Twistorial Methods in General Relativity
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Twistorial Methods in General Relativity

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Contents

1 Introduction **5**

2 Preliminaries on manifolds and bundles **7**
2.1 Basic concepts 7
2.2 Lie groups and algebras 9
2.3 Principal and associated vector bundles 12

3 The construction of spinors **15**
3.1 Pseudo-Riemannian vector spaces 16
3.2 The Clifford algebra 19
3.3 Gamma matrices 22
3.4 Spin groups 24
3.5 The spinor representation 28

4 Spin structures **33**
4.1 Abstract index tensor algebra 33
4.2 Connections 39
4.3 The spinor bundle 46

5 Two-spinor calculus **51**
5.1 Basic properties 51
5.2 Association between spinors and tensors 54
5.3 The geometry of spinors 58
5.4 Curvature and the Einstein equations 60

6 Twistor space **67**
6.1 Conformal transformations 67
6.2 The twistor equation 70
6.3 Dual twistors 72
6.4 Null twistors and space-time points 76
6.5 The twistor Klein correspondence 79

7 Twistor cosmology **83**
7.1 The Plücker embedding and (anti-)de Sitter space 83

7.2 Infinity twistors 85
7.3 Twistors of Friedmann-Robertson-Walker space-times 89

8 Twistors in curved space 97
8.1 Sheaf cohomology and the Penrose transform 98
8.2 The non-linear graviton and the googly problem 102
8.3 Quantisation and palatial twistor theory 104

9 Outlook 107

A Compactification of flat space 109

Chapter 1

Introduction

In 1967, Roger Penrose introduced twistor theory [25, 28], with the aim of finding a way to unify general relativity and quantum mechanics. It is a well-known fact that the local, deterministic force of gravity is incompatible with the stochastic decay of the quantum wave function. Since both quantum mechanics and general relativity have had great successes in predicting different physical phenomena, but are mutually incompatible, it is suggested that both theories are fundamentally flawed. Penrose's idea was to construct twistor space as the complex manifold from which the laws of physics originate, with space-time being a secondary object determined entirely by twistor space. If one could then find a way of unifying general relativity and quantum mechanics in twistor space, one has found a way to unify the theories in regular space-time.

The theory of twistors is built upon the spinor formalism introduced by Élie Cartan, where the symmetry groups $\mathrm{SO}^+(p,q)$ of a pseudo-Riemannian vector space of signature (p,q) are doubly covered by their spin groups $\mathrm{Spin}^+(p,q)$. Via representations of the Clifford algebra one can then regard transformations of real coordinates using complex coordinates.

For 1+3-dimensional Minkowski space, the spin group $\mathrm{Spin}^+(1,3)$ acts on two pairs of two-dimensional complex vectors. The tangent bundle of a general 1+3-dimensional manifold M can then, under a few assumptions, be seen as consisting of two independent (left- and right-handed) spinor bundles, taking complex vector values. This structure can be considered as being more fundamental than the real structure on M.

For spaces that are conformally Minkowski, twistor space $\mathbb{T}$ is a 4 complex-dimensional subspace of the set of spinor fields, where the Klein correspondence can be used to associate lines in the projectivisation $\mathbb{PT}$ with points in complexified compactified Minkowski space $\mathbb{CM}^\bullet$. The philosophy behind twistor theory is to consider twistor space as given, and to see physical space as a derived object. Quantum effects are then supposed to originate from twistor space and manifest themselves in physical space. Twistor theory suggests a deep connection between space-time curvature and the quantum mechanical commutation relations.

In this theses, we will explore some of the powerful tools that twistor theory brings to the table. In particular, we will see how twistor theory is able to describe conformally flat space-times with great ease, allowing an elegant way to describe conformal infinities and singularities. We will also see how the complex-valuedness of twistors is used to encode properties of physical fields in cohomology groups. We will, however, also encounter some of the major shortcomings of twistor theory; we will find that twistor theory can not be obviously adapted to space-times with arbitrary curvature, something which has thus far precluded the theory from describing quantum gravity in a satisfying way.

One of the aims of this thesis is to rigorously build up twistor theory from the spinor formalism. Some prior knowledge of quantum mechanics, general relativity, basic abstract algebra and differential geometry is assumed.

In Chapter 2, some basic notions from differential geometry, which will be used throughout the thesis, are discussed. In Chapter 3, we will use Clifford algebras to define the spin groups of pseudo-Riemannian vector spaces, leading to a description of constant spinors. Combining these notions, we will use Chapter 4 to develop some of the basic tools for using spinor fields. First we will formalise the abstract index tensor algebra, extensively used in physics, after which connections on associated vector bundles will be discussed. These concepts will then be combined to define the spin structure on the tangent bundle.

Chapter 5 will further develop the theory of spinors, now for 1+3-dimensional manifolds. This will eventually lead to a description of the Einstein equations in spinorial terms.

From Chapter 6 onwards, we will be concerned with twistor space. We will first discuss conformal rescalings and construct twistor space from the conformally invariant twistor equation. After this, the Klein correspondence between twistor space and Minkowski space will be discussed. Chapter 7 will subsequently treat the twistorial description of other spaces which are conformally Minkowski, in particular focusing on (anti-)de Sitter space and Friedmann-Robertson-Walker cosmological models.

Finally, in Chapter 8, we will discuss some interesting non-linear features of twistor theory. The Penrose transform for massless free fields and the non-linear graviton construction for anti-self-dual space-times are treated. The final paragraph deals with the quantisation of twistors and the potential role of this in palatial twistor theory, which aims to describe arbitrary solutions to the Einstein equations in twistorial terms.

Chapter 2

Preliminaries on manifolds and bundles

This chapter is aimed at providing some standard results in differential geometry which will be used throughout this work, but which are not typically treated in an undergraduate course. The discussion here is loosely based on [11, 21]. The claims made in this chapter are generally not hard to prove, and proofs can be found in these references.

Our main goal in this chapter will be, beyond providing basic definitions of (co)tangent bundles and derivatives, to introduce the notions of *Lie groups* and *associated vector bundles*. The symmetry groups of pseudo-Riemannian vector spaces and their associated spin groups will turn out to be examples of such Lie groups. Lie groups allow us, given a vector bundle and a right-action of a Lie group on the underlying base manifold, to define associated vector bundles. We will give a construction of the tangent bundle as an associated vector bundle, which in Chapter 4 will be used for defining the notion of the covariant derivative. The spinor bundle, which we will introduce in that chapter, will also be an example of an associated vector bundle.

Throughout this and all subsequent chapters, when referring to a *manifold*, we mean a C^∞ *differentiable* manifold. Similarly, any *smooth* function is to mean a C^∞ function.

2.1 Basic concepts

In this paragraph, we briefly outline some of the basic concepts of differential geometry that will be used throughout this work, with the goal of fixing the notation used. We take M to be an arbitrary n-dimensional manifold.

We denote the set of ***smooth scalar functions*** on M by $C^\infty(M)$, consisting of all smooth functions

$$f : M \to \mathbb{R}.$$

Given a second manifold N, the set $C^\infty(M, N)$ similarly denotes the set of all smooth functions from M to N.

Given a vector bundle $\pi : E \to M$, we denote the set of smooth sections of this bundle by $\Gamma(E)$.

An important bundle structure is that of the ***tangent bundle*** of M, denoted by TM, which we define by

$$TM = \bigsqcup_{p \in M} T_pM,$$

where T_pM is the n-dimensional vector space of derivations at p. This set has a canonical projection map and with the usual manifold structure gives a vector bundle over M.

Elements of $\Gamma(TM)$ are called ***vector fields*** on M, which can equivalently be seen as the space of derivations

$$X : C^\infty(M) \to C^\infty(M),$$

satisfying the ***Leibniz rule***, i.e., for all $f, g \in C^\infty(M)$ we have that

$$X(fg) = fX(g) + gX(f),$$

where the multiplication is point-wise. We write X_p for the vector field X evaluated at the point $p \in M$. The set $\Gamma(TM)$ is often denoted by $\mathfrak{X}(M)$.

Similarly, we have the ***cotangent bundle*** T^*M, given by

$$T^*M = \bigsqcup_{p \in M} T_p^*M,$$

where T_p^*M is the vector space dual of T_pM. The space of sections $\Gamma(T^*M)$ can alternatively be characterized by the set of $C^\infty(M)$-linear functions

$$f : \mathfrak{X}(M) \to C^\infty(M),$$

and is also denoted by $\Omega^1(M)$, called the set of ***differential 1-forms on M***.

Physical fields will almost always be sections of tensor products of the tangent and cotangent bundle, collectively called *tensor fields*. Take the example of a magnetic field; at any point on the manifold, the strength and direction of the magnetic field is given by a smooth section of TM. We will develop the theory of such tensor fields in Paragraph 4.1, where we also introduce the *abstract index formalism*, which will make computations with these objects much more convenient.

Similar as to how smooth functions can take values in a manifold, differential 1-forms can take values in an arbitrary vector space V. We define the set of ***differential 1-forms on M with values in V***, $\Omega^1(M, V)$, as the set of $C^\infty(M)$-linear maps

$$f : \mathfrak{X}(M) \to C^\infty(M, V).$$

Another important notion is the ***commutator*** of two vector fields X, $Y \in \Gamma(TM)$, which is defined by

$$[X, Y]f = (X \circ Y)(f) - (Y \circ X)(f) \tag{2.1}$$

for all $f \in C^\infty(M)$.

We know, given some function with a real domain and codomain, how to define the derivative of this function. In a similar vein, we want to define a concept of differentiation of smooth functions between manifolds and smooth scalar functions on a manifold. For this, we consider the following two similar, but slightly different, definitions:

Definition 2.1.1. Let M, N be manifolds and $F : M \to N$ a smooth map. Then for every point $p \in M$, given $X_p \in T_pM$ and $f \in C^\infty(M)$, we define the ***derivative*** of F in p by

$$(dF)_p : T_pM \to T_{F(p)}N$$
$$(dF)_p(X_p)(f) = X_p(f \circ F).$$

This map is well defined. This induces a map

$$\mathrm{d}F : \Gamma(TM) \to \Gamma(TN)$$
$$(\mathrm{d}F(X))_p = (dF)_p(X_p)$$

called the ***differential*** of F.

Definition 2.1.2. Let M be a manifold and $f \in C^\infty(M)$. Then we define the ***differential*** of f by

$$\mathrm{d}f : \mathfrak{X}(M) \to C^\infty(M)$$
$$X \mapsto X(f).$$

Note that $\mathrm{d}f \in \Omega^1(M)$. The operator d satisfies the Leibniz rule

$$\mathrm{d}(fg) = f\mathrm{d}(g) + f\mathrm{d}(g).$$

Since the notation is very similar to that Definition 2.1.1, one must be careful no to confuse the two, although from context it is often clear which differential operator is meant.

Given a chart (U, ϕ) of M, a $C^\infty(U)$-basis of $\mathfrak{X}(U)$ is given by the differentials

$$\left\{\frac{\partial}{\partial x^i}\right\}_{i \leq n}, \tag{2.2}$$

where the x^i denote the coordinate functions of ϕ.

Similarly, a $C^\infty(U)$-basis of $\Omega^1(U)$ is given by

$$\{\mathrm{d}x_i\}_{i \leq n}. \tag{2.3}$$

2.2 Lie groups and algebras

This paragraph focuses on *Lie groups*, manifolds that are endowed with a smooth group structure. These groups will be used to define associated vector bundles in the next paragraph. We will also discuss a closely related notion, namely that of *Lie algebras*, which will be defined as the tangent space of a Lie group at the unit element. Lie algebras will have a role in the definition of the covariant derivative on an associated vector bundle.

Definition 2.2.1. A manifold G endowed with a group structure $\circ$ such that the maps

$$\circ : G \times G \to G : (g, h) \mapsto g \circ h,$$
$$\_^{-1} : G \to G : g \mapsto g^{-1}$$

are smooth is called a ***Lie group***. As usual, we often write gh instead of $g \circ h$.

Given two Lie groups G, H we say that a map

$$\phi : G \to H$$

is a ***Lie group homomorphism*** if it is a smooth group homomorphism.

We also have a notion of Lie subgroups:

Definition 2.2.2. An ***embedded Lie subgroup*** of G is defined as the image of a group homomorphism

$$\phi : H \to G$$

that is homeomorphic onto its image, where H is a Lie group and ϕ is an immersion, i.e., $\mathrm{d}_h\phi$ is injective for all $h \in H$. In this case, we say ϕ is a ***Lie group embedding***.

A very important theorem due to Cartan [6, pp. 22–24] states the following:

Theorem 2.2.3 (Cartan). *Let G be a Lie group and $H \subset G$ a subgroup of G. Then H is an embedded Lie subgroup of G if and only if H is closed in G.*

Since a Lie group is a group, it acts on itself in a smooth way. The following three maps, which are induced by this action, will be used in the future, so we will briefly define them here:

Definition 2.2.4. Let $h \in G$. We define the ***left, right, and conjugation actions*** of h on G by

$$l_h : G \to G : g \mapsto hg,$$
$$r_h : G \to G : g \mapsto gh,$$
$$c_h : G \to G : g \mapsto hgh^{-1},$$

respectively. The conjugation action is a Lie group homomorphisms.

An important family of Lie groups, which we will see frequently in our upcoming discussions, are *matrix groups*, which are defined as the closed subgroups of the so-called *general linear groups*. In fact, the symmetry groups of pseudo-Riemannian vector space which we will encounter in the next chapter, and even some of the spin groups, are of this form.

Definition 2.2.5. Let K be a field and $n \geq 1$. We define the ***general linear group*** of dimension n over K by

$$\mathrm{GL}(n, K) := \{A \in \mathrm{Mat}(n \times n, K) : \det A \neq 0\}.$$

When V is an n-dimensional K-vector space, we also write

$$\mathrm{GL}(V) := \{A \in \mathrm{End}(V) : \det A \neq 0\} \cong \mathrm{GL}(n, K).$$

Proposition 2.2.6. *For $n \geq 1$ and $K = \mathbb{R}$ or $K = \mathbb{C}$, we have that $\mathrm{GL}(n, K)$ is a Lie group of dimension n^2.*

Proof. $\mathrm{GL}(n, K)$ has a well-defined group structure, where multiplication and inversion of matrices are given by polynomial maps in the entries. Hence, these maps are smooth. Since $\mathrm{GL}(n, K) = \det^{-1}(K \setminus \{0\})$, where

$$\det : \mathrm{Mat}(n \times n) \cong K^{n^2} \to K$$

is the smooth determinant map, we see that $\mathrm{GL}(n, K)$ is an open subset of $\mathrm{Mat}(n \times n)$ and hence a smooth manifold of dimension n^2. $\square$

Now that we know some basic results about Lie groups, we are in a position to define the *Lie algebra* associated to such a Lie group.

Definition 2.2.7. Let G be a Lie group. We then define its ***Lie algebra*** by

$$\mathfrak{g} = T_e G,$$

where $e \in G$ is the unit element. The Lie algebra has a natural structure of an algebra, with multiplication given by the commutator bracket of Equation (2.1) in the unit element.

Note that a Lie algebra is not necessarily associative, since, in general, the commutator bracket need not be associative.

Given two Lie algebras $\mathfrak{g}, \mathfrak{h}$, with multiplication given by the commutator brackets $[\_,\_]_{\mathfrak{g}}, [\_,\_]_{\mathfrak{h}}$, we say that a map

$$\psi : \mathfrak{g} \to \mathfrak{h}$$

is a ***Lie algebra homomorphism*** if for all $X_e, Y_e \in \mathfrak{g}$ the relation

$$\psi([X_e, Y_e]_{\mathfrak{g}}) = [\psi(X_e), \psi(Y_e)]_{\mathfrak{h}}$$

holds.

Since the Lie algebra can be identified with the tangent space at the unit element, a Lie group homomorphism $\phi : G \to H$ defines a Lie algebra homomorphism induced by the differential $\mathrm{d}\phi$. For simplicity, we write

$$\phi_* := (\mathrm{d}\phi)_e : \mathfrak{g} \to \mathfrak{h}.$$

This map is called the ***push-forward*** of ϕ.

Push-forwards of Lie group homomorphisms allow us to define a useful operator:

Definition 2.2.8. Let G be a group and $\mathfrak{g}$ its Lie algebra. We define the ***adjoint representation*** by

$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{Aut}(\mathfrak{g}) \\ g &\mapsto (c_g)_*, \end{aligned}$$

where c_g is the conjugation action of g on G. Ad defines a representation on $\mathfrak{g}$.

We can also let a Lie group G act on a manifold M, giving us a notion of a *fundamental vector field*. This definition will turn out to be useful in our definition of a connection in Chapter 4.

Definition 2.2.9. Let G be a Lie group with associated Lie algebra $\mathfrak{g}$ and M a manifold. Let $X \in \mathfrak{g}$ and $p \in M$. Suppose G acts smoothly on M via a right action. Consider the smooth orbit map

$$\begin{aligned} \phi_p : G &\to M \\ g &\mapsto pg. \end{aligned}$$

The ***fundamental vector field*** $\widetilde{X} \in \Gamma(TM)$ of X is then defined by

$$\widetilde{X}_p = (\mathrm{d}\phi_p)_e(X). \tag{2.4}$$

The mountain of definitions in this paragraph briefly outline the structure of Lie groups and algebras. The additional structure provided by these objects will be important for some of the concepts that we will introduce later in this thesis, including the associated vector bundles of the next paragraph.

2.3 Principal and associated vector bundles

In this paragraph, we will discuss some additional bundle structures on manifolds, ultimately constructing a way of simultaneously having a vector bundle structure, alongside a Lie group action on a manifold. This will be used in Chapter 4 to define the notion of spin structures and the concept of a connection.

Definition 2.3.1. Let G be a Lie group, P and M manifolds and $\pi : P \to M$ a smooth surjection. Suppose G acts smoothly on P via a right-action. We say that P is a ***principal G-bundle*** when the following properties are satisfied:

1. For all $x \in M$ there exists a neighbourhood $U_x \ni x$ of M such that $\pi^{-1}(U_x)$ is homeomorphic to $U_x \times G$ in a G-equivariant way. In other words, there exists a homeomorphism
$$\phi : \pi^{-1}(U_x) \to U_x \times G,$$
satisfying, for all $g \in G$ and $p \in \pi^{-1}(U_x)$ that
$$\phi(pg) = \phi(p)g,$$
where the Lie group action on $U_x \times G$ is defined by
$$(y, a)g = (y, ag),$$
for all $y \in U_x$ and a, $g \in G$. Such a pair (U_x, ϕ) is called a ***bundle chart***.
2. For all $g \in G$ and $x \in M$ we have that
$$gP_x = P_x,$$
and that the map
$$\begin{aligned} G &\to P_x \\ g &\mapsto pg \end{aligned}$$
is a homeomorphism for all $p \in P_x$. In other words, G preserves the fibres of the bundle and acts freely and transitively on them.

We also have a notion of maps between principal bundles:

Definition 2.3.2. Let $\pi : P \to M$ be a principal G-bundle and $\pi' : P' \to M$ be a principal G'-bundle. Let $\phi : G \to G'$ be a Lie group homomorphism. We define a ***bundle morphism*** H to be a map
$$H : P \to P',$$
satisfying, for all $p \in P$ and $g \in G$, the relations
$$\begin{aligned} H(pg) &= H(p)\phi(g), \\ \pi(p) &= (\pi' \circ H)(p). \end{aligned}$$

When ϕ is a Lie group embedding, we say that H is a ***G-reduction*** of P', and we say that the image of H is a sub-bundle of P'.

An important example of a principal bundle is the ***frame bundle***. Given an n-dimensional manifold M, we consider the Lie group $\mathrm{GL}(n, \mathbb{R})$ and the set
$$\mathrm{Fr}(M) := \bigsqcup_{p \in M} \mathrm{Fr}(M)_p,$$

where

$$\mathrm{Fr}(M)_p = \{\text{real bases for } T_pM\}.$$

This set is provided with the canonical choice of projection map. The group $\mathrm{GL}(n, \mathbb{R})$ acts freely and transitively on the fibres in an obvious way, namely by right-multiplication of a vector with a matrix, i.e.,

$$(e_1, \ldots, e_n) \cdot A = (e_1, \ldots, e_n)A,$$

for some $A \in \mathrm{GL}(n, \mathbb{R})$. The set $\mathrm{Fr}(M)$ can then be given a bundle structure; given a chart (U, ϕ) of M, we consider the map

$$\begin{aligned} \tilde{\phi}^{-1} : U \times \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Fr}(U) \\ (p, A) &\mapsto \left(\left(\frac{\partial}{\partial x^i}\Big|_p \right)_{i \leq n} A, p \right), \end{aligned}$$

which is a G-equivariant isomorphism, providing a bundle chart for $\mathrm{Fr}(M)$. The induced topology provides $\mathrm{Fr}(M)$ with a smooth manifold structure and defines a bundle over M.

We want to combine the concept of a principal vector bundle with that of a vector bundle. For this we first need the following result:

Proposition 2.3.3. *Let $\pi : P \to M$ be a principal G-bundle and V a K-vector space. Let $\rho : G \to GL(V)$ be a representation of G. Then the map*

$$\begin{aligned} \phi : P \times V \times G &\to P \times V \\ (p, v, g) &\mapsto (p, v)g := (pg, \rho(g)^{-1}v) \end{aligned}$$

defines a free right action on $P \times V$ and is closed.

In particular, the set

$$P \times_\rho V := (P \times V)/G$$

can be given a smooth manifold structure such that the quotient map

$$P \times V \to P \times_\rho V$$

is a submersion.

Proof. A proof can be found in [11, p. 239]. □

Theorem 2.3.4. *Let $\pi : P \to M$ be a principal G-bundle and V a K-vector space. Let $\rho : G \to GL(V)$ be a representation of G. Then $P \times_\rho V$ has the structure of a vector bundle with well-defined projection map*

$$\begin{aligned} \pi' : P \times_\rho V &\to M \\ [p, v] &\mapsto \pi(p). \end{aligned}$$

The fibres of this bundle are isomorphic to V. We call this bundle the **associated vector bundle** *of P to ρ.*

Proof. The projection map is well-defined since G acts on the fibres, hence

$$\pi'(pg) = \pi'(p)$$

for all $p \in P$, $g \in G$.

Let (U_x, ϕ) be a bundle chart for P, defined by

$$\phi(p) = (\pi(p), \sigma(p)).$$

Then the map

$$\begin{aligned} \psi : \left(\pi^{-1}(U_x)\right) \times_\rho V &\to U \times V \\ [p, v] &\mapsto (\pi(p), \rho(\sigma(p))v) \end{aligned}$$

is a well-defined smooth map, since the quotient map $P \times V \to P \times_\rho V$ is a submersion, and it can be verified that it has a smooth inverse. Hence, it provides a vector bundle chart for $P \times_\rho V$, with fibres isomorphic to V. □

Now that we have properly defined the notion of an associated vector bundle, we can see how we can give the tangent bundle of Paragraph 2.1 such a structure. To achieve this, we return to the frame bundle of an n-dimensional manifold M. We consider the defining representation

$$\begin{aligned} \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{Aut}(\mathbb{R}^n) \\ A &\mapsto (x \mapsto Ax). \end{aligned}$$

Then the map

$$\begin{aligned} H : \mathrm{Fr}(M) \times_\rho \mathbb{R}^n &\to TM \\ [(e_i)_{i \le n}, p, (x_i)_{i \le n}] &\mapsto \left(\sum_{i=0}^{n} e_i x_i, p \right) \end{aligned}$$

gives a well-defined bundle isomorphism. Hence we have

$$\mathrm{Fr}(M) \times_\rho \mathbb{R}^n \cong TM, \tag{2.5}$$

so we have provided TM with the structure of an associated vector bundle. This will turn out to be of great use in Paragraph 4.2, where the associated vector bundle structure will give rise to the definition of a connection.

Chapter 3

The construction of spinors

This chapter will be dedicated to introducing the spinor formalism for pseudo-Riemannian vector spaces. Roughly speaking, the idea behind the spinor formalism is to regard transformations of real vectors as transformations of (pairs of) complex vectors, called *spinors*. In order to obtain this correspondence, we will need to go through a number of steps.

First, in Paragraph 3.1, we will introduce the notion of a *pseudo-Riemannian vector space*, which is a finite dimensional real vector space endowed with a non-degenerate symmetric bilinear form of arbitrary signature. We will then construct the symmetry groups of this space, which will be the groups consisting of linear transformations keeping the symmetric bilinear form invariant. We will further specialise these symmetry groups, in order for them to also preserve orientation and *time-orientation*.

Subsequently, in Paragraph 3.2, we will define the *Clifford algebra* associated to these spaces. These Clifford algebras can be seen as encoding the geometry of the pseudo-Riemannian spaces. In Paragraph 3.3 we will introduce the *gamma matrices*, which will allow us to characterise the Clifford algebras. In particular, it turns out that the complex Clifford algebra can be seen as a matrix algebra acting on complex spaces.

In Paragraph 3.4, we will construct distinguished subsets of the Clifford algebra, called the *spin groups*. Under some conditions, these spin groups doubly cover the symmetry groups of a pseudo-Riemannian vector space, and thus can be seen as giving 'more fundamental' transformations of the pseudo-Riemannian vector space. In Paragraph 3.5, we combine this with the classifications of the complex Clifford algebras to construct the *spinor representation*, which will show that transformations of real coordinates can be covered by the more fundamental spinor transformations of complex coordinates. These complex coordinates will be called *spinors*.

We will see, for even-dimensional spaces, that the spinor representation consists of two irreducible representations, elements of which will be called *right-* and *left-handed spinors*. We will further explore what the spinor representation looks like in 1+3-dimensional Minkowski space.

We will extend the spinor formalism in Chapter 4, where we will consider *spinor fields*, which will allow us to regard the real tensor fields on the tangent bundle as consisting of complex vector-valued spinor fields. The spinor formalism is one of the most important elements of twistor theory, as a twistor will be seen in Chapter 6 to be a certain kind of spinor field.

The approach we take in this chapter is a combination of [4, 11, 20] and [30, Appendix]. Throughout this chapter, we take K to be the field $\mathbb{R}$ or $\mathbb{C}$.

3.1 Pseudo-Riemannian vector spaces

We are used to considering spaces endowed with a positive semi-definite inner product. However, to describe the structure of space-time, we must also be able to deal with *negative* distances. A basic example of this is 4-dimensional Minkowski space $\mathbb{M}$, consisting of one temporal dimension and three spatial dimensions. This dimensionality is believed to be the same as the one of the universe we currently reside in.

We set the speed of light c to be equal to 1. In Minkowski space, the 'distance' between two points (t, x, y, z) and (t', x', y', z') is given by

$$(t - t')^2 - (x - x')^2 - (y - y')^2 - (z - z')^2,$$

so that the ***null cone***, the points at a distance of 0 from the origin, are precisely the points that light can reach, given it passes through the origin.

In order for us to describe these types of spaces, we will lay out the basics of so-called *pseudo-Riemannian spaces* and describe some of their symmetry groups.

Definition 3.1.1. Let V be a K-vector space. A map $q : V \times V \to K$ is called a ***symmetric bilinear form*** if q is symmetric and K-bilinear.

The map q is called ***non-degenerate*** if for all $v \in V$ there exists some $w \in V$ such that

$$q(v, w) \neq 0.$$

Henceforth, we will only be considering non-degenerate symmetric bilinear forms.

In the physical case, we will mostly be interested in real vector spaces $\mathbb{R}^{p,q}$ of dimension $n = p+q$, endowed with the symmetric bilinear form η defined on a standard basis $\{e_i\}_{i\geq 1}$ of $\mathbb{R}^{p,q}$ by

$$\eta(e_i, e_j) = \begin{cases} \delta_{ij}, & i \leq p, \\ -\delta_{ij}, & p < i \leq p+q, \end{cases} \tag{3.1}$$

where δ_{ij} is the standard Kronecker delta. Such a space is called a ***pseudo-Riemannian vector space***.

Note that η is non-degenerate. We say that this bilinear form η has ***signature*** (p, q), which we can also write as

$$\mathrm{sg}(\eta) = (\overbrace{+, \ldots, +}^{p \text{ times}}, \overbrace{-, \ldots, -}^{q \text{ times}}).$$

In this notation, the standard 4-dimensional Minkowski space $\mathbb{M}$ we considered a moment ago is the vector space $\mathbb{R}^{1,3}$, with corresponding symmetric bilinear form of signature $(+ - - -)$.

For the complex vector space $\mathbb{C}^n$, we can choose a basis $\{e_i\}_{i\geq 1}$ and set

$$\eta(e_i, e_j) = \delta_{ij}. \tag{3.2}$$

With these two examples we have in fact described all possibilities of finite dimensional K-vector spaces endowed with a (non-degenerate) symmetric bilinear form:

Proposition 3.1.2. *A non-degenerate symmetric bilinear form q on a finite dimensional $\mathbb{R}$- or $\mathbb{C}$-vector space can be obtained from one of the bilinear forms η of Equations* (3.1), (3.2) *through a change of basis. Consequently, any non-degenerate symmetric $\mathbb{R}$-bilinear form has a well-defined signature.*

Proof. This follows immediately from applying an adapted Gram-Schmidt algorithm. We can see that in the case of $\mathbb{C}$ we have only one non-degenerate bilinear form, since if $q(e_j, e_j) = -1$, we can set $e'_j = ie_j$, yielding $q(e'_j, e'_j) = 1$. □

A basis of $\mathbb{R}^{p,q}$ or $\mathbb{C}^n$ satisfying Equation (3.1) or (3.2), respectively, is called an ***orthonormal basis***. A basis $\{e_i\}$ only satisfying $q(e_i, e_j) = 0$ for $i \neq j$ is called ***orthogonal***. Note the similarity with an orthonormal basis in a regular inner product space, only here we do not require the 'inner product' q to satisfy $q(v, v) \geq 0$ for all $v \in V$.

Similar to the regular case of inner products, we can define the symmetry groups preserving orthogonality or orthonormality.

Definition 3.1.3. Let (V, q) be a K-vector space endowed with a symmetric bilinear form. We define the ***orthogonal group*** of (V, q) by

$$\mathrm{O}(V, q) = \{A \in \mathrm{GL}(V) : q(Av, Aw) = q(v, w) \text{ for all } v, w \in V\}.$$

The ***special orthogonal group*** of (V, q) is then defined by

$$\mathrm{SO}(V, q) = \{A \in \mathrm{O}(V, q) : \det A = 1\}.$$

For $V = \mathbb{R}^{p,q}$, we will only consider the bilinear form η of Equation (3.1), and accordingly write $\mathrm{O}(p, q) := \mathrm{O}(\mathbb{R}^{p,q}, \eta)$ and $\mathrm{SO}(p, q) := \mathrm{SO}(\mathbb{R}^{p,q}, \eta)$.

Note that with respect to the standard basis we can alternatively write

$$\mathrm{O}(p, q) = \left\{ A \in \mathrm{GL}(p+q) : A^T \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} A = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right\}.$$

Taking determinants on both sides of the equations defining $A \in \mathrm{O}(p, q)$, we immediately find that $\det A = \pm 1$.

Consequently, these groups are closed subgroups of $\mathrm{GL}(V)$, so that by Theorem 2.2.3 they can be given the structure of embedded Lie subgroups of $\mathrm{GL}(V)$.

Definition 3.1.4. Let $A \in \mathrm{O}(p, q)$. Writing $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with respect to the standard basis, where

$$A_{11} \in \mathrm{Mat}(p \times p, \mathbb{R}),$$

$$A_{12} \in \mathrm{Mat}(p \times q, \mathbb{R}),$$

$$A_{21} \in \mathrm{Mat}(q \times p, \mathbb{R}),$$

$$A_{22} \in \mathrm{Mat}(q \times q, \mathbb{R}),$$

we define the ***time-orientation*** $\tau(A)$ of A to be ± 1 whenever $\pm \det A_{11} > 0$, and the ***space-orientation*** $\sigma(A)$ of A to be ± 1 whenever $\pm \det A_{22} > 0$.

Note that by definition of $\mathrm{O}(p,q)$ we cannot have that $\det A_{11} = 0$ or $\det A_{22} = 0$, so any $A \in \mathrm{O}(p,q)$ has a well-defined time- and space-orientation.

With this notion, we can define the following two groups:

Definition 3.1.5. The ***orthochronous Lorentz group*** of signature (p,q) is defined by

$$\mathrm{O}^{+}(p,q) = \{A \in \mathrm{O}(p,q) : \tau(A) = 1\}.$$

The ***special orthochronous Lorentz group*** of signature (p,q) is defined by

$$\mathrm{SO}^{+}(p,q) = \mathrm{SO}(p,q) \cap \mathrm{O}^{+}(p,q).$$

Note that these sets are again embedded Lie subgroups of $\mathrm{GL}(V)$.

For Minkowski space, we only had one time coordinate, so requiring time-orientability is to say that we do not consider transformations which reverse the direction of time. Physically, this is quite desirable, as in nature there is evidence for time having a canonical direction, as demonstrated by e.g. the second law of thermodynamics. However, this unfortunately precludes the possibility of time-travel.

Another desirable property of $\mathrm{SO}^{+}(p,q)$ is that it is a connected component of $\mathrm{O}(p,q)$.

Proposition 3.1.6. *Let $p,q \geq 0$. The group $\mathrm{SO}^{+}(p,q)$ is the connected component of the identity in $\mathrm{O}(p,q)$.*

Proof. First we must show that there is no connected component of $\mathrm{O}(p,q)$ strictly containing $\mathrm{SO}^{+}(p,q)$. Suppose there exists some $A \in \mathrm{O}(p,q) \setminus \mathrm{SO}^{+}(p,q)$ sharing a connected subset X with $\mathrm{SO}^{+}(p,q)$. Then either $\det A = -1$ or $\det A_{11} < 0$. However, since there are no elements $B \in \mathrm{O}(p,q)$ with $\det B = 0$ or $\det B_{11} = 0$, one of the sets

$$X|_{-} := \{B \in X : \det B = -1\},$$
$$X|_{<0} := \{B \in X : \det B_{11} < 0\},$$

is non-empty and open by the continuity of the determinant and disjoint with $\mathrm{SO}^{+}(p,q)$. Hence X is not connected, giving a contradiction.

Note we have that $I \in \mathrm{SO}^{+}(p,q)$. Now we must show that $\mathrm{SO}^{+}(p,q)$ is connected. Suppose $A, B \in \mathrm{SO}^{+}(p,q)$. Consider the path

$$\gamma : [0,1] \to \mathrm{SO}^{+}(p,q)$$
$$t \mapsto At + B(t-1).$$

Let $t \in [0,1]$. By $\mathbb{R}$-linearity of q we see that, for all $v, w \in \mathbb{R}^{p,q}$

$$q\left((At + B(t-1))v, (At + B(t-1))w\right) = q(v,w).$$

Additionally, by linearity of the determinant, we find

$$\det(At + B(t-1)) = t\det A + (1-t)\det B = 1,$$
$$\det(At + B(t-1))_{11} = t\det A_{11} + (1-t)\det B_{11} > 0.$$

We can hence conclude by that γ is well defined, so γ is a path connecting A and B, so we find that $\mathrm{SO}^{+}(p,q)$ is connected. □

The symmetry groups defined here will be essential for defining the notion of spinors, as we will see in the next few paragraphs. Especially the connectedness properties of $\mathrm{SO}^{+}(p,q)$ will prove to be very important.

3.2 The Clifford algebra

In order to define the concept of spinors, we need to generalise the geometry of the pseudo-Riemannian spaces. We will see that, provided a vector space V and a symmetric bilinear form q, we can construct the *Clifford algebra*, $\mathrm{Cl}(V,q)$, which will serve this purpose. The spin groups we will define in Paragraph 3.4 will turn out to be subsets of the Clifford algebra.

Ultimately, the Clifford algebra will serve as the mediator between the real symmetry groups of pseudo-Riemannian spaces and the complex spin representations. In this way, the definition and classification of the Clifford algebra will be a big step in the construction of the spinor formalism, one of the major work horses of twistor theory.

Definition 3.2.1. Let (V,q) be a K-vector space endowed with a symmetric bilinear form. The ***Clifford algebra*** of (V,q) is an associative algebra $\mathrm{Cl}(V,q)$ with a linear map $\gamma : V \to \mathrm{Cl}(V,q)$ satisfying

1. The ***Clifford relation***; for all $v, w \in V$, we have that
$$\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2q(v,w).$$
2. The ***universal property***; if A is an associative algebra with a linear map $\delta : V \to A$ satisfying
$$\delta(v)\delta(w) + \delta(w)\delta(v) = -2q(v,w)$$
for all $v, w \in V$, then there exists a unique homomorphism
$$\phi : \mathrm{Cl}(V,q) \to A$$
such that
$$\phi \circ \gamma = \delta,$$
i.e., the following diagram commutes:
$$\begin{array}{ccc} V & \xrightarrow{\gamma} & \mathrm{Cl}(V,q) \\ & \searrow^{\delta} & \downarrow{\scriptstyle \exists!\ \phi} \\ & & A \end{array}$$

Note we have that $\gamma(v)^2 = -q(v,v)$ for all $v \in V$, so if we see $q(v,v)$ as representing the 'length' of v, we can see γ as the 'square root' of $-q$. For this reason, the Clifford algebra is seen as representing the 'square root' of the underlying geometry.

We refer to *the* Clifford algebra in Definition 3.2.1, even though it is not yet clear that the Clifford algebra is in any sense unique, or even exists. We shall justify this language now.

Theorem 3.2.2 (**Existence of Clifford algebras**)**.** *Let V be a finite dimensional K-vector space with associated symmetric bilinear form q. Let*
$$T(V) = \bigoplus_{k \geq 0} V^{\otimes k}$$
denote the **tensor algebra** *of V, and let $I(V)$ be the ideal generated by*
$$\{v \otimes v + q(v,v) : v \in V\}.$$
Then $T(V)/I(V)$, endowed with the multiplication $a \cdot b = a \otimes b$ and the canonical inclusion $\gamma : V \to T(V)/I(V)$ defines a Clifford algebra of (V,q).

Proof. Note that $T(V)/I(V)$ is clearly a K-vector space. Suppose $a, b \in T(V)/I(V)$ and $v \in V$. Then

$$\begin{aligned}(a + v \otimes v + q(v,v)1) \cdot b &= (a + v \otimes v + q(v,v)1) \otimes b \\ &= a \otimes b + v \otimes v \otimes b + q(v,v)1 \otimes b \\ &= a \otimes b - q(v,v)1 \otimes b + q(v,v)1 \otimes b \\ &= a \otimes b = a \cdot b.\end{aligned}$$

A similar manipulation can be done with b, showing that the multiplication on $T(V)/I(V)$ is well defined.

Further note that for all $x \in V$ we have that

$$\gamma^2(x) = x \otimes x = -q(x,x),$$

hence for all $v, w \in V$ we get

$$\begin{aligned}-2q(v,w) &= -q(v+w, v+w) + q(v,v) + Q(w,w) \\ &= \gamma^2(v+w) - \gamma^2(v) - \gamma^2(w) \\ &= \gamma^2(v) + \gamma(v)\gamma(w) + \gamma(w)\gamma(v) + \gamma^2(w) - \gamma^2(v) - \gamma^2(w) \\ &= \gamma(v)\gamma(w) + \gamma(w)\gamma(v),\end{aligned}$$

so the Clifford relation is satisfied. Furthermore, since V generates $T(V)$, we have that $\gamma(V)$ generates $T(V)/I(V)$ multiplicatively.

Finally, suppose A is an associative algebra and $\delta : V \to A$ is a linear map satisfying the Clifford relation. This map canonically extends to the tensor algebra by applying δ to each component and mapping $\delta(a \otimes b)$ to $\delta(a) \cdot \delta(b)$ in A. Call this extended map $\tilde{\delta}$. Note by construction, we have

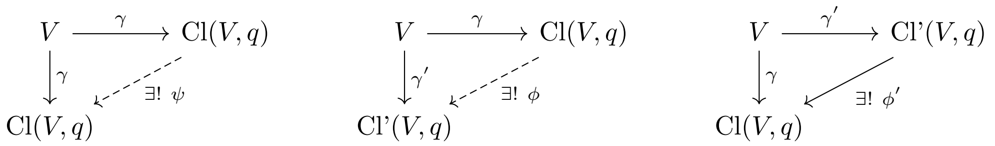
$$I(V) \subset \ker \tilde{\delta}$$

so the quotient induces a map $\phi : T(V)/I(V) \to A$ as required. Uniqueness follows from the fact that $\delta = \phi \circ \gamma$ and the fact that $T(V)/I(V)$ is generated by $\gamma(V)$, with ϕ being determined entirely by δ on $\gamma(V) \cong V$. □

Now that we have shown the Clifford algebra exists, we still need to determine in which way it is unique.

Proposition 3.2.3 (Uniqueness of Clifford algebras). *If* $(\mathrm{Cl}(V,q), \gamma)$, $(\mathrm{Cl}'(V,q), \gamma')$ *are two Clifford algebras of* (V,q), *there exists a unique algebra isomorphism between them.*

Proof. Consider the following commutative diagrams obtained from the universal property



By the first diagram, there exists a unique homomorphism ψ such that

$$\gamma = \psi \circ \gamma,$$

and since $\mathrm{id}_{\mathrm{Cl}(V,q)}$ satisfies this, we must have $\psi = \mathrm{id}_{\mathrm{Cl}(V,q)}$. From the second diagram, we find that there is a unique homomorphism ϕ such that

$$\gamma' = \phi \circ \gamma,$$

and from the third we find

$$\gamma = \phi' \circ \gamma'.$$

Combining these two, we get

$$\gamma = \phi' \circ \phi \circ \gamma,$$

so the first diagram yields $\phi' \circ \phi = \mathrm{id}_{\mathrm{Cl}(V,q)}$ and by symmetry $\phi \circ \phi' = \mathrm{id}_{\mathrm{Cl}'(V,q)}$.

Hence $\mathrm{Cl}(V,q) \cong \mathrm{Cl}'(V,q)$ via ϕ, and the uniqueness of the isomorphism follows from the uniqueness requirement of the universal property. □

We now see that *the* Clifford algebra of (V,q) is given by $T(V)/I(V)$, which we saw was generated multiplicatively by $\gamma(V)$. In what follows, we shall often leave γ implicit, and write elements of $\mathrm{Cl}(V,q)$ as

$$ab \cdots d := \gamma(a)\gamma(b)\cdots\gamma(d)$$

for some $a, b, \ldots, d \in V$.

We can subdivide the Clifford algebra even further:

Definition 3.2.4. Let

$$T^0(V) = \bigoplus_{k\geq 0} V^{\otimes 2k}, \quad T^1(V) = \bigoplus_{k\geq 0} V^{\otimes(2k+1)}$$

denote the ***even*** and ***odd part*** of the tensor algebra of V, respectively. Define

$$\mathrm{Cl}^0(V,q) := T^0(V)/(I(V) \cap T^0(V)),$$
$$\mathrm{Cl}^1(V,q) := T^1(V)/(I(V) \cap T^1(V)),$$

to be the ***even*** and ***odd part*** of the Clifford algebra, respectively.

One can easily verify that

$$\mathrm{Cl}(V,q) = \mathrm{Cl}^0(V,q) \oplus \mathrm{Cl}^1(V,q)$$

and that $\mathrm{Cl}^0(V,q)$ is a sub-algebra of $\mathrm{Cl}(V,q)$, generated by products $a_1 \cdots a_{2n}$ for $a_i \in V$ and $n \in \mathbb{Z}_{>0}$. Using the unique identification of $\mathrm{Cl}(V,q)$ with $T(V)/I(V)$, we further obtain the following results:

Theorem 3.2.5. *For any K-vector space V and associated symmetric bilinear form q,* $\mathrm{Cl}(V,q)$ *is naturally isomorphic as a vector space to the exterior algebra Λ^*V of V*

Proof. It can easily be shown that the map

$$\phi : \Lambda^*V \to \mathrm{Cl}(V,q)$$
$$v_1 \wedge \cdots \wedge v_i \mapsto v_1 \cdots v_i$$

is a K-isomorphism of vector spaces (see [20, pp. 10–11]). □

Corollary 3.2.6. *Suppose* $\dim_k V = n$. *As vector spaces, we have*

$$\dim_k \mathrm{Cl}(V,q) = 2^n,$$
$$\dim_k \mathrm{Cl}^0(V,q) = \dim_k \mathrm{Cl}^1(V,q) = 2^{n-1}.$$

Proof. The first part follows from Theorem 3.2.5. The second part follows from realising that $\mathrm{Cl}^j(V,q)$ is the $(-1)^j$ eigenspace of the algebra endomorphism $\mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q) : v \mapsto -v$. □

Returning to the standard $\mathbb{R}$- and $\mathbb{C}$-vector spaces from Paragraph 3.1, we can consider the following 'standard' Clifford algebras in which we will be most interested moving forward.

Definition 3.2.7. The ***standard Clifford algebras*** are defined as follows:

1. For $V = \mathbb{R}^{p,q}$ and $q = \eta$ as in Equation (3.1), we define $\mathrm{Cl}(p,q) := \mathrm{Cl}(V,q)$.
2. For $V = \mathbb{C}^n$ and $q = \eta$ as in Equation (3.2), we define $\mathbb{C}\mathrm{l}(n) := \mathrm{Cl}(V,q)$.

Even when working with the real pseudo-Riemannian spaces, we will want to consider the classification of the complex Clifford algebra. Thankfully, we can consider the real Clifford algebras as subalgebras of the complex Clifford algebras in the following sense:

Lemma 3.2.8. *For all $p, q \geq 0$ there exists an algebra isomorphism such that*

$$\mathrm{Cl}(p,q) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\mathrm{l}(p+q).$$

Proof. By considering $\mathbb{C}^{p+q}$ as $\mathbb{R}^{p+q} \otimes_{\mathbb{R}} \mathbb{C}$, we can consider the map

$$\begin{aligned} \delta : \mathbb{C}^{p,q} &\to \mathrm{Cl}(p,q) \otimes_{\mathbb{R}} \mathbb{C} \\ x \otimes z &\mapsto \gamma(x) \otimes z, \end{aligned}$$

where $x \in \mathbb{R}^{p+q}$, $z \in \mathbb{C}$, and γ is the embedding $\mathbb{R}^{p,q} \hookrightarrow \mathrm{Cl}(p,q)$. By considering the multiplication on $\mathrm{Cl}(p,q) \otimes_{\mathbb{R}} \mathbb{C}$ induced by multiplication on $\mathrm{Cl}(p,q)$ we get that for all $x \otimes z \in \mathbb{C}^{p+q}$

$$\begin{aligned} \delta(x \otimes z)^2 = x \cdot x \otimes z^2 &= -\eta(x,x) z^2 \\ &= -q(x \otimes z, x \otimes z), \end{aligned}$$

where η is the standard bilinear product on $\mathbb{R}^{p,q}$ and q the standard bilinear product on $\mathbb{C}^{p+q}$. Since the image of δ multiplicatively spans $\mathrm{Cl}(p,q) \otimes_{\mathbb{R}} \mathbb{C}$ and the δ satisfies the Clifford relation, we find, by virtue of $\mathbb{C}\mathrm{l}(p+q)$ and $\mathrm{Cl}(p,q) \otimes_{\mathbb{R}} \mathbb{C}$ having the same $\mathbb{C}$-dimension, that $\mathrm{Cl}(p,q) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}\mathrm{l}(p+q)$ by the uniqueness of the Clifford algebra. □

In the next paragraphs, we will classify the Clifford algebras in a more practical way, without involving the somewhat abstract tensor algebra $T(V)$. We will then use the result obtained above to regard the real Clifford algebras as subalgebras of the corresponding complex Clifford algebras in a concrete way.

3.3 Gamma matrices

In the previous paragraph, we constructed the Clifford algebra in the general case using a rather abstract construction. However, a more appropriate way is to regard the Clifford algebras as matrix algebras, by considering *representations* on vector spaces. To this end, we will introduce the *gamma matrices*. These will hint at how we can classify the Clifford algebra in terms of matrix algebras, which will turn out to be more functional than the abstract definition given in the previous paragraph. From this classification, it will also be clear why the spinor formalism is most useful when dealing with spaces of *even* dimension.

Definition 3.3.1. Let $V = \mathbb{R}^{p,q}$ be an n-dimensional $\mathbb{R}$-vector space with standard bilinear form η with basis $\{e_i\}_{i \leq n}$. Let W be an m-dimensional K-vector space. Consider a representation $\rho : \mathrm{Cl}(p, q) \to \mathrm{End}(W)$. For $i \leq n$, we define the ***gamma-matrices*** by

$$\gamma_i = \rho(e_i).$$

Note that by the Clifford relation we have that $\gamma_i\gamma_j + \gamma_j\gamma_i = -2\eta(e_i, e_j)I_m$ for all $i, j \leq n$, where I_m is the $m \times m$ identity matrix.

For even-dimensional spaces, we can define a special operator, whose eigenspaces will correspond with left-handed and right-handed spinors.

Definition 3.3.2. For $p + q = n = 2k$, where $k \in \mathbb{Z}_{>0}$, we define the ***chirality operator*** by

$$\Gamma := -i^{3k+t}\gamma_1 \cdots \gamma_n.$$

Note that we only defined chirality operators corresponding to pseudo-Riemannian vector spaces of *even* dimension, and not for odd dimensions. As it will turn out, this is because only in the even-dimensional case, the spinors, which we will define shortly, can be split up into two disjoint classes of right-handed and left-handed spinors, whereas such a distinction is not possible for odd dimensional spaces.

For this reason, some of our upcoming definitions and results will only consider the even-dimensional situation. Thankfully, this obstruction is no big hurdle for our discussion of twistor theory, since our universe has four dimensions.

We can use γ-matrices to explicitly compute Clifford algebras. We will illustrate this procedure for $\mathrm{Cl}(1, 3)$. The ***Pauli matrices*** σ_k are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3}$$

For the standard Minkowski space $\mathbb{M} = \mathbb{R}^{1,3}$ with standard bilinear form η we can consider the ***chiral representation*** $\rho : \mathrm{Cl}(1, 3) \to \mathrm{End}(\mathbb{C}^4)$ given by the gamma-matrices

$$\gamma_1 = \begin{pmatrix} 0 & iI_2 \\ iI_2 & 0 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & i\sigma_{k-1} \\ -i\sigma_{k-1} & 0 \end{pmatrix} \text{ for } k \in \{2, 3, 4\}.$$

We can verify that

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\eta(e_i, e_j)I_4$$

for all $i, j \leq 4$, and that these gamma-matrices induce a faithful representation on $\mathbb{C}^4$. As such, we can see $\mathrm{Cl}(1, 3)$ as the 16 real-dimensional vector space generated by products of the γ_i's.

Using Definition 3.3.2, we can calculate the chirality operator to be given by

$$\Gamma = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}.$$

We end this paragraph by giving two particularly important results concerning the classification of the standard Clifford algebras. Proofs can be found in [4, Chs. 6–7], following a similar approach to the one illustrated above.

Theorem 3.3.3. *For even n, let $N = 2^{n/2}$. Then the complex Clifford algebras satisfy*

$$\begin{aligned}\mathbb{C}\mathrm{l}(n) &\cong \mathrm{End}(\mathbb{C}^N),\\ \mathbb{C}\mathrm{l}^0(n) &\cong \mathrm{End}(\mathbb{C}^{N/2}) \oplus \mathrm{End}(\mathbb{C}^{N/2}).\end{aligned}$$

For odd n, let $N = 2^{(n-1)/2}$. Then the complex Clifford algebras satisfy

$$\begin{aligned}\mathbb{C}\mathrm{l}(n) &\cong \mathrm{End}(\mathbb{C}^N) \oplus \mathrm{End}(\mathbb{C}^N),\\ \mathbb{C}\mathrm{l}^0(n) &\cong \mathrm{End}(\mathbb{C}^N).\end{aligned}$$

Theorem 3.3.4. *For the real Clifford algebras* $\mathrm{Cl}(p,q)$ *and* $\mathrm{Cl}^0(p,q)$*, we set $\rho = p - q$ and $n = p + q$. Then the structure of the real Clifford algebras is given in Table 3.3.5.*

ρ mod 8	N	N'	$\mathrm{Cl}(p,q)$	$\mathrm{Cl}^0(p,q)$
0	$2^{n/2}$	$2^{(n-2)/2}$	$\mathrm{End}(\mathbb{R}^N)$	$\mathrm{End}(\mathbb{R}^{N'}) \oplus \mathrm{End}(\mathbb{R}^{N'})$
1	$2^{(n-1)/2}$	$2^{(n-1)/2}$	$\mathrm{End}(\mathbb{C}^N)$	$\mathrm{End}(\mathbb{R}^{N'})$
2	$2^{(n-2)/2}$	$2^{(n-2)/2}$	$\mathrm{End}(\mathbb{H}^N)$	$\mathrm{End}(\mathbb{C}^{N'})$
3	$2^{(n-3)/2}$	$2^{(n-3)/2}$	$\mathrm{End}(\mathbb{H}^N) \oplus \mathrm{End}(\mathbb{H}^N)$	$\mathrm{End}(\mathbb{H}^{N'})$
4	$2^{(n-2)/2}$	$2^{(n-4)/2}$	$\mathrm{End}(\mathbb{H}^N)$	$\mathrm{End}(\mathbb{H}^{N'}) \oplus \mathrm{End}(\mathbb{H}^{N'})$
5	$2^{(n-1)/2}$	$2^{(n-3)/2}$	$\mathrm{End}(\mathbb{C}^N)$	$\mathrm{End}(\mathbb{H}^{N'})$
6	$2^{n/2}$	$2^{(n-2)/2}$	$\mathrm{End}(\mathbb{R}^N)$	$\mathrm{End}(\mathbb{C}^{N'})$
7	$2^{(n-1)/2}$	$2^{(n-1)/2}$	$\mathrm{End}(\mathbb{R}^N) \oplus \mathrm{End}(\mathbb{R}^N)$	$\mathrm{End}(\mathbb{R}^{N'})$

Table 3.3.5: *Classification of the real Clifford algebras.*

Here $\mathbb{H}$ denotes the algebra of quaternions.

In practice, we will only use the classification of the complex Clifford algebras. Spinors will then be the complex vectors on which the Clifford algebra acts, in the sense of Theorem 3.3.3.

3.4 Spin groups

Now that we have defined and classified the Clifford algebras, we can consider specific subgroups of the Clifford algebras which we can study in conjunction with the symmetry groups defined in Paragraph 3.1. These groups will be called the *spin groups*. These groups will, under special circumstances, be the *double covers* of the symmetry groups discussed in Paragraph 3.1, in the sense that they are Lie groups that via a 2-to-1 covering map provide a universal cover, preserving the group structure. The spin groups can then be regarded as providing 'deeper' symmetries of the underlying pseudo-Riemannian spaces.

Definition 3.4.1. Let $(\mathbb{R}^{p,q}, \eta)$ be the standard pseudo-Riemannian vector space and bilinear form of Equation (3.1). We then define the following subsets:

$$\begin{aligned}S^{p,q}_{+} &= \{v \in \mathbb{R}^{p,q} : \eta(v,v) = 1\},\\ S^{p,q}_{-} &= \{v \in \mathbb{R}^{p,q} : \eta(v,v) = -1\},\\ S^{p,q}_{\pm} &= \{v \in \mathbb{R}^{p,q} : \eta(v,v) = \pm 1\}.\end{aligned}$$

We further define the ***pin group*** $\mathrm{Pin}(p,q)$, the ***spin group*** $\mathrm{Spin}(p,q)$ and the ***orthochronous spin group*** $\mathrm{Spin}^+(p,q)$ by

$$\begin{aligned}\mathrm{Pin}(p,q) &= \{v_1 v_2 \cdots v_s : v_i \in S_{\pm}^{p,q}, s \geq 0\},\\ \mathrm{Spin}(p,q) &= \{v_1 v_2 \cdots v_{2s} : v_i \in S_{\pm}^{p,q}, s \geq 0\},\\ \mathrm{Spin}^+(p,q) &= \{v_1 v_2 \cdots v_{2s} w_1 w_2 \cdots w_{2t} : v_i \in S_{+}^{p,q}, w_i \in S_{-}^{p,q}, s,t \geq 0\}.\end{aligned}$$

Proposition 3.4.2. *The sets* $\mathrm{Pin}(p,q)$*,* $\mathrm{Spin}(p,q)$ *and* $\mathrm{Spin}+(p,q)$ *are subgroups of* $\mathrm{Cl}^{\times}(p,q)$*.*

Proof. We will only show this for the group $\mathrm{Pin}(p,q)$, the other cases follow analogously. Note that $\mathrm{Pin}(p,q)$ is endowed with an associative multiplication from the Clifford algebra, and is clearly closed under this multiplication. It remains to be shown that $\mathrm{Pin}(p,q)$ is closed under taking inverses.

Note, given some $v \in S_{\pm}^{p,q}$, we have that

$$\eta(v,v) = \pm 1 \implies v^{-1} = \mp v,$$

so given

$$u = v_1 \cdots v_n \in \mathrm{Pin}(p,q),$$

for $v_1, \dots, v_n \in S_{\pm}^{p,q}$, we find that

$$u^{-1} = (-1)^k v_n \cdots v_1 \in \mathrm{Pin}(p,q),$$

where $k = 0$ if an even number of the v_i are in $S_{+}^{p,q}$ and $k = 1$ otherwise. □

It is not a priori clear why we are interested in these groups. However, we can embed these groups in the complex Clifford algebras, which via the classification Theorem 3.3.3 provides us with canonical complex representations of these groups. First though, we will want to see how the pin and (orthochronous) spin groups relate to the symmetry groups of pseudo-Riemannian spaces.

To make this more precise, we first need to consider the following well-known algebraic theorem:

Theorem 3.4.3 (Cartan-Dieudonné). *For all* $p, q \geq 0$*, consider the standard bilinear form* η *on* $\mathbb{R}^{p,q}$*. Suppose* $n = p + q$*. We define a* **reflection** *through the hyperplane orthogonal to* $v \in \mathbb{R}^{p,q}$ *as a map*

$$\begin{aligned}\rho_v : \mathbb{R}^{p,q} &\to \mathbb{R}^{p,q}\\ w &\mapsto w - 2\frac{\eta(v,w)}{\eta(v,v)} v,\end{aligned}$$

provided that v *is not null, i.e.,* $\eta(v,v) \neq 0$*.*

We have that $\rho_v \in \mathrm{O}(p,q)$*. Conversely, every element of* $\mathrm{O}(p,q)$ *can be written as a product of at most* $2n$ *such reflections through planes orthogonal to elements in* $S_{\pm}^{p,q}$*.*

Proof. For a proof in a more general case, see [7, pp. 10–12]. □

We can use this theorem to associate elements in the pin group with rotations of the orthogonal group.

Definition 3.4.4. For all $p, q \geq 0$ we define the map

$$\begin{aligned} R : \mathrm{Pin}(p,q) \times \mathbb{R}^{p,q} &\to \mathbb{R}^{p,q} \\ (v,x) &\mapsto \mathrm{sgn}(v) \cdot v \cdot x \cdot v^{-1}. \end{aligned}$$

where for $u = v_1 \cdots v_r$ with $v_i \in S^{p,q}_{\pm}$ we have that

$$\mathrm{sgn}(u) = \begin{cases} 1, & u \in \mathrm{Spin}(p,q), \\ -1, & \text{otherwise}, \end{cases}$$

and $\cdot$ denotes the multiplication in $\mathrm{Cl}(p,q)$.

Theorem 3.4.5. *For all $p, q \geq 0$ we have the following:*

1. *The map R of Definition 3.4.4 is well-defined.*
2. *For all $v \in S^{p,q}_{\pm}$, we have that the map $R_v := R(v, \_) : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q}$ is a reflection through the hyperplane orthogonal to v.*
3. *The map*
$$\begin{aligned} \lambda : \mathrm{Pin}(p,q) &\to \mathrm{O}(p,q) \\ v &\mapsto R_v \end{aligned}$$
is a well-defined surjective continuous group homomorphism.
4. *We have that $\lambda(v) \in \mathrm{SO}(p,q)$ if and only if $v \in \mathrm{Spin}(p,q)$.*
5. *We have that $\lambda(v) \in \mathrm{SO}^+(p,q)$ if and only if $v \in \mathrm{Spin}^+(p,q)$.*

Proof. For any $v \in S^{p,q}_{\pm}$ we have $\eta(v,v) = \pm 1$, so $v^{-1} = \mp 1$ by the Clifford relation. As such, we obtain for all $x \in \mathbb{R}^{p,q}$ that $R(v,x) = \pm v \cdot x \cdot v$.

If x is orthogonal to v, we get that $\eta(x,v) = 0$, so that $x \cdot v = -v \cdot x$ by the Clifford relation, so $R(v,x) = \pm \eta(v,v) x = x$.

For x parallel to v, we write $x = x_v v$, for $x_v \in \mathbb{R}$ so that

$$\pm v \cdot x \cdot v = \pm v \cdot x_v v \cdot v = \pm x_v v \cdot -\eta(v,v) = -x.$$

By linearity of $R(v, \_)$ we then find that $R(v, \_)$ is a reflection to the hyperplane $v^{\perp}$.

For arbitrary $w = v_1 \cdots v_j \in \mathrm{Pin}(p,q)$ we find that

$$R_w = R(w, \_) = R(v_1, \_) \circ \cdots \circ R(v_j, \_),$$

so $R(w, \_)$ is well-defined and a combination of reflections through planes. Combining this with Cartan-Dieudonné, we find that the map λ is well-defined and surjective. This proves points 1, 2 and 3.

For points 4 and 5, we can write the matrices associated to R_v. Then simply taking determinants gives the desired result. □

We can now show that the induced map λ has kernel $\{\pm 1\}$.

Lemma 3.4.6. *Let λ be the map of Theorem 3.4.5(3). Then λ, along with the restricted homomorphisms*

$$\lambda|_{\mathrm{Spin}(p,q)} : \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q),$$
$$\lambda|_{\mathrm{Spin}^+(p,q)} : \mathrm{Spin}^+(p,q) \to \mathrm{SO}^+(p,q),$$

have kernel $\mathbb{Z}/2\mathbb{Z}$.

Proof. We clearly have $\lambda(\pm 1) = I$. Now suppose $v \in \mathrm{Pin}(p,q)$ such that $\lambda(v) = I$. Then clearly $v \in \mathrm{Spin}^+(s,t)$ since $I \in \mathrm{SO}^+(s,t)$, using Theorem 3.4.5(5).

We hence find for all $u \in \mathbb{R}^{p,q}$ that $\lambda(v)(u) = v \cdot u \cdot v^{-1} = u$. Multiplying both sides with u on the right and v on the left, we obtain

$$u \cdot v \cdot u = -\eta(u,u)v.$$

Suppose $v \notin \mathbb{R}$. We then write $v = ae_{i_1} \cdots e_{i_{2k}}$ where the e_j are distinct basis elements of $\mathbb{R}^{s,t}$, $k \geq 1$ and $a \in \mathbb{R}$. Note we used that $v \in \mathrm{Spin}(p,q)$. Substituting this in the expression obtained above, substituting u for $e_{i_{2k}}$, we find

$$e_{i_{2k}} \cdot ae_{i_1} \cdots e_{i_{2k}} \cdot e_{i_{2k}} = -\eta(e_{i_{2k}}, e_{i_{2k}}) \cdot ae_{i_1} \cdots e_{i_{2k}}$$
$$\implies -\eta(e_{i_{2k}}, e_{i_{2k}})e_{i_{2k}} \cdot ae_{i_1} \cdots e_{i_{2k-1}} = -\eta(e_{i_{2k}}, e_{i_{2k}}) \cdot ae_{i_1} \cdots e_{i_{2k}}.$$

Using the cyclical identity $e_ie_j = -e_je_i$ for $i \neq j$ following from the Clifford relation on the left hand site of this expression $2k-1$ times to get the $e_{i_{2k}}$ all the way on the right, we obtain

$$\eta(e_{i_{2k}}, e_{i_{2k}})ae_{i_1} \cdots e_{i_{2k-1}} \cdot e_{i_{2k}} = -\eta(e_{i_{2k}}, e_{i_{2k}}) \cdot ae_{i_1} \cdots e_{i_{2k}},$$

so, comparing the two sides, we find $a = 0$, giving a contradiction. Hence $u \in \mathbb{R}$, and since $\lambda(a)(v) = |a|^2 v$ for $a \in \mathbb{R}$ and $v \in \mathbb{R}^{s,t}$, we find $u = \pm 1$ as required. ☐

One can give the groups $\mathrm{Spin}(p,q)$ and $\mathrm{Spin}^+(p,q)$ the structure of embedded Lie subgroups of $\mathrm{Cl}^\times(p,q)$, so that the maps of Lemma 3.4.6 are double covers of Lie groups. The most natural way of showing this is by giving an equivalent construction of these groups. This construction can be found in [1].

Finally, we complete this line of reasoning by showing that $\mathrm{Spin}^+(p,q)$ is indeed the double cover of $\mathrm{SO}^+(p,q)$ in the sense that it is in fact the topological universal cover of this group, assuming some conditions on the dimension.

Proposition 3.4.7. *For $n \geq 3$ the homomorphisms from Lemma 3.4.6 given by*

$$\begin{aligned} \lambda &: \mathrm{Spin}^+(0,n) \to \mathrm{SO}^+(0,n), \\ \lambda &: \mathrm{Spin}^+(1,n) \to \mathrm{SO}^+(1,n), \end{aligned} \tag{3.6}$$

are universal covers.

Proof. This follows from the fact that the mentioned spin groups are connected, and that the fundamental groups of the mentioned symmetry groups are $\mathbb{Z}/2\mathbb{Z}$, in combination with Lemma 3.4.6. For a detailed computation of the fundamental group, refer to [3, pp. 58–59]. ☐

In particular, the second map of Proposition 3.4.7 will be of great use to us moving forward. Summarising the results of this paragraph, we found that the group $\mathrm{Spin}^+(p,q)$, under special circumstances, doubly covers the group $\mathrm{SO}^+(p,q)$ and that the covering map is in fact a Lie group homomorphism. As such, we have a very natural way to regard transformations of the real coordinates of pseudo-Riemannian space as coming from the 'more fundamental' transformations of $\mathrm{Spin}^+(p,q)$. It is interesting to note that $\mathbb{M}$ is the space of smallest non-Riemannian dimension satisfying Proposition 3.4.7.

3.5 The spinor representation

We can now finally combine the discussions of the previous paragraphs to define spinors at a point. We will only be looking at pseudo-Riemannian vector spaces $\mathbb{R}^{p,q}$ of *even* dimension $n = p + q$, since only in this case the even complexified Clifford algebra $\mathbb{C}\mathrm{l}^0(p+q)$ is a direct sum of two matrix algebras over the complex numbers. As in Theorem 3.3.3, we take $N = 2^{n/2}$.

We will combine the classification of the complex Clifford algebra of Paragraph 3.3 and the double covering of $\mathrm{SO}^+(p,q)$ by $\mathrm{Spin}^+(p,q)$ of Paragraph 3.4 to see what we mean by complex spin transformations.

Definition 3.5.1. Let $p, q \geq 0$, and $n = p + q$ even. We consider the map

$$\kappa : \mathbb{C}\mathrm{l}(p+q) \to \mathrm{End}(\Delta_n)$$

induced from the unique isomorphism of Theorem 3.3.3, where $\Delta_n = \mathbb{C}^N$ as in Theorem 3.3.3. By considering $\mathrm{Cl}(p,q)$ as a subset of $\mathbb{C}\mathrm{l}(p+q)$ via Lemma 3.2.8 and the orthochronous spin group $\mathrm{Spin}^+(p,q)$ as a subset of $\mathrm{Cl}^\times(p,q)$, we obtain, by restricting κ, the ***spin representation***

$$\kappa^+ : \mathrm{Spin}^+(p,q) \to \mathrm{GL}(\Delta_n). \tag{3.7}$$

The set Δ_n will be referred to as the set of ***constant Dirac spinors***. Since κ is an isomorphism, we have that κ^+ is injective.

We now see that transformations of $\mathrm{SO}^+(p,q)$ correspond with particular transformations of complex numbers obtained from the spin representation. These transformations will be called ***spin transformations***.

The main advantage of the spinor representation for even dimensions is that it is reducible; by the classification of complex Clifford algebras, we have that

$$\mathbb{C}\mathrm{l}^0(p,q) \cong \mathrm{End}(\Delta_n^+) \oplus \mathrm{End}(\Delta_n^-),$$

where $\Delta_n^\pm \cong \mathbb{C}^{N/2}$ as in Theorem 3.3.3. As such, the spin representation is reducible, in the following sense:

Proposition 3.5.2. *Let Γ be the chirality operator for the spin representation (cf. Definition 3.3.2). Then the spaces $\Delta_n^\pm$ correspond with the ± 1 eigenspaces of Γ.*

Proof. Note $\Gamma^2 = 1$ by definition, so the eigenvalues of Γ are ± 1. It is then easy to show that the eigenspaces of Γ are invariant under the γ-matrices of the spin representation. □

As a result, we can regard the map κ^+ as a map

$$\kappa^+ : \mathrm{Spin}^+(p,q) \to \mathrm{GL}(\Delta_n^+) \oplus \mathrm{GL}(\Delta_n^-).$$

We will now specialise to the specific case of $\mathbb{M} = \mathbb{R}^{1,3}$. In this case, we call the elements of Δ_n^+ the ***left-handed Weyl spinors***, which we will denote by $\gamma^A \in \Delta_n^+$, and the elements of Δ_n^- the ***right-handed Weyl spinors***, denoted by $\psi^{A'} \in \Delta_n^-$. Alternatively, we will write $\Delta_n^{\pm} = \mathbb{S}^{\pm}$. Note that both $\mathbb{S}^+$ and $\mathbb{S}^-$ are two-dimensional complex vector spaces.

Any Dirac spinor $\chi^a \in \Delta_n$ can then be written as a combination of a left- and a right-handed Weyl spinor. We will write this as

$$\chi^a = \phi^A \psi^{A'}.$$

The motivation for this notation, along with the significance of the indices A, A' and a will be discussed in Paragraph 4.1, where we will not only consider constant spinors, but also the notion of *spinor fields*.

In the case of 4-dimensional Minkowski space, we have the following identification of the spin group:

Theorem 3.5.3. *Let $x^{\boldsymbol{a}} = (x^0, x^1, x^2, x^3) \in \mathbb{R}^{1,3}$. We consider the correspondence*

$$\begin{aligned} \mathbb{R}^{1,3} &\to \mathrm{Herm}(2, \mathbb{C}) \\ x^{\boldsymbol{a}} &\mapsto x^{\boldsymbol{\mu}} \sigma_{\boldsymbol{\mu}} = \frac{1}{\sqrt{2}} \begin{pmatrix} x^0 + x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 - x^3 \end{pmatrix} := x^{\mathbf{AA}'}, \end{aligned}$$

where $\sigma_{\boldsymbol{\mu}}$ are the Pauli matrices of Equation (3.3) *and $\sigma_0 = I_2$.*

We also consider the map

$$\begin{aligned} \phi : \mathrm{SL}(2, \mathbb{C}) \times \mathbb{R}^{1,3} &\to \mathbb{R}^{1,3} \\ (M, x) &\mapsto M x^{\boldsymbol{AA'}} M^\dagger. \end{aligned}$$

The induced map $M \mapsto \phi(M, \_)$ induces a transformation of $\mathrm{SO}^+(1,3)$*. The hence obtained map $\psi : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}^+(1,3)$ is surjective, with kernel $\mathbb{Z}/2\mathbb{Z}$.*

Then, since $\mathrm{SL}(2, \mathbb{C})$ *is simply connected, we have that* $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2, \mathbb{C})$.

Proof. One can show that the induced map ψ is given by

$$\psi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto$$
$$\frac{1}{2} \begin{pmatrix} a\bar{a} + b\bar{b} + c\bar{c} + d\bar{d} & a\bar{b} + b\bar{a} + c\bar{d} + d\bar{c} & i(a\bar{b} - b\bar{a} + c\bar{d} - d\bar{c}) & a\bar{a} - b\bar{b} + c\bar{c} - d\bar{d} \\ a\bar{c} + c\bar{a} + b\bar{d} + d\bar{b} & a\bar{d} + d\bar{a} + b\bar{c} + c\bar{b} & i(a\bar{d} - d\bar{a} + b\bar{c} - c\bar{b}) & a\bar{c} + c\bar{a} - b\bar{d} - d\bar{b} \\ i(c\bar{a} - a\bar{c} + d\bar{b} - b\bar{d}) & i(d\bar{a} - a\bar{d} + c\bar{b} - b\bar{c}) & a\bar{d} + d\bar{a} - b\bar{c} - c\bar{b} & i(c\bar{a} - a\bar{c} + b\bar{d} - d\bar{b}) \\ a\bar{a} + b\bar{b} - c\bar{c} - d\bar{d} & a\bar{b} + b\bar{a} - c\bar{d} - d\bar{c} & i(a\bar{b} - b\bar{a} + d\bar{c} - c\bar{d}) & a\bar{a} - b\bar{b} - c\bar{c} + d\bar{d} \end{pmatrix}.$$

We can readily verify that $\psi(M) = I_4 \iff M = \pm I_2$ and that ψ is surjective. □

We will call $x^{\boldsymbol{AA'}}$ the ***spin-vector*** or ***spinor*** of $x^{\boldsymbol{a}}$. The association of real coordinates with complex spinors gives us direct way to translate a real vector into a (sum of) tensor products of left-handed and right-handed constant spinors. Further note that the *length* of the vector $x^{\boldsymbol{a}}$, corresponds with taking the *determinant* of the spin-vector $x^{\boldsymbol{AA'}}$ as a matrix. More precisely, we have that

$$\eta(x^{\boldsymbol{a}}, x^{\boldsymbol{a}}) = 2 \det(x^{\boldsymbol{AA'}}). \tag{3.8}$$

We can define some simple operations on the space of spinors:

Definition 3.5.4. Let $\kappa, \mu, \nu \in \mathbb{S}^+$, and $\lambda \in \mathbb{C}$. Let $\{0, 1\}$ denote a complex basis for $\mathbb{S}^+$. We define the following operations on $\mathbb{S}^+$:

1. ***Scalar multiplication*** is a map $\mathbb{C} \times \mathbb{S}^+ \to \mathbb{S}^+$ given by
$$\lambda(\kappa^0, \kappa^1) = (\lambda\kappa^0, \lambda\kappa^1).$$

2. ***Addition*** is a map $\mathbb{S}^+ \times \mathbb{S}^+ \to \mathbb{S}^+$ given by
$$(\kappa^0, \kappa^1) + (\mu^0, \mu^1) = (\kappa^0 + \mu^0, \kappa^1 + \mu^1).$$

3. ***The symplectic product*** is a map $\langle \cdot, \cdot \rangle : \mathbb{S}^+ \times \mathbb{S}^+ \to \mathbb{S}^+$ satisfying
$$\begin{aligned} \langle \kappa, \mu \rangle &= -\langle \mu, \kappa \rangle, \\ \lambda\langle \kappa, \mu \rangle &= \langle \lambda\kappa, \mu \rangle, \\ \langle \kappa + \nu, \mu \rangle &= \langle \kappa, \mu \rangle + \langle \nu, \mu \rangle, \end{aligned}$$
i.e., the symplectic product is a skew $\mathbb{C}$-bilinear map. Note that this does not correspond with the regular definition of an inner product.

In the same way, we can define scalar multiplication, addition and the inner product on $\mathbb{S}^-$.

The anti-symmetry of the symplectic product is a logical consequence of Equation (3.8), since, whereas the bilinear form η is symmetric, the determinant is skew under interchange of rows or columns. This skewness will also be a result of the anti-symmetry of the epsilon spinors, which we will encounter in Theorem 3.5.6, which will be used later for raising and lowering spinor indices.

In the Minkowski spinor characterisation, we get the spinor representation
$$\kappa^+ : \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(\mathbb{S}^+) \oplus \mathrm{GL}(\mathbb{S}^-),$$
which, under the isomorphism of Theorem 3.5.3 and the identification of the even complex Clifford algebra of Theorem 3.3.3 can be verified to act on the reduced spaces $\mathbb{S}^\pm$ via
$$\begin{aligned} \kappa^+_+ : \mathrm{SL}(2, \mathbb{C}) &\to \mathrm{GL}(\mathbb{S}^+) \\ M &\mapsto (\psi^A \mapsto M\psi^A) \end{aligned}$$
and
$$\begin{aligned} \kappa^+_- : \mathrm{SL}(2, \mathbb{C}) &\to \mathrm{GL}(\mathbb{S}^+) \\ M &\mapsto (\psi^{A'} \mapsto \bar{M}\psi^{A'}). \end{aligned} \tag{3.9}$$

As such, we can regard elements of $\mathbb{S}^-$ as complex conjugates of elements of $\mathbb{S}^+$ and vice versa.

Additionally, we can consider the dual representations associated to the spin representation:

Definition 3.5.5. We define the ***dual spaces*** of $\mathbb{S}^+$ and $\mathbb{S}^-$ to be $\mathbb{S}_+$ and $\mathbb{S}_-$, respectively. We denote elements of $\mathbb{S}_+$ by ϕ_A and elements of $\mathbb{S}_-$ by $\phi_{A'}$

The same notational conventions apply for the indices of dual spinors as for regular spinors.

We can now consider the dual representations induced by the spinor representation κ^+:

$$\begin{aligned}\kappa_+^{*+} : \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{GL}(\mathbb{S}_+)\\ M &\mapsto (\psi_A \mapsto \psi_A M^{-1})\end{aligned}$$

and

$$\begin{aligned}\kappa_-^{*+} : \mathrm{SL}(2,\mathbb{C}) &\to \mathrm{GL}(\mathbb{S}_-)\\ M &\mapsto (\psi_{A'} \mapsto \psi_{A'} \bar{M}^{-1}).\end{aligned}$$

We obtain the following association between the regular and dual Weyl spinors:

Theorem 3.5.6. *We define* ε *to be the linear map corresponding to the matrix* $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *in the standard basis. The maps*

$$\begin{aligned}\varepsilon_{AB} : \psi^A &\mapsto \psi_B := \left(\psi^A\right)^T \varepsilon,\\ \varepsilon_{A'B'} : \psi^{A'} &\mapsto \psi_{B'} := \left(\psi^{A'}\right)^T \varepsilon,\end{aligned}$$

define isomorphisms of the representations κ_+^+ *and* κ_+^{*+}*, and* κ_-^+ *and* κ_-^{*+}*, respectively. These maps will be called the* **epsilon spinors***.*

Proof. $\mathrm{SL}(2,\mathbb{C})$-equivariance of the map ε_{AB} follows from the fact that for all $M \in \mathrm{SL}(2,\mathbb{C})$ the identity

$$M^T \varepsilon M = \varepsilon$$

holds. Hence for all $\psi^A \in \mathbb{S}^+$, $M \in \mathrm{SL}(2,\mathbb{C})$ we have that

$$\begin{aligned}\varepsilon_{AB}\left(\kappa_+^+(M)\left(\psi^A\right)\right) &= \varepsilon_{AB}(M\psi^A)\\ &= (\psi^A)^T M^T \varepsilon\\ &= (\psi^A)^T \varepsilon M^{-1}\\ &= \varepsilon_{AB}(\psi^A) M^{-1} = \kappa_+^{*+}(M)\left(\varepsilon_{AB}\left(\psi^A\right)\right).\end{aligned}$$

We can then define an inverse

$$\varepsilon^{BA} : \psi_B \mapsto \psi^A := (\psi_B)^T \varepsilon,$$

which can similarly be shown to be $\mathrm{SL}(2,\mathbb{C})$-equivariant. Hence ε_{AB} is an isomorphism of representations.

Similar reasoning shows that $\varepsilon_{A'B'}$ is also an isomorphism of representations. □

We will take the ε-spinors defined in the previous theorem as the operators which will lower indices. Furthermore, it can be seen that the map $\{\cdot,\cdot\} : \mathbb{S}^+ \times \mathbb{S}^+ \to \mathbb{C}$ defined by

$$(\psi^B, \phi^A) \mapsto \varepsilon_{AB}\left(\psi^B\right)\phi^A = \psi_A \phi^A$$

defines a symplectic product consistent with Definition 3.5.4 on $\mathbb{S}^+$. We will discuss this further in Paragraph 5.1.

To briefly summarise some of the results given in this paragraph, we give the following commutative diagram

$$\begin{array}{ccc} \mathbb{S}^{+} & \leftrightarrows & \mathbb{S}_{+} \\ \updownarrow & & \updownarrow \\ \mathbb{S}^{-} & \leftrightarrows & \mathbb{S}_{-} \end{array}$$

where the horizontal arrows correspond with the epsilon spinors given in Theorem 3.5.6, and the dotted vertical lines correspond with complex conjugation.

Chapter 4

Spin structures

This chapter is dedicated to three seemingly unrelated topics, which will turn out be essential in the forthcoming discussion of spinor fields, which will be the building blocks of the theory of twistors developed in later chapters.

In Paragraph 4.1, we will introduce the so-called *abstract index tensor algebra*, which is able to describe both tensor and spinor fields with great efficiency, using indices describing *global objects*. This description of fields will turn out to be invaluable for our computational purposes.

Paragraph 4.2 introduces *connections* on associated vector bundles. The notion of a connection allows us to talk about *differentiation* of vector fields in the direction of another vector field. The associated differential operator will be called a *covariant derivative*. We will define and construct a particular type of covariant derivative, called the *Levi-Civita derivative*, which we will take as defining a canonical connection on a manifold. This concept will prove to be very important, as the defining twistor equation of Chapter 6 will be a differential equation using this Levi-Civita derivative.

In the final paragraph of this section, we can finally discuss the notion of the *spinor bundle*. This bundle structure can be regarded as giving, at a point, a tangent structure of Weyl spinors, instead of real vectors. We do this by considering the local vector space structure of the tangent space as that of a pseudo-Riemannian vector space, which, using the double covering of the orthochronous symmetry group by the spin group, allows us to regard the real coordinates of the tangent space as complex coordinates. We can also use this spin structure to uniquely lift the Levi-Civita connection to define differentiation of spinor fields, which are in themselves sections of the spinor bundle.

4.1 Abstract index tensor algebra

In this paragraph, we will discuss and construct a useful formalism that will be used extensively in the discussion of tensor and spinor fields, called the *abstract index formalism*. This formalism gives a very functional way of doing computations with tensor fields without explicit reference to the underlying charts or bases. Throughout this paragraph, we take M to be some fixed manifold of dimension n. We will follow the conventions of [29].

Definition 4.1.1. Let M be a manifold. By $\mathfrak{T}$ (Fraktur T), we will denote the commutative ring $C^\infty(M)$, the ring of smooth real scalar functions on M.

Similarly, the commutative ring $C^\infty(M, \mathbb{C})$ of smooth complex scalar functions on M is denoted by $\mathfrak{S}$ (Fraktur S).

Additionally, we define the $\mathfrak{T}$-module

$$\mathfrak{T}^\bullet = \mathfrak{X}(M)$$

of vector fields. By $\mathfrak{S}^\bullet$ we will denote the $\mathfrak{S}$-module of spinor fields, which will be defined in Paragraph 4.3.

Furthermore, we define a countably infinite ***labelling set***

$$\mathscr{L} = \{\alpha, \beta, \ldots, \alpha_0, \ldots, \alpha_1, \ldots\}$$

of abstract labels.

Finally, for some $\alpha \in \mathscr{L}$, we define

$$\mathfrak{T}^\alpha = \mathfrak{T}^\bullet \times \{\alpha\},$$
$$\mathfrak{S}^\alpha = \mathfrak{S}^\bullet \times \{\alpha\}.$$

These can then be canonically identified as $\mathfrak{T}$- and $\mathfrak{S}$-modules isomorphic to $\mathfrak{T}^\bullet$ and $\mathfrak{S}^\bullet$, respectively.

In the forthcoming discussion, the behaviour of $\mathfrak{T}^\bullet$ and $\mathfrak{S}^\bullet$ will be very similar, so for simplicity, we will only discuss $\mathfrak{T}$, with results for $\mathfrak{S}$ following analogously. Although our choice of notation might seem confusing at first, it will be useful for distinguishing between vector fields and spinor fields, which will be very important in the next chapter.

Definition 4.1.2. Let $\alpha \in \mathscr{L}$. Elements of $\mathfrak{T}^\alpha$ are called ***contravariant tensors*** or ***tensors of valence*** $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

We define the ***dual*** of $\mathfrak{T}^\alpha$ by

$$\mathfrak{T}_\alpha = \mathrm{Hom}_{\mathfrak{T}}(\mathfrak{T}^\alpha, \mathfrak{T}),$$

where $\mathrm{Hom}_{\mathfrak{T}}(\mathfrak{T}^\alpha, \mathfrak{T})$ is the module of $\mathfrak{T}$-linear maps from $\mathfrak{T}^\alpha$ to $\mathfrak{T}$. Elements of $\mathfrak{T}_\alpha$ are called ***covariant tensors*** or ***tensors of valence*** $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$.

Note that $\mathfrak{T}_\alpha$ is canonically isomorphic to the dual $\mathfrak{T}_\bullet$ of $\mathfrak{T}^\bullet$. For notational simplicity, we will henceforth write $Q_\alpha A^\alpha := Q_\alpha(A^\alpha)$ for an element $Q_\alpha \in \mathfrak{T}_\alpha$ acting on an element $A^\alpha \in \mathfrak{T}^\alpha$. Note that $\mathfrak{T}_\bullet \cong \Omega^1(M)$, as defined in Paragraph 2.1.

The abstract indices of $\mathscr{L}$ mostly serve an administrative purpose. The goal of these indices is to denote different copies of the same modules. We want to be able to add together elements within $\mathfrak{T}^\alpha$ and $\mathfrak{T}^\beta$ separately, but we cannot add together elements of $\mathfrak{T}^\alpha$ with elements of $\mathfrak{T}^\beta$. In a sense, this labelling is arbitrary; if we have some equation

$$Q_\alpha(A^\alpha + B^\alpha) = W_\alpha C^\alpha,$$

we can equally well write

$$Q_\beta(A^\beta + B^\beta) = W_\beta C^\beta,$$

for some $\beta \in \mathscr{L}$, without changing the veracity of the expression.

We are now in a position to define multi-valence tensor fields.

Definition 4.1.3. Let $\{\alpha, \dots, \gamma\}$ and $\{\lambda, \dots, \nu\}$ be two disjoint subsets of $\mathscr{L}$ of cardinality p and q, respectively.

A ***tensor field*** $A^{\alpha\dots\gamma}_{\lambda\dots\nu}$ of ***valence*** $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ is an element of an abstract tensor product of $\mathfrak{T}$-modules, called $\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu}$, which is given by

$$\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu} := \underbrace{\mathfrak{T}^{\alpha} \otimes \dots \otimes \mathfrak{T}^{\gamma}}_{p \text{ occurrences}} \otimes \underbrace{\mathfrak{T}_{\lambda} \otimes \dots \otimes \mathfrak{T}_{\nu}}_{q \text{ occurrences}}.$$

As a result, the tensor field $A^{\alpha\dots\gamma}_{\lambda\dots\nu}$ can be canonically identified with an element in

$$\Gamma(TM^{\otimes p} \otimes T^*M^{\otimes q}).$$

Note that $\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu}$ is again a $\mathfrak{T}$-module. In particular, we can regard $\mathfrak{T}$ as the space of valence $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ tensors.

In general, we will denote an element of $\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu}$ using the same indices as those of the module. For example, the tensor field X^{α}_{β} will be an element of $\mathfrak{T}^{\alpha}_{\beta}$.

For some, a more familiar way to regard the tensor field $A^{\alpha\dots\gamma}_{\lambda\dots\nu}$ of Definition 4.1.3 is as a $\mathfrak{T}$-multilinear map

$$A^{\alpha\dots\gamma}_{\lambda\dots\nu} : \underbrace{\mathfrak{T}_{\alpha} \times \dots \times \mathfrak{T}_{\gamma}}_{p \text{ occurrences}} \times \underbrace{\mathfrak{T}^{\lambda} \times \dots \times \mathfrak{T}^{\nu}}_{q \text{ occurrences}} \to \mathfrak{T}, \tag{4.1}$$

with the space of all such maps denoted by $\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu}$. It turns out that these two identifications of tensor fields are equivalent in our case, a property called ***total reflexivity***. When we are dealing with finite-dimensional vector fields, this correspondence is immediate, since we can identify the dual of the vector space with the vector space itself.

However, in our case, this equivalence is not at all obvious, since it relies on the module $\mathfrak{T}^{\bullet}$ being sufficiently 'nice'. It turns out that the existence of a partition of unity on the manifold M is the 'niceness' property we are looking for. For a proof that this implies total reflexivity, refer to [29, pp. 98-102].

Since the tensor (and spinor) fields we will be dealing with in this thesis will always come from sections of bundles, we will use the interpretation of tensor fields in the sense of Definition 4.1.3 and Equation (4.1) interchangeably.

In Definition 4.1.3, we use the tensor product of the $\mathfrak{T}$-modules. However, when we are tensoring elements of modules $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$ with different labelling indices, we can, without ambiguity, write

$$A^{\alpha}B^{\beta} := A^{\alpha} \otimes B^{\beta}. \tag{4.2}$$

This allows us to multiply together tensors of arbitrary valence $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$, giving us a tensor of valence $\left[\begin{smallmatrix} p+s \\ q+t \end{smallmatrix}\right]$. However, we do need to ensure that the labelling sets of the corresponding tensors are disjoint; if they are not, we can make them disjoint by choosing a new labelling set for one of the tensors.

Using the correspondence between Definition 4.1.3 and Equation (4.1), we can now unambiguously define the following operations:

Definition 4.1.4. Suppose $\mathfrak{T}^{\alpha\dots}_{\gamma\dots}$ and $\mathfrak{T}^{\lambda\dots}_{\nu\dots}$ are tensor spaces of valence $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$, respectively. Suppose their labelling sets are disjoint.

1. ***Addition*** is a map $\mathfrak{T}^{\alpha\dots}_{\gamma\dots} \times T^{\alpha\dots}_{\gamma\dots} \to \mathfrak{T}^{\alpha\dots}_{\gamma\dots}$ mapping two tensors to their sum as a tensor product. Addition is associative.
2. ***Outer multiplication*** is a map $\mathfrak{T}^{\alpha\dots}_{\gamma\dots} \times \mathfrak{T}^{\lambda\dots}_{\nu\dots} \to \mathfrak{T}^{\alpha\dots\lambda\dots}_{\gamma\dots\nu\dots}$ mapping two tensors to their tensor product in the sense of Equation (4.2).

   Since the modules $\mathfrak{T}^{\alpha\dots}_{\gamma\dots}$ and $\mathfrak{T}^{\lambda\dots}_{\nu\dots}$ have disjoint labelling sets, we can regard this form of multiplication to be commutative, by presupposing some fixed order in which the tensor product is taken. Multiplication is associative.
3. ***(ϕ,ψ)-contraction*** is a $\mathfrak{T}$-multilinear map $\mathfrak{T}^{\alpha\dots\beta\phi}_{\gamma\dots\delta\psi} \to \mathfrak{T}^{\alpha\dots\beta}_{\gamma\dots\delta}$ defined on tensors by

$$D^\alpha \cdots G^\beta H^\phi J_\gamma \cdots K_\delta L_\psi \mapsto \left(H^\eta L_\eta\right) D^\alpha \cdots G^\beta J_\gamma \cdots K_\delta,$$

   where we consider the tensors to be multilinear maps in the sense of Equation (4.1).

Evaluation of a tensor $A^{\alpha\dots\gamma}_{\lambda\dots\nu}$ in a tensor B_γ may then be regarded as first doing outer multiplication, resulting in the tensor $A^{\alpha\dots\gamma}_{\lambda\dots\nu} B_\delta$, and subsequently performing (γ,δ)-contraction. Outer multiplication followed with contraction is often called ***transvection***.

We do need to ensure that when we are doing outer multiplication of tensors, no two labels may appear more than once as either a subscript or a superscript. For example, the expression

$$A^{\alpha\beta}_\gamma B^\gamma_{\beta\delta} + C^\alpha_\delta$$

is perfectly well defined, since we can first do outer multiplication of the two tensors on the left with disjoint labelling sets, perform two contractions, after which we perform addition. However, the expression

$$A^\alpha B_\alpha C^\alpha$$

is not well defined, since in general

$$A^\alpha (B_\alpha C^\alpha) \neq (A^\alpha B_\alpha) C^\alpha.$$

We will now briefly discuss how we can describe a tensor field if we have some (local) basis of $\mathfrak{T}^\alpha$. Suppose that (U,ϕ) is a chart for M. Let $f \in \mathfrak{T}$ be some map, satisfying $U = \{p \in M : f(p) \neq 0\}$. Then we define an equivalence relation $\sim$ on $\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu}$ by

$$A^{\alpha\dots\gamma}_{\lambda\dots\nu} \sim B^{\alpha\dots\gamma}_{\lambda\dots\nu} \iff f A^{\alpha\dots\gamma}_{\lambda\dots\nu} = f B^{\alpha\dots\gamma}_{\lambda\dots\nu}.$$

We denote

$$\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu}(f) := \mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu}(U) := \mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu} / \sim .$$

Note in particular that $\mathfrak{T}^\alpha(f) \cong \Gamma(TU)$ and $\mathfrak{T}_\alpha(f) \cong \Gamma(T^*U)$. Note these are free $\mathfrak{T}$-modules, with a basis given by Equations (2.2) and (2.3), respectively.

We can now let the *bold* index $\boldsymbol{\alpha}$ denote the basis corresponding to U, and write

$$\delta_{\boldsymbol{\alpha}} = (\delta_1, \dots, \delta_n) = \left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) \tag{4.5}$$

and

$$\delta^{\boldsymbol{\alpha}} = (\delta^1, \dots, \delta^n) = \left(\mathrm{d}x^1, \dots, \mathrm{d}x^n\right).$$

The elements associated with $\delta_{\boldsymbol{\alpha}}$ and $\delta^{\boldsymbol{\alpha}}$ in $\mathfrak{T}^{\alpha}(f)$ and $\mathfrak{T}_{\alpha}(f)$, respectively, will be denoted by $\delta^{\alpha}_{\boldsymbol{\alpha}}$ and $\delta^{\boldsymbol{\alpha}}_{\alpha}$. These are both arrays of length n, corresponding with the n-valuedness of the bold index, and are defined, respectively, as the ***basis*** and ***dual basis*** of $\mathfrak{T}^{\alpha}(f)$.

So, the difference between bold indices and regular indices is that bold indices have a relation to some concrete basis, but regular indices have no particular relation to a basis. Consequently, $V^{\boldsymbol{\alpha}}$ is an array of elements of $\mathfrak{T}$, which in combination with a basis gives a tensor, whereas the element V^{α} gives the entirety of the tensor.

In particular, on the open set U, we have that

$$V^{\alpha} = V^{\boldsymbol{\alpha}}\delta^{\alpha}_{\boldsymbol{\alpha}},$$

where the Einstein summation convention is used over the repeated bold index.

To make some of the forthcoming notation more tractable, we can consider ***compound indices*** of the labelling sets. When we have two disjoint index sets $\{\alpha, \dots, \gamma\}$ and $\{\lambda, \dots, \nu\}$, we can introduce the compound index $\mathcal{A} = \alpha \dots \gamma\lambda^* \dots \nu^*$, and denote $\mathfrak{T}^{\alpha\dots\gamma}_{\lambda\dots\nu} = \mathfrak{T}^{\mathcal{A}}$, where the stars denote which indices occur in the bottom.

So, if we have, for example, two compound indices $\mathcal{A} = \alpha\beta\lambda^*$ and $\mathcal{B} = \theta\phi^*\chi^*$, we have that

$$A^{\mathcal{AB}} = A^{\alpha\beta}{}_{\lambda}{}^{\theta}{}_{\phi\chi}, \tag{4.6}$$

where staggering of indices is now necessary to allow for consistent composition of composite indices. We will tacitly assume that when two different composite indices are used, their underlying index sets are disjoint.

We will see later, in Paragraph 4.3, that the index a of a real tensors in $\mathfrak{T}^{a}$ can be seen as a composite index of the two spinor indices A and A'.

Consequently, just as we had the equivalence between the notions of Definition 4.1.3 and Equation (4.1), we obtain the following, more general, result:

Theorem 4.1.5. *Let $\mathcal{A}, \dots \mathcal{C}, \mathcal{D}$ be disjoint composite indices. Then the set of $\mathfrak{T}$-linear maps*

$$\mathfrak{T}^{\mathcal{A}} \times \dots \times \mathfrak{T}^{\mathcal{C}} \to \mathfrak{T}^{\mathcal{D}}$$

is canonically isomorphic to $\mathfrak{T}^{\mathcal{D}}{}_{\mathcal{A}\dots\mathcal{C}}$.

A useful example of tensors we now obtain is the ***Kronecker delta***, defined by

$$\delta^{\beta}_{\alpha} : (A^{\alpha}, B_{\beta}) \mapsto A_{\beta}B^{\beta} = A_{\alpha}B^{\alpha}, \tag{4.7}$$

which is an element of $\mathfrak{T}^{\alpha}_{\beta}$, alternatively defined by $\delta^{\beta}_{\alpha}A^{\alpha} = A^{\beta}$, so this map is given by the canonical isomorphism between $\mathfrak{T}^{\alpha}$ and $\mathfrak{T}^{\beta}$.

Another important example is the epsilon spinor ε_{AB} of Paragraph 3.5, which turns out to be an element of $\mathfrak{S}_{AB}$, justifying our use of indices in its definition.

We finish this paragraph by discussing symmetry and anti-symmetry of tensors.

Definition 4.1.6. Let $A^{\mathcal{A}B\dots D\mathcal{E}} \in \mathfrak{T}^{\mathcal{A}B\dots D\mathcal{E}}$ be a tensor, where $B \dots D$ denote m distinct indices. Denote by S_m the symmetry group of order $m!$ acting on the index set $\{B, \dots, D\}$, and let $\varepsilon : S_m \to \{\pm 1\}$ be the sign map on S_m.

1. We define the ***symmetrisation*** of $A^{\mathcal{A}B\ldots D\mathcal{E}}$ in the indices $B\ldots D$ to be
$$A^{\mathcal{A}(B\ldots D)\mathcal{E}} = \frac{1}{m!}\sum_{\sigma\in S_m} A^{\mathcal{A}\sigma(B)\ldots\sigma(D)\mathcal{E}}.$$
2. We define the ***anti-symmetrisation*** of $A^{\mathcal{A}B\ldots D\mathcal{E}}$ in the indices $B\ldots D$ to be
$$A^{\mathcal{A}[B\ldots D]\mathcal{E}} = \frac{1}{m!}\sum_{\sigma\in S_m} \varepsilon(\sigma) A^{\mathcal{A}\sigma(B)\ldots\sigma(D)\mathcal{E}}.$$

Similarly we can define (anti-)symmetrisation operations on sets of dual indices.

Note, since regular and dual indices do not mix, we can first perform (anti-)symmetrisation operations on regular indices and then on dual indices; or vice versa. Note that (anti-)symmetrisation on disjoint sets of indices commutes.

We can use these operations to define a notion of *(anti-)symmetric* tensors.

Definition 4.1.7. Let $A^{A\ldots C} \in \mathfrak{T}^{A\ldots C}$ be a tensor. We say that $A^{A\ldots C}$ is

1. ***symmetric*** if $A^{A\ldots C} = A^{(A\ldots C)}$.
2. ***anti-symmetric*** or ***skew*** if $A^{A\ldots C} = A^{[A\ldots C]}$.

We similarly define (anti-)symmetric dual tensors.

We list, without proof, the following properties of (anti-)symmetrisation:

Proposition 4.1.8. *Let $A^{\cdots}_{\cdots}$ and $B^{\cdots}_{\cdots}$ be arbitrary tensors. Then the following expressions hold:*

$$\begin{aligned}
A_{\ldots ab\ldots} &= A_{\ldots(ab)\ldots} + A_{\ldots[ab]\ldots}.\\
A_{\ldots(a\ldots(b\ldots c)\ldots d)\ldots} &= A_{\ldots(a\ldots b\ldots c\ldots d)\ldots}.\\
A_{\ldots[a\ldots[b\ldots c]\ldots d]\ldots} &= A_{\ldots[a\ldots b\ldots c\ldots d]\ldots}.\\
A_{\ldots(a\ldots[b\ldots c]\ldots d)\ldots} &= A_{\ldots[a\ldots(b\ldots c)\ldots d]\ldots} = 0.\\
A_{\mathcal{A}[ab]}{}^{(ab)} &= A_{\mathcal{A}(ab)}{}^{[ab]} = 0.
\end{aligned}$$

These properties hold equally well when applied to upper indices.

The notation introduced in this paragraph may seem confusing and obtuse at first, but it turns out to be a necessary evil to cover the abstract index formalism here. The notion of a 'tensor' looks very different when employed by a mathematician or by a physicist. Physicists like to see tensors as linear maps or arrays, simply summing over repeated indices to perform computations, whereas a mathematician views a tensor as an element of a tensor product. However, these two viewpoints are in fact equivalent, using the formalism introduced here.

We would like to preserve the computational power of the physicists' concept of tensors, as we will have to do many calculations involving indices, while also keeping the mathematician's definition of tensors in mind. This will be especially important in Paragraph 4.3, where we would like to see real tensor fields as tensor products of complex spinor fields.

4.2 Connections

In this paragraph, our aim will be to define *connections* on associated vector bundles. A connection will serve to define extra structure on the bundle, so we can speak of derivatives of smooth sections of this associated vector bundle. As usual, we let M be an n-dimensional manifold and p, $q \geq 0$ be such that $n = p + q$.

Along the way, we will also define the notion of *pseudo-Riemannian metrics*, and we will introduce another way of considering the tangent bundle as an associated vector bundle, by using the Lie group $\mathrm{SO}^+(p,q)$ instead of $\mathrm{GL}(n,\mathbb{R})$. Ultimately, we will combine all these concepts to define the *Levi-Civita covariant derivative*, which will satisfy some desirable properties. This derivative will turn out to be unique. Moving forward, this derivative will be the canonical choice of derivative on the tangent bundle.

Definition 4.2.1. Let $G \to P \to M$ be a principal G-bundle, with projection map $\pi : P \to M$. For any $x \in M$ and $P_x = \pi^{-1}(x)$ with $p \in P_x$, we define the ***vertical tangent space*** at p to be

$$V_p = \ker(\mathrm{d}\pi)_p.$$

The space

$$V = \bigsqcup_{p \in P} V_p$$

is called the ***vertical tangent bundle*** of P.

A ***horizontal tangent space*** at $p \in P$ is a space $H_p \subset T_pP$ such that $T_pP = V_p \oplus H_p$. We similarly define the ***horizontal vector bundle*** H.

We can interpret the horizontal tangent spaces as the 'level sets' of the bundle.

A simple example is the trivial principal bundle $G \to M \times G \to M$. Here, the vertical tangent spaces are given by $V_{(x,g)} = T_gG$. We can then choose horizontal tangent spaces to be given by $H_{(x,g)} = T_xM$.

This choice for horizontal tangent spaces is known as the ***flat connection***.

Definition 4.2.2. Let $G \to P \to M$ be a principal G-bundle. A ***connection*** on this bundle is a map $A \in \Omega^1(P, \mathfrak{g})$ satisfying the following properties:

1. $A \circ (\mathrm{d}r_g) = \mathrm{Ad}_{g^{-1}} \circ A$ for all $g \in G$
2. $A(\tilde{X}) = X$ for all $X \in \mathfrak{g}$, where $\tilde{X}$ is defined as in Equation (2.4).

Note for every $p \in P$ we hence obtain a map $A_p : T_pP \to \mathfrak{g}$.

There is a correspondence between the notion of a connection and the notion of horizontal bundles, as given in the theorem below:

Theorem 4.2.3. *Let $\pi : P \to M$ be a principal G-bundle. Then the following hold:*

1. *Let H be a horizontal vector bundle on P. Then the map A defined by*

$$A_p(\widetilde{X_p} + Y_p) = X$$

for all $X \in \mathfrak{g}$, $Y_p \in H_p$, defines a connection on P.

2. *Let A be a connection on P. Then*

$$H_p = \ker A_p$$

defines a horizontal vector bundle on P.

Proof. Refer to [11, pp. 262–263]. □

It turns out that, having defined a connection on our bundle, this will give rise to our desired notion of differentiation. To construct a derivative, we will first have to define *parallel transport* of a curve.

Definition 4.2.4. Let $\pi : P \to M$ be a principal G-bundle, with a connection defined by the horizontal bundle H.

Consider a smooth curve $\gamma : [0,1] \to M$. A smooth curve $\gamma^* : [0,1] \to P$ is called a ***lift*** of γ if the following properties are satisfied.

1. The equality $\pi \circ \gamma^* = \gamma$ holds.
2. For all $t \in (0,1)$ we have that
$$\frac{d}{dt}\Big|_{t=0} \gamma^*(t) \in H_{\gamma^*(t)}.$$

Theorem 4.2.5. *Let $\gamma : [0,1] \to M$ be a smooth curve and $p \in \pi^{-1}(\gamma(0))$. Given a connection, there exists a unique lift γ_p^* of γ in the sense of Definition 4.2.4, such that $\gamma^*(0) = p$.*

Proof. From the local triviality of P, there exists some δ that locally satisfies Property 1 of Definition 4.2.4. We can then find a function g such that $\gamma^*(t) = \delta(t)g(t)$ by solving a differential equation. This solution turns out to be unique. For further details, refer to [11, p. 287]. □

Using these tools, we can straightforwardly define parallel transport:

Definition 4.2.6. Suppose $\pi : P \to M$ is a principal G-bundle. Let $\gamma : [0,1] \to M$ be a smooth curve in M. We define ***parallel transport*** in P with respect to the connection A as a map

$$\begin{aligned} \Pi_\gamma^A : P_{\gamma(0)} &\to P_{\gamma(1)} \\ p &\mapsto \gamma_p^*(1), \end{aligned}$$

where γ_p^* is the lifted curve of Theorem 4.2.5.

Parallel transport can be seen as taking a curve on M, lifting it to a point $p \in P$, and seeing where this new curve ends, provided that the curve 'travels' along the horizontal tangent bundle.

We will need slightly more structure on our bundle to connect the notion of parallel transport to the concept of a covariant derivative. This structure will turn out to that of an associated vector bundle, which we covered in Paragraph 2.3. From here on out, we will consider a principal bundle $G \to P \to M$ and a representation $\rho : G \to \mathrm{GL}(V)$, where V is a finite dimensional K-vector space (with $K = \mathbb{R}$ or $\mathbb{C}$). We will then consider the associated vector bundle $E = V \times_\rho P$ (cf. Thm. 2.3.4). We then obtain the following:

Theorem 4.2.7. *We can define parallel transport on the associated vector bundle E as the map*

$$\begin{aligned}\Pi_\gamma^{E,A} : E_{\gamma(0)} &\to E_{\gamma(1)}\\ [p,v] &\mapsto [\Pi_\gamma^A(p), v].\end{aligned}$$

This is a well-defined linear isomorphism.

Using this concept of parallel transport, we can finally define the *covariant derivative* on an associated vector bundle. From here on out, we assume $\Phi \in \Gamma(E)$ to be some section of the associated bundle, and $p \in M$ a point. Let $X_p \in T_pM$ be an element of the tangent space at p.

We consider a curve $\gamma : (-\varepsilon, \varepsilon) \to M$, with ε sufficiently small, satisfying

$$\begin{aligned}\gamma(0) &= p,\\ \frac{d}{dt}\Big|_{t=0} \gamma'(t) &= X_p.\end{aligned}$$

Then for all $t \in (-\varepsilon, \varepsilon)$, we can parallel transport $\Phi(\gamma(t)) \in E_{\gamma(t)}$ back to E_p along the lift $\gamma^*_{\Phi(p)}$, giving us an element of E_p given by

$$\psi_\gamma^\Phi(t) := \left(\Pi_{\gamma_t}^{E,A}\right)^{-1} (\Phi(\gamma(t))) \in E_p,$$

where γ_t is the curve obtained by restricting γ to $[t, 0]$ for $t \leq 0$ or $[0, t]$ for $t > 0$. Note that the map ψ_γ^Φ is smooth.

Definition 4.2.8. Let Φ, p, γ and ψ_γ^Φ be as above and A a connection on E. Then we define the ***covariant derivative*** by

$$D(\Phi, p, \gamma, A) = \frac{d}{dt}\Big|_{t=0} \psi_\gamma^\Phi(t).$$

Note that $D(\Phi, x, \gamma, A) \in T_{\Phi(p)}E_p$.

An interpretation of the covariant derivative is to which extent the section Φ varies from a 'constant' section along a path γ, where a 'constant' section has tangents which lie entirely in a horizontal vector space. This procedure is visualised in Figure 4.1.

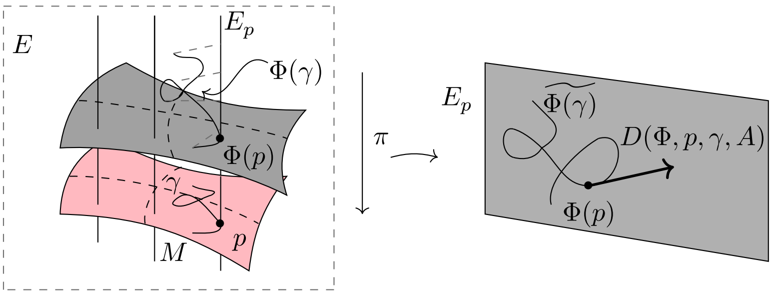


Figure 4.1: *On the left, an associated bundle $\pi : E \to M$ is shown, where π projects on the manifold M, shown in red. Copies of M lie horizontally above M, whereas the fibres E_x protrude vertically. A curve γ with $\gamma(0) = p$ is lifted to E, after which the curve is parallel transported back to E_p.*
On the left, one more dimension of E_p is shown, where $D(\Phi, \gamma, p, A)$ is now given by the tangent vector of the parallelly transported curve in $\Phi(p)$.

It is now not difficult to prove the following result:

Proposition 4.2.9. *The covariant derivate $D(\Phi, p, \gamma, A)$ does not depend on the parametrization of γ, but only on $\frac{d}{dt}\Big|_{t=0}\gamma(t) = X_p$.*

As such, we can regard the covariant derivative as a map that, given a section $\Phi \in \Gamma(E)$ and a vector field $X \in \mathfrak{X}(M)$, returns a section, which at every point $p \in M$ gives a value of the derivative of the section Φ along the direction of X. To make this more concrete, we can consider the following definition:

Definition 4.2.10. Let $\Phi \in \Gamma(E)$ be a smooth section, and $X \in \Gamma(TM)$. Let $p \in M$ and γ be a curve with $\gamma(0) = p$ and $\gamma'(0) = X_p$ as above. Let A be a connection on E. The covariant derivative can alternatively be defined as a smooth map

$$\begin{aligned} \nabla^A : \Gamma(E) \times \Gamma(TM) &\to \Gamma(E) \\ \nabla^A(\Phi, X)(p) &= D(\Phi, p, \gamma, A). \end{aligned} \tag{4.8}$$

Equivalently, we can describe ∇^A as a map

$$\begin{aligned} \nabla^A : \Gamma(E) &\to \Gamma(T^*M \otimes E) \\ \Phi &\mapsto \nabla^A(\Phi, \_). \end{aligned} \tag{4.9}$$

Often the connection A is implicit, and we simply write $\nabla^A = \nabla$.

The covariant derivative has some useful, and desirable, properties:

Proposition 4.2.11. *The map ∇ as in (4.8) is $\mathbb{R}$-linear in both entries. For all $X \in \Gamma(TM)$ and $f \in \mathfrak{T}$ we have that*

$$\nabla(\Phi, fX) = f\nabla(\Phi, X).$$

Furthermore, the map ∇ as in (4.9) satisfies the **Leibniz rule***, i.e.,*

$$\nabla(f\Phi) = f\nabla\Phi + \mathrm{d}f \otimes \Phi.$$

Proof. The proof of K-linearity and linearity in $\mathfrak{T}$ with respect to the second coordinate follow immediately from the definition of the covariant derivative. The final point is not so easy to show with our definition of a covariant derivative. An equivalent, but slightly different definition of the covariant derivative is stated in [19, p. 114], from which the Leibniz rule immediately follows. □

Conversely, it can be shown that any operator $\tilde{\nabla}$ with the properties of Proposition 4.2.11 defines a unique connection (cf. [17, p. 36]).

We will now look specifically at the tangent bundle as an associated vector bundle, as discussed in Paragraph 2.3. In this case, using the notation of Paragraph 4.1, the covariant derivative is a map

$$\nabla : \mathfrak{T}^b \to \mathfrak{T}^b_\bullet,$$

or, if we also give an index to the covariant derivative, ensuring that the indices at the bottom and at the top correspond within expressions containing the covariant derivative, we write

$$\nabla_a : \mathfrak{T}^b \to \mathfrak{T}^b_a.$$

Note however, that the covariant derivative is not a $\mathfrak{T}$-linear map, so in particular, it is not an element of $\mathfrak{T}_a$, as this notation may suggest.

We would like to extend the covariant derivative to a map that takes on values in arbitrary valence $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ tensors, subject to a Leibniz rule as in Proposition 4.2.11. For a valence $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ tensor, the construction is obvious; we simply let

$$\begin{aligned}\nabla_a : \mathfrak{T} &\to \mathfrak{T}_a\\ f &\mapsto (\mathrm{d}f)_a,\end{aligned}$$

where $(\mathrm{d}f)_a$ is the element in $\mathfrak{T}_a$ associated with $\mathrm{d}f$ in $\mathfrak{T}_\bullet = \Gamma(T^*M)$ (cf. Defs. 2.1.2, 4.1.2). The Leibniz rule of Proposition 4.2.11 can then be rewritten as

$$\nabla_a(fA^b) = A^b\nabla_a f + f\nabla_a A^b.$$

For elements of $\mathfrak{T}_b$, the Leibniz rule suggests the following form for the covariant derivative:

$$\begin{aligned}\nabla_a(A_bB^b) &= A_b\nabla_a B^b + (\nabla_a A_b)B^b\\ \implies (\nabla_a A_b)B^b &= \nabla_a(A_bB^b) - A_b\nabla_a B^b,\end{aligned}$$

which uniquely defines the covariant derivative $\nabla_a A_b$.

For general valence $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ tensors, we can apply a similar algorithm, giving the following expression:

$$\begin{aligned}(\nabla_a T^{b\ldots d}_{l\ldots n})B_b\cdots D_dL^l\cdots N^n =& \nabla_a(T^{b\ldots d}_{l\ldots n}B_b\cdots D_dL^l\cdots N^n)\\ &- T^{b\ldots d}_{l\ldots n}(\nabla_a B_b)\cdots D_dL^l\cdots N^n - \ldots\\ &- T^{b\ldots d}_{l\ldots n}B_b\cdots D_dL^l\cdots(\nabla_a N^n).\end{aligned} \tag{4.10}$$

Then, by construction, these covariant derivatives all satisfy the properties of Proposition 4.2.11, and commute with index substitutes and contractions not involving the index a of ∇_a. We can now regard the covariant derivative as a *collection* of maps $\mathfrak{T}^{\mathcal{B}} \to \mathfrak{T}^{\mathcal{B}}_a$, and which particular one we are using follows from context.

As it turns out, there will always be a canonical choice of covariant derivative on the tangent bundle, namely the *Levi-Civita derivative*, which will turn out to be *torsion-free* and *metric invariant*. For this, we first need to introduce the notion of *directional derivatives*.

Definition 4.2.12. Let $X_a \in \mathfrak{T}^a$. We define the ***directional covariant derivative*** in the direction X^a to be the map given by

$$\begin{aligned}X^a\nabla_a := \underset{X}{\nabla} : \mathfrak{T}^{\mathcal{B}} &\to \mathfrak{T}^{\mathcal{B}}\\ A^{\mathcal{B}} &\mapsto X^a\nabla_a A^{\mathcal{B}}.\end{aligned}$$

The second concept that we will need is that of a *pseudo-Riemannian metric*. This concept is not just important for constructing the Levi-Civita derivative, but will be of vital importance for much of our discussions in the upcoming chapters.

Definition 4.2.13. A ***pseudo-Riemannian metric*** on a manifold M is a valence $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$ tensor $g_{ab} \in \mathfrak{T}_{ab}$ such that at every point of M, g_{ab} induces a non-degenerate symmetric bilinear form. We say the ***signature*** (p,q) of g_{ab} is the signature (p,q) of the induced bilinear form

Note, by continuity, that any pseudo-Riemannian metric has a well-defined signature. Furthermore, since up to local basis transformations on the tangent bundle, any non-degenerate symmetric bilinear form is of the standard form η as in Equation (3.1), we can identify T_xM with $\mathbb{R}^{p,q}$ for all $x \in M$.

Using this metric, we can define a notion of an 'inner product' between tensor fields $X^a, Y^a \in \mathfrak{T}^a$, by setting

$$X_aY^a = g_{ba}X^bY^a = X^ag_{ab}Y^b = X^aY_a \in \mathfrak{T}.$$

More particularly, we see that in general

$$X_a = g_{ab}X^b,$$

so that g_{ab} can be regarded as the canonical isomorphism between $\mathfrak{T}^b$ and $\mathfrak{T}_a$. By $g^{ab} : \mathfrak{T}_a \to \mathfrak{T}^b$ we denote the inverse of g_{ab}, which we use to get that

$$X^b = g^{cb}g_{ac}X^a,$$

so that

$$\delta_a^b = g^{cb}g_{ac},$$

where δ_a^b is the Kronecker delta of Equation (4.7). In particular,

$$g^{ab}g_{ab} = g_{ab}g^{ab} = n,$$

where n is the dimension of the manifold M.

For Minkowski space $\mathbb{M}$, we have an obvious flat choice for the metric. Given the coordinate functions (t, x, y, z), we can set

$$g_{ab} = \mathrm{d}t_a\mathrm{d}t_b - \mathrm{d}x_a\mathrm{d}x_b - \mathrm{d}y_a\mathrm{d}y_b - \mathrm{d}z_a\mathrm{d}z_b \tag{4.11}$$

where $\mathrm{d}t_a$ is the element in $\mathfrak{T}_a$ associated with $\mathrm{d}t$ in $\Omega^1(\mathbb{M})$, following the notation of Equation (2.3) (cf. Eq. (A.1)).

As the final puzzle piece, we will consider another construction of the tangent bundle as an associated vector bundle, using the more prohibitive orthochronous symmetry Lie group $\mathrm{SO}^+(p, q)$, which we discussed extensively in Chapter 3.

Definition 4.2.14. Let $x \in M$, with associated bilinear form η on T_xM of signature (p, q). Let $\{e_i(x)\}_{i\leq n}$ be an orthonormal basis of T_xM with respect to η. Then the ***orthochronous frame bundle*** is the set

$$\mathrm{SO}^+(M) := \bigsqcup_{x\in M} \left\{(e_i(x))_{i\leq n}A : A \in \mathrm{SO}^+(p, q)\right\}. \tag{4.12}$$

We have a canonical projection map $\pi_{\mathrm{SO}^+} : \mathrm{SO}^+(M) \to M$. If ρ_{SO^+} is the defining representation of $\mathrm{SO}^+(p, q)$ on $\mathbb{R}^{p,q}$, then, similar as to in Equation (2.5), we have that

$$TM \cong \mathrm{SO}^+(M) \times_{\rho_{\mathrm{SO}^+}} \mathrm{SO}^+(p, q)$$

if and only if there exists a $\mathrm{SO}^+(p, q)$-reduction of the principal tangent bundle (cf. Def. 2.3.2). If this is the case, M is called ***orthochronous***. From here on out, all manifolds M we will investigate will be assumed to be orthochronous, as a 'reasonable' physical condition.

When we are talking about a covariant derivative on the manifold M with metric g_{ab}, we distinctly mean a covariant derivative on the associated vector bundle induced by the orthochronous symmetry group.

We are now in a position to define, and prove the existence of, the type of covariant derivative on this bundle that we are interested in:

Theorem 4.2.15 (**Fundamental theorem of Riemannian Geometry**). *Let M be a manifold and g_{ab} a pseudo-Riemannian metric. Then there exists a unique covariant derivative ∇_c on TM called the* **Levi-Civita derivative***, such that*

1. *∇_c is* **torsion-free***, i.e., for all tensor fields X^α, $Y^\alpha \in \mathfrak{T}^\bullet$, we have that*
$$\underset{X}{\nabla} Y^\alpha - \underset{Y}{\nabla} X^\alpha - [X, Y]^\alpha = 0, \tag{4.13}$$
where $[X, Y]^\alpha$ is the commutator of Equation (2.1)*.*
2. *∇_c is* **metric-invariant***, the metric tensor g_{ab} satisfies*
$$\nabla_c g_{ab} = 0.$$

Proof. Let X^a, Y^a, Z^a be arbitrary valence $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ tensors. Note we can rewrite Equation (4.10) as
$$\nabla_c(g_{ab}Y^a Z^b) = (\nabla_c g_{ab})Y^a Z^b + g_{ab}(\nabla_c Y^a)Z^b + g_{ab}Y^a(\nabla Z^b).$$
Since $\nabla_c g_{ab} = 0$, and transvecting with X^a, we obtain the following expression:
$$X^c \mathrm{d}(g_{ab}Y^a Z^b)_c = g_{ab}\left(\underset{X}{\nabla} Y^a\right) Z^b + g_{ab}Y^a \left(\underset{X}{\nabla} Z^b\right).$$
This computation can be repeated, permuting X, Y and Z, giving us the following expression:
$$\begin{aligned}
&X^c \mathrm{d}(g_{ab}Y^a Z^b)_c + Y^c \mathrm{d}(g_{ab}X^a Z^b)_c - Z^c \mathrm{d}(g_{ab}X^a Y^b)_c \\
&= \left(g_{ab}\left(\underset{X}{\nabla} Y^a\right) Z^b + g_{ab}Y^a\left(\underset{X}{\nabla} Z^b\right)\right) + \left(g_{ab}\left(\underset{Y}{\nabla} X^a\right) Z^b + g_{ab}X^a\left(\underset{Y}{\nabla} Z^b\right)\right) \\
&\quad - \left(g_{ab}\left(\underset{Z}{\nabla} X^a\right) Y^b + g_{ab}X^a\left(\underset{Z}{\nabla} Y^b\right)\right) \\
&= g_{ab}\left(2\underset{X}{\nabla} Y^a + [Y, Z]^a\right) Z^b + g_{ab}[X, Z]^a Y^b + g_{ab}[Y, Z]^a X^b,
\end{aligned}$$
where the final equality from the torsion-free property. Using the $\mathfrak{T}$-linearity of g_{ab}, we then find
$$\begin{aligned}
2g_{ab}\left(\underset{X}{\nabla} Y^a\right) Z^b =& X^c \mathrm{d}(g_{ab}Y^a Z^b)_c + Y^c \mathrm{d}(g_{ab}X^a Z^b)_c - Z^c \mathrm{d}(g_{ab}X^a Y^b)_c \\
&- g_{ab}[X, Z]^a Y^b - g_{ab}[Y, Z]^a X^b - g_{ab}[Y, X]^a Z^b,
\end{aligned} \tag{4.14}$$
so since g_{ab} is non-degenerate and X^a and Z^a are arbitrary, this uniquely defines $\nabla_c Y^a$ if it exists. By further calculation it can be shown that the covariant derivative induced by (4.14) is metric-invariant and torsion-free, completing the proof. □

The Levi-Civita derivative we have defined in this paragraph will be our go-to choice for a differential operator moving forward. This operator will allow us to define the *curvature tensor* in Paragraph 5.4, which will tell us something about the flatness of the spaces we are dealing with. The twistor equation, which will define twistor space for conformally flat spaces, will be a differential equation involving the Levi-Civita derivative. In short, this derivative will be of much utility for us in the future.

4.3 The spinor bundle

We will now combine the theory of Chapters 2 and 3 and the previous two paragraphs to define a *spin structure* on an orthochronous manifold M with pseudo-Riemannian metric g_{ab} of signature (p, q). The spin structure will allow us to extend the spinor formalism of Chapter 3 to tensor fields, allowing us to regard real tensor fields on M as consisting of complex spinor fields, which will be defined as sections of the spinor bundle.

We will also encounter the highly-consequential *Geroch's theorem*, which will give us a necessary and sufficient condition for a non-compact signature $(1, 3)$ manifold to be spin.

Definition 4.3.1. Let M be an orthochronous manifold with a pseudo-Riemannian metric g_{ab} of signature (p, q). A ***spin structure*** on M is defined as a principal $\mathrm{Spin}^+(p, q)$-bundle $\pi_{\mathrm{Sp}^+} : \mathrm{Spin}^+(M) \to M$, alongside a double cover

$$\Lambda^+ : \mathrm{Spin}^+(M) \to \mathrm{SO}^+(M),$$

such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Spin}^+(p,q) \times \mathrm{Spin}^+(M) & \longrightarrow & \mathrm{Spin}^+(M) \\
\Big\downarrow {\scriptstyle \lambda^+ \times \Lambda^+} & & \Big\downarrow {\scriptstyle \Lambda^+} \\
\mathrm{SO}^+(p,q) \times \mathrm{SO}^+(M) & \longrightarrow & \mathrm{SO}^+(M) \\
 & & \Big\downarrow {\scriptstyle \pi_{\mathrm{SO}^+}} \\
 & & M
\end{array}
\qquad \left(\mathrm{Spin}^+(M) \xrightarrow{\pi_{\mathrm{Sp}^+}} M\right)$$

where λ^+ is the map of Equation (3.6) (also see Eq. (4.12)), and the horizontal arrows denote the projection maps.

We define $\mathrm{Spin}^+(M)$ to be the ***spinor manifold*** of M. We call the manifold M ***spin*** if there exists a spin structure on M.

It turns out that a spin structure exists on a manifold if and only if its *second Stiefel-Whitney class* vanishes, denoted by $w_2(M) = 0$. For details about the Stiefel-Whitney class, refer to [23, p. 50]. A proof of this statement can be found in [3, pp. 70–81]. We will not use this result in detail, instead simply assuming that the manifolds we are dealing with are spin.

A very particular case of spin manifolds, which will be of great interest to us moving forward, comes from the following result by Robert Geroch [10]:

Theorem 4.3.2 (Geroch). *Let M be an orthochronous non-compact 4-dimensional manifold with metric g_{ab} of signature $(1, 3)$.*

Then M is spin if and only if there exist four global vector fields (T^a, X^a, Y^a, Z^a) of the orthochronous tangent bundle of M, such that at every point in M, (T^a, X^a, Y^a, Z^a) forms an orthonormal basis for the induced symmetric bilinear form of g_{ab}.

Such a set of vector fields is called a **Minkowski tetrad**.

This is quite a remarkable result, since if we assume that M is non-compact and spin, this means that the vector bundle admits a trivial structure. Instead of having to deal with local bases of the module $\mathfrak{T}^\bullet$, we can now consider a global basis, which will make many computations much more straightforward. Another interesting detail is that this theorem only holds for this particular choice of dimension and signature, since its proof relies heavily on the fact that $\mathrm{Spin}^+(1,3) \cong \mathrm{SL}(2,\mathbb{C})$, which we showed in Theorem 3.5.3.

If we take the existence of a Minkowski tetrad to be a reasonable physical assumption, this is equivalent to requiring a spin structure as a reasonable physical assumption. When dealing with manifolds of this metric, we will henceforth assume that the conditions of Geroch's theorem hold.

Given a spin manifold $\mathrm{Spin}^+(M)$ on a manifold M, we can define a *spinor bundle* on M. Just as the spin manifold is the equivalent of the frame bundle in spinor terms, the spinor bundle will be the spinor analogue of the tangent bundle.

Definition 4.3.3. Let $\mathrm{Spin}^+(p,q) \to \mathrm{Spin}^+(M) \to M$ be a spin structure on M. Let

$$\kappa^+ : \mathrm{Spin}^+(p,q) \to \mathrm{GL}(\Delta_n)$$

denote the spinor representation (cf. Eq. (3.7)).

The associated vector bundle

$$S(M) := \mathrm{Spin}^+(M) \times_{\kappa^+} \Delta_n$$

is called the ***spinor bundle***.

Similar to in Proposition 3.5.2, when $p+q=n$ is even, the spinor bundle is a direct sum of a ***left-handed*** and ***right-handed spinor bundle***, each defined by

$$S_\pm(M) = \mathrm{Spin}^+(M) \times_{\kappa^+} \Delta_n^\pm.$$

We have now finally reached one of our first major goals; defining the spinor bundle. Sections of the spinor bundle will be the spinor fields, which we have been talking about so much. These spinor fields will have the structure of $\mathfrak{S}$-modules, recalling that we defined that

$$\mathfrak{S} = C^\infty(M,\mathbb{C}).$$

Just as we discussed for tensor fields in Paragraph 4.1, the modules of spinor fields are also totally reflexive. We will employ similar labelling conventions as in Paragraph 4.1 for spinor fields.

We denote sections of the spinor bundle with *lowercase Latin* indices, i.e.,

$$\Gamma(S(M)) \cong \mathfrak{S}^a.$$

These indices are also called ***world-tensor indices***. Elements of $\mathfrak{S}^a$ are called ***spinor fields***, or just ***spinors***. We can define multi-valence spinors in the obvious way, following Definition 4.1.3.

When $p+q=n$ is even, we denote sections of the left- and right-handed spinor bundle with *unprimed uppercase Latin* and *primed uppercase Latin* indices, respectively, i.e.,

$$\begin{aligned} \Gamma(S_+(M)) &\cong \mathfrak{S}^A. \\ \Gamma(S_-(M)) &\cong \mathfrak{S}^{A'}. \end{aligned} \tag{4.15}$$

These indices are also called ***spinor indices***. Elements of these sets are called ***left-*** and ***right-handed spinor fields***, respectively, or just ***spinors***.

Note we have that $\mathfrak{S}^a \cong \mathfrak{S}^A \otimes \mathfrak{S}^{A'}$, so we can regard a as the composite index $a = AA'$.

As such, we can regard a left-handed spinor $\psi^A \in \mathfrak{S}^A$ as a smooth function, giving at every point of the manifold a left-handed Weyl-spinor, whereas a right-handed spinor $\phi^{A'} \in \mathfrak{S}^{A'}$ returns a right-handed Weyl-spinor at every point. Using the identification of real vectors with complex spinors, any real tensor field X^a can be regarded as an element of $\mathfrak{S}^a = \mathfrak{S}^A \otimes \mathfrak{S}^{A'}$.

The general space of ***valence*** $\left[\begin{smallmatrix} r & s \\ u & t \end{smallmatrix}\right]$ ***spinor fields*** is the space

$$\mathfrak{S}^{A\ldots CA'\ldots D'}_{L\ldots NL'\ldots M'} := \mathfrak{S}^{A\ldots C}_{L\ldots N} \otimes \mathfrak{S}^{A'\ldots D'}_{L'\ldots M'},$$

where we have r upper unprimed indices, u lower unprimed indices, s upper primed indices and t lower primed indices. Ultimately, this construction allows us to identify any tensor field of arbitrary valence $\left[\begin{smallmatrix} s \\ t \end{smallmatrix}\right]$ as consisting of spinor fields of valence $\left[\begin{smallmatrix} s & s \\ t & t \end{smallmatrix}\right]$.

We conclude this paragraph by defining a specific covariant derivative on the spinor bundles induced by the Levi-Civita covariant derivative, called the *spin covariant derivative.*

Proposition 4.3.4. *Suppose M is spin, with spin structure given by $\Lambda^+ : \mathrm{Spin}^+(M) \to \mathrm{SO}^+(M)$ Let A_{LC} denote the connection corresponding to the Levi-Civita derivative on the orthochronous tangent bundle. Then the map*

$$A_{\mathrm{Spin}} := (\mathrm{d}\lambda^+)^{-1} \circ (A_{\mathrm{LC}} \circ \mathrm{d}\Lambda^+) \tag{4.16}$$

defines a connection on $\mathrm{Spin}^+(M)$.

Proof. We need to check the conditions of Definition 4.2.2. In order to check the first property, let $g \in \mathrm{Spin}^+(p,q)$ and X an element of the tangent space of $\mathrm{Spin}^+(M)$. We then have that

$$\begin{aligned} A_{\mathrm{Spin}}(r_{g*}X) &= (\mathrm{d}\lambda^+)^{-1} \circ (A_{\mathrm{LC}}(\mathrm{d}\Lambda^+ \mathrm{d}r_g X)) \\ &= (\mathrm{d}\lambda^+)^{-1} \circ (A_{\mathrm{LC}}(\mathrm{d}r_{\lambda^+(g)} \mathrm{d}\Lambda^+ X)) \\ &= (\mathrm{d}\lambda^+)^{-1} \circ \mathrm{Ad}_{\lambda^+(g)^{-1}} \circ \mathrm{d}\lambda^+ \circ A_{\mathrm{Spin}}(X) \\ &= \mathrm{Ad}_{g^{-1}} \circ A_{\mathrm{Spin}}(X), \end{aligned}$$

where we used Equation (4.16), Definitions 2.2.8, 4.2.2(1) and the properties of the spin structure (cf. Def. 4.3.1).

The second property follows immediately from Definition 4.2.2(2). ☐

Definition 4.3.5. The covariant derivative associated with the connection A_{Spin} on the spinor bundle is called the ***spin covariant derivative***. Provided that our underlying manifold M is even-dimensional, we denote the spin covariant derivative by $\nabla_{AA'}$. Adjusting the discussion of Paragraph 4.2, this map can be uniquely extended to a collection of maps

$$\nabla_{AA'} : \mathfrak{S}^{\mathcal{B}} \to \mathfrak{S}^{\mathcal{B}}_{AA'},$$

where $\mathcal{B}$ is a composite index consisting of *uppercase Latin spinor indices* (cf. Eq. (4.15)) not involving A or A'.

All these maps then satisfy, with needed adjustments, the $\mathfrak{S}$-linearity and Leibniz rule of Proposition 4.2.11.

In this paragraph, we have obtained one of this thesis' most important results: the identification of tensor fields with spinor fields, provided that our underlying manifold is spin. Not only have we given a way in which we can regard a tensor field as being 'covered' by a complex spinor field, we also saw how, for even-dimensional manifolds, these spinor fields split up in left- and right-handed spinor fields, just as how Dirac spinors split in left- and right-handed Weyl spinors. We also lifted the Levi-Civita covariant derivative of Theorem 4.2.15 to the spinor bundle, so we got a notion of derivatives of spinors.

In the next chapter, we will look at special properties of these spinors for 4-dimensional spaces of signature $(1, 3)$, assuming that the conditions of Geroch's theorem hold. It will turn out that spinors have far more structure than tensors, which make them very attractive to work with. Ultimately, in Chapter 6, we can then define *twistors* as the right-handed spinor fields satisfying the *twistor equation*.

Chapter 5

Two-spinor calculus

In the previous chapter, we defined a deeper structure on the tangent space of M; the spin structure. From our construction, it was clear that any real tensor in $\mathfrak{T}^{a\ldots c}_{l\ldots n}$ corresponds to some spinor in $\mathfrak{S}^{a\ldots c}_{l\ldots n}$. In particular, when the dimension of the underlying manifold M is even, such a tensor should correspond to some spinor in $\mathfrak{S}^{AA'\ldots CC'}_{LL'\ldots NN'}$.

In the subsequent discussion, we will only consider non-compact 4-dimensional manifolds M with a pseudo-Riemannian metric g_{ab} of signature $(+---)$, which we will further assume to be spin. In other words, we assume that the conditions of Geroch's theorem hold. Although some of our results will also hold for general spin manifolds of even dimension, most will be particular to these specific conditions.

When we are talking about such manifolds, any spinor splits into a left-handed and right-handed spinor part, whose values will be 2-complex dimensional. As such, these spinors are collectively called ***two-spinors***.

In Paragraph 5.1, we will discuss some important basic properties of two-spinors, in particular touching upon *spinor dyads*, which will be bases for the module of spinor fields, and the *epsilon spinors*, which serve the role of the metric g_{ab} in raising and lowering spinor indices. A remarkable property of these epsilon spinors is that they are unique in this signature. We will use these properties in Paragraph 5.2 to give an algorithm for translating any 1-valent and 2-valent tensor field into its spinor components.

In Paragraph 5.3, we will treat a geometrical interpretation of constant 1-valent spinor fields on Minkowski space, which we will use extensively when discussing the twistor Klein correspondence in Chapter 6. Finally, in Paragraph 5.4, we will use our definition of the Levi-Civita derivative and the theory developed in the rest of this chapter to derive the Riemann curvature tensor, which will allow us to state the Einstein equations and describe them using spinor fields.

5.1 Basic properties

As we saw in Equation (3.9), elements of $\mathbb{S}^+$ could be regarded as complex conjugates of elements of $\mathbb{S}^-$, and vice versa. We can take this reasoning further to apply general to spinor fields, i.e., we have that $\overline{\mathfrak{S}^A} = \mathfrak{S}^{A'}$ and $\overline{\mathfrak{S}^{A'}} = \mathfrak{S}^A$.

As such, for any arbitrary $\kappa^A \in \mathfrak{S}^A$, we have that

$$\overline{\kappa^A} := \overline{\kappa}^{A'} \in \mathfrak{S}^{A'}$$

and for $\omega^{A'} \in \mathfrak{S}^{A'}$, we have that

$$\overline{\omega^{A'}} := \overline{\omega}^A \in \mathfrak{S}^A.$$

Complex conjugation must also match the other properties of complex conjugation on $\mathbb{C}^4$, so for any λ, $\mu \in \mathfrak{S}$, κ^A, $\chi^A \in \mathfrak{S}^A$ we have

$$\overline{\lambda\kappa^A + \mu\chi^A} = \overline{\lambda\kappa}^{A'} + \overline{\mu\chi}^{A'} \tag{5.1}$$

and of course, these rules need to hold for arbitrary valence $\left[\begin{smallmatrix} p & s \\ q & t \end{smallmatrix}\right]$ spinors, i.e.,

$$\overline{\xi^{A\ldots BC'\ldots D'}_{E\ldots FG'\ldots H'}} = \overline{\xi}^{A'\ldots B'C\ldots D}_{E'\ldots F'G\ldots H}.$$

Similar to how we had a metric g_{ab} on M to lower tensor indices, we have a similar concept called *epsilon spinors*, for lowering spinor indices, as we already saw in Theorem 3.5.6.

Definition 5.1.1. The ***epsilon spinor*** is a skew-symmetric tensor $\varepsilon_{AB} \in \mathfrak{S}_{AB}$.

We define the ***dual*** of κ^A to be

$$\kappa_B = \kappa^A \varepsilon_{AB},$$

so ε_{AB} is the canonical isomorphism between $\mathfrak{S}^A$ and $\mathfrak{S}_B$. We call its inverse ε^{AB}, such that

$$\kappa^A = \varepsilon^{AB}\kappa_B.$$

Furthermore, we define the normalisation of ε_{AB} to be such that

$$\varepsilon_{AB}\varepsilon^{AB} = 2,$$

which uniquely defines ε_{AB}. Similarly, we can define $\varepsilon_{A'B'}$ and $\varepsilon^{A'B'}$.

This definition matches with the one encountered in Paragraph 3.5; to make this more precise, we first need to introduce the concept of a *spinor dyad*:

Definition 5.1.2. A set $\{o^A, \iota^A\} \subset \mathfrak{S}^A$ is called a ***spinor dyad*** if any spinor κ^A can be written as

$$\kappa^A = \kappa^0 o^A + \kappa^1 \iota^A, \tag{5.2}$$

with $\kappa^0, \kappa^1 \in \mathfrak{S}$ and

$$o_A \iota^A = 1. \tag{5.3}$$

A dual dyad is defined similarly.

In other words, a spinor dyad is a basis for the module of spinor fields, satisfying a sort of 'orthogonality condition', provided by Equation (5.3). It turns out that in the specific case of a non-compact 4-dimensional space-time M of signature $(1, 3)$, the existence of a global spinor dyad is equivalent with M being spin, similar to the result of Theorem 4.3.2 [10]. Since we assumed the conditions of Geroch's theorem to hold, we will henceforth assume that we can always construct such a global spinor dyad.

By the skewness of ε_{AB}, it immediately follows that

$$o_A o^A = \iota_A \iota^A = 0.$$

As such, we find that

$$\kappa^0 = \iota^A \kappa_A, \quad \kappa^0 = o_A \kappa^A.$$

Note that the description in a spinor dyad yields that

$$\omega_A \kappa^A = \omega^0 \kappa^1 - \omega^1 \kappa^0.$$

We can set the 'Kronecker delta' symbols with respect to this spinor dyad to be (cf. Eq. (4.5))

$$\varepsilon_0{}^A = o^A, \quad \varepsilon_1{}^A = \iota^A, \quad \varepsilon_{\boldsymbol{A}}{}^A = (\varepsilon_0{}^A, \varepsilon_1{}^A),$$

so with respect to the spinor basis $\{o, \iota\}$ we get that

$$\varepsilon_{\boldsymbol{AB}} = \varepsilon_{AB} \varepsilon_{\boldsymbol{A}}{}^A \varepsilon_{\boldsymbol{B}}{}^B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{5.4}$$

which, unsurprisingly, is exactly the same matrix ε describing the isomorphism between $\mathbb{S}^+$ and $\mathbb{S}_+$ in Theorem 3.5.6. This justifies our use of the term 'dual' that we used in the definition of the ε-spinor.

Similarly, we can write the 'Kronecker delta' for the isomorphism between $\mathfrak{S}^A$ and $\mathfrak{S}^B$ as

$$\delta_A^{\ B} := \varepsilon_A^{\ B} = \varepsilon_{AC} \varepsilon^{CB} = -\varepsilon_{CA} \varepsilon^{BC} = -\varepsilon^B{}_A.$$

Note that we need to be careful regarding the position of indices (cf. Eq. (4.6)) due to the skewness of ε_{AB}; in general we have that

$$\chi^{\mathcal{A}B}{}_{C\mathcal{E}} = -\chi^{\mathcal{A}}{}_C{}^B{}_{\mathcal{E}},$$

a property sometimes referred to as the *spinor see-saw.* (B and C are regular spinor indices, $\mathcal{A}$ and $\mathcal{E}$ are composite indices).

As the spinor system $\mathfrak{S}^A$ is two dimensional, the epsilon spinor ε_{AB} is, up to scaling, unique, by virtue of its being skew. So, since we require $\varepsilon_{AB}\varepsilon^{AB} = 2$, we see that ε_{AB} is actually unique; a result that only holds in a 4-dimensional space-time of signature $(1, 3)$.

Of particular interest are the following two simple results, which follow directly from the properties described above:

Proposition 5.1.3. *For any $\phi_{\mathcal{F}AB} \in \mathfrak{S}_{AB}$ we have that*

$$\phi_{\mathcal{F}AB} - \phi_{\mathcal{F}BA} = \phi_{\mathcal{F}C}{}^C \varepsilon_{AB}.$$

Proof. Note for all $\tau^A, \omega^A, \kappa^A \in \mathfrak{S}^A$ we have that

$$\begin{aligned}\tau_A\omega^A\kappa^B + \omega_A\kappa^A\tau^B + \kappa_A\tau^A\omega^B &= 3\ \varepsilon_{CA}\tau^{(A}\kappa^B\omega^{C)} \\ &= 3\ \varepsilon_{[CA]}\tau^{(A}\kappa^B\omega^{C)} = 0,\end{aligned}$$

by skewness of ε_{CA} and Proposition 4.1.8(5).

We can rewrite the left-hand side of this equation as

$$(\varepsilon_{AB}\varepsilon_C{}^D + \varepsilon_{BC}\varepsilon_A{}^D + \varepsilon_{CA}\varepsilon_B{}^D)\tau^A\omega^B\kappa^C = 0,$$

so since this holds for *all* $\tau^A, \omega^A, \kappa^A \in \mathfrak{S}^A$, we get

$$\varepsilon_{AB}\varepsilon_C{}^D + \varepsilon_{BC}\varepsilon_A{}^D + \varepsilon_{CA}\varepsilon_B{}^D = 0. \tag{5.5}$$

Now transvecting with ε^{EC} and using the spinor see-saw, we obtain

$$\varepsilon_B{}^E\varepsilon_A{}^D - \varepsilon_A{}^E\varepsilon_B{}^D = \varepsilon_{AB}\varepsilon^{ED}, \tag{5.6}$$

which, when transvected with $\phi_{\mathcal{F}ED}$, yields the wanted expression. ☐

Proposition 5.1.4. *Let ϕ^A, $\chi^A \in \mathfrak{S}^A$. Then the following are equivalent:*

1. $\phi_A\chi^A = 0$.
2. $\phi^A = \lambda\chi^A$ *for some* $\lambda \in \mathfrak{S}$.

Proof. This follows immediately by writing ϕ^A and χ^A in terms of a spinor dyad. ☐

These two results will be used a lot when analysing tensorial equations in spinorial terms. They are again particular to our choice of dimension and signature, and they give a lot more structure to spinors than we would have with ordinary tensor fields. Ultimately, this is one of the great powers of the spinor formalism, many expressions become much nicer to deal with.

5.2 Association between spinors and tensors

In this paragraph, our aim will be to explicitly describe the relationship between *real* tensors, i.e. elements of $\mathfrak{T}^{a\ldots c}_{d\ldots f}$, and *complex* spinors; i.e. elements of $\mathfrak{S}^{A\ldots BC'\ldots E'}_{F\ldots GH'\ldots I'}$.

In the previous paragraph, we saw that complex conjugation carries a spinor of $\mathfrak{S}^A$ into $\mathfrak{S}^{A'}$. Since complex conjugation must leave real tensors invariant, we must have that $\mathfrak{T}^a \subseteq \mathfrak{S}^a := \mathfrak{S}^{AA'}$ (The $\subseteq$ only means an injection of modules here).

So, we require that for all $\chi^{a\ldots}_{b\ldots} \in \mathfrak{T}^{a\ldots}_{b\ldots}$ that

$$\overline{\chi^{a\ldots}_{b\ldots}} = \chi^{a\ldots}_{b\ldots} \tag{5.7}$$

From this observation follows the following important theorem:

Theorem 5.2.1. *Suppose M satisfies the conditions of Geroch's theorem (see 4.3.2); i.e., M is non-compact and spin, such that there exists a globally defined Minkowski tetrad and spinor dyad. Then the set of all $\chi^{a\ldots}_{b\ldots} \in \mathfrak{S}^{a\ldots}_{b\ldots}$ satisfying Equation* (5.7) *can be canonically identified with* $\mathfrak{T}^{a\ldots}_{b\ldots}$.

Proof. We will show that this holds for $\mathfrak{T}^a$, the general case then follows. Define

$$\mathrm{Re}(\mathfrak{S}^a) := \{\chi^{AA'} \in \mathfrak{S}^a : \overline{\chi^{AA'}} = \chi^{AA'}\}.$$

Note that Equation (5.1) gives us that

$$\mathfrak{S}^a = \mathrm{Re}(\mathfrak{S}^a) \oplus i\,\mathrm{Re}(\mathfrak{S}^a). \tag{5.8}$$

Note that both $\mathrm{Re}(\mathfrak{S}^a)$ and $i\,\mathrm{Re}(\mathfrak{S}^a)$ have the structure of a $\mathfrak{T}$-module.

By the existence of the spin structure, and the reasoning above, there exists some (canonical) $\mathfrak{T}$-injection

$$\phi : \mathfrak{T}^a \hookrightarrow \mathrm{Re}(\mathfrak{S}^a),$$

stemming from the commutative diagram of Definition 4.3.1.

We have that $\mathfrak{S}^a$ is a free $\mathfrak{S}$-module of dimension 4, since it is of the form $\mathfrak{S}^A \otimes \mathfrak{S}^{A'}$, each component being generated by a spinor dyad. Hence, since a free $\mathfrak{S}$ module of rank m is a free $\mathfrak{T}$ module of rank $2m$, by

$$\mathfrak{S} = \mathfrak{T} \oplus i\mathfrak{T},$$

we have that $\mathrm{Re}(\mathfrak{S}^a)$ is a $\mathfrak{T}$-module of rank 4 by Equation (5.8), but since $\mathfrak{T}^a$ is a free 4-dimensional $\mathfrak{T}$-module generated by a Minkowski tetrad, we find that by existence of the injection ϕ that

$$\mathfrak{T}^a \cong \mathrm{Re}(\mathfrak{S}^a)$$

as required. □

From here on out, we will simply write that $\mathfrak{T}^{a\cdots}_{c\cdots} = \mathrm{Re}(\mathfrak{S}^{a\cdots}_{c\cdots})$. We can make the isomorphism between these two modules more explicit, so we can transfer between tensor and spinor descriptions at will.

For this, we first 'define' the real tensors

$$\begin{aligned} g_{ab} &= \varepsilon_{AB}\varepsilon_{A'B'}, \\ g^{ab} &= \varepsilon^{AB}\varepsilon^{A'B'}, \\ g_a{}^b &= g_{ac}g^{cb}. \end{aligned} \tag{5.9}$$

Note that the following equations hold:

$$g_{ab}g^{ab} = 4, \quad g_a{}^b = \delta_a^b, \quad g_{ab} = g_{ba},$$

so g_{ab} has the same properties as the metric defined in Paragraph 4.2.

Furthermore, following [29, p. 119], we can define a tetrad of spinors by

$$l^a = o^A\overline{o}^{A'}, \quad n^a = \iota^A\overline{\iota}^{A'}, \quad m^a = o^A\overline{\iota}^{A'}, \quad \overline{m}^a = \overline{m^a}. \tag{5.10}$$

It can easily be verified that these form a basis for $\mathfrak{S}^a$, so this tetrad can be taken as the basis in the proof of Theorem 5.2.1.

We can then define the *real* tensor fields

$$t^a = \frac{1}{\sqrt{2}}(l^a + n^a), \quad x^a = \frac{1}{\sqrt{2}}(m^a + \overline{m}^a)$$
$$y^a = \frac{i}{\sqrt{2}}(m^a - \overline{m}^a), \quad z^a = \frac{1}{\sqrt{2}}(l^a - n^a).$$

Now using g_{ab} as in (5.9) to raise and lower indices, we find that (t^a, x^a, y^a, z^a) forms a Minkowski tetrad relative to the metric g_{ab}. As such, we see that, given a spinor dyad, we can reconstruct a Minkowski tetrad and the metric g_{ab}, using the uniqueness of the epsilon spinors.

Taking the inverse of the relations of Equation (5.10), we can write any real tensor

$$B^a = B^0 t^a + B^1 x^a + B^2 y^a + B^3 z^a$$

in spinor coordinates with respect to the dyad $\{o^A, \iota^A\}$ as

$$B^{\boldsymbol{AA'}} = \begin{pmatrix} B^{00'} & B^{01'} \\ B^{10'} & B^{11'} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} B^0 + B^3 & B^1 + iB^2 \\ B^1 - iB^2 & B^0 - B^3 \end{pmatrix}, \tag{5.11}$$

which, unsurprisingly, is the same identification as for spinors at a point described in Theorem 3.5.3.

Now, using the correspondence between g_{ab} and ε_{AB} and Proposition 5.1.4, we obtain the following result:

Proposition 5.2.2. *A real tensor field B^a is null if and only if*

$$B^a = q\kappa^A \overline{\kappa}^{A'}$$

for some $\kappa^A \in \mathfrak{S}^A$ and $q \in \mathfrak{T}$.

Proof. Let $B^a \in \mathfrak{T}^a$ be a real tensor field. Decomposing B^a in spinors, we obtain

$$B^a = \sum_{i=0}^{n} \mu_i^A \omega_i^{A'},$$

where i is not a spinor index, but the index over which we are summing. Using the reality of B^a, this reduces to

$$B^a = \sum_{i=0}^{n} \mu_i^A \overline{\mu}_i^{A'}.$$

Note that B^a is null if and only if

$$g_{ab} B^a B^b = 0 \iff \sum_{i,j} \mu_{A,i} \mu_j^A \overline{\mu}_{A',i} \overline{\mu}_j^{A'} = 0.$$

This reduces to

$$\sum_{i,j} |\mu_{A,i} \mu_j^A|^2 = 0,$$

where $|\cdot|^2$ denotes the complex norm map. By Proposition 5.1.4, this equation holds if and only if $\mu_i^A = \lambda_{ij} \mu_j^A$ for all $i, j \leq n$ and some $\lambda_{ij} \in \mathfrak{S}$. Combining this with the reality of B^a, the claim immediately follows. □

We now know how to convert valence $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ tensors to valence $\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]$ spinors, and vice versa. By simply lowering indices, we also get a construction for dual tensors of valence $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. We will now illustrate an algorithm to convert general tensors with two indices to spinors of a more tractable form.

Consider a symmetric tensor $C_{ab} = C_{ba} \in \mathfrak{T}_{ab}$. Note we can write, using spinor indices, that

$$C_{AA'BB'} = \frac{1}{2}(C_{ABA'B'} + C_{ABB'A'}) + \frac{1}{2}(C_{BAB'A'} - C_{ABB'A'}),$$

so twice applying Proposition 5.1.3 to the second pair of parentheses, we obtain

$$C_{AA'BB'} = \widehat{C}_{ABA'B'} + \frac{1}{4}\varepsilon_{AB}\varepsilon_{A'B'}C_{DD'}{}^{DD'},$$

where

$$\widehat{C_{ab}} := \frac{1}{2}(C_{ABA'B'} + C_{ABB'A'}) = C_{(AB)(A'B')} = C_{ab} - \frac{1}{4}C_a{}^a g_{ab}$$

is called the ***trace-free*** part of C_{ab} (note we have $C_d{}^d = 0$ by Prop. 5.1.3).

Going one step further, we also have the ***trace-reversal*** of C_{ab}, given by

$$\widehat{\widehat{C_{ab}}} = C_{ab} - \frac{1}{2}C_c{}^c g_{ab}, \tag{5.13}$$

which satisfies $\widehat{\widehat{C_a{}^a}} = -C_a{}^a$.

Similarly, we can factor skew 2-index tensors. For this, we first need to define a specific tensor:

Definition 5.2.3. The ***completely skew e-tensor*** is given by

$$e_{abcd} = i\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'D'}\varepsilon_{B'C'} - i\varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'C'}\varepsilon_{B'D'},$$

and satisfies

$$e_{abcd} = e_{[abcd]}, \qquad e_{abcd}e^{abcd} = -24. \tag{5.14}$$

Similar to how ε_{AB} is unique, since a spinor basis has 2 elements, Equation (5.14) uniquely defines e_{abcd}, since a Minkowski tetrad has 4 elements.

Now let $D_{ab} = -D_{ba} \in \mathfrak{S}_{ab}$ be an arbitrary skew valence $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$ spinor. Similar to above, we can write

$$D_{AA'BB'} = \frac{1}{2}(D_{ABA'B'} - D_{ABB'A'}) + \frac{1}{2}(D_{ABB'A'} - D_{BAB'A'}),$$

which, using Proposition 5.1.3 on each of the parentheses yields

$$D_{AA'BB'} = \phi_{AB}\varepsilon_{A'B'} + \psi_{A'B'}\varepsilon_{AB}, \tag{5.15}$$

where

$$\phi_{AB} := \frac{1}{2}D_{ABC'}{}^{C'}, \quad \psi_{A'B'} := \frac{1}{2}D_C{}^C{}_{A'B'},$$

are both symmetric.

If we further require that $D_{ab} \in \mathfrak{T}_{ab}$, we get $\overline{\phi_{AB}} = \psi_{A'B'}$. As such, we see that any symmetric spinor in $\mathfrak{S}_{AB}$ uniquely determines a skew tensor in $\mathfrak{T}_{ab}$ and vice-versa.

We now define the notion of dualisation of real tensors of valence $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$:

Definition 5.2.4. The ***dual*** of $G_{ab} \in \mathfrak{T}_{ab}$ is defined by

$$^*G_{ab} = \frac{1}{2} e_{abcd} G^{cd}.$$

We say that G_{ab} is

1. ***anti-self-dual*** if
$$^*G_{ab} = -iG_{ab}.$$

2. ***self-dual*** if
$$^*G_{ab} = iG_{ab}.$$

(The terminology *dual* is different from the dual we have encountered thus far, namely $(G_{ab})^* = G^{ab}$. Unfortunately, both terms share the same name, but from context it will clear which one we mean.) Note, in general, we have that $^{**}G_{ab} = -G_{ab}$.

When we are dealing with 4-valent tensors, we also write

$$^*H_{abcd} = \frac{1}{2} e_{abef} H^{ef}{}_{cd}, \quad H^*_{abcd} = \frac{1}{2} e_{cdef} H_{ab}{}^{ef}. \tag{5.16}$$

This notation is slightly non-standard, but we will need to use it in the Paragraph 5.4.

Returning to the decomposition of (5.15), we find that

$$^*G_{ab} = -i\phi_{AB}\varepsilon_{A'B'} + i\psi_{A'B'}\varepsilon_{AB}.$$

So, we have that

$$^-G_{ab} := \phi_{AB}\varepsilon_{A'B'}, \quad ^+G_{ab} := \psi_{A'B'}\varepsilon_{AB}$$

are anti-self-dual and self-dual, respectively, so the decomposition of (5.15) gives us a decomposition of G_{ab} in a anti-self-dual and self-dual part, i.e.,

$$G_{ab} = {}^-G_{ab} + {}^+G_{ab}.$$

Now, since by Proposition 4.1.8(1), any tensor $A_{ab} \in \mathfrak{T}_{ab}$ can be decomposed in a symmetric and skew part, we can use the procedures outlined above to decompose any real tensor of valence $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$ in a sum of products of epsilon-spinors and symmetric spinors. By raising indices, we can extend this programme to arbitrary tensors with 2 indices.

In fact, we can use these steps to translate *any* arbitrary valence tensor into sums of products of epsilon spinors and symmetric spinors, but for the purposes we have in mind, the procedure outlined above is sufficient. See [29, p. 153] for the algorithm in full.

5.3 The geometry of spinors

In this paragraph, we will briefly shift our focus from an arbitrary manifold M, satisfying Geroch's theorem, to the specific case of flat Minkowski space. We will discuss a geometrical interpretation of Equation (5.11) and Proposition 5.2.2, by giving an association between constant spinors and so-called *null flags* in Minkowski space. We will use this description extensively when discussing twistor theory in Chapter 6.

This construction presented here can be seen as the first little step in giving the twistor Klein correspondence, to which much of the latter part of Chapter 6 will be dedicated. Since the construction presented here does not yet require the definition of a twistor, this is a natural point to treat this geometry of spinors.

When we have a Minkowski tetrad (t^a, x^a, y^a, z^a), we can associate the point $r = (r^0, r^1, r^2, r^3)$ of Minkowski space with the (constant) field

$$R^a = r^0 t^a + r^1 x^a + r^2 y^a + r^3 z^a.$$

Supposing that R^a is future-null, i.e., that $g_{ab}R^a R^b = 0$ and $r^0 > 0$, we have by Proposition 5.2.2 and the constancy of R^a that there exists some constant spinor κ^A such that

$$R^a = \kappa^A \overline{\kappa}^{A'}$$

We can take $q = 1$ in Proposition 5.2.2 since R^a is constant and future-pointing. Thus, any constant spinor determines a unique future pointing null vector. We call the vector R^a the ***flagpole*** of κ^A.

Note, however, that different spinors may have the same flagpole; we have the gauge freedom of multiplying κ^A with an arbitrary phase factor, keeping the flagpole constant.

To sidestep this issue, we define a (constant) symmetric tensor

$$P^{ab} = \kappa^A \kappa^B \varepsilon^{A'B'} + \kappa^{A'} \kappa^{B'} \varepsilon^{AB}.$$

Now if ϕ^A is some constant spinor such that ϕ^A, κ^A constitute a spinor dyad, i.e. (cf. Def. 5.1.2),

$$\kappa_A \phi^A = 1, \tag{5.17}$$

we find, using Equation (5.4), that

$$P^{ab} = R^a S^b - S^a R^b,$$

where

$$S^a = \kappa^A \overline{\phi}^{A'} + \phi^A \overline{\kappa}^{A'}$$

is a real constant tensor, which is orthogonal to R^a and has length $S^a S_a = -2$.

Note we now have the freedom to add complex multiples of κ^A to ϕ^A without changing the validity of Equation (5.17), and we have the following relation for some $\lambda \in \mathbb{C}$:

$$\phi^A \mapsto \phi^A + \lambda \kappa^A \implies S^a \mapsto S^a + 2\,\mathrm{Re}(\lambda) R^a,$$

so the set of all possible S^a's forms a line parallel to R^a in the Minkowski vector space. The plane spanned by this line and R^a is called the ***flag plane*** of κ^A. The ***flag*** of κ^A is defined to be the set containing both the flag plane and the flagpole.

Note that the map $\kappa^A \mapsto e^{i\theta}\kappa$ rotates the flag plane through an angle of 2θ about R^a, so we now have a 2-to-1 correspondence between spinors and flags, since a rotation through π in spin space leaves the flag invariant, also in accordance with Proposition 3.4.7.

In Figure 5.1, the results of this paragraph are briefly summarised, showing what a null flag corresponding to a spinor looks like in 1+2 dimensional Minkowski space.

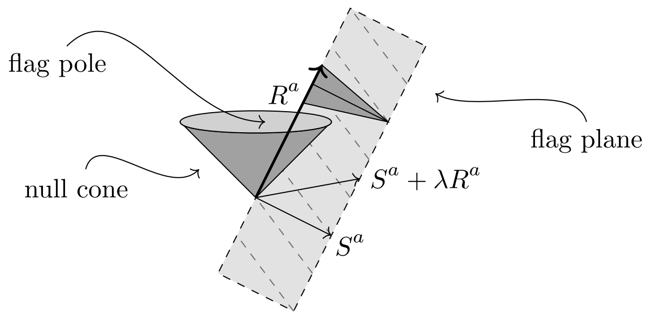


Figure 5.1: *Geometric representation of a constant spinor κ^A with flagpole R^a and flag plane spanned by S^a. The flagpole lies on the null cone at the origin, the flag plane rotates about R^a with an angle 2θ as the phase of κ^A is changed by an angle θ.*

5.4 Curvature and the Einstein equations

In this paragraph, we will further examine the spin covariant derivative of Paragraph 4.3. In particular, one of our goals will be to define the *Riemann curvature tensor*, which will tell us something about the flatness of the manifold M. Using this notion, we can formulate the *Einstein equations*, and describe them in spinor form. For the translation of the Einstein equations in spinor form, we follow [29, Chapter 4].

Our description of the Einstein equations will turn out to give conditions for when we can apply twistor theory to a space-time manifold, either using the procedures of Chapter 7 for conformally Minkowski spaces, or using the non-linear graviton construction of Paragraph 8.2 for patch-wise conformally flat spaces.

Recall that for the Levi-Civita derivative, we had the condition of *metricity*; i.e., we have that (following Thm. 4.2.15)

$$\nabla_a g_{bc} = 0.$$

As such, we see that index raising or lowering commutes with taking the derivative, in other words,

$$\nabla_a X^{\dots}_{c\dots} = \nabla_a g_{bc} X^{b\dots}_{\dots} = g_{bc} \nabla_a X^{b\dots}_{\dots} + X^{b\dots}_{\dots} \nabla_a g_{bc} = g_{bc} \nabla_a X^{b\dots}_{\dots}, \tag{5.18}$$

so we can unambiguously define

$$\nabla^a := g^{ab} \nabla_b.$$

We can show that a similar property holds for the epsilon spinors:

Proposition 5.4.1. *Let $\nabla_{AA'}$ denote the spinor covariant derivative. We then have*

$$\nabla_{AA'} \varepsilon_{BC} = 0.$$

Proof. For all ϕ^B, ψ^C we have that

$$(\nabla_{AA'} \varepsilon_{BC}) \phi^B \psi^C = \nabla_{AA'} (\varepsilon_{BC} \phi^B \psi^C) - \varepsilon_{BC} \phi^B \nabla_{AA'} \psi^C - \varepsilon_{BC} \psi^C \nabla_{AA'} \phi^B$$

by the Leibniz rule (cf. Eq. (4.10)). We now transvect this equation with $2\varepsilon_{B'C'}\chi^{B'}\xi^{C'}$ for some $\chi^{B'}, \xi^{C'}$ and substitute the following three identities coming from the Leibniz rule

$$2\varepsilon_{B'C'}\chi^{B'}\xi^{C'}\nabla_{AA'} = \varepsilon_{B'C'}\chi^{B'}\nabla_{AA'}(\psi^C\xi^{C'}) + \varepsilon_{B'C'}\xi^{C'}\nabla_{AA'}(\psi^C\chi^{B'}) \\ - \psi^C\nabla_{AA'}(\varepsilon_{B'C'}\chi^{B'}\xi^{C'}),$$

$$\psi^C\xi^{C'}\nabla_{AA'}(\phi^B\chi^{B'}) + \phi^B\chi^{B'}\nabla_{AA'}(\psi^C\xi^{C'}) \\ = \nabla_{AA'}(\phi^B\psi^C\chi^{B'}\xi^{C'}) = \psi^C\chi^{D'}\nabla_{AA'}(\phi^B\xi^{C'}) + \phi^B\xi^{C'}\nabla_{AA'}(\psi^C\chi^{B'}),$$

$$\nabla_{AA'}(\phi^B\psi^C\chi^{B'}\xi^{C'}\varepsilon_{BC}\varepsilon_{B'C'}) - \varepsilon_{BC}\varepsilon_{B'C'}\nabla_{AA'}(\phi^B\psi^C\chi^{B'}\xi^{C'}) \\ = \phi^B\psi^C\chi^{B'}\xi^{C'}\nabla_{AA'}(\varepsilon_{BC}\varepsilon_{B'C'}) \\ = \phi^B\psi^C\chi^{B'}\xi^{C'}\nabla_{AA'}(g_{bc}) = 0,$$

where the second to last equality follows from $g_{bc} = \varepsilon_{BC}\varepsilon_{B'C'}$ and the final equality follows from metricity of the derivative.

We hence obtain

$$2\varepsilon_{B'C'}\chi^{B'}\xi^{C'}(\nabla_{AA'}\varepsilon_{BC})\phi^B\psi^C = 0,$$

whence, since $\phi^B, \psi^C, \chi^{B'}, \xi^{C'}$ are arbitrary, we get

$$\nabla_{AA'}\varepsilon_{BC} = 0,$$

as required. □

A very desirable property of the 'metricity' of the spin covariant derivative is that raising and lowering spinor indices, similar to Equation (5.18), commutes with taking the spin covariant derivative. Consequently, we can again, unambiguously, define

$$\nabla^{AA'} := \varepsilon^{AB}\varepsilon^{A'B'}\nabla_{BB'}, \quad \nabla^{A}_{B'} := \varepsilon^{AB}\nabla_{BB'}, \quad \nabla^{A'}_{B} := \varepsilon^{A'B'}\nabla_{BB'}.$$

We will use these identities at length when discussing twistors in the next chapter.

Next, we will derive the *curvature tensor*, using which we can state the Einstein equations in spinorial terms. We first need to define the *commutator* of a covariant derivative:

Definition 5.4.2. Let ∇_a be a covariant derivative on the orthochronous tangent bundle. Then the ***commutator*** is defined as the map

$$\Delta_{ab} := \nabla_a\nabla_b - \nabla_b\nabla_a : \mathfrak{T}^{\mathcal{A}} \to \mathfrak{T}^{\mathcal{A}}{}_{[ab]}.$$

Note that when we are dealing with the Levi-Civita derivative, which is torsion-free (see Eq. (4.13)), we have that $X^{ab}\Delta_{ab}f = 0$ for all $X^{ab} \in \mathfrak{T}^{ab}$ and $f \in \mathfrak{T}$, so in particular

$$\Delta_{ab}f = 0.$$

We can then easily see that the map

$$\begin{aligned} R_{abc}{}^d : \mathfrak{T}^c &\to \mathfrak{T}_{ab}{}^d \\ A^c &\mapsto \Delta_{ab}A^d \end{aligned} \tag{5.20}$$

is a $\mathfrak{T}$-linear map, and hence, by Theorem 4.1.5, we have that $R_{abc}{}^{d} \in \mathfrak{T}_{abc}{}^{d}$. We call the tensor

$$R_{abcd} := g_{ed} R_{abc}{}^{e} \tag{5.21}$$

the ***Riemann curvature tensor***. For Minkowski space $\mathbb{M}$ with the Levi-Civita covariant derivative, where we provide Minkowski space with the flat metric g_{ab} of Equation (4.11), we find by Schwarz's theorem that

$$R_{abcd} = 0. \tag{5.22}$$

Many of the other tensors and spinors we will derive in the following will also be identically zero for flat space.

Some tedious computations (see [29, pp. 194, 209]) yield the following (Bianchi-)identities:

Proposition 5.4.3. *Let R_{abcd} be the Riemann curvature tensor. Then the following four identities hold:*

1. *The* **first Bianchi identity***:*
$$R_{abcd} + R_{bcad} + R_{cabd} = 0.$$
2. *The* **second Bianchi identity***:*
$$\nabla_{[a} R_{bc]de} = 0.$$
3. **Interchange symmetry***:*
$$R_{abcd} = R_{cdab}.$$
4. **Skew-symmetry***:*
$$R_{abcd} = R_{[ab][cd]}.$$

We will now use the machinery introduced at the end of Paragraph 5.2 to give a twistor description of the Riemann curvature tensor.

From the anti-symmetry in ab and cd we obtain, using the decomposition of Equation (5.15) and the reality of R_{abcd} that

$$\begin{aligned} R_{abcd} = R_{AA'BB'CC'DD'} = {} & X_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} \\ & + \overline{X}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} + \overline{\Phi}_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'}, \end{aligned} \tag{5.23}$$

where

$$X_{ABCD} = \frac{1}{4} R_{AE'B}{}^{E'}{}_{CF'D}{}^{F'}, \qquad \Phi_{ABC'D'} = \frac{1}{4} R_{AE'B}{}^{E'}{}_{FC'}{}^{F}{}_{D'}$$

are called the ***curvature spinors***. Note that these two spinors are symmetric in AB and CD or AB and $C'D'$, respectively. We call the spinor $\Phi_{ABC'D'}$ the ***Ricci spinor***.

By the interchange symmetry (Prop. 5.4.3(3)), we find

$$X_{ABCD} = X_{CDAB}, \qquad \overline{\Phi}_{ABC'D'} = \Phi_{ABC'D'}, \tag{5.24}$$

so in particular, we have that Φ_{ab} is real, whereas the symmetry of $\Phi_{ABC'D'}$ in AB and $C'D'$, combined with Proposition 5.1.3 gives that

$$\Phi_{ab} = \Phi_{(ab)}, \quad \Phi_{a}{}^{a} = 0,$$

and the symmetry of X_{ABCD} implies that

$$X_{A(BC)}{}^{A} = 0.$$

We can now observe that Proposition 5.4.3(1) is equivalent to

$$R^{*}{}_{ab}{}^{bc} = 0,$$

where the star is to mean dualisation on the last two indices in the sense of Equation (5.16). Writing this out in full, one can verify that we hence obtain

$$X_{AB}{}^{B}{}_{C}\varepsilon_{A'C'} = \overline{X}_{A'B'}{}^{B'}{}_{C'}\varepsilon_{AC},$$

which, after raising C and C' and performing (C, A) and (C', A')-contraction, yields

$$X_{AB}{}^{AB} = \overline{X_{A'B'}{}^{A'B'}}.$$

We can now introduce a very important real quantity, given by

$$\Lambda := \frac{1}{6}X_{AB}{}^{AB} = \overline{\Lambda}. \tag{5.25}$$

We further introduce the ***Ricci tensor***, given by

$$R_{ab} := R_{acb}{}^{c},$$

so that Equation (5.23) can be rewritten as

$$R_{ab} = 6\Lambda\varepsilon_{AB}\varepsilon_{A'B'} - 2\Phi_{ABA'B'} = 6\Lambda g_{ab} - 2\Phi_{ab}.$$

We also have a notion of ***scalar curvature***, also called the ***Ricci scalar***, given by

$$R := R_{a}{}^{a} = 24\Lambda.$$

Finally, we define the ***Einstein tensor***, given by (cf. Eq. (5.13))

$$G_{ab} := \widehat{\widehat{R_{ab}}} = R_{ab} - \frac{1}{2}Rg_{ab} = -6\Lambda\varepsilon_{AB}\varepsilon_{A'B'} - 2\Phi_{ABA'B'},$$

or, equivalently,

$$G_{ac} = {}^{*}R^{*}{}_{abc}{}^{b},$$

where the star operators, coming from Equation (5.16), denote that we first dualize over the first two components and then over the second two components of R_{abcd}, after which raise the d index and perform (b, d)-contraction. We are now in a position to define the *Einstein equations*, which are some of the most important objects of study in general relativity.

Definition 5.4.4. The ***Einstein equation*** is defined by

$$G_{ab} + \lambda g_{ab} = -8\pi G T_{ab}, \tag{5.26}$$

where $\lambda \in \mathbb{R}$ is the ***cosmological constant***, T_{ab} is the ***energy-momentum tensor*** and $G \in \mathbb{R}$ is the ***gravitational constant***.

We say the underlying manifold M is an ***Einstein manifold*** when $T_{ab} = 0$, so that the Einstein equation becomes

$$R_{ab} = \lambda g_{ab}.$$

We can rewrite the Einstein equation for Einstein manifolds in spinor terms as

$$\Phi_{ABA'B'} = 0, \quad \Lambda = \frac{1}{6}\lambda$$

and the general Einstein equation as

$$\Phi_{ABA'B'} = 4\pi G(T_{ab} - \frac{1}{4}T_c{}^c g_{ab}), \quad \Lambda = \frac{1}{3}\pi T_q{}^q + \frac{1}{6}\lambda.$$

We can also further decompose X_{ABCD}, namely as follows:

$$\begin{aligned} X_{ABCD} &= \frac{1}{3}(X_{ABCD} + X_{ACDB} + X_{ADBC}) + \frac{1}{3}(X_{ABCD} - X_{ACDB}) + \frac{1}{3}(X_{ABCD} - X_{ADCB}) \\ &= X_{(ABCD)} + \frac{1}{3}\varepsilon_{BC}X_{AE}{}^{E}{}_{D} + \frac{1}{3}\varepsilon_{BD}X_{AEC}{}^{E}, \end{aligned}$$

where for the second equality, we used the symmetry of X_{ABCD} and Equation (5.24) on the first pair of parentheses, and we used Proposition 5.1.3 on the other pairs of parentheses. Comparing this expression with Equation (5.25), we obtain

$$X_{ABCD} = \Psi_{ABCD} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}),$$

where

$$\Psi_{ABCD} := X_{(ABCD)} \tag{5.27}$$

is called the ***gravitational spinor*** or ***Weyl conformal spinor***. This spinor will turn out to be of great importance in twistor theory.

We want to substitute this expression into the decomposition of R_{abcd} of Equation (5.23). Note this entails replacing X_{ABCD} with $\Psi_{ABCD} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC})$, so, additionally making use of Equation (5.6), which we derived in the proof of Proposition 5.1.3, and lowering the D and E components in this expression, we finally obtain the following expression:

$$\begin{aligned} R_{abcd} = R_{AA'BB'CC'DD'} = \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} &+ \Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} \\ &+ \overline{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD} + \overline{\Phi}_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'} \\ &+ 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}). \end{aligned} \tag{5.28}$$

We can further simplify this by defining the following tensors:

$$\begin{aligned} {}^{-}C_{abcd} &= \Psi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'}, \\ {}^{+}C_{abcd} &= \overline{\Psi}_{A'B'C'D'}\varepsilon_{AB}\varepsilon_{CD}, \\ C_{abcd} &= {}^{-}C_{abcd} + {}^{+}C_{abcd}, \\ E_{abcd} &= \Phi_{ABC'D'}\varepsilon_{A'B'}\varepsilon_{CD} + \overline{\Phi}_{A'B'CD}\varepsilon_{AB}\varepsilon_{C'D'}, \\ g_{abcd} &= \varepsilon_{AC}\varepsilon_{BD}\varepsilon_{A'C'}\varepsilon_{B'D'} - \varepsilon_{AD}\varepsilon_{BC}\varepsilon_{A'D'}\varepsilon_{B'C'}, \end{aligned} \tag{5.29}$$

where ${}^{-}C_{abcd}$ and ${}^{+}C_{abcd}$ are anti-self-dual and self-dual, respectively. The tensor C_{abcd} is also called the ***Weyl conformal tensor***. This tensor is exactly the part of the curvature tensor that stays invariant under conformal rescaling (also see Prop. 6.1.3).

As such, the equation for R_{abcd} becomes

$$R_{abcd} = C_{abcd} + E_{abcd} + 2\Lambda g_{abcd} = {}^{-}C_{abcd} + {}^{+}C_{abcd} + E_{abcd} + 2\Lambda g_{abcd}.$$

This relatively simple expressions in spinor terms governing curvature are a result of the fact that our manifold M has a metric of signature $(+---)$. If, for example, we were dealing with a space of signature $(++--)$, this expression would be far more complicated.

However, this simplicity also has a major disadvantage. The twistor theory we will develop in the next chapters will only give a meaningful description for spaces where the Weyl tensor is anti-self-dual, which due to Equation (5.29) is the case only if $\Psi_{ABCD} = 0$, which we will see to mean that the space is conformally flat. We will see why this is the case in Paragraph 8.2.

Since one would like to be able to describe any arbitrary solution to the Einstein equations in twistorial terms, this obstruction is very undesirable. In fact, in the presence of any mass, or any other form of energy, the Weyl spinor will almost always be non-zero. This is one of the major shortcomings of twistor theory, which as of yet remains unresolved. We will look at this in a little more detail in Paragraph 8.3.

Chapter 6

Twistor space

In the previous few chapters, we have had to introduce a lot of different concepts. In this chapter, this will finally pay its dividends; we will be able to define *twistors*, which are the objects we are most interested in in this thesis.

In Paragraph 6.1, we will first discuss the idea behind *conformal rescalings*, which alter the pseudo-Riemannian metric on a manifold M in a specific way. Twistor theory is in its core a conformal theory; the twistor space related to a Riemannian manifold will remain unchanged under such a conformal rescaling. We will see this in Paragraph 6.2, where we construct twistor space for 4-dimensional Minkowski space using the *twistor equation*, which will turn out to be conformally invariant. We will show that the solution space of the twistor equation is a 4-complex dimensional vector space, and we will describe the elements of this space in terms of spinors.

In Paragraph 6.3, we will use the description of the solutions to the twistor equation that we obtained in the previous paragraph to define some important manipulations of twistors. In particular, we will define duals and complex conjugates of twistors. We will also define multivalent twistors and consider a type of norm on twistor space, called the *helicity*.

All these notions will then be combined to describe the *twistor Klein correspondence*. This correspondence tells us something about how we can reconstruct Minkowski space, given the structure of twistor space. As such, we can see twistor space as the more fundamental object, with Minkowski space being derived from it. In Paragraph 6.4, we will first discuss a correspondence between null twistors and lines in Minkowski space, which we will extend in Paragraph 6.5 to derive a full correspondence between *projective twistor space* $\mathbb{PT}$ and *complexified compactified Minkowski space* $\mathbb{CM}^{\bullet}$.

6.1 Conformal transformations

Before we can discuss the twistor equation and twistor space proper, we must first consider the concept of *conformal rescalings*. As usual, we consider a 4-dimensional manifold M satisfying the conditions of Geroch's theorem.

Definition 6.1.1. Let M be a manifold with a metric g_{ab} of signature $(+---)$ satisfying

Geroch's theorem. A ***conformal rescaling*** of the metric is a map

$$g_{ab} \mapsto \widehat{g}_{ab} := \Omega^2 g_{ab},$$

where $\Omega \in \mathfrak{T}$ is a strictly positive real scalar function. We call the Ω the ***conformal transformation*** or ***conformal map*** associated to the rescaling.

We say that a space M with metric g_{ab} is ***(patch-wise) conformally flat*** if at every point $p \in M$, there exists an open neighbourhood $U_p \ni p$ and a conformal map Ω such that $\Omega^2 g_{ab}$ constrained to U_p is the metric of flat Minkowski space, as defined in Equation (4.11).

We say a space M with metric g_{ab} is ***conformally Minkowski*** if the neighbourhood U_p defined above can be taken to be the whole of M.

After conformal rescaling of the metric, index raising and lowering will be achieved through the new metric $\widehat{g}_{ab}$. We also want to know what happens when raising and lowering indices of spinors. Note that by the equation $g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}$, the only logical choice is to choose the map

$$\varepsilon_{AB} \mapsto \widehat{\varepsilon}_{AB} := \Omega\varepsilon_{AB}, \tag{6.1}$$

whence complex conjugation immediately yields

$$\varepsilon_{A'B'} \mapsto \widehat{\varepsilon}_{A'B'} := \Omega\varepsilon_{A'B'}.$$

Additionally, when we have a spinor dyad $\{o^A, \iota^A\}$, we want to convert this into another spinor dyad $\{\widehat{o}^A, \widehat{\iota}^A\}$. Since we need to preserve Equation (5.4), we cannot just take $o^A = \widehat{o}^A$ and $\iota^A = \widehat{\iota}^A$. We can however make the following (somewhat arbitrary) choice

$$\widehat{o}^A = \Omega^{-1} o^A, \quad \widehat{\iota}^A = \iota^A. \tag{6.2}$$

Definition 6.1.2. We say a spinor $\chi^{\cdots}_{\cdots} \in \mathfrak{S}^{\cdots}_{\cdots}$ is of ***conformal weight k*** if under conformal rescaling the relation

$$\widehat{\chi}^{\cdots}_{\cdots} = \Omega^k \chi^{\cdots}_{\cdots}$$

holds.

Note that by our choice of spinor dyad, some spinor $\kappa^A \in \mathfrak{S}^A$ can have have conformal weight 0 or -1, but need not have a well-defined conformal weight by the asymmetry in the choice of spinor dyad. However, any one-index spinor is the sum of two spinors of well-defined conformal weight. The asymmetry in the scaling of Equation (6.2) is very useful, since if we have some spinor κ^A, we can always choose a dyad such that κ^A has conformal weight 0.

Further note, by Equation (6.1), and supposing κ^A is of conformal weight 0, we have that

$$\widehat{\kappa}_B = \widehat{\varepsilon}_{AB}\widehat{\kappa}^A = \Omega\kappa_B,$$

so lowering of spinor indices *raises* the conformal weight by 1. Conversely, raising of spinor indices *lowers* the conformal weight by 1. Similarly, lowering and raising *real* indices, raises and lowers the conformal weight by 2, respectively.

We now want to construct a new covariant derivative $\widehat{\nabla}_{AA'}$ on the conformally rescaled manifold that is torsion free and satisfies our stronger version of metricity, i.e.,

$$\widehat{\nabla}_{AA'}\widehat{\varepsilon}_{AB} = 0,$$

as stated in Proposition 5.4.1. Such a covariant must exist by Theorem 4.2.15, but we are interested in how this new covariant derivative relates to the old covariant derivative $\nabla_{AA'}$ under conformal rescaling.

Following our reasoning in Paragraph 5.4, were we defined the curvature tensor, the map given by

$$\begin{aligned}\Xi_{AA'B}{}^{C} : \mathfrak{S}^{B} &\mapsto \mathfrak{S}_{AA'}{}^{C}\\ \chi^{B} &\mapsto (\widehat{\nabla}_{AA'} - \nabla_{AA'})\chi^{C}\end{aligned}$$

is $\mathfrak{S}$-linear, so $\Xi_{AA'B}{}^{C} \in \mathfrak{S}_{AA'B}{}^{C}$ (cf. Eq. (5.20), Thm. 4.1.5).

Note by metricity and the Leibniz rule, we have that

$$\begin{aligned}0 = (\widehat{\nabla}_{AA'} - \nabla_{AA'})\varepsilon_{BC} = -\Xi_{AA'B}{}^{D}\varepsilon_{DC} + \Xi_{AA'B}{}^{D}\varepsilon_{BD} = -\Xi_{AA'BC} + \Xi_{AA'CB}\\ \implies \Xi_{AA'BC} = \Xi_{AA'CB}\end{aligned}$$

Vanishing of torsion can be shown to imply, in combination with the previous result, that

$$\Xi_{A'(ABC)} = 0,$$

so that, using the procedures of Paragraph 5.2, we can write Ξ as a sum of symmetric spinors multiplied with epsilon spinors:

$$\Xi_{A'ABC} = \kappa_{AA'}\varepsilon_{BC} + \chi_{A'B}\varepsilon_{AC} + \psi_{A'C}\varepsilon_{AB}.$$

Then applying Equation (5.5) to the final term, giving

$$\psi_{A'C}\varepsilon_{AB} = \psi_{A'A}\varepsilon_{CB} + \psi_{A'B}\varepsilon_{AB},$$

we obtain

$$\Xi_{A'ABC} = i\Pi_{AA'}\varepsilon_{BC} + \Upsilon_{A'B}\varepsilon_{AC},$$

where $\Pi_{AA'}, \Upsilon_{A'A}$ can be verified to be real, following [29, p. 217].

However, by metricity of the covariant derivative, we find

$$\begin{aligned}0 = \widehat{\nabla}_{AA'}\widehat{\varepsilon}_{BC} = \widehat{\nabla}_{AA'}(\Omega\varepsilon_{BC}) &= \varepsilon_{BC}(\nabla_{AA'}\Omega - \Omega\Xi_{AA'B}{}^{B})\\ &= \varepsilon_{BC}(\nabla_{AA'}\Omega - 2i\Omega\Pi_{AA'} - \Omega\Upsilon_{AA'}),\end{aligned}$$

which, by reality of Ω, yields

$$\Pi_{AA'} = 0, \quad \nabla_{AA'}\log\Omega = \Upsilon_{AA'}.$$

As such, we find that by the Leibniz rule, we have in general that

$$\begin{aligned}\widehat{\nabla}_{AA'}\psi_{D\ldots E'\ldots}^{B\ldots C'\ldots} = \nabla_{AA'}\psi_{D\ldots E'\ldots}^{B\ldots C'\ldots} - \Upsilon_{DA'}\psi_{A\ldots E'\ldots}^{B\ldots C'\ldots} - \cdots - \Upsilon_{AE'}\psi_{D\ldots A'\ldots}^{B\ldots C'\ldots} - \ldots\\ + \varepsilon_{A}{}^{B}\Upsilon_{FA'}\psi_{D\ldots E'\ldots}^{F\ldots C'\ldots} + \cdots + \varepsilon_{A'}{}^{C'}\Upsilon_{AF'}\psi_{D\ldots E'\ldots}^{B\ldots F'\ldots} + \ldots.\end{aligned} \tag{6.4}$$

Using the machinery introduced here, it is straightforward, although very tedious, to show that the Weyl conformal spinor Ψ_{ABCD}, as defined in Equation (5.27), is conformally invariant (see [30, pp. 120–121]). In fact, we get the following result:

Proposition 6.1.3. *Let (M, g_{ab}) be a space-time of signature $(+---)$. Then M is patch-wise conformally flat if and only if*

$$\Psi_{ABCD} = 0.$$

Proof. Since Ψ_{ABCD} is conformally invariant, it immediately follows that any patch-wise conformally flat space-time satisfies $\Psi_{ABCD} = 0$. The proof of the converse uses some of the techniques of local twistors introduced at the start of Chapter 8. A full proof of this can be found in [30, pp. 137–139]. □

6.2 The twistor equation

In this paragraph, we are finally in the position to define *twistors*. For conformally flat spaces, twistor space is defined as the solution space to the twistor equation, which will be introduced here. We will see that the twistor equation is conformally invariant, so if we have some space that is conformally Minkowski, its twistor space will be the same as that of Minkowski space. As such, in this paragraph, we will describe the solutions to the twistor equation in Minkowski space, so that we know what twistor space looks like for all conformally Minkowski spaces. For conformally flat spaces that are not conformally Minkowski, one must use the *non-linear graviton construction*, outlined in Paragraph 8.2.

Definition 6.2.1. The ***twistor equation*** is given by

$$\nabla_{A'}^{(A}\omega^{B)} = 0. \tag{6.5}$$

Spinors ω^B satisfying the twistor equation are called ***twistors***.

The form of the twistor equation might seem arbitrary, but it has some very appealing properties that we will explore in this chapter. One of its advantages, as previously mentioned, is that the twistor equation is conformally invariant.

Lemma 6.2.2. *The twistor equation is conformally invariant.*

Proof. Let ω^B be a twistor and Ω a conformal map. In accordance with Equation (6.2), we can choose a spinor dyad such that ω^A is of conformal weight 0. Then, by Equation (6.4), we obtain

$$\widehat{\nabla}_{AA'}\widehat{\omega}^B = \nabla_{AA'}\omega^B + \varepsilon_A{}^B\Upsilon_{CA'}\omega^{A'}. \tag{6.6}$$

Raising the A index and symmetrising over A and B yields

$$\widehat{\nabla}_{A'}^{(A}\widehat{\omega}^{B)} = \Omega^{-1}\left(\nabla_{A'}^{(A}\omega^{B)} + \frac{1}{2}\varepsilon^{AB}\Upsilon_{CA'}\omega^{A'} + \frac{1}{2}\varepsilon^{BA}\Upsilon_{CA'}\omega^{A'}\right) = \Omega^{-1}\nabla_{A'}^{(A}\omega^{B)},$$

using the asymmetry of ε_{AB}. The Ω^{-1} term comes from noting that raising of spinor indices lowers the conformal weight by 1.

As such, we see that

$$\widehat{\nabla}_{A'}^{(A}\widehat{\omega}^{B)} = 0 \iff \nabla_{A'}^{(A}\omega^{B)} = 0,$$

establishing conformal invariance. □

We will henceforth look at solutions to the twistor equation in flat *Minkowski space* $\mathbb{M} = \mathbb{R}^{1,3}$, noting that the solution space of the twistor equation will remain the same in all conformally Minkowski space-times.

As in Paragraph 5.3, we can label *points* of Minkowski space with constant *vector fields*. Alternatively, we can construct a vector field $x^a \in \mathfrak{T}^a$, which at a point $y \in \mathbb{M}$ is assigned the vector y in the tangent space at y. This can be done canonically, since $\mathbb{M}$ is a vector space; the tangent space at any point can be identified with $\mathbb{M}$ itself.

Note that this vector field satisfies the equation

$$\nabla_{AA'} x^b = g_a{}^b. \tag{6.7}$$

To solve the twistor equation in flat space, we consider a twistor ω^C. We will look at the equation

$$\nabla^B_{B'} \nabla^A_{A'} \omega^C = \nabla^A_{A'} \nabla^B_{B'} \omega^C,$$

resulting from the vanishing of the Riemann curvature tensor (cf. Eq. (5.21), Eq. (5.22)). Combining this with the fact that ω^C is a twistor, we obtain by Equation (6.5) that

$$\nabla^B_{B'} \nabla^A_{A'} \omega^C = -\nabla^B_{B'} \nabla^C_{A'} \omega^A$$

so that $\nabla^B_{B'} \nabla^A_{A'} \omega^C$ is skew in AC and BC. Consequently, $\nabla^B_{B'} \nabla^A_{A'} \omega^C$ is skew in ABC, so it must vanish, by the two-valuedness of spinor indices.

As a result, we see that $\nabla^A_{A'} \omega^C$ must be constant, so by skewness in AC must be a constant multiple of ε^{AC}. We hence write

$$\nabla_{AA'} \omega^C = -i \varepsilon_A{}^C \pi_{A'} \tag{6.9}$$

for some constant spinor $\pi_{A'} \in \mathbb{S}_-$. The choice for the factor $-i$ is arbitrary, but will turn out to be convenient later.

Now considering Equation (6.7), we obtain the general solution to the twistor equation in flat space

$$\omega^A = -i x^{AA'} \pi_{A'} + \mathring{\omega}^A, \tag{6.10}$$

where $\mathring{\omega}^A \in \mathbb{S}^+$ is the constant spinor field satisfying

$$\omega^A(O) = \mathring{\omega}^A(O). \tag{6.11}$$

This yields the following result:

Theorem 6.2.3. *The solution space of the twistor equation in flat space is called the* **twistor space***, denoted by* $\mathbb{T}$*. Twistor space is a 4-dimensional* $\mathbb{C}$*-vector space.*

Proof. From the reasoning above, we see that any solution ω^B to the twistor equation is uniquely determined by two constant spinors $\mathring{\omega}^B$ and $\pi_{B'}$.

Conversely, by direct verification, any spinor $\tilde{\omega}^B$ satisfying

$$\tilde{\omega}^B = -i x^{BB'} \tilde{\pi}_{B'} + \mathring{\tilde{\omega}}^B$$

for some $\tilde{\pi}_{B'} \in \mathbb{S}_-$, $\mathring{\tilde{\omega}}^B \in \mathbb{S}^+$ is a solution to the twistor equation. Since both $\mathbb{S}_-$ and $\mathbb{S}^+$ are two-dimensional complex vector spaces, the result follows. □

We write $\mathbb{T}^\alpha$ for the space of spinors in $\mathfrak{S}^A$ satisfying the twistor equation, which is canonically isomorphic to $\mathbb{T}$. Elements of $\mathbb{T}^\alpha$ are denoted by

$$\mathsf{Z}^\alpha = (\omega^A, \pi_{A'}),$$

where ω^A and $\pi_{A'}$ satisfy Equation (6.10). Note that we write twistors using sans-serif uppercase Latin characters with lowercase Greek indices.

We can then also define

$$\mathsf{Z}^A = \omega^A, \quad \mathsf{Z}_{A'} = \pi_{A'}. \tag{6.12}$$

Given a spinor dyad $\{o^A, \iota^A\}$, we can now define the ***twistor basis*** given by

$$\delta_0^\alpha = o^A, \ \ \delta_1^\alpha = \iota^A, \ \ \delta_2^\alpha = o_{A'}, \ \ \delta_3^\alpha = \iota_{A'},$$

such that

$$\mathsf{Z}^\alpha = \mathsf{Z}^{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}^\alpha,$$

where we have

$$\mathsf{Z}^0 = \omega^0, \ \ \mathsf{Z}^1 = \omega^1, \ \ \mathsf{Z}^2 = \pi_{0'} = \mathsf{Z}_{0'}, \ \ \mathsf{Z}^3 = \pi_{1'} = \mathsf{Z}_{1'}. \tag{6.13}$$

(The components of ω^A and $\pi_{A'}$ are defined as in Eq. (5.2)).

6.3 Dual twistors

We start the extension of twistor space by considering *dual* and *multi-valent twistors*, similar to how we defined multi-valent tensors in Paragraph 4.1. However, since twistor space is not the set of sections of some bundle, we need a slightly different approach, which will be outlined here.

In particular, it will turn out that 2-valent twistors are of great importance, with the *infinity twistor* being one example of such a twistor, which we will encounter in Paragraph 7.2.

Definition 6.3.1. The ***dual twistor space*** $\mathbb{T}_\alpha$ consists of elements $\mu^{B'} \in \mathfrak{S}^{B'}$ satisfying the ***dual twistor equation***, given by

$$\nabla_A^{(A'} \mu^{B')} = 0.$$

Following similar reasoning as in the previous paragraph, we can alternatively write elements of $\mathbb{T}_\alpha$ as

$$\mathsf{W}_\alpha = (\lambda_A, \mu^{A'}),$$

where $\mu^{A'}$ is a solution to the dual twistor equation, and $\lambda_A \in \mathbb{S}_+$ is such that

$$\mu^{A'} = ix^{AA'}\lambda_A + \mathring{\mu}^{A'}. \tag{6.14}$$

where $\mathring{\mu}^{A'}$ is defined similarly as in Equation (6.11).

We define a ***twistor inner product*** between $\mathsf{W}_\alpha = (\lambda_A, \mu^{A'})$ and $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ by

$$\mathsf{W}_\alpha \mathsf{Z}^\alpha = \lambda_A(O)\omega^A(O) + \mu^{A'}(O)\pi_{A'}(O), \tag{6.15}$$

where O denotes the origin of $\mathbb{M}$.

We want this notion of twistor inner product to be Poincaré invariant, i.e., independent of the choice of arbitrary origin. This is indeed the case:

Lemma 6.3.2. *The twistor inner product between* $\mathsf{W}_\alpha = (\lambda_A, \mu^{A'})$ *and* $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ *satisfies*

$$\mathsf{W}_\alpha \mathsf{Z}^\alpha = \lambda_A(p)\omega^A(p) + \mu^{A'}(p)\pi_{A'}(p),$$

for all $p \in \mathbb{M}$

Proof. Note that this is equivalent to showing that at any point of $\mathbb{M}$, we have that

$$\lambda_A \omega^A + \mu^{A'}\pi_{A'} = \lambda_A \mathring{\omega}^A + \mathring{\mu}^{A'}\pi_{A'}.$$

since λ_A, $\pi_{A'}$ are constant. Substituting Equations (6.10), (6.14) on the left-hand-side indeed gives the right-hand-side when written out. □

Note that complex conjugation of a solution to the twistor equation yields a solution to the dual twistor equation and vice versa. We hence get a notion of complex conjugates of twistors defined as follows:

Definition 6.3.3. The ***complex conjugate*** of a twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ and of a dual twistor $\mathsf{W}_\alpha = (\lambda_A, \mu^{A'})$ are defined by

$$\overline{\mathsf{Z}^\alpha} := \overline{\mathsf{Z}}_\alpha := (\overline{\pi}_A, \overline{\omega}^{A'}),$$
$$\overline{\mathsf{W}_\alpha} := \overline{\mathsf{W}}^\alpha := (\overline{\mu}^A, \overline{\lambda}_{A'}),$$

respectively.

Inspired by Equation (6.12), we see a very simple way to define multi-valence twistors. We will only look at bitwistors here, twistors with two twistor indices, but generalisations to more twistor indices is straightforward.

Given two twistors

$$\mathsf{X}^\alpha = (\phi^A, \psi_{A'}), \quad \mathsf{Z}^\alpha = (\omega^A, \pi_{A'}),$$

we can define

$$\mathsf{S}^{\alpha\beta} = \mathsf{X}^\alpha \mathsf{Z}^\beta, \tag{6.16}$$

by observing that Equation (6.12) should yield

$$\mathsf{S}^{AB} = \phi^A\omega^B, \;\; \mathsf{S}^A{}_{B'} = \phi^A\pi_{B'}, \;\; \mathsf{S}_{A'}{}^B = \psi_{A'}\omega^B, \;\; \mathsf{S}_{A'B'} = \psi_{A'}\pi_{B'},$$

which we can write more concisely as

$$\mathsf{S}^{\alpha\beta} = \begin{pmatrix} \mathsf{S}^{AB} & \mathsf{S}^A{}_{B'} \\ \mathsf{S}_{A'}{}^{B'} & \mathsf{S}_{A'B'} \end{pmatrix} = \begin{pmatrix} \phi^A\omega^B & \phi^A\pi_{B'} \\ \psi_{A'}\omega^B & \psi_{A'}\pi_{B'} \end{pmatrix}. \tag{6.17}$$

Similarly, we can define valence $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ and valence $\left[\begin{smallmatrix}0\\2\end{smallmatrix}\right]$ twistors. This construction also works for twistors of general valence $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$, in which case the array of Equation (6.17) would need to be $(p+q)$-dimensional, with 2^{p+q} spinor parts. We also allow finite sums of terms of the form of Equation (6.16).

Similarly to tensors and spinors, the space of general valence $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ twistors is denoted by $\mathbb{T}^{\alpha\ldots\beta}_{\gamma\ldots\delta}$, following the notation of Definition 4.1.3.

Using the twistor inner product of Equation (6.15), we can define contraction of multi-valence tensor in the sense of Definition 4.1.4(3). For example, if we want to contract some twistor R_α with the twistor $\mathsf{S}^{\alpha\beta}$ of Equation (6.16), we can simply set

$$\mathsf{S}^{\alpha\beta}\mathsf{R}_\alpha = \mathsf{X}^\alpha \mathsf{R}_\alpha \mathsf{Z}^\beta$$

where $\mathsf{X}^\alpha \mathsf{R}_\alpha$ is given by Equation (6.15). The hence obtained twistor will thus be an element of $\mathbb{T}^\beta$.

To find the complex conjugate of a multi-valent twistor, we first write the twistor as a sum of outer products of valence 1 twistors and dual twistors, and perform complex conjugation on all these twistors separately.

It turns that, if we require some extra structure, some bitwistors will be uniquely defined by only one of their spinor parts. We will consider the example of a *symmetric twistor*. Other twistors can similarly be shown to be uniquely determined by some of their properties, such as the infinity twistors of Paragraph 7.2. To show this, Equations (6.18) and (6.19) will turn out to be essential.

Definition 6.3.4. Given a twistor $\mathsf{R}^{\alpha\ldots\beta}_{\gamma\ldots\delta} \in \mathbb{T}^{\alpha\ldots\beta}_{\gamma\ldots\delta}$, the ***primary part*** of $\mathsf{R}^{\alpha\ldots\beta}_{\gamma\ldots\delta}$ is defined to be the spinor part with all indices upright. Using the notation of Equation (6.17), the primary part of $\mathsf{R}^{\alpha\ldots\beta}_{\gamma\ldots\delta}$ is $\mathsf{R}^{A\ldots BC'\ldots D'}$.

We say that a 2-valent twistor $\mathsf{S}^{\alpha\beta}$ is ***symmetric*** if it satisfies

$$\mathsf{S}^{\alpha\beta} = \mathsf{S}^{\beta\alpha},$$

and ***skew*** if it satisfies

$$\mathsf{S}^{\alpha\beta} = -\mathsf{S}^{\beta\alpha}.$$

These definitions give rise to the following result:

Proposition 6.3.5. *The following properties hold:*

1. *A twistor $\mathsf{S}^{\alpha\beta}$ is symmetric if and only if its primary part is symmetric.*
2. *If $\mathsf{S}^{\alpha\beta}$ is symmetric, it is uniquely determined by its primary part.*

Proof. Note that if $\mathsf{S}^{\alpha\beta}$ is symmetric, we immediately get from Equation (6.17) that its primary part is symmetric. Conversely, suppose

$$\mathsf{S}^{\alpha\beta} = \begin{pmatrix} \phi^{AB} & \chi^{A}{}_{B'} \\ \psi_{A'}{}^{B} & \omega_{A'B'} \end{pmatrix},$$

and let V_α, W_α be two dual twistors. By Lemma 6.3.2, we have that

$$\mathsf{S}^{\alpha\beta}\mathsf{V}_\alpha\mathsf{W}_\beta \in \mathbb{C}.$$

We hence obtain a system of 4 equations, which by substituting the twistor equations associated with V_α, W_α gives the general solution

$$\begin{aligned} \phi^{AB} &= \mathring{\phi}^{AB} - ix^{AA'}\mathring{\psi}_{A'}{}^{B} - ix^{BB'}\mathring{\chi}^{A}{}_{B'} - ix^{BB'}x^{AA'}\mathring{\omega}_{A'B'}, \\ \chi^{A}{}_{B'} &= \mathring{\chi}^{A}{}_{B'} - ix^{AA'}\mathring{\omega}_{A'B'}, \\ \psi_{A'}{}^{B} &= \mathring{\psi}_{A'}{}^{B}, -ix^{BB'}\mathring{\omega}_{A'B'} \\ \omega_{A'B'} &= \mathring{\omega}_{A'B'}, \end{aligned} \tag{6.18}$$

which can be re-expressed in differential form by comparison with the twistor equation (6.5). In particular, the first three equations can be rewritten as

$$\begin{aligned}\nabla_{CC'}\phi^{AB} &= -i\varepsilon_C^{\ B}\chi^A_{\ C'} - i\varepsilon_C^{\ A}\psi_{C'}^{\ B},\\ \nabla_{CC'}\chi^A_{\ B'} &= -i\varepsilon_C^{\ B}\omega_{A'C'},\\ \nabla_{CC'}\psi_{A'}^{\ B} &= -i\varepsilon_C^{\ A}\omega_{C'B'}.\end{aligned} \tag{6.19}$$

All these equations hold generally, but if ϕ^{AB} is symmetric, the symmetry of (6.19)(1) requires that $\psi_{A'}^{\ B} = \chi^B_{\ A'}$, from which (6.19)(2-3) establish symmetry of $\omega_{A'B'}$, proving part 1.

Further note that (6.19)(1) then gives

$$\begin{aligned}\nabla_{CC'}\phi^{CB} &= -2i\psi_{C'}^{\ B} - i\chi^B_{\ C'},\\ \nabla_{CC'}\phi^{BC} &= -i\psi_{C'}^{\ B} - 2i\chi^B_{\ C'},\end{aligned}$$

whence we can uniquely determine $\chi^A_{\ B'}, \psi_{A'}^{\ B}$, and obtain $\omega_{A'B'}$ from (6.19)(3), so ϕ^{AB} uniquely determines $\mathsf{S}^{\alpha\beta}$, proving part 2. □

We finish this paragraph by defining a type of norm on twistor space, called the *helicity* of a twistor.

Definition 6.3.6. The ***helicity*** of a twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ is defined by

$$s := \frac{1}{2}\mathsf{Z}^\alpha\overline{\mathsf{Z}}_\alpha = \frac{1}{2}(\omega^A\overline{\pi}_A + \pi_{A'}\overline{\omega}^{A'}). \tag{6.20}$$

We say that Z^α is ***null*** if $s = 0$, ***right-handed*** if $s > 0$ and ***left-handed*** if $s < 0$.

We denote by $\mathbb{T}^0$, $\mathbb{T}^+$ and $\mathbb{T}^-$ the spaces of null, right-handed and left-handed twistors, respectively.

Lemma 6.3.7. *Let Z^α be a twistor. Then its helicity is conformally invariant.*

Proof. Let $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ and Ω be a conformal transformation. Suppose without loss of generality that ω^A is of conformal weight 0. By Equations (6.6), (6.9) we obtain

$$\widehat{\pi}_{A'} = \pi_{A'} + i\Upsilon_{AA'}\omega^A. \tag{6.21}$$

We hence obtain

$$\begin{aligned}\widehat{s} &= \frac{1}{2}\left(\widehat{\omega}^A\overline{\widehat{\pi}}_A + \widehat{\pi}_{A'}\overline{\widehat{\omega}}^{A'}\right)\\ &= \frac{1}{2}\left(\omega^A(\overline{\pi}_A - i\overline{\Upsilon}_{AA'}\overline{\omega}^{A'}) + (\pi_{A'} + i\Upsilon_{AA'}\omega^A)\overline{\omega}^{A'}\right)\\ &= \frac{1}{2}(\omega^A\overline{\pi}_A + \pi_{A'}\overline{\omega}^{A'}) = s\end{aligned}$$

using the reality of $\Upsilon_{AA'}$. So, since $s = \widehat{s}$, we find that the helicity is conformally invariant. □

In conclusion, just as twistor space $\mathbb{T}$ is unchanged under conformal rescalings, so are the spaces $\mathbb{T}^0$, $\mathbb{T}^+$ and $\mathbb{T}^-$.

6.4 Null twistors and space-time points

In this paragraph, we will make the first steps towards making an association between twistors and Minkowski space. In the previous chapters, we have taken the manifold M to be given, and derived the spin structure and twistor space associated with it from the geometric description of M. However, we can also take a different approach. If we see spinor space, or more specifically twistor space, as given, and more fundamental than space-time points, we should be able to derive the properties of the underlying space-time from its twistor description.

We will make a first step in this direction, which we shall expand upon in Paragraph 6.5. The construction here will only apply when we take $M = \mathbb{M}$. We will see some procedures for describing other space-times in terms of their twistor spaces in Chapter 7.

We consider twistor space $\mathbb{T} := \mathbb{T}^\alpha$ as in the previous paragraph. We suppose $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'}) \in \mathbb{T}$ is a null twistor, i.e., a twistor satisfying $\mathsf{Z}^\alpha \overline{\mathsf{Z}}_\alpha = 0$, following Definition 6.3.6.

In order to make the description in this paragraph work, we must impose some additional conditions on Z^α, which will be lifted in the next paragraph. As such, we suppose for now that $\pi_{A'} \neq 0$, and, without loss of generality, that $\overset{\circ}{\omega}{}^A$ and $\overline{\pi}^A = \varepsilon^{AB}\overline{\pi}_B$ are not proportional at the origin. If they are proportional, we can simply choose another point as origin, since Equation (6.10) yields

$$\omega^A \overline{\pi}_A = \overset{\circ}{\omega}{}^A \overline{\pi}_A - i x^{AA'} \overline{\pi}_A \pi_{A'},$$

so, using Proposition 5.1.4, we see that $\overset{\circ}{\omega}{}^A$ and $\overline{\pi}^A$ are not proportional when we choose the origin to be a point satisfying $x^{AA'}\overline{\pi}_A \pi_{A'} \neq 0$ with respect to O, which must exist since $\pi_{A'} \neq 0$.

We wish to look for the locus of points in $\mathbb{M}$ where $\omega^A = 0$. Note from Equation (6.10) we are hence looking for the *points* $x^{AA'}$ satisfying

$$\overset{\circ}{\omega}{}^A = i x^{AA'} \overline{\pi}_{A'}, \tag{6.22}$$

so we get one solution given by

$$x^a = (i \overset{\circ}{\overline{\omega}}{}^{B'} \pi_{B'})^{-1} \overset{\circ}{\omega}{}^A \overset{\circ}{\overline{\omega}}{}^{A'}.$$

Note this is well-defined, since Proposition 5.1.4 yields that $\overset{\circ}{\overline{\omega}}{}^{B'} \pi_{B'} \neq 0$ using our assumption of non-parallelity. In fact, we get that x^a is real, since Z^α being null yields, using Equation (6.20), that

$$\overline{\omega}^{B'} \pi_{B'} = -\omega^B \overline{\pi}_B,$$

from which it follows that $i\overset{\circ}{\overline{\omega}}{}^{B'} \pi_{B'}$ is real. We call Equation (6.22) the ***incidence relation***.

To find all the other real solutions to $\omega^A = 0$, we note that when $\tilde{x}^a$ and x^a are both solutions to (6.22), that

$$(x^a - \tilde{x}^a)\pi_{A'} = 0,$$

so by, Proposition 5.1.4, we must have $x^a - \tilde{x}^a \propto \pi^{A'}$, so, by reality, we get

$$x^a - \tilde{x}^a = \lambda \overline{\pi}^A \pi^{A'},$$

for some $\lambda \in \mathbb{R}$. As a result, we find the general solution in $\mathbb{M}$ for $\omega^A = 0$ to be

$$x^a = (i \overset{\circ}{\overline{\omega}}{}^{B'} \pi_{B'})^{-1} \overset{\circ}{\omega}{}^A \overset{\circ}{\overline{\omega}}{}^{A'} + \lambda \overline{\pi}^A \pi^{A'}. \tag{6.23}$$

Recalling the relation between spinors and points in Minkowski space of Paragraph 5.3, we see that the locus of points in $\mathbb{M}$ where $\omega^A = 0$ describes a *null line* in $\mathbb{M}$, whose direction is along the flagpole of $\overline{\pi}^A$ and which goes through a point on the flagpole of $\mathring{\omega}^A$. We denote the null line in $\mathbb{M}$ defined by Z^α by Z.

Conversely, any point in Minkowski space P satisfying $\omega^A(P) \nparallel \overline{\pi}^A$ lies in the null cone of some point on the null line described by Z^α. So, at an arbitrary point outside of the null line, the direction of the flagpole of ω^A is exactly such that it intersects the null line.

These results are graphically summarised in the Figure 6.1.

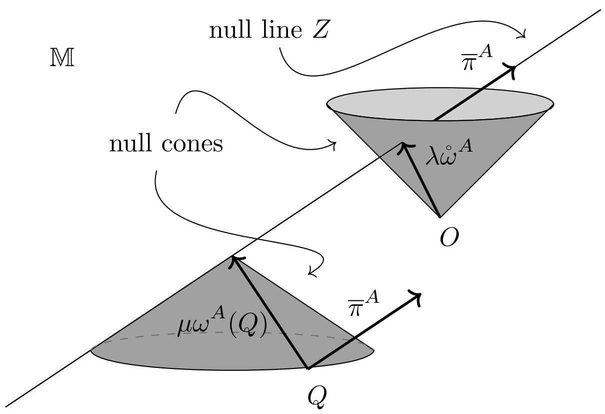


Figure 6.1: *Construction of the null line Z in $\mathbb{M}$ associated with the twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$. The line intersects the point of the flagpole of $\lambda\mathring{\omega}^A$ protruding from the origin, and is in the direction of $\overline{\pi}^A$. At any point Q where $\omega^A \nparallel \overline{\pi}^A$, this construction yields the same line. Here $\lambda = (i\overline{\mathring{\omega}}^{B'}\pi_{B'})^{-1}$ and $\mu = (i\overline{\omega}^{B'}(Q)\pi_{B'})^{-1}$.*

It turns out that our description of null lines in terms of twistors only depends on the twistor Z^α up to proportionality. This leads us to the following result:

Lemma 6.4.1. *Suppose $\mathsf{X}^\alpha, \mathsf{Z}^\alpha$ are twistors with associated null lines X and Z in $\mathbb{M}$. Then the following are equivalent:*

1. $X = Z$.
2. $\mathsf{Z}^\alpha = \lambda \mathsf{X}^\alpha$ *for some* $\lambda \in \mathbb{C}^\times$.

Proof. This follows immediately from Equation (6.23). □

As a result, we only need to look at the one-dimensional linear subspace generated by Z^α to construct the null line associated with Z^α. Since the projectivisation of a vector space is defined as the set of one-dimensional linear subspaces, we can alternatively look at the projectivisation $\mathbb{P}(\mathbb{T})$ of twistor space. We can alternatively define $\mathbb{P}(\mathbb{T})$ as the set $(\mathbb{T} \setminus \{0\})/\sim$, where the equivalence relation $\sim$ is defined by

$$\mathsf{Z}^\alpha \sim \mathsf{X}^\alpha \iff \mathsf{Z}^\alpha = \lambda \mathsf{X}^\alpha \text{ for some } \lambda \in \mathbb{C}^\times. \tag{6.24}$$

We will denote the equivalence class of Z^α by $[\mathsf{Z}^\alpha] = [(\omega^A, \pi_{A'})]$, or, given a spinor dyad, as the ratio $[\mathsf{Z}^\alpha] = [\mathsf{Z}^0 : \mathsf{Z}^1 : \mathsf{Z}^2 : \mathsf{Z}^3]$, in accordance with Equation (6.13).

We will further consider the notion of projective twistor space, alongside the situation where $\pi_{A'} = 0$ or Z^α is not null in the next paragraph.

We see from our construction that when two null lines in Minkowski space intersect, they define a unique point in $\mathbb{M}$. When two twistors define intersecting null lines, we say they are ***incident***. By intersect, we mean intersections in the projective sense, which is to say that two parallel lines can intersect at infinity.

Proposition 6.4.2. *Two twistors $\mathsf{X}^\alpha = (\phi^A, \psi_{A'})$, $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ with different associated null lines X and Z are incident if and only if*

$$\mathsf{X}^\alpha \overline{\mathsf{Z}}_\alpha = \mathsf{Z}^\alpha \overline{\mathsf{X}}_\alpha = 0.$$

Proof. We label the intersection point of X and Z by the constant field y^a. By Equation (6.22), we then have

$$\begin{aligned} iy^{AA'}\pi_{A'} &= \mathring{\omega}^A, \\ iy^{AA'}\psi_{A'} &= \mathring{\phi}^A, \end{aligned} \tag{6.25}$$

where, again, $\mathring{\phi}^A$ is defined as in Equation (6.11). Choosing a spinor dyad $\{o^A, \iota^A\}$, we find that this equation translates to

$$iy^{\boldsymbol{AA'}} \begin{pmatrix} \pi_{0'} & \psi_{0'} \\ \pi_{1'} & \psi_{1'} \end{pmatrix} = \begin{pmatrix} \omega^0 & \phi^0 \\ \omega^1 & \phi^1 \end{pmatrix},$$

which has a unique solution if and only if

$$\begin{pmatrix} \pi_{0'} & \psi_{0'} \\ \pi_{1'} & \psi_{1'} \end{pmatrix} \tag{6.26}$$

is invertible, i.e., if the directions of X and Z are not parallel.

We first assume that the matrix of (6.26) is not invertible. In this case we have $\pi_{A'} \parallel \psi_{A'}$, so that X and Z intersect at infinity. We rescale Z such that $\pi_{A'} = \psi_{A'}$, leaving Z unchanged. Note we then get

$$\begin{aligned} \mathsf{Z}^\alpha \overline{\mathsf{X}}_\alpha &= \omega^A \overline{\psi}_A + \pi_{A'} \overline{\phi}^{A'} \\ &= \omega^A \overline{\pi}_A + \psi_{A'} \overline{\phi}^{A'} \\ &= -\pi_{A'} \overline{\omega}^{A'} - \phi^A \overline{\psi}_A \\ &= -\psi_{A'} \overline{\omega}^{A'} - \phi^A \overline{\pi}_A = -\mathsf{X}^\alpha \overline{\mathsf{Z}}_\alpha, \end{aligned}$$

where the third equality uses Equation (6.20) and the nullity of X^α and Z^α. We hence find $\mathrm{Re}(\mathsf{Z}^\alpha \overline{\mathsf{X}}_\alpha) = 0$. But since

$$\overline{\mathsf{Z}^\alpha \overline{\mathsf{X}}_\alpha} = \mathsf{X}^\alpha \overline{\mathsf{Z}}_\alpha,$$

this means that $\mathsf{Z}^\alpha \overline{\mathsf{X}}_\alpha = 0$.

We now assume the matrix of Equation (6.26) is invertible, so that $\psi_{A'} \nparallel \pi_{A'}$. In this case, solving Equation (6.25) in the given spinor dyad and converting back to abstract indices we obtain

$$y^a = -\frac{i}{\pi_{B'}\psi^{B'}} (\omega^A \psi^{A'} - \phi^A \pi^{A'}), \tag{6.27}$$

which, using (6.20), is verified to be real if and only if $\mathsf{Z}^\alpha\overline{\mathsf{X}}_\alpha = 0$. This completes the proof. □

In this paragraph, we have derived a correspondence between null lines in Minkowski space and null twistors satisfying certain properties. We have also found conditions for when two such null lines intersect in twistorial terms, so we can associate a point of $\mathbb{M}$ with a pair of incident null twistors. The correspondence between $\mathbb{M}$ and $\mathbb{T}$ can be extended to be much more general, as we will explore in the next paragraph.

6.5 The twistor Klein correspondence

In projective geometry, there exists a well known correspondence between lines in a 3-dimensional projective space and points in an associated 5-dimensional projective space, called the *Klein correspondence* [18]. In this paragraph, we will look at a similar correspondence between *projective twistor space* $\mathbb{PT}$ and an extension of Minkowski space, called *compactified complexified Minkowski space*, denoted by $\mathbb{CM}^\bullet$. We can canonically extend the construction of the previous paragraph to twistors that are not necessarily null.

Throughout this paragraph, we will consider compactified Minkowski space $\mathbb{M}^\#$, which is topologically of the form $S^1 \times S^3$. The construction of compactified Minkowski space can be found in Appendix A.

The basic idea of the twistor Klein correspondence is that we can associate points in projective twistor space $\mathbb{PT}$ with complex null lines in complexified compactified Minkowski $\mathbb{CM}^\bullet$, a limited case of which we have already seen in the previous paragraph. Conversely, points in $\mathbb{CM}^\bullet$ can be associated with lines in $\mathbb{PT}$. This correspondence is not necessarily straightforward and uses much of the setup of the last few paragraphs. However, this is one of the most profound and beautiful results in twistor theory, connecting the theory of spinor fields with the inherently abstract world of projective geometry.

Definition 6.5.1. Let V be a $\mathbb{C}$-vector space and $D = \{d_0, \ldots, d_k\} \subseteq \mathbb{Z}$ be a finite set satisfying $1 \leq d_0 < \cdots < d_k < \dim V$.

We define the ***flag manifold*** of type D with respect to V to be the complex manifold given by

$$\mathbb{F}_{d_0,\cdots,d_k}(V) := \{(S_0, \ldots, S_k) : S_0 \subset \cdots \subset S_k,\ S_i \text{ is a } d_i\text{-dimensional linear subspace of } V\}.$$

We will mostly be concerned with the flag manifolds of twistor space, which we will define thusly:

Definition 6.5.2. We define the following flag manifolds of twistor space:

1. ***Projective twistor space*** is defined by
$$\mathbb{PT} := \mathbb{F}_1(\mathbb{T}).$$
2. ***Complexified compactified Minkowski space*** is defined by
$$\mathbb{CM}^\bullet := \mathbb{F}_2(\mathbb{T}).$$
3. ***The correspondence space*** is defined by
$$\mathbb{F}_{12} := \mathbb{F}_{1,2}(\mathbb{T}).$$

There then exists a so-called *double fibration* given by

$$
\begin{array}{ccc}
 & \mathbb{F}_{12} & \\
 {}^{\mu}\swarrow & & \searrow^{\nu} \\
\mathbb{PT} & & \mathbb{CM}^{\bullet}
\end{array}
\tag{6.28}
$$

where μ and ν are the canonical projection maps.

Note that the definition of $\mathbb{PT}$ corresponds with the usual definition of the projectivisation of a vector space, i.e., $\mathbb{PT} = \mathbb{P}(\mathbb{T})$, which we discussed in the previous paragraph. As such, we can label coordinates of $\mathbb{PT}$ by the ratio $[\mathsf{Z}^\alpha] = [\mathsf{Z}^0 : \mathsf{Z}^1 : \mathsf{Z}^2 : \mathsf{Z}^3]$, where Z^α is a twistor.

We can further define the spaces $\mathbb{PT}^0$, $\mathbb{PT}^+$ and $\mathbb{PT}^-$ as the spaces of projective twistors that are null, right-handed, and left-handed, by requiring

$$[\mathsf{Z}^\alpha] \in \mathbb{PT}^0 \iff \mathsf{Z}^\alpha \in \mathbb{T}^0,$$

which is well defined, since for some $\lambda \in \mathbb{C}^\times$ we have that

$$\mathsf{Z}^\alpha \overline{\mathsf{Z}}_\alpha = 0 \iff \lambda \mathsf{Z}^\alpha \overline{\lambda \mathsf{Z}}_\alpha = 0$$

A similar construction yields $\mathbb{PT}^+$ and $\mathbb{PT}^-$.

In this definition, $\mathbb{CM}^\bullet$ is the set of 2-planes in $\mathbb{T}$, which in projective geometry is also defined as the ***Grassmannian*** $\mathrm{Gr}_2(\mathbb{T})$. Note that elements of $\mathbb{CM}^\bullet$ are hence determined by two non-proportional twistors Z^α, X^α, with the plane being spanned by these two twistors giving an element in $\mathbb{CM}^\bullet$. We would like to somehow get a connection between $\mathbb{CM}^\bullet$, and the more familiar compactification of Minkowski space, which has been constructed in Appendix A.

Note that $\mathbb{M}$ is canonically contained in $\mathbb{CM}^\bullet$; if we consider two non-proportional incident null twistors X^α and Z^α whose associated null lines X and Z are non-parallel, we can identify the 2-plane

$$Y = \{\beta \mathsf{X}^\alpha + \gamma \mathsf{Z}^\alpha : \beta, \gamma \in \mathbb{C}\} \in \mathbb{CM}^\bullet$$

with the point $y^a \in \mathbb{M}$ given by Equation (6.27).

If we do allow parallelity of null lines and drop the condition that $\pi_{A'} \neq 0$ or $\psi_{A'} \neq 0$ for $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ and $\mathsf{X}^\alpha = (\phi^A, \psi_{A'})$, the null planes defined through two null twistors in $\mathbb{CM}^\bullet$ correspond to either points in $\mathbb{M}$, or points 'at infinity', so we get a correspondence between compactified Minkowski space $\mathbb{M}^\#$ and the subset of $\mathbb{CM}^\bullet$ defined by

$$\mathbb{M}^\bullet := \{Y \in \mathbb{CM}^\bullet : Y = \{\beta \mathsf{X}^\alpha + \gamma \mathsf{Z}^\alpha : \beta, \gamma \in \mathbb{C}\},\ \mathsf{X}^\alpha,\ \mathsf{Z}^\alpha \in \mathbb{T}^0\}.$$

For the construction of compactified Minkowski space $\mathbb{M}^\#$, see Appendix A.

If we now further drop the nullity of Z^α and X^α, we see, comparing with Equation (6.23), that an arbitrary twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ satisfying $\pi_{A'} \neq 0$ defines a null line in complexified Minkowski space $\mathbb{CM}$, which is the regular Minkowski space, where we allow the coordinates to be complex in addition to real. This line is given by

$$Z = \left\{ (i\overline{\mathring{\omega}}^{B'} \pi_{B'})^{-1} \mathring{\omega}^A \overline{\mathring{\omega}}^{A'} + \lambda \overline{\pi}^A \pi^{A'} : \lambda \in \mathbb{C} \right\}, \tag{6.29}$$

which hence also describes a line in $\mathbb{CM}^\bullet$, namely the line in $\mathbb{CM}^\bullet$ consisting of the *planes* in $\mathbb{T}$ containing Z^α. This description might be a little confusing, it is important to note here that planes in $\mathbb{T}$ correspond with points in $\mathbb{CM}^\bullet$, so lines in $\mathbb{T}$, defined by a single point in $\mathbb{PT}$, correspond with lines in $\mathbb{CM}^\bullet$.

Such a line is called an ***α-plane*** (The name *plane* is a little confusing, since we are talking about a complex line, but this line is isomorphic to a real plane). One can refer to Figure 6.2 for a visualisation of the correspondence between $\mathbb{PT}$ and $\mathbb{CM}^\bullet$ in terms of α-planes.

When $\pi_{A'} = 0$, Z^α defines a line at infinity in the compactification of complexified Minkowski $\mathbb{CM}^\#$, which are also called α-planes. So, given two arbitrary non-proportional twistors Z^α, X^α, these have a unique point of intersection in $\mathbb{CM}^\#$, which when their α-planes are non-parallel, is given by the finite point

$$y^{AA'} = -\frac{i}{\pi_{B'}\psi^{B'}}(\omega^A\psi^{A'} - \phi^A\pi^{A'}) \in \mathbb{CM} \tag{6.30}$$

or defines a point at infinity when the α-planes are parallel, or when at least one of the α-planes is at infinity.

Combining all these results, we see that there exists an identification between $\mathbb{CM}^\#$ and $\mathbb{CM}^\bullet$ given by

$$\{\beta\mathsf{X}^\alpha + \gamma\mathsf{Z}^\alpha\} \in \mathbb{CM}^\bullet \leftrightarrow (\text{Intersection of the associated } \alpha\text{-planes } X \text{ and } Z) \in \mathbb{CM}^\#$$

so that the definition of $\mathbb{CM}^\bullet$ indeed warrants its name of *complexified compactified Minkowski space*.

Similarly, we can regard incidence of dual twistors in $\mathbb{CM}^\bullet$. As it turns out, through a very similar calculation, a dual twistor W_α defines a complex line W in $\mathbb{CM}^\bullet$, called a ***β-plane***. We can alternatively regard W_α as the complex 2-plane W in $\mathbb{PT}$ defined by the points $[\mathsf{Z}^\alpha]$ satisfying

$$\mathsf{W}_\alpha\mathsf{Z}^\alpha = 0.$$

Combining all these results and referring back to the fibration of Diagram (6.28), we get the full correspondence:

Theorem 6.5.3 (**The twistor Klein correspondence**). *Let μ, ν be the projection maps of Diagram* (6.28). *Then the following statements hold:*

1. *The space $\mathbb{CM}^\bullet$ can be naturally identified with $\mathbb{CM}^\#$.*
2. *For all $[\mathsf{Z}^\alpha] \in \mathbb{PT}$ we have that*

$$\nu \circ \mu^{-1}([\mathsf{Z}^\alpha]) \subset \mathbb{CM}^\bullet$$

defines an α-plane in $\mathbb{CM}^\bullet$

3. *For all $y^{AA'} \in \mathbb{CM}^\bullet$, we have that*

$$\mu \circ \nu^{-1}(y^{AA'}) \subset \mathbb{PT}$$

defines a line in $\mathbb{PT}$.

4. *For all 2-planes* $\mathsf{W} \subset \mathbb{PT}$*, we have that*

$$\nu \circ \mu^{-1}(\mathsf{W}) \subset \mathbb{CM}^\bullet$$

defines a β*-plane in* $\mathbb{CM}^\bullet$*.*

Figure 6.2 now paints the full picture of the correspondence between $\mathbb{PT}$ and $\mathbb{CM}^\bullet$ (cf. [30, p. 312]).

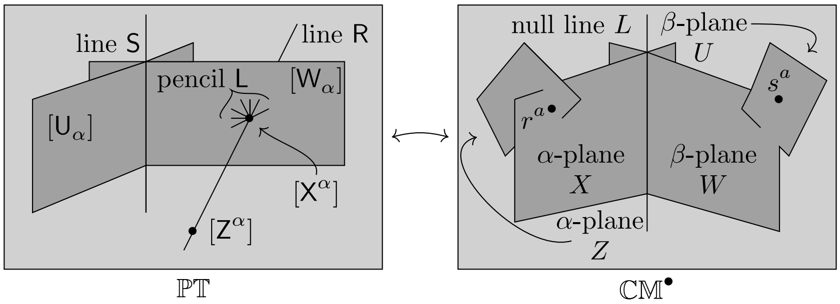


Figure 6.2: *Visualisation of the twistor Klein correspondence between* $\mathbb{PT}$ *and* $\mathbb{CM}^\bullet$*. Twistors correspond to* α*-planes, dual twistors correspond to* β*-planes, intersections of dual twistor planes and lines through* $\mathbb{PT}$ *correspond with points in* $\mathbb{CM}^\bullet$*. Additionally, a pencil of lines* L *on a dual twistor corresponds with a null line* L*, which we have not discussed here.*

We complete this chapter by using the findings above to define a new notion of coordinates on $\mathbb{CM}^\bullet$ in twistorial terms, which will prove to be very convenient in the next chapter.

Definition 6.5.4. A non-zero skew valence $\left[\begin{smallmatrix}2\\0\end{smallmatrix}\right]$ twistor $\mathsf{R}^{\alpha\beta}$ is called ***simple skew*** if it is of the form

$$\mathsf{R}^{\alpha\beta} = \mathsf{X}^\alpha \mathsf{Z}^\beta - \mathsf{Z}^\alpha \mathsf{X}^\beta,$$

where $\mathsf{X}^\alpha, \mathsf{Z}^\beta$ are non-proportional.

We define the ***totally skew twistor*** $\varepsilon_{\alpha\beta\gamma\delta}$ to be the unique skew twistor satisfying

$$\varepsilon_{\alpha\beta\gamma\delta} = \varepsilon_{[\alpha\beta\gamma\delta]}, \quad \varepsilon_{0123} = 1.$$

We can then define the ***dual*** of $\mathsf{R}^{\alpha\beta}$ to be given by

$$\mathsf{R}_{\alpha\beta} := \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}\mathsf{R}^{\gamma\delta}, \tag{6.31}$$

noting that this yields

$$\mathsf{R}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\mathsf{R}_{\gamma\delta}.$$

From our discussion of the Klein correspondence, it follows that a simple skew twistor, up to a non-zero complex scalar, uniquely defines a point in $\mathbb{CM}^\bullet$. We will see in the next chapter that this freedom in choosing a scaling will be very useful, and that the choice of scaling of the simple skew twistors representing points will serve an equivalent role to choosing a metric on a (conformally Minkowski) space-time.

Chapter 7

Twistor cosmology

In the previous chapter, we looked at the twistor Klein correspondence for Minkowski space. In this chapter, we will extend this construction to look at twistorial description of other conformally Minkowski space-times.

In Paragraph 7.1, we will extend the reasoning of Appendix A by considering a second way of constructing compactified Minkowski space, by embedding it into the light-cone of 6-dimensional flat space. Via a similar construction we obtain *(anti)-de Sitter space*, giving us all maximally symmetric solutions to the Einstein equations. We will connect this construction with the notion of twistors in Paragraph 7.2, where we define *infinity twistors*, which can be used for giving the structure of conformal infinity for these spaces, while also acting as the equivalent of a metric in the twistor formalism.

Finally, in Paragraph 7.3, we will look at some slightly less neat space-times, namely the *Friedmann-Robertson-Walker space-times*, which give simple cosmological solutions to the Einstein equations. We will explore how we can describe these models in twistorial terms, which at the same time shows how we can extend this to any conformally Minkowski space-time. Along the way, we will encounter the notion of *Bang* and *Crunch twistors*. In particular, we will look at a matter-dominated universe, and derive the relationship between twistor space and the metric in this case.

7.1 The Plücker embedding and (anti-)de Sitter space

In Appendix A, the standard construction for compactified Minkowski space $\mathbb{M}^{\#}$ through conformal maps is given. Although the embedding of Minkowski space $\mathbb{M}$ in the Einstein cylinder $\mathcal{E}$ gives a direct way of constructing $\mathbb{M}^{\#}$, we can consider an alternative construction, which also yields all the other maximally symmetric 4-dimensional signature $(+---)$ spaces. One can refer to Appendix A for the description of the conformal infinities adjoined to $\mathbb{M}$.

We will consider the 6-dimensional flat pseudo-Riemannian space $\mathbb{R}^{2,4}$ endowed with a constant orthonormal hexad (T, V, W, X, Y, Z) of signature $(2, 4)$. Hence, we have a metric of the form

$$\mathrm{d}s^2 = \mathrm{d}T^2 + \mathrm{d}V^2 - \mathrm{d}W^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2. \tag{7.1}$$

The null cone of $\mathbb{R}^{2,4}$ is given by the solution space of the homogeneous equation

$$T^2 + V^2 - W^2 - X^2 - Y^2 - Z^2 = 0. \tag{7.2}$$

Since this is a homogeneous expression, it defines a 4-dimensional projective algebraic subvariety $\mathbb{P}^\bullet$ of $\mathbb{P}^5 := \mathbb{P}(\mathbb{R}^{2,4})$. We can choose various conic sections of the null cone, which will give us an embedding of real 4-dimensional spaces into the 5-dimensional projective space $\mathbb{P}^5$.

We first consider the intersection of the null cone with the hyperplane defined by

$$V - W = 1, \tag{7.3}$$

in which case the induced metric on the intersection is of the form

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2,$$

which is the metric of Minkowski space. By identifying the Minkowski tetrad (t, x, y, z) with the coordinates (T, X, Y, Z) of $\mathbb{R}^{2,4}$ on the intersection, we get an identification between $\mathbb{M}$ and the intersection of the hyperspaces defined by Equations (7.2) and (7.3). Noting that the points of the intersection are defined by

$$2W + 2 = 2V = \frac{1}{2}(1 - T^2 + X^2 + Y^2 + Z^2), \tag{7.4}$$

it can easily be verified that any element $[T : V : W : X : Y : Z] \in \mathbb{P}^\bullet$ satisfying $V \neq W$ uniquely identifies a point of the intersection, and can hence be identified with $\mathbb{M}$. The inclusion of $\mathbb{M}$ into $\mathbb{P}^5$ hence obtained is called the ***Plücker embedding***.

The remaining points of $\mathbb{P}^\bullet$, where $V = W$, can be identified with the boundary $\mathscr{I}$ of $\mathbb{M}^\#$, as defined in Appendix A. Note that this clearly defines a compactification of $\mathbb{M}$, since $\mathbb{P}^\bullet$ is the projective closure of $\mathbb{M}$. (Refer to [9, Chs. 5 and 12], [16]).

In this description, the point $[T : V : W : X : Y : Z] = [0 : 1 : 1 : 0 : 0 : 0] \in \mathbb{P}^\bullet$ can be identified with the points $\imath^\pm, \imath^0$ of Equation (A.3).

In this description, it is somewhat easier to see that $\mathbb{M}^\#$ is conformally related to $\mathbb{M}$. On the null cone, we can rewrite the metric (7.1) as

$$\mathrm{d}s^2 = W^2(\mathrm{d}(T/W)^2 + \mathrm{d}(V/W)^2 - \mathrm{d}(X/W)^2 - \mathrm{d}(Y/W)^2 - \mathrm{d}(Y/W)^2),$$

where the part inside the bracket is independent of the representative of $[T : V : W : X : Y : Z]$. Then, using Equation (7.4), we can write this as a metric depending on 4 variables, which is conformally equivalent to the metric of Minkowski space, and hence conformally equivalent to the metric of Equation (A.2), which is to be expected, since our two constructions of compactified Minkowski should give the same result.

We can construct even more spaces using this procedure. First we consider the ***de Sitter space***. Instead of intersecting the null cone in $\mathbb{R}^{2,4}$ with the hyperplane defined by $V - W = 1$, we now consider the intersection with the hyperplane defined by

$$T = Q,$$

where $Q > 0$ is a real constant, so that our space is defined by the equation

$$V^2 - W^2 - X^2 - Y^2 - Z^2 = -Q^2,$$

also called the ***de Sitter sphere***. We will denote the manifold associated with the de Sitter space by $\mathrm{d}\mathcal{S}$. Now any point $[T : V : W : X : Y : Z] \in \mathbb{P}^{\bullet}$ such that $T \neq 0$ uniquely determines a point in $\mathrm{d}\mathcal{S}$, so the compactification of $\mathrm{d}\mathcal{S}$ is achieved by adjoining a single time-like point to $\mathrm{d}\mathcal{S}$. Since using similar reasoning as above shows that $\mathrm{d}\mathcal{S}$ is conformally equivalent to $\mathbb{P}^{\bullet}$, we find that $\mathrm{d}\mathcal{S}$ is conformally equivalent to $\mathbb{M}$, so in particular we see that de Sitter space is conformally flat.

The final canonical choice we can make for intersecting the null cone of $\mathbb{R}^{2,4}$, results in the ***anti-de Sitter space***, which we obtain by intersecting the null cone with the hyperplane satisfying

$$W = Q,$$

where $Q > 0$ is again a real constant. The in $\mathbb{R}^{2,4}$ resulting space is then defined by

$$T^2 + V^2 - X^2 - Y^2 - Z^2 = Q^2.$$

which is also called the ***anti-de Sitter sphere***. We will denote this space by $\mathcal{A}\mathrm{d}\mathcal{S}$. Now, similar to Minkowski and regular de Sitter space, we obtain $\mathbb{P}^{\bullet}$ by adjoining the points satisfying $W = 0$. Anti-de Sitter space is again conformally flat.

The three choices we made for slicing the null cones, are, up to real linear transformations of the coordinates, the only three. Hence, the spaces we obtain out of this construction are the only ones which can be embedded into a four dimensional projective algebraic variety describing the null cone of a higher-dimensional flat space. We hence say that these spaces are the only 4-dimensional signature $(+ - - -)$ spaces that are ***maximally symmetric***. The symmetry group of $\mathbb{R}^{2,4}$ turns out to have a deep connection to twistors, which we will further discuss at the start of Chapter 8.

The de Sitter space is particularly useful in that it can be used to construct cosmological models corresponding with arbitrary space-time curvature by choosing appropriate constant-time slicings. The hence obtained *Friedmann-Robinson-Walker space-times* are then conformally flat. We will consider these space-times in more detail in Paragraph 7.3.

Finally, note that our description does not change significantly if we allow the coordinates of $\mathbb{R}^{2,4}$ to take on *complex* values, instead of only real values. In this case, we have that

$$\mathbb{P}^{\bullet} \cong \mathbb{CM}^{\#},$$

and we can similarly talk about complexified (anti-)de Sitter space. We have now obtained a description of $\mathbb{CM}^{\#}$ which is consistent with our previous discussion of Paragraph 6.5.

7.2 Infinity twistors

We are now in a position to discuss the concept of an *infinity twistor*, which will be the analogue of the space-time metric in the twistor formulation.

Definition 7.2.1. Given a conformally flat space-time M, we define the ***infinity twistor*** as a simple skew 2-valence twistor $\mathsf{I}_{\alpha\beta}$, which given a simple skew twistor $\mathsf{R}^{\alpha\beta}$ representing a *finite point* satisfies

$$\mathsf{R}^{\alpha\beta}\mathsf{I}_{\alpha\beta} \neq 0,$$

and when $\mathsf{R}^{\alpha\beta}$ represents an *infinite point* satisfies

$$\mathsf{R}^{\alpha\beta}\mathsf{I}_{\alpha\beta} = 0.$$

Furthermore, we define the ***dual*** of $\mathsf{I}_{\alpha\beta}$ by

$$\mathsf{I}^{\alpha\beta} := \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}\mathsf{I}_{\gamma\delta}. \tag{7.5}$$

We further require infinity twistors to be ***real***, i.e.,

$$\overline{\mathsf{I}^{\alpha\beta}} = \mathsf{I}_{\alpha\beta}.$$

First, we will compute the infinity twistor for flat Minkowski space. If we consider two twistors $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$, $\mathsf{X}^\alpha = (\phi^A, \psi_{A'})$ and construct the simple skew twistor

$$\mathsf{R}^{\alpha\beta} = \mathsf{Z}^\alpha\mathsf{X}^\beta - \mathsf{X}^\alpha\mathsf{Z}^\beta, \tag{7.6}$$

we obtain, recalling Equations (6.30), (6.31), that

$$\mathsf{R}_{\alpha\beta} = \pi_{D'}\psi^{D'}\begin{pmatrix} \varepsilon_{AB} & iy_A{}^{B'} \\ -iy^{A'}{}_B & -\frac{1}{2}\varepsilon^{A'B'}y_{CC'}y^{CC'} \end{pmatrix}. \tag{7.7}$$

If we now consider the twistor

$$\mathsf{I}_{\alpha\beta} = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix} \tag{7.8}$$

(this is indeed a twistor, since it is of the form of (7.6)), we find that

$$\mathsf{R}^{\alpha\beta}\mathsf{I}_{\alpha\beta} = 2\pi_{A'}\psi^{A'},$$

which is 0 if and only if the α-planes Z and X are parallel or either of the α-planes are at infinity, hence $\mathsf{I}^{\alpha\beta}$ is an infinity twistor for Minkowski space.

We want to give a description of $\mathsf{I}^{\alpha\beta}$ and $\mathsf{R}^{\alpha\beta}$ in the coordinates of $\mathbb{R}^{2,4}$ discussed in the previous paragraph, so we can extend the discussion of infinity twistors to (anti-)de Sitter space.

We now choose some arbitrary spinor dyad $\{o^A, \iota^A\}$. Note that by skewness of $\mathsf{R}^{\alpha\beta}$, a (possibly complex) space-time point $y^{AA'}$ is uniquely determined by the ratio

$$\left[\mathsf{R}^{01} : \mathsf{R}^{02} : \mathsf{R}^{03} : \mathsf{R}^{12} : \mathsf{R}^{13} : \mathsf{R}^{23}\right] = \left[-\frac{1}{2}y_a y^a : -iy^{01'} : iy^{00'} : -iy^{11'} : iy^{10'} : 1\right] \in \mathbb{P}(\mathbb{C}^6) \tag{7.9}$$

(cf. Equations (6.13), (7.7)). Recall that, in accordance with (6.24), this equality holds when the coordinates of the ratio on the left-hand site are a constant non-zero multiple of the coordinates of the ratio on the right-hand side. We can pick the standard Minkowski coordinates

$$y^0 = t, \quad y^1 = x, \quad y^2 = y, \quad y^3 = z,$$

which corresponds to an element in $\mathbb{P}^\bullet$, which by Equation (7.4) is given by

$$[T : V : W : X : Y : Z] = \left[t : \frac{1}{2}(1 - y_a y^a) : \frac{1}{2}(-1 - y_a y^a) : x : y : z\right] \in \mathbb{P}^\bullet. \tag{7.10}$$

This allows us to express, up to proportionality, elements of the form $[T : V : W : X : Y : Z]$ as simple skew twistors.

We need to make an important distinction, so that we can use the infinity twistor as an analogue for the metric of our space-time. Recall that we defined simple skew twistors in Definition 6.5.4 to represent space-time points. However, these simple skew twistors only represented points in Minkowski space up to proportionality; if we scale a simple skew twistor a non-zero scalar, the point it represents is unchanged. Since we want to deduce the distance between two space-time points by the distance between simple skew twistors, we must distinguish which simple skew twistors we take to 'canonically' represent our space-time points. Our choice will be to take the simple skew twistors $\mathsf{R}^{\alpha\beta}$ which satisfy

$$\mathsf{R}^{\alpha\beta}\mathsf{I}_{\alpha\beta} = 2 \tag{7.11}$$

to be these distinguished twistors, since we will see that this choice results in a very simple form of the metric. As such, $\mathsf{I}_{\alpha\beta}$ can be seen as the object that defines the notion of distance in twistorial terms.

To make this more concrete, comparing Equations (7.9) and (7.10), and setting $X = x$, $Y = y$, $Z = z$, $T = t$, we obtain, using the spinor correspondence of (5.11), that in this particular dyad we have that

$$\begin{aligned} \mathsf{R}^{01} &= \frac{1}{2}(V+W), \quad \mathsf{R}^{02} = \frac{1}{2}\sqrt{2}(Y-iX), \quad \mathsf{R}^{03} = \frac{i}{2}\sqrt{2}(T+Z), \\ \mathsf{R}^{12} &= \frac{i}{2}\sqrt{2}(Z-T), \quad \mathsf{R}^{13} = \frac{1}{2}\sqrt{2}(Y+iX), \quad \mathsf{R}^{23} = V-W, \end{aligned} \tag{7.12}$$

which can be verified to satisfy

$$\mathsf{R}^{\boldsymbol{\alpha\beta}}\mathsf{I}_{\boldsymbol{\alpha\beta}} = 2$$

whenever $V = W + 1$ and

$$\mathsf{R}^{\boldsymbol{\alpha\beta}}\mathsf{I}_{\boldsymbol{\alpha\beta}} = 0$$

whenever $V = W$.

It can then easily be verified that the metric of Minkowski space is given by

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{d}\mathsf{R}^{\boldsymbol{\alpha\beta}}\mathrm{d}\mathsf{R}_{\boldsymbol{\alpha\beta}}, \tag{7.13}$$

where we only consider the skew twistors satisfying the scaling relation (7.11) to represent points of Minkowski space. Recall how we defined $\mathsf{R}_{\alpha\beta}$ in Definition 6.5.4. As promised, this form of the metric is very simple, and the information about the distance between two points is not so much contained in Equation (7.13), but in (7.11).

We now see, as we discussed at the end of Paragraph 6.5, that the infinity twistor $\mathsf{I}^{\alpha\beta}$ plays the role of the metric in twistor theory, since if we have some (conformally flat) space-time, described by an infinity twistor $\mathsf{I}^{\alpha\beta}$, the corresponding metric will always be of the form of Equation (7.13). In essence, the metric (7.13) is fixed, but the choice of the infinity twistor corresponding to the space actually encodes the information of the metric. If we choose not only to look at the distinguished twistors satisfying Equation (7.11), the metric would become

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{d}\left(\frac{2\mathsf{R}^{\boldsymbol{\alpha\beta}}}{\mathsf{R}^{\boldsymbol{\alpha\beta}}\mathsf{I}_{\alpha\beta}}\right)\mathrm{d}\left(\frac{2\mathsf{R}^{\boldsymbol{\alpha\beta}}}{\mathsf{R}^{\boldsymbol{\alpha\beta}}\mathsf{I}_{\alpha\beta}}\right),$$

which shows the role of the infinity twistor in the metric more explicitly. From this form, it is also clear why the distance between a finite point and a point at infinity, which satisfies $\mathsf{R}^{\alpha\beta}\mathsf{I}_{\alpha\beta} = 0$, is infinite.

We can easily extend this description to (anti-)de Sitter space. Recalling from Paragraph 7.1, we obtain de Sitter space by considering the points of $\mathbb{P}^\bullet$ satisfying $T = 0$ as the conformal boundary of d$\mathcal{S}$. If we choose the same spinor dyad $\{o^A, \iota^A\}$ as we did for Minkowski space, and again consider the parametrization (7.12), we find by inversion of the third and fourth equality in (7.12) that

$$T = \frac{i}{2}(\mathsf{R}^{12} - \mathsf{R}^{03}).$$

If we now set

$$\mathsf{I}_{\boldsymbol{\alpha\beta}} = \frac{i}{Q}\sqrt{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

it follows by direct verification that

$$\mathsf{R}^{\boldsymbol{\alpha\beta}}\mathsf{I}_{\boldsymbol{\alpha\beta}} = 2 \iff T = Q,$$
$$\mathsf{R}^{\boldsymbol{\alpha\beta}}\mathsf{I}_{\boldsymbol{\alpha\beta}} = 0 \iff T = 0,$$

where we take $\mathsf{R}^{\alpha\beta}$ to satisfy (7.12). (Note this twistor is not real, contrary to the requirements, see Eq. (7.15)).

Similarly, for anti-de Sitter space, this procedure leads to the infinity twistor given by

$$\mathsf{I}_{\boldsymbol{\alpha\beta}} = \frac{1}{Q}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}.$$

These infinity twistors are all given with respect to an arbitrary choice of spinor dyad, even though we would like the description to be coordinate free, as in all of our previous discussions. In fact, these infinity twistors can alternatively be uniquely identified by the relations

$$\mathsf{I}_{\alpha\beta}\mathsf{I}^{\alpha\beta} = 0, \quad \mathsf{I}_{\alpha\beta}\mathsf{I}^{\alpha\beta} = \frac{2}{Q^2}, \quad \mathsf{I}_{\alpha\beta}\mathsf{I}^{\alpha\beta} = -\frac{2}{Q^2}, \tag{7.14}$$

for Minkowski space $\mathbb{M}$, de Sitter space d$\mathcal{S}$ and anti-de Sitter space $\mathcal{A}$d$\mathcal{S}$, respectively. Uniqueness follows from the simple skewness and reality of the infinity twistors (cf. Def. 7.2.1), where the proof follows a similar procedure as the one used to prove Proposition 6.3.5.

From this it is clear why these three spaces are the only maximally symmetric flat spaces, since, by varying Q, they are the only three spaces with constant infinity twistors, so these are the only three spaces of signature $(+ - - -)$ that are entirely isotropic.

As it turns out, the de Sitter space is a solution to the Einstein vacuum equation

$$R_{ab} = \lambda g_{ab},$$

which we discussed in Paragraph 5.4, where the cosmological constant is given by

$$\lambda = \frac{3}{Q^2}$$

(see [14, p. 124]).

For a negative cosmological constant, the anti-de Sitter space is the solution to the vacuum equation where the cosmological constant is given by

$$\lambda = -\frac{3}{Q^2}.$$

As such, by Equation (7.14), we can characterise solutions to the Einstein vacuum equations by the infinity twistor

$$\mathsf{I}_{\alpha\beta}\mathsf{I}^{\beta\gamma} = -\frac{\lambda}{6}\delta_\alpha{}^\gamma,$$

or, equivalently

$$\mathsf{I}_{\alpha\beta} = \begin{pmatrix} \frac{\lambda}{6}\varepsilon_{AB} & 0 \\ 0 & \varepsilon^{A'B'} \end{pmatrix}. \tag{7.15}$$

in an entirely coordinate-free manner. This showcases the great efficiency of the twistor formalism for providing solutions to this particular form of the Einstein equations.

7.3 Twistors of Friedmann-Robertson-Walker space-times

In this paragraph, we consider some conformally Minkowski cosmological space-time models, called *Friedmann-Robertson-Walker space-times.* These are relatively simple models, describing the evolution of the universe, provided some information about the energy-densities of matter, radiation, and vacuum. We will follow and fill some of the holes of the discussion found in [30, Paragraph 9.5]. Along the way, we will have to do some tedious calculations, but the upshot is that we obtain a more elegant, twistorial, description of the FRW spaces.

The general form of the ***Friedmann-Robertson-Walker metric*** is given by (following [12, p. 387])

$$\mathrm{d}s^2 = \mathrm{d}u^2 - a^2(u)\left[\frac{\mathrm{d}r^2 + r^2\mathrm{d}\omega^2}{(1+\frac{1}{4}kr^2)^2}\right], \tag{7.16}$$

where

$$\mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2,$$

and $k = 1,\ 0,\ -1$ is a parameter denoting whether the underlying manifold M is said to be ***closed***, ***flat***, or ***open***. The topology of the part of the manifold represented in the brackets in Equation (7.16) is either a 3-sphere, Euclidean 3-space or Lobachevsky hyperbolic 3-space (see [8]) in the closed, flat, and open cases, respectively.

Here $a \in \mathfrak{T}$ is called the ***scale factor***, a function only dependent on the u-coordinate of the manifold, satisfying the ***Friedmann equations***, which follow directly from the Einstein equations:

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\lambda}{3}, \\ \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\lambda}{3}, \end{aligned} \tag{7.17}$$

(for a derivation, see [5, pp. 332–336]), where λ is the cosmological constant, G is the gravitational constant of (5.26), $\dot{a} = \partial_u a$, $p \in \mathfrak{T}$ is the pressure and $\rho \in \mathfrak{T}$ is the energy density, which is given by

$$\rho = \rho_{\text{crit}}\left(\Omega_v + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4}\right),$$

where ρ_{crit} is a real constant, and Ω_v, Ω_m and Ω_r denote the relative density of vacuum, mass, and radiation energy, respectively. All three of these are real constants, satisfying

$$\Omega_v + \Omega_m + \Omega_r = \Omega_t$$

where $\Omega_t > 1$ for $k = 1$, $\Omega_t = 1$ for $k = 0$ and $\Omega_t < 1$ for $k = -1$.

The FRW metrics are conformally equivalent to the metrics given by

$$\begin{aligned} \mathrm{d}s^2 &= a(\eta)^2 \left[\mathrm{d}\eta^2 - \mathrm{d}\nu^2 - \sin^2 \nu \mathrm{d}\omega^2\right], && (k = 1), \\ \mathrm{d}s^2 &= a(t)^2 \left[\mathrm{d}t^2 - \mathrm{d}r^2 - r^2 \mathrm{d}\omega^2\right], && (k = 0), \\ \mathrm{d}s^2 &= a(\kappa)^2 \left[\mathrm{d}\kappa^2 - \mathrm{d}\mu^2 - \sinh^2 \mu \mathrm{d}\omega^2\right], && (k = -1), \end{aligned} \tag{7.18}$$

effected by the conformal maps

$$\begin{aligned} \eta &= \int \frac{\mathrm{d}u}{a(u)}, & \nu &= 2\arctan\left(\frac{1}{2}r\right), && (k = 1), \\ t &= \int \frac{\mathrm{d}u}{a(u)}, & r &= r, && (k = 0), \\ \kappa &= \int \frac{\mathrm{d}u}{a(u)}, & \mu &= 2\arctan\left(\frac{1}{2}\mu\right), && (k = -1). \end{aligned}$$

Note the metric we hence obtain in the brackets for $k = 1$ is in fact the metric for the Einstein cylinder $\mathscr{E}$, a region of which we showed to be conformally Minkowski in Appendix A, which is given by the metric corresponding to $k = 0$. The third metric, corresponding to $k = -1$, is sometimes called the ***anti-Einstein cylinder***, denoted by $\mathscr{A}$.

Full conformal equivalence of the three metrics between the brackets of Equation (7.18) is given by the relations

$$\begin{aligned} t &= \frac{\sin\eta}{\cos\eta + \cos\nu} &&= \frac{\sinh\kappa}{\cosh\kappa + \cosh\mu}, \\ r &= \frac{\sin\nu}{\cos\eta + \cos\nu} &&= \frac{\sinh\mu}{\cosh\kappa + \cosh\mu}, \\ \tan\eta &= \frac{2t}{r^2 - t^2 + 1} &&= \frac{\sinh\kappa}{\cosh\mu}, \\ \tan\nu &= \frac{2r}{t^2 - r^2 + 1} &&= \frac{\sinh\mu}{\cosh\kappa}, \\ \tanh\kappa &= \frac{2t}{t^2 - r^2 + 1} &&= \frac{\sin\eta}{\cos\nu}, \\ \tanh\mu &= \frac{2r}{r^2 - t^2 + 1} &&= \frac{\sin\nu}{\cos\eta}, \end{aligned} \tag{7.19}$$

where the domains of validity are such that these relations remain well-defined.

We will use this long list of expressions in due course to calculate the analogues of the infinity twistors for some models corresponding to solutions to the Friedmann equations, but we must note that, contrary to the flat and (anti-)de Sitter models, we can also have *singularities* in FRW space-times.

If we look at the regions where $\mathscr{A}$ and $\mathbb{M}$ are conformal to $\mathcal{E}$, we see, following similar reasoning as the one that led to Figure A.2, that $\mathbb{M}$ is conformal to the region of $\mathcal{E}$ bounded by

$$\eta \pm \nu = \pi, \tag{7.20}$$

and $\mathscr{A}$ is conformal to the region bounded by

$$\eta \pm \nu = \frac{\pi}{2}. \tag{7.21}$$

Showcasing this in a similar way as in Figures A.1, A.2, we obtain Figure 7.1.

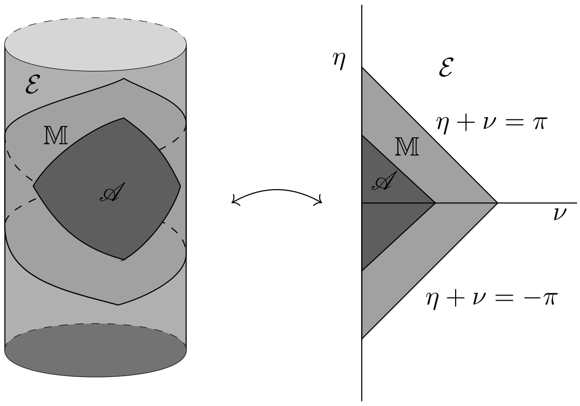


Figure 7.1: *On the left, a section of the Einstein cylinder is shown, with the regions conformal to $\mathbb{M}$ and $\mathscr{A}$ embedded in $\mathcal{E}$. On the right, a diagram similar to Figure A.1 is shown, where the regions conformal to $\mathbb{M}$ and $\mathscr{A}$ using the conformal coordinates η, ν of $\mathcal{E}$ is illustrated.*

So, if we have a universe with a constant scale factor, the infinity twistors will correspond to the conformal boundaries given by Equations (7.20), (7.21).

Before we can start finding solutions to different universe models, we may first consider how the coordinates for these FRW models correspond to the coordinates in $\mathbb{P}^5$ discussed in Paragraph 7.1. By straightforward verification, we find the relations

$$\begin{aligned} [T : V] &= [\sin\eta : \cos\eta], && (k = 1), \\ [T : V - W] &= [t : 1], && (k = 0), \\ [T : W] &= [\sin\kappa : -\cosh\kappa], && (k = -1). \end{aligned} \tag{7.22}$$

Since the FRW metrics of Equation (7.18) are (partially) conformal to Minkowski space, we consider the 'unphysical' Minkowski metric

$$\mathrm{d}\widehat{s}^2 = \mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2, \tag{7.23}$$

where the metrics of (7.18) are conformally related by

$$\mathrm{d}\widehat{s}^2 = \Omega^2 \mathrm{d}s^2,$$

for which Equations (7.18) and (7.19) give the conformal rescalings

$$\begin{aligned} &\Omega = ((\cos\eta + \cos\nu)a(\eta))^{-1}, && (k = 1) \\ &\Omega = a(t)^{-1}, && (k = 1) \\ &\Omega = ((\cosh\kappa + \cosh\mu)a(\kappa))^{-1}, && (k = 1) \end{aligned} \tag{7.24}$$

which, comparing to (7.22) and choosing the scaling of points in $\mathbb{P}^\bullet$ corresponding to (7.10), gives a description of the FRW space-times in terms of sections of $\mathbb{P}^\bullet$, although the sections of $\mathbb{P}^\bullet$ resulting in the FRW space-times are no longer hyperplanes.

The exact correspondence between the coordinates is then given by

$$\begin{aligned} \cos^2\eta &= \frac{V^2}{V^2+T^2}, \quad \cos^2\nu = \frac{W^2}{V^2+T^2}, \\ \cosh^2\kappa &= \frac{W^2}{W^2-T^2}, \quad \cosh^2\mu = \frac{W^2}{W^2-T^2}. \end{aligned} \tag{7.25}$$

Referring back to Equation (7.12), we can give a simple skew twistor description of the FRW space-times, where the relevant parameters are related by

$$T = \frac{i}{2}\sqrt{2}(\mathsf{R}^{12}-\mathsf{R}^{03}), \quad V = \mathsf{R}^{01}+\frac{1}{2}\mathsf{R}^{23}, \quad W = \mathsf{R}^{01}-\frac{1}{2}\mathsf{R}^{23}. \tag{7.26}$$

When we have a solution to the Friedmann Equations (7.17), the nature of the scale factor a might restrict the valid values for the time coordinates η, t, κ of the FRW models of Equation (7.18). The locus of points in the space time satisfying

$$a(u) = \infty$$

correspond to the conformal infinities we already discussed, whereas points where

$$a(u) = 0$$

correspond with ***singularities***. We accordingly call a past singularity of this type a ***Big Bang*** and a future singularity a ***Big Crunch***.

We set the ***Hubble constant*** to be

$$H_0 := \frac{\dot{a}(u_0)}{a(u_0)},$$

with u_0 corresponding to some fixed η_0, t_0 or κ_0, representing the present. We consider a *matter-dominated* universe, i.e., a universe where

$$\rho = \rho_{\text{crit}}\frac{\Omega_m}{a^3},$$

i.e., $\Omega_m = \Omega_t$. For now, we assume the cosmological constant λ to be 0. We assume the mass to come from a pressure-less cosmological dust, so we set $p = 0$ in (7.17)(1). Solving the second of the Friedmann Equations (7.17)(2), we obtain, by choosing appropriate integration constants, the following solutions for the scale factor:

$$\begin{aligned} a(\eta) &= \frac{\Omega_t}{2H_0(\Omega_t-1)^{2/3}}(1-\cos\eta), && (k=1) \\ a(t) &= \left(\frac{t}{t_0}\right)^{2/3}, && (k=0) \\ a(\kappa) &= \frac{\Omega_t}{2H_0(1-\Omega_t)^{2/3}}(\cosh\kappa-1), && (k=-1) \end{aligned} \tag{7.27}$$

where the freedom in the integration was used to ensure that we have Big Bangs at $\eta = \kappa = t = 0$. Note that the open and flat models expand indefinitely, and hence have two future conformal infinities. For the closed model, we see that there is maximum expansion at $\eta = \frac{\pi}{2}$ and a Big Crunch for $\eta = \pi$.

To specify a twistor to describe these singularities, we consider the concept of a ***Bang twistor*** $\mathsf{B}^{\alpha\beta}$, which is a skew 2-valent twistor which, when $\mathsf{R}^{\alpha\beta}$ is a twistor representing a point at the Big Bang, satisfies

$$\mathsf{B}^{\alpha\beta}\mathsf{B}_{\beta\gamma} = \delta_\gamma{}^\alpha, \quad \mathsf{B}_{\alpha\beta}\mathsf{R}^{\alpha\beta} = 0.$$

(The dual is defined in the obvious way, cf. Eq. (7.5)). Choosing a spinor dyad so that in $\mathbb{P}^\bullet$ the twistors corresponding to Minkowski coordinates have the form (7.12), we see that the Big Bang (and the Big Crunch of the closed model) correspond to $T = 0$ (cf. Eq. (7.25)), i.e., in this basis, the Bang twistor is of the form

$$\mathsf{B}_{\boldsymbol{\alpha\beta}} = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

In the closed model, we can also define a ***Crunch twistor*** $\mathsf{C}^{\alpha\beta}$, satisfying similar relations as the Bang twistor at the Big Crunch, but from our discussion it is clear that $\mathsf{B}^{\alpha\beta} = \mathsf{C}^{\alpha\beta}$.

In order to describe the conformal infinities of the FRW models, we need a slightly different approach to the one taken for Minkowski and (anti-)de Sitter space in Paragraph 7.2.

For $k = 1$, this consists of selecting some (not-necessarily real) simple skew 2-valence twistor $\mathsf{I}_{\alpha\beta}$. For $k = 0$, we can select a real simple skew 2-valence twistor $\mathsf{I}^{\alpha\beta}$ and for $k = -1$, we select two real simple skew twistors $\mathsf{I}^{\alpha\beta}$, $\mathsf{J}^{\alpha\beta}$.

For a general FRW solution where $\lambda = 0$, by comparison to Figure (7.1), we let these twistors describe the conformal infinities for $\mathbb{M}$ and $\mathscr{A}$, whereas for $\mathcal{E}$, these infinities correspond to 'virtual' infinities, which are realised in the matter-dominated solution we have been discussing. We require the normalisations

$$\begin{aligned} \mathsf{I}^{\alpha\beta}\bar{\mathsf{I}}_{\alpha\beta} &= 2, \quad (k = 1), \\ \mathsf{I}^{\alpha\beta}\mathsf{J}_{\alpha\beta} &= 2, \quad (k = -1), \end{aligned} \tag{7.28}$$

where for $k = 0$ the twistor $\mathsf{I}^{\alpha\beta}$ acts as a regular infinity twistor.

Comparing to Equations (7.27), (7.25), (7.26), we see that these conformal infinities must have a correspondence given by

$$\begin{aligned} T - iV &= \frac{1}{2}\sqrt{2}\ \mathsf{I}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, \quad T + iV = \frac{1}{2}\sqrt{2}\ \bar{\mathsf{I}}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, && (k = 1) \\ T + W &= \frac{1}{2}\sqrt{2}\ \mathsf{J}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, \quad T - W = \frac{1}{2}\sqrt{2}\ \mathsf{I}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, && (k = 0) \\ T &= \frac{1}{4}\sqrt{2}\ \mathsf{B}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, \quad V - W = \frac{1}{2}\mathsf{I}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, && (k = -1) \end{aligned} \tag{7.29}$$

where it is worth noting that in presence of a Big Bang, we have that

$$\mathsf{B}^{\alpha\beta} = \mathsf{I}^{\alpha\beta} + \bar{\mathsf{I}}^{\alpha\beta}\ (k = 1), \quad \mathsf{B}^{\alpha\beta} = \mathsf{I}^{\alpha\beta} + \mathsf{J}^{\alpha\beta},\ (k = -1).$$

Clearly, for $k = 0$, the infinity twistor provided in (7.8) suffices. For the readers interest, the exact forms of the other infinity twistors in this dyad are given by

$$\mathsf{I}_{\boldsymbol{\alpha\beta}} = \frac{i}{4}\sqrt{2}\begin{pmatrix} 0 & -2 & 0 & -\sqrt{2} \\ 2 & 0 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & 0 & -1 \\ \sqrt{2} & 0 & 1 & 0 \end{pmatrix}, \qquad (k=1),$$

$$\mathsf{I}_{\boldsymbol{\alpha\beta}} = \frac{1}{4}\sqrt{2}\begin{pmatrix} 0 & -2 & 0 & -\sqrt{2} \\ 2 & 0 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & 0 & -1 \\ \sqrt{2} & 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{J}_{\boldsymbol{\alpha\beta}} = \frac{1}{4}\sqrt{2}\begin{pmatrix} 0 & 2 & 0 & -\sqrt{2} \\ -2 & 0 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & 0 & 1 \\ \sqrt{2} & 0 & -1 & 0 \end{pmatrix}, \qquad (k=-1),$$

but this form has no particular meaning, since it is based upon our (somewhat arbitrary) choice of dyad so that the coordinates of the slicing of $\mathbb{P}^\bullet$ are in terms of the Plücker embedding of $\mathbb{M}$ of Paragraph 7.1. In fact, all necessary information about these twistors is contained in Equation (7.28), which alongside the simple-skewness and reality conditions uniquely define these infinity twistors.

This construction works for arbitrary solutions to the FRW space-times and will describe the conformal infinities independent of which solution for the Friedmann equations is found. However, one piece of information is lost, namely the relation between the metric and the infinity twistors. We conclude this paragraph by briefly discussing how to recover this component of the theory.

Instead of a skew twistor, the scaling is now done by a function I, satisfying

$$I(\mathsf{R}^{\alpha\beta}) = 2$$

for twistors corresponding to 'physical points' (cf. Eq. (7.11)), which for Minkowski, de Sitter, and anti-de Sitter space corresponds to the linear map $\mathsf{I}_{\alpha\beta}$. Again, we take Minkowski space as the 'unphysical' hatted space, as in (7.23), so that

$$\mathsf{I}_{\alpha\beta} = \widehat{I}.$$

Recalling Equation (7.15) and the conformal scaling of the epsilon spinors (cf. Eq. (6.1)), we must get that

$$I = \Omega\widehat{I}. \tag{7.30}$$

Substituting Equation (7.29) into (7.24) via Equation (7.25) then gives the result. For $k = 0$, this is about the most we can do, but for $k = \pm 1$, the form of the resulting equation is quite elegant, so we shall outline their derivations.

We introduce the variables

$$c^2 := \frac{i}{2}\sqrt{2}\ \mathsf{I}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, \quad d^2 := -\frac{i}{2}\sqrt{2}\ \bar{\mathsf{I}}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, \quad (k=1)$$
$$c^2 := \frac{1}{2}\sqrt{2}\ \mathsf{I}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, \quad d^2 := -\frac{1}{2}\sqrt{2}\ \mathsf{J}_{\alpha\beta}\mathsf{R}^{\alpha\beta}, \quad (k=-1)$$

where $\mathsf{I}^{\alpha\beta}, \mathsf{J}^{\alpha\beta}$ are now the infinity twistors of (7.29). Note by Equations (7.29), (7.22) we obtain

$$\eta = -i\log(c/d), \qquad (k=1)$$
$$\kappa = \log(c/d), \qquad (k=-1)$$

which, when substituted in (7.24), gives us, via (7.30) that

$$I(\mathsf{R}^{\alpha\beta}) = \frac{2cd}{a\left(i^{\frac{1}{2}(3k+3)}\log(c/d)\right)},$$

where a is the scale factor. This function now takes the role of the metric; similar as in Paragraph 7.1, the metric of the space-time is now given in twistorial terms by

$$\mathrm{d}s^2 = \frac{1}{2}\mathrm{d}\mathsf{R}_{\alpha\beta}\mathsf{R}^{\alpha\beta},$$

where, as in the previous paragraph, the simple skew twistors $\mathsf{R}^{\alpha\beta}$ are the distinguished twistors representing real points satisfying

$$I(\mathsf{R}^{\alpha\beta}) = 2. \tag{7.31}$$

If we solely consider the matter-dominated universe, as we have done previously, we obtain by substituting in (7.27) that

$$I(\mathsf{R}^{\alpha\beta}) = -\frac{8kH_0(\Omega_t - 1)^{2/3}}{\Omega_t}\left(\frac{cd}{c-d}\right)^2. \tag{7.32}$$

where this equality now holds for arbitrary simple skew twistors, contrary to just the twistors $\mathsf{R}^{\alpha\beta}$ satisfying (7.31).

The algorithm for explicitly computing Bang and Crunch twistors for the matter-dominated model extends to any FRW solution, following similar reasoning as seen here.

Note that the twistorial description of the Friedmann-Robertson-Walker space-times is much more concise than the regular form; we can describe the entirety of the space using the infinity twistors, Bang and Crunch twistors and the scaling function, which we derived for the matter dominated universe as being given by Equation (7.32). All these twistors and functions have relatively simple coordinate-free forms, which stands in stark contrast with the usual descriptions, where one always relies on local coordinate systems and transformations are usually given using hyperbolic or trigonometric expressions.

Chapter 8

Twistors in curved space

In this chapter, we will discuss some of the more interesting aspects of twistor theory in curved spaces. In the previous chapter, we saw how to describe a number of spaces that were conformally equivalent to Minkowski space in twistorial terms. However, for general spaces, the translation to a twistor description is not so obvious.

A naive way to approach this problem would be to consider *local twistors*. Orthochronous transformations of the symmetry group $\mathrm{SO}^+(2,4)$ of the space $\mathbb{R}^{2,4}$ considered in Paragraph 7.1 keep the linear structures of the light cone invariant and hence induce *conformal* transformations of Minkowski space. As it turns out, the group $\mathrm{SO}^+(2,4)$ is isomorphic to the path-connected component of the conformal group $\mathrm{C}(1,3)$ of Minkowski space.

The associated spin group $\mathrm{Spin}^+(2,4)$ is isomorphic to $\mathrm{SU}(2,2)$, which is the group leaving the twistor inner product of Equation (6.15) invariant. In fact, the spin representation of Definition 3.5.1 acts on an 8-dimensional vector space, and the ± 1 eigenspaces of the chirality operator of Proposition 3.5.2 turn out to be isomorphic to the 4-dimensional twistor space $\mathbb{T}^\alpha$ and dual twistor space $\mathbb{T}_\alpha$.

Then, on an arbitrary signature $(+---)$ spin manifold, there exists a bundle structure, called the ***tractor bundle*** (see [2]), so that a section of this bundle assigns to any point $p \in M$ a constant twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ in a smooth way, which under conformal rescaling effected by a map $\Omega \in \mathfrak{T}$, in accordance with (6.21), is mapped to

$$\widehat{\mathsf{Z}}^\alpha = (\omega^A, \pi_{A'} + i\Upsilon_{AA'}\omega^A).$$

An appropriate form of ***local twistor transport*** can be defined on the resulting twistor space, uniquely determined by the Levi-Civita covariant derivative (see [30, p. 134]).

However, there are a number of problems with this approach. First of all, the resulting twistor transport is not integrable. Furthermore, this twistorial description assumes we already have some manifold with certain curvature properties, and tries to derive a twistor space from this. However, we prefer to take the approach where the twistor description is treated as 'given', from which we can then derive the properties of the underlying space-time. Also, the locality of this description is a major blockade, the power of the twistor description in Chapters 6 and 7 was mostly based upon the global nature of twistors.

In this chapter, we will explore some alternative approaches, some of which turn out to work quite generally. However, even these descriptions can only deal with fields that are either self-dual or anti-self-dual, and the resolution to this obstruction is still being investigated as of the time of writing.

In this chapter, we will take a less rigorous approach and only sketch the ideas behind the procedures used, since an entirely thorough treatment of these methods would take us too far astray.

8.1 Sheaf cohomology and the Penrose transform

In this paragraph, we shall explore the *Penrose transform*, which gives a description of certain spinor fields in terms of contour integrals of holomorphic functions on twistor space. The ideas presented here are mainly based on [30, Paragraph 6.10]. The construction here is only for flat Minkowski space $\mathbb{M}$, but can be generalised to arbitrary conformally Minkowski spaces.

Definition 8.1.1. A spinor $\phi_{ABC\ldots D} \in \mathfrak{S}_{ABC\ldots D}$ on Minkowski space $\mathbb{M}$ is called a ***massless free field*** if it satisfies

$$\phi_{ABC\ldots D} = \phi_{(ABC\ldots D)},$$
$$\nabla^{AA'}\phi_{ABC\ldots D} = 0.$$

We similarly have massless free fields of the form $\phi_{A'\ldots D'} \in \mathfrak{S}_{A'\ldots D'}$.

A ***massless scalar field*** $\phi \in \mathfrak{S}$ satisfies the ***wave equation***

$$\nabla_{AA'}\nabla^{AA'}\phi = 0.$$

We denote the number of indices of ϕ by the integer n, where we take primed indices to correspond to positive values of n and unprimed indices to correspond to negative values. We say that n is the ***helicity*** of the massless free field.

Now let $r = r^a \in \mathbb{M}$ be an arbitrary point in $\mathbb{M}$ and f a homogeneous holomorphic function that is analytic on a region of $\mathbb{T}$. We will call such a function a ***twistor wave function***. We then consider the operator, defined on twistors *incident* with r^a (see Eq. (6.22)), given by

$$\rho_r : \mathsf{Z}^\alpha \mapsto (ir^{AA'}\pi_{A'}, \pi_{A'}),$$

i.e., we evaluate Z^α at the origin. Since the origin is arbitrary, we can alternatively regard twistor wave functions as being holomorphic functions from $\mathbb{C}^4$ to $\mathbb{C}$ and regard ρ_r as a function

$$\rho_r : f(\mathsf{Z}^\alpha) \mapsto f(ir^{AA'}\pi_{A'}, \pi_{A'}).$$

We consider the line $\mathsf{R} \subset \mathbb{PT}$ associated to $r^a \in \mathbb{M}$. We take a twistor wave function f, homogeneous of degree $-n-2$. Then for $n \geq 0$, we set

$$\phi_{A'\ldots D'}(r) = \oint_\Gamma \pi_{A'}\ldots\pi_{D'}\rho_r f(\mathsf{Z}^\gamma)\pi_{E'}\mathrm{d}\pi^{E'} \tag{8.1}$$

and for $n < 0$, we set

$$\phi_{A\ldots D}(r) = \oint_\Gamma \frac{\partial}{\partial\omega^A}\cdots\frac{\partial}{\partial\omega^D}\rho_r f(\mathsf{Z}^\gamma)\pi_{E'}\mathrm{d}\pi^{E'}. \tag{8.2}$$

Here Γ is a closed contour on the projective line R in $\mathbb{PT}$. Note that the integrals are well defined, since by the homogeneity degree of f, the homogeneity of the integrand is 0, so that the integrand is well defined on $\mathbb{PT}$, instead of just on $\mathbb{T}$. It can easily be seen that the expression on the right-hand side of Equations (8.1), (8.2) are symmetric.

The expressions on the right-hand side are indeed massless free fields; for $n > 0$, we have by

$$\frac{\partial}{\partial r^z}\rho_r f(\mathsf{Z}^\alpha) = \frac{\partial}{\partial r^z} f(ir^{ZZ'}\pi_{Z'}, \pi_{Z'}) = i\pi_{Z'}\rho_r\frac{\partial f}{\partial\omega^Z}$$

that

$$\frac{\partial}{\partial r^{ZZ'}}\phi_{A'\dots D'}(r) = i\oint_\Gamma \pi_{A'}\dots\pi_{D'}\pi_{Z'}\rho_r\frac{\partial f}{\partial\omega^Z}\pi_{E'}\mathrm{d}\pi^{E'},$$

which, after raising the Z' index and performing (A', C')-contraction yields

$$\nabla_Z{}^{A'}\phi_{A'\dots D'}(r) = 0,$$

so that $\phi_{\dots}$ is a massless free field. The cases for $n \leq 0$ follow similarly.

In fact, it can be shown that *any* massless free field can be obtained by such an integral over a contour in twistor space. This transformation between massless free fields and twistor wave functions is called the ***Penrose transform***.

In order for the integrals of Equations (8.1), (8.2) to give non-zero solutions, the corresponding twistor wave functions need to have singularities in appropriate regions of the line R. Since this is a 1-dimensional complex projective line, being diffeomorphic to the Riemann sphere S^2, we take the contour Γ to be over this Riemann sphere in an appropriate sense. We further assume that the function $f(\mathsf{Z}^\alpha)$ can be extended holomorphically to a region of twistors incident with a region of complexified Minkowski space, defined by

$$\mathbb{CM}^+ := \{x^a - iy^a \in \mathbb{CM} : x^a, y^a \in \mathbb{M},\ y^a \text{ future time-like}\},$$

also called the ***forward tube***. It can be shown using Equation (6.27) that these points correspond to lines lying entirely in the closure of the right-handed projective twistor space $\overline{\mathbb{PT}^+}$. Figure 8.1 shows how we can visualise the singular region S of f and how we can see a contour over a complex line R as a contour over the Riemann sphere.

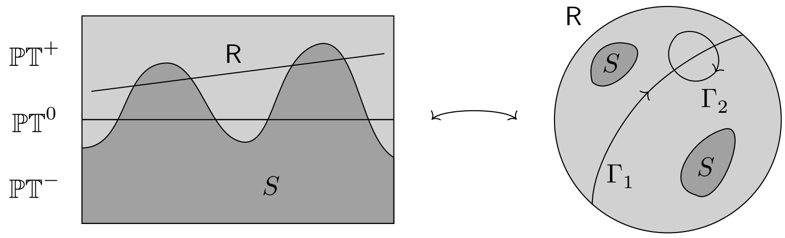


Figure 8.1: *On the left, the domain of singularity S of a twistor wave function f is shown in $\mathbb{PT}$. A line in projective twistor space R through $\overline{\mathbb{PT}^+}$ intersects the region of singularity. On the right, R is portrayed by a Riemann sphere, and we integrate over closed contours Γ_1, Γ_2, avoiding the singular regions. Note that the contour Γ_2 can be continuously deformed to a point, giving a trivial field, whereas the integral over Γ_1 should yield a non-zero massless free field.*

Note there is considerable freedom in Γ and f, leaving the resulting massless free field unchanged. Firstly, Γ can be continuously deformed on the sphere R, as long as no singular regions are crossed.

Secondly, we can add functions h to f whose singular regions are such that the contour Γ can be contracted to a point when integrating over h. To find what the 'truly' different twistor wave functions are, we must invoke the language of *Čech cohomology.* The definitions used here are slightly non-standard, but suffice for our present purposes.

We consider the complex manifold $\overline{\mathbb{PT}^+}$ and the ***sheaf*** $\mathcal{O}(n)$ of ***germs of twisted homogeneous holomorphic functions of degree n***. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a locally finite cover of $\overline{\mathbb{PT}^+}$. We consider

$$F = \{f_1, f_2, \ldots, f_{12}, f_{21}, f_{13}, \ldots, f_{123}, \ldots\},$$

a collection of elements assigned to each of the open sets U_i and their non-empty finite intersections $U_i \cap \cdots \cap U_j$. Here $f_{i\ldots j}$ is a holomorphic homogeneous twistor wave function of degree n defined on $U_i \cap \cdots \cap U_j$, satisfying

$$f_{i\ldots j} = f_{[i\ldots j]}.$$

We define the ***k-cochains*** associated to this F to be the sequences of elements of F having k lexicographically ordered increasing indices. Hence, we have

$$\begin{aligned}
&\text{0-cochain } \alpha = (f_1, f_2, f_3, \ldots),\\
&\text{1-cochain } \beta = (f_{12}, f_{13}, f_{23}, \ldots),\\
&\text{2-cochain } \gamma = (f_{123}, f_{124}, f_{134}, \ldots),\\
&\qquad\qquad\ddots
\end{aligned}$$

We can index elements of a k-cochain by $\zeta_{ij\ldots l}$, where $ij\ldots l$ number k indices and $i < j < \cdots < l$. In the above, we then have that $\zeta_{ij\ldots l} = f_{ij\ldots l}$.

We then define a ***coboundary operator*** ∂ from k-cochains to $(k+1)$-cochains defined by

$$\begin{aligned}
\partial\alpha &= (f_2 - f_1, f_3 - f_1, f_3 - f_2, \ldots),\\
\partial\beta &= (f_{12} - f_{13} + f_{23}, f_{12} - f_{14} + f_{24}, \ldots),\\
\partial\gamma &= (f_{124} - f_{123} + f_{234} - f_{134}, \ldots),\\
&\ \vdots\\
(\partial\sigma)_{ij\ldots l} &= \sum_{n\in\{i,j,\ldots,l\}} (-1)^{n+1}\sigma_{ij\ldots\widehat{n}\ldots l}.
\end{aligned}$$

(here $\widehat{n}$ denotes the omission of the index n). One can then easily verify that $\partial^2 = 0$. Note that $(\partial\alpha)_{12} = f_2 - f_1$ is holomorphic on $U_1 \cap U_2$, since f_2 and f_1 are holomorphic when restricted to $U_1 \cap U_2$, respectively. Consequently, since we can restrict holomorphic functions in this way, we see that the coboundary operator is well-defined.

We say that a k-cochain ζ is a ***cocycle*** when

$$\partial\zeta = 0,$$

and we say it is a ***coboundary*** when there exists some $(k-1)$-cocycle η such that

$$\partial\eta = \zeta.$$

Note that the sets of k-cocycles and k-coboundaries have an abelian group structure. We denote these groups by $\mathcal{Z}^k(\mathcal{U}, \mathcal{O}(n))$ and $\mathcal{B}^k(\mathcal{U}, \mathcal{O}(n))$, respectively. Note that

$$\mathcal{B}^k(\mathcal{U}, \mathcal{O}(n)) \subset \mathcal{Z}^k(\mathcal{U}, \mathcal{O}(n)).$$

We are now in a position to define the Čech cohomology groups:

Definition 8.1.2. Let $\mathcal{U}$ a locally finite cover of $\overline{\mathbb{PT}^+}$. We define the kth ***Čech cohomology group*** subject to $\mathcal{U}$ to be the group

$$\check{\mathrm{H}}^k_{\mathcal{U}}(\overline{\mathbb{PT}^+}, \mathcal{O}(n)) := \mathcal{Z}^k(\mathcal{U}, \mathcal{O}(n))/\mathcal{B}^k(\mathcal{U}, \mathcal{O}(n)). \tag{8.3}$$

Then the general kth ***Čech cohomology group*** is defined by taking the direct limit of refinements of covers $\mathcal{U}$ of $\overline{\mathbb{PT}^+}$, i.e.,

$$\check{\mathrm{H}}^k(\overline{\mathbb{PT}^+}, \mathcal{O}(n)) = \lim_{\mathcal{U}} \check{\mathrm{H}}^k_{\mathcal{U}}(\mathcal{U}, \mathcal{O}(n)).$$

(For details on the direct limit, refer to [13, p. 224]).

We can now regard elements of $\check{\mathrm{H}}^1\left(\overline{\mathbb{PT}^+}, \mathcal{O}(n)\right)$ as twistor wave functions. To illustrate this, we consider a covering of $\overline{\mathbb{PT}^+}$ with 2 open sets U_1, U_2, such that any projective line R through $\overline{\mathbb{PT}^+}$ meets $U_1 \cap U_2$, and we consider a twistor wave function f of degree n that is holomorphic on $U_1 \cap U_2$.

As a result, f defines a 1-cochain $\beta = (f)$, which satisfies

$$\partial\beta = 0 \implies \beta \in \mathcal{Z}^1(\mathcal{U}, \mathcal{S}).$$

Note that the freedom we described in choosing f to represent a massless free field, is equivalent to adding a coboundary $(h_1 - h_2)$ to f, where h_1, h_2 are holomorphic homogeneous functions of degree n defined on U_1, U_2. But this is exactly the equivalence relation we divided out in Equation (8.3).

For an arbitrary (locally) finite cover $\mathcal{U} = \{U_i\}_i$, we can similarly define a twistor wave function that is homogeneous of degree n and holomorphic on the (non-empty) intersections $U_i \cap U_j$, where we take a branched contour through these intersections. The cocycle condition is then equivalent to allowing us to 'glue' together the functions in triple intersections, so we can deform the contour in these regions. (Recall that the cocycle condition on the intersection $U_1 \cap U_2 \cap U_3$ states that

$$f_{12} - f_{13} + f_{23} = 0,$$

which can be seen to mean that the function f can be extended holomorphically to the entire region where covers intersect one another). This situation is sketched for covers with 2 and 3 sets in Figure 8.2.

The machinery introduced here can be used to prove our previous claim that any massless free field can be obtained via a Penrose transform.

Theorem 8.1.3. *There is an equivalence between the group* $\check{\mathrm{H}}^1\left(\overline{\mathbb{PT}^+}, \mathcal{O}(-n-2)\right)$ *and the set of massless free fields of helicity* n *holomorphic on a region of* $\mathbb{CM}^+$.

Proof. For a full proof, see [15, Ch. 10]. □

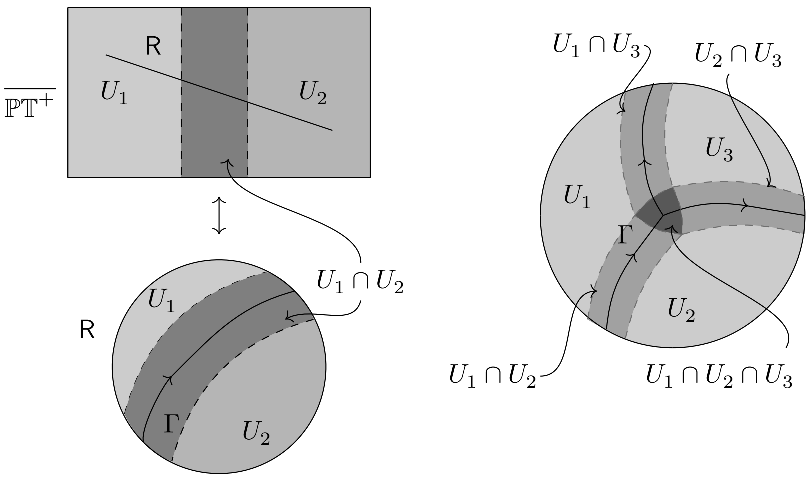


Figure 8.2: *Left:* $\overline{\mathbb{PT}^+}$ *is covered by two open sets* U_1, U_2, *such that a twistor wave function* f *is holomorphic on the intersection of the open sets. We then integrate over a closed contour* Γ *on the Riemann sphere* R, *with* Γ *contained entirely in* $U_1 \cap U_2$.
Right: For a cover with 3 sets, we integrate over a branched contour integral, where the vertices of the contour can be freely moved in $U_1 \cap U_2 \cap U_3$ *and the contour can be deformed homotopically in the other intersections.*

8.2 The non-linear graviton and the googly problem

In the previous paragraph, we considered how linear fields on Minkowski space can be expressed in twistorial terms. In this paragraph, we will give a procedure to describe non-linear anti-self-dual gravitational fields using a non-linear analogue of projective twistor space, denoted by $\mathbb{P}\mathcal{T}$. This construction is called the *non-linear graviton construction.* We will only give an outline of the construction, since we have not developed the tools to treat this construction rigorously. We follow some of the reasoning of [26], [32, Chapter 9] and [15, Chapter 12].

As always, we consider a 4-dimensional manifold M with a metric of signature $(1,3)$ satisfying the conditions of Geroch's theorem. In order to give a twistorial description of this manifold, we want, as in Chapter 6, to get a collection of α-planes at every point, which will play the role of $\mathbb{P}\mathcal{T}$. Suppose $p \in M$ is an arbitrary point. Then two tangents of the same α-plane v^a, w^a at p must be of the form

$$v^a = \lambda^a \pi^{A'}, \quad w^a = \mu^A \pi^{A'} \tag{8.4}$$

for some (not-necessarily constant) spinor $\pi^{A'}$ (see Equation (6.29)). Locally, we want the α-plane to span the tangent space to a sub-manifold N of M. Frobenius' theorem (see [21, p. 496]), then asserts that

$$v^a \nabla_a w^b - w^a \nabla_a v^b = \alpha v^b + \beta w^b$$

for constants α, β. After substitution of Equation (8.4), we obtain

$$\pi^{A'} \nabla_a \pi_{B'} = \zeta_A \pi_{B'} \tag{8.5}$$

for some ζ_A. After transvecting with $\pi^{B'} \pi^{C'} \nabla_A{}^{C'}$ and using the symmetries of the Riemann curvature tensor and the decomposition in spinor terms (cf. Prop. 5.4.3, Eq. (5.28)), this can be seen to be rewritten as

$$\overline{\Psi}_{A'B'C'D'}\pi^{A'}\pi^{B'}\pi^{C'}\pi^{D'} = 0,$$

which by symmetry of $\overline{\Psi}_{A'B'C'D'}$ implies

$$\overline{\Psi}_{A'B'C'D'} = 0,$$

i.e., M must have an anti-self-dual Weyl tensor (see Eq. (5.29)), which means that M is patchwise conformally flat by Proposition 6.1.3. For real spaces with metrics of signature $(++++)$ or $(++--)$ the condition of anti-self-dual Weyl tensor is less restrictive, so one can encode a larger family of spaces using twistor theory. However, these signatures have other less appealing properties, in particular, Geroch's theorem (see Thm. 4.3.2) no longer holds.

Now we want to construct the non-linear twistor space $\mathbb{P}\mathcal{T}$ in the case of an anti-self-dual Weyl tensor. Considering a point $p \in M$, then given some spinor field $\xi_{A'}$, the equations

$$\pi^{A'}(p) = \xi^{A'}(p), \quad \pi^{A'}\nabla_{AA'}\pi_{B'} = 0 \tag{8.6}$$

have solutions by virtue of the vanishing of $\overline{\Psi}_{A'B'C'D'}$ (note the similarity with Equation (8.5)). The solutions $\pi^{A'}$ to this equation defines a 2-space

$$\mathsf{P} = \{\lambda^A\pi^{A'} : \lambda^A \in \mathfrak{S}^A\}$$

satisfying Frobenius' theorem. The tangent vectors at p are then of the form $\lambda^A\pi^{A'}$. The α-planes we thus constructed vary smoothly over M. By considering the dimensionality, we hence obtain a 3-complex dimensional space $\mathbb{P}\mathcal{T}$ of α-planes, which we will call the ***curved projective twistor space*** of M.

In general, $\mathbb{P}\mathcal{T}$ need not be Hausdorff, so it is not even necessarily a manifold. Since it can be shown that Equation (8.6) is conformally invariant, the construction of $\mathbb{P}\mathcal{T}$ is conformally invariant, so it is unsurprising that the spaces in Chapter 7 could all be described in terms of the regular, flat twistor space $\mathbb{PT}$. In general, we would like to impose some extra structure on $\mathbb{P}\mathcal{T}$.

Definition 8.2.1. We say a 4-dimensional complex manifold M is ***civilised*** if its curved projective twistor space $\mathbb{P}\mathcal{T}$ is homeomorphic to $\mathbb{PT} \setminus \left\{[(\omega^A, \pi_{A'})] : [\pi_{A'}] = [0:0]\right\}$.

We deliberately exclude the twistors representing α-planes at infinity. Note that any conformally flat manifold M by definition consists of open sets which are conformally equivalent to $\mathbb{M}$, so any such manifold can be covered by civilised subsets.

Conversely, we want any space $\mathbb{P}\mathcal{T}$ sufficiently resembling a curved projective twistor space to define, up to conformal rescaling, a unique anti-self-dual space-time manifold. The sufficient condition turns out to be that there exists a 4-parameter family of compact holomorphic curves, each having normal bundle $N = \mathcal{O}(1) + \mathcal{O}(1)$. We have not discussed what this entails, one can refer to [32, pp. 436–439] for further details. This gives rise to the following theorem:

Theorem 8.2.2 (Penrose). *There is a one-to-one correspondence between*

1. *the equivalence classes of civilised manifolds (M, g_{ab}) with anti-self-dual Weyl tensor modulo conformal rescalings.*
2. *3-dimensional complex manifolds $\mathbb{P}\mathcal{T}$ homeomorphic to $\mathbb{PT} \setminus \left\{[(\omega^A, \pi_{A'})] : [\pi_{A'}] = [0:0]\right\}$ such that there exists a 4-parameter family of compact holomorphic curves, each having normal bundle $N = \mathcal{O}(1) + \mathcal{O}(1)$.*

Further structure can be given to $\mathbb{P}\mathcal{T}$ to ensure that the corresponding manifold M is an Einstein manifold, i.e., that it gives a solution to the Einstein vacuum equation (cf. Def. (5.4.4))

$$R_{ab} = \lambda g_{ab}.$$

The precise nature of these conditions and a proof showing equivalence can be found in [15, pp. 107-109].

Throughout this paragraph, we could alternatively have taken the approach to consider β-planes in M. This would have turned out to give a description for manifolds with *self-dual* Weyl tensor, in a description mirroring that of the dual twistor space. A similar construction exists for describing (anti-)self-dual solutions to the Yang-Mills field equations, called the *Ward construction* [31].

If we are interested in incorporating the entirety of general relativity, we need some way to also describe solutions to the Einstein equations that do not have anti-self-dual Weyl tensor. Thus far, there has been no satisfying solution to this problem, which has now stood for over 40 years. In the literature, this problem is called the ***googly problem***, and it is considered one of the biggest hurdles for twistor theory. (Googly is a term in cricket, where a ball is thrown in a left-handed manner, but results in a right-handed spinning ball).

8.3 Quantisation and palatial twistor theory

In this paragraph, we will briefly outline a potential solution to the googly problem. For this, the twistor space first needs to be quantized in an appropriate way. For our present purposes, we assume $\hbar = c = 1$. As in the previous two paragraphs, the reasoning here is far from complete. The purpose here is giving an idea of how quantisation may be introduced in the twistor formalism, and how this might lead to a description of general curved spaces in twistorial terms.

In Chapter 6, we saw that a twistor $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$ satisfying

$$s = \frac{1}{2}\mathsf{Z}^\alpha \overline{\mathsf{Z}}_\alpha = 0$$

described a null line in $\mathbb{M}$. As such, we can regard Z^α as representing a *photon* in Minkowski space. Contrary to how we constructed α-planes in complexified compactified Minkowski space, we will now consider a construction where general non-null twistors can be considered as massless particles possessing spin.

Given $\mathsf{Z}^\alpha = (\omega^A, \pi_{A'})$, we define the two real tensors

$$p^a = \overline{\pi}^A \pi^{A'}, \quad M^{ab} = i\omega^{(A}\overline{\pi}^{B)}\varepsilon^{A'B'} - i\overline{\omega}^{(A'}\pi^{B')}\varepsilon^{AB}, \tag{8.7}$$

satisfying, by Equation (6.10), the identities

$$p^a \equiv p^a(O), \qquad M^{ab} - x^b p^a + x^a p^b \equiv M^{ab}(O),$$

where $O \in \mathbb{M}$ is the origin. Note these are precisely the transformation properties of momentum and angular momentum. Massless helicity $\tilde{s}$ particles further satisfy (by [24, p. 11])

$$^*M_{ab}p^b = \tilde{s}p_a, \tag{8.8}$$

which implies (cf. Eq. (8.7))

$$\frac{1}{2}(\omega^B \overline{\pi}_B + \overline{\omega}^{B'} \pi_{B'}) \overline{\pi}_A \pi_{A'} = \tilde{s} p_a,$$

which, comparing to Equation (6.20) yields

$$\tilde{s} = s,$$

so that Z^α describes a helicity s massless particles, with (angular) momentum given by (8.7).

We want to use this description to consider a twistor Z^α and its complex conjugate $\overline{\mathsf{Z}}_\alpha$ as non-commuting operators acting on twistor wave functions. We know, from quantum mechanics, the canonical commutation relation

$$p_a x^b - x^b p_a = i \delta_a{}^b,$$

which, upon substitution of (8.7) and (6.10) suggests the form for the twistor commutation relation given by

$$\mathsf{Z}^\alpha \overline{\mathsf{Z}}_\beta - \overline{\mathsf{Z}}_\beta \mathsf{Z}^\alpha = \delta_\alpha{}^\beta. \tag{8.9}$$

Furthermore, Equation (8.8) can be rewritten as

$$s = \frac{1}{4}(\mathsf{Z}^\alpha \overline{\mathsf{Z}}_\alpha + \overline{\mathsf{Z}}_\alpha \mathsf{Z}^\alpha).$$

We suppose that the twistor wave functions these operators act on must be holomorphic in Z^α. Then the obvious choice for the operator Z^α is given by

$$\mathsf{Z}^\alpha : f \mapsto \mathsf{Z}^\alpha f,$$

and inspired by the quantum mechanical form for the momentum operator

$$p_a : f \mapsto -i \frac{\partial f}{\partial x^a},$$

we make the choice

$$\overline{\mathsf{Z}}_\alpha : f \mapsto -\frac{\partial f}{\partial \mathsf{Z}^\alpha},$$

noting that with this choice Equation (8.9) holds. We can see this choice of operators as the axiom defining the quantisation of twistor theory, just as in quantum mechanics the choice for momentum operator is an axiom defining the theory.

Since we obtain from Equation (8.9) that

$$\mathsf{Z}^\alpha \overline{\mathsf{Z}}_\alpha - \overline{\mathsf{Z}}^\alpha \mathsf{Z}^\alpha = 4,$$

we can alternatively write the helicity operator as

$$s = -\frac{1}{2}\left(\mathsf{Z}^\alpha \frac{\partial}{\partial \mathsf{Z}^\alpha} + 2\right),$$

from which we see that the eigenstates of s are given by homogeneous holomorphic functions, where an eigenvalue q corresponds with a function homogeneous of degree $2q - 2$.

In these cases, the eigenstates of the helicity operator are precisely the twistor wave functions discussed in Paragraph 8.1, so if we are only interested in the massless free fields resulting from twistor wave functions, we can regard the operators Z^α, $\overline{\mathsf{Z}}_\alpha$ as acting on the representatives of the group $\check{\mathrm{H}}^1\left(\overline{\mathbb{PT}^+}, \mathcal{O}(2q-2)\right)$. In this way, elements of $\check{\mathrm{H}}^1\left(\overline{\mathbb{PT}^+}, \mathcal{O}(2q-2)\right)$ are considered to be 1-particle wave functions, whereas elements of $\check{\mathrm{H}}^k\left(\overline{\mathbb{PT}^+}, \mathcal{O}(2q-2)\right)$ describe systems of k particles.

The idea behind ***palatial twistor theory*** is now to see the twistor structure of Minkowski space as being modelled by the non-commutative algebra $\mathbb{A}$, being generated by Z^α and $\overline{\mathsf{Z}}_\alpha$, while also, in some sense, allowing infinite combinations of these elements. Then just as we got a curved twistor space $\mathbb{P}\mathcal{T}$ for anti-self-dual space-times, we should now, in some way, deform the flat algebra $\mathbb{A}$ to some curved algebra $\mathcal{A}$ describing a general curved space.

In this way, one could see the quantisation of twistor space as a requirement for describing curved spaces, potentially leading to a satisfying theory of quantum gravity. These ideas have not yet been worked out fully, although some progress is being made with regards to how the deformation of this algebra should work [22, 27].

Chapter 9

Outlook

In this thesis, we constructed Penrose's twistor theory from scratch. Along the way, we found how twistor space is able to encode the physical properties of Minkowski space. This programme could be extended to give a description of (anti-)de Sitter space using infinity twistors. In Chapter 7, we also saw a procedure to describe Friedmann-Robertson-Walker space-times in twistorial terms, allowing us to describe asymptotics, such as the Big Bang, in very simple bitwistor forms. We explicitly calculated the scaling of the metric for a matter dominated cosmological model, leading to the simple form of Equation (7.32).

In Chapter 8, we discussed some non-linear aspects of twistor theory, such as the powerful Penrose transform, which is able to describe massless free fields in terms of elements of Čech cohomology groups. The non-linear graviton construction allowed us to also encode the geometry of an anti-self-dual solution to the Einstein equations in curved twistor space $\mathbb{P}\mathcal{T}$.

Throughout this thesis, we have mostly followed Penrose's original mission statement of twistor theory, namely to provide a theory of quantum gravity. We have largely ignored many other developments in twistor theory, such as the use of the Penrose transform in the analysis of non-linear differential equations, or the extension to twistor string theory, which is used extensively for determining scattering amplitudes in particle collision problems.

There is also some mathematical interest in twistor theory for spaces of arbitrary dimension and signature. Another particularly interesting generalisation is that of ambitwistor theory, which, contrary to our current twistor formalism, can describe non-anti-self-dual space-times. However, this theory loses much of the mathematical simplicity of the twistor formalism discussed here.

It is interesting to see whether the twistor quantisation and palatial twistor theory briefly touched upon in Paragraph 8.3 may lead to a more elegant description of general curved spaces. The introduction of quantisation is very desirable, since it may hint at a connection between non-trivial space-time curvature and quantum mechanical effects, suggesting that the introduction of quantum mechanics in twistor theory is a necessary step to describe arbitrary solutions to the Einstein equations. Twistor theory might then imply that the cause of the collapse of the wave function stems from gravitational effects. Although research is still ongoing, it may be of interest to see whether twistors may eventually lead to a convincing theory of quantum gravity.

Appendix A

Compactification of flat space

In this appendix, we will consider a way of constructing compactified Minkowski space $\mathbb{M}^{\#}$, which will be used in the twistor Klein correspondence of Chapter 6. The approach taken here will be to use conformal rescalings of the metric of Minkowski space, after which we adjoin points representing infinity. In this way, we can consider Minkowski space as being contained in the *Einstein cylinder* $\mathcal{E}$. We will follow the construction of [14, Paragraph 5.1].

Given a manifold M with pseudo-Riemannian g_{ab}, a conformal map Ω gives rise to a new metric tensor

$$\widehat{g}_{ab} = \Omega^2 g_{ab}.$$

The notions of conformal rescalings are discussed in Paragraph 6.1.

In physics, the metric of Minkowski space is often written

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2, \tag{A.1}$$

where (t^a, x^a, y^a, z^a) is a Minkowski tetrad and $\mathrm{d}t$, $\mathrm{d}x$, ... are the associated basis vectors for $\mathfrak{T}_a$. By re-parametrising Minkowski space in spherical coordinates, this metric becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

at a point parametrised by (t, r, θ, ϕ). Using this form of the metric will be more appropriate for performing direct computations than the abstract index approach of Paragraph 4.1.

Further introducing ***Eddington-Finkelstein***-coordinates, defined by $u = t - r$ and $v = t + r$, we obtain the metric

$$\mathrm{d}s^2 = \mathrm{d}u\mathrm{d}v - \frac{1}{4}(v-u)^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2),$$

where Minkowski space corresponds with the region $v - u \geq 0$, $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$.

We can now introduce the conformal map

$$\Omega = 2(1+u^2)^{-\frac{1}{2}}(1+v^2)^{-\frac{1}{2}},$$

which yields the conformally scaled metric

$$\mathrm{d}\widehat{s}^2 = ((1+u^2)(1+v^2))^{-1}(4\mathrm{d}u\mathrm{d}v - (v-u)^2)(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2).$$

Note this metric can be analytically extended to the region $u, v \to \pm\infty$, which can be made more precise by making the substitutions

$$p = \arctan u, \quad q = \arctan v,$$

with associated metric

$$d\hat{s}^2 = 4dpdq - \sin^2(p-q)(d\theta^2 + \sin^2\theta d\phi^2), \tag{A.2}$$

where Minkowski space now corresponds to the region $0 \leq q - p \leq \pi,\ \ p,\, q \in (-\pi/2, \pi/2)$.

This metric can be extended to the regions where p or q equal $\pm\pi/2$. Hence, we can adjoin points representing these regions to Minkowski space, thus obtaining a set of *points at infinity*, which we will denote by $\mathscr{I}$. We then define $\overline{\mathbb{M}} = \mathbb{M} \cup \mathscr{I}$. The 2-spheres

$$\begin{aligned} \imath^0 &= \left\{(p,q,\theta,\phi) \in \overline{\mathbb{M}} : (p,q) = \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)\right\}, \\ \imath^+ &= \left\{(p,q,\theta,\phi) \in \overline{\mathbb{M}} : (p,q) = \left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)\right\}, \\ \imath^- &= \left\{(p,q,\theta,\phi) \in \overline{\mathbb{M}} : (p,q) = \left(-\frac{1}{2}\pi, -\frac{1}{2}\pi\right)\right\}, \end{aligned} \tag{A.3}$$

are called ***spatial infinity***, ***future-temporal infinity*** and ***past-temporal infinity***, respectively. Furthermore, ***future*** and ***past null infinity*** $\mathscr{I}^+$ and $\mathscr{I}^-$ are defined as the portions of $\mathscr{I}$ where $q = \frac{1}{2}\pi$ and $p = -\frac{1}{2}\pi$, respectively.

Figure A.1 shows the structure of $\overline{\mathbb{M}}$ using this construction.

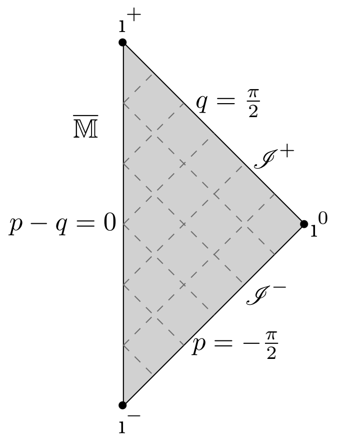


Figure A.1: *The region conformal to $\mathbb{M}$ using the coordinates p, q. The dotted lines denote regions of constant p and q.*

An alternative way to regard $\mathbb{M}$ is as embedded in the ***Einstein cylinder*** $\mathscr{E}$, which we obtain by considering the entire strip $0 \leq p - q \leq \pi$. When we then consider the coordinate transformations

$$\tau = p + q, \quad \rho = q - p,$$

giving the metric

$$d\hat{s}^2 = d\tau^2 - [d\rho^2 + \sin^2\rho(d\theta^2 + \sin^2\theta d\phi^2)],$$

so the Einstein cylinder has the structure of a product of an infinite line, representing time, with a 3-sphere, representing space. We can embed the conformally rescaled Minkowski space in this cylinder, being a patch of the cylinder wrapping around itself, meeting itself in $\imath^0$, as illustrated in Figure A.2.

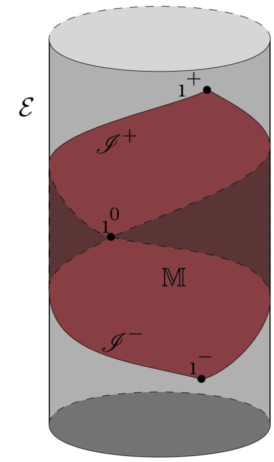


Figure A.2: *A section of the Einstein cylinder $\mathcal{E}$. Every point on the cylinder shown here corresponds to a 2-sphere. The region conformal to $\mathbb{M}$ is shown in red, wrapping around the cylinder, and meeting itself in the point $\imath^0$ at the back.*

We see that the topology of the Einstein cylinder is $\mathbb{R} \times S^3$. The topology of the region of Minkowski space conformal to the Einstein cylinder, which lies open in $\overline{\mathbb{M}}$, is $\mathbb{R}^4$. However, when adding the boundaries $\mathscr{I}^+$ and $\mathscr{I}^-$ and the points $\imath^{\pm}, \imath^0$, which are topologically not 4-dimensional, we see that we must make some further identification to ensure that $\overline{\mathbb{M}}$ has the structure of a manifold.

Inspired by the fact that we can take the patches $p, q \in (0, \pi)$ instead of $p, q \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$, the natural choice would be to appropriately identify $\mathscr{I}^+$ and $\mathscr{I}^-$ with each other, and consider $\imath^{\pm}, \imath^0$ as a single point. This also makes sense physically; any null line with infinite future $A \in \mathscr{I}^-$ will go through a future point on $B \in \mathscr{I}^+$ uniquely determined by A, so we get a canonical identification, which turns $\overline{\mathbb{M}}$ into a 4-dimensional manifold called ***compactified Minkowski space***, denoted by $\mathbb{M}^{\#}$, which topologically is of the form

$$\mathbb{M}^{\#} = S_1 \times S_3.$$

This definition of $\mathbb{M}^{\#}$ is used in Paragraph 6.5; twistors $\mathsf{Z}^{\alpha} = (\omega^A, 0)$ correspond to null lines lying entirely in the region $\mathscr{I}$, and two twistors Z^{α} and X^{α} with parallel α-planes Z and X now intersect somewhere in $\mathscr{I}$.

In Paragraph 7.1, another way of obtaining $\mathbb{M}^{\#}$ is discussed, by embedding $\mathbb{M}$ in the null cone of a 6-dimensional vector space. The construction discussed there is conformally equivalent to the construction discussed here, since $\mathbb{M}^{\#}$ can be embedded in the Einstein cylinder, so these two construction will lead to the same topology for $\mathbb{M}^{\#}$.

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