically equivalent to $\mathbb{R}^3$ minus a line and a point not on that line. We choose a basepoint $x_0 \in X$ and consider two maps $\alpha, \alpha' : (S^2, s_0) \to (X, x_0)$ enclosing the monopole as shown in Figure 7.3. We also define γ to be a loop based at x_0 winding around the string, see Figure 7.3(a).

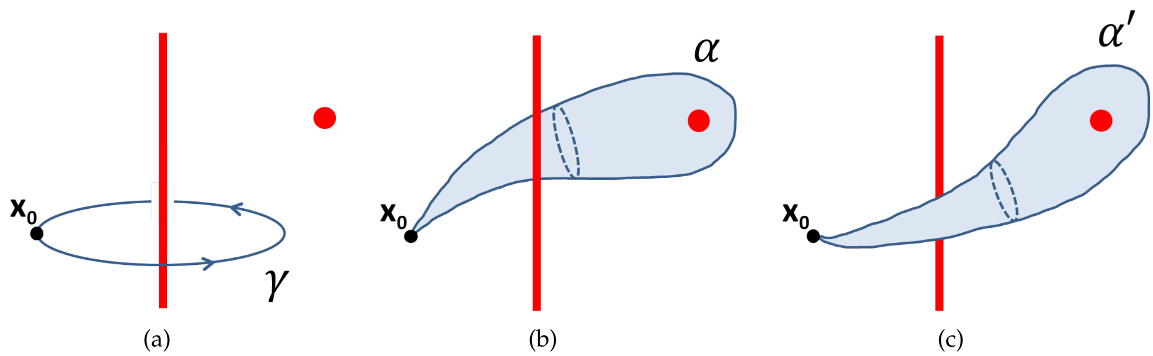
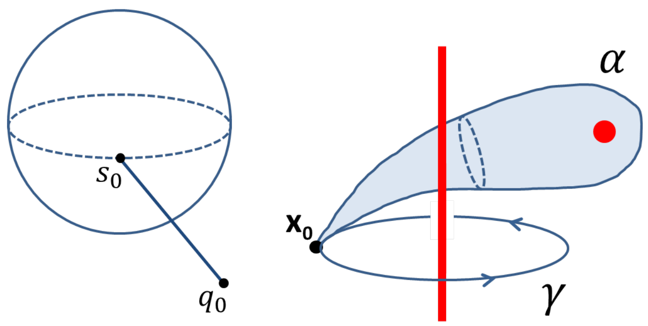


Figure 7.3: The maps γ, α and α' into X. The red dot and line represent the monopole and string respectively.

Note that α and α' are homotopic in $[S^2, X]$ by rotating the basepoint around the string. If we keep the basepoint fixed, however, the maps α and α' cannot be deformed into each other. We see that X is another example of a space that is not 2-simple. Proposition A.1 implies that α and α' are in the same orbit of the π_1-action on $\pi_2(X, x_0)$. In fact, we will show that $[\alpha'] = [\gamma\alpha]$ in $\pi_2(X, x_0)$.

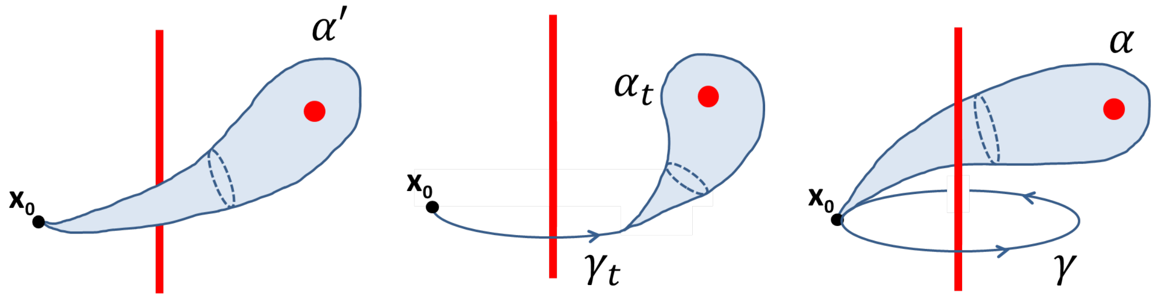
Viewed as a map $(I^2, \partial I^2) \to (X, x_0)$, the map $\gamma\alpha$ is constructed as in Figure A.1. Let us temporarily reparametrise the square as $I^2 = [-1, 1]^2$. The map $\gamma\alpha$ is then constant on each subspace $K_r = \{(s_1, s_2) \in I^2 \mid \max\{|s_i|\} = r\}$ for $r \geq \frac{1}{2}$. Hence, $\gamma\alpha$ induces a map from the quotient space Q obtained from I^2 by identifying all the points on K_r for all $r \geq \frac{1}{2}$. This map sends the point q_0 corresponding to $\partial I^2 = K_1$ to x_0. Under this identification, the map $\gamma\alpha$ is shown in Figure 7.4.



(a) (b)

Figure 7.4: (a): The quotient space Q obtained from I^2 by identifying points on K_r for $r \geq \frac{1}{2}$. The region $\max\{|s_i|\} \leq \frac{1}{2}$ becomes a sphere, and the points s_0 and q_0 correspond to $K_{\frac{1}{2}}$ and K_1 respectively. (b): The image of $\gamma\alpha : (Q, q_0) \to (X, x_0)$. Both s_0 and q_0 get mapped to the basepoint x_0.

We now construct a basepoint-preserving homotopy Γ from α' to $\gamma\alpha$ as follows. For each $t \in I$ let $\alpha_t : (S^2, s_0) \to (X, \gamma(t))$ be a map with image a deformed sphere based at $\gamma(t)$ enclosing the monopole (without winding around the string). We take $\alpha_0 = \alpha'$ and $\alpha_1 = \alpha$. Let γ_t be the loop γ restricted to the interval $[0, t]$. The homotopy Γ is now given at time t by $\Gamma_t = \gamma_t\alpha_t$. Viewing the domain of $\gamma_t\alpha_t$ as Q, the homotopy is visualised in Figure 7.5.



(a) (b) (c)

Figure 7.5: The homotopy Γ_t from α' to $\gamma\alpha$ for three values of t. The basepoint of α_t winds around the string, but $\gamma_t\alpha_t$ is always based at x_0. The map $\gamma_t\alpha_t$ is viewed as a map of the form $(Q, q_0) \to (X, x_0)$.

It follows that $[\alpha'] = [\gamma\alpha]$. Suppose we define the topological charge of the monopole to be the element $[\Phi \circ \alpha] \in \pi_2(\mathcal{M}, \Phi(x_0)) \cong \mathbb{Z}$. Note that this charge depends on both α and the isomorphism $\pi_2(\mathcal{M}, \Phi(x_0)) \cong \mathbb{Z}$. We now transport the monopole counter-clockwise around the string and deform α (preserving the basepoint) such that it encloses the monopole at any time. The map $\Phi \circ \alpha$ is then continuously deformed into $\Phi \circ \alpha'$. We have seen that $\Phi \circ \alpha$ and $\Phi \circ \alpha'$ are related by the action of $\Phi \circ \gamma$, which is a nontrivial loop in $\mathcal{M}$. We find $[\Phi \circ \alpha] = -[\Phi \circ \alpha']$ in $\pi_2(\mathcal{M}, \Phi(x_0))$.

Just like for the magnetic charge, the sign of the topological charge flips as the monopole winds around the string. Note that this last statement does not depend on any of the choices we made while defining the topological charge. The topological influence of a string on a monopole winding around it is the physical manifestation of the π_1-action on $\pi_2(\mathcal{M})$.

Chapter 8

Outlook

We have seen in the previous sections how to classify defects using homotopy groups by assigning a topological charge to them. This topological charge is an element of a homotopy group $\pi_n(\mathcal{M})$ of the vacuum manifold, and whenever this group is nontrivial the model contains topological defects. We used this criterion to show that many Grand Unified Theories predict monopoles. Subsequently, we introduced a magnetic charge for monopoles and showed how to interpret this as an element of $\pi_1(H)$ in a model with gauge group H. The link between the topological and magnetic charges is provided by the long exact sequence of homotopy groups, and in particular the connecting homomorphism $\delta : \pi_2(\mathcal{M}) \to \pi_1(H)$.

Most of these results were derived under the assumptions that $\mathcal{M}$ is n-simple for $n = 1, 2$ and H is connected. We considered a model in section 7 where this is not the case and our definitions of topological and magnetic charge are ambiguous. In addition, the model contained Alice strings that can change the topological and magnetic charge of a monopole. Mathematically, we can describe this phenomenon for the topological charge using the π_1-action on the homotopy groups. An interesting next step would be to generalise these observations to obtain a topological classification of monopoles suitable for any model.

One possible approach is to use *Abe homotopy groups* [1]. The n-th Abe homotopy group $\kappa_n(X, x_0)$ of a space (X, x_0) is isomorphic to the semidirect product $\pi_n(X, x_0) \rtimes \pi_1(X, x_0)$, where the product is defined using the π_1-action on π_n. The group κ_2 therefore encodes the influence of strings on monopoles. In [21] it is shown how to use the conjugacy classes of κ_n to classify topological defects. Using this approach, the possible topological charges of a monopole in the model of section 7 are reduced from $\mathbb{Z}$ to $\mathbb{Z}_2$ by the influence of strings. Only the value of the charge modulo 2 is retained.

The same result is obtained using the *orbit group of Trebin* [40]. This group is defined as $\pi_n(X, x_0)/D_n$, where D_n is the normal subgroup generated by $(\gamma\alpha) \odot \alpha^{-1}$ for $\gamma \in \pi_1(X, x_0)$ and $\alpha \in \pi_n(X, x_0)$. For $X = \mathbb{R}P^2$ we indeed find $D_2 \cong 2\mathbb{Z}$ and $\pi_2(X, x_0)/D_2 \cong \mathbb{Z}_2$. Trebin interpreted this result as follows: a monopole of even charge can split into two monopoles of half this charge, which annihilate after transporting one of them around the string.

Recently a description has been proposed that retains more information than just the value of the topological charge modulo 2 [2]. The vacuum manifold $\mathcal{M}$ is replaced by a covering space $\widetilde{\mathcal{M}}$ with a trivial π_1-action on the homotopy groups. The topological charge can then unambiguously be defined in $\pi_2(\widetilde{\mathcal{M}})$. Moreover, defects are classified using $[X, \mathcal{M}]$, where $X \subset \mathbb{R}^3$ is the region outside of the cores of the defects.

We have not encountered a generalisation of the correspondence between topological and magnetic charge found in Proposition 6.2 to the case where $\mathcal{M}$ is not (necessarily) 2-simple and H is not connected. Based on the similarities in the behaviour of these charges in the model of section 7, this could be interesting to investigate further.

Appendix A

Supporting homotopical constructions

We discuss the importance of the basepoint in the definition of the homotopy group, leading to the notion of an n-simple space that becomes particularly relevant in section 7. Moreover, the connecting homomorphism δ in the long exact sequence of homotopy groups is constructed. This construction is specialised for fibre bundles involving Lie groups.

A.1 Basepoints and homotopy

In physics we naturally encounter maps from S^n to a space X without any basepoint requirement that we want to classify up to homotopy. In this section, we study under what conditions we can omit the basepoint in the definition of $\pi_n(X, x_0)$ following [16].

Let $x_0, x_1 \in X$ and suppose there exists a path $\gamma : I \to X$ from $\gamma(0) = x_0$ to $\gamma(1) = x_1$. Then to any map $f : (I^n, \partial I^n) \to (X, x_1)$ we can associate a map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ by shrinking the domain of f to a concentric cube and inserting γ radially in the cleared space, see Figure A.1. An explicit formula for this map can be written down as follows [19]. We temporarily reparametrise the n-cube as $I^n = [-1, 1]^n$. For $s = (s_1, \ldots, s_n) \in I^n$ we define $m = \max\{|s_i|\}$. The map γf is then given by

$$(\gamma f)(s) = \begin{cases} f(2s), & m \leq \frac{1}{2} \\ \gamma(2(1-m)), & m \geq \frac{1}{2}. \end{cases} \tag{A.1}$$

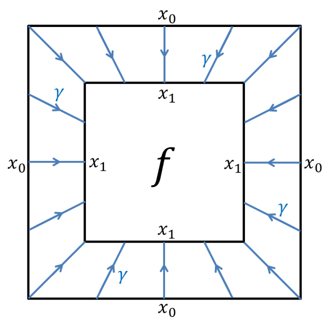


Figure A.1: The map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ in the case $n = 2$. It is obtained by shrinking the domain of f to a smaller square and inserting γ radially outside of this square.

It can be shown that this procedure induces an isomorphism $\beta_\gamma : \pi_n(X, x_1) \xrightarrow{\sim} \pi_n(X, x_0)$ given by $\beta_\gamma([f]) = [\gamma f]$ with inverse $\beta_{\gamma^{-1}}$ (where γ^{-1} is the inverse path). This implies that the homotopy groups of a path-connected space do not depend on the basepoint up to isomorphism, and so we often write $\pi_n(X)$ instead of $\pi_n(X, x_0)$ in this case.

The isomorphisms β_γ can be used to show that homotopy equivalent spaces have isomorphic homotopy groups. Recall that a map $\varphi : X \to Y$ is a homotopy equivalence if there exists a map $\psi : Y \to X$ such that $\psi \circ \varphi$ is homotopic to id_X (through maps $X \to X$) and $\varphi \circ \psi$ is homotopic to id_Y.

Lemma A.1. *Let $\varphi : (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Then the induced maps $\varphi_* : \pi_n(X, x_0) \xrightarrow{\sim} \pi_n(Y, y_0)$ are isomorphisms for all $n \geq 0$.*

Proof. Let $\psi : Y \to X$ be such that $\psi \circ \varphi$ is homotopic to id_X and $\varphi \circ \psi$ is homotopic to id_Y. For $n = 0$ we find $(\psi \circ \varphi)_* = \mathrm{id}_{\pi_0(X)}$ and $(\varphi \circ \psi)_* = \mathrm{id}_{\pi_0(Y)}$, implying the result by the functoriality properties. For $n > 0$ more care is needed because the homotopy $\Gamma : Y \times I \to Y$ from $\varphi \circ \psi$ to id_Y may not be basepoint-preserving. Let $\gamma : I \to Y$, $\gamma(t) = \Gamma(y_0, t)$ and let γ_t be the path obtained by restricting γ to the interval $[0, t]$. We also define $\Gamma_t : Y \to Y$ to be $\Gamma|_{Y \times \{t\}}$. Given $[f] \in \pi_n(Y, y_0)$ we can consider the basepoint-preserving homotopy $\Gamma' : (I^n \times I, \partial I^n \times I) \to (Y, \varphi \circ \psi(y_0))$ given by $\Gamma'_t = \gamma_t(\Gamma_t \circ f)$. Restricting this homotopy to $t = 0$ and $t = 1$, we obtain the equality $(\varphi \circ \psi)_*([f]) = \beta_\gamma([f])$. Hence, $(\varphi \circ \psi)_* = \beta_\gamma$ is an isomorphism. We similarly find that $(\psi \circ \varphi)_*$ is an isomorphism, and the result again follows by the functoriality properties. □

If we take γ to be a loop with basepoint x_0, we obtain an automorphism β_γ of $\pi_n(X, x_0)$ sending $[f]$ to $[\gamma f]$. It can be verified that this map only depends on the homotopy class of γ and even defines a group homomorphism $\pi_1(X, x_0) \to \mathrm{Aut}(\pi_n(X, x_0))$, called the *$\pi_1$-action* on π_n. Note that for $n = 1$ this is the action of π_1 on itself by inner automorphisms $[f] \mapsto [\gamma \odot f \odot \gamma^{-1}]$.

The homotopy group $\pi_n(X)$ of a path-connected space X still contains a reference to the basepoint, because homotopies are required to be basepoint-preserving (Definition 3.1). The key to trying to get rid of this basepoint is to use the π_1-action [16, Prop. 4A.2]. Let us denote by $[S^n, X]$ the set of homotopy classes of maps $S^n \to X$, where homotopies are through maps of the same form.

Proposition A.1. *Let X be a path-connected space with basepoint $x_0 \in X$. Then there is a canonical bijection between the orbit space $\pi_n(X, x_0)/\pi_1(X, x_0)$ of the π_1-action on π_n and the set $[S^n, X]$.*

Proof. Let $f : S^n \to X$ and let γ be a path from x_0 to $f(s_0)$. Then we have $[\gamma f] \in \pi_n(X, x_0)$, and f is homotopic to γf in $[S^n, X]$ by enlarging the domain of f in Figure A.1 and restricting γ to the interval $[t, 1]$ for $t \in I$. This shows that the natural map $\varphi : \pi_n(X, x_0) \to [S^n, X]$ is surjective. If $[\gamma] \in \pi_1(X, x_0)$ and $[f] \in \pi_n(X, x_0)$ we similarly find $[f] = [\gamma f]$ in $[S^n, X]$, so φ induces a map $\pi_n(X, x_0)/\pi_1(X, x_0) \to [S^n, X]$. If $f, g : (S^n, s_0) \to (X, x_0)$ are homotopic in $[S^n, X]$ through $\Gamma : S^n \times I \to X$, then we have $[f] = [\gamma g]$ in $\pi_n(X, x_0)$ with $\gamma(t) = \Gamma(s_0, t)$. The homotopy between f and γg is constructed by viewing Γ as a map $I^n \times I \to X$ and restricting it to a family of n-cubes starting with $I^n \times \{0\}$ and ending with $I^n \times \{1\} \cup \partial I^n \times I$, all the cubes having the same boundary. It follows that the induced map is injective. □

If the π_1-action is trivial we find that we can identify $\pi_n(X, x_0)$ with $[S^n, X]$, leaving out any reference to a basepoint. Moreover, in this case there is a canonical isomorphism between the homotopy groups at different points (in general the isomorphism depends on the chosen path).

Definition A.1. A path-connected space X is *n-simple* $(n \geq 1)$ if the π_1-action on $\pi_n(X, x_0)$ is trivial for all $x_0 \in X$.

Note that the π_1-action is trivial for some $x_0 \in X$ if and only if it is trivial for all $x_0 \in X$ because X is path-connected.

Example A.1. A simply connected space is obviously n-simple for all n.

Example A.2. Let G be a path-connected *topological group*. This is a group G that is also topological space such that the group operation $(g, h) \mapsto g \cdot h$ and the inversion map $g \mapsto g^{-1}$ are continuous. Let $e \in G$ be the unit element. For $[f] \in \pi_n(G, e)$ and $[\gamma] \in \pi_1(G, e)$ there is a homotopy $\Gamma(x, t) = f(x) \cdot \gamma(t)$ from f to f satisfying $\Gamma(s_0, t) = \gamma(t)$. This gives a basepoint-preserving homotopy from f to γf as described in the proof of Proposition A.1. It follows that G is n-simple for all n.

For $n = 1$, an n-simple space is the same thing as a path-connected space with abelian fundamental group because the action is given by inner automorphisms. It follows from Example A.2 that the fundamental group of a path-connected topological group is abelian. This is in fact also true if the group is not path-connected, as follows from the following Lemma [15, Prop. 1.2.21].

Lemma A.2. *Let G be a topological group with unit element e. Then all path components of G are homeomorphic to the path component G_e of e.*

Proof. Let $g \in G$ and let G_g be the path component of g. The map $L_g : G_e \to G$ given by $L_g(h) = gh$ has a path-connected image containing g and hence gives a map $G_e \to G_g$. Similarly, the map $L_{g^{-1}} : G_g \to G_e$ is well-defined and the inverse of $L_g : G_e \to G_g$. It follows that L_g is a homeomorphism, and therefore $G_e \cong G_g$. □

Lemma A.2 implies that we can write $\pi_n(G)$ without specifying the basepoint even if G is not path-connected. Note that Lie groups are always locally path-connected, so being path-connected is equivalent to being connected in that case.

Corollary A.1. *The fundamental group $\pi_1(G)$ of a topological group G is abelian.*

Example A.3. Consider the real projective space $\mathbb{RP}^n$ obtained by identifying antipodal points on S^n for $n > 1$. The quotient map $p : S^n \to \mathbb{RP}^n$ is a two-sheeted covering map, and because S^n is simply connected we find $\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ by Example 3.3. Moreover, the map $p_* : \pi_n(S^n) \to \pi_n(\mathbb{RP}^n)$ is an isomorphism and hence Theorem 3.1 gives $\pi_n(\mathbb{RP}^n) \cong \mathbb{Z}$.

Note that S^n is simply connected and therefore n-simple. The antipodal map $\alpha : S^n \to S^n$ is not homotopic to the identity map for even n [23, Prop. 13.31], so it must be in the homotopy class of $[S^n, S^n]$ corresponding to $-1 \in \mathbb{Z}$ because α is a homeomorphism. This implies that there exists a homotopy $\Gamma : S^n \times I \to S^n$ from α to a map $f : (S^n, s_0) \to (S^n, s_0)$ representing -1 in $\pi_n(S^n)$. We also find a homotopy $p \circ \Gamma$ from $p \circ \alpha = p$ to $p \circ f$. However, since p_* is an isomorphism we know that $p = p_*[\mathrm{id}_{S^n}]$ represents 1 in $\pi_n(\mathbb{RP}^n)$ and $p \circ f = p_*[f]$ represents -1. Hence, the classes 1 and -1 are equal in $[S^n, \mathbb{RP}^n]$. It follows that these classes are in the same orbit of the π_1-action, implying that the nontrivial element of $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}_2$ acts on $\pi_n(\mathbb{RP}^n) \cong \mathbb{Z}$ by changing the sign. We see that $\mathbb{RP}^n$ is not n-simple for even n.

A.2 The connecting homomorphism

The construction of the connecting homomorphism $\delta : \pi_n(B, b_0) \to \pi_{n-1}(F, e_0)$ for a fibre bundle $F \to E \xrightarrow{p} B$ and the proof of Theorem 3.3 rely on the fact that the map p has the homotopy lifting property with respect to all disks $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ [16, Prop. 4.48].

Definition A.2. A map $p : E \to B$ has the *homotopy lifting property* with respect to a space X if for any homotopy $\Gamma : X \times I \to B$ and any lift $\tilde{f}_0$ of $f_0 = \Gamma|_{X\times\{0\}}$ (satisfying $p \circ \tilde{f}_0 = f_0$) there exists a homotopy $\tilde{\Gamma} : X \times I \to E$ lifting Γ such that $\tilde{\Gamma}|_{X\times\{0\}} = \tilde{f}_0$. In other words, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\tilde{f}_0} & E \\ {\scriptstyle X\times\{0\}}\downarrow & \overset{\tilde{\Gamma}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times I & \xrightarrow[\Gamma]{} & B \end{array}$$

We can now define the connecting homomorphism δ. Our construction is adapted from [26] to the S^n-viewpoint. Let $f : (S^n, s_0) \to (B, b_0)$ be a representative of $[f] \in \pi_n(B, b_0)$. We choose our basepoint s_0 to be the north pole $s_0 = (0, 0, \ldots, 1)$. Let $q : D^n \to S^n$ be the map obtained by identifying all points on the boundary ∂D^n and sending them to s_0, and let $j : S^{n-1} \to D^n$ be the inclusion mapping S^{n-1} onto ∂D^n. By the homotopy lifting property, we can lift $f \circ q$ to a map $\tilde{f} : (D^n, j(s_0)) \to (E, e_0)$. This follows by viewing the domain of $f \circ q$ as $D^{n-1} \times I \cong D^n$ and choosing a lift with initial condition e_0. Because $p \circ \tilde{f} \circ j = f \circ q \circ j$ is constant and equal to b_0, the map $\tilde{f} \circ j$ has image in $F = p^{-1}(b_0)$ and can be restricted to a map $r : (S^{n-1}, s_0) \to (F, e_0)$. We set $\delta([f]) = [r]$. The construction is summarised in the diagram below.

$$\begin{array}{ccccc} (S^{n-1}, s_0) & \xrightarrow{j} & (D^n, j(s_0)) & \xrightarrow{q} & (S^n, s_0) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ (F, e_0) & \xrightarrow{i} & (E, e_0) & \xrightarrow{p} & (B, b_0) \end{array}$$

See [26] for a proof that δ is a well-defined group homomorphism.

For certain fibre bundles of the form $H \to G \to G/H$ with G a Lie group and H a closed subgroup (see Lemma B.2) there exists an alternative description of the map δ. We have not found the following proof that this description coincides with the definition of δ given above anywhere in the literature.

Proposition A.2. *Let G be a connected Lie group and H a connected closed subgroup such that G/H is n-simple. Let $\Phi : S^n \to G/H$ be a map and let $\Psi^{\pm}$ be lifts of Φ along the quotient map $\pi : G \to G/H$ on the sets $W^{\pm} = \{x \in S^n \mid \pm x_{n+1} \geq -\frac{1}{2}\}$. Define $\Omega : S^{n-1} \to H$ on the equator $x_{n+1} = 0$ by $\Omega = (\Psi^+)^{-1}\Psi^-$. Then under the identifications $[S^n, G/H] = \pi_n(G/H, H)$ and $[S^{n-1}, H] = \pi_{n-1}(H, e)$ we have $\delta([\Phi]) = [\Omega]$.*

Proof. The existence of $\Psi^\pm$ follows from the homotopy lifting property of π ($W^\pm$ is homeomorphic to D^n), and Ω takes values in H because $\Psi^+(x)H = \Psi^-(x)H = \Phi(x)$ for all $x \in W^+ \cap W^-$ and hence $\Omega(x)H = H$ for $x \in S^{n-1}$. We may assume without loss of generality that Φ maps the north pole s_0 to H: this can be accomplished by replacing Φ by $g\Phi$ for some $g \in G$ satisfying $g^{-1}H = \Phi(s_0)$. Since G is connected, there exists a path γ in G from e to g. The homotopy $\Gamma(x,t) = \gamma(t)\Phi(x)$ then shows that $[\Phi] = [g\Phi]$. We also have to multiply the lifts $\Psi^\pm$ by g from the left, which leaves Ω unchanged.

We will construct a lift χ of $\Phi \circ q$ sending $j(s_0)$ to the unit element e, where q and j are the maps introduced during the construction of δ above. Set χ equal to $\Psi^- \circ q$ on the region $D^n_S = \{x \in D^n \mid \|x\| \leq \frac{1}{2}\}$ corresponding to the southern hemisphere under q. On $D^n \setminus D^n_S$ we define χ to be $\Psi^+ \circ q$ multiplied from the right by Ω on each shell S^{n-1} of constant $\|x\|$. The map χ in the case $n = 2$ is shown in Figure A.2. Note that the two definitions of χ coincide on the boundary ∂D^n_S corresponding to the equator. Moreover, χ is a lift of $\Phi \circ q$ because $\Psi^\pm$ are lifts and Ω takes values in H. On the boundary ∂D^n we have $\chi = \Psi^+(s_0)\Omega$, which is a product of elements in H. Multiplying χ by an appropriate element $h \in H$ from the right, we obtain a lift satisfying $\chi(j(s_0)) = e$.

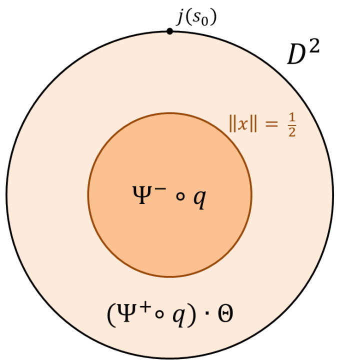


Figure A.2: The lift $\chi : D^n \to G$ for $n = 2$. On D^2_S (dark area) it equals $\Psi^- \circ q$ and on $D^2 \setminus D^2_S$ (light area) it is $\Psi^+ \circ q$ multiplied from the right by Θ, given in polar coordinates as $\Theta(r,\theta) = \Omega(\theta)$. On the brown circle $\|x\| = \frac{1}{2}$ these two maps coincide. The map χ satisfies the basepoint condition after multiplication by $\chi(j(s_0))^{-1} \in H$ from the right.

Let r be the map $\chi \circ j$ with codomain restricted to H. From the construction of δ it follows that $\delta([\Phi]) = [r] = [\Psi^+(s_0)\Omega h]$. Because H is connected, there exist paths in H from e to h and from e to $\Psi^+(s_0)$. Just like before we can use these paths to construct homotopies showing that $[\Psi^+(s_0)\Omega h] = [\Omega]$, and therefore $\delta([\Phi]) = [\Omega]$. □

Appendix **B**

Differential geometry

In this section we review the necessary background on Lie groups, Lie group actions and connections for this thesis. We assume some familiarity with the theory of smooth manifolds as covered in [24]. The section is based on the first five chapters of [15].

B.1 Lie groups and Lie algebras

Let us begin by defining Lie groups and discussing some examples.

Definition B.1. A *Lie group* is a group G which is at the same time a smooth manifold so that the maps $G \times G \to G$, $(g, h) \mapsto g \cdot h$ and $G \to G$, $g \mapsto g^{-1}$ are smooth.

In other words, Lie groups are groups that have a smooth structure that is compatible with the group operations. Notions of manifolds and groups such as the dimension or being abelian naturally carry over to Lie groups.

Example B.1. Let $\mathbb{F}$ be a field that is either $\mathbb{R}$ or $\mathbb{C}$. The simplest example of a Lie group is the additive group $\mathbb{F}^n$ with the usual smooth structure: addition and multiplication by -1 are indeed smooth maps. The general linear group $\mathrm{GL}(n, \mathbb{F})$ of $n \times n$ invertible matrices over $\mathbb{F}$ can also be given the structure of a Lie group by viewing it as the open subset of $\mathrm{Mat}(n, \mathbb{F}) = \mathbb{F}^{n^2}$ given by the inverse image of the group of units $\mathbb{F}^*$ under the (smooth) determinant map $\mathbb{F}^{n^2} \to \mathbb{F}$. The group $\mathrm{GL}(n, \mathbb{F})$ then inherits a smooth structure from $\mathbb{F}^{n^2}$, and it can be verified that matrix multiplication and inversion are smooth maps.

If G is a Lie group, an *embedded Lie subgroup* of G is a subgroup H such that H is an embedded submanifold of G. The Closed subgroup theorem characterises embedded Lie subgroups: a proof can be found in section 1.8 of [15].

Theorem B.1. (Closed subgroup theorem) *Let G be a Lie group and suppose $H \subset G$ is a subgroup. Then H is an embedded Lie subgroup if and only if H is closed in G.*

In particular, every closed subgroup of a Lie group is again a Lie group. We are particularly interested in matrix groups, which are closed subgroups of $\mathrm{GL}(n, \mathbb{F})$.

Example B.2. The special linear group $\mathrm{SL}(n, \mathbb{F}) \subset \mathrm{GL}(n, \mathbb{F})$ consisting of all $n \times n$ matrices over $\mathbb{F}$ with determinant 1 is a closed subgroup since it is equal to the inverse image of 1 under the determinant map. We also define the orthogonal and unitary groups

$$\mathrm{O}(n) = \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid AA^T = I\}\,, \tag{B.1}$$
$$\mathrm{U}(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid AA^\dagger = I\}\,. \tag{B.2}$$

These are closed subgroups of general linear groups given by the inverse image of I under the smooth maps $\mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R}), A \mapsto AA^T$ and $\mathrm{Mat}(n,\mathbb{C}) \to \mathrm{Mat}(n,\mathbb{C}), A \mapsto A\,A^\dagger$ respectively. The special orthogonal and special unitary groups are $\mathrm{SO}(n) = \mathrm{O}(n) \cap \mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n,\mathbb{C})$. By the closed subgroup theorem all of these groups are Lie groups. The (special) orthogonal and unitary groups are compact for all $n \geq 1$. All the matrix groups defined above are connected except for $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{O}(n)$: these groups have two components separated by the sign of the determinant.

Maps between Lie groups are required to respect both the manifold and group structures.

Definition B.2. Let G and H be Lie groups. A *Lie group homomorphism* is a map $\varphi : G \to H$ that is both smooth and a group homomorphism. A *Lie group isomorphism* is a Lie group homomorphism that is also a diffeomorphism (and hence a group isomorphism).

Example B.3. The Lie group $\mathrm{U}(1)$ consists of all complex numbers of norm 1 and therefore is equal to the unit circle S^1 in $\mathbb{C}$. We can map it into the group $\mathrm{SO}(2)$ of rotations in $\mathbb{R}^2$ using

$$\varphi : \mathrm{U}(1) \to \mathrm{SO}(2)\,, \qquad e^{i\alpha} \mapsto \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{B.3}$$

This map is a Lie group isomorphism.

Example B.4. We have

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix} \in \mathrm{Mat}(2,\mathbb{C}) \,\middle|\, x,y \in \mathbb{C}, |x|^2 + |y|^2 = 1 \right\}. \tag{B.4}$$

Viewing S^3 as the unit sphere in $\mathbb{C}^2$, it can be checked that the map

$$\varphi : S^3 \to \mathrm{SU}(2)\,, \qquad (x,y) \mapsto \begin{pmatrix} x & -\overline{y} \\ y & \overline{x} \end{pmatrix} \tag{B.5}$$

is a diffeomorphism. This map becomes a Lie group isomorphism if we give S^3 the structure of a Lie group by viewing it as the set of unit quaternions.

Besides Lie groups, the notion of a Lie algebra is very important for our applications in physics.

Definition B.3. A vector space V together with a map $[\cdot,\cdot] : V \times V \to V$ is a *Lie algebra* if $[\cdot,\cdot]$ is bilinear, antisymmetric and satisfies the Jacobi identity:

$$[u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0 \tag{B.6}$$

for all $u, v, w \in V$. The map $[\cdot,\cdot]$ is called the *Lie bracket*.

Example B.5. The real vector space $V = \mathbb{R}^3$ with Lie bracket $[x,y] = x \times y$ equal to the cross product is a Lie algebra.

There is a natural way to make the tangent space T_eG at the unit element e of a Lie group G into a Lie algebra, as described in section 1.5 of [15]. We will denote this Lie algebra as $\mathfrak{g}$. Note that $\mathfrak{g}$ is a real vector space of dimension equal to the dimension of G. If G is a matrix group, we can view T_eG as a subset of $T_e\mathrm{GL}(n,\mathbb{F}) = \mathrm{Mat}(n,\mathbb{F})$, where the last step follows from the fact that $\mathrm{GL}(n,\mathbb{F})$ is open in $\mathrm{Mat}(n,\mathbb{F})$. Under this identification, the Lie algebras of the matrix groups introduced above are as follows [15, Thm. 1.5.22, Thm. 1.5.27].

Proposition B.1. *The Lie algebras corresponding to the matrix groups from Example B.2 are*

$$\mathfrak{gl}(n,\mathbb{F}) = \mathrm{Mat}(n,\mathbb{F})\,, \tag{B.7}$$
$$\mathfrak{sl}(n,\mathbb{F}) = \{A \in \mathrm{Mat}(n,\mathbb{F}) \mid \mathrm{tr}(A) = 0\}\,, \tag{B.8}$$
$$\mathfrak{o}(n) = \mathfrak{so}(n) = \{A \in \mathrm{Mat}(n,\mathbb{R}) \mid A + A^T = 0\}\,, \tag{B.9}$$
$$\mathfrak{u}(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A + A^\dagger = 0\}\,, \tag{B.10}$$
$$\mathfrak{su}(n) = \{A \in \mathrm{Mat}(n,\mathbb{C}) \mid A + A^\dagger = 0,\ \mathrm{tr}(A) = 0\}\,. \tag{B.11}$$

The Lie bracket is given by the commutator of matrices $[A,B] = AB - BA$.

Just like for Lie groups, we can consider maps between Lie algebras.

Definition B.4. Let $(V,[\cdot,\cdot]_V)$ and $(W,[\cdot,\cdot]_W)$ be Lie algebras over the same field. A *Lie algebra homomorphism* is a linear map $\psi : V \to W$ satisfying $[\psi(u),\psi(v)]_W = \psi([u,v]_V)$ for all $u,v \in V$.

Example B.6. Let G and H be Lie groups and $\varphi : G \to H$ a Lie group homomorphism. Then the differential $D_e\varphi : \mathfrak{g} \to \mathfrak{h}$ at the unit element $e \in G$ is a Lie algebra homomorphism [15, Thm. 1.5.18]. We write φ_* for this induced homomorphism.

For any Lie group G there is a smooth map $\exp : \mathfrak{g} \to G$ sending 0 to e and satisfying

$$\exp((x+y)X) = \exp(xX)\exp(yX) \quad \text{and} \quad \exp(-X) = (\exp(X))^{-1} \tag{B.12}$$

for all $x,y \in \mathbb{R}$ and $X \in \mathfrak{g}$, called the *exponential map* (see section 1.7 in [15]). Another important property is that the tangent vector to the curve $\gamma(t) = \exp(tX)$ at the identity is X. We can characterise this map for matrix groups:

Proposition B.2. *If G is a matrix group, the exponential map* $\exp : \mathfrak{g} \to G$ *is given by*

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k\,. \tag{B.13}$$

This series converges for any matrix $A \in \mathrm{Mat}(n,\mathbb{F})$.

Example B.7. For $G = \mathrm{U}(1)$ we find from Proposition B.1 that $\mathfrak{u}(1) = i\mathbb{R}$. The exponential map exp: $\mathfrak{u}(1) \to \mathrm{U}(1)$ sends ix to e^{ix} and hence is equal to the complex exponential map $i\mathbb{R} \to S^1$. In particular, this map is surjective and a local diffeomorphism.

In some of our applications in gauge theory, Lie groups are required to be (semi)simple. This notion becomes relevant when considering representations of Lie groups (see section B.2). We define an *ideal* of a Lie algebra $(V,[\cdot,\cdot])$ to be a subspace $W \subset V$ such that $[v,w] \in W$ for all $v \in V$ and $w \in W$. We call $(V,[\cdot,\cdot])$ *abelian* if the bracket $[\cdot,\cdot]$ is zero.

Definition B.5. A Lie algebra $(V, [\cdot, \cdot])$ is *simple* if it is nonabelian has no nontrivial ideals (different from 0 and V). A *semisimple* Lie algebra is a Lie algebra with no nonzero abelian ideals. A connected Lie group G is (semi)simple if its Lie algebra $\mathfrak{g}$ is (semi)simple.

Note that any simple Lie algebra is also semisimple. Moreover, a (finite) product of semisimple Lie groups is again semisimple. The groups $\mathrm{SU}(n \geq 2)$ and $\mathrm{SO}(n \geq 3)$ are simple except for $\mathrm{SO}(4)$, which is only semisimple. $\mathrm{SO}(2)$ and $\mathrm{U}(n)$ are not (semi)simple for any $n \geq 1$.

B.2 Lie group representations and actions

In this section we study Lie group actions starting with representations, which are linear actions on vector spaces. From now on, any vector space V is a finite dimensional real or complex vector space equipped with the standard smooth structure defined in [24, Ex. 1.24]. We can use this smooth structure to make the group $\mathrm{GL}(V)$ of linear automorphisms of V into a Lie group, just like we did for $V = \mathbb{F}^n$. The tangent space $T_e\mathrm{GL}(V)$ can then be identified with the vector space $\mathrm{End}(V)$ of linear endomorphisms of V. Note that we view $\mathrm{End}(V)$ as a real vector space even if V is complex.

Definition B.6. Let G be a Lie group and V a vector space. A *representation* of G on V is a Lie group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Example B.8. Any matrix subgroup of $\mathrm{GL}(n, \mathbb{F})$ has a representation on $\mathbb{F}^n$ given by matrix multiplication on column vectors from the left. This is called a *fundamental representation*.

Example B.9. Let G be a Lie group and $g \in G$. The conjugation map $c_g : G \to G, h \mapsto ghg^{-1}$ is a Lie group isomorphism (with inverse $c_{g^{-1}}$), so the induced map $(c_g)_* : \mathfrak{g} \to \mathfrak{g}$ is also an isomorphism. The map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ sending g to $(c_g)_*$ is a Lie group homomorphism and defines the *adjoint representation* of G on $\mathfrak{g}$. For matrix groups, the adjoint representation is given by $\mathrm{Ad}(A)(B) = ABA^{-1}$ for matrices $A \in G$ and $B \in \mathfrak{g}$. If G is abelian, the adjoint representation is trivial.

We can also define representations of real Lie algebras:

Definition B.7. Let $\mathfrak{g}$ be a real Lie algebra and V a vector space. A *representation* of $\mathfrak{g}$ on V is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathrm{End}(V)$.

Example B.10. If $\rho : G \to \mathrm{GL}(V)$ is a representation of a Lie group G, the differential $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ is a representation of $\mathfrak{g}$. If ρ is a fundamental representation, then $\rho_*(A)$ is again given by multiplication by A on column vectors. If $V = \mathfrak{g}$ and $\rho = \mathrm{Ad}$ is the adjoint representation, it can be shown [15, Thm. 2.1.52] that the induced representation is $\mathrm{Ad}_*(X)(Y) = [X, Y]$ for $X, Y \in \mathfrak{g}$.

In gauge theory Lie groups are required to be compact because G-invariant inner products on a representation V are needed to build the theory. The existence of such inner products is guaranteed for compact groups by the following theorem [15, Thm. 2.1.39].

Theorem B.2. *Let G be a compact Lie group and $\rho : G \to \mathrm{GL}(V)$ a representation. Then there exists a G-invariant inner product $\langle \cdot, \cdot \rangle_V$ on V, i.e. $\langle \rho(g)(v), \rho(g)(w) \rangle_V = \langle v, w \rangle_V$ for all $v, w \in V$ and $g \in G$.*

Example B.11. If G is one of the compact matrix groups in Example B.2 and $\rho = \mathrm{Ad}$ is the adjoint representation of G, an Ad-invariant inner product on $\mathfrak{g}$ is $\langle A, B \rangle_{\mathfrak{g}} = -\mathrm{Tr}(AB)$. The Ad-invariance follows from the cyclicity of the trace. If in addition G is simple, this is the only Ad-invariant inner product up to multiplication by a positive constant. For semisimple Lie groups a choice of an Ad-invariant inner product is determined by a number of positive constants, called *coupling constants* [15, Thm. 2.5.3, Thm. 2.5.4]. These coupling constants describe the strength of interactions in gauge theories.

We now turn to general actions of Lie groups on a manifold $\mathcal{M}$.

Definition B.8. Let G be a Lie group and $\mathcal{M}$ a smooth manifold. A *smooth left action* of G on $\mathcal{M}$ is a smooth map $\phi : G \times \mathcal{M} \to \mathcal{M}$, $(g, p) \mapsto g \cdot p$ satisfying $e \cdot p = p$ and $g \cdot (h \cdot p) = gh \cdot p$ for all $g, h \in G$ and $p \in \mathcal{M}$.

We similarly define smooth right actions. Smooth Lie group actions are a special case of regular group actions, and so notions defined for regular group actions still make sense. For a left action ϕ and any $p \in \mathcal{M}$ the orbit map $\phi_p : G \to \mathcal{M}$, $g \mapsto g \cdot p$ is smooth. The stabiliser $G_p = \{g \in G \mid g \cdot p = p\}$ of p is a closed subgroup of G given by the inverse image of $\{p\}$ under ϕ_p and hence a Lie subgroup by Theorem B.1.

Example B.12. If $\rho : G \to \mathrm{GL}(V)$ is a representation, then $\phi : G \times V \to V$, $(g, v) \mapsto \rho(g)(v)$ is a smooth left action. If G is a (special) orthogonal or unitary group and ρ is the fundamental representation, the action preserves the norm and hence can be restricted to the unit sphere. This gives us transitive actions of $\mathrm{SO}(n)$ and $\mathrm{O}(n)$ on S^{n-1} and of $\mathrm{SU}(n)$ and $\mathrm{U}(n)$ on S^{2n-1}. For $\mathrm{SU}(n)$, the stabiliser of $e_1 = (1, 0, 0, \dots)$ consists of matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{SU}(n-1) \end{pmatrix} \cong \mathrm{SU}(n-1)\,. \tag{B.14}$$

Analogous results hold for $\mathrm{O}(n), \mathrm{SO}(n)$ and $\mathrm{U}(n)$.

We have the following useful description of the Lie algebra of a stabiliser G_p.

Lemma B.1. *Let ϕ be a smooth left action of a Lie group G on a manifold $\mathcal{M}$ and let $\phi_p : G \to \mathcal{M}$ be the orbit map of $p \in \mathcal{M}$. The kernel of the differential $D_e\phi_p : \mathfrak{g} \to T_p\mathcal{M}$ is the Lie algebra $\mathfrak{g}_p$ of the stabiliser G_p.*

Proof. If $X \in \mathfrak{g}_p$ we have $\exp(tX) \in G_p$ for all $t \in \mathbb{R}$ and so $\exp(tX) \cdot p = p$. We see that $D_e\phi_p(X)$ equals the velocity vector of a constant curve, which is zero. Hence $X \in \ker(D_e\phi_p)$. The other inclusion is proven in [15, Prop. 3.2.10]. □

It is well-known for regular actions of a group G on a set X that there exists a bijection between the set of cosets G/G_x of the stabiliser of $x \in X$ and the orbit O_x of x. We want to adjust this statement to the smooth setting, for which we need the following theorem. This is a nontrivial result given as Corollary 3.7.35 in [15].

Theorem B.3. *Let G be a Lie group and H a closed subgroup. Then there exists a unique smooth structure on the quotient G/H such that the quotient map $\pi : G \to G/H$ is a smooth submersion. The dimension of G/H is* $\dim G - \dim H$.

Recall that a submersion is a smooth map $f : M \to N$ such that for all $p \in M$ the differential $D_pf : T_pM \to T_{f(p)}N$ is surjective. Similarly, f is an immersion if D_pf is injective for all $p \in M$.

Proposition B.3. *Let ϕ be a smooth left action of a Lie group G on a manifold $\mathcal{M}$ and fix a point $p \in \mathcal{M}$. The map $f : G/G_p \to \mathcal{M}$, $[g] \mapsto g \cdot p$ is an injective immersion whose image is the orbit $O_p = \{g \cdot p \mid g \in G\}$. If G is compact, then the orbit O_p is an embedded submanifold of $\mathcal{M}$ diffeomorphic to G/G_p.*

Proof. From $g \cdot p = h \cdot p \iff g^{-1}h \in G_p$ it easily follows that f is well-defined and injective with image O_p. If G is compact, then so is G/G_p and hence f is a continuous map from a compact space into a Hausdorff space. It follows that f is closed and restricts to a homeomorphism $G/G_p \xrightarrow{\sim} O_p$. It remains to be shown that f is an immersion, which is proven in [15, Thm. 3.8.8]. □

Example B.13. Consider the action of $\mathrm{SU}(n)$ on $\mathbb{C}^n$ defined in Example B.12. The unit sphere S^{2n-1} forms an orbit of the action and the stabiliser of e_1 is isomorphic to $\mathrm{SU}(n-1)$. By Proposition B.3 we find $\mathrm{SU}(n)/\mathrm{SU}(n-1) \cong S^{2n-1}$. Analogous results hold for $\mathrm{O}(n), \mathrm{SO}(n)$ and $\mathrm{U}(n)$.

B.3 Connections

Connections are defined on principal fibre bundles, which are fibre bundles that are compatible with a Lie group action. First of all, we can adjust the definition of a fibre bundle (Definition 3.3) to the smooth setting by requiring that the spaces E, B, F are manifolds, p is smooth and the trivialisations h are diffeomorphisms. We then define a principal fibre bundle as follows.

Definition B.9. Let G be a Lie group acting from the right on a manifold P. A fibre bundle $G \to P \xrightarrow{\pi} M$ (in the smooth setting) is a *principal G-bundle* if

1. G preserves the fibres and acts freely and transitively on them, i.e. for all $x \in M$ and $p \in \pi^{-1}(x)$ the map $G \to \pi^{-1}(x)$, $g \mapsto p \cdot g$ is a well-defined bijection.
2. The local trivialisations $h : \pi^{-1}(U) \xrightarrow{\sim} U \times G$ satisfy $h(p \cdot g) = h(p) \cdot g$, where g acts on $U \times G$ by $(x, a) \cdot g = (x, ag)$.

The fibres $\pi^{-1}(x)$ for $x \in M$ are embedded submanifolds of P diffeomorphic to G. A *section* of a principal bundle is a smooth map $s : U \to P$ satisfying $\pi \circ s = \mathrm{id}_U$, where $U \subset M$ is open. The section is *global* if $U = M$ and *local* otherwise. A principal bundle admitting a global section is called *trivial*.

Example B.14. Consider the Hopf bundle $S^1 \to S^3 \xrightarrow{\pi} \mathbb{C}\mathrm{P}^1$ from Example 3.2. This is also a fibre bundle in the smooth setting, and if we identify S^1 with the Lie group $\mathrm{U}(1)$ there is a smooth right action $S^3 \times \mathrm{U}(1) \to S^3, ((z_1, z_2), \lambda) \mapsto (z_1\lambda, z_2\lambda)$ given by complex multiplication. This action restricts to a free and transitive action on the fibres. The trivialisations h_i satisfy

$$h_i((z_1, z_2) \cdot \lambda) = \left([z_1\lambda : z_2\lambda], \frac{z_i\lambda}{|z_i\lambda|}\right) = \left([z_1 : z_2], \frac{z_i\lambda}{|z_i|}\right) = h_i(z_1, z_2) \cdot \lambda \tag{B.15}$$

for $\lambda \in \mathrm{U}(1)$ and $(z_1, z_2) \in S^3$. It follows that the Hopf bundle is a principal $\mathrm{U}(1)$-bundle.

If G is a Lie group and H is a closed subgroup, then H acts on G by right multiplication. This yields an important example of a principal bundle.

Lemma B.2. *Let G be a Lie group, H a closed subgroup and $\pi : G \to G/H$ the quotient map. Then $H \to G \xrightarrow{\pi} G/H$ is a principal H-bundle.*

Proof. Note that H acts freely and transitively on each fibre gH. Since π is a surjective submersion, around each point of G/H there exists a neighborhood U and a smooth local section $s : U \to G$ [15, Lem. 3.7.4]. We define $\phi : \pi^{-1}(U) \to U \times H$, $g \mapsto ([g], s([g])^{-1}g)$. Note that $s([g])^{-1}g$ is indeed an element of H because $s([g])H = gH$. The map ϕ is a diffeomorphism with inverse $\phi^{-1}([g], h) = s([g])h$. The composition $\text{proj}_1 \circ \phi$ obviously equals π on $\pi^{-1}(U)$, so ϕ is a local trivialisation. The second requirement of Definition B.9 is also satisfied since $\phi(gh) = ([g], s([g])^{-1}gh) = ([g], s([g])^{-1}g) \cdot h$. □

The infinitesimal behaviour of the action of G on the total space P can be described using fundamental vector fields.

Definition B.10. Let G be a Lie group acting from the right on a manifold $\mathcal{M}$. For $X \in \mathfrak{g}$, the *fundamental vector field* $X^\# \in \mathfrak{X}(\mathcal{M})$ associated to X is defined by

$$X^\#_p = (D_e\phi_p)(X) = \left.\frac{d}{dt}\right|_{t=0} (p \cdot \exp(tX)) \,, \tag{B.16}$$

where ϕ_p is the orbit map of $p \in \mathcal{M}$.

The last ingredient we need to define connections is the notion of vector valued differential forms. Let V be a finite dimensional real vector space (with the standard smooth structure) and M a smooth manifold. We denote the set of smooth maps from M to V by $C^\infty(M, V)$. Similarly, we define the set $\Omega^k(M, V)$ of V-valued k-forms on M: this can be viewed as the set $\Gamma^\infty((M \times V) \otimes_{\mathbb{R}} \Lambda^k T^*M)$ of smooth sections of the tensor product of the trivial bundle $M \times V$ with the k-th exterior power of the cotangent bundle of M. Equivalently, a k-form ω with values in V is an alternating map $\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M, V)$ taking k vector fields as input that is multilinear over $C^\infty(M, \mathbb{R})$, i.e.

$$\omega(X_1, \ldots, f \cdot X_i + g \cdot Y, \ldots, X_k) = f \cdot \omega(X_1, \ldots, X_k) + g \cdot \omega(X_1, \ldots, X_{i-1}, Y, \ldots, X_k) \tag{B.17}$$

for $X_1, \ldots, X_k, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M, \mathbb{R})$, where $\cdot$ is defined pointwise. The set $\Omega^k(M, V)$ can be identified with $\Omega^k(M) \otimes_{\mathbb{R}} V$, where $\Omega^k(M) = \Omega^k(M, \mathbb{R})$ is the set of all ordinary k-forms on M. We can also define $\Omega^k(M, V)$ for complex vector spaces by viewing V as real vector space. Operations like the pullback or exterior derivative of a k-form naturally extend to V-valued forms.

We now turn to connections. A connection allows us to compare fibres of principal bundles $G \to P \to M$ over nearby points. In particular, a connection is used in section 4.1 to define the covariant derivative. Let $R_g : P \to P$ be the map given by $R_g(p) = p \cdot g$ for $g \in G$.

Definition B.11. A *connection* or *gauge field* on a principal G-bundle $G \to P \xrightarrow{\pi} M$ is a $\mathfrak{g}$-valued 1-form $\tilde{A} \in \Omega^1(P, \mathfrak{g})$ on P satisfying

1. $R_g^*\tilde{A} = \text{Ad}(g^{-1}) \circ \tilde{A}$ for all $g \in G$, where $R_g^*\tilde{A}$ is the pullback of $\tilde{A}$ by R_g.
2. $\tilde{A}(X^\#) = X$ identically on P for all $X \in \mathfrak{g}$, where $X^\#$ is the fundamental vector field associated to X.

Example B.15. We will construct a connection on the Hopf bundle $\mathrm{U}(1) \to S^3 \to S^2$. By viewing S^3 as the unit sphere in $\mathbb{C}^2$, the tangent space $T_{(z_1,z_2)}S^3$ at the point $(z_1, z_2) \in S^3$ becomes a subspace of $T_{(z_1,z_2)}\mathbb{C}^2 \cong \mathbb{C}^2$. Consider the 1-forms $\alpha_i, \overline{\alpha}_i \in \Omega^1(S^3, \mathbb{C})$ for $i = 1, 2$ given by $\alpha_{i,(z_1,z_2)}(x_1, x_2) = x_i$ and $\overline{\alpha}_{i,(z_1,z_2)}(x_1, x_2) = \overline{x}_i$ for $(x_1, x_2) \in T_{(z_1,z_2)}S^3 \subset \mathbb{C}^2$. We define a 1-form $\tilde{A}$ on S^3 by

$$\tilde{A}_{(z_1,z_2)} = \tfrac{1}{2}(\overline{z}_1\alpha_1 - z_1\overline{\alpha}_1 + \overline{z}_2\alpha_2 - z_2\overline{\alpha}_2)\,. \tag{B.18}$$

Note that we are writing α_i instead of $\alpha_{i,(z_1,z_2)}$ to simplify notation. The 1-form $\tilde{A}$ takes values in $i\mathbb{R} = \mathfrak{u}(1)$ because the complex conjugate of (B.18) is $-\tilde{A}_{(z_1,z_2)}$. It is shown in [15, Prop. 5.2.4] that $\tilde{A}$ is a connection. We call it the *natural connection* on the Hopf bundle.

A connection gives us a decomposition of the tangent space T_pP at a point p into a "vertical" and "horizontal" subspace. The vertical subspace V_p is the tangent space of the fibre of $\pi(x)$ at p, which equals the kernel of $D_p\pi$. A horizontal subspace is a subspace H_p such that $T_pP = V_p \oplus H_p$. A connection defines a horizontal subspace for all $p \in P$ by $H_p = \ker \tilde{A}_p$. This choice of a horizontal direction in T_pP varies smoothly with p and is compatible with the action of G. In fact, it can be shown that specifying a family of horizontal tangent spaces in the following way is equivalent to giving a connection $\tilde{A}$ [15, Thm. 5.2.2].

Definition B.12. Let $G \to P \xrightarrow{\pi} M$ be a principal G-bundle. A subbundle H of TP consisting of horizontal tangent spaces is an *Ehresmann connection* if $D_pR_g(H_p) = H_{p\cdot g}$ for $p \in P, g \in G$.

We have the following result on the existence of (Ehresmann) connections [36, Prop. 1.3.7].

Theorem B.4. *Every principal fibre bundle admits an Ehresmann connection.*

We will use Ehresmann connections to prove a result supporting claims made in section 5.2. Let G be a compact Lie group and $\rho : G \to \mathrm{GL}(V)$ a representation. Let $\mathcal{M} \subset V$ be an orbit of the G-action induced by ρ (see Example B.12). For $r > 0$ we write $U_r = \{x \in \mathbb{R}^n \mid \|x\| > r\}$ with closure $\overline{U}_r$. Suppose $f : \mathbb{R}^n \to V$ is a smooth map such that $f(U_1) \subset \mathcal{M}$. Let e_i be the i-th standard basis vector of $\mathbb{R}^n$ and let $\partial_i f$ denote the derivative of f in direction e_i.

Proposition B.4. *Let $\varepsilon > 1$. There exists a smooth map $A_i : \mathbb{R}^n \to \mathfrak{g}$ such that*

$$\partial_i f(x) + \rho_*(A_i(x))(f(x)) = 0 \tag{B.19}$$

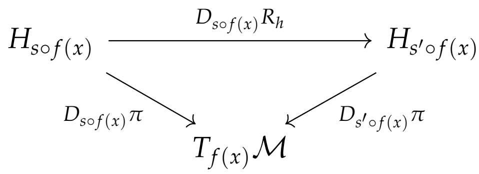
for all $x \in \overline{U}_\varepsilon$, where ρ_* is the representation of $\mathfrak{g}$ induced by ρ.

Proof. Let $v \in \mathcal{M}$ and let $G_v \subset G$ be the stabiliser of v under the G-action. Recall from Lemma B.2 that $G_v \to G \to G/G_v$ is a principal bundle, and Proposition B.3 implies that $\pi : G \to \mathcal{M}$ given by $\pi(g) = \rho(g)(v)$ induces a diffeomorphism $G/G_v \xrightarrow{\sim} \mathcal{M}$. We find that $G_v \to G \xrightarrow{\pi} \mathcal{M}$ is a principal G_v-bundle. By Theorem B.4 there exists an Ehresmann connection H on this bundle. That is, we have a subspace H_g of T_gG that is complementary to the kernel of $D_g\pi$ and has dimension $\dim G - \dim G_v = \dim \mathcal{M}$ for all $g \in G$. This means that $D_g\pi$ restricts to an isomorphism $H_g \to T_{\pi(g)}\mathcal{M}$. We restrict the differential of π to H.

Let $x \in U_1$. Because the bundle is locally trivial, there exists an open neighborhood $W \subset \mathcal{M}$ of $f(x)$ such that there is a local trivialisation $\phi : \pi^{-1}W \to W \times G_v$. This gives us a local section $s(p) = \phi^{-1}(p, e)$ of π. Let $K = U_1 \cap f^{-1}W$. Note that for all $x \in K$ the vector $\partial_i f(x)$ is tangent to $\mathcal{M}$. Since $D_{s\circ f(x)}\pi$ (restricted to $H_{s\circ f(x)}$) is an isomorphism, there is a smooth map $q : K \to H$ such that $q(x) \in H_{s\circ f(x)}$ and $(D_{s\circ f(x)}\pi)(q(x)) = -\partial_i f(x)$. Let $R_g : G \to G$ be the map given by multiplication by $g \in G$ from the right. We define A_i on K by

$$A_i(x) = (D_{s\circ f(x)} R_{(s\circ f(x))^{-1}})(q(x))\,. \tag{B.20}$$

Note that A_i takes values in $T_e G = \mathfrak{g}$. We claim that this definition does not depend on the choice of trivialisation ϕ and satisfies (B.19). Let ψ be another trivialisation around $f(x)$ and $s'(p) = \psi^{-1}(p, e)$ the associated section. Then there exists an element $h \in G_v$ such that $s'(f(x)) = s(f(x))h$. We have the following commutative diagram:



Let us write $u = s(f(x))$. The diagram shows that

$$D_{uh} R_{(uh)^{-1}} \circ (D_{uh}\pi)^{-1} = D_{uh} R_{(uh)^{-1}} \circ \left(D_u \pi \circ (D_u R_h)^{-1}\right)^{-1} = D_u R_{u^{-1}} \circ (D_u \pi)^{-1}\,, \tag{B.21}$$

implying the first claim. For the second claim we note that

$$\rho_*(A_i(x))(f(x)) = D_e(\pi \circ R_u)(A_i(x)) = D_e(\pi \circ R_u) \circ (D_u R_{u^{-1}})(q(x)) \tag{B.22}$$

$$= (D_u \pi)(q(x)) = -\partial_i f(x)\,. \tag{B.23}$$

It follows that we have found a map $A_i : U_1 \to \mathfrak{g}$ with the required properties. It is shown in [24, Lem. 2.26] that the restriction $A_i|_{\overline{U}_\varepsilon}$ can be extended to a smooth map $A_i : \mathbb{R}^n \to \mathfrak{g}$. $\square$

Bibliography

[1] M. Abe. Über die stetigen Abbildungen der n-Sphäre in einen metrischen Raum. *Japanese journal of mathematics*, 16:169–176, 1940.

[2] T. Annala and M. Möttönen. Charge ambiguity and splitting of monopoles. *Physical Review Research*, 4, 06 2022.

[3] A. Baarsma. Magnetic monopoles. Student Seminar Theoretical Physics, Universiteit Utrecht, 2009.

[4] K. Bauer, F. Deloup, and P. Zvengrowski. Base points in homotopy theory and the fundamental theorem of algebra. *Morfismos*, 13:1–15, 01 2009.

[5] T. Bröcker and T. Dieck. *Representations of Compact Lie Groups*. Springer Berlin Heidelberg, 1985. ISBN: 978-3-662-12918-0.

[6] S. Chatrchyan and V. K. et al. Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC. *Physics Letters B*, 716(1):30–61, Sep 2012.

[7] S. Coleman. *The Magnetic Monopole Fifty Years Later*, pages 21–117. Springer US, 1983. ISBN: 978-1-4613-3655-6.

[8] S. Coleman. *Aspects of Symmetry: Selected Erice Lectures*. Cambridge University Press, 1985.

[9] P. Dirac. Quantised Singularities in the Electromagnetic Field. *Proceedings of the Royal Society of London Series A*, 133(821):60–72, Sep 1931.

[10] M. Gabella. Non-abelian gauge theories with spontaneous symmetry breaking: Higgs mechanism. Monograph, 2006.

[11] H. Georgi and S. L. Glashow. Unified weak and electromagnetic interactions without neutral currents. *Physical Review Letters*, 28(22):1494–1497, 1972.

[12] J. Goldstone. Field theories with superconductor solutions. *Il Nuovo Cimento (1955-1965)*, 19:154–164, 1961.

[13] D. Griffiths. *Introduction to Electrodynamics*. Pearson Education, 2013. ISBN: 978-0-321-85656-2.

[14] A. H. Guth. Inflationary universe: A possible solution to the horizon and flatness problems. *Phys. Rev. D*, 23:347–356, Jan 1981.

[15] M. Hamilton. *Mathematical Gauge Theory: With Applications to the Standard Model of Particle Physics*. Springer International Publishing, 2017. ISBN: 978-3-319-68439-0.

[16] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2001.

[17] P. W. Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Lett.*, 13:508–509, Oct 1964.

[18] G. Hooft. Magnetic monopoles in unified gauge theories. *Nuclear Physics B*, 79(2):276–284, 1974.

[19] M. Hutchings. Introduction to higher homotopy groups and obstruction theory. Lecture notes UC Berkely, 2011.

[20] T. Kibble. Topology of cosmic domains and strings. *Journal of Physics A: Mathematical and General*, 9(8):1387–1398, Aug 1976.

[21] S. Kobayashi, M. Kobayashi, Y. Kawaguchi, M. Nitta, and M. Ueda. Abe homotopy classification of topological excitations under the topological influence of vortices. *Nuclear Physics B*, 856(2):577–606, Mar 2012.

[22] S. Kobayashi, N. Tarantino, and M. Ueda. Topological influence and backaction between topological excitations. *Phys. Rev. A*, 89:033603, Mar 2014.

[23] J. Lee. *Introduction to Topological Manifolds*. Springer New York, 2011. ISBN: 978-1-4419-7940-7.

[24] J. Lee. *Introduction to Smooth Manifolds*. Springer New York, 2013. ISBN: 978-1-4419-9982-5.

[25] M. Marian. A study of the dirac monopole. Bachelor thesis, McGill University, 2018.

[26] A. Moore. Long exact homotopy sequence construction. Monograph, 2020.

[27] H. Nielsen and P. Olesen. Vortex-line models for dual strings. *Nuclear Physics B*, 61:45–61, 1973.

[28] A. M. Polyakov. Particle spectrum in quantum field theory. *Soviet Journal of Experimental and Theoretical Physics Letters*, 20:194, Sep 1974.

[29] M. Postma. Alice electrodynamics. MSc thesis, Universiteit van Amsterdam, 1997.

[30] J. Preskill. Cosmological production of superheavy magnetic monopoles. *Phys. Rev. Lett.*, 43:1365–1368, Nov 1979.

[31] J. Preskill. Magnetic monopoles. *Annual Review of Nuclear and Particle Science*, 34(1):461–530, 1984.

[32] J. Preskill. Vortices and monopoles. In *Architecture of fundamental interactions at short distances*, volume 1, pages 235–337, Jan 1987.

[33] A. Rajantie. Introduction to magnetic monopoles. *Contemporary Physics*, 53(3):195–211, May 2012.

[34] W. Rossmann. *Lie Groups: An Introduction Through Linear Groups*. Oxford graduate texts in mathematics. Oxford University Press, 2002. ISBN: 0-19-859683-9.

[35] V. Rubakov. *Classical Theory of Gauge Fields*. Princeton University Press, 2002. ISBN: 0-691-05927-6.

[36] G. Rudolph and M. Schmidt. *Differential Geometry and Mathematical Physics: Part II. Fibre Bundles, Topology and Gauge Fields*. Springer Netherlands, 2017. ISBN: 978-94-024-0959-8.

[37] A. Schwarz. Field theories with no local conservation of the electric charge. *Nuclear Physics B*, 208(1):141–158, 1982.

[38] J. Schwinger. A magnetic model of matter. *Science*, 165(3895):757–761, 1969.

[39] C. Torre. Introduction to Classical Field Theory. *All complete monographs*. 3. URL: `https://digitalcommons.usu.edu/lib_mono/3` (version: 07-06-2022).

[40] H.-R. Trebin. The topology of non-uniform media in condensed matter physics. *Advances in Physics - ADVAN PHYS*, 31:195–254, 06 1982.

[41] H. Urbantke. The hopf fibration seven times in physics. *Journal of Geometry and Physics*, 46(2):125–150, 2003.

[42] A. Vilenkin and E. Shellard. *Cosmic Strings and Other Topological Defects*. Cambridge University Press, 1994. ISBN: 0-521-39153-9.

[43] T. T. Wu and C. N. Yang. Concept of nonintegrable phase factors and global formulation of gauge fields. *Phys. Rev. D*, 12:3845–3857, Dec 1975.

[44] C. N. Yang and R. L. Mills. Conservation of isotopic spin and isotopic gauge invariance. *Phys. Rev.*, 96:191–195, Oct 1954.

Figures

[45] gj255 (`https://math.stackexchange.com/users/119238/gj255`). Visualising $\pi_2(S^2)$ and $\pi_2(\mathbb{R}P^2)$. Mathematics Stack Exchange. URL: `https://math.stackexchange.com/q/2634214` (version: 07-06-2022).

[46] physics_maths (`https://tex.stackexchange.com/users/39323/physics-maths`). How to draw this particular mexican hat potential? TeX-LaTeX Stack Exchange. URL: `https://tex.stackexchange.com/q/229178` (version: 07-06-2022).