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Positivity bounds for scalar-tensor theories

THESIS

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Positivity bounds for scalar-tensor theories

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Abstract

Recently a new set of conditions for effective field theories has been found, called *positivity bounds*. Positivity bounds are conditions on the scattering amplitudes of particles in the low-energy EFT coming from the existence of a viable UV completion (with properties such as Poincaré-invariance, crossing-symmetry, analyticity, unitarity and locality). In this research we have derived the positivity bounds for Horndeski gravity on a Minkowski background and under the assumption that we can transport them to the cosmological background we have implemented them in the EFTCAMB code in order to study their impact on the viable parameter space (defined by the usual ghost, gradient and tachyonic stability conditions). In the numerical analysis we focused on the K-mouflage subclass and found that the positivity bounds exclude certain regions of parameter space. Finally, we have considered how the positivity bounds for Horndeski gravity (with $G_5 = G_5(\phi)$) change when considering a cosmological background under the assumption that only boosts are broken.

Contents

1 Introduction

General relativity (GR) was formulated by Einstein in 1916. It provided a rigorous framework in which one could describe gravity as a smooth spacetime manifold. Moreover, GR also provided a rigorous description of cosmology: the study of the evolution of the Universe. Before Einstein it could partially be done using Newtonian gravity but this was much more restricted. It was found that under the assumption that there is some unknown matter content in the Universe (dark matter) GR could be used to describe the formation of light elements in the very early stages of the Universe, called Big Bang Nucleosynthesis (BBN), and that it can also be used to explain the formation of structure due to growing matter perturbations and gravitational instabilities in the later stages of the Universe, called Large-Scale Structure (LSS). It also predicted the relic radiation coming from the era of recombination (i.e. when $pe^- \to H\gamma$ happened so that photons could travel freely) called the Cosmic Microwave Background (CMB). Most importantly GR predicted the ΛCDM model for cosmology which was successfully tested in several ways, such as by looking at the power spectrum of CMB, matter power spectrum of LSS and BBN.

GR turned out to be quite successful on small scales such as our galaxy and the Solar system but on large (cosmological) scales GR seems to be incomplete since it cannot account for the cosmic acceleration. A good reason for this is that the Λ in the ΛCDM model (in which there is approximately 30% ordinary matter and 70% of Λ today) is not consistent since its theoretical value does not match with the observational value. Therefore people think that either GR should be modified (and ignoring the existence of dark energy) or that one should add another (scalar) field to ordinary GR. In either case GR is thought to be incomplete. And with the upcoming data related to cosmological scales, it is useful to have models which can be compared with this data. Various models have been proposed such as quintessence, K-essence, $f(R)$-theories and scalar-tensor theories of dark energy. Most of these theories are contained in the Horndeski gravity theory but extensions are possible such as the DHOST theory. Generally, there are two ways in how to deal with the cosmological constant problem: introduce new particles responsible for cosmic acceleration and assume GR is valid or do not introduce new particles and assume that GR is modified at large distances such that self-acceleration occurs.

The problem with all these models however is that it is unfeasible to test each of them observationally. Therefore it is natural to provide a common language for these models in select only the relevant theories, which is called effective field theory of dark energy. In such an effective description of modified gravity, it is possible to restrict the parameter space by the so-called theoretical priors approach. With this method one can exclude certain parameters in a model or even exclude certain models as a whole. This will therefore help in the search for the best model of modified gravity/dark energy. Any unphysical behavior should not be allowed in the best description of dark energy. Basically, a candidate theory of modified gravity often introduces new propagating degrees of freedom (apart from the graviton in GR) and one needs to check whether these are physical are not and whether they introduce instabilities. Furthermore, a good candidate theory of modified gravity should be consistent with the well-known observations of e.g. the Solar system, BBN, LSS and CMB. And as a consequence it should reduce to GR in some limit. In this research we will in particular look at so-called positivity bounds. These refer to conditions on the scattering amplitudes of interactions in the considered models. Positivity bounds originate from quantum field theory/particle physics. The idea is that a modified theory of gravity is a low-energy effective field theory (EFT) and the positivity bounds are conditions which show whether it is possible to extend the theory to high-energies (UV completion) with certain conditions such as: Poincaré invariance, unitarity (well-defined probabilities), analyticity (causality), crossing symmetry (invariance under exchange of Mandelstam variables) and polynomial boundedness (locality). Such requirements yield conditions on the coefficients in the scattering amplitudes of the corresponding low-energy EFT, which we call positivity bounds. A UV completion of gravity is needed because it is known that GR is only valid up the (order of the) Planck scale (as it is an effective non-normalizable theory [1]), above which quantum gravity would be required. The advantage of this method of positivity bounds is that even without knowing the exact UV completion, it is still possible to require general conditions on this UV completion, which give raise to conditions on the coefficients in the scattering amplitudes in the low-energy EFT. The point is that the high-energy theory (UV completion) involves both heavy and light particles, however in the low-energy EFT one has basically integrated out these heavy fields such that only light fields occur [2], and of course there will be corrections to the theory which can be ignored if you consider low enough energies. The point is that the UV theory, which contains fundamental physics of the low-energy EFT, cannot be tested experimentally since the observations do not have

[1] It cannot be quantized properly as it is non-normalizable at 2-loop level or higher order loop corrections.

[2] In the sense that $e^{iS[\Phi_{\rm L}]} = \int \mathcal{D}\Phi_{\rm H} e^{iS[\Phi_{\rm L},\Phi_{\rm H}]}$ where $\Phi_{\rm L}$, $\Phi_{\rm H}$ are light and heavy fields respectively. The meaning of 'light' and 'heavy' is defined by the energy up to which the low-energy EFT has been tested, which is often taken as the cut-off of the low-energy EFT.

access to such high energies. However, in many cases the low-energy EFT models can be tested experimentally. This method of positivity bounds allows to check whether some low-energy EFT model can be the low-energy limit of some UV completion, but at the same time the low-energy EFT model data gives a way of checking whether maybe one (or more) of the general assumptions about the UV completion, such as causality, should be violated. In the past theoretical priors coming from instabilities of EFT's of dark energy were found using the EFTCAMB analysis applied to Horndeski gravity (and generalizations), which included ghosts, tachyons, gradient instabilities [3]. This method gave some theoretical priors which could be used to constrain the dark energy models [38]. Taking into account the observational constraints, the positivity bounds of EFT's of dark energy can help in further reducing the allowed parameter space.

One goal of the project is to find the correct positivity bounds for Horndeski theory on a Minkowski background, which is motivated by the fact that the two references [17], [21] disagree with each other. Another goal is to implement the positivity bounds in EFTCAMB using the reconstruction method (under the assumption that the positivity bounds can be transported to the cosmological background) and study how the viable parameter space of the K-mouflage full model changes under the inclusion of the positivity bounds. And we will comment on how to investigate the phenomenology describing the large-scale structure (Σ, μ) and the equation-of-state of dark energy, which will follow in an upcoming article [43]. The last goal of the project is to consider whether it is possible to compute the positivity bounds for Horndeski theory on the cosmological background (under the assumption that only boosts are broken) and to check how the expressions might differ from those on the Minkowski background.

The structure of the thesis will be as follows. In section 2 a short introduction to background cosmology will be provided. Section 3 will address the main observational evidences of cosmic acceleration. Perturbation theory in GR and degrees of freedom will be discussed in section 4. In section 5 modified gravity and viability conditions will be discussed (including examples such as Horndeski gravity and Lovelock gravity). The EFT formalism of dark energy will be introduced in section 6. In section 7 the positivity bound formalism will be discussed with the focus on Horndeski gravity. Section 7.1 contains a short introduction to some (advanced) quantum field theory. In section 7.2 the concept of positivity bounds will be motivated and in section 7.3 the formalism of positivity bounds of EFT's on a Minkowski background will be addressed. The example of positivity bounds for Horndeski theory on a Minkowski background is discussed in section 7.4. In section 7.4, under the assumption that the positivity bounds remain to hold (at least approximately) on a cosmological background, we explain how to implement the positivity bounds in EFTCAMB using the reconstruction method for Horndeski theory and we discuss about the numerical analysis of the positivity bounds for the K-mouflage full model and LSS phenomenology (in the light of an upcoming article [43]). In section 7.5 we address how the formalism of positivity bounds and the results for Horndeski theory change when going from a Minkowski background to a cosmological background. Lengthy calculations of the results presented in the main text can be found in the Appendix.

We will assume units in which $\hbar = c = 1$ and the metric convention $(-, +, +, +)$. In the Appendix section 9.1 futher details and basic formulas will be provided as well.

[3] The Ostrogradski instability, i.e. the Hamiltonian is unbounded from below if the equation of motion is higher than second order, is also taken into account [9].

2 Background cosmology [1]

In general relativity it is well-known that spacetime tells matter how to move and matter tells spacetime how to curve. This was understood by Einstein in 1916 who formulated the Einstein field equations (ignoring the cosmological constant Λ):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ the metric tensor, R the Ricci scalar, G is Newton's gravitational constant and $T_{\mu\nu}$ is the energy-momentum tensor.

Equation (2.1) can be found by the variation of the Einstein-Hilbert action S_{EH} with respect to the inverse metric $g^{\mu\nu}$ and setting it equal to zero:

$$S_{\text{EH}} = \int d^4x\sqrt{-g}\Big(\frac{R}{16\pi G} + \hat{\mathcal{L}}_M\Big), \tag{2.2}$$

where $g = \det(g_{\mu\nu})$ and $\hat{\mathcal{L}}_M$ is the matter Lagrangian. In this convention it holds that the energy-momentum is defined via:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{-g}\hat{\mathcal{L}}_M \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{2.3}$$

2.1 FRW metric

The observations by Hubble showed that all galaxies move away from us according to the Hubble law $v = H_0 d$. This could either mean that we are in the center of the Universe or that the Universe started with a Big Bang, where the latter more feasible and therefore this is assumed to be the case. The Universe is homogeneous and isotropic on large scales. This became clear from the observations of distribution of matter and the CMB. The CMB has an almost uniform blackbody temperature of 2.7 K with fluctuations (anisotropies) of order 10^{-5}. Similarly, the distribution of matter in the Universe is quite uniform on large scales. Since we assume that we are not in the center of the Universe, we can assume that the Universe is not only isotropic but homogeneous as well. Therefore it is expected that the metric which describes the cosmology of the Universe should be homogeneous and isotropic as well. It turns out that the only possible geometry is that the Universe consists of maximally symmetric spacelike surfaces with time moving orthogonally between them and that there are only three possible geometries defined by the curvature $\kappa \in \{-1, 0, 1\}$. $\kappa = -1$ corresponds to an hyperbolic geometry, $\kappa = 0$ to a flat geometry and $\kappa = 1$ to a spherical geometry. The metric which describes all of this is the Friedmann–Robertson–Walker (FRW) metric:

$$ds^2 = -dt^2 + a^2(t)\Big[\frac{dr^2}{1-\kappa r^2} + r^2(d\theta^2 + \sin^2(\theta)d\phi^2)\Big]. \tag{2.4}$$

In this equation $a(t)$ is the scale factor (which describes the expansion of the Universe), $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ are the angular spherical coordinates and t is the cosmic time (time measured by an observer that moves along the expansion of the Universe).

Observations of for instance the anisotropies of the CMB showed that the Universe is spatially flat, i.e. $\kappa = 0$. Therefore the FRW metric takes a much simpler form in Cartesian coordinates:

$$ds^2 = -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2]. \tag{2.5}$$

Sometimes it is useful to write this in the conformal time coordinate $d\tau := dt/a(t)$ such that:

$$ds^2 = a^2(\tau)[-d\tau^2 + dx^2 + dy^2 + dz^2]. \tag{2.6}$$

2.2 Friedmann and continuity equation

The energy-momentum tensor in the background cosmology is given by the perfect fluid approximation:

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}. \tag{2.7}$$

Here ρ is the energy density of the fluid in the rest frame, p is the pressure of the fluid in the rest frame, U_μ is the 4-velocity of the fluid with respect to the expanding Universe and $g_{\mu\nu}$ is the FRW metric. Under the assumption that the fluid moves along the expansion of the Universe it holds that $U^\mu = (1, 0, 0, 0)$. From this it can be seen that the energy-momentum tensor takes the form:

$$T^\mu_\nu = \mathrm{diag}(-\rho, p, p, p). \tag{2.8}$$

The local energy-momentum conservation $\nabla_\mu T^{\mu\nu} = 0$ can be used to derive the continuity equation:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{2.9}$$

Upon defining the equation of state $w := p/\rho$ it follows that $\rho \propto a^{-3(1+w)}$ if w is constant. Relativistic matter (radiation) has the equation of state $w = 1/3$ and non-relativistic matter (dust) has neglibible pressure and thus $w = 0$.

The Friedmann equations can be derived by plugging (2.8) and (2.4) into (2.1):

$$\begin{aligned} H^2 := \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p). \end{aligned} \tag{2.10}$$

In these equations ρ, p are understood as the total energy density and pressure respectively, e.g. $\rho = \sum_i \rho_i$ where i sums over the different species.

It is often convenient to introduce the density parameter Ω_i for each species i:

$$\Omega_i = \frac{8\pi G}{3H^2}\rho_i =: \frac{\rho_i}{\rho_{\mathrm{crit}}}, \tag{2.11}$$

where ρ_{crit} is called the critical (energy) density. In this notation the first Friedmann equation takes the following form:

$$\sum_i \Omega_i - 1 = \frac{\kappa}{H^2 a^2}. \tag{2.12}$$

It should be mentioned that in the literature Ω_i often also means $\Omega_i(t_0)$ at present time t_0. The distinction is then clear from the specific context.

2.3 Cosmological constant

As will be discussed in section 3, there are several observations which reveal that the expansion of the Universe accelerates. This phenomenon is called the cosmic acceleration. GR with only radiation and matter cannot explain this phenomenon since a Universe with matter or radiation will decelerate. Obviously the Universe seems to be dominated by matter at the moment. Most of this matter is in the form of (cold) dark matter (DM), i.e. some particles which gravitate but interact very weakly through other forces. The existence of DM became clear from several observations: gravitational lensing, rotational curves of galaxies, X-ray clusters and LSS. So one would expect the Universe to decelerate nowadays whereas the observations reveal that it accelerates. The first attempt to deal with the cosmic acceleration is to introduce the cosmological constant Λ, which was originally introduced by Einstein in order to explain why the Universe is static (however such a solution turned out to be unstable). This can be done by modifying the Einstein-Hilbert action:

$$\tilde{S}_{\text{EH}} = \int d^4x\sqrt{-g}\Big(\frac{R-2\Lambda}{16\pi G} + \hat{\mathcal{L}}_M\Big). \tag{2.13}$$

The Einstein field equations are therefore also modified:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}. \tag{2.14}$$

From the above equation it is suggestive to regard Λ as a species with the energy-momemtum tensor:

$$T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu}. \tag{2.15}$$

Therefore it follows under the assumption of a perfect fluid that the energy density and the pressure of Λ is given by:

$$-\,p_\Lambda = \rho_\Lambda = \frac{\Lambda}{8\pi G}. \tag{2.16}$$

From this relation it is clear that Λ has a constant energy density. Therefore the first Friedmann equation (2.10) shows that $a \propto e^{Ht}$ where $H = \sqrt{\Lambda/3}$. Thus the cosmological constant Λ indeed predicts that the expansion of the Universe is accelerating ($\ddot{a} > 0$) if Λ dominates over matter at the moment. The fact that Λ now dominates over matter, whereas earlier on matter dominated, means that very precise fine-tuning conditions of Λ are needed for this to happen. The reason lies in that Λ is an example of a static dark energy (DE) candidate and cannot track the matter density (for that it would need to be dynamical scalar field) [2].

3 Observational evidences of cosmic acceleration

There are several independent observational evidences which show that the Universe undergoes accelerating expansion. These observational evidences also allow us to constrain or determine the values of the parameters Ω_m, Ω_Λ, Ω_r. In this section we will briefly explain some observational evidences of cosmic acceleration, heavily based on the discussion in [2].

3.1 Type Ia supernovae

At the end of the life of a star it may become a white dwarf. White dwarfs accrete matter from the surroundings up to the Chandrasekhar limit of $\sim 1.4\, M_\odot$ (where $M_\odot$ is the mass of the Sun). Afterwards the temperature is so large that carbon and oxygen will be burnt and this will lead to a very bright event which we call a supernova. Particular type of supernovae which are useful for observations of cosmic acceleration are the type Ia supernovae (which are characterised by certain spectral lines). These types of supernovae are called standardizable, i.e. after some technical procedure the light curves of different supernovae are averaged in such a way that they have the same absolute magnitude M. Their brightness (apparent magnitude m) is therefore only determined by the (luminosity) distance d_L:

$$m = M + 5\ \log\left(\frac{d_L}{10\ \text{pc}}\right) + K, \tag{3.1}$$

where K is a correction term (which accounts for that observations take place in a certain wavelength range).

The distance d_L can also be written as a function of the redshift z (defined by $1/a =: 1 + z$):

$$d_L(z) = (1 + z) \cdot \int_0^z \frac{dz'}{H(z')}. \tag{3.2}$$

Remarkable is that this equation fully depends on the expansion history of the Universe via the Hubble parameter $H(z')$.

In the observations one could determine the apparent magnitude of several type Ia supernovae. Since M and K are known it was possible in observations to infer the distances d_L to such objects. Also the distance d_L could be found from measuring the redshift of the supernovae (by looking at its host galaxy) and by assuming a certain expansion history of the Universe via $H(z')$. The result was that a flat Universe with only radiation at early stages and only matter at later stages does not give a consistent result between the equations (3.1) and (3.2). The best fit was found by assuming that the Universe nowadays consists of approximately 70% of Λ, 30% matter and almost neglibible radiation of 0.01% (ΛCDM model). So this means that the Universe undergoes accelerated expansion.

3.2 CMB and LSS

Observations of type Ia supernovae showed that the Universe undergoes cosmic acceleration. It is useful to have different independent observations which point in the same direction. Furthermore, these other observations will also be useful for constraining the cosmological parameters Ω_m and Ω_Λ. It is even possible to determine the density of baryonic matter Ω_b (i.e. the Standard Model particles) but we will not discuss this here.

In the early Universe photons could interact with other species such as electrons and positrons. As the Universe cooled down, the energy of the photons decreased via $T \sim 1/a$. And therefore the amount of interactions between photons and other species decreased. At some redshift of $z \sim 1100$, the photons could travel freely since neutral hydrogen was formed through recombination $pe^- \to H\gamma$. This is what is meant with the Cosmic Microwave Background (CMB). It is like blackbody radiation with a temperature of 2.7 K with anisotropies of order 10^{-5}. These anisotropies are described by a so-called (angular) power spectrum, see the subfigure on the left in Figure 1 below. The different features in the spectrum allow us to learn about the curvature κ of the Universe and the cosmological parameters Ω_m and Ω_Λ.

The acoustic oscillations are caused by gravitational instabilities which counteract the outward pressure in overdensity regions, which are present at the beginning of the structure formation. The ISW plateau is caused by CMB photons which move through a non-constant potential, by which photons can gain or lose energy. The maximum in the CMB power spectrum (indicated as first peak in Figure 1) has been used to show that the Universe is spatially flat ($\kappa = 0$). The spectrum as a whole can be used to fit the best cosmological model. This again turned out to be the ΛCDM model.

Similarly, the structure formation is characterised by the matter power spectrum, which tells us how matter clusters. The acoustic oscillations in the CMB are also present as baryon acoustic oscillations (BAO) in the matter power spectrum (right subfigure in Figure 1). By measuring a typical length scale of these oscillations (acoustic scale) at different redshift, angular diameter distance d_A and the Hubble parameter H could be derived as a function of redshift. Like with the type Ia supernovae, the only way these results could be made consistent is by introducing the ΛCDM model.

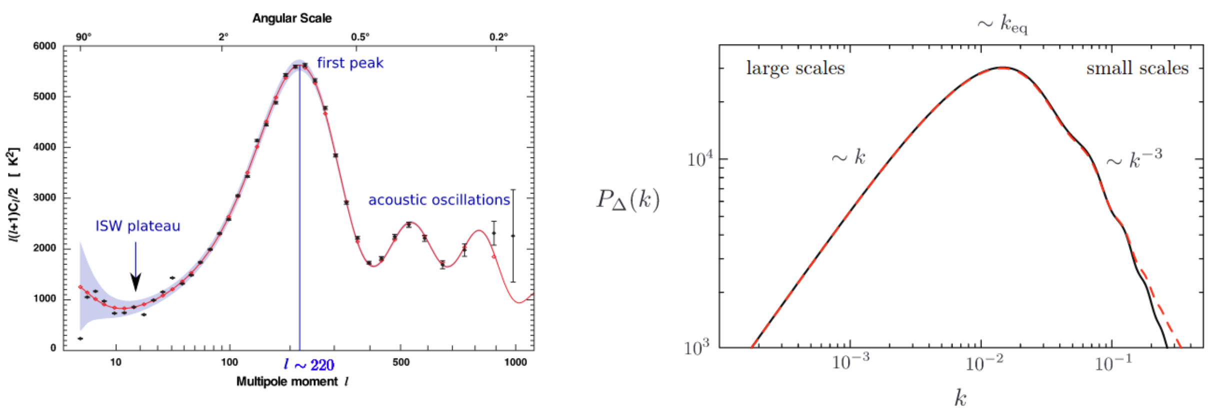


Figure 1: Left: CMB power spectrum as a function of the angular separation. The solid line indicates the best-fit of the data. Indicated in the figure are certain features of the power spectrum. Reference: NASA/WMAP Science team. Right: Matter power spectrum as a function of wave vector k. Δ indicates the comoving density contrast. The solid line is the spectrum without non-linear corrections. The dashed line is the spectrum with non-linear corrections. On small scales the BAO are visible. Reference: Daniel Baumann, Cosmology notes.

3.3 Constraining the parameter space

The different observations leading to the conclusion that there should be cosmic acceleration, also provide a way of constraining the parameter space $\{(\Omega_m, \Omega_\Lambda)\}$. Combining the results of the above observational methods one obtains Figure 2. From Figure 2 it is clear that the observations allow for a best-fit value of $(\Omega_m, \Omega_\Lambda)$, namely where the data of different observations intersects. The values for the cosmological parameters from these constraints are: $\Omega_m h^2 = 0.1358^{+0.0037}_{-0.0036}$ and $\Omega_\Lambda = 0.725 \pm 0.015$ [3].

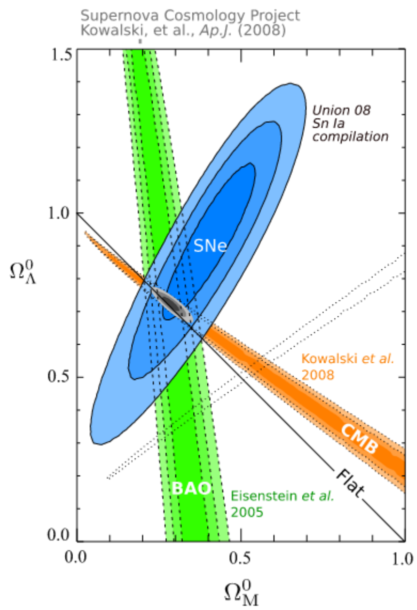


Figure 2: Indicated are the observational constraints on the cosmological parameters Ω_m and Ω_Λ. The different contours indicate the 68.3%, 95.4% and 99.7% confidence intervals. Reference: [2] and the references in the Figure.

4 Perturbation theory in GR and discussion of degrees of freedom [1]

In GR the only propagating degree of freedom is the gravitational wave. Modifying GR boils down to modifying the Einstein-Hilbert action (2.2). This amounts to a field equation different from (2.1) and possibly the introduction of equations of motion for the additional (propagating) degrees of freedom. To make this more clear, we will discuss what is meant with a degree of freedom and when a degree of freedom is said to be propagating.

Generally, a degree of freedom is some parameter or function that can be chosen freely in a theory. As an example to illustrate this, we will focus on the case of metric degrees of freedom in linearized GR around a flat spacetime (can be generalized to curved backgrounds such as FRW). The metric is given by $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $|h_{\mu\nu}| \ll 1$ [4]. The subtle point in discussing degrees of freedom is whether they are physical or not. If a degree of freedom is physical this means that it does not change under an active spacetime diffeomorphism or gauge transformation (i.e. the map $h_{\mu\nu} \mapsto h_{\mu\nu} + (\mathcal{L}_\xi \eta)_{\mu\nu}$ where $\mathcal{L}_\xi$ is the Lie derivative of η along the vector field ξ defined by $(\mathcal{L}_\xi \eta)_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)}$ and $|\xi^\mu| \ll 1$). Such a transformation leaves the Riemann tensor invariant, i.e. $\delta R_{\mu\nu\alpha\beta} = 0$, therefore it does not change the physics of the spacetime. The metric $h_{\mu\nu}$ seems to have 10 degrees of freedom, however it turns out that this gauge freedom only allows for 6 physical degrees of freedom. These can be studied by the scalar-vector-tensor (SVT) decomposition of the metric components. In the context of the ΛCDM model, these are known as the Bardeen variables [4]. For linearized GR on a flat background the SVT decomposition is as follows:

$$\begin{aligned} h_{00} &= -2\Phi \\ h_{0i} &= w_i \\ h_{ij} &= 2s_{ij} - 2\Psi\delta_{ij}, \end{aligned} \tag{4.1}$$

where $\Psi := -\frac{1}{6}\delta^{ij}h_{ij}$ is the gravitational potential and $s_{ij} := \frac{1}{2}(h_{ij} - \frac{1}{3}\delta^{kl}h_{kl}\delta_{ij})$ the strain. s_{ij} is used to describe gravitational waves.

From the full metric $g_{\mu\nu}$ the following Einstein equations follow (00, $0j$, ij equations from top to bottom):

$$\begin{aligned} \nabla^2\Psi &= 4\pi G T_{00} - \frac{1}{2}\partial_k\partial_l s^{kl} \\ (\delta_{jk}\nabla^2 - \partial_j\partial_k)w^k &= -16\pi G T_{0j} + 4\partial_0\partial_j\Psi + 2\partial_0\partial_k s_j^{\ k} \\ (\delta_{ij}\nabla^2 - \partial_i\partial_j)\Phi &= 8\pi G T_{ij} + (\delta_{ij}\nabla^2 - \partial_i\partial_j - 2\delta_{ij}\partial_0^2)\Psi - \delta_{ij}\partial_0\partial_k w^k + \partial_0\partial_{(i}w_{j)} \\ &\quad + \Box s_{ij} - 2\partial_k\partial_{(i}s_{j)}^{\ k} - \delta_{ij}\partial_k\partial_l s^{jl}. \end{aligned} \tag{4.2}$$

From these equations it is clear that the only metric component needed to solve for the other metric components at all t is the strain s_{ij} since there are only time-derivatives of s_{ij}, provided the spatial boundary conditions are known. This means that only the time-evolution of s_{ij} has to be known. Therefore s_{ij} is said to be propagating degree of freedom.

This hints at that in ordinary GR the only propagating degree of freedom is the gravitational waves, which is a massless spin-2 (tensor) field. In modifications of GR there are typically additional degrees of freedom which may be propagating as well. Examples of this are discussed in section 5.

[4] Indices are raised and lowered with the Minkowski metric. And only terms linear in $h_{\mu\nu}$ will be considered.

5 Modified gravity and viability conditions

The late-time cosmic acceleration is an indication that standard GR is not applicable at cosmological scales which means that GR must be modified. The Weinberg-Deser theorem tells us that the only local Lorentz invariant theory of a massless spin-2 particle (graviton) must be GR. Therefore we could introduce new fields (apart from the graviton), break Lorentz invariance, give up locality, make the graviton massive or something else. The general point is that these modifications of GR introduce new degrees of freedom.

In modified gravity models it is essential to check whether the introduced degrees of freedom are not ghosts (i.e. they correspond to negative energy excitation) or have superluminal speeds (i.e. their propagation speed exceeds the speed of light) or the introduction of other instabilities. The existence of ghosts would imply that the vacuum cannot be stable since it will decay into positive and negative energy particles [1]. A degree of freedom with superluminal velocity is not consistent with causality. Furthermore, a modified gravity theory needs to be consistent with the well-known observations (the tests of GR): Solar system, our galaxy, LSS, CMB and BBN. In the following, some examples of modifications of GR are discussed. Of course the list of modifications of GR is endless, therefore we will only restrict to the ones which are relevant for this thesis.

5.1 Cosmological constant

The most obvious modification of ordinary Einstein gravity in order to account for the cosmic acceleration is the cosmological constant (equation (2.13)). Since the energy density of the cosmological constant is constant, it is natural to regard this as the vacuum energy density. In particle physics it is known that the zero-point energy of an excitation of mass m is given by $\frac{1}{2}\omega_{\mathbf{k}} = \frac{1}{2}\sqrt{m^2+\mathbf{k}^2}$ at momentum $\mathbf{k}$. The total ground-state energy density is therefore given by the following expression [5]:

$$\rho_{\text{vac}} = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2}\sqrt{m^2+\mathbf{k}^2}. \tag{5.1}$$

It is evident that this integral will diverge since the integrand diverges for $|\mathbf{k}| \to \infty$. This is an indication that QFT is only valid up to some cut-off scale k_Λ such that the integral is only performed up to this scale [5]. It is often assumed that this cut-off scale is at the Planck scale so $k_\Lambda \sim M_P$ where $M_P^{-2} := 8\pi G$ is the reduced Planck mass [2]. Therefore the vacuum energy density is given by $\rho_{\text{vac}} \sim M_P^4$. The value from observations $\rho_{\text{vac}}^{(\text{obs})}$ however is found by the expression $\Omega_\Lambda = \frac{\Lambda}{8\pi G\rho_{\text{crit}}} = \frac{3\Lambda}{H_0}$. By $\Omega_\Lambda \sim 0.7$ [3], $H_0 \sim 70\ \text{km s}^{-1}\ \text{Mpc}^{-1}$ and $M_P = 1.22\cdot 10^{19}$ GeV one finds that:

$$\rho_{\text{vac}}^{(\text{obs})} \sim 10^{-120}\rho_{\text{vac}}. \tag{5.2}$$

The above calculation shows that the observational and theoretical values for the vacuum energy density do not agree. This is called the cosmological constant problem.

There are some theories which try to predict the observational value in order to get agreement. That some theory in principle could exist, is perfectly reasonable, since GR only applies to the low-energy limit, whereas this cosmological constant problem may be solved in the UV completion of GR (which contains energies above the cut-off as well). However, there are no promising theories at the moment [2]. There are some theories which try to address the problem such as supersymmetry, however there are several issues in this theory (in particular in relation to experiments).

Another problem is that Ω_m and Ω_Λ are of the same order of magnitude nowadays. Since $\Omega_m/\Omega_\Lambda \propto a^3$ it is not so obvious why we measure dark energy precisely in the era where these density parameters are of the same order of magnitude. This is called the coincidence problem [2]. If Λ would be a time-dependent scalar field rather than a constant scalar, then under certain conditions we might be able to choose it such that it 'tracks' the matter [7]. This means that $\rho_m \gg \rho_\Lambda$ earlier on and that $\rho_\Lambda \sim \rho_m$ now and eventually $\rho_m \ll \rho_\Lambda$. This observation, together with the cosmological constant problem, suggests that a more promising description of cosmic acceleration requires a dynamical scalar field (or degree of freedom in general).

5.2 Quintessence and K-essence [7]

A simple way to introduce cosmic acceleration is via a dynamical real scalar field ϕ through the Lagrangian:

$$\hat{\mathcal{L}} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \tag{5.3}$$

where $V(\phi)$ is some potential.

Such a model is called quintessence. From the energy-momentum tensor associated to this Lagrangian it is straightforward to see that the equation of state $w = \rho/P$ in the slow-roll regime ($\dot{\phi}^2 \ll |V|$) obeys $w \approx -1$. For cosmic acceleration it is required that the Universe is dominated by a species with equation of state $w < -1/3$. This is clearly the case and therefore ϕ can drive the cosmic acceleration. The two largest issues with this simple theory however is that the potential needs to be chosen and that the theory needs to be stable in the particle physics sense. E.g. in the case of a potential $V = \frac{1}{2} m_\phi^2 \phi^2$ it follows that $m_\phi \sim 10^{-33}$ eV. Such energy scales are very low and hard to explain from the point of particle physics. For this one needs to introduce other physics such as treating ϕ as a pseudo-Nambu-Goldstone boson such as the axion field, in which $U(1)$ symmetry is broken such that the particle obtains a small mass by oscillations. Therefore such a theory is not really promising. A natural generalization of quintessence is K-essence in which the Lagrangian takes the form:

$$\hat{\mathcal{L}} = K(X) - V(\phi), \tag{5.4}$$

where $X := -\frac{1}{2}(\partial_\mu \phi)^2$ for some scalar field ϕ and $K(X) \geq 0$.

This theory is however prone to various instabilities.

5.3 Scalar-tensor theory [1]

As a more advanced step in modifying the Einstein-Hilbert action, it is natural to introduce a scalar field λ in the following way:

$$S = \int d^4x \sqrt{-g} \Big[f(\lambda) R - \frac{1}{2} h(\lambda) g^{\mu\nu} (\nabla_\mu \lambda)(\nabla_\nu \lambda) - U(\lambda) + \hat{\mathcal{L}}_M[g_{\mu\nu}, \psi_i] \Big], \tag{5.5}$$

where $f(\lambda)$, $h(\lambda)$ and $U(\lambda)$ are general functions of λ and $\{\psi_i\}$ are matter fields.

From this action it is clear that λ is a degree of freedom, confirming our claim that modifications of GR lead to introducing new degrees of freedom.

The dynamics of the scalar field λ is determined by $\delta S / \delta \lambda = 0$:

$$h \Box \lambda + \frac{1}{2} h' g^{\mu\nu} (\nabla_\mu \lambda)(\nabla_\nu \lambda) - U' + f' R = 0, \tag{5.6}$$

where e.g. $f' := \partial_\lambda f$.

The frame with the metric $g_{\mu\nu}$ is called the Jordan frame (or string frame). However, in this frame it is not so obvious that the term $f(\lambda)R$ actually contains a kinetic term for λ. This can be illustrated by going to the Einstein frame via the conformal transformation $g_{\mu\nu} \mapsto \tilde{g}_{\mu\nu} = 16\pi \tilde{G} f(\lambda) g_{\mu\nu} \equiv \omega^2 g_{\mu\nu}$ where $\tilde{G}$ is the Newton's constant in the Einstein frame. This can be seen by that in this frame the Einstein-Hilbert part of the action (equation (5.5)) takes the following form:

$$S_{fR} := \int d^4x\sqrt{-g}f(\lambda)R = \int d^4x\sqrt{-\tilde{g}}(16\pi\tilde{G})^{-1}\Big[\tilde{R} - \frac{3}{2}\tilde{g}^{\rho\sigma}f^{-2}\Big(\frac{df}{d\lambda}\Big)^2(\tilde{\nabla}_\rho\lambda)(\tilde{\nabla}_\sigma\lambda)\Big]. \tag{5.7}$$

So in the Einstein frame it becomes evident that without explicitly introducing a kinetic term for the scalar field λ, the scalar field still has a kinetic term involving λ. Therefore multiplying R by $f(\lambda)$ introduces the propagating degree of freedom λ.

Another advantage of the Einstein frame is that the field equation takes the ordinary form $\tilde{G}_{\mu\nu} = 8\pi\tilde{G}\tilde{T}_{\mu\nu}$. So in the Einstein frame, the field equation is just like the one derived from the Einstein-Hilbert action. However, it is not just GR in different notation, this equation does really describe a modified version of GR since particles that move along geodesics of the metric $g_{\mu\nu}$ will not move along geodesics of $\tilde{g}_{\mu\nu}$.

In the Einstein frame unlike in the Jordan frame, matter can be non-minimally coupled to gravity. This means that we have that the integrand of S_M is $\sqrt{-\tilde{g}}\omega^4\hat{\mathcal{L}}_M[\omega^{-2}\tilde{g}_{\mu\nu},\psi_i]$ rather than $\sqrt{-g}\hat{\mathcal{L}}_M[g_{\mu\nu},\psi_i]$ [8], i.e. the volume element is not mere $\sqrt{-\tilde{g}}$ but multiplied by ω^4 which depends on the scalar field λ. The consequences of non-minimal coupling can be that the energy-momentum tensor and its conservation law may change compared to ordinary GR [8].

5.4 f(R) theory [6]

The scalar-tensor theory (equation (5.5)) can be modified by replacing $f(\lambda)R$ by $f(R)$. It can be illustrated that $f(R)$ theory can be seen as a special case of scalar-tensor theory in which $h(\lambda) = U(\lambda) = 0$ and $f(\lambda) = \lambda/(16\pi G)$. $f(R)$ theory is usually an interesting example in the study of instabilities. Ignoring the explicit scalar field terms the action for $f(R)$ theory reads:

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}f(R) + \int d^4x\sqrt{-g}\hat{\mathcal{L}}_M[g_{\mu\nu},\psi_i], \tag{5.8}$$

where $\kappa^2 := 8\pi G$.

The field equation corresponding to this action are found to be:

$$f_R R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f_R + g_{\mu\nu}\Box f_R = \kappa^2 T^{(M)}_{\mu\nu}, \tag{5.9}$$

where we defined $f_R := \partial f/\partial R$ and $\Box = \nabla^\mu\nabla_\mu$.

The trace of this equation gives:

$$3\Box f_R + f_R R - 2f(R) = \kappa^2 T, \tag{5.10}$$

where $T = g^{\mu\nu}T^{(M)}_{\mu\nu}$.

From this expression it can be seen that f_R is an additional degree of freedom coming from modifying the Einstein-Hilbert action [5]. This scalar field is called the 'scalaron'. Again this scalar field can be used to construct another scalar field ϕ which is non-minimally coupled to matter and this can be studied in the Einstein frame.

Not all $f(R)$ models are valid models for describing DE. If a model can describe DE, it is called *viable*. In particular, a viable $f(R)$ DE model must obey the conditions $f_R > 0$ and $f_{RR} > 0$ for $R \geq R_0$ where $f_{RR} = \partial f_R/\partial R$ and R_0 is the Ricci scalar today. The first requirement makes sure that there are no ghosts in the theory. These

[5] The equation (5.9) contains fourth derivatives of the metric. However, they are 'distributed' over the scalar and tensor degree of freedom which can be seen by SVT decomposition and the linearized Einstein equations. Each containing two derivatives of the metric tensor. Ostrogradski theorem shows that this does not lead to instabilities. But e.g. two tensor degrees of freedom would lead to an instability.

are negative energy excitations in the theory, which would make the vacuum unstable since pairs of particles with opposite energy can be created without violating energy conservation. And the second requirement makes sure that there is no tachyonic instability. This means that there cannot be modes with a negative mass squared $M^2 < 0$.

The $f(R)$ theory may be further generalized by replacing $f(R)$ by $f(R, \phi)$ for some scalar field ϕ. Obviously, this introduces an extra degree of freedom. And it turns out that in such a theory again various instabilities may occur such as the presence of ghosts, gradient instability or tachyonic instability. These instabilities will be discussed further in the following subsection.

5.5 Viability conditions in a general context [6],[9]

A physical theory of DE needs to be viable. This means that cannot contain instabilities and that it should be consistent with standard GR in some limit and that it is consistent with well-known observational tests of GR. Thus not every theory which predicts late-time cosmic acceleration is a valid theory. In general there will be certain conditions on parameters in each theory. These conditions are often called *theoretical priors* and provide a physically allowed region of parameter space of a given theory. In particular, in this section we will only focus on the viability conditions of the EFT formulation. The section on positivity bounds will contain other viability conditions on the EFT formulation which are based on whether the EFT admits a UV completion which obeys the usual requirements such as causality and Poincaré-invariance. Below we summarize the main requirements on a viable DE theory in the EFT formulation.

5.5.1 Ghost instabilities

A ghost is a negative energy excitation. Such a mode can be present in a theory if the kinetic term has the opposite sign. The reason that ghosts cannot be present in a theory is because they predict that the vacuum is unstable, since a ghost plus particle can be created without energy cost. The way how ghosts can be detected in a theory is as follows. The idea is to write the action up to second order in perturbation theory, write it in terms of gauge-invariant fields (denoted by a vector $\vec{\phi}$) and make sure that the second order perturbation of the Lagrangian is of the form $\mathcal{L} = \dot{\vec{\phi}}^T A \dot{\vec{\phi}} + ...$ where A is some matrix. The first term is the kinetic term of the Lagrangian and the no-ghost requirement is that the matrix A should not have negative eigenvalues. In practice this often means that we just require that $\det(A) > 0$. Clearly, this condition provides us with a bound on the parameter space. This is an example of a theoretical prior in the EFT description of DE.

5.5.2 Gradient instabilities

Modes in a theory travel at a certain speed. The idea is that at second order perturbation theory, the Lagrangian in the action will contain a term proportional to $-\frac{1}{2}(\nabla\vec{\psi})^T c_S^2 (\nabla\vec{\psi})$ where c_S^2 is some matrix and $\vec{\psi}$ are gauge-invariant fields. In order to avoid the gradient instability, the matrix c_S^2 cannot contain negative eigenvalues in the regime of large momenta.

5.5.3 Tachyonic instabilities

Scalar modes in an EFT description of DE are have a certain mass matrix M, which can be identified by writing the action at second order perturbation theory and by recognizing the term of the type $-\frac{1}{2}\vec{u}^T M^2 \vec{u}$ where $\vec{u}$ contains gauge-invariant fields. Again to avoid tachyonic instabilities one needs M^2 to not contain negative eigenvalues for small momenta. The origin of a tachyonic instability lies in that the perturbation around the vacuum is not performed well although the theory might be correct [9].

5.5.4 Ostrogradski instability

Ostrogradski showed that a theory with an equation of motion containing higher than second order derivatives is unstable. Unstable in this context means that the corresponding Hamiltonian will be unbounded from below. The

Ostrogradski instability therefore also exhibits the presence of ghosts. Therefore the Lagrangian in a modified theory of gravity should generally be such that the equations of motion are at most second order.

5.5.5 Local gravity tests

Standard GR has been tested on scales such as the Solar system and our galaxy. It turned out to be successful in describing physical processes such as CMB, LSS and BAO. A consistent theory of modified gravity needs to reduce to GR on smaller scales and it needs to respect the parts of the thermal and cosmic history of the Universe that have been tested by observations.

5.5.6 Example of instabilities

For the case of a single gauge-invariant scalar field ϕ one finds $\mathcal{L} = \frac{A}{2}\dot{\phi}^2 - \frac{c_S^2}{2}(\nabla\phi)^2 - \frac{1}{2}M^2\phi^2$. Then no ghosts will occur if $A > 0$. There will be no gradient instability if for $c_S^2 k^2 \gg M^2$ it holds that $c_S^2 > 0$. And there will be no tachyonic instability if for $c_S^2 k^2 \ll M^2$ it holds that $M^2 > 0$. In Fourier space the equation of motion is $A\ddot{\phi} + (c_S^2 k^2 + M^2)\phi = 0$. From this we can identify $\omega^2 = (c_S^2 k^2 + M^2)/A$. Since $\phi \sim e^{\pm i\omega t}$, it follows that if $A < 0$ and the numerator positive we find $\phi \sim e^{\pm\omega t}$, so the presence of ghosts lead to exponentially growing and decaying solutions (similarly for the other instabilities if you consider them separately). A subtle point however is that one could have multiple instabilities at the same time, making $\omega^2 > 0$. Therefore this argument is only valid for the individual instabilities.

One remedy for the presence of ghosts is that it might be that the mass of the ghost is (much) larger than the EFT cut-off [9]. In such a case the EFT is well-defined and we assume that the UV completion can be constructed such that at high-energies the ghost is not present as well. Consider the above Lagrangian with $c_S^2 = -1$, $M = 0$ and $A = 1$. The solution of the equation of motion in Fourier space is $\phi_k(t) \propto e^{\pm kt}$, so it has a timescale $1/k$ over which the gradient instability grows (considering the growing mode) [9]. This illustrates that no matter the value of k, i.e. whether it is above or below the EFT cut-off Λ, the theory will have a gradient instability after some time. It however might be that you consider $k \ll \Lambda$ which have cosmological timescales, then gradient instabilities are not that important. Consider the above Lagrangian with $A = 1$ and $c_S^2 = 1$. Then the solutions for $k \to 0$ are $\phi(t) \sim e^{\pm Mt}$, such that the timescale of the tachyonic instability is $1/M$. This means that if one considers modes with $\Lambda \gg k \gg M$ in the EFT that the tachyonic instability is not so important.

5.6 Gauss-Bonnet and Lovelock gravity [6],[7]

A natural step is to include other invariants in the $f(R)$ theory such as the Kretschmann scalar $P_1 \equiv R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ or another scalar such as $P_2 \equiv R_{\alpha\beta}R^{\alpha\beta}$. However, these contractions typically yield higher than second order derivatives in the equation of motion. Therefore, the Gauss-Bonnet term $\mathcal{G} := R^2 - 4P_2 + P_1$ was introduced instead. This term does not introduce these type of terms in the equation of motion. It is well-known that the Gauss-Bonnet term is a topological invariant, i.e. $\sqrt{-g}\mathcal{G} = \partial_\alpha \mathcal{D}^\alpha$ for some vector field $\mathcal{D}^\alpha$. This means that under integration this is simply a boundary term and will therefore not contribute to the equations of motion. It turns out that the only way in which spin-2 ghosts can be avoided (when considering the infinite amount of invariants), while obtaining second order equations of motion, is by introducing the Gauss-Bonnet term in the following fashion:

$$S = \int d^4x\sqrt{-g}f(R,\mathcal{G}). \tag{5.11}$$

Such a theory however may still contain scalar ghost modes.

This conclusion originates from the observation by Lovelock, who stated that general 4D metric theory which gives second order equations of motion which are diffeomorphism invariant [6] is given by the Einstein-Hilbert action (up

[6]For instance the Einstein equations are diffeomorphism invariant. This means that the pull-back of the Einstein equations again obeys the Einstein equations for the metric $g' = \phi_\star g$ for some diffeomorphism ϕ.

to boundary terms involving $\mathcal{G}$). This was formulated in the Lovelock gravity theory. Generally, Lovelock stated that the action (5.11) does not introduce any extra tensorial degrees of freedom, only extra scalar degrees of freedom such as f_R and $f_{\mathcal{G}}$.

5.7 Generalized Galileons and Horndeski gravity [6],[7]

Then a symmetry called Galileon symmetry on Minkowski spacetime was discovered for a scalar field Lagrangian. This originated from the DGP model (see section 5.8) in which they considered the 5D metric for a massive graviton and projected this on 4D spacetime with the metric for massless graviton together with a scalar field π. An action is said to be Galileon invariant if the equation of motion does not change under the transformation $\pi \mapsto \pi + b_\mu x^\mu + c$ where b_μ and c are constants. π is called a Galileon (field). This boils down to a constraint on the allowed Lagrangians. Then it was shown that there are only fives terms allowed in the Lagrangian which respect the Galileon symmetry. It was also shown that the expressions can be made covariant in such a way that the resulting equations of motion are still second order. The Lagrangian for generalized Galileons for a generic 4D metric is given:

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{8\pi G}\sum_{i=2}^{5}\hat{\mathcal{L}}_i + \hat{\mathcal{L}}_M\Big], \tag{5.12}$$

where the $\hat{\mathcal{L}}_i$ are defined by:

$$\begin{aligned}
\hat{\mathcal{L}}_1 &= \pi,\\
\hat{\mathcal{L}}_2 &= \nabla_\mu\pi\nabla^\mu\pi,\\
\hat{\mathcal{L}}_3 &= \Box\pi\nabla_\mu\pi\nabla^\mu\pi,\\
\hat{\mathcal{L}}_4 &= (\nabla_\mu\pi)(\nabla^\mu\pi)[2(\Box\pi)^2 - 2(\nabla_{\alpha\beta}\pi)(\nabla^{\alpha\beta}\pi) - (1/2)R\nabla_\mu\pi\nabla^\mu\pi],\\
\hat{\mathcal{L}}_5 &= (\nabla_\lambda\pi)(\nabla^\lambda\pi)[(\Box\pi)^3 - 3\Box\pi(\nabla_{\alpha\beta}\pi)(\nabla^{\alpha\beta}\pi) + 2(\nabla_\mu\nabla^\nu\pi)(\nabla_\nu\nabla^\rho\pi)(\nabla_\rho\nabla^\mu\pi)\\
&\quad - 6(\nabla_\mu\pi)(\nabla^\mu\nabla^\nu\pi)(\nabla^\rho\pi)G_{\nu\rho}].
\end{aligned} \tag{5.13}$$

Horndeski gravity is the generalization of Lovelock gravity. Horndeski considered a 4D gravitational action with an additional scalar field and investigated what the most general action [7] is which leads to second order equations of motion and diffeomorphism invariance. The general Lagrangian which Horndeski found to obey these requirements turned out to be equivalent to the one of generalized Galileons. Let ϕ be some scalar field and $X := -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The Lagrangian in Horndeski theory [8] is given by:

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{8\pi G}\sum_{i=2}^{5}\hat{\mathcal{L}}_i + \hat{\mathcal{L}}_M\Big], \tag{5.14}$$

where $\hat{\mathcal{L}}_i$ are now defined by:

$$\begin{aligned}
\hat{\mathcal{L}}_2 &= G_2(\phi, X),\\
\hat{\mathcal{L}}_3 &= G_3(\phi, X)\Box\phi \qquad (5.15)\\
\hat{\mathcal{L}}_4 &= G_4(\phi, X)R + G_{4,X}(\phi, X)[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}],\\
\hat{\mathcal{L}}_5 &= G_5(\phi, X)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)[(\Box\phi)^3 + 2\phi_{;\mu}^{\ \ \nu}\phi_{;\nu}^{\ \ \alpha}\phi_{;\alpha}^{\ \ \mu} - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\Box\phi].
\end{aligned} \tag{5.16}$$

[7] Ignoring DHOST which was discovered in the EFT of dark energy.

[8] In the modern formulation. The original formulation by Horndeski was a bit different.

In this equation G_i are general functions of X and ϕ, semicolon indicates covariant derivative, comma indicates partial derivative and $\Box\phi := g^{\mu\nu}\phi_{;\mu\nu}$.

Horndeski gravity contains many DE models such as quintessence, K-essence, scalar-tensor theory for certain choice of functions G_2, G_3, G_4 and G_5. Therefore Horndeski gravity is a promising model in the context of DE.

5.8 Other approaches of modified gravity [1],[6]

Another way of addressing the problem of cosmic acceleration is by introducing the concept of extra dimensions. For instance in the Dvali-Gabadadze-Porrati model (DGP model). The idea is to consider the Universe as being embedded in a 5D Minkowski space. Cosmic acceleration can be seen as the leakage of gravity into the fifth dimension on cosmological scales. These models are often inspired by string theoretic approaches. But in the effective field theory they allow for self-accelerating solutions without the need to introduce some extra field. Models like DGP model are quite elegant, however they often contain various issues.

In the formalism we discussed so far, it was implicitly assumed that the connection coefficients are the Christoffel connection coefficients coming from the metric. In a more general context it is possible to regard the metric and connection as independent, and even to not require the connection to be torsion-free or metric-compatible. However, it turns out that leaving out these assumptions only amounts to introducing GR plus some extra tensor fields. This is often not regarded as modified GR.

GR is known to be non-renormalizable which means that it cannot be quantized properly [9]. Therefore GR plus QFT only cannot describe quantum gravity. Therefore we think that GR and QFT should break down somewhere around the Planck mass $M_P \sim 10^{19}$ GeV and are therefore only effective theories of some unknown UV completion. One way to cope with the non-renormalizability of GR is to introduce an infinite amount of terms in the Lagrangian:

$$\hat{\mathcal{L}} = R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 \nabla^\mu R \nabla_\mu R + ..., \tag{5.17}$$

where α_i are coupling constants and the Lagrangian contains all invariants made out of the curvature tensor (and their derivatives).

It turns out that such a theory is renormalizable but it introduces several issues such as the presence of ghosts. Therefore it is fair to say that we do not know an effective field theory of gravity which can be properly quantized.

[9] It turns out that GR starts becomes non-renormalizable at 2-loop level and higher order loop corrections.

6 ADM formalism and effective field theory formalism

6.1 Perturbation theory [8]

The cosmology of the Universe is often described by the FRW metric (2.4) with $\kappa = 0$ since it is spatially homogeneous and isotropic on large scales. However, from e.g. the existence of galaxies it becomes clear that inhomogeneities play an important role in the history of the Universe. This motivates why the study of perturbation theory is relevant. Also, perturbation theory is the main language of the EFT of DE and therefore it is relevant to briefly review it.

The main idea of perturbation theory is that every tensor field T can be split into a background part T_0 and a perturbation part δT:

$$T(\tau, x^i) = T_0(\tau) + \delta T(\tau, x^i). \tag{6.1}$$

Note that by isotropy and homogeneity the background part can only depend on conformal time τ. The perturbation δT can be studied at different orders. Let $\epsilon \ll 1$, then we can write δT as a sum over perturbations of different order n (denoted δT_n):

$$\delta T(\tau, x^i) = \sum_{n=1}^{\infty} \frac{\epsilon^n}{n!} \delta T_n(\tau, x^i). \tag{6.2}$$

In linear (or first order) perturbation theory we only consider the first term in this expansion and in second order perturbation theory take into account the term with ϵ^2 as well.

It is possible to study perturbations of scalars, 4-vectors and tensors. This is usually done by studying how the components change under a transformation on the constant-τ hypersurfaces. Then via the SVT decomposition these components can be decomposed further. The key question in perturbation theory is whether a perturbation degree of freedom is spurious or not. Due to the decomposition of the metric in background plus perturbation, there is some arbitrariness in the time slicing. In order to resolve this problem, one often studies gauge-invariant quantities, which are quantities that do not change under a gauge (or coordinate) transformation. There are two approaches to studing gauge transformations: active or passive approach. The active approach can be viewed as a way of relating two different physical points under a gauge transformation, whereas the passive approach describes the same physical point in two different coordinate systems. In equations this means that at second order we have for the active and passive approach respectively:

$$\begin{aligned} x^\mu(q) &= x^\mu(p) + \epsilon \xi_1^\mu(p) + \frac{\epsilon^2}{2}(\xi^\mu_{1,\nu}(p)\xi_1^\nu(p) + \xi_2^\mu(p)), \\ \tilde{x}^\mu(q) &= x^\mu(q) - \epsilon \xi_1^\mu(q) + \frac{\epsilon^2}{2}(\xi^\mu_{1,\nu}(q)\xi_1^\nu(q) - \xi_2^\mu(q)). \end{aligned} \tag{6.3}$$

The active and passive approach are mathematically quite different, however the physics will of course not depend on the chosen picture. Let T be a tensor. In the active approach given a vector field ξ^μ (which we assume to be small), we can write the result of the gauge transformation as:

$$\tilde{T} = e^{\mathcal{L}_\xi} T, \tag{6.4}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to ξ. Assuming ξ^μ to be small, we can expand $\exp(\mathcal{L}_\xi) = 1 + \epsilon\mathcal{L}_{\xi_1} + \frac{1}{2}\epsilon^2\mathcal{L}^2_{\xi_1} + \ldots$ where $\xi^\mu = \epsilon\xi_1^\mu + \frac{\epsilon^2}{2}\xi_2^\mu + \ldots$.

This definition allows us to find how perturbations change under a gauge transformation and in turn this can be used to construct gauge-invariant quantities. The choice of gauge is arbitrary and in practice on picks the gauge which is most convenient for a given problem in cosmology. Examples are the longitudinal gauge, spatially flat gauge and the synchronous gauge.

6.2 ADM formalism and effective field theory for scalar-tensor theories [11]

Unlike the usual treatment, GR can also be studied via the Hamiltonian treatment, which underlies the ADM formalism. The ADM formalism turns out to be suitable for the EFT formulation of DE [10], but it should be noted that the ADM formalism is much more general. One motivation for this formalism is that it allows to describe different modified gravity models in a common language, which makes it simpler to compare the different models. The other reason is that this formalism allows for discovering new modified gravity models that could otherwise not easily be constructed such as beyond Horndeski models [10]. The basic notion of the ADM formalism for scalar-tensor theories is to require that constant time hypersurfaces coincide with the hypersurfaces on which the scalar field is uniform. This requirement amounts to requiring that the scalar field has a time-like spacetime gradient, i.e. $\nabla_\mu \nabla^\mu \phi < 0$. In the ADM formalism one thus picks a certain coordinate system. Therefore the diffeomorphism invariance is broken, which will cause the theory to have an additional scalar degree of freedom compared to standard GR. A general action is written in terms of geometrical quantities which are related to the above mentioned hypersurfaces. Let n^μ be the future-oriented time-like unit vector normal to these hypersurfaces. Then we have that $n^\mu n_\mu = -1$ and we can define the pull-back metric on these hypersurfaces as:

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu. \tag{6.5}$$

One can describe the the intrinsic curvature of the hypersurfaces by the ordinary Ricci tensor ${}^{(3)}R_{\mu\nu}$, where (3) indicates we consider the (three-dimensional) hypersurfaces. The extrinsic curvature tensor is defined via:

$$K^\mu_{\ \nu} = h^{\mu\rho} \nabla_\rho n_\nu. \tag{6.6}$$

In the context of DE we often deal with a scalar field ϕ. Let $X := g^{\rho\sigma} \nabla_\rho \phi \nabla_\sigma \phi$. Then the unit vector is easily related to the field ϕ via:

$$n_\mu = -\frac{1}{\sqrt{-X}} \nabla_\mu \phi. \tag{6.7}$$

These geometrical quantities have all been defined without the reference to a particular coordinate system. Let us focus on these geometrical quantities in terms of ADM coordinates, i.e. the coordinates for which constant time hypersurfaces coincide with uniform scalar field hypersurfaces. The metric in the ADM formulation is given by:

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{6.8}$$

where N is called the lapse and N^i the shift.

In these coordinates the above geometrical quantities are expressed as:

$$n_\mu = (-N, \mathbf{0}), K_{ij} = \frac{1}{2N}(\dot{h}_{ij} - D_i N_j - D_j N_i), \tag{6.9}$$

where a dot means derivative with respect to t, spatial indices are raised and lowered with h_{ij} and D_i denotes the covariant derivative with respect to h_{ij}.

The idea of the ADM formalism is to study general actions of the form:

[10] It turns out that these are related to standard Horndeski via a disformal transformation $g_{\mu\nu} \mapsto \Omega^2(\phi) g_{\mu\nu} + \Gamma(\phi, X) \partial_\mu \phi \partial_\nu \phi$.

$$S_g = \int d^4x\sqrt{-g}L(N, K_{ij}, {}^{(3)}R_{ij}, h_{ij}, D_i; t). \tag{6.10}$$

Models such as standard GR, quintessence, K-essence, Horndeski and beyond Horndeski models can all be casted into this form. For instance the ADM Lagrangian for standard GR is of the form:

$$L_{\rm GR} = \frac{1}{16\pi G}[K_{ij}K^{ij} - K^2 + {}^{(4)}R]. \tag{6.11}$$

The dynamics of perturbations in this formalism is controlled by the so-called α-parameters. Let us discuss how these arise for the case of perturbations around the flat FRW metric of the form:

$$ds^2 = -\bar{N}^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \tag{6.12}$$

From this it follows that ${}^{(3)}R_{ij} = 0$, $K^i_j = H\delta^i_j \equiv \frac{\dot{a}}{\bar{N}a}$ such that the background Lagrangian is a function $\bar{L}(a, \dot{a}, \bar{N})$. By variation of the action with respect to a and N the background equations of motion can be obtained. However, for the study of α-parameters it is important to consider the full Lagrangian to quadratic order. Define $\delta N := N - \bar{N}$ and $\delta K^j_i := K^j_i - H\delta^j_i$. Then the Lagrangian can be expanded (we will omit the (3) in ${}^{(3)}R^i_j$ for briefness):

$$L(N, K^i_j, R^i_j, ...) = \bar{L} + L_N\delta N + \frac{\partial L}{\partial K^i_j}\delta K^i_j + \frac{\partial L}{\partial R^i_j}\delta R^i_j + L^{(2)} + ..., \tag{6.13}$$

where $L^{(2)}$ indicates the quadratic order Lagrangian. The derivatives are evaluated at the background. After some algebraic manipulations and by introducing coefficients the second order part $\sqrt{-g}L^{(2)}$ can be written as:

$$\begin{aligned}\sqrt{-g}L^{(2)} = {} & \bar{N}\mathcal{G}^\star\delta_1 R\delta\sqrt{h} + a^3\Big(L_N + \frac{1}{2}\bar{N}L_{NN}\Big)\delta N^2 + \bar{N}a^3\Big[\mathcal{G}^\star\delta_2 R + \frac{1}{2}\hat{\mathcal{A}}_K\delta K^2 + \mathcal{C}^\star\delta K\delta R \\ & + \mathcal{A}_K\delta K^i_j\delta K^j_i + \mathcal{A}_R\delta R^i_j\delta R^j_i + \frac{1}{2}\hat{\mathcal{A}}_R\delta R^2 + \Big(\frac{\mathcal{G}^\star}{\bar{N}} + \mathcal{B}^\star_R\Big)\delta N\delta R\Big] + ...,\end{aligned} \tag{6.14}$$

In this expression there are some coefficients that depend on the particular theory and δ_1, δ_2 indicate the first and second order perturbations.

This formula allows us to study tensor, vector and scalar perturbations separately. The tensor modes correspond at linear order to the metric $h_{ij} = a^2(t)(\delta_{ij} + \gamma_{ij})$ where $\gamma^i_i = \partial^i\gamma_{ij} = 0$. The quadratic action for these tensor modes reads (by putting $\bar{N} = 1$):

$$S^{(2)}_\gamma = \int d^3x dt a^3\Big[\frac{\mathcal{A}_K}{4}\dot{\gamma}^2_{ij} - \frac{\mathcal{G}^\star}{4a^2}(\partial_k\gamma_{ij})^2\Big]. \tag{6.15}$$

From this expression we note that it is suggestive to define the mass by $M^2 = 2\mathcal{A}_K$. To avoid the presence of ghosts we clearly need $\mathcal{A}_K > 0$. The action can be rewritten:

$$S^{(2)}_\gamma = \int d^3x dt a^3\frac{M^2}{8}\Big[\dot{\gamma}^2_{ij} - \frac{c^2_T}{a^2}(\partial_k\gamma_{ij})^2\Big], \tag{6.16}$$

where $c^2_T := \mathcal{G}^\star/\mathcal{A}_K$ can be understood as the graviton propagator speed squared.

The parameter α_T describes the deviation of the speed of gravitational waves compared to standard GR:

$$\alpha_T := c_T^2 - 1. \tag{6.17}$$

The mass M is in the general case time-dependent. The time-dependence of this quantity is characterised by the parameter α_M:

$$\alpha_M := \frac{1}{H}\frac{d}{dt}\ln(M^2). \tag{6.18}$$

The evolution equation for tensor modes can then be described in terms of these two α-parameters. Similarly, one can study the vector modes which are characterised by $N^i = N_V^i$ with $\partial_i N_V^i = 0$. The quadratic action of these modes do not give rise to new α-parameters. However, to describe the dynamics of the scalar perturbations one introduces three other α-parameters called α_K, α_B and α_H. To describe these scalar modes, three scalar perturbations $\delta N, \psi, \zeta$ are introduced: $N = 1 + \delta N$, $N^i = \delta^{ij}\partial_j\psi$ and $h_{ij} = a^2(t)e^{2\zeta}\delta_{ij}$. The quadratic action reveals that only ζ will be a propagating degree of freedom. The α-parameters are defined as [11]

$$\alpha_B := \frac{\mathcal{B}}{4H\mathcal{A}_K}, \alpha_K = \frac{2L_N + L_{NN}}{2H^2\mathcal{A}_K}, \alpha_H := \frac{\mathcal{G}^\star + \mathcal{B}_R^\star}{\mathcal{A}_K} - 1. \tag{6.19}$$

The Lagrangian for a general metric $g_{\mu\nu}$ with the condition on ζ can be written as follows:

$$S^{(2)} = \int d^3x dt a^3 \frac{M^2}{2}\Big[\delta K_{ij}\delta K^{ij} - \delta K^2 + (1+\alpha_T)\Big(R\frac{\delta\sqrt{h}}{a^3} + \delta_2 R\Big) + \alpha_K H^2 \delta N^2 + 4\alpha_B H \delta K \delta N + (1+\alpha_H)\delta R \delta N\Big]. \tag{6.20}$$

This form can be casted into the standard EFT form by which the α-parameters are related to EFT functions (see e.g. [10],[11]), which we will also encounter in the section on positivity bounds for the case of Horndeski theory (equation (7.58)) [26].

In the ADM formalism we picked a particular coordinate system in which the constant time hypersurfaces coincide with the uniform scalar field hypersurfaces. To study the evolution of cosmological perturbations one often performs the Stückelberg trick, which is the introduction of a time diffeomorphism $t \mapsto t + \pi(t, \mathbf{x})$ with π a scalar field perturbation (called the Stückelberg field or Goldstone boson). With this transformation the action of the ADM formalism can be made covariant by introducing a new scalar perturbation π.

[11] This assumes that there not more than two spatial derivatives in terms of ζ only in the quadratic action.

7 Positivity bounds for scalar-tensor theories

Positivity bounds originate from the study of particle physics. Therefore it is important to first introduce some basic notions of (advanced) quantum field theory (QFT) that will be used in the derivations of the positivity bounds.

7.1 Review of QFT on Minkowski spacetime [12],[13]

QFT can be studied in two main ways. Namely, via the canonical quantization method or via the path integral formalism. In the context of positivity bounds of cosmological EFT models it turns out that the path integral formalism is more appropriate, e.g. because of the presence of the graviton field $h_{\mu\nu}$. All the fields we discuss are assumed to be in the Heisenberg picture. Let $|\Omega\rangle$ denote the vacuum state of some theory (including interactions). In QFT we are often interested in computing correlation functions since these are related to the scattering amplitudes and they give us the Feynman rules of a theory. For instance the 2-point correlation function of a scalar field ϕ is found by:

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)e^{iS}}{\int \mathcal{D}\phi e^{iS}}, \tag{7.1}$$

where $S = \int \mathcal{L}(x)d^4x$ is the action and $\mathcal{D}\phi$ indicates that we integrate over all field configurations. In similar fashion one can define the relation for spinor fields, gauge fields and gravitons.

The key notion of the advanced QFT formalism is to express all correlation functions in terms of a so-called generating function $Z[J]$. The physical idea behind this is that an arbitrary external source $J(x)$ gets introduced which couples to the field ϕ and creates intermediate states of a physical process, which is equivalent to integrating over all field configurations. In a more general context this source can carry indices, depending on the field you consider, for instance $J^{\mu\nu}$ in the case of a graviton $h_{\mu\nu}$. We will however first focus on a scalar field. The generating function for a scalar field $Z[J]$ is defined via:

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x[\mathcal{L}(\phi)+J(x)\phi(x)]}. \tag{7.2}$$

From this definition it can be seen that the two-point correlation function can be written in terms of the generating function using the functional derivative (see Appendix, section 9.2 for more details):

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \frac{1}{Z[0]}\Big(-i\frac{\delta}{\delta J(x_1)}\Big)\Big(-i\frac{\delta}{\delta J(x_2)}\Big)Z[J]\Big|_{J=0}. \tag{7.3}$$

Similarly, one can define the n-point correlation function:

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|\Omega\rangle = \frac{1}{Z[0]}\prod_{i=1}^{n}\Big(-i\frac{\delta}{\delta J(x_i)}\Big)Z[J]\Big|_{J=0} = \frac{\int \mathcal{D}\phi\phi(x_1)\phi(x_2)...\phi(x_n)e^{iS}}{\int \mathcal{D}\phi e^{iS}}. \tag{7.4}$$

An important remark is that the propagator of some theory can be found by evaluating the two-point correlation for the free part of the theory. For instance consider the case of the Klein-Gordon Lagrangian for a scalar field $\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2$. The action in this case can be written as follows:

$$S_0 = \int d^4x\Big[\frac{1}{2}\phi(\partial^2 - m^2 + i\epsilon)\phi + J\phi\Big], \tag{7.5}$$

where we introduced a small term $i\epsilon$ like in the Feynman description and the subscript 0 indicates that we consider the free theory. The first term suggests that the propagator $D(x-y)$ should obey the following equation:

$$(\partial^2 - m^2 + i\epsilon)D(x-y) = i\delta(x-y). \tag{7.6}$$

In Fourier space this equation can be solved easily:

$$\tilde{D}(k) = \frac{-i}{k^2 + m^2 - i\epsilon}, \tag{7.7}$$

such that the real space propagator is written as

$$D(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + m^2 - i\epsilon} e^{-ik\cdot(x-y)}. \tag{7.8}$$

By defining the field $\phi'(x) = \phi(x) - i\int d^4y D(x-y)J(y)$, the free field generating function can be written as:

$$Z_0[J] = Z_0[0] e^{-\frac{1}{2}\int d^4x d^4y J(x) D(x-y) J(y)}. \tag{7.9}$$

A simple calculation then shows that $\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = D(x_1 - x_2)$.

Interactions of the scalar field can be taken into account by adding a interaction potential $V(\phi)$ to the Lagrangian. Since for a generic analytic functional F we have $F(\phi)e^{i\int d^4xJ\phi} = F(-i\frac{\delta}{\delta J})e^{i\int d^4xJ\phi}$ it follows that:

$$Z[J] = \int \mathcal{D}\phi e^{i\int d^4x(\mathcal{L}_0 + V(\phi) + J\phi)} = e^{i\int d^4x V(-i\frac{\delta}{\delta J(x)})} Z_0[J]. \tag{7.10}$$

In general this is quite complicated to compute or take functional derivatives of. Therefore it is often assumed that the interaction potential can be treated as a perturbation [12]. In that case the exponent can be expanded to leading-order as:

$$Z[J] \approx \Big(1 + i\int d^4x V\Big(-i\frac{\delta}{\delta J(x)}\Big)\Big) Z_0[J]. \tag{7.11}$$

The correlation functions can be found from this expression by computing functional derivatives. These correlation functions simply provide the Feynman rules (of vertices) of a given theory. In similar fashion gauge fields and spinors can be described. However, for the first gauge fixing is an important complication whereas the latter is difficult because it requires the notion of Grassmann numbers.

Another important field in the context of cosmology is the graviton field. This is interpreted as the first-order perturbation of the metric in perturbation theory. Let $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$. The field $h_{\mu\nu}$ can be interpreted as the graviton. The generating function of a theory containing both a graviton and a scalar field is given by [13]:

$$Z[J, J^{\alpha\beta}] = \int \mathcal{D}\phi \mathcal{D}h e^{i\int d^4x[\mathcal{L} + J(x)\phi(x) + h_{\alpha\beta}(x)J^{\alpha\beta}(x)]}, \tag{7.12}$$

where $\mathcal{D}h$ integrates over all graviton field configurations and $J^{\alpha\beta}$ is a generalised current with indices (which satisfies $J^{\alpha\beta} = J^{\beta\alpha}$ due to $h_{\alpha\beta} = h_{\beta\alpha}$). Let us focus on the pure graviton case (i.e. $\phi = J = 0$). However, we know that gravity theories such as standard GR and Horndeski are diffeomorphism invariant [14]. This means that the action does not change under a gauge transformation $h_{\mu\nu} \mapsto h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu + h_{\mu\sigma}\partial_\nu\xi^\sigma + h_{\nu\sigma}\partial_\mu\xi^\sigma + \xi^\sigma\partial_\sigma h_{\mu\nu}$ where ξ^μ is an infinitesimal vector field. Typically, the path integral (7.12) diverges since the integral does not take into account the fact that some field configurations are related by a gauge transformation and are therefore physically equivalent. Therefore we need to fix the gauge. The way to do this is by adding a gauge fixing term to the Lagrangian $\mathcal{L}$ of the type $\mathcal{L}_{\mathrm{GF}} = \frac{1}{2\alpha}C_\nu(h)C^\nu(h)$ for some constant α and field $C_\nu(h)$, which comes from the Faddeev-Popov method [15]. In the literature the convenient gauge is often de Donder gauge for which $C^\nu(h) = \partial_\mu h^{\mu\nu} - \frac{1}{2}\eta^{\alpha\nu}\eta^{\sigma\beta}\partial_\alpha h_{\sigma\beta}$. Once the gauge has been fixed it is possible to compute correlation functions in the usual fashion, for instance the 2-point correlation function is found by:

[12] A typical example is ϕ^4 theory for which $V(\phi) = -\frac{\lambda}{4!}\phi^4$ with $\lambda \ll 1$ a small parameter.

$$\langle\Omega|T\{h_{\alpha\beta}(x_1)h_{\sigma\lambda}(x_2)\}|\Omega\rangle = \frac{\int \mathcal{D}h h_{\alpha\beta}(x_1)h_{\sigma\lambda}(x_2)e^{iS}}{\int \mathcal{D}h e^{iS}} = \frac{1}{Z[0]}\Big(-i\frac{\delta}{\delta J^{\alpha\beta}(x_1)}\Big)\Big(-i\frac{\delta}{\delta J^{\sigma\lambda}(x_2)}\Big)Z[J^{\kappa\nu}]\Big|_{J^{\kappa\nu}=0}. \tag{7.13}$$

The propagator of a graviton can just like for the scalar field be found by looking at the quadratic free part of the Lagrangian of some theory (after gauge fixing has been performed). In section 7.4 (and details in the Appendix) we will derive the graviton propagator in case of Horndeski theory on Minkowski space (with $\bar{G}_4 = 1/2$ being the special case of GR).

If we consider both gravitons and scalar fields then we can define for instance the correlation function for $\phi\phi h$-interactions as follows:

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)h_{\alpha\beta}(x_3)\}|\Omega\rangle = \frac{1}{Z[0,0]}\Big(-i\frac{\delta}{\delta J(x_1)}\Big)\Big(-i\frac{\delta}{\delta J(x_2)}\Big)\Big(-i\frac{\delta}{\delta J^{\alpha\beta}(x_3)}\Big)Z[J,J^{\mu\nu}]\Big|_{J=J^{\mu\nu}=0}. \tag{7.14}$$

Interactions can be taking into account in the Lagrangian like for the pure scalar field case (7.11).

7.2 Motivation behind positivity bounds

In many modified gravity theories, the Lagrangian describes the physics up to a certain energy scale which we call the cut-off. Such a theory is called an effective field theory (EFT). Typically this cut-off is taken to be of order the Planck mass since above those energies QFT and GR does not apply. To describe such energies one would require a quantum gravity theory (or grand unified theory) or at least some UV completion consistent with the known physics at low energies. The viability conditions on the EFT theory have been discussed. But there are additional viability conditions coming from the question whether the EFT admits a UV completion with certain general properties, i.e. whether there is some theory which describes the physics above the cut-off and which reduces to the EFT at low energies. The approach that will be taken in this thesis is that the UV completion should be Wilsonian (called the standard UV completion), i.e. it must be causal, unitary, Poincaré invariant [13], local and crossing-symmetric [16]. These requirements yield conditions on the scattering amplitudes in the EFT, which are called positivity bounds. In the context of cosmology these are additional theoretical priors apart from the usual ghost, gradient and tachyonic instabilities. Therefore it is expected that these positivity bounds constrain the known EFT models further, which makes it easier to rule out models and to compare them with upcoming cosmological data. Furthermore, these positivity bounds can also reveal whether a certain EFT can be constructed from a standard UV completion or not. If the positivity bounds are not consistent with the cosmological data, this is an indication that the EFT does not admit a standard UV completion. In that case it might be that some assumptions have to be omitted such as Poincaré invariance or causality.

7.3 Formalism of positivity bounds for scalar-tensor theories on Minkowski spacetime

In this subsection the formalism of positivity bounds of scalar theories will be summarized following the literature [21],[23]. In this discussion we will first focus on positivity bounds for massive scalar particles in Minkowski space. This however immediately gives rise to problems in the case of Horndeski theory since the graviton admits a massless t-pole, which means that the scattering amplitude scales as $\mathcal{A} \sim 1/t$ and thus diverges in the forward limit $t \to 0$ [17]. Hence the scattering amplitude cannot be analytic, whereas this should be the case in order to apply the positivity bounds formalism [23]. In [17] a possible solution for this has been presented, namely it is said that we can work in the decoupling limit $M_{\rm pl} \to \infty$ (with Λ_3 fixed) in which the massless t-pole is suppressed by $\sim 1/(M_{\rm pl}t)$. A more rigorous argument for the fact that positivity bounds remain to hold in gravitational theories is provided in [21].

[13] This is only true for a Minkowski background. In the case of a cosmological background, time-translation symmetry and boost symmetry are broken.

Positivity bounds are conditions on the EFT scattering amplitudes coming from taking the low-energy limit of conditions on the scattering amplitudes of the underlying UV theory which is left completely general apart from these requirements. Consider 2-to-2 scattering of massive scalar fields. Let p_1, p_2 be the incoming momenta and p_3, p_4 the outgoing momenta. Then we define the Mandelstam variables by $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_3)^2$ and $u = 4m^2 - t - s = -(p_1 - p_4)^2$ where m is the mass of a scattered particle. We assume that all particles have the same mass m. The first requirement is that the theory should be **Poincaré invariant** such that the scattering amplitude A can be written as a function of the Mandelstam variables (i.e. $A = A(s,t)$) and that the coefficients in the scattering amplitude are constant so that one can apply ordinary QFT [21]. The second requirement on a Wilsonian UV theory is that it should satisfy **unitarity** [23]. Intuitively this tells us that probabilities should be conserved. Unitarity requires that the optical theorem $\mathrm{Im}[A(s,0)] = \sqrt{s(s-4m^2)}\sigma(s)$ with σ the cross-section should be satisfied and that the partial wave expansion (P_l are Legendre polynomials and $a_l \in \mathbb{C}$):

$$A(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{l=0}^{\infty}(2l+1)P_l(\cos(\theta))a_l(s), \tag{7.15}$$

should obey $\mathrm{Im}(a_l(s)) = |a_l(s)|^2 + ...$ [14] with $0 \leq |a_l(s)|^2 \leq \mathrm{Im}(a_l(s)) \leq 1$ for $s \geq 4m^2$. Or by $\partial_t^n P_l(1+t)|_{t=0} \geq 0$ one can also write the condition (for an interacting theory) as:

$$\frac{\partial^n}{\partial t^n}\mathrm{Im}[A(s,t)]\Big|_{t=0} > 0 \; \forall n \geq 0, \; s \geq 4m^2.$$

Another requirement is that the scattering amplitude in the UV theory should be **analytic**. Usually it is required that $A(s,t)$ is analytic [15] in the whole (s,t)-plane apart from poles and branch cuts which can be explained by unitarity and crossing symmetry (i.e. between s-channel, t-channel and u-channel of a given process). A weaker notion of analyticity however has been applied to derive the positivity bounds. Namely, one assumes that the scattering amplitude is analytic (up to poles) for $0 \leq s,t,u < 4m^2$. In scalar theories it is possible to extend the positivity property from unitarity to $0 \leq t < 4m^2$ since $A(s,t)$ has a t-channel pole at $t = m^2$ with s-independent real residue and therefore $\mathrm{Im}[A(s,t)]$ is analytic without poles for $|t| < 4m^2$. In the derivation of the positivity bounds this means that Cauchy's integral theorem can be applied to the scattering amplitude. From analyticity and unitary one could derive the Froissart-Martin bound, which describes the scattering amplitude in the limit $s \to \infty$ while keeping t fixed, in order to ensure that the theory is local. The Froissart-Martin bound is given by:

$$\lim_{s\to\infty}|A(s,t)| < Cs^{1+\epsilon(t)}, \tag{7.16}$$

where C is some constant, $\epsilon(t)$ is some function of the Mandelstam variable t. Technically, the Froissart-Martin bound is needed in the derivation of the positivity bounds in order to be able to neglect arc parts of contour integrals in the complex (upper- and lower-half) plane.

Another important notion in the development of positivity bounds is the so-called dispersion relation. But first it is convenient to define the pole-subtracted scattering amplitude $B(s,t)$ via:

$$B(s,t) := A(s,t) - \frac{\lambda}{m^2 - s} - \frac{\lambda}{m^2 - u} - \frac{\lambda}{m^2 - t}, \tag{7.17}$$

where $\lambda := \mathrm{Res}_{u=m^2}A(s,t)$. Define $\bar{x} := x - 4m^2/3$ where x can be any quantity and $v := \bar{s} + \bar{t}/2$. Then it follows that $B(s,t)$ should be given by:

$$B(s,t) = b(t) + \int_{4m^2}^{\infty}\frac{d\mu}{\pi(\bar{\mu} + \bar{t}/2)}\frac{2v^2\mathrm{Im}[A(\mu,t)]}{(\bar{\mu} + \bar{t}/2)^2 - v^2}.$$

[14] In this formula ... indicate terms which come from inelastic scattering and these are proportional to the cross-section and turn out to be positive as well.

[15] I.e. the Cauchy equations $\partial_s\mathrm{Re}[A(s,t)] = \partial_t\mathrm{Im}[A(s,t)]$ and $\partial_t\mathrm{Re}[A(s,t)] = -\partial_s\mathrm{Im}[A(s,t)]$ are satisfied.

The function $b(t)$ is not determined by analyticity properties and depends on the process.

Define for $N \geq 1$:

$$B^{(2N,M)}(t) := \frac{1}{M!}\partial_v^{2n}\partial_t^M B(v,t)|_{v=0}. \tag{7.18}$$

These functions can be expressed in terms of positive integrals:

$$B^{(2N,M)}(t) = \sum_{k=0}^{M} \frac{(-1)^k}{k!2^k} I^{(2N+k,M-k)}, \tag{7.19}$$

where

$$I^{(q,p)}(t) := \frac{q!}{p!}\frac{2}{\pi}\int_{4m^2}^{\infty} \frac{d\mu \partial_t^p \mathrm{Im}[A(\mu,t)]}{(\bar{\mu}+\bar{t}/2)^{q+1}} > 0. \tag{7.20}$$

The positivity of the integrals $I^{(q,p)}(t) > 0$ is precisely where the positivity bounds come from. Upon defining α_k, β_k and c_k as:

$$\begin{aligned}
\alpha_k &= \sum_{r=0}^{k} \frac{2^{2(r-k)}}{(2k-2r)!} c_r, \\
\beta_k &= (-1)^k \sum_{r=0}^{k} \frac{2^{2(r-k)-1}}{(2k-2r-1)!} c_r, \\
c_0 &= 1, \\
c_k &= -\sum_{r=0}^{k-1} \frac{2^{2(r-k)}}{(2k-2r)!} c_r \ \forall k \geq 1.
\end{aligned} \tag{7.21}$$

Define $\mathcal{M} := (t+4m^2)/2$ and $Y^{(2N,M)}(t)$ recursively:

$$Y^{(2N,M)}(t) = \sum_{r=0}^{M/2} c_r B^{(2N+2r,M-2r)} + \frac{1}{\mathcal{M}^2} \sum_{k \text{ even}}^{(M-1)/2} (2(N+k)+1)\beta_k Y^{(2(N+k),M-2k-1)}.$$

From which the positivity bounds $Y^{(2N,M)}(t) \geq I^{(2N,M)} > 0$ for all $N \geq 1$, $M \geq 0$ and $0 \leq t < 4m^2$ are found.

In the low-energy (EFT) limit, under the assumption that loop corrections can be ignored (i.e. if tree-level amplitude suffices), the scattering amplitude $B(s,t)$ can be written in terms of $x = -(\bar{s}\bar{t} + \bar{t}\bar{u} + \bar{s}\bar{u})$ and $y = -\bar{s}\bar{t}\bar{u}$ with $\bar{s} + \bar{t} + \bar{u} = 0$ as:

$$B(s,t) = \sum_{n,m \in \mathbb{N}} \frac{a_{nm}}{\Lambda^{4n+6m}} x^n y^m, \tag{7.22}$$

where Λ is some energy scale which makes the a_{nm} dimensionless. The first few positivity bounds imply that:

$$
\begin{aligned}
&Y^{(2,0)} : a_{10} > 0 \\
&Y^{(2,1)} : a_{01} > -\frac{3\Lambda^2}{2\Lambda_{\text{th}}^2} a_{10} \\
&Y^{(4,0)} : a_{20} > 0.
\end{aligned}
\tag{7.23}
$$

Λ_{th} is defined as the first mass scale above the cut-off of the EFT.

7.4 Positivity bounds for Horndeski theory on a Minkowski background

As discussed in section 5.7 the Horndeski gravity theory (or generalised Galileons) contains a scalar field ϕ as a dark energy candidate and this model contains many other models such as $f(R)$ theory, standard GR, quintessence. Since Horndeski theory is an EFT we can expect that the theory breaks down at some cut-off $\Lambda_3 \ll M_{\text{pl}}$. Because of this reason, one defines the mass scales $\Lambda_1 = M_{\text{pl}}$, $\Lambda_2^2 = M_{\text{pl}} H_0$ and $\Lambda_3^3 = M_{\text{pl}} H_0^2$ where H_0 is the Hubble parameter today such that the Horndeski Lagrangian can be written as [17]:

$$
S = \int d^4x \sqrt{-g} \sum_{i=2}^{5} \mathcal{L}_i, \tag{7.24}
$$

where the Lagrangians $\mathcal{L}_i$ are defined by

$$
\begin{aligned}
\mathcal{L}_2 &= \Lambda_2^4 G_2, \\
\mathcal{L}_3 &= \Lambda_2^4 G_3[\Phi], \\
\mathcal{L}_4 &= M_{\text{pl}}^2 G_4 R + \Lambda_2^4 G_{4,X}([\Phi]^2 - [\Phi^2]), \\
\mathcal{L}_5 &= M_{\text{pl}}^2 G_5 G_{\mu\nu} \Phi^{\mu\nu} - \frac{1}{6} \Lambda_2^4 G_{5,X}([\Phi]^3 - 3[\Phi][\Phi^2] + 2[\Phi^3]),
\end{aligned}
$$

Here G_2, G_3, G_4 and G_5 are functions of the dimensionless quantities ϕ/Λ_1 and $X = -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi/\Lambda_2^4$. We will also allow an explicit mass term $-\frac{1}{2}m^2\phi^2$ in the action. The functions G_i are different from the formulation in section 5.7 as the energy scales are written explicitly. The notation is such that $\Phi^\mu_\nu := \nabla^\mu\nabla_\nu\phi/\Lambda_3^3$ and square brackets indicate the trace, e.g. $[\Phi^2] = \nabla^\mu\nabla_\nu\phi\nabla^\nu\nabla_\mu\phi/\Lambda_3^6$ and $,\phi$ indicates the partial derivative with respect to ϕ/Λ_1 and $,X$ indicates the partial derivative with respect to X.

The tree-level scattering amplitudes in Horndeski theory take the following form for an elastic $2 \to 2$ scattering in the limit $M_{\text{pl}} \to \infty$ (with Λ_3 fixed) [17]:

$$
\mathcal{A}(s,t) = c_{ss} \frac{s^2}{\Lambda_2^4} + c_{sst} \frac{s^2 t}{\Lambda_3^6} + ..., \tag{7.25}
$$

where s, t are the Mandelstam variables (see previous subsection for the definitions). This equation counts in powers of s and t. It assumes the high-s limit (s is large but below the cut-off scale) and the forward limit $t \to 0$ and the equation is obtained in the center-of-mass frame. The assumption of the forward limit and center-of-mass frame are without loss of generality since the amplitude is Poincaré invariant as we consider a Minkowski background spacetime. Loop corrections for bounds up to order $\mathcal{O}(\Lambda_3^{-6})$ can be neglected to good approximation [21]. The positivity bounds for such a theory are of the following form [17]:

$$
c_{ss} \geq 0, c_{sst} \geq -\frac{3\Lambda_3^4}{2\Lambda_2^4} c_{ss}. \tag{7.26}
$$

In the reference [17], the action (7.24) is expanded about a Minkowski spacetime via $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}/M_{\text{pl}}$ and $\phi = \langle\phi\rangle + \varphi$ where $\langle\phi\rangle = 0$ and M_{pl} is taken to be large with Λ_3 fixed (so the order of perturbation is counted in $1/M_{\text{pl}}$). From this it is possible to study the scattering amplitudes of the processes $\varphi\varphi \to \varphi\varphi$, $\varphi h \to \varphi h$ and $hh \to hh$. Using these scattering amplitudes it is then possible to derive the positivity bounds. In this section we will provide some detailed calculations of the process $\varphi\varphi \to \varphi\varphi$. The main reason is that there is a disagreement in the literature about the exact positivity bounds for Horndeski gravity [17], [21]. The method provided can also be used to find positivity bounds for other dark energy theories such as vector-tensor theory or DHOST. Also by making the method in [17] more explicit it becomes easier to study positivity bounds for cosmological backgrounds [22]. We start by deriving the propagators of the scalar field ϕ and the graviton $h_{\alpha\beta}$. Afterwards we will compute the vertices of the theory in momentum space. Indices on perturbations and partial derivatives are raised using the Minkowski metric. Indices on covariant derivatives are raised and lowered with the full metric.

7.4.1 Propagators in Horndeski gravity

The scalar propagator can be found by considering the Lagrangian which contain terms quadratic in φ after performing the expansion around the Minkowski background. In order to derive this we notice that the G_i functions can be expanded about the background by [18]:

$$G_i(\phi/\Lambda_1, X) = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \left(\frac{\partial^{n+m} G_i}{\partial^n (\phi/M_{\text{pl}}) \partial^m X} \right)_{\phi=0, X=0} \left(\frac{\varphi}{M_{\text{pl}}} \right)^n Y^m, \tag{7.27}$$

where $Y = -\frac{1}{2\Lambda_2^4} \nabla_\mu \varphi \nabla^\mu \varphi$. Evaluation of quantities at the background spacetime are indicated by bars on top.

The quadratic order Lagrangian for φ, denoted $\mathcal{L}_{\varphi\varphi}$, can be found from the Lagrangian (7.24) to be:

$$\begin{aligned} \mathcal{L}_{\varphi\varphi} = &\frac{1}{2}\Lambda_2^4 \bar{G}_{2,\phi\phi} \frac{\varphi^2}{M_{\text{pl}}^2} - \Lambda_2^4 \bar{G}_{2,X} \left(\frac{1}{2\Lambda_2^4} \partial_\mu \varphi \partial^\mu \varphi \right) + \frac{\Lambda_2^4}{\Lambda_3^3 M_{\text{pl}}} \bar{G}_{3,\phi} \varphi \partial^\mu \partial_\mu \varphi \\ &+ \frac{\Lambda_2^4}{\Lambda_3^6} \bar{G}_{4,X} (\partial^\mu \partial_\mu \varphi \partial^\nu \partial_\nu \varphi - \partial^\mu \partial_\nu \varphi \partial_\mu \partial^\nu \varphi) - \frac{1}{2} m^2 \varphi^2. \end{aligned} \tag{7.28}$$

Integration by parts and neglecting boundary terms shows that the second last term vanishes and the third term can be rewritten such that:

$$\mathcal{L}_{\varphi\varphi} = \frac{1}{2M_{\text{pl}}^2} \Lambda_2^4 \bar{G}_{2,\phi\phi} \varphi^2 - \frac{1}{2} \bar{G}_{2,X} (\partial^\mu \varphi)(\partial_\mu \varphi) - \bar{G}_{3,\phi} (\partial^\mu \varphi)(\partial_\mu \varphi) - \frac{1}{2} m^2 \varphi^2. \tag{7.29}$$

The propagator can be normalised in such a way that it is that of a massless scalar field by letting $\bar{G}_{2,\phi\phi} = \frac{m^2 M_{\text{pl}}^2}{\Lambda_2^4} = m^2/H_0^2$ and $\bar{G}_{3,\phi} + \frac{1}{2}\bar{G}_{2,X} = 1/2$ such that the Lagrangian becomes that of a free massless scalar field:

$$\mathcal{L}_{\varphi\varphi} = -\frac{1}{2} (\partial^\mu \varphi)(\partial_\mu \varphi). \tag{7.30}$$

Therefore the scalar propagator is in momentum space given by (ignoring the $i\epsilon$ term):

$$\mathcal{P}(p) = -\frac{i}{p^2}. \tag{7.31}$$

The graviton propagator is much less trivial to derive. The part of the Lagrangian (7.24) quadratic in the graviton field, denoted $\mathcal{L}_{hh}$, can be found by the following expansion at quadratic order: [16]

$$\mathcal{L}_{hh} = \delta_1\sqrt{-g}M_{\rm pl}^2\bar{G}_4\delta_1 R + M_{\rm pl}^2\bar{G}_4\delta_2 R, \tag{7.32}$$

where δ_1, δ_2 indicate the first and second order part (the latter includes the factor $1/2!$).

It can be shown that the first and second order part of the Ricci scalar are given by (Appendix, section 9.3):

$$\begin{aligned} M_{\rm pl}\delta_1 R &= \partial^\alpha\partial^\beta h_{\alpha\beta} - \eta^{\mu\nu}\partial_\alpha\partial^\alpha h_{\mu\nu}, \\ M_{\rm pl}^2\delta_2 R &= \frac{1}{2}(\partial^\mu h^\lambda_\mu)(\partial_\nu h^\nu_\lambda) - \frac{1}{4}(\partial^\mu h^\alpha_\nu)(\partial_\mu h^\nu_\alpha) - \frac{1}{4}(\partial_\alpha h)(\partial^\alpha h), \end{aligned} \tag{7.33}$$

where $h := \eta^{\mu\nu}h_{\mu\nu}$ and $\partial^\alpha := \eta^{\alpha\beta}\partial_\beta$.

From equation (2.13) of [15] it follows that $\sqrt{-g} = 1 + \frac{h^{\mu\nu}\eta_{\mu\nu}}{2M_{\rm pl}} + \frac{1}{8M_{\rm pl}^2}\left(\frac{1}{8}(\eta_{\mu\nu}h^{\mu\nu})^2 - \frac{1}{4}h_{\mu\nu}^2\right) + \mathcal{O}(1/M_{\rm pl}^3)$. With this expression it can be seen that $\mathcal{L}_{hh}$ can be written as (by performing integration by parts):

$$\mathcal{L}_{hh} = \frac{\bar{G}_4}{4}\delta^{\alpha\beta\gamma}_{\rho\sigma\nu}(\partial_\beta h^\rho_\alpha)(\partial^\sigma h^\nu_\gamma), \tag{7.34}$$

where $\delta^{\alpha\beta\gamma}_{\rho\sigma\nu}$ is the generalised Kronecker delta (see Appendix, section 9.3 for details).

The graviton propagator $\mathcal{P}^{\nu\beta}_{\mu\alpha}$ without gauge fixing in real space satisfies (in analogy with the scalar propagator):

$$\frac{\bar{G}_4}{2}\delta^{\alpha\rho\mu'}_{\beta\sigma\nu'}\partial_\rho\partial^\sigma\mathcal{P}^{\nu\beta}_{\mu\alpha} = i\delta^{\mu'}_\mu\delta^\nu_{\nu'}\delta(x-y). \tag{7.35}$$

So we can then define the graviton propagator (in momentum space) $\mathcal{P}^{\nu\beta}_{\mu\alpha}(p)$ by:

$$\frac{\bar{G}_4}{2}\mathcal{P}^{\nu\beta}_{\mu\alpha}(p)\delta^{\alpha\rho\mu'}_{\beta\sigma\nu'}p_\rho p^\sigma = -i\delta^{\mu'}_\mu\delta^\nu_{\nu'}. \tag{7.36}$$

So we find the expression for the graviton propagator given in the Feynman rules of reference [17] where there has been no gauge fixing applied yet. Horndeski gravity is diffeomorphism invariant so gauge fixing is necessary. Applying a gauge fixing in this gauge amounts to adding the following Lagrangian to $\mathcal{L}_{hh}$:

$$\mathcal{L}_{\rm GF} = -\frac{\bar{G}_4}{2}\left(\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_\nu h\right)^2. \tag{7.37}$$

The gauge-fixed Lagrangian $\mathcal{L}_{hh} + \mathcal{L}_{\rm GF}$ then becomes:

$$\mathcal{L}_{hh} + \mathcal{L}_{\rm GF} = \frac{\bar{G}_4}{8}\partial^\alpha h\partial_\alpha h - \frac{\bar{G}_4}{4}(\partial_\lambda h^\mu_\nu)(\partial^\lambda h^\nu_\mu). \tag{7.38}$$

This expression can be written in the following form after integration by parts [15]:

[16] The term with $\delta_2\sqrt{-g}\Lambda_2^4\bar{G}_2$ will be ignored since it is suppressed by $\delta_2\sqrt{-g}\Lambda_2^4\bar{G}_2 \propto \mathcal{O}(\Lambda_3^3/M_{\rm pl})$ in the decoupling limit (compared to the other terms in $\mathcal{L}_{hh}$ it is small). This term plays the role of the mass of the graviton (or cosmological constant for $\phi = 0$), which is ignored in the decoupling limit, following the literature [17], [21].

$$\mathcal{L}_{hh} = \frac{\bar{G}_4}{4} h_{\mu\nu}\partial_\sigma\partial^\sigma (I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta})h_{\alpha\beta}, \tag{7.39}$$

where $I^{\mu\nu\alpha\beta} = \frac{1}{2}(\eta^{\mu\beta}\eta^{\alpha\nu} + \eta^{\mu\alpha}\eta^{\nu\beta})$.

The (gauge-fixed) propagator $D_{\alpha\beta\gamma\delta}(x-y)$ can be found by [15]:

$$\frac{\bar{G}_4}{2}\Big(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\Big)\partial^\sigma\partial_\sigma D_{\alpha\beta\gamma\delta}(x-y) = iI^{\mu\nu}{}_{\gamma\delta}\delta(x-y). \tag{7.40}$$

In Fourier space the propagator satisfies:

$$\frac{\bar{G}_4}{2}\Big(I^{\mu\nu\alpha\beta} - \frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta}\Big)(-k^2)D_{\alpha\beta\gamma\delta}(k) = iI^{\mu\nu}{}_{\gamma\delta}. \tag{7.41}$$

From this it follows that the propagator in its explicit form is given by (see Appendix, section 9.3 for the proof):

$$D_{\mu\nu\alpha\beta}(k) = \frac{-2i}{\bar{G}_4 k^2}\Big(I_{\mu\nu\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Big) = \frac{-i}{\bar{G}_4 k^2}\Big(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\Big). \tag{7.42}$$

So it follows that the graviton propagator is like in ordinary GR, which is expected since the only part that contributes to it comes from the GR term $\propto \bar{G}_4$ in Horndeski theory.

From these propagators it is possible to write down the free generating function, which will be used later to compute vertices in Fourier space. Obviously the scalar field part will take the form of equation (7.9). The graviton propagator is more complicated. Consider the graviton part of the free generating function, i.e. equation (7.12) with $J = 0$ and no integration over ϕ. The free generating function for gravitons only can be written as (see Appendix, section 9.3 for the proof):

$$Z_0[J^\mu_\nu] = Z_0[0]e^{-\frac{1}{2}\int d^4x d^4y J^{\alpha\beta}(x)D_{\alpha\beta\sigma\kappa}(x-y)J^{\sigma\kappa}(y)}. \tag{7.43}$$

Thus it follows that the free generating function including both h and ϕ is given by:

$$Z_0[J^\mu_\nu, J] = Z_0[0,0]e^{-\frac{1}{2}\int d^4x d^4y J(x)D(x-y)J(y)}e^{-\frac{1}{2}\int d^4x d^4y J^{\alpha\beta}(x)D_{\alpha\beta\sigma\kappa}(x-y)J^{\sigma\kappa}(y)}. \tag{7.44}$$

7.4.2 Vertices in Horndeski theory

In this section we will derive the tree-level Lagrangian vertices up to order $1/M_{\rm pl}$ in the Horndeski theory to derive results like in the Appendix of [17] (up to some little mismatches in the numerical coefficients but consistent with the findings of [21]). This can be done by expanding the action (7.24) to a particular order in the fields φ and $h_{\alpha\beta}/M_{\rm pl}$. In this section we will work out the example of the $\varphi\varphi\varphi$-vertex in detail and the rest of the derivations will be in the Appendix, section 9.4. The Lagrangians derived in this section and the Appendix, section 9.4 will include the $\sqrt{-g}$ in the action evaluated up to a certain relevant order.

To derive the $\varphi\varphi\varphi$-vertex, we expand the action (7.24) up to cubic order in φ and consider only the parts which contain $\varphi\varphi\varphi$. This yields the following Lagrangian:

$$\mathcal{L}_{\varphi\varphi\varphi} = \Lambda_2^4\bar{G}_{2,X\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)Y + \frac{1}{6}\Lambda_2^4\bar{G}_{2,\phi\phi\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)^3 + \frac{\Lambda_2^4}{2\Lambda_3^3}\bar{G}_{3,\phi\phi}(\partial^\mu\partial_\mu\varphi)\Big(\frac{\varphi}{M_{\rm pl}}\Big)^2 + \frac{\Lambda_2^4}{\Lambda_3^3}\bar{G}_{3,X}(\partial^\mu\partial_\mu\varphi)Y$$
$$+ \frac{\Lambda_2^4}{\Lambda_3^6}\bar{G}_{4,X\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)\Big[\partial^\mu\partial_\mu\varphi\partial^\nu\partial_\nu\varphi - \partial^\mu\partial_\nu\varphi\partial_\mu\partial^\nu\varphi\Big]$$
$$- \frac{\Lambda_2^4}{6\Lambda_3^9}\bar{G}_{5,X}\Big[2\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\alpha\varphi\partial^\alpha\partial_\mu\varphi - 3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi + \partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial_\beta\partial^\beta\varphi\Big]. \tag{7.45}$$

From definitions $\Lambda_2^2 = M_{\rm pl}H_0$, $\Lambda_3^3 = M_{\rm pl}H_0^2$ and $Y = -\frac{1}{2\Lambda_2^4}\nabla^\mu\varphi\nabla_\mu\varphi$ (and writing partial derivatives instead of covariant ones since we are not interested in gravitons for the $\varphi\varphi\varphi$-vertex) it follows that the Lagrangian can be written as:

$$\mathcal{L}_{\varphi\varphi\varphi} = -\frac{1}{2M_{\rm pl}}\bar{G}_{2,X\phi}\varphi(\partial^\mu\varphi)(\partial_\mu\varphi) + \frac{\Lambda_3^3}{6M_{\rm pl}^2}\bar{G}_{2,\phi\phi\phi}\varphi^3 + \frac{1}{2M_{\rm pl}}\bar{G}_{3,\phi\phi}(\partial^\mu\partial_\mu\varphi)\varphi^2$$
$$- \frac{1}{2\Lambda_3^3}\bar{G}_{3,X}(\partial^\mu\partial_\mu\varphi)(\partial_\nu\varphi)(\partial^\nu\varphi) + \frac{1}{\Lambda_3^3}\bar{G}_{4,X\phi}\varphi\Big[\partial^\mu\partial_\mu\varphi\partial^\nu\partial_\nu\varphi - \partial^\mu\partial_\nu\varphi\partial_\mu\partial^\nu\varphi\Big]$$
$$- \frac{\Lambda_2^4}{6\Lambda_3^9}\bar{G}_{5,X}\Big[2\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\alpha\varphi\partial^\alpha\partial_\mu\varphi - 3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi + \partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial_\beta\partial^\beta\varphi\Big]. \tag{7.46}$$

The order of the vertices will be counted in powers of $1/M_{\rm pl}$ (with $M_{\rm pl}$ large) and Λ_3 fixed. This Lagrangian contains leading-order terms which scale as $1/\Lambda_3^3$, subleading order terms which go as $1/M_{\rm pl}$ and lower order terms. Following the Appendix in [17] we will only consider the positivity bounds up to $1/M_{\rm pl}$. This means we can ignore the second term in the Lagrangian since it is $\mathcal{O}(M_{\rm pl}^{-2})$. The claim is also that the last term in $\mathcal{L}_{\varphi\varphi\varphi}$ vanishes under integration by parts and neglecting the boundary terms. Neglecting all boundary terms the Lagrangian can be written in a compact notation (see Appendix, section 9.4 for the proof):

$$\mathcal{L}_{\varphi\varphi\varphi} = \frac{\bar{G}_{2,X\phi} + 2\bar{G}_{3,\phi\phi}}{4M_{\rm pl}}\varphi^2\partial_\mu\partial^\mu\varphi + \frac{\bar{G}_{3,X} + 3\bar{G}_{4,X\phi}}{3\Lambda_3^3}\delta^{\mu\nu}_{\alpha\beta}\varphi\partial_\mu\partial^\alpha\varphi\partial_\nu\partial^\beta\varphi. \tag{7.47}$$

The expressions for the $\varphi\varphi\varphi\varphi$-vertex, $\varphi\varphi h$-vertex, hhh-vertex, φhh-vertex are derived as well in the Appendix, section 9.4.

In momentum space such a Lagrangian corresponds to the Feynman rule for the $\varphi\varphi\varphi$-vertex. Define $c^{(m)}_{\varphi\varphi\varphi}$ and $c^{(\Lambda)}_{\varphi\varphi\varphi}$ such that:

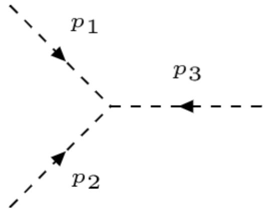
$$\mathcal{L}_{\varphi\varphi\varphi} = c^{(m)}_{\varphi\varphi\varphi}\varphi^2\partial_\mu\partial^\mu\varphi + c^{(\Lambda)}_{\varphi\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}\varphi\partial_\mu\partial^\alpha\varphi\partial_\nu\partial^\beta\varphi. \tag{7.48}$$

The $\varphi\varphi\varphi$-vertex can be written as (let p_1, p_2, p_3 denote the ingoing momenta of the scalar fields at the vertex with $p_1 + p_2 + p_3 = 0$):

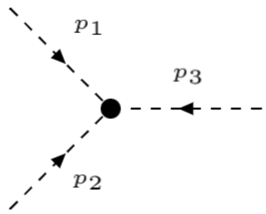
$$V_{\varphi\varphi\varphi} = V^{(m)}_{\varphi\varphi\varphi} + V^{(\Lambda)}_{\varphi\varphi\varphi} = -2ic^{(m)}_{\varphi\varphi\varphi}(p_1^2 + p_2^2 + p_3^2) + 2ic^{(\Lambda)}_{\varphi\varphi\varphi}[(p_1\cdot p_1)(p_2\cdot p_2)$$
$$+ (p_2\cdot p_2)(p_3\cdot p_3) + (p_1\cdot p_1)(p_3\cdot p_3) - (p_1\cdot p_2)^2 - (p_1\cdot p_3)^2 - (p_2\cdot p_3)^2]. \tag{7.49}$$

Diagramatically we will represent this as the Feynman diagrams in Figure 3.

The complete set of Feynman rules relevant for $\varphi\varphi \to \varphi\varphi$ scattering for Horndeski theory are derived in the Appendix, section 9.5.



$$= 2ic^{(\Lambda)}_{\varphi\varphi\varphi}[p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 - (p_1 \cdot p_2)^2 - (p_1 - p_3)^2 - (p_2 \cdot p_3)^2]$$



$$= -2ic^{(m)}_{\varphi\varphi\varphi}[p_1^2 + p_2^2 + p_3^2]$$

Figure 3: Feynman diagrams $\varphi\varphi\varphi$-vertex for Horndeski gravity on a Minkowski background. The dot indicates that the Feynman diagram is sub-leading order (i.e. $\mathcal{O}(1/M_{\text{pl}})$).

7.4.3 Positivity bounds for $\varphi\varphi \to \varphi\varphi$ in Horndeski theory

The scattering amplitude for $\varphi\varphi \to \varphi\varphi$ in Horndeski theory is like in equation (7.25). We will assume that $\Lambda_2 \gg \Lambda_3$ such that the positivity bounds (ignoring loop corrections) are of the form:

$$c_{ss} \geq 0, c_{sst} \geq 0. \tag{7.50}$$

Let us comment on the diagrams contribute to the scattering amplitude. First of all diagrams consisting of two sub-leading three-point vertices can be neglected since they will be sub-leading compared to the other diagrams (i.e. diagrams consisting of two leading three-point vertices and diagrams consisting of one leading and one sub-leading three-point vertex). Each diagram consisting of two three-point vertices can be present in the t-channel, s-channel and u-channel. Diagrams involving one sub-leading three-point vertex and one leading-order vertex come with an additional symmetry factor of 2 from the exchange of vertices [17]. Therefore in computing the scattering amplitude we add the diagrams in Figure 4.

As an example we illustrate how the scattering amplitude of diagrams in Figure 5 is determined.

Let $q := p_1 + p_2$ in the s-channel. From the Feynman rules we have that the s-channel diagram has the amplitude (including the symmetry factor 2 from the exchange of vertices):

$$\begin{aligned} 2V^{(\Lambda)}_{\varphi\varphi\varphi}(p_1, p_2, -q)D(q)V^{(\Lambda)}_{\varphi\varphi\varphi}(q, -p_3, -p_4) &= -\frac{4i}{s}c^{(m)}_{\varphi\varphi\varphi}c^{(\Lambda)}_{\varphi\varphi\varphi}(2m^4 + 4m^2 s - 2(p_1 \cdot p_2)^2 - 4(p_1 \cdot q)^2)(2m^2 + s) \\ &= -\frac{4i}{s}c^{(\Lambda)}_{\varphi\varphi\varphi}c^{(m)}_{\varphi\varphi\varphi}(2m^4 + 4m^2 s - 2(-s/2 + m^2)^2 - s^2)(2m^2 + s) \\ &= -\frac{4i}{s}c^{(\Lambda)}_{\varphi\varphi\varphi}c^{(m)}_{\varphi\varphi\varphi}(6m^2 s - \frac{3}{2}s^2)(2m^2 + s) \\ &= -2ic^{(\Lambda)}_{\varphi\varphi\varphi}c^{(m)}_{\varphi\varphi\varphi}(24m^6 + 6m^2 s - 3s^2). \end{aligned} \tag{7.51}$$

[17] For the diagrams involving twice the same three-point vertex this is not true since in the Dyson series we get something like $-(1/2!)\int d^4x d^4y(\mathcal{L}^{(m)} + \mathcal{L}^{(\Lambda)})^2$ so terms with $\mathcal{L}^{(m)}\mathcal{L}^{(\Lambda)}$ come with an additional factor 2 compared to diagrams with two three-point vertices of the same order.

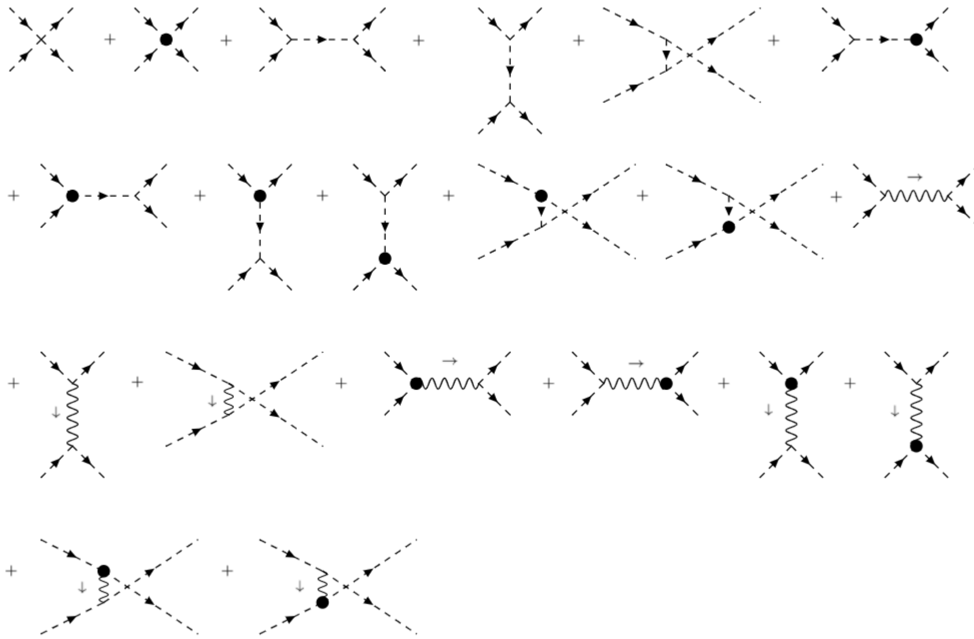


Figure 4: Tree-level diagrams for $\varphi\varphi \to \varphi\varphi$ scattering in Horndeski theory.

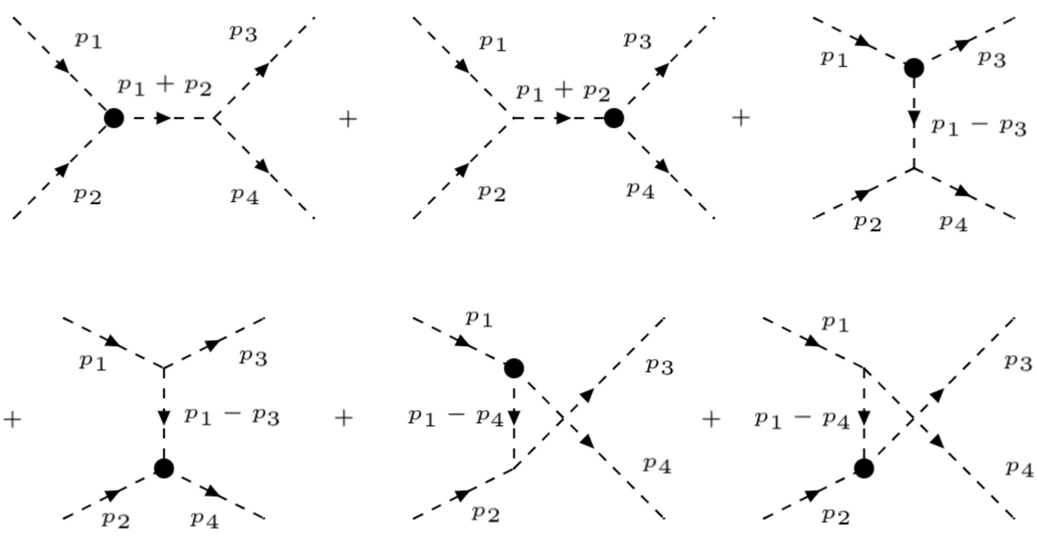


Figure 5: Feynman diagrams involving a scalar propagator and two three-point scalar vertices of different order.

In this computation we used that $s = -q^2 = -(p_1+p_2)^2$ and that the external scalar fields are on-shell. The corresponding u-channels and t-channels are clearly found by $s \to u$ and $s \to t$ (by crossing symmetry), so that the total scattering amplitude for diagrams of the above type is:

$$-2ic^{(\Lambda)}_{\varphi\varphi\varphi}c^{(m)}_{\varphi\varphi\varphi}[72m^4+6m^2(s+t+u)-3(s^2+t^2+u^2)] = -2ic^{(\Lambda)}_{\varphi\varphi\varphi}c^{(m)}_{\varphi\varphi\varphi}[96m^4-3(s^2+t^2+u^2)]. \quad (7.52)$$

In the last equality we used that $4m^2 = u+t+s$. The scattering amplitude of the form (7.25) will be derived in the high-s limit and forward limit (t small) in the center-of-mass frame. In this frame it holds that for elastic scattering:

$$\begin{aligned} s &= 4(p^2+m^2) \\ t &= -2p^2(1-\cos(\theta)) \\ u &= -2p^2(1+\cos(\theta)). \end{aligned} \quad (7.53)$$

In this expression θ is the scattering angle, defined as the angle between $\mathbf{p}_1$ and $\mathbf{p}_3$ (i.e. $\mathbf{p}_1 \cdot \mathbf{p}_3 = |\mathbf{p}_1||\mathbf{p}_3|\cos(\theta)$). The result then directly follows by assuming that the scattering is elastic and that we work in the center-of-mass frame, i.e. it holds that $|\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{p}_3| = |\mathbf{p}_4| =: p$. Note that the high-$s$ limit corresponds to $p^2 \gg m^2$ and the forward limit corresponds to small θ. Clearly, in these limits we have that $u/s \to -1$ and $|t| \ll |u|, s$. Hence the amplitude (7.52) in this limit is found by:

$$12ic^{(\Lambda)}_{\varphi\varphi\varphi}c^{(m)}_{\varphi\varphi\varphi}s^2 = \frac{i(\bar{G}_{2,X\phi}+2\bar{G}_{3,\phi\phi})(\bar{G}_{3,X}+3\bar{G}_{4,X\phi})}{\Lambda_2^4}s^2. \quad (7.54)$$

We ignored terms $\mathcal{O}(m^4/\Lambda_2^4)$ since $\Lambda_2 \gg \Lambda_3$ and Λ_3 is the cut-off of the EFT (so $m \ll \Lambda_3$, hence $m \ll \Lambda_2$). In the future we will not mention this explicitly when evaluating scattering amplitudes in the high-s limit. Notice that this term indeed appears in the desired total scattering amplitude (7.25). Notice that scattering amplitude we compute is $i\mathcal{A}$.

The total scattering amplitude is computed in the Appendix, section 9.6. The result is that it takes the form (7.25) and therefore we find the positivity bounds:

$$\begin{aligned} \frac{2}{3}c_{sst} &= -\bar{G}_{4,XX}+\frac{2}{3}\bar{G}_{5,\phi X}+\frac{1}{2}(\bar{G}_{3,X}+3\bar{G}_{4,X\phi})^2-\frac{1}{\bar{G}_4}(\bar{G}_{4,X}-\bar{G}_{5,\phi})^2 \geq 0, \\ c_{ss} &= \frac{1}{2}\bar{G}_{2,XX}-2\bar{G}_{4,X\phi\phi}+\bar{G}_{2,\phi\phi}(\bar{G}_{3,X}+3\bar{G}_{4,\phi X})^2+(\bar{G}_{2,X\phi}+2\bar{G}_{3,\phi\phi})(\bar{G}_{3,X}+3\bar{G}_{4,\phi X}) \\ &\quad + \frac{1}{\bar{G}_4}(\bar{G}_{5,\phi}-\bar{G}_{4,X})(2\bar{G}_{4,\phi\phi}-1) \geq 0. \end{aligned} \quad (7.55)$$

In this expression the normalisation $2\bar{G}_{3,\phi}+\bar{G}_{2,X} = 1$ has been applied in the last term. The corresponding positivity bounds are precisely the ones stated in [21]. To see this we note that they have put $\bar{G}_4 = 1/2$ and their convention for G_3 has opposite sign. And that $\bar{G}_{2,\phi\phi}$ in our case is defined differently since we use a different convention for the mass m. This originates from the fact that we added an explicit mass term $-\frac{1}{2}m^2\varphi^2$ to the Lagrangian whereas this is not done in the literature [21] [18]. Hence the results that we found are consistent with the ones provided in [21] but not with those in [17]. This basically shows that there are some incorrect numerical

[18] Notice that $m_\phi^2 = 2m^2$ with m_ϕ the mass which appears in [21], since $\mathcal{L}_{\varphi\varphi}$ in their convention with an explicit mass term would yield $-m_\phi^2\varphi^2$. Therefore the result of [21] can be written as $-\frac{1}{2}\bar{G}_{2,\phi\phi} = \frac{m_\phi^2}{2H_0^2} = m^2/H_0^2$ where we used that in their convention $\bar{G}_{2,\phi\phi} = -m_\phi^2/H_0^2$. And m^2/H_0^2 is precisely $\bar{G}_{2,\phi\phi}$ in our calculation, which confirms that the results are consistent.

coefficients in the literature [17], confirming the claims by [21]. For future reference it is useful to stick to the convention of [21], so without an explicit mass term $-\frac{1}{2}m^2\varphi^2$, since we would like to study positivity bounds for the pure Horndeski action. Therefore we will use the above discussed mass convention for which (we will however not change the convention of the sign of G_3):

$$\begin{aligned}
\frac{2}{3}c_{sst} &= -\bar{G}_{4,XX} + \frac{2}{3}\bar{G}_{5,\phi X} + \frac{1}{2}(\bar{G}_{3,X} + 3\bar{G}_{4,X\phi})^2 - 2(\bar{G}_{4,X} - \bar{G}_{5,\phi})^2 \geq 0, \\
c_{ss} &= \frac{1}{2}\bar{G}_{2,XX} - 2\bar{G}_{4,X\phi\phi} - \frac{1}{2}\bar{G}_{2,\phi\phi}(\bar{G}_{3,X} + 3\bar{G}_{4,\phi X})^2 + (\bar{G}_{2,X\phi} + 2\bar{G}_{3,\phi\phi})(\bar{G}_{3,X} + 3\bar{G}_{4,\phi X}) \\
&\quad + 2(\bar{G}_{5,\phi} - \bar{G}_{4,X})(2\bar{G}_{4,\phi\phi} - 1) \geq 0,
\end{aligned} \tag{7.56}$$

where we now define $\bar{G}_{2,\phi\phi} = -m^2/H_0^2$ and we do not add an explicit mass term to the Horndeski Lagrangian.

7.4.4 Future improvements of positivity bounds in Horndeski theory

The positivity bounds we derived are the ones corresponding to tree-level diagrams for the process $\varphi\varphi \to \varphi\varphi$ at leading and sub-leading order. These do not take into account loop corrections. Loop corrections will yield more accurate positivity bounds. It is possible to study scattering processes at even lower order in the cut-off scales which will yield additional positivity bounds [23]. Loop corrections at that order cannot be neglected [21].

Another interesting question to study is whether changing the background metric from Minkowski to FRW metric will change the positivity bounds [17]. This question is non-trivial since cosmological backgrounds in general yield scattering amplitudes which are not Lorentz invariant [22], such that the usual positivity formalism for scalar theories does not apply [23]. For this alternative positivity bounds are needed [22]. This problem will be considered in section 7.5 for Horndeski gravity (with $G_5 = G_5(\phi)$) under the assumption that only boosts are broken.

Finally, we could ask ourselves the question whether the statement is true that the amplitudes of the processes $h\varphi \to h\varphi$ and $hh \to hh$ indeed vanish at order $\mathcal{O}(1/M_{\rm pl})$ we are working [17]. We question this because the scattering amplitudes for $\varphi\varphi \to \varphi\varphi$ mentioned in [17] also already contained several mismatches with our computations and the ones provided in [21].

7.4.5 Examples of Horndeski positivity bounds and mapping to the EFT

Let us consider positivity bounds for some subclasses of Horndeski theory and map the positivity bounds to the EFT of DE. First let us mention that the positivity bounds we derived assumed that $\bar{G}_{2,X} + 2\bar{G}_{3,\phi} = 1$ and $\bar{G}_4 = 1/2$. In order words the positivity bounds are not as general as suggested by [17],[21], as they only apply to a subcase of Horndeski theories satisfying these conditions. The first condition followed from canonically normalizing the scalar propagator. However, we can lighten these assumptions and generalize the applicability of these positivity bounds by requiring only $\bar{G}_{2,X} + 2\bar{G}_{3,\phi} > 0$ and $\bar{G}_4 \neq 0$. The first condition ensures that the scalar field is not a ghost, i.e. the kinetic term has the right sign. And $\bar{G}_4 \neq 0$ ensures that the graviton propagator does not diverge (and that there are gravitational waves propagating at all). The reason we can do this is that in equation (7.29) we can define $\tilde{\varphi} := \varphi\sqrt{\bar{G}_{2,X} + 2\bar{G}_{3,\phi}}$. Then we can treat $\tilde{\varphi}$ as the perturbation on the field (and hence consider $\tilde{\varphi}\tilde{\varphi} \to \tilde{\varphi}\tilde{\varphi}$ scattering) with mass $m_\phi^2/(\bar{G}_{2,X} + 2\bar{G}_{3,\phi})$ [19]. The vertices in real space need to be rewritten in terms of $\tilde{\varphi}$. As a result the positivity bounds obtain the following form:

[19] Then we have that $\mathcal{L}_{\varphi\varphi} = -\frac{1}{2}(\partial\tilde{\varphi})^2 - \frac{1}{2}\tilde{m}^2\tilde{\varphi}^2 + \frac{1}{2M_{\rm pl}^2}\Lambda_2^4\frac{\bar{G}_{2,\phi\phi}}{\bar{G}_{2,X}+2\bar{G}_{3,\phi}}\tilde{\varphi}^2$ where $\tilde{m}^2/H_0^2 = \bar{G}_{2,\phi\phi}/(\bar{G}_{2,X} + 2\bar{G}_{3,\phi})$. In the convention without an explicit mass term, we have the same scaling for the mass and the field.

$$
\begin{aligned}
&-\bar{G}_{4,XX}+\frac{2}{3}\bar{G}_{5,\phi X}+\frac{1}{2}\frac{(\bar{G}_{3,X}+3\bar{G}_{4,X\phi})^2}{2\bar{G}_{3,\phi}+\bar{G}_{2,X}}-\frac{1}{\bar{G}_4}(\bar{G}_{4,X}-\bar{G}_{5,\phi})^2\geq 0\\
&\frac{1}{2}\bar{G}_{2,XX}-2\bar{G}_{4,X\phi\phi}-\frac{1}{2}\bar{G}_{2,\phi\phi}\frac{(\bar{G}_{3,X}+3\bar{G}_{4,\phi X})^2}{(2\bar{G}_{3,\phi}+\bar{G}_{2,X})^2}+\frac{(\bar{G}_{2,X\phi}+2\bar{G}_{3,\phi\phi})(\bar{G}_{3,X}+3\bar{G}_{4,X\phi})}{2\bar{G}_{3,\phi}+\bar{G}_{2,X}}\\
&+\frac{1}{\bar{G}_4}(\bar{G}_{5,\phi}-\bar{G}_{4,X})(2\bar{G}_{4,\phi\phi}-2\bar{G}_{3,\phi}-\bar{G}_{2,X})\geq 0.
\end{aligned}
\tag{7.57}
$$

Let us now consider some famous examples of Horndeski theory (assuming a Minkowski background) [24]:

- **General relativity (GR):** $G_2 = G_3 = G_5 = 0$ and $G_4 = 1/2$. Obviously, the scalar propagator is not well-defined since there is no scalar field in GR. However, we expect that the positivity bounds should be trivial ($0 \geq 0$).

- **K-essence:** $G_2 = G_2(X, \phi)$, $G_3 = G_5 = 0$, $G_4 = 1/2$. It follows easily that $0 \geq 0$ and $\bar{G}_{2,XX} \geq 0$.

- **$f(R)$-theory:** $f(R)$-theory has no kinetic term for the scalar field ϕ, since $\Lambda_2^4 G_2 = V(\phi)$ for some potential V, $G_3 = G_5 = 0$ and G_4 like in Brans-Dicke theory (see below). The propagator is again not well-defined in such a theory.

- **Brans-Dicke theory:** Consider for a field $\tilde{\phi}$ with $\bar{\tilde{\phi}} = M_{\mathrm{pl}}$ instead of $\bar{\phi} = 0$ to ensure that quantities do not diverge (note that the positivity bounds do not depend on the choice of $\bar{\phi} := \langle\phi\rangle$ by Poincaré invariance of the Minkowski background). Define $\Lambda_2^4 G_2 = -\frac{M_{\mathrm{pl}}}{2\tilde{\phi}}\omega_{\mathrm{BD}}(\nabla_\mu\tilde{\phi})^2 - \Lambda_2^4 V(\tilde{\phi})$ ($\omega_{\mathrm{BD}} = 0$ is the $f(R)$-theory case so let $\omega_{\mathrm{BD}} \neq 0$), $G_3 = G_5 = 0$ and $M_{\mathrm{pl}}^2 G_4 = \frac{1}{2}M_{\mathrm{pl}}\tilde{\phi}$. We see that the bounds both become trivial: $0 \geq 0$.

- **Covariant Galileon:** $G_2 = \beta_2 X$, $G_3 = -\beta_3 X$, $G_4 = \frac{1}{2} + \beta_4 X^2$, $G_5 = \beta X^2$ where $\beta_2, \beta_3, \beta_4, \beta$ are some coefficients. The positivity bounds reduce to $-2\beta_4 + \beta_3^2/\beta_2 \geq 0$.

- **Shift-symmetric $\mathcal{L}_2, \mathcal{L}_4$:** $G_2 = G_2(X)$, $G_4 = G_4(X)$, $G_3 = G_5 = 0$. The positivity bounds become $-\bar{G}_{4,XX} - (1/\bar{G}_4)\bar{G}_{4,X}^2 \geq 0$ and $\frac{1}{2}\bar{G}_{2,XX} + (1/\bar{G}_4)\bar{G}_{4,X}\bar{G}_{2,X} \geq 0$. This agrees with the studied example in [17].

- **Gravitational wave (GW) speed constraints [25]:** The speed of the gravitational waves has been measured to be approximately equal to the speed of light. The most general Horndeski Lagrangian which takes this into account is: $G_2 = G_2(\phi, X)$, $G_3 = G_3(\phi, X)$, $G_4 = G_4(\phi)$, $G_5 = 0$. One of the bounds becomes trivial: $\frac{1}{2}\bar{G}_{3,X}^2/(2\bar{G}_{3,\phi} + \bar{G}_{2,X}) \geq 0$. The other one becomes: $\frac{1}{2}\bar{G}_{2,XX}(2\bar{G}_{3,\phi} + \bar{G}_{2,X})^2 - \frac{1}{2}\bar{G}_{2,\phi\phi}\bar{G}_{3,X}^2 + \bar{G}_{3,X}(\bar{G}_{2,\phi X} + 2\bar{G}_{3,\phi\phi})(2\bar{G}_{3,\phi} + \bar{G}_{2,X}) \geq 0$. Notice that the bound provided in [25] is based on [17] and therefore contains an error. Also in [25] they adopted the canonical normalization of the scalar field.

The positivity bounds we derived assume a Minkowski background. However, if we want to apply the formalism of positivity bounds to cosmology, we should consider a cosmological background (i.e. a metric with the FRW metric as the background). It has been postulated in [17], [26] that the Minkowski positivity bounds are approximately true on a cosmological background up to corrections $\mathcal{O}(H^2/\Lambda_3^2)$ where Λ_3 is the EFT cut-off. In [26] this has been shown for a specific subcase of Horndeski gravity (shift-symmetric $c_T = 1$) but we may expect that this should be true for general Horndeski theory [17]. The idea is based on the fact that Horndeski gravity is a covariant theory and its corresponding EFT on a cosmological background breaks time-translation symmetry and boost symmetry, therefore the scattering amplitude is not relativistic anymore (i.e. it depends on the choice of coordinates), however we can assume that up to corrections it should be possible to transport the positivity bounds from Minkowski to the FRW background ([17],[26]). The reason that we do not start directly on the FRW background is that there are major complications when defining positivity bounds on curved backgrounds [22]. There is no full rigorous formalism of positivity bounds on a cosmological background yet. The main challenges are the time-dependence of the coefficients in the Lagrangian (making it non-trivial to apply usual QFT) and the presence of gravitons (and the corresponding t-pole issue). Under the assumption of being able to transport the positivity bounds on Minkowski space to the cosmological background, $\bar{G}_i$ will now mean that G_i is evaluated at the FRW background with possibly time-dependent background field $\bar{\phi}(t)$. We cannot set it equal to zero in this case, since not all time slices are equivalent like in Minkowski space, i.e. the vacuum is not Poincaré invariant anymore.

In general it is not so simple to constrain the EFT parameters using the above positivity bounds but for the shift-symmetric Horndeski action it could be shown by considering the α-parameters in the ADM formalism that the allowed parameter space (by instability conditions) could be reduced by taking into account the positivity bounds [17]. In order to incorporate more general subcases of Hordeski models, we consider reconstructed Horndeski models, which allow us to write the functions G_i in terms of EFT functions [26].

In the EFT formalism of scalar-tensor theories of DE, the action takes the following form [26]:

$$
\begin{aligned}
S = \frac{M_\star^2}{2}\int d^4x\sqrt{-g}[\Omega(t)R - 2\Lambda(t) - \Gamma(t)\delta g^{00}] + \frac{1}{2}\int d^4x\sqrt{-g}\Big[M_4^2(t)(\delta g^{00})^2 - \bar{M}_1^3(t)\delta K\delta g^{00} \\
- 2\bar{M}_2^2(t)\Big(\delta K^2 - \delta K^{\mu\nu}\delta K_{\mu\nu} - \frac{1}{2}\delta R^{(3)}\delta g^{00}\Big)\Big] + S_M[g_{\mu\nu}, \Psi_m] + \Delta S,
\end{aligned}
\tag{7.58}
$$

where ΔS indicates higher order terms of the action, Ψ_m describe matter fields. In this action the coefficients $\Omega(t)$, $\Gamma(t)$, $\Lambda(t)$, $M_2^4(t)$, $\bar{M}_1^3(t)$ and $\bar{M}_2^2(t)$ are the EFT functions. $M_\star$ indicates the (time-independent) Planck mass. This expression already assumes a spatially flat FRW background. The idea of reconstructed Horndeski theories is to match the G_i functions in the covariant formalism to the EFT functions [26]. This matching is done up to quadratic order and higher order corrections are taken into account via variations ΔG_i, i.e. at the background and linear level theories with different ΔG_i are equivalent [26]. The convention of X, $G_{i,X}$, $G_{i,\phi}$ in [26] is different from our convention. In the convention where $X = (\nabla\phi)^2$, $G_{i,X} = \partial G_i/\partial X$ and $G_{i,\phi} = \partial G_i/\partial\phi$[20] we have the following relations between the functions G_i and EFT parameters in the unitary gauge ($\delta\phi = 0$) [26]:

$$
\begin{aligned}
&G_2(\phi, X) = -M_\star^2 U(\phi) - \frac{1}{2}M_\star^2 Z(\phi)X + a_2(\phi)X^2 + \Delta G_2 \\
&G_3(\phi, X) = b_1(\phi)X + \Delta G_3 \\
&G_4(\phi, X) = \frac{1}{2}M_\star^2 F(\phi) + c_1(\phi)X + \Delta G_4 \\
&G_5(\phi, X) = \Delta G_5.
\end{aligned}
\tag{7.59}
$$

where the functions $U(\phi), Z(\phi), a_2(\phi), b_1(\phi), F(\phi), c_1(\phi)$ are defined by:

[20] In this section we will adopt these conventions.

$$
\begin{aligned}
U(\phi) &= \Lambda + \frac{\Gamma}{2} - \frac{M_2^4}{2M_\star^2} - \frac{9H\bar{M}_1^3}{8M_\star^2} - \frac{(\bar{M}_1^3)'}{8} \\
&\quad + \frac{M_\star^2(\bar{M}_2^2)''}{4} + \frac{7(\bar{M}_2^2)'H}{4} + \bar{M}_2^2 H' + \frac{9H^2\bar{M}_2^2}{2M_\star^2} \\
Z(\phi) &= \frac{\Gamma}{M_\star^4} - \frac{2M_2^4}{M_\star^6} - \frac{3H\bar{M}_1^3}{2M_\star^6} + \frac{(\bar{M}_1^3)'}{2M_\star^4} \\
&\quad - \frac{(\bar{M}_2^2)''}{M_\star^2} - \frac{H(\bar{M}_2^2)'}{M_\star^4} - \frac{4H'\bar{M}_2^2}{M_\star^4} \\
a_2(\phi) &= \frac{M_2^4}{2M_\star^8} + \frac{(\bar{M}_1^3)'}{8M_\star^6} - \frac{3H\bar{M}_1^3}{8M_\star^8} - \frac{(\bar{M}_2^2)''}{4M_\star^4} \\
&\quad + \frac{H(\bar{M}_2^2)'}{4M_\star^6} + \frac{H'\bar{M}_2^2}{M_\star^6} - \frac{3H^2\bar{M}_2^2}{2M_\star^8} \\
b_1(\phi) &= \frac{2H\bar{M}_2^2}{M_\star^6} - \frac{(\bar{M}_2^2)'}{M_\star^4} + \frac{\bar{M}_1^3}{2M_\star^6} \\
F(\phi) &= \Omega + \frac{\bar{M}_2^2}{M_\star^2} \\
c_1(\phi) &= \frac{\bar{M}_2^2}{2M_\star^4}.
\end{aligned}
\tag{7.60}
$$

and the functions ΔG_i are defined by:

$$
\begin{aligned}
\Delta G_{2,3} &= \sum_{n=3}^{\infty} \xi_n^{(2,3)}(\phi)\Big(1 + \frac{X}{M_\star^4}\Big)^n, \\
\Delta G_{4,5} &= \sum_{n=4}^{\infty} \xi_n^{(4,5)}(\phi)\Big(1 + \frac{X}{M_\star^4}\Big)^n.
\end{aligned}
\tag{7.61}
$$

The functions $\xi_n^{(i)}(\phi)$ are free functions which do not change the linear theory but they can be used to reconstruct (under possibily some redefinitions of fields) subcases of the covariant Horndeski theory [26]. Primes in the expressions indicate differentation with respect to the field ϕ. In the unitary gauge we have that $X = (-1 + \delta g^{00})M_\star^4$ and therefore $\bar{X} = -M_\star^4$ at the background, such that ΔG_i vanish at background level (since we can take the background field $\phi = tM_\star^2$ without loss of generality) [26]. This is useful since the positivity bounds are evaluated at the cosmological background. The positivity bounds can be written in the convention of reference [26] [21]:

$$
\begin{aligned}
&-4G_{4,XX} - \frac{4}{3}G_{5,\phi X} + \frac{(G_{3,X} + 3G_{4,\phi X})^2}{G_{3,\phi} - G_{2,X}} - \frac{1}{G_4}(2G_{4,X} + G_{5,\phi})^2 \geq 0 \\
&2G_{2,XX} + 4G_{4,X\phi\phi} - \frac{1}{2}G_{2,\phi\phi}\frac{(G_{3,X} + 3G_{4,X\phi})^2}{(G_{3,\phi} - G_{2,X})^2} \\
&+ \frac{2(G_{2,X\phi} - G_{3,\phi\phi})(G_{3,X} + 3G_{4,X\phi})}{G_{3,\phi} - G_{2,X}} + \frac{2}{G_4}(G_{5,\phi} + 2G_{4,X})(G_{4,\phi\phi} - G_{3,\phi} + G_{2,X}) \geq 0.
\end{aligned}
\tag{7.62}
$$

[21] Via $\bar{G}_2 = \frac{1}{\Lambda_2^4}G_2$, $\bar{G}_3 = \frac{\Lambda_3^3}{\Lambda_2^4}G_3$, $\bar{G}_4 = \frac{1}{M_{\rm pl}^2}G_4$, $\bar{G}_5 = \frac{\Lambda_3^3}{M_{\rm pl}^2}G_5$, $\bar{G}_{i,X} = -2\Lambda_2^4 G_{i,X}$ and $\bar{G}_{i,\phi} = M_{\rm pl}G_{i,\phi}$ where a bar indicates our convention and without bar indicates the convention of [26]. So X will mean $(\nabla_\mu\phi)^2$ in this section.

Evaluate the relevant functions and derivatives appearing in the positivity bounds yields (recalling they are evaluated at the background where $\bar{X} = -M_\star^4$):

$$
\begin{aligned}
&\frac{(b_1+3c_1')^2}{\frac{1}{2}M_\star^2 Z + 2M_\star^4 a_2 - b_1' M_\star^4} - \frac{4c_1^2}{\frac{1}{2}M_\star^2 F - c_1 M_\star^4} \geq 0, \\
&4(a_2 + c_1'') - \frac{1}{2}(-M_\star^2 U'' + \frac{1}{2}M_\star^6 Z'' + a_2'' M_\star^8)\frac{(b_1+3c_1')^2}{(\frac{1}{2}M_\star^2 Z + 2M_\star^4 a_2 - b_1' M_\star^4)^2} \\
&+ 2\frac{(b_1+3c_1')(-\frac{1}{2}M_\star^2 Z' - 2a_2' M_\star^4 - b_1'' M_\star^4)}{\frac{1}{2}M_\star^2 Z + 2M_\star^4 a_2 - b_1' M_\star^4} \\
&+ \frac{4c_1}{\frac{1}{2}M_\star^2 F - c_1 M_\star^4}\Big(\frac{1}{2}F'' M_\star^2 - c_1'' M_\star^4 + b_1' M_\star^4 - \frac{1}{2}M_\star^4 Z - 2M_\star^4 a_2\Big) \geq 0.
\end{aligned}
\tag{7.63}
$$

Important is to note that unlike in the case of Minkowski background, the left-hand side of the positivity bound is a function of time and therefore needs to be satisfied at all times in order that the EFT admits a viable UV completion. The EFT functions can be written in terms of the ADM parameters $\{\alpha_T, \alpha_M, \alpha_B, \alpha_K\}$ as well [26]. In order to check whether the positivity bound is satisfied one has to make a choice for the parametrization of α-parameters or EFT functions. Examples of $\alpha_i(a) = c_i \Omega_\Lambda(a)$, $\alpha_i(a) = c_i a$ where $i \in \{M, B, K\}$ have been considered for the case of $c_{\rm GW} = 1$ and it was shown that the allowed parameter space containing points of the form (c_M, c_B, c_K) depends on this choice of parametrization [25]. The positivity bounds we provided also take into account the possibility that $c_{\rm GW} \neq 1$, so they are more general. It is useful to relate the EFT functions in the reference [26] to the ones used in EFTCAMB [27],[28] [22]:

EFT functions in [26]	EFT functions in [27]
$M_\star$	m_0
Ω	$1+\Omega$
$-\Lambda M_\star^2$	Λ
$-(M_\star^2/2)\Gamma$	$-c$
M_2^4	M_2^4
$\bar{M}_1^3$	$\bar{M}_1^3$
$\bar{M}_2^2$	$\frac{1}{2}\bar{M}_2^2$

Table 1: Conversion of EFT conventions between references [26], [27].

The positivity bounds in the convention of EFTCAMB are usually written in code notation in which EFT functions are derived with respect to a, the Hubble parameter $H = \frac{1}{a}\frac{da}{dt}$ is converted to the Hubble parameter in conformal time $\mathcal{H} = \frac{1}{a}\frac{da}{d\tau}$, $'$ indicates $\frac{d}{da}$ (which should not be confused with the prime defined earlier which meant $\frac{d}{d\phi}$) and $\dot{\mathcal{H}} := \frac{d\mathcal{H}}{d\tau}$ in this convention. Furthermore, it is useful to introduce $\{\gamma_i\}$ with $i = 1, ..., 6$ via $M_2^4 = \gamma_1 m_0^2 H_0^2$, $\bar{M}_1^3 = \gamma_2 m_0^2 H_0$ and $\bar{M}_2^2 = m_0^2\gamma_3$, $2\gamma_5 = -\gamma_4 = \gamma_3$ and $\gamma_6 = 0$ (in EFTCAMB convention) for Horndeski theory [27]. Let us write the above positivity bounds in EFTCAMB convention and in this code notation (see Appendix, section 9.8, for details). Using the above table to convert to the EFTCAMB convention and writing the expression in code notation yields the following positivity bounds ($'$ will mean $\frac{d}{da}$ instead of $\frac{d}{d\phi}$):

$$
\begin{aligned}
&-\frac{4a^2(1+\Omega)H_0^2\gamma_2^2 + 4(\mathcal{H}^2 + 2\dot{\mathcal{H}})\gamma_3^3 + 8a\mathcal{H}^2(1+\Omega)\gamma_3\gamma_3' + a^2\mathcal{H}^2(1+\Omega)(\gamma_3')^2}{8(1+\Omega)m_0^6[-2\frac{ca^2}{m_0^2} + 3a\mathcal{H}H_0\gamma_2 + (\mathcal{H}^2 + 2\dot{\mathcal{H}})\gamma_3 + 2a\mathcal{H}^2\gamma_3']} \\
&-\frac{4a\mathcal{H}H_0\gamma_2(4(1+\Omega)\gamma_3 + 3\gamma_3^2 + a(1+\Omega)\gamma_3') + 8\gamma_3^2(-\frac{ca^2}{m_0^2} + \mathcal{H}^2(2(1+\Omega) + a\gamma_3'))}{8(1+\Omega)m_0^6[-2\frac{ca^2}{m_0^2} + 3a\mathcal{H}H_0\gamma_2 + (\mathcal{H}^2 + 2\dot{\mathcal{H}})\gamma_3 + 2a\mathcal{H}^2\gamma_3']} \geq 0,
\end{aligned}
\tag{7.64}
$$

[22] Reference [27] and [28] only differ by using Ω in [28] compared to $1+\Omega$ in [27].

$$
\begin{aligned}
&\frac{1}{2}\Bigg\{ \frac{2\gamma_3\Big[-2\frac{ca^2}{m_0^2} + 3aH_0\mathcal{H}\gamma_2 + \gamma_3\Big(\mathcal{H}^2 + 2\dot{\mathcal{H}}\Big) + 2a\mathcal{H}^2\gamma_3' + a(\dot{\mathcal{H}}\Omega' + a\mathcal{H}^2\Omega'')\Big]}{a^2m_0^6(1+\Omega)} + \frac{2(\dot{\mathcal{H}}\gamma_3' + a\mathcal{H}^2\gamma_3'')}{am_0^6} \\
&+ \frac{4a^2H_0^2\gamma_1 + aH_0\mathcal{H}(-3\gamma_2 + a\gamma_2') + \dot{\mathcal{H}}(4\gamma_3 - a\gamma_3') + \mathcal{H}^2(-10\gamma_3 + a(\gamma_3' - a\gamma_3''))}{a^2m_0^6} \\
&+ \frac{2aH_0\gamma_2 + \mathcal{H}(4\gamma_3 + a\gamma_3')}{a^2m_0^6\Big(2\frac{ca^2}{m_0^2} - 3aH_0\mathcal{H}\gamma_2 - 2\gamma_3\dot{\mathcal{H}} - \mathcal{H}^2(\gamma_3 + 2a\gamma_3')\Big)}\Bigg[-\frac{2\dot{c}a^2}{m_0^2} + 3aH_0(\gamma_2\dot{\mathcal{H}} + \mathcal{H}^2(-\gamma_2 + a\gamma_2')) \\
&- 2\mathcal{H}\dot{\mathcal{H}}(\gamma_3 - 3a\gamma_3') + 2\gamma_3\ddot{\mathcal{H}} - \mathcal{H}^3(2\gamma_3 + a(\gamma_3' - 2a\gamma_3''))\Bigg] \\
&- \frac{(2aH_0\gamma_2 + \mathcal{H}(4\gamma_3 + a\gamma_3'))^2}{8a^2m_0^6\Big(-\frac{2ca^2}{m_0^2} + 3aH_0\mathcal{H}\gamma_2 + (\mathcal{H}^2 + 2\dot{\mathcal{H}})\gamma_3 + 2a\mathcal{H}^2\gamma_3'\Big)^2}\Bigg[\frac{2a^2}{m_0^2}(-\mathcal{H}\dot{\Lambda} + \ddot{\Lambda}) - \mathcal{H}(a^2H_0\dot{\mathcal{H}}(\gamma_2' - 3a\gamma_2'') \\
&+ \ddot{\mathcal{H}}(-2\gamma_3 + 5a\gamma_3' + 4a^2\gamma_3'')) - 2\gamma_3(2\dot{\mathcal{H}}^2 + \dddot{\mathcal{H}}) - a(-aH_0\gamma_2'\ddot{\mathcal{H}} + \dot{\mathcal{H}}^2(5\gamma_3' + 3a\gamma_3'') + \gamma_3'\dddot{\mathcal{H}}) \\
&+ a^4H_0\mathcal{H}^3\gamma_2''' - 2\mathcal{H}^2\dot{\mathcal{H}}(-18\gamma_3 + a^2(4\gamma_3'' + 3a\gamma_3''')) - \mathcal{H}^4(24\gamma_3 - 12a\gamma_3' + 2a^3\gamma_3''' + a^4\gamma_3'''')\Bigg]\Bigg\} \geq 0.
\end{aligned} \tag{7.65}
$$

Furthermore, recall that the positivity bounds are only valid when the condition $-G_{2,X} + G_{3,\phi} > 0$ is satisfied. However, it turns out that this condition does not coincide with the ghost/gradient condition on the FRW background. This has been checked by comparing the usual ghost condition coming from the field π on the FRW background in the EFTCAMB code [27] compared to the ghost condition coming from the field φ on the Minkowski background (and writing this on the FRW background with the reconstruction method). The consideration of a simple pure EFT model with $\gamma_i = 0$, $\Omega(a) = \exp(\Omega_0 a^s) - 1$, $w_{\mathrm{DE}} = -1$, could already reveal this phenomenon by performing stability checks for points $(\Omega_0, s) \in [-2, 2] \times [-2, 2]$. We checked that in such a model the two ghost conditions do not coincide (the Minkowski requirement was more stringent), which indicates that the expression $-\bar{G}_{2,X} + G_{3,\phi} > 0$ cannot be covariant and hence does not apply on the FRW background. So although we expect the positivity bounds to be (at least approximately) valid on the FRW background (which has been proven at least for $c_T = 1$ with shift-symmetry to be correct, see [30]), the usual EFT stability conditions on the field φ are not the same as the ones on the field π. In similar fashion the tachyonic stability condition $\tilde{m}^2/H_0^2 = -G_{2,\phi\phi}/(G_{3,\phi} - G_{2,X}) \geq 0$ is not valid on the FRW background either. This means that for the application of the positivity bounds a healthy Minkowski limit has to be required (see also [42]). However, defining such a limit from a given EFT on the FRW background is not straightforward since EFT functions depend on time whereas on the Minkowski background all functions are constant. For a certain subcase of Horndeski theory with constant coefficients it has also been illustrated that the positivity bounds cannot be applied to any of the stable models [42]. However, in our approach the coefficients will not be constant and therefore we will instead have to assume that the usual ghost, gradient and tachyonic stability conditions from EFTCAMB are a necessary requirement for the positivity bounds on the FRW background. This assumption is reasonable under the assumption that the bounds can be transported to good approximation since then the stability conditions of π on the FRW background should be that of φ on the Minkowski background. However it is fair to say that the assumption of transportation of the positivity bounds from the FRW background to the Minkowski background is not rigorously proven yet (see [22], [30] and [44]).

The condition $G_4 \neq 0$ turns out to be valid on the FRW background and was already incorporated in the code by the fact that $\Omega = -1$ leads to an unstable theory.

In the EFTCAMB code we will multiply the second positivity bound by $a^2m_0^6$ in order to ensure that the expression is well-defined when a is small (numerically a is never zero, however it can produce NaN for the lowest value of a). And the first bound will be multiplied by m_0^6. This will not change the bounds since these quantities are all non-negative. Similarly, the tachyonic stability condition for the existence of a healthy Minkowski limit will be multiplied by $a^4m_0^2$.

As an illustration, in Appendix section 9.7, we show that (in the convention of [29]) that for shift-symmetric Horndeski theory $\alpha_B \leq \frac{2\alpha_T}{1+\alpha_T}$, α_K is unconstrained and α_M is some function of α_B [17]. This example has been used in literature to illustrate that positivity bounds help us to constrain cosmological parameter space [17].

7.4.6 EFTCAMB implementation of positivity bounds

The idea of EFTCAMB is to implement instability conditions such as the ghost, gradient and tachyonic instability in order to investigate what modified gravity models satisfy such conditions. EFTCAMB has a structure which consists of flags (i.e. a tree structure, see Figure 1 in [27]), labelled $\mathrm{EFTFLAG} \in \{0, 1, 2, 3, 4\}$. $\mathrm{EFTFLAG} = 0$ corresponds to GR, for which the code is just CAMB rather than EFTCAMB. $\mathrm{EFTFLAG} = 1$ corresponds to pure EFT models which has as the most important subflag $\mathrm{PureEFTmodel} = 1$. For this one needs to specify a certain parametrization of w_{DE}, Ω, γ_1, γ_2, γ_3, $\gamma_4, \gamma_5, \gamma_6$. $\mathcal{H}$, Λ, c are then computed internally in the code using the background and constraint equations. There is also a possibility to restrict to Horndeski models. $\mathrm{EFTFLAG} = 2$ corresponds to alternative EFT parametrizations. The subflag of interest for this research will be the $\mathrm{AltParEFTmodel} = 2$ which is the so-called $(\Omega, \Lambda, \gamma_i)$-parametrization. In such a model one specifies the parametrization of Ω, Λ, γ_i and solve for the background quantities $\mathcal{H}$, w_{DE}, c. The study of α-parameters can be done via the subflag $\mathrm{AltParEFTmodel} = 1$. $\mathrm{EFTFLAG} = 3$ corresponds to designer models, which are specific EFT models such as $f(R)$ theory for which the background needs to be specified. Finally, the flag $\mathrm{EFTFLAG} = 4$ corresponds to the full mapping EFT procedure which includes models such as Hořava gravity. More details regarding the implementation of the usual instability conditions and the structure of the EFTCAMB code can be found in [27].

For the study of LSS phenomenology it is important to focus on the $(\Omega, \Lambda, \gamma_i)$-parametrization [38]. For the purpose of this research, the positivity bounds (7.64), (7.65) have been implemented in EFTCAMB for the $(\Omega, \Lambda, \gamma_i)$-parametrization. We fix the parametrized form of $(\Omega, \Lambda, \gamma_i)$ and solve for $c, \mathcal{H}, w_{\mathrm{DE}}$ ([40] for details). For completeness the positivity bounds have also been implemented for the flag $\mathrm{EFTFLAG} = 1$.

In the previously studied instabilities there were no terms involving $\dddot{\mathcal{H}}$ and $\ddot{\Lambda}$ in general, however these appear in the positivity bounds, therefore we need to define them. However, the precise expressions depends on the $\mathrm{EFTFLAG}$.

In case of $\mathrm{EFTFLAG} = 1$, from the expressions (5), (6) and (10) in the reference [27], it follows easily by performing an additional derivative that [23]

$$\begin{aligned}
\frac{\ddot{\Lambda}a^2}{m_0^2} = &-\frac{2\mathcal{H}\dot{\Lambda}a^2}{m_0^2} - 2a\mathcal{H}\Omega'(\ddot{\mathcal{H}} - \mathcal{H}\dot{\mathcal{H}} - \mathcal{H}^3) - 2\Omega(\dddot{\mathcal{H}} - \dot{\mathcal{H}}^2 - \mathcal{H}\ddot{\mathcal{H}} - 3\mathcal{H}^2\dot{\mathcal{H}}) \\
&- a\mathcal{H}(\Omega' + a\Omega'')(5\mathcal{H}\dot{\mathcal{H}} + \ddot{\mathcal{H}} - \mathcal{H}^3) - a\Omega'(5\dot{\mathcal{H}}^2 + 5\mathcal{H}\ddot{\mathcal{H}} + \dddot{\mathcal{H}} - 3\mathcal{H}^2\dot{\mathcal{H}}) \\
&- a^2(2\mathcal{H}^2\Omega'' + a\mathcal{H}^2\Omega''' + \dot{\mathcal{H}}\Omega'')(2\mathcal{H}^2 + 3\dot{\mathcal{H}}) - a^2\Omega''\mathcal{H}(4\dot{\mathcal{H}}\mathcal{H} + 3\ddot{\mathcal{H}}) \\
&- 3a^3\mathcal{H}^4\Omega''' - 3a^3\dot{\mathcal{H}}\mathcal{H}^2\Omega''' - a^4\mathcal{H}^4\Omega'''' \\
&- \frac{1}{m_0^2}(3\mathcal{H}^2a^2\rho_{\mathrm{DE}}(1 + w_{\mathrm{DE}}) - 2a^2\mathcal{H}^2\rho_{\mathrm{DE}} - \rho_{\mathrm{DE}}a^2\dot{\mathcal{H}})(aw'_{\mathrm{DE}} - 3w_{\mathrm{DE}}(1 + w_{\mathrm{DE}})) \\
&+ \frac{\rho_{\mathrm{DE}}a^2}{m_0^2}\mathcal{H}(a\mathcal{H}w'_{\mathrm{DE}} + a^2\mathcal{H}w''_{\mathrm{DE}} - 3a\mathcal{H}w'_{\mathrm{DE}}(1 + w_{\mathrm{DE}}) - 3w_{\mathrm{DE}}(1 + a\mathcal{H}w'_{\mathrm{DE}})),
\end{aligned} \tag{7.66}$$

[23] And converting $\frac{d}{d\tau} = \frac{da}{d\tau}\frac{d}{da} = a\mathcal{H}\frac{d}{da}$ for derivatives on some of the functions in equation (10) in [27].

$$
\begin{aligned}
\dddot{\mathcal{H}} =& \Big(2\mathcal{H}^2\tilde{\rho}_m + \dot{\mathcal{H}}\tilde{\rho}_m - 3\mathcal{H}^2\tilde{\rho}_m(1+w_m)\Big)\Big(\frac{1}{6} + w_m + \frac{3}{2}w_m^2\Big) \\
&+ \Big(2\mathcal{H}^2\tilde{\rho}_{\rm DE} - 3\mathcal{H}^2(1+w_{\rm DE})\tilde{\rho}_{\rm DE} + \tilde{\rho}_{\rm DE}\dot{\mathcal{H}}\Big)\Big(\frac{1}{6} + w_{\rm DE} + \frac{3}{2}w_{\rm DE}^2 - \frac{1}{2}aw'_{\rm DE}\Big) \\
&+ a\mathcal{H}^2\tilde{\rho}_{\rm DE}\Big(\frac{1}{2}w'_{\rm DE} + 3w_{\rm DE}w'_{\rm DE} - \frac{a}{2}w''_{\rm DE}\Big) \\
&+ \frac{\mathcal{H}^2}{3}\tilde{\rho}_\nu - \mathcal{H}^2\tilde{P}_\nu - \frac{3}{2}\mathcal{H}\dot{\tilde{P}}_\nu + \frac{\dot{\mathcal{H}}}{6}\tilde{\rho}_\nu + \frac{\mathcal{H}}{6}\dot{\tilde{\rho}}_\nu - \frac{\dot{\mathcal{H}}}{2}\tilde{P}_\nu - \frac{1}{2}\ddot{\tilde{P}}_\nu .
\end{aligned}
\tag{7.67}
$$

In the last expression a sum over matter species has been performed, to be precise it includes cold dark matter and baryons for which $w_m = 0$ and photons and relativistic neutrinos for which $w_m = 1/3$. The last terms are coming from contributions of non-relativistic neutrinos. The notation is such that tilde on top means e.g. $\tilde{\rho}_m := \rho_m a^2/m_0^2$, $\dot{\tilde{\rho}}_\nu = \dot{\rho}_\nu a^2/m_0^2$ (like how it is implemented in the EFTCAMB code).

And in the case of $\mathrm{EFTFLAG} = 2$ (with $\mathrm{AltParEFTmodel} = 2$) we only need to specify the expression for $\dddot{\mathcal{H}}$ since Λ is parametrized. From equation (4) of [27] it easily follows by applying an additional derivative that:

$$
\begin{aligned}
\dddot{\mathcal{H}} =& -\frac{a\mathcal{H}(3\Omega' + a\Omega'')}{2\Big(1+\Omega+\frac{a}{2}\Omega'\Big)^2}\Big[-\frac{1}{2}a\mathcal{H}^3(3\Omega' + 4a\Omega'' + a^2\Omega''') - \mathcal{H}\dot{\mathcal{H}}\Big(1+\Omega+\frac{7}{2}a\Omega' + \frac{3}{2}a^2\Omega''\Big) \\
&- \frac{1}{2}\Big(\frac{\dot{P}_{m,\nu}a^2}{m_0^2} + 2\mathcal{H}\frac{P_{m,\nu}a^2}{m_0^2}\Big) - \frac{1}{2}\Big(\frac{\dot{\Lambda}a^2}{m_0^2} + 2\mathcal{H}\frac{\Lambda a^2}{m_0^2}\Big)\Big] \\
&+ \frac{1}{1+\Omega+\frac{1}{2}a\Omega'}\Big[-\frac{a\mathcal{H}^2}{2}(\mathcal{H}^2 + 3\dot{\mathcal{H}})(3\Omega' + 4a\Omega'' + a^2\Omega''') - \frac{1}{2}a^2\mathcal{H}^4(7\Omega'' + 6a\Omega''' + a^2\Omega'''') \\
&- (\dot{\mathcal{H}}^2 + \mathcal{H}\ddot{\mathcal{H}})\Big(1+\Omega+\frac{7}{2}a\Omega' + \frac{3}{2}a^2\Omega''\Big) - \frac{a\mathcal{H}^2\dot{\mathcal{H}}}{2}(9\Omega' + 13a\Omega'' + 3a^2\Omega''') \\
&- \frac{1}{2}\Big(\frac{\ddot{P}_{m,\nu}a^2}{m_0^2} + 4\mathcal{H}\frac{\dot{P}_{m,\nu}a^2}{m_0^2} + 2\dot{\mathcal{H}}\frac{P_{m,\nu}a^2}{m_0^2} + 4\mathcal{H}^2\frac{P_{m,\nu}a^2}{m_0^2}\Big) \\
&- \frac{1}{2}\Big(\frac{\ddot{\Lambda}a^2}{m_0^2} + 4\mathcal{H}\frac{\dot{\Lambda}a^2}{m_0^2} + 2\dot{\mathcal{H}}\frac{\Lambda a^2}{m_0^2} + 4\mathcal{H}^2\frac{\Lambda a^2}{m_0^2}\Big)\Big].
\end{aligned}
\tag{7.68}
$$

7.4.7 Case study of positivity bounds: K-mouflage model

A simple example to investigate is when $\Omega = \gamma_2 = \gamma_3 = 0$, called the K-mouflage model. In this case the first positivity bound becomes trivial and the second positivity bound reduces to $\gamma_1 \geq 0$. Consider the linear parametrization $\gamma_1 = \gamma_1^0 a$, $\Lambda = \Lambda_0 a$ (in the $(\Omega, \Lambda, \gamma_i)$-parametrization) [24]. Let $\gamma_1^0, \Lambda_0 \in [-4, 4]$ and consider 10^6 points (equidistantly distributed on a square grid) in the (Λ_0, γ_1^0)-plane and check each of the points for stability. Clearly, from the positivity bounds we expect that $\gamma_1^0 \geq 0$. The following figures illustrate how the positivity bounds constrain the allowed parameter space compared to the usual stability conditions (ghost, gradient and tachyonic):

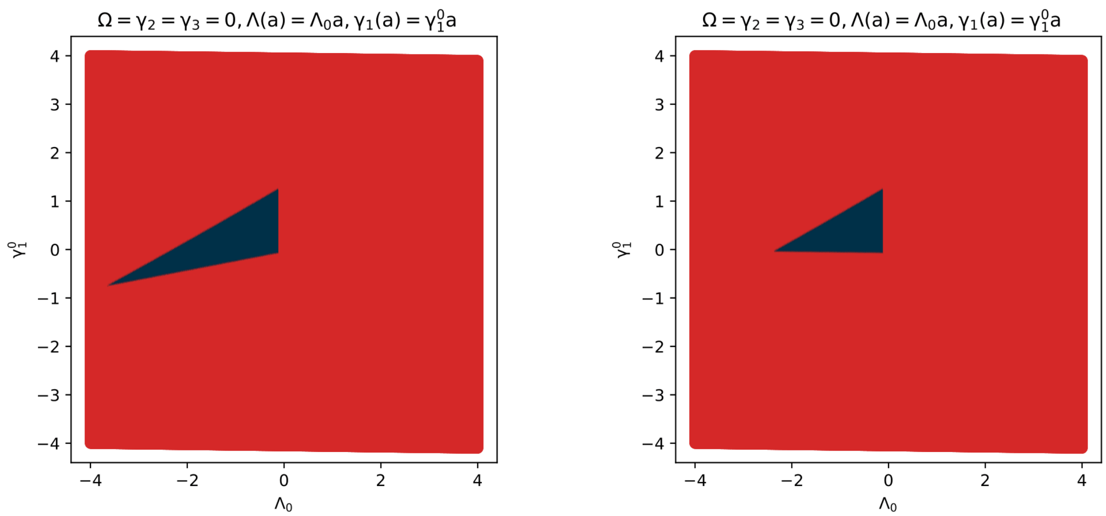


Figure 6: K-mouflage with linear parametrization on a grid consisting of 10^6 points. On the left is the stability check including the usual ghost, gradient and tachyonic conditions (from EFTCAMB) and on the right the positivity bounds have been superimposed. Red points indicate unstable models and blue points indicate stable models.

Notice from the figure that although positivity bounds only constrain γ_1^0 that the combination with other stability conditions excludes also values of Λ_0. In order to study the impact of positivity bounds on the K-mouflage model in a more general way, one can also consider the model specific K-mouflage model which is characterised by 5 parameters $\{\epsilon_{2,0}, \alpha_U, \gamma_A, \gamma_U, m\}$ [45]. And this model has been implemented in EFTCAMB as a full model [45]. We want to have an idea how the positivity bound $\gamma_1 \geq 0$ impacts the allowed parameter space. Recall the fiducial values $(\alpha_U, \gamma_U, m, \epsilon_{2,0}, \gamma_A) = (0.1, 1, 3, -10^{-8}, 0.2)$ for a ΛCDM-like K-mouflage model [45]. In order to gain some insight in how positivity bounds impact the allowed parameters, we let each combination of two parameters deviate from the fiducial value (and fix the remaining three parameters to be the fiducial value). This will be done in such a way that the parameters can take the following ranges [45]: $\epsilon_{2,0} \in [-1, 0]$, $\gamma_A \in [0.2, 25]$, $m \in [1, 10]$, $\gamma_U \in [1, 10]$ and $\alpha_U \in [0, 2]$. The motivations for these ranges are discussed in [45]. In order to fully specify the K-mouflage model we have to fix the cosmological parameters as well, which we take to be $(\Omega_b h^2, \Omega_c h^2, H_0, n_s, \tau, A_s) = (0.02226, 0.1193, 67.51, 0.9653, 0.063, 2.1306 \cdot 10^{-9})$ (in the usual units) [45]. The arbitrary combination of two parameters that we vary will be considered in a rectangular grid of $100 \times 100 = 10^4$ points and each point will undergo a stability check. We will consider ghost and gradient stability conditions and the same with the inclusion of positivity bounds. In the main text we will show some interesting figures and the rest will be put in the Appendix, section 9.9.

[24]This serves as a simple illustration of the impact of positivity bounds and a consistency check of the implementation in EFTCAMB.

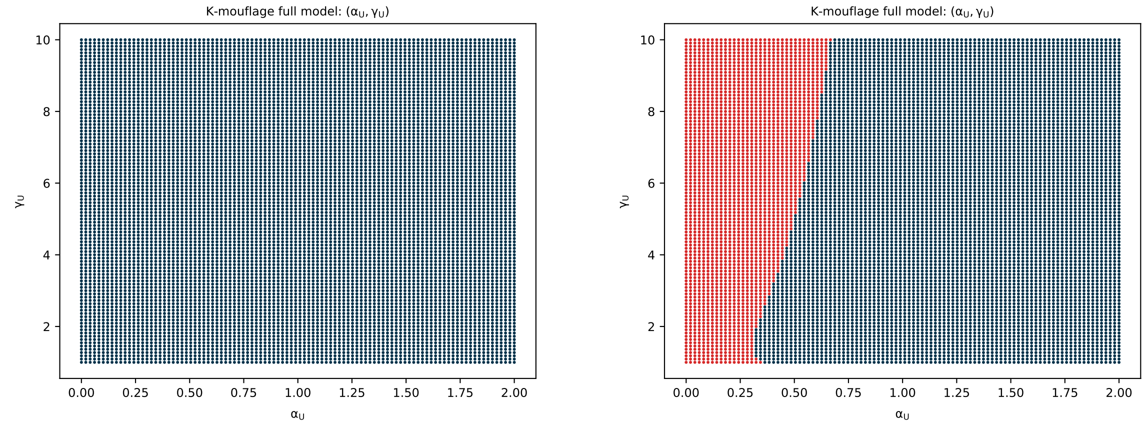


Figure 7: K-mouflage full model: α_U, γ_U. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\epsilon_{2,0} = -10^{-8}$, $m = 3$ and $\gamma_A = 0.2$.

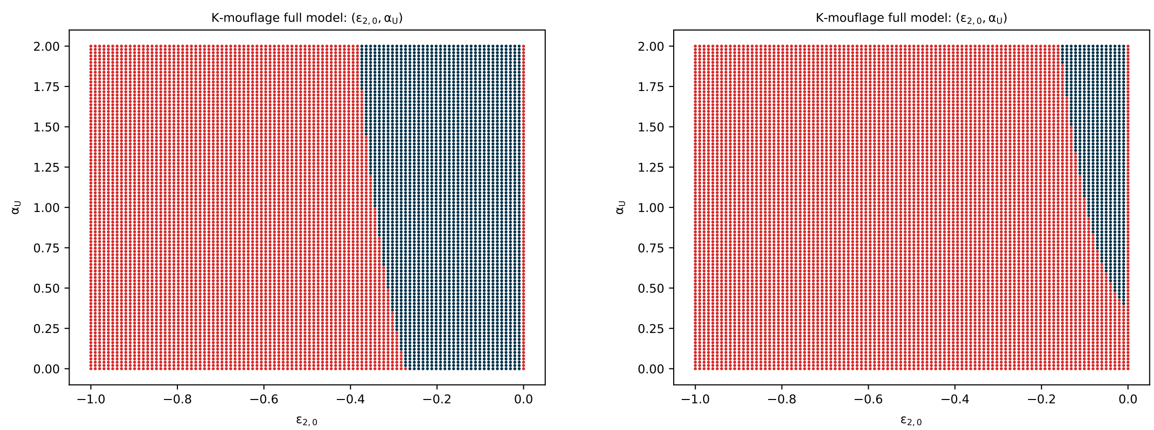


Figure 8: K-mouflage full model: $\epsilon_{2,0}$, α_U. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $m = 3$, $\gamma_U = 1$ and $\gamma_A = 0.2$.

The plots clearly show that the positivity bounds impact the allowed parameter space of K-mouflage models. Some of the instabilities present in the case without imposing positivity bounds such as at $m = 1$ are easily explained by theoretical considerations, see [45]. It is also possible to vary the cosmological parameters which is most easily done by performing a MCMC over the K-mouflage model parameters and the cosmological parameters, i.e. $\{\epsilon_{2,0}, m, \alpha_U, \gamma_A, \gamma_U\} \cup \{H_0, \Omega_b h^2, \Omega_c h^2, n_s, A_s, \tau\}$ [25]. To explore the parameter space in an efficient way we include SNIa data [41] to constrain the expansion history $H(z)$ since it turns out that without this the effect of positivity bounds is neglibible. The results of the simulation are the following contour plots (see the figures below), which again illustrate that positivity bounds constrain the allowed parameter space, mostly the plots involving parameters m and $\epsilon_{2,0}$ are affected by the positivity bounds. We also show how some of the cosmological parameters are affected by the inclusion of positivity bounds. And we will focus on a section of the contour plot to highlight the effect of positivity bounds on the allowed parameter space involving $\epsilon_{2,0}$ or m, since this for these parameters the effect of positivity bounds is most evident.

The general conclusion we can draw from these contour plots is that positivity bounds indeed constrain the parameter space of the K-mouflage model and that this happens mostly in the parameters m and $\epsilon_{2,0}$. It seems that the way

[25] I am grateful to Fabrizio Renzi helping me with this numerical part of the research.

positivity bounds constrain the allowed parameter space depends on the data set under consideration. Therefore we conclude that positivity bounds are mostly powerful in constraining parameter space with the inclusion of some existing observational data. In our case we used SNIa data, but other data may be used as well in order to see how the constraining of positivity bounds get influenced.

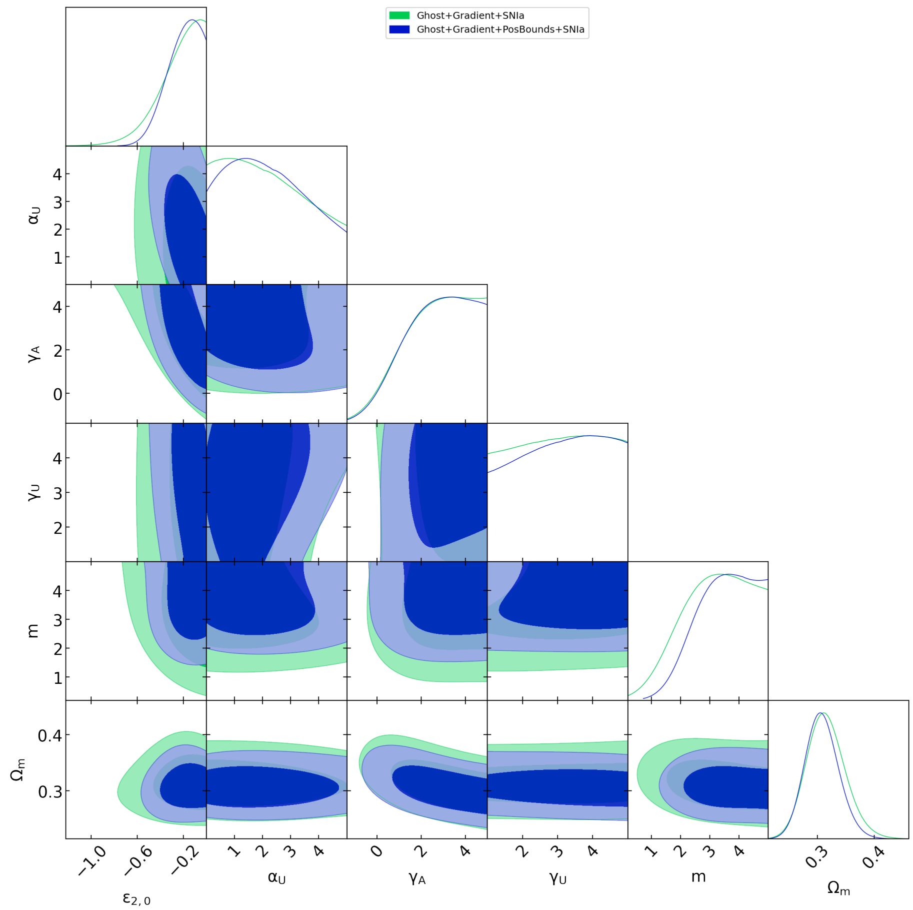


Figure 9: Contour plot of K-mouflage full model parameters where the cosmological parameters are sampled as well (we only showed Ω_m here). In green are the regions which pass the ghost and gradient stability check and the blue regions include the positivity bounds. The model parameters in this plot are sampled from $[-5, 5]$ and $H_0 \in [20, 100]$. The dark contours indicate the 0.95-confidence regions and the lighter contours indicate 0.68-confidence regions.

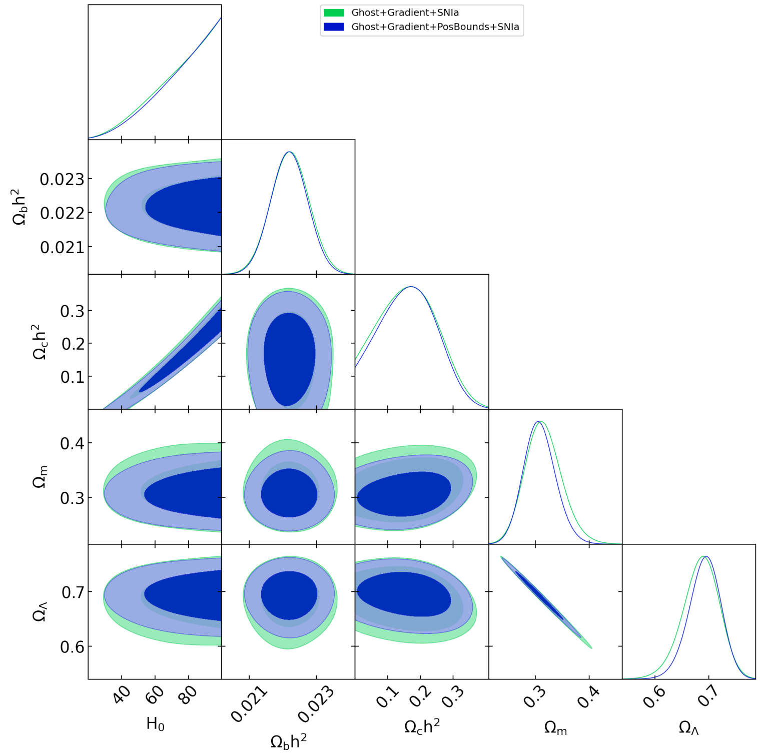


Figure 10: Contour plot of some cosmological parameters for the K-mouflage full model. In green are the regions which pass the ghost and gradient stability check and the blue regions include the positivity bounds. The model parameters in this plot are sampled from $[-5, 5]$ and $H_0 \in [20, 100]$. The dark contours indicate the 0.95-confidence regions and the lighter contours indicate 0.68-confidence regions.

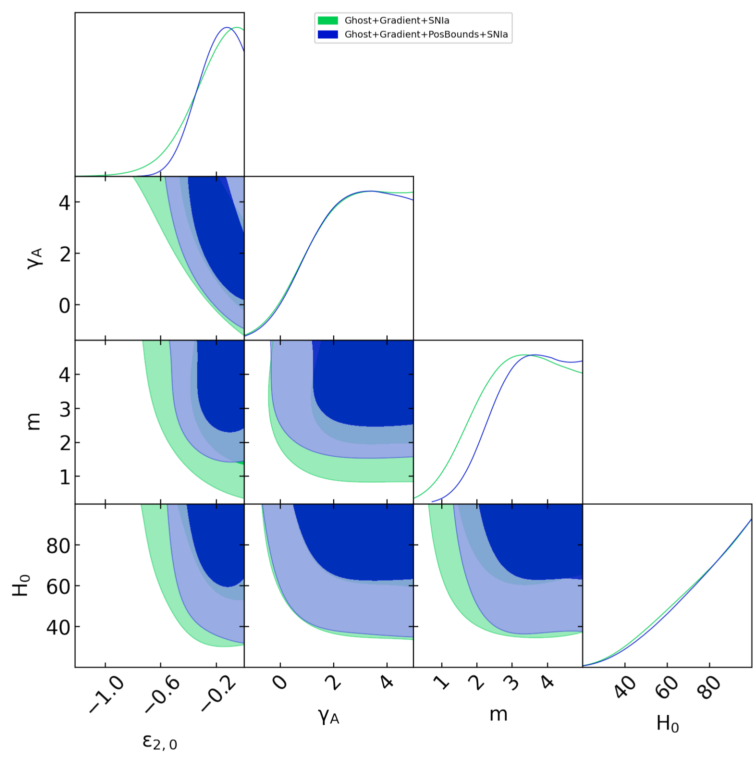


Figure 11: Section of the contour plot of K-mouflage full model that is affected a lot by positivity bounds (including H_0). In green are the regions which pass the ghost and gradient stability check and the blue regions include the positivity bounds. The model parameters in this plot are sampled from $[-5, 5]$ and $H_0 \in [20, 100]$. The dark contours indicate the 0.95-confidence regions and the lighter contours indicate 0.68-confidence regions.

7.4.8 Large-scale structure phenomenology under positivity bounds

In order to gain insight in the more general constraining power of the derived positivity bounds, it is useful to consider the large-scale structure (LSS) phenomenology of Horndeski theories [38]. In particular, we are interested in how the allowed phenomenology parameters (Σ, μ) changes under the inclusion of the positivity bounds in the EFTCAMB code. In the Newtonian gauge, the FRW metric for scalar perturbations takes the form [38]:

$$ds^2 = -(1+2\Psi)dt^2 + a^2(1-2\Phi)d\mathbf{x}^2, \tag{7.69}$$

where Ψ, Φ are small scalar perturbations and a is the scale factor. Let $\Delta := \delta + 3aHv/k$ be the comoving density contrast, where $\delta := \delta\rho/\rho$ is the overdensity, v is the irrotational component of the peculiar velocity, H is the Hubble parameter and k the wavenumber. The Einstein equations in the Newtonian gauge can be expressed as follows [38]:

$$\begin{aligned} k^2\Psi &= -4\pi G\mu(a,k)a^2\rho\Delta, \\ k^2(\Psi+\Phi) &= -8\pi G\Sigma(a,k)a^2\rho\Delta, \end{aligned} \tag{7.70}$$

where (Σ, μ) are the two functions describing the LSS phenomenology.

The EFTCAMB code allows to reconstruct (Σ, μ) at any (a, k) without relying on a quasi-static approximation. The analysis will gradually address the subcases like in Table 1 of [38] and additionally the case of Horndeski with $\Omega = 0$:

- GBD: Ω, Λ, $\gamma_1 = \gamma_2 = \gamma_3 = 0$,
- H_s: Ω, Λ, γ_1, γ_2, $\gamma_3 = 0$,
- Horndeski: Ω, Λ, γ_1, γ_2, γ_3,
- Horndeski with $\Omega = 0$, $\gamma_2 = \gamma_3 = 0$ (called K-mouflage).

We can immediately notice that the case GBD is not of interest for us, since the positivity bounds become trivial $(0 \geq 0)$, and therefore the LSS phenomenology does not change compared to previous work [38]. Therefore we will focus on the other three cases.

For the parametrization of the non-zero EFT functions we will adopt the Padé expansion [38]:

$$f(a) = \frac{\sum_{n=0}^{N} \alpha_n (a-a_0)^n}{1 + \sum_{m=1}^{M} \beta_m (a-a_0)^m}, \tag{7.71}$$

where we will take $(M, N) = (4, 5)$ in order to ensure convergence and we will assume that α_n, β_m are uniformly distributed in $[-1, 1]$. a_0 can be drawn from $[0, 1]$, the results do not depend so much on this choice. The coefficients in the Padé expansions of non-zero EFT functions in each subcase will be sampled using the implemented MCMC in Cobaya [39]. The stability check [26] allows us to select the allowed models and only for those the phenomenology (Σ, μ) will be computed at the following values $a \in \{0.25, 0.575, 0.9\}$, $k \in \{0.001, 0.05, 0.1\}$ and $(a, k) = (0.9, 0.01)$ where k has units of h/Mpc. We have taken into account that the number of allowed models should be large enough ($> 10^4$) in order to have a sufficient statistical representative ensemble of allowed models [38]. The convergence of the MCMC runs are determined by Gelman-Rubin statistic $R-1$, which by default requires $R - 1 \leq 0.01$ for convergence [39].

In order to ensure an efficient exploration of the parameter space, it useful to impose some priors [27] coming from observations [38]:

[26] We will take into account the ghost and gradient stability conditions and superimpose positivity bounds to study their impact.
[27] These priors will be made quite loose in order to admit for a reasonably full exploration of the parameter space.

- GW speed at low redshift is equal to the speed of light, i.e. $\alpha_T(z=0)=0$ (or equivalently $\gamma_3(z=0)=0$).

- The fifth force is not deteceted on Earth: $|\Omega(z=0)-1|<0.1$.

- Consistency with the BBN and CMB: $|\Omega(z=1100)-1|<0.1$.

- Impose data of luminosity diameter distance $d_L(z)$ of supernovae (SNIa) coming from observational data in order to constrain $H(z)$ [41].

Furthermore, the we impose the following ranges for some physical quantities: $H_0 \in [20,100]$ (in $\mathrm{km/s/Mpc}$), sum of neutrino masses equal to $0.06\ \mathrm{eV}$ and $\Omega_m \in [0.1, 0.9]$.

The equation-of-state w_{DE} can also be computed using the following expression [40]:

$$w_{\mathrm{DE}} = \frac{-2\dot{\mathcal{H}} - \mathcal{H}^2 - P_m a^2/m_0^2}{3\mathcal{H}^2 - \rho_m a^2/m_0^2}, \tag{7.72}$$

where P_m, ρ_m are the pressure and energy density of all particle species (i.e. sum over CDM, baryons, photons, and massless and massive neutrinos).

In the case of K-mouflage we found that $\Sigma = \mu$, i.e. the points in the phenomenology clouds all lie on a line, which is expected given the fact that $\Omega = \gamma_3 = 0$ such that $\alpha_T = \alpha_M = 0$ [27], so there is no anistropic stress in K-mouflage. The impact of positivity bounds in the phenomenology is not so clear in this context, therefore we opted instead to study the full K-mouflage model (as discussed above) and the reconstruction of the EFT functions Λ, γ_1 and equation-of-state w_{DE}. However, this reconstruction does not give so much difference with and without the inclusion of positivity bounds, apart from that $\gamma_1 \geq 0$ with the inclusion of positivity bounds.

The other results of this analysis will soon be published in an upcoming article [42].

7.5 Positivity bounds for Horndeski theory on a cosmological background

The above calculations assumed an expansion around the Poincaré invariant Minkowski background. However, we would like to investigate how the positivity bounds will change if one considers perturbations around a flat cosmological background (2.5). It turns out that the above procedure in the covariant formalism is less appropriate for cosmological backgrounds since the graviton propagator becomes non-trivial. Therefore it is much more natural to address the problem in the EFT of DE. The positivity bounds for a shift-symmetric beyond-Horndeski model with $c_T = 1$ have already been derived under the assumption that the Universe is slowly expanding compared to the EFT cut-off of the theory (i.e. $(H/\Lambda)^2 \ll 1$, $\dot{H}/\Lambda^2 \ll 1$ and $|\ddot{\phi}/(H\dot{\phi})| \ll 1$) [30]. The positivity bounds under these assumptions are found to be similar to the ones evaluated at a Minkowski background up to corrections of leading-order $(H/\Lambda)^2$ and a slight difference in the Minkowski limit $H/\Lambda \to 0$. On the Minkowski background the $c_{ss} \geq 0$ positivity bound for $c_T = 1$ (i.e. $\alpha_T = 0$, see [29] for the conditions on the covariant functions) becomes $\bar{G}_{2,XX} \geq 0$, wheres on a cosmological background the positivity bound reads [30]:

$$\frac{\dot{\phi}^2}{2M_{\mathrm{pl}}^2 H^2}\bar{G}_{2,XX} \geq 0. \tag{7.73}$$

Note that the two results are of course consistent. The derivation of this positivity bound assumed that gravitons are decoupled from Goldstone bosons so that the coupling between metric perturbations and Goldstone bosons could be neglected. Such an approximation is well understood in the case $c_T = 1$, however the case $c_T \neq 1$ becomes more subtle and one has to study constraint equations [31].

In this part we will derive the scattering amplitude for the process $\pi\pi \to \pi\pi$ in the case of Horndeski theory with $G_5 = G_5(\phi)$ under the assumption that only boosts are broken. The reason for this choice is that the corresponding free theory has already been fully work out in literature since the free theory corresponds to studying the quadratic part of the action [32].

The corresponding EFT action in the unitary gauge expanded around a flat cosmological background is given by [33] [28]

$$S = \int d^4x\sqrt{-g}\Big[\frac{M_\star^2}{2}f(t)^{(4)}R - \Lambda(t) - c(t)g^{00} + \frac{m_2^4(t)}{2}(\delta g^{00})^2 - \frac{m_3^3(t)}{2}\delta K\delta g^{00} - m_4^2(t)\Big(\delta\mathcal{K}_2 - \frac{1}{2}\delta g^{00(3)}R\Big) - \frac{m_5^2(t)}{2}\delta g^{00}\delta\mathcal{K}_2\Big] \tag{7.74}$$

where $\delta\mathcal{K}_2 := (\delta K)^2 - \delta K^\mu_\nu\delta K^\nu_\mu$, $\delta g^{00} := 1 + g^{00}$, $\delta K_{\mu\nu} := K_{\mu\nu} - Hh_{\mu\nu}$ [36]. $M_\star$ is the Planck mass. The other coefficients are EFT functions. We will assume the EFT functions to be slowly varying in time such that we can treat them as constants but taking into account the slight time dependence is straightforward [33]. The reason why this assumption is needed is that it is complicated to compute scattering amplitudes on a general cosmological background and that there are still no robust positivity bounds defined on general cosmological background [30]. We will also ignore the presence of an effective matter fluid, however this can simply be incorporated in the relevant constraint equations for Φ and Ψ in the Newtonian gauge [32]. We will assume the high-energy limit (but below the usual cut-off Λ_3), i.e. $\dot{\pi} \gg H\pi$, such that we can keep terms which contain at least one derivative. This allows for finding the scalar propagator in its canonical form and it will help in circumventing the issue with the massless t-pole in the final scattering amplitude [17]. Another additional assumption we will do is that we work on sub-Hubble scales ($k/(aH) \gg 1$). All of these assumptions will be the first step toward computing positivity bounds for a Horndeski theory for which the decoupling limit is not well-defined (since $c_T \neq 1$). In the future (some of) these assumptions might be lightened or dropped completely for sake of generality.

In the derivation we will assume the Newtonian gauge defined by the metric [33]:

$$ds^2 = -(1 + 2\Phi)dt^2 + a^2(t)(1 - 2\Psi)(e^\gamma)_{ij}dx^i dx^j. \tag{7.75}$$

The metric γ_{ij} satisfies $\partial_i\gamma_{ij} = \gamma_{ii} = 0$ [29]. We will often write $N^2 := 1 + 2\Phi$ and $h_{ij} := a^2(t)(1 - 2\Psi)(e^\gamma)_{ij}$ in order to make contact with the ADM formalism.

The general idea of studying positivity bounds for the above theory is to study the $\pi\pi \to \pi\pi$ scattering amplitude, where π is the Goldstone boson coming from performing the Stückelberg trick in order to restore covariance of the EFT action. Such an interaction can be given by three type of vertices at tree-level: $\pi\pi\pi$, $\pi\pi\gamma$ and $\pi\pi\pi\pi$. The idea is to first determine the propagators of the theory (free theory) and then find the vertices by performing a Stückelberg trick on the EFT action.

7.5.1 Free Horndeski theory on a cosmological background

The graviton propagator is well-known from the literature, see equation (6.16). The speed of gravitational waves computed around the cosmological background in terms of the EFT functions is given by [32]:

$$c_T^2 = 1 - \frac{2m_4^2}{M_\star^2 f + 2m_4^2}. \tag{7.76}$$

The time-dependent Planck mass M is given by [33]:

[28]Note that we have used that $m_6 = \tilde{m}_6 = m_7 = 0$ and $m_4^2 = \tilde{m}_4^2$.
[29]With γ_{ii} we mean $\delta^{ij}\gamma_{ij}$ and $\partial_i\gamma_{ij}$ means $\delta^{ik}\partial_k\gamma_{ij}$.

$$M^2 = M_\star^2 f + 2m_4^2. \tag{7.77}$$

Under redefinition of the spatial coordinates $x^i \mapsto x^i a/c_T$, the graviton propagator action can be written as:

$$S_{\gamma\gamma} = \int d^4x \frac{M^2}{8} a^3 [\dot{\gamma}_{ij}^2 - \frac{c_T^2}{a^2}(\partial_k \gamma_{ij})^2] = \int d^4x \frac{M^2}{8} c_T^3 [\dot{\gamma}_{ij}^2 - (\partial_k \gamma_{ij})^2]. \tag{7.78}$$

Define the canonically normalized graviton as $\gamma_{ij}^{(c)} := \frac{1}{\sqrt{2}} M c_T^{3/2} \gamma_{ij}$ so that the graviton action reads:

$$S_{\gamma\gamma} = \int d^4x \frac{1}{4} [\dot{\gamma}_{ij}^{(c)2} - (\partial_k \gamma_{ij}^{(c)})^2]. \tag{7.79}$$

The corresponding graviton propagator is given by [33],[34]:

$$D_{ijpq}(k) = -\frac{i \sum_\sigma \epsilon_{ij}^\sigma(\mathbf{k}) \epsilon_{pq}^{\sigma\star}(\mathbf{k})}{k^2} \equiv \frac{-i(\lambda_{ip}\lambda_{jq} + \lambda_{iq}\lambda_{jp} - \lambda_{ij}\lambda_{pq})}{k^2}, \tag{7.80}$$

where $\lambda_{ij} := \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}$.

In order to find the scalar propagator $S_{\pi\pi}$ one needs to solve constraint equations of Φ, Ψ [33]. It turns out that under in the sub-Hubble and high-energy limit there is a simple relation between Φ, Ψ and the Goldstone boson field π. The idea is to introduce the field π via the Stückelberg trick $t \mapsto t + \pi(t, \mathbf{x})$ such that geometrical quantities such as the extrinsic curvature tensor K_{ij} change under such a transformation. Performing such a transformation in the EFT action and fixing the Newtonian gauge gives two constraint equations in the sub-Hubble limit [32] [30]

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \Phi}\Big|_{\gamma_{ij}=0} \implies A_\Psi^{(2)} \Psi + A_\pi^{(2)} \pi = 0, \\ 0 &= \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta \Psi}\Big|_{\gamma_{ij}=0} \implies C_\Phi^{(2)} \Phi + C_\Psi^{(2)} \Psi + C_\pi^{(2)} \pi + C_{\dot{\pi}}^{(2)} \dot{\pi} = 0, \end{aligned} \tag{7.81}$$

where the functions appearing in this expression are defined by (treating the EFT functions as constant [33]):

$$\begin{aligned} A_\Psi^{(2)} &= -2fM_\star^2 - 4m_4^2 \\ A_\pi^{(2)} &= -m_3^3 \\ C_\Phi^{(2)} &= -2fM_\star^2 - 4m_4^2 \\ C_\Psi^{(2)} &= 2fM_\star^2 \\ C_\pi^{(2)} &= -4Hm_4^2 \\ C_{\dot{\pi}}^{(2)} &= -2(2m_4^2 - 2m_4^2) = 0. \end{aligned} \tag{7.82}$$

Therefore the constraint equations are easily found by:

$$\Psi = -\frac{A_\pi^{(2)}}{A_\Psi^{(2)}} \pi = -\frac{m_3^3}{2fM_\star^2 + 4m_4^2} \pi, \tag{7.83}$$

[30] Note that matter fluctuations are ignored in this treatment but can be incorporated straightforwardly.

$$\Phi = -\frac{C_\Psi^{(2)}}{C_\Phi^{(2)}}\Psi - \frac{C_\pi^{(2)}}{C_\Phi^{(2)}}\pi = \frac{2fM_\star^2}{2fM_\star^2+4m_4^2}\Psi - \frac{4Hm_4^2}{2fM_\star^2+4m_4^2}\pi = -\frac{2fM_\star^2 m_3^3 + 4Hm_4^2(2fM_\star^2+4m_4^2)}{(2fM_\star^2+4m_4^2)^2}\pi. \tag{7.84}$$

These constraint equations together with relevant Stückelberg tricks will suffice to express the EFT action in terms of π and γ.

The scalar propagator can also be easily found by exploiting the expression for $\frac{1}{\sqrt{-g}}\delta S/\delta\pi\Big|_{\gamma_{ij}=0}$ in [32]. In the sub-Hubble limit and taking into account terms with leading-order time derivatives of π one finds easily that (recall that in the variation Ψ, Φ, π are treated as independent fields):

$$S_{\pi\pi} = \int d^4x\sqrt{-g}\Big[E_{\ddot\Psi}\ddot\Psi\pi - \frac{1}{2}E_{\ddot\pi}\dot\pi^2 - \frac{1}{a^2}\Big(E_\Phi^{(2)}\pi\partial_i\partial_i\Phi + E_\Psi^{(2)}\pi\partial_i\partial_i\Psi - \frac{1}{2}E_\pi^{(2)}(\partial_i\pi)^2\Big)\Big], \tag{7.85}$$

where the functions appearing in this expression are found by:

$$\begin{aligned}
E_{\ddot\Psi} &= 3m_3^3 \\
E_{\ddot\pi} &= -2(c+4m_2^4) \\
E_\Phi^{(2)} &= -m_3^3 \\
E_\Psi^{(2)} &= -4Hm_4^2 \\
E_\pi^{(2)} &= -(2c - 4m_4^2\dot H + 4H^2m_4^2 + Hm_3^3).
\end{aligned} \tag{7.86}$$

In the action $S_{\pi\pi}$ we can replace $\sqrt{-g}$ by a^3 at quadratic order. Recall that $\Psi \propto \pi$, so the first term in the action satisfies $\ddot\Psi\pi \propto \ddot\pi\pi$. Under an integration by parts one would pick up a term $H\dot\pi\pi$, however under the assumption $\dot\pi \gg H\pi$ we can neglect this. Therefore we are allowed to perform integration by parts with time derivatives. Taking this into account and treating the EFT functions as constant, the action takes the form:

$$S_{\pi\pi} = \int d^4x a^3\Big[-E_{\ddot\Psi}\dot\Psi\dot\pi - \frac{1}{2}E_{\ddot\pi}\dot\pi^2 - \frac{1}{a^2}\Big(-E_\Phi^{(2)}\partial_i\pi\partial_i\Phi - E_\Psi^{(2)}\partial_i\Psi\partial_i\Psi - \frac{1}{2}E_\pi^{(2)}(\partial_i\pi)^2\Big)\Big]. \tag{7.87}$$

Rescaling the spatial coordinates by $x^i \mapsto x^i/a$ allows us to write the action in the form:

$$S_{\pi\pi} = \int d^4xU[\dot\pi^2 - c_s^2(\partial_i\pi)^2] =: \int d^4x\frac{1}{2}[\dot\pi^{(c)2} - (\partial_i\pi^{(c)})^2], \tag{7.88}$$

where we defined the canonically normalized field $\pi^{(c)} := \sqrt{2c_s^3U}\pi$ and the speed of propagation $c_s^2 = V/U$. The functions U, V are given by:

$$\begin{aligned}
U &= \frac{m_3^3}{2fM_\star^2+4m_4^2}E_{\ddot\Psi} - \frac{1}{2}E_{\ddot\pi} = \frac{3(m_3^3)^2}{2fM_\star^2+4m_4^2} + c + 4m_4^2 \\
V &= -E_\Phi^{(2)} - E_\Psi^{(2)} - \frac{1}{2}E_\pi^{(2)} = -m_3^3\frac{2fM_\star^2m_3^3 + 4Hm_4^2(2fM_\star^2+4m_4^2)}{(2fM_\star^2+4m_4^2)^2} + 4Hm_4^2\frac{m_3^3}{2fM_\star^2+4m_4^2} \\
&\quad + \frac{1}{2}(2c - 4m_4^2\dot H + 4H^2m_4^2 + Hm_3^3).
\end{aligned} \tag{7.89}$$

The canonically normalized field $\pi^{(c)}$ thus corresponds to the standard Lagrangian $\mathcal{L}_{\pi\pi} = -\frac{1}{2}(\partial_\mu \pi^{(c)})^2$ with the usual propagator $-i/k^2$ in momentum space.

The scattering process will be considered with the canonically normalized fields. So when we write $\pi\pi \to \pi\pi$ this is strictly speaking $\pi^{(c)}\pi^{(c)} \to \pi^{(c)}\pi^{(c)}$.

7.5.2 Interaction vertices in Horndeski theory on a cosmological background

In order to derive interaction vertices up to quartic order in π it is needed to compute Stückelberg trick up to cubic order. The reason we only need to consider cubic order Stückelberg tricks and not quartic order is because the relevant quantities are always multiplied by a perturbation which is at least first order. And note that the Ricci scalar ${}^{(4)}R$ and the volume element $d^4x\sqrt{-g}$ are diffeomorphism invariant, so they do not change under a Stückelberg trick. In the literature the Stückelberg tricks are only derived up to quadratic order [34], therefore it is needed to extend these to cubic order transformations. The full derivations of the Stückelberg tricks up to cubic order can be found in the Appendix, section 9.10. Let us summarize the main results [31]

[31] These Stückelberg tricks assume a metric $g_{\mu\nu}$ with $N_i = 0$ such as the metric in the Newtonian gauge.

$$
\begin{aligned}
g^{00} &\to -\frac{1}{N^2}(1+\dot\pi)^2 + h^{ij}\partial_i\pi\partial_j\pi,\\
\delta g^{00} &\to 1-\frac{1}{N^2}(1+\dot\pi)^2 + h^{ij}\partial_i\pi\partial_j\pi,\\
g^{0i} &\to h^{ij}\partial_j\pi,\\
g^{ij} &\to h^{ij},\\
\partial_i &\to \partial_i - (1-\dot\pi+\dot\pi^2)\partial_i\pi\partial_0 + \mathcal{O}(4)\\
\partial_0 &\to (1-\dot\pi+\dot\pi^2-\dot\pi^3)\partial_0 + \mathcal{O}(4)\\
g_{00} &\to -N^2(1-2\dot\pi+3\dot\pi^2-4\dot\pi^3) + \mathcal{O}(4)\\
g_{0i} &\to N^2(1-2\dot\pi+3\dot\pi^2)\partial_i\pi + \mathcal{O}(4)\\
g_{ij} &\to h_{ij} - N^2\partial_i\pi\partial_j\pi + 2\dot\pi\partial_i\pi\partial_j\pi + \mathcal{O}(4)\\
K^i_j &\to K^i_j - (1-2\dot\pi)h^{ik}\partial_j N\partial_k\pi + Nh^{ik}\partial_k\pi\partial_j\dot\pi(1-2\dot\pi) - \frac{1}{a^4}\partial_i\pi\partial_j\partial_k\pi\partial_k\pi + \frac{1}{a^2}\partial_0 N\partial_i\pi\partial_j\pi\\
&\quad - \frac{\ddot\pi}{a^2}\partial_i\pi\partial_j\pi - Nh^{ik}\partial_j\partial_k\pi(1-\dot\pi+\dot\pi^2) + Nh^{ik}\partial_j\pi\partial_k\dot\pi(1-2\dot\pi) - \frac{1}{2a^4}(\partial_k\pi)^2\partial_i\partial_j\pi\\
&\quad + \frac{N}{2}h^{ik}h^{lm}(1-\dot\pi)(\partial_m\pi)(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk}) - \frac{N}{2}h^{ik}h^{lm}\dot h_{kl}\partial_m\pi\partial_j\pi(1-2\dot\pi)\\
&\quad - h^{ik}(1-\dot\pi)\partial_k N\partial_j\pi + \frac{N}{4}(1-2\dot\pi)h^{ik}h^{mn}\dot h_{jk}\partial_m\pi\partial_n\pi + \mathcal{O}(4),\\
\delta K^i_j &\to \delta K^i_j - \Big(\dot H\pi + \frac{1}{2}\ddot H\pi^2 + \frac{1}{6}\dddot H\pi^3\Big)\delta^i_j - (1-2\dot\pi)h^{ik}\partial_j N\partial_k\pi + Nh^{ik}\partial_k\pi\partial_j\dot\pi(1-2\dot\pi) - \frac{1}{a^4}\partial_i\pi\partial_j\partial_k\pi\partial_k\pi\\
&\quad + \frac{1}{a^2}\partial_0 N\partial_i\pi\partial_j\pi - \frac{\ddot\pi}{a^2}\partial_i\pi\partial_j\pi - Nh^{ik}\partial_j\partial_k\pi(1-\dot\pi+\dot\pi^2) + Nh^{ik}\partial_j\pi\partial_k\dot\pi(1-2\dot\pi)\\
&\quad - \frac{1}{2a^4}(\partial_k\pi)^2\partial_i\partial_j\pi + \frac{N}{2}h^{ik}h^{lm}(1-\dot\pi)(\partial_m\pi)(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk}) - \frac{N}{2}h^{ik}h^{lm}\dot h_{kl}\partial_m\pi\partial_j\pi(1-2\dot\pi)\\
&\quad - h^{ik}(1-\dot\pi)\partial_k N\partial_j\pi + \frac{N}{4}(1-2\dot\pi)h^{ik}h^{mn}\dot h_{jk}\partial_m\pi\partial_n\pi + \mathcal{O}(4),\\
K &\to K + \frac{1}{a^2}\partial_0 N(\partial_k\pi)^2 + \frac{N}{2}\dot h^{ij}\partial_i\pi\partial_j\pi(1-2\dot\pi) + 2Nh^{ij}\partial_i\dot\pi\partial_j\pi(1-2\dot\pi) - \frac{\ddot\pi}{a^2}(\partial_k\pi)^2 - N(1-\dot\pi)\partial_i h^{ik}\partial_k\pi\\
&\quad - Nh^{ik}\Big(1-\dot\pi+\dot\pi^2 + \frac{3}{2a^2}(\partial_k\pi)^2\Big)\partial_i\partial_k\pi - 2h^{ik}(1-\dot\pi)\partial_k\pi\partial_i N\\
&\quad + \frac{N}{4}h^{ij}h^{kl}\dot h_{ij}(1-4\dot\pi)\partial_k\pi\partial_l\pi - \frac{N}{2}h^{il}h^{kj}(1-\dot\pi)\partial_l\pi\partial_i h_{kj} + \mathcal{O}(4),\\
\delta K &\to \delta K - 3\Big(\dot H\pi + \frac{1}{2}\ddot H\pi^2 + \frac{1}{6}\dddot H\pi^3\Big) + \frac{1}{a^2}\partial_0 N(\partial_k\pi)^2 + \frac{N}{2}\dot h^{ij}\partial_i\pi\partial_j\pi(1-2\dot\pi) + 2Nh^{ij}\partial_i\dot\pi\partial_j\pi(1-2\dot\pi)\\
&\quad - \frac{\ddot\pi}{a^2}(\partial_k\pi)^2 - N(1-\dot\pi)\partial_i h^{ik}\partial_k\pi - Nh^{ik}\Big(1-\dot\pi+\dot\pi^2+\frac{3}{2a^2}(\partial_k\pi)^2\Big)\partial_i\partial_k\pi\\
&\quad - 2h^{ik}(1-\dot\pi)\partial_k\pi\partial_i N + \frac{N}{4}h^{ij}h^{kl}\dot h_{ij}(1-4\dot\pi)\partial_k\pi\partial_l\pi - \frac{N}{2}h^{il}h^{kj}(1-\dot\pi)\partial_l\pi\partial_i h_{kj} + \mathcal{O}(4),\\
K^i_0 &\to \frac{N}{2}(1-2\dot\pi)h^{ij}h^{kl}\dot h_{jk}\partial_l\pi - \frac{1}{a^4}\partial_j\pi\partial_i\partial_j\pi + \mathcal{O}(3)\\
\delta K^i_0 &\to \frac{N}{2}h^{ij}h^{kl}\dot h_{jk}\partial_l\pi - \frac{1}{a^4}\partial_j\pi\partial_i\partial_j\pi - N^2Hh^{ij}\partial_j\pi - \frac{\dot H}{a^2}\pi\partial_i\pi + \mathcal{O}(3)\\
K^0_i &\to \frac{1}{N^4}(1-N^2+2\dot\pi)(\partial_i N - N\partial_i\dot\pi) + \mathcal{O}(3)\\
\delta K^0_i &\to \frac{1}{N^4}(1-N^2+2\dot\pi)(\partial_i N - N\partial_i\dot\pi) + \mathcal{O}(3)\\
K^0_0, \delta K^0_0 &\to \mathcal{O}(3)\\
{}^{(3)}R &\to {}^{(4)}R + 2\,{}^{(4)}R^{\mu\nu}n_\mu n_\nu + 2\,{}^{(4)}R_{00}\Big[h^{kl}\partial_k\pi\partial_l\pi - \frac{2\dot\pi}{a^2}(\partial_k\pi)^2\Big] - 4h^{ij\,(4)}R_{0j}\partial_i\pi\Big(1-2\dot\pi+3\dot\pi^2+\frac{1}{a^2}(\partial_k\pi)^2\Big)\\
&\quad + 2N^2h^{ik}h^{jl\,(4)}R_{kl}\partial_i\pi\partial_j\pi(1-2\dot\pi) - \tilde K^2 + \frac{2H}{a^2}\partial_i\pi(1-N^2+2\dot\pi)(\partial_i N - N\partial_i\dot\pi) + \tilde K^i_j\tilde K^j_i + \mathcal{O}(4),
\end{aligned}
\tag{7.90}
$$

where the components of the 4D Ricci tensor are given by

$$\begin{aligned}
{}^{(4)}R_{00} &= \frac{1}{a^2}\partial_i^2 N - \frac{1}{2}(\dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij}) + 3H\partial_0 N - \frac{1}{4}h^{ik}h^{jl}\dot{h}_{jk}\dot{h}_{il} + \mathcal{O}(2), \\
{}^{(4)}R_{0i} &= \frac{1}{2}\dot{h}_{ij}\partial_k h^{kj} + \frac{1}{2}h^{kj}\partial_k\dot{h}_{ij} - \frac{1}{2}\dot{h}_{kl}\partial_i h^{kl} - \frac{1}{2}h^{kl}\partial_i\dot{h}_{kl} + \frac{1}{2N}(h^{jk}\dot{h}_{jk}\partial_i N - h^{jk}\dot{h}_{ij}\partial_k N) \\
&\quad + \frac{1}{4}h^{jm}h^{kn}\dot{h}_{in}\partial_k h_{jm} - \frac{1}{4}h^{jm}h^{kn}\dot{h}_{jn}(\partial_i h_{km} + \partial_k h_{im} - \partial_m h_{ik}) + \mathcal{O}(3) \\
{}^{(4)}R_{ij} &= \frac{1}{2a^2}(\partial_i\partial_k h_{jk} + \partial_j\partial_k h_{ik} - \partial_k^2 h_{ij} - \partial_i\partial_j h_{kk}) - a\dot{a}\delta_{ij}\partial_0 N - \partial_i\partial_j N \\
&\quad + \frac{1}{4N^2}\left(h^{kl}\dot{h}_{kl}\dot{h}_{ij} + 2\ddot{h}_{ij} - h^{kl}\dot{h}_{jl}\dot{h}_{ik} - h^{kl}\dot{h}_{il}\dot{h}_{jk}\right) + \mathcal{O}(2).
\end{aligned} \tag{7.91}$$

We have to consider $\delta\tilde{K}_0^i$, $\delta\tilde{K}_i^0$ only up to quadratic order since these appear in the action only via multiplication and both do not contain a zeroth order part. Notice that the expressions are only valid up to a certain given order although there are metric components present.

These Stückelberg tricks allow us to move out of the unitary gauge and compute the $\pi\pi\pi$-vertex, $\pi\pi\pi\pi$-vertex and $\gamma\pi\pi$-vertex.

Plugging in the Stückelberg tricks into the action and expanding the action to fourth order in the perturbations (in the Appendix, section 9.13, we provide details) we find the following interaction Lagrangian under the high-energy assumption and on sub-Hubble scales [32]

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= \alpha_1\dot{\gamma}_{ij}\partial_i\dot{\pi}\partial_j\pi + \alpha_2\partial_i\partial_j\pi\partial_l\pi\partial_l\gamma_{ij} + \beta_1\dot{\pi}^3 + \beta_2\dot{\pi}\partial_i\dot{\pi}\partial_i\pi + \beta_3\dot{\pi}(\partial_k^2\pi)^2 + \beta_4(\partial_k^2\pi)\partial_i\dot{\pi}\partial_i\pi \\
&\quad + \beta_5\partial_i\dot{\pi}\partial_j\pi\partial_i\partial_j\pi + \beta_6\dot{\pi}(\partial_i\partial_j\pi)^2 + \beta_7\ddot{\pi}\dot{\pi}^2 + \lambda_1\dot{\pi}^4 + \lambda_2\dot{\pi}^2\partial_i\dot{\pi}\partial_i\pi + \lambda_3\dot{\pi}(\partial_k^2\pi)\partial_i\dot{\pi}\partial_i\pi \\
&\quad + \lambda_4\partial_i\dot{\pi}\partial_j\pi\partial_j\dot{\pi}\partial_i\pi + \lambda_5\ddot{\pi}(\partial_k^2\pi)(\partial_k\pi)^2 + \lambda_6(\partial_k^2\pi)^2(\partial_k\pi)^2 + \lambda_7\dot{\pi}^2(\partial_k^2\pi)^2 + \lambda_8\dot{\pi}\partial_i\pi\partial_j\dot{\pi}\partial_i\partial_j\pi \\
&\quad + \lambda_9\partial_j\pi\partial_i\partial_k\pi\partial_k\pi\partial_i\partial_j\pi + \lambda_{10}\ddot{\pi}\partial_i\pi\partial_j\pi\partial_i\partial_j\pi + \lambda_{11}\dot{\pi}^2(\partial_i\partial_j\dot{\pi})^2 + \lambda_{12}(\partial_k\pi)^2(\partial_i\partial_j\pi)^2.
\end{aligned} \tag{7.92}$$

$\{\alpha_i, \beta_i, \lambda_i\}$ indicate coefficients which depend on the EFT functions which appear in the original action. The exact expressions can be found in the Appendix, section 9.12. Taking into account the typical scalings $m_2^4 \sim M_{\text{pl}}^2 H_0^2$, $m_3^3 \sim M_{\text{pl}}^2 H_0$ and $m_4^2 \sim \tilde{m}_4^2 \sim m_5^2 \sim M_{\text{pl}}^2$ [33], it follows that in the interaction Lagrangian can further be approximated as [33]

$$\begin{aligned}
\mathcal{L}_{\text{int}} &= \beta_3\dot{\pi}(\partial_k^2\pi)^2 + \beta_4(\partial_k^2\pi)\partial_i\dot{\pi}\partial_i\pi + \beta_5\partial_i\dot{\pi}\partial_j\pi\partial_i\partial_j\pi \\
&\quad + \beta_6\dot{\pi}(\partial_i\partial_j\pi)^2 + \lambda_3\dot{\pi}(\partial_k^2\pi)\partial_i\dot{\pi}\partial_i\pi + \lambda_4\partial_i\dot{\pi}\partial_j\pi\partial_j\dot{\pi}\partial_i\pi \\
&\quad + \lambda_5\ddot{\pi}(\partial_k^2\pi)(\partial_k\pi)^2 + \lambda_6(\partial_k^2\pi)^2(\partial_k\pi)^2 + \lambda_7\dot{\pi}^2(\partial_k^2\pi)^2 \\
&\quad + \lambda_8\dot{\pi}\partial_i\pi\partial_j\dot{\pi}\partial_i\partial_j\pi + \lambda_9\partial_j\pi\partial_i\partial_k\pi\partial_k\pi\partial_i\partial_j\pi + \lambda_{10}\ddot{\pi}\partial_i\pi\partial_j\pi\partial_i\partial_j\pi \\
&\quad + \lambda_{11}\dot{\pi}^2(\partial_i\partial_j\dot{\pi})^2 + \lambda_{12}(\partial_k\pi)^2(\partial_i\partial_j\pi)^2.
\end{aligned} \tag{7.93}$$

[32] The way the approximation is done by picking the largest amount of time derivatives given a fixed amount of spatial derivatives and vice versa, following the approach of [33] and like in the example of inflation in [22]. Furthermore, the transformation $x_i \mapsto x_i/a$ has been applied already in this Lagrangian. And the Lagrangian does already take into account the factor $\sqrt{-g}$.

[33] This approximation relies on the assumption that e.g. $|\alpha_1\dot{\gamma}_{ij}\partial_i\dot{\pi}\partial_j\pi/\beta_3\dot{\pi}(\partial_k^2\pi)^2| \sim (H/E) \ll 1$ where E displays the energy scale of the particles. This approximation is not very rigorous, however it is needed in order to be able to apply the formalism of positivity bounds without boosts (gravitons cannot be treated well in that formalism due to t-pole divergences). And also in Appendix D.1 of [33] they provide an argument for their analysis about decay rates involving scalars and gravitons that relies on a similar approximation.

The field π can be canonically normalized, which amounts to a redefinition of the EFT coefficients, which we will indicate with a tilde on top. The resulting field $\pi^{(c)}$ we will (with abuse of notation) call π again. The scattering amplitude of $\pi\pi \to \pi\pi$ can be computed from this interaction Lagrangian. Before we compute the tree-level scattering amplitude we first address a complication of applying the previously mentioned positivity bound formalism to cosmology.

7.5.3 Positivity bounds without boosts

The positivity bound formalism of reference [23] was developed for scalar particles in Minkowski spacetime. The scattering amplitude in that formalism was assumed to be Poincaré invariant. This means that the scattering amplitude is invariant under Lorentz transformations (rotations and boosts), spatial translations and time translations. The last two notions imply that the coefficients in the Lagrangian should be constant and the first notion implies that the amplitude can only be a function of Lorentz invariant quantities. In fact, the amplitude could only be a function of s, t, which are indeed Lorentz invariant quantities. In cosmology however this is not the case. Obviously in cosmology time translation symmetry and boosts are broken, by the fact that the Universe expands in time. This means that there is no uniformity and homogeneity in space-time, only on the spatial 3D hypersurfaces. Therefore it is natural to consider the (1+3)-decomposition ADM formalism in cosmology. In cosmology we expect therefore that time translations and boosts are broken. Only spatial translations and (spatial) rotations are expected to be a symmetry of the scattering amplitude. This requires that the scattering amplitude only contains inner products of 3-vectors and that EFT coefficients are invariant under spatial translations. But in cosmology time-translation invariance for general EFT functions is broken due to the expansion of the Universe, however under the slowly expanding Universe approximation we can treat the EFT functions as roughly constant so that we only have to consider broken boost symmetry and not broken time translation symmetry. The reason that this is desired is that no positivity bounds with broken time translation symmetry together with broken boost symmetry have been developed yet as this is rather complicated and may require other formalisms such as wave function coefficients rather than scattering amplitudes since these may become ill-defined [22]. Let us discuss how positivity bounds for EFT's with broken boost symmetry can be developed [22]. We require the following five requirements on the underlying UV completion [22]:

- Respect the unbroken symmetries (rotations and translations),
- Crossing symmetry,
- Unitarity (via the optical theorem),
- Analyticity (causality),
- Polynomial boundedness (locality).

Unbroken symmetries:

Let π be a scalar field with mass m. Assume that the Poincaré invariance is broken by some normal vector n^μ (like in the ADM formalism). The system is assumed to have rotational and translational symmetry and that the low-energy EFT dispersion relation is of the form $\omega^2 = c_\pi^2|\mathbf{k}|^2 + m^2$ (under canonical normalization and assuming $\omega^2 \gg m^2$ it simply becomes $\omega^2 \approx |\mathbf{k}|^2$). The Fourier convention will be $\partial_\mu \to -ik_\mu$ where $k_\mu = (\omega, \mathbf{k})$. We will adopt a convention in which $\omega < 0$ for outgoing particles and $\omega > 0$ for ingoing particles (such that in a Feynman diagram all external particles move inward) and we define $s_{ij} := (\omega_i + \omega_j)^2 - c_\pi^2|\mathbf{k}_i + \mathbf{k}_j|^2$ and $\omega_{ij} := \omega_i + \omega_j$. By energy conservation one has that $\omega_{12} = -\omega_{34}$. The scattering amplitude can be expressed in terms of the complete basis: $s = s_{12}$, $t = s_{13}$, $u = s_{14}$, $\omega_t = \omega_{13}$, $\omega_u = \omega_{14}$ where $u = s_{14}$ is fixed by $s + t + u = 4m^2$. Note that this definition of Mandelstam variables is indeed consistent with $\omega < 0$ for outgoing fluctuations.

Crossing symmetry:

Scattering amplitudes can be studied in six regions (s-, t-, u-, $\bar{s}$-, $\bar{t}$- and $\bar{u}$-channel regions). For instance the s-channel region corresponds to $s > -t > 0$ and $\omega_s > \sqrt{s + (\omega_u - \omega_t)^2}$ and the u-channel region corresponds to $u > -t > 0$ and $\omega_u > \sqrt{u + (\omega_s - \omega_t)^2}$. Each of these come with an associated scattering amplitude, $\mathcal{A}_s(s, t, \omega_s, \omega_t, \omega_u)$ and $\mathcal{A}_u(u, t, \omega_u, \omega_t, \omega_s)$ respectively. Since we consider identical scalar particles the channels represent the same physics. They are related by crossing symmetries such as $\mathcal{A}_s(s, t, \omega_s, \omega_t, \omega_u) = \mathcal{A}_u(u, t, \omega_u, \omega_t, \omega_s)$ by $s \leftrightarrow u$ crossing symmetry.

Unitarity:

Requiring the S-matrix to be unitary leads to the optical theorem:

$$2\text{Disc}(\mathcal{A}_s) = \sum_n \mathcal{A}_{\pi_1\pi_2 \to n} \mathcal{A}^\star_{\pi_3\pi_4 \to n}, \tag{7.94}$$

where $\text{Disc}(\mathcal{A}_s) := \frac{1}{2i}(\mathcal{A}_s - \mathcal{A}^\star_{\bar{s}})$ where $\bar{s}$ corresponds to the $\bar{s}$-channel (this is the process $\bar{s} : \bar{\pi}_3\bar{\pi}_4 \to \bar{\pi}_1\bar{\pi}_2$). In the forward limit the optical theorem yields that $\text{Disc}(\mathcal{A}_s) \geq 0$ in the UV completion. The optical theorem is technically important in constructing the positivity bounds, see the Appendix of [22].

Analyticity:

In Poincaré-invariant theories the scattering amplitude $\mathcal{A}_s(s, t)$ should be analytic in the whole complex s-plane at fixed t (if $\text{Im}(s) \neq 0$, i.e. the real line may contain poles and branch cuts allowed by unitarity). This argument allowed for deriving positivity bounds in Poincaré-invariant theories since it allowed to relate the EFT amplitude at low energies to the UV amplitude at high energies via the Cauchy residue theorem. In such Poincaré-invariant theories, you can freely choose a frame such as the center-of-mass frame. However, in the case of cosmology boosts are broken and this is not possible anymore. In previous literature [30] the center-of-mass amplitude was computed in a context where boosts were broken. In the formalism of [22] this would amount to $\omega_s = \sqrt{s}$, $\omega_t = \omega_u = 0$. However, the corresponding scattering amplitude $\mathcal{A}_s(s, t) = \mathcal{A}_s(s, t, \sqrt{s}, 0, 0)$ is not analytic (because the function $\sqrt{s}$ can only be analytic in the upper or lower half complex plane, not the whole complex plane) and the amplitude is not $s \leftrightarrow u$ crossing symmetric. To overcome this issue one has to work in the Breit parametrization [22]. In a Poincaré-invariant theory this would amount to boosting to a frame in which $\mathbf{k}_1 - \mathbf{k}_3 = \mathbf{0}$, i.e. it holds that:

$$\omega_s + \omega_u = \sqrt{4m^2 - t}, \omega_s - \omega_u = \frac{s - u}{2\sqrt{4m^2 - t}}, \omega_t = 0, \tag{7.95}$$

assuming that m is small. The scattering amplitude written in this frame is analytic and crossing symmetric. However, in cosmology boosts are broken so we cannot simply work in this frame. The solution is to parametrize the kinematical quantities as follows [22]:

$$\omega_s + \omega_u = 2M\gamma, \omega_s - \omega_u = \frac{s - u}{4M}, \tag{7.96}$$

where γ, M are called Breit variables. The amplitude in e.g. the s-channel can then be written as $\hat{\mathcal{A}}_s(s, t, M, \omega_t, \gamma)$ (which is $\mathcal{A}_s$ with the energies fixed as in the parametrization). This scattering amplitude is analytic and crossing symmetric.

Polynomial boundedness:

The last requirement is that the UV scattering amplitude should be bounded at high-energies:

$$\lim_{s \to \infty} |\hat{\mathcal{A}}_s(s, t, M, \omega_t, \gamma)| < s^2, \tag{7.97}$$

where (t, M, ω_t, γ) are all taken fixed. In Poincaré-invariant theories this is guaranteed by the Froissart-Martin bound, but it turns out that the bound is even satisfied for theories in which boosts are broken.

Given the above five requirements it is possible to construct positivity bounds which can be used to constrain any low-energy EFT in which boosts are broken [22]. $\hat{\mathcal{A}}_s(s, t, M, \omega_t, \gamma)$ at fixed (t, M, ω_t, γ) is analytic in the whole complex s-plane apart from the real line required by unitarity. This means that are allowed to use the Cauchy residue theorem (at fixed (t, M, ω_t, γ)):

$$\frac{1}{n!}\partial_s^n \hat{\mathcal{A}}_s(s) = \oint_{|\mu|\to\infty} \frac{d\mu}{\pi\mu^{n+1}} \hat{\mathcal{A}}_s(\mu) + \int_{-\infty}^{\infty} \frac{d\mu}{\pi} \frac{\text{Im}(\hat{\mathcal{A}}_s(s))}{(\mu - s - i\epsilon)^{n+1}}. \tag{7.98}$$

The first integral on the right can be neglected for all $n \geq 2$ by the assumption of polynomial boundedness. The crossing symmetry allows us to rewrite the second integral and the right side. Let s_b be some scale at which the EFT starts to break down. Then the n-th derivative of the EFT amplitude can be written as:

$$\hat{\mathcal{A}}_s^{(n)}(s) := \frac{1}{n!}\partial_s^n \hat{\mathcal{A}}_s - \int_{2m^2 - t/2}^{s_b} \frac{d\mu}{\pi} P_n(\mu, s), \tag{7.99}$$

where $P_n(\mu, s)$ is defined by:

$$P_n(\mu, s) := \frac{\text{Im}(\hat{\mathcal{A}}_s(\mu))}{(\mu - s)^{n+1}} - \frac{\text{Im}(\hat{\mathcal{A}}_u(\mu))}{(u - \mu)^{n+1}}. \tag{7.100}$$

The n-th derivative can be related to the UV completion via:

$$\hat{\mathcal{A}}_s^{(n)}(s) = \int_{s_b}^{\infty} \frac{d\mu}{\pi} P_n(\mu, s). \tag{7.101}$$

However, since we want to constrain the low-energy EFT of some unknown UV completion the above expression will never be used. We can only use the first definition of the n-th derivative since for this you only need information about the low-energy EFT. It turns out that a set of simple positivity bounds can be found by [22]:

$$\lim_{t\to 0} \lim_{\omega_t \to 0} \hat{\mathcal{A}}_s^{(2N)} \geq 0, \tag{7.102}$$

for all $N \geq 1$, any values of $s, M > 0, \gamma \geq 1$ which are allowed in the low-energy EFT.

It is important to note that the limits in the above expression do not commute (to ensure a well-defined EFT amplitude which has no divergent t-pole) and that in the forward limit the different channels coincide [22].

In fact, in this formalism of positivity bounds there exists an infinite family of other positivity bounds, originating from t and $\omega_1\omega_3$ derivatives of the EFT amplitude, can be derived and used to constrain the EFT functions of some theory whose underlying UV completion satisfies the above requirements [22].

7.5.4 Scattering amplitude and positivity bounds of Horndeski theory on a cosmological background

The tree-level scattering amplitude for the interaction Lagrangian (7.93) can be shown to be (see Appendix, section 9.13, for details of the proof):

$$\mathcal{A} = \sum_{n=1}^{5} s^n \chi_n(\omega_s^2 + (-1)^n \omega_u^2) + f_2 s^2 + f_4 s^4 + f(t) + \text{constant}, \tag{7.103}$$

where $\forall n \in \mathbb{N} : \lim_{t\to 0} \lim_{\omega_t \to 0} f^{(n)}(t) = 0$ and χ_i, f_j are expressed in terms of the EFT coefficients $\{\lambda_i, \beta_j\}$ (see Appendix, section 9.13, for the expressions). For simplicity, we will not take into account loop corrections here as they are suppressed by s/Λ^2 with Λ the cut-off of the EFT and yield corrections to the positivity bounds [22]. This expression gives raise to three positivity bounds in the forward limit (but there are more positivity bounds beyond this):

$$
\begin{aligned}
\lim_{t,\omega_t\to 0} \partial_s^2 \mathcal{A} &= \frac{1}{4M^2}\Big[16\gamma^4 M^4(\chi_2 + 6\chi_4 s^2) + 8\gamma M^2(\chi_1 + 6s^2\chi_3 + 15s^4\chi_4) + 6\chi_2 s^2 + 15\chi_4 s^4\Big] \\
&\quad + 2f_2 + 12 f_4 s^2 \geq 0 \\
\lim_{t,\omega_t\to 0} \partial_s^4 \mathcal{A} &= \frac{1}{3M^2}[16\gamma^2 M^4 \chi_4 + 8M^2\gamma(\chi_3 + 15s^2\chi_5) + \chi_2 + 15\chi_4 s^2] + 24 f_4 \geq 0 \\
\lim_{t,\omega_t\to 0} \partial_s^6 \mathcal{A} &= \frac{90}{M^2}\chi_4 + 720\gamma\chi_5 \geq 0.
\end{aligned}
\tag{7.104}
$$

The latter bound is the most constraining bound in general since by $\chi_5 \sim 1/M^2$ and the fact that $M^2 > 0$ the bound only depends on γ and the EFT coefficients. Intuitively one would expect to recover the bounds in the center-of-mass frame when $\gamma \to 1$. In this case the last positivity bound becomes trivial and is satisfied for all EFT functions. Therefore only the two first bounds are constraining, which is consistent with the result on the Minkowski background spacetime. However, the results are quite different in general since these positivity bounds are s-dependent rather than s-independent, which makes it hard to compare the Minkowski positivity bounds with the ones on a cosmological background. The reason probably lies in the fact that we did not apply the decoupling limit in the case $c_T \neq 1$ is not as trivial as in the case $c_T = 1$ [30]. The derivation of Horndeski positivity bounds on Minkowski background relied on the decoupling limit (which is well-defined since $c_T = 1$ is always true in such a spacetime) by which terms that would lead to higher than $\sim s^2$ in the scattering amplitude are neglected. This could explain why in the case of shift-symmetric $c_T = 1$ (in which the decoupling limit is well-defined) [30] it was possible to recover the Minkowski limit results. However, it could also be the case due to consideration of a shift-symmetric model. Also, in this derivation we made several other assumptions in order to apply the formalism of [22]. Let us summarize some caveats and suggestions of the derivation of positivity bounds on a cosmological background which have to be considered in future research:

- The derivation assumes a slowly expanding Universe which was needed since the positivity bounds did not incorporate that time translation symmetry is broken. In order to overcome this obstacle one would need to construct positivity bounds in which boost and time translation symmetry are both broken. Such a construction is however quite difficult since scattering amplitudes may become awkward and ill-defined and other formalisms may be required such as wave function coefficients [22]. The latter however is still under development and not mature enough for the computation of (positivity) bounds on EFT functions.

- The derivation assumed the high-energy/sub-horizon limit in order to be able to define well-defined scattering amplitudes and propagators and to make sure that the dispersion relation is of the type $\omega^2 = c_\pi^2|\mathbf{k}|^2 + m^2$ as needed for the formalism of positivity bounds without boosts. It also ensured that the t-pole could be dealt with easily, otherwise one would need to compute other positivity bounds (like in the Appendix of [22]). A more rigorous derivation would include: loop corrections, all interaction terms and would need to drop the high-energy and sub-Hubble limit assumption. The latter however is quite difficult as then scattering amplitudes are not well-defined anymore and again one would need another formalism such as wave function coefficients [22]. Another complication is that one would need to incorporate the massless spin-2 particle γ in the formalism of positivity bounds without boosts, something which has not been done yet.

- These positivity bounds are valid for Horndeski theory with the condition that $G_5 = G_5(\phi)$. This assumption was done in order to have a simple free theory, however the extension to a X-dependent G_5 is straightforward. Also other theories such as beyond Horndeski and DHOST could be investigated.

- The mass $m_{\pi^{(c)}}$ of the scalar field has been ignored since we worked in the high-energy limit. However, formally one would need a non-zero mass for the Froissart-Martin bound [22]. The mass can be easily incorporated via $m^2_{\pi^{(c)}} = -(2c_s^3 U)(E_\Phi c_\Phi + E_\Psi c_\Psi + E_\pi)$ [34] where $\Phi =: c_\Phi \pi$ and $\Psi =: c_\Psi \pi$ and the expressions for the other coefficients can be found in [32].

- The relation between $c_T = 1$ and $c_T \neq 1$, the role of the decoupling limit and the Minkowski limit should be investigated further.

- The positivity bounds do not incorporate any matter effective matter fluid, whereas the late Universe contains approximately 30% matter as its content. Matter can be included in the constraint equations (see [32]) and it is expected that including it will lead to more constraining positivity bounds.

[34] By considering the term $-\frac{1}{2}m_\pi^2\pi^2$ in the action $S_{\pi\pi}$.

8 Conclusion

Positivity bounds provide a way of constraining parameters of EFT models by requiring the existence of some unknown UV completion with certain properties. What sets this apart from the previous literature (like in EFTCAMB) is that these positivity bounds are deduced from some arbitrary UV completion given an EFT with certain properties rather than from the EFT itself (which only gives the usual ghost, gradient and tachyonic stability conditions like in EFTCAMB). Positivity bounds are requirements on the scattering amplitudes of particle scatterings. The properties required on the UV completion depend on the symmetries of the EFT under consideration and the particles present in the EFT. In case of an EFT defined on a Minkowski background the symmetries are [23]: Poincaré symmetry, crossing symmetry, analyticity, unitarity and locality. Whereas in the case of a cosmological (FRW) background the Poincaré-symmetry is partially broken since the time-translation symmetry and boost symmetry are broken in this case [22]. In particular, in this thesis we have studied the positivity bounds for scalar-tensor theories with the focus on Horndeski gravity. Following the approach of [17] we have derived the positivity bounds for $\varphi\varphi \to \varphi\varphi$ scattering for Horndeski gravity expanded about a Minkowski background. This calculation allowed us to resolve the disagreement in the literature (see [17],[21]) of the exact positivity bounds for Horndeski gravity. We confirmed that the positivity bounds provided in the reference [21] are the correct ones.

Another goal was to study the impact of the positivity bounds in a cosmological setting. However, QFT with gravitons on cosmological backgrounds is non-trivial since it leads to the t-pole divergence in the scattering amplitude. The presence of gravitons together with the breaking of time-translation symmetry are the reasons why there is no full rigorous formalism of positivity bounds on a cosmological background yet. Therefore we adopted the approach to assume that the positivity bounds derived on the Minkowski background can be transported to good approximation to the cosmological background (see [17],[25], [30]). Under this assumption we have used the method of reconstructed Horndeski models to write the positivity bounds in EFTCAMB notation [26]. And we have implemented the positivity bounds in the EFTCAMB code, allowing us to check subclasses of Horndeski theory for stability together with positivity bounds in order to study the impact of positivity bounds. We have studied the impact of positivity bounds for a subclass of Horndeski gravity, called K-mouflage, which is defined by the EFT functions $\Omega = 0, \Lambda, \gamma_1, \gamma_2 = \gamma_3 = 0$. We have studied this for the example of a linear parametrization, as well as a model specific study given by the five parameters $\{\epsilon_{2,0}, \alpha_U, \gamma_U, \gamma_A, m\}$ [45]. The outcome of this analysis showed that positivity bounds indeed constrain the parameter space, in agreement with what was found in [17] for shift-symmetric Horndeski. A more general study of how positivity bounds impact the allowed parameter space for Horndeski gravity will be studied by considering the LSS phenomenology functions (Σ, μ) and the results of this will soon appear in an upcoming paper [43].

As a final direction for this research we considered how the positivity bounds may be impacted by considering a cosmological background rather than a Minkowski background. So far there are only positivity bounds developed for cosmology under certain stringent conditions: Universe should expand slowly enough compared to the cut-off of the EFT, sub-horizon scales (or high-energy limit), EFT functions vary slowly in time (compared to the Hubble rate) and gravitons cannot be included in the formalism [22]. As regards the symmetries, the only symmetry which can be broken compared to the Minkowski background formalism is the boost symmetry [22]. This means that time-translation symmetry will assumed to be present (which explains the mentioned assumptions), since otherwise the formalism of scattering amplitudes can become ill-defined and one has to rely on other formalisms such as wavefunction coefficients [22]. However, in cosmology those constraints are not always satisfied for any EFT of DE. Under the above assumptions we have performed a calculation of the positivity bounds for Horndeski gravity (with $G_5 = G_5(\phi)$) on a cosmological background by working in the EFT of DE and considering the scattering $\pi\pi \to \pi\pi$ of Stückelberg fields. The result of the calculation are in principle an infinite collection of positivity bounds, but we focused on the positivity bounds in the forward limit $(t, \omega_t \to 0)$. As a result we have found three positivity bounds expressed in terms of EFT functions. In the limit $\gamma \to 1$ we find two non-trivial positivity bounds like in the case of a Minkowski background. On the other hand, we find that the positivity bounds are different since they are s-dependent on the cosmological background. The conclusion is that positivity bounds on the cosmological background are not understood yet and that the limit to a Minkowski background is also not clear yet.

Suggestions for future research are the following:

- Develop a formalism of positivity bounds for cosmology without any assumptions. For instance, it should include the breaking of time-translation symmetry and the presence of gravitons.

- It would be interesting to apply this formalism to Horndeski gravity and compare the result with the one found by the derivation on the Minkowski background and the reconstruction method. And then investigate the limit from a cosmological background to a Minkowski background and check whether the assumption of transporting the positivity bounds is valid (like [44] but applied to the EFT of DE).

- The influence of ordinary matter on the positivity bounds should be taken into account.

- Positivity bounds for other modified gravity theories such as vector-tensor theory and DHOST could be constructed on both a Minkowski and cosmological background.

- Classify which subclasses of Horndeski theory are influenced by positivity bounds.

9 Appendix

9.1 Basic formulas

Let $g_{\mu\nu}$ be some metric which will be taken with $(-,+,+,+)$ signature. Assuming a torsion-free metric compatible connection, we can express the Christoffel symbols in terms of the metric:

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \tag{9.1}$$

The Riemann tensor is defined by:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{9.2}$$

From the Riemann tensor the Ricci tensor is defined by contracting over the indices ρ and μ:

$$R_{\sigma\nu} = R^{\rho}_{\sigma\rho\nu}. \tag{9.3}$$

And the Ricci scalar is defined as the trace of the Ricci tensor:

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}. \tag{9.4}$$

The covariant derivative ∇ of a tensor field T is defined by:

$$\begin{aligned}\nabla_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} = {} & \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} + \ldots + \Gamma^{\mu_k}_{\sigma\lambda} T^{\mu_1\mu_2\ldots\lambda}{}_{\nu_1\nu_2\ldots\nu_l} \\ & - \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} - \ldots - \Gamma^{\lambda}_{\sigma\nu_l} T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\lambda}.\end{aligned} \tag{9.5}$$

For a scalar field ϕ, the covariant derivative coincides with the partial derivative: $\nabla_\nu \phi = \partial_\nu \phi$.

The Lie-derivative of a tensor field T with respect to a vector field V is defined by:

$$\begin{aligned}\mathcal{L}_V T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} = {} & V^\sigma \partial_\sigma T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} - (\partial_\lambda V^{\mu_1}) T^{\lambda\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\nu_l} - \ldots - (\partial_\lambda V^{\mu_k}) T^{\mu_1\mu_2\ldots\lambda}{}_{\nu_1\nu_2\ldots\nu_l} \\ & + (\partial_{\nu_1} V^\lambda) T^{\mu_1\mu_2\ldots\mu_k}{}_{\lambda\nu_2\ldots\nu_l} + \ldots + (\partial_{\nu_l} V^\lambda) T^{\mu_1\mu_2\ldots\mu_k}{}_{\nu_1\nu_2\ldots\lambda}.\end{aligned} \tag{9.6}$$

9.2 Functional derivatives

Let $Z[J]$ be a generating functional for a scalar field. Instead of giving the formal definition we just mention that the functional derivative $\frac{\delta}{\delta J(x)}$ has the following useful properties:

$$\frac{\delta}{\delta J(x)} J(y) = \delta(x-y), \tag{9.7}$$

$$\frac{\delta}{\delta J(x)} \int d^4y J(y) f(y) = f(x), \tag{9.8}$$

$$\frac{\delta}{\delta J(x)} F[J(y)] = F[\delta(x-y)], \tag{9.9}$$

$$\frac{\delta}{\delta J(x)} e^{i\int d^4y J(y)\phi(y)} = i\phi(x), \tag{9.10}$$

$$\frac{\delta}{\delta J(x)}\int d^4y(\partial_\mu J(y))\phi(y) = -\partial_\mu\phi(x). \tag{9.11}$$

The results for the graviton generating function $Z[J^{\alpha\beta}]$ are that one needs to take into account the fact that:

$$\frac{\delta J^{\alpha\beta}(x)}{\delta J^{\kappa\gamma}(y)} = \frac{1}{2}I^{\alpha\beta}{}_{\kappa\gamma}\delta(x-y). \tag{9.12}$$

9.3 Graviton propagator of Horndeski theory on a Minkowski background

The inverse metric of $g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\rm pl}}h_{\mu\nu}$ follows from the requirement $g_{\mu\nu}g^{\nu\sigma} = \delta^\sigma_\mu$. To first order the result is that $g^{\mu\nu} = \eta^{\mu\nu} - \frac{1}{M_{\rm pl}}h^{\mu\nu} + \mathcal{O}(1/M_{\rm pl}^2)$. Therefore the Christoffel symbols can be found by:

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2M_{\rm pl}}g^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}). \tag{9.13}$$

From the definition of the Ricci scalar it follows that:

$$\begin{aligned}
R = &-\frac{1}{2M_{\rm pl}^2}g^{\sigma\nu}(\partial_\mu h^{\mu\lambda})(\partial_\nu h_{\sigma\lambda} + \partial_\sigma h_{\nu\lambda} - \partial_\lambda h_{\nu\sigma})\\
&+ \frac{1}{2M_{\rm pl}}g^{\sigma\nu}g^{\mu\lambda}(\partial_\mu\partial_\nu h_{\sigma\lambda} + \partial_\mu\partial_\sigma h_{\nu\lambda} - \partial_\mu\partial_\lambda h_{\nu\sigma})\\
&+ \frac{1}{2M_{\rm pl}^2}g^{\sigma\nu}(\partial_\nu h^{\mu\lambda})(\partial_\mu h_{\sigma\lambda} + \partial_\sigma h_{\mu\lambda} - \partial_\lambda h_{\mu\sigma})\\
&- \frac{1}{2M_{\rm pl}}g^{\mu\lambda}g^{\sigma\nu}(\partial_\nu\partial_\mu h_{\sigma\lambda} + \partial_\nu\partial_\sigma h_{\mu\lambda} - \partial_\nu\partial_\lambda h_{\mu\sigma})\\
&+ \frac{1}{4M_{\rm pl}^2}g^{\mu\kappa}g^{\lambda\alpha}g^{\sigma\nu}(\partial_\mu h_{\lambda\kappa} + \partial_\lambda h_{\mu\kappa} - \partial_\kappa h_{\mu\lambda})(\partial_\nu h_{\sigma\alpha} + \partial_\sigma h_{\nu\alpha} - \partial_\alpha h_{\nu\sigma})\\
&- \frac{1}{4M_{\rm pl}^2}g^{\mu\kappa}g^{\lambda\alpha}g^{\sigma\nu}(\partial_\nu h_{\lambda\kappa} + \partial_\lambda h_{\nu\kappa} - \partial_\kappa h_{\nu\lambda})(\partial_\mu h_{\sigma\alpha} + \partial_\sigma h_{\mu\alpha} - \partial_\alpha h_{\mu\sigma}).
\end{aligned} \tag{9.14}$$

Only the second and fourth terms do contribute to $\delta_1 R$:

$$\delta_1 R = \frac{1}{M_{\rm pl}}(\partial^\alpha\partial^\beta h_{\alpha\beta} - \eta^{\mu\nu}\partial_\alpha\partial^\alpha h_{\mu\nu}). \tag{9.15}$$

The second order contribution of the Ricci scalar is found by:

$$
\begin{aligned}
M_{\rm pl}^2\delta_2 R = &-\frac{1}{2}(\partial_\mu h^{\mu\lambda})\eta^{\sigma\nu}(\partial_\nu h_{\sigma\lambda}+\partial_\sigma h_{\nu\lambda}-\partial_\lambda h_{\nu\sigma})\\
&-\frac{1}{2}(h^{\mu\lambda}\eta^{\sigma\nu}+h^{\sigma\nu}\eta^{\mu\lambda})(\partial_\mu\partial_\nu h_{\sigma\lambda}+\partial_\mu\partial_\sigma h_{\nu\lambda}-\partial_\mu\partial_\lambda h_{\nu\sigma})\\
&+\frac{1}{2}(\partial_\nu h^{\mu\lambda})\eta^{\sigma\nu}(\partial_\mu h_{\sigma\lambda}+\partial_\sigma h_{\mu\lambda}-\partial_\lambda h_{\mu\sigma})\\
&+\frac{1}{2}(h^{\mu\lambda}\eta^{\sigma\nu}+\eta^{\mu\lambda}h^{\sigma\nu})(\partial_\nu\partial_\mu h_{\sigma\lambda}+\partial_\sigma\partial_\sigma h_{\mu\lambda}-\partial_\nu\partial_\lambda h_{\mu\sigma})\\
&+\frac{1}{4}\eta^{\mu\kappa}\eta^{\lambda\alpha}\eta^{\sigma\nu}(\partial_\mu h_{\lambda\kappa}+\partial_\lambda h_{\mu\kappa}-\partial_\kappa h_{\mu\lambda})(\partial_\nu h_{\sigma\alpha}+\partial_\sigma h_{\nu\alpha}-\partial_\alpha h_{\nu\sigma})\\
&-\frac{1}{4}\eta^{\mu\kappa}\eta^{\lambda\alpha}\eta^{\sigma\nu}(\partial_\nu h_{\lambda\kappa}+\partial_\lambda h_{\nu\kappa}-\partial_\kappa h_{\nu\lambda})(\partial_\mu h_{\sigma\alpha}+\partial_\sigma h_{\mu\alpha}-\partial_\alpha h_{\mu\sigma}).
\end{aligned}
\tag{9.16}
$$

Therefore the second order contribution to the Ricci scalar is given by:

$$
\begin{aligned}
M_{\rm pl}^2\delta_2 R = &-\frac{1}{2}(\partial^\mu h^\lambda_\mu)[2(\partial_\nu h^\nu_\lambda)-\partial_\lambda h]\\
&-\frac{1}{2}h^{\mu\lambda}[\partial_\mu\partial_\nu h^\nu_\lambda+\partial_\mu\partial_\nu h^\nu_\lambda-\partial_\mu\partial_\nu h]\\
&-\frac{1}{2}h^{\sigma\nu}[\partial_\mu\partial_\nu h^\mu_\sigma+\partial_\mu\partial_\sigma h^\mu_\nu-\partial_\mu\partial^\mu h_{\nu\sigma}]\\
&+\frac{1}{2}(\partial_\nu h^{\mu\lambda})[\partial_\nu h^{\mu\lambda}+\partial^\nu h_{\mu\lambda}-\partial_\lambda h^\nu_\mu]\\
&+\frac{1}{2}h^{\mu\lambda}[\partial_\nu\partial_\mu h^\nu_\lambda+\partial^\nu\partial_\nu h_{\mu\lambda}-\partial_\nu\partial_\lambda h^\nu_\mu]\\
&+\frac{1}{2}h^{\sigma\nu}[\partial_\nu\partial_\mu h^\mu_\sigma+\partial_\nu\partial_\sigma h-\partial_\nu\partial_\lambda h^\lambda_\sigma]\\
&+\frac{1}{4}(\partial_\mu h^{\mu\alpha}+\partial^\alpha h-\partial_\mu h^{\mu\alpha})(\partial_\nu h^\nu_\alpha+\partial_\nu h^\nu_\alpha-\partial_\alpha h)\\
&-\frac{1}{4}(\partial_\nu h^{\alpha\mu}+\partial^\alpha h^\mu_\nu-\partial^\mu h^\alpha_\nu)(\partial_\mu h^\nu_\alpha+\partial^\nu h_{\mu\alpha}-\partial_\alpha h^\nu_\mu).
\end{aligned}
\tag{9.17}
$$

After integrating by parts and neglecting boundary terms and collecting terms of the same type it follows that this can be simplified to:

$$
M_{\rm pl}^2\delta_2 R = \frac{1}{2}(\partial^\mu h^\lambda_\mu)(\partial_\nu h^\nu_\lambda)-\frac{1}{4}(\partial^\mu h^\alpha_\nu)(\partial_\mu h^\nu_\alpha)-\frac{1}{4}(\partial_\alpha h)(\partial^\alpha h). \tag{9.18}
$$

The second order graviton Lagrangian $\mathcal{L}_{hh}$ can therefore be written as:

$$
\mathcal{L}_{hh} = \frac{\bar{G}_4}{2}h(\partial_\mu\partial_\nu h^{\mu\nu}-\partial_\alpha\partial^\alpha h)+\bar{G}_4\Big(\frac{1}{2}(\partial^\mu h^\lambda_\mu)(\partial_\nu h^\nu_\lambda)-\frac{1}{4}(\partial^\mu h^\alpha_\nu)(\partial_\mu h^\nu_\alpha)-\frac{1}{4}(\partial_\alpha h)(\partial^\alpha h)\Big). \tag{9.19}
$$

After integration by parts this Lagrangian becomes:

$$
\mathcal{L}_{hh} = \frac{\bar{G}_4}{4}\Big(\partial_\alpha h\partial^\alpha h-(\partial_\lambda h^\mu_\nu)(\partial^\lambda h^\nu_\mu)-2(\partial_\mu h)(\partial^\sigma h^\mu_\sigma)+2(\partial_\lambda h^\lambda_\mu)(\partial^\sigma h^\mu_\sigma)\Big). \tag{9.20}
$$

By the definition of the generalised Kronecker delta:

$$\delta^{\mu_1 \dots \mu_p}_{\nu_1 \dots \nu_p} = p! \delta^{\mu_1}_{[\nu_1} \dots \delta^{\mu_p}_{\nu_p]}, \tag{9.21}$$

it follows that the Lagrangian can be written in the form:

$$\mathcal{L}_{hh} = \frac{\bar{G}_4}{4} \delta^{\alpha\beta\gamma}_{\rho\sigma\nu} (\partial_\beta h^\rho_\alpha)(\partial^\sigma h^\nu_\gamma). \tag{9.22}$$

We already derived the expression for the gauge-fixed version of this Lagrangian (7.39). Let us show that the propagator $D_{\mu\nu\alpha\beta}(k)$ has the desired form of equation (7.42) starting from equation (7.41). We will propose an Ansatz like in [15]:

$$D_{\alpha\beta\gamma\delta}(k) = \frac{1}{\bar{G}_4 k^2} (a I_{\alpha\beta\gamma\delta} + b \eta_{\alpha\beta} \eta_{\gamma\delta}), \tag{9.23}$$

for some $a, b \in \mathbb{C}$ to be determined.

We find from equation (7.41) with our Ansatz:

$$\frac{1}{2} (I^{\mu\nu\alpha\beta} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta})(a I_{\alpha\beta\gamma\delta} + b \eta_{\alpha\beta} \eta_{\gamma\delta}) = -i I^{\mu\nu}{}_{\gamma\delta}. \tag{9.24}$$

From the simple findings $I^{\mu\nu\alpha\beta} I_{\alpha\beta\gamma\delta} = I^{\mu\nu}{}_{\gamma\delta}$ and $I^{\mu\nu\alpha\beta} \eta_{\delta\gamma} \eta_{\alpha\beta} = \eta^{\mu\nu} \eta_{\delta\gamma}$ and $\eta^{\alpha\beta} \eta_{\alpha\beta} = 4$ it follows that:

$$\frac{1}{2} (a I^{\mu\nu}{}_{\gamma\delta} + b \eta^{\mu\nu} \eta_{\delta\gamma} - \frac{a}{2} \eta^{\mu\nu} \eta_{\gamma\delta} - 2b \eta^{\mu\nu} \eta_{\gamma\delta}) = -i I^{\mu\nu}{}_{\gamma\delta}. \tag{9.25}$$

Therefore the Ansatz yields the solution $(a, b) = (-2i, i)$. Therefore it follows that indeed the graviton propagator is given by equation (7.42).

Define the shifted graviton field:

$$h'_{\alpha\beta}(x) = h_{\alpha\beta}(x) - i \int d^4 y D_{\alpha\beta\kappa\sigma}(x - y) J^{\kappa\sigma}(y). \tag{9.26}$$

With this it follows that using the Lagrangian $\mathcal{L}_{hh}$ (equation (7.39)):

$$\begin{aligned} i \int d^4 x (\mathcal{L}_{hh} + J^{\alpha\beta} h_{\alpha\beta}) = & \frac{i\bar{G}_4}{4} \int d^4 x h'_{\mu\nu}(x) \partial_\sigma \partial^\sigma (I^{\mu\nu\alpha\beta} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta}) h'_{\alpha\beta}(x) \\ & + i \int d^4 x J^{\alpha\beta}(x) h'_{\alpha\beta}(x) \\ & - \int d^4 x d^4 y J^{\alpha\beta}(x) D_{\alpha\beta\kappa\sigma}(x - y) J^{\kappa\sigma}(y) \\ & - \frac{i\bar{G}_4}{4} \int d^4 x d^4 y d^4 z D_{\mu\nu\kappa\sigma}(x - y) J^{\kappa\sigma}(y) \partial_\sigma \partial^\sigma (I^{\mu\nu\alpha\beta} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta}) D_{\alpha\beta\gamma\lambda}(x - z) J^{\gamma\lambda}(z) \\ & - \frac{2\bar{G}_4}{4} \int d^4 x d^4 y h'_{\mu\nu}(x) \partial_\sigma \partial^\sigma (I^{\mu\nu\alpha\beta} - \frac{1}{2} \eta^{\mu\nu} \eta^{\alpha\beta}) D_{\alpha\beta\kappa\sigma}(x - y) J^{\kappa\sigma}(y). \end{aligned} \tag{9.27}$$

The factor 2 in the last line comes from integration by parts so that we get two identical terms. Using the definition of the propagator as in equation (7.40) and the fact that $J^{\alpha\beta} = J^{\beta\alpha}$ (due to $D_{\alpha\beta\kappa\sigma} = D_{\alpha\beta\sigma\kappa}$) it follows directly that:

$$i\int d^4x(\mathcal{L}_{hh}+J^{\alpha\beta}h_{\alpha\beta})=\frac{i\bar{G}_4}{4}\int d^4xh'_{\mu\nu}(x)\partial_\sigma\partial^\sigma(I^{\mu\nu\alpha\beta}-\frac{1}{2}\eta^{\mu\nu}\eta^{\alpha\beta})h'_{\alpha\beta}(x)$$
$$-\frac{1}{2}\int d^4xd^4yJ^{\alpha\beta}(x)D_{\alpha\beta\kappa\sigma}(x-y)J^{\kappa\sigma}(y). \tag{9.28}$$

Therefore the free generating function for gravitons only can be written as:

$$Z_0[J^\mu_\nu]=Z_0[0]e^{-\frac{1}{2}\int d^4xd^4yJ^{\alpha\beta}(x)D_{\alpha\beta\sigma\kappa}(x-y)J^{\sigma\kappa}(y)}, \tag{9.29}$$

Thus it follows that the free generating function including both h and ϕ is given by:

$$Z_0[J^\mu_\nu,J]=Z_0[0,0]e^{-\frac{1}{2}\int d^4xd^4yJ(x)D(x-y)J(y)}e^{-\frac{1}{2}\int d^4xd^4yJ^{\alpha\beta}(x)D_{\alpha\beta\sigma\kappa}(x-y)J^{\sigma\kappa}(y)}. \tag{9.30}$$

9.4 Vertices of Horndeski theory on a Minkowski background

9.4.1 $\varphi\varphi\varphi$-vertex

The Lagrangian for the $\varphi\varphi\varphi$-vertex up to order $1/M_{\rm pl}$ was found to be:

$$\mathcal{L}_{\varphi\varphi\varphi}=-\frac{1}{2M_{\rm pl}}\bar{G}_{2,X\phi}\varphi(\partial^\mu\varphi)(\partial_\mu\varphi)+\frac{1}{2M_{\rm pl}}\bar{G}_{3,\phi\phi}(\partial^\mu\partial_\mu\varphi)\varphi^2-\frac{1}{2\Lambda_3^3}\bar{G}_{3,X}(\partial^\mu\partial_\mu\varphi)(\partial_\nu\varphi)(\partial^\nu\varphi)$$
$$+\frac{1}{\Lambda_3^3}\bar{G}_{4,X\phi}\varphi\Big[\partial^\mu\partial_\mu\varphi\partial^\nu\partial_\nu\varphi-\partial^\mu\partial_\nu\varphi\partial_\mu\partial^\nu\varphi\Big]$$
$$-\frac{\Lambda_2^4}{6\Lambda_3^9}\bar{G}_{5,X}\Big[2\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\alpha\varphi\partial^\alpha\partial_\mu\varphi-3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi+\partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial_\beta\partial^\beta\varphi\Big]. \tag{9.31}$$

The first term can be simplified by noting that:

$$\partial_\mu(\varphi^2\partial^\mu\varphi)=2\varphi\partial_\mu\varphi\partial^\mu\varphi+\varphi^2\partial_\mu\partial^\mu\varphi \implies \varphi\partial_\mu\varphi\partial^\mu\varphi=-\frac{1}{2}\varphi^2\partial_\mu\partial^\mu\varphi. \tag{9.32}$$

In the implication arrow we neglected the boundary term. We will adopt the notation in which we will just set the total derivatives equal to zero (since they will drop out after integration in the action).

Notice first that $\partial^\mu\partial_\mu\varphi\partial^\nu\partial_\nu\varphi-\partial^\mu\partial_\nu\varphi\partial_\mu\partial^\nu\varphi=\delta^{\mu\nu}_{\alpha\beta}\partial_\mu\partial^\alpha\varphi\partial_\nu\partial^\beta\varphi$. The term $\propto\bar{G}_{3,X}$ can also be casted in this form. Let us rewrite $\varphi\partial^\mu\partial_\nu\varphi\partial_\mu\partial^\nu\varphi$ in order to see this:

$$\begin{aligned}\varphi\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\mu\varphi&=-\partial_\nu\varphi\partial^\mu(\varphi\partial_\mu\partial^\nu\varphi)\\&=-\partial_\nu\varphi\partial^\mu\varphi\partial_\mu\partial^\nu\varphi-\varphi\partial_\nu\varphi\partial^\mu\partial_\mu\partial^\nu\varphi\\&=\partial_\mu(\partial_\nu\varphi\partial^\mu\varphi)\partial^\nu\varphi-\varphi\partial_\nu\varphi\partial^\mu\partial_\mu\partial^\nu\varphi\\&=-\varphi\partial_\nu\varphi\partial^\mu\partial_\mu\partial^\nu\varphi+\partial_\mu\partial_\nu\varphi\partial^\mu\varphi\partial^\nu\varphi+\partial_\nu\varphi\partial_\mu\partial^\mu\varphi\partial^\nu\varphi\\&=-\varphi\partial_\nu\varphi\partial^\mu\partial_\mu\partial^\nu\varphi+\partial_\mu\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi-\varphi\partial^\mu(\partial_\mu\partial_\nu\varphi\partial^\nu\varphi)\\&=-2\varphi\partial_\nu\varphi\partial^\mu\partial_\mu\partial^\nu\varphi-\varphi\partial_\mu\partial_\nu\varphi\partial^\mu\partial^\nu\varphi+(\partial_\mu\partial^\mu\varphi)(\partial_\nu\varphi)(\partial^\nu\varphi).\end{aligned} \tag{9.33}$$

So therefore it follows that:

$$\varphi\partial^\mu\partial_\nu\varphi\partial_\mu\partial^\nu\varphi = -\varphi\partial_\nu\varphi\partial^\mu\partial_\mu\partial^\nu\varphi + \frac{1}{2}(\partial_\mu\partial^\mu\varphi)(\partial_\nu\varphi)(\partial^\nu\varphi). \tag{9.34}$$

Similarly we can rewrite $\varphi\partial^\mu\partial_\mu\varphi\partial^\nu\partial_\nu\varphi$:

$$\begin{aligned}\varphi\partial^\mu\partial_\mu\varphi\partial^\nu\partial_\nu\varphi &= -\partial_\mu\varphi\partial^\mu(\varphi\partial^\nu\partial_\nu\varphi)\\ &= -\partial_\mu\varphi\partial^\mu\varphi\partial^\nu\partial_\nu\varphi - \varphi\partial_\mu\varphi\partial^\mu\partial^\nu\partial_\nu\varphi.\end{aligned} \tag{9.35}$$

Thus by combining the results from equations (9.34) and (9.35) one can obtain that:

$$\partial_\mu\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi = -\frac{2}{3}\delta^{\mu\nu}_{\alpha\beta}\partial_\mu\partial^\alpha\varphi\partial_\nu\partial^\beta\varphi. \tag{9.36}$$

Let us now illustrate why the term $\propto \bar{G}_{5,X}$ is equal to zero if we neglect boundary terms. Consider first the term with the -3 in front:

$$\begin{aligned}\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi &= -\partial_\mu(\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi)\partial^\mu\varphi\\ &= -2\partial_\mu\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi\partial^\mu\varphi\\ &= 2\partial^\alpha(\partial^\beta\partial_\alpha\varphi\partial^\mu\varphi)\partial_\mu\partial_\beta\varphi\\ &= 2\partial^\beta\partial_\alpha\partial^\alpha\varphi\partial^\mu\varphi\partial_\mu\partial_\beta\varphi + 2\partial^\beta\partial_\alpha\varphi\partial^\alpha\partial^\mu\varphi\partial_\mu\partial_\beta\varphi\\ &= -2\partial^\beta(\partial^\mu\varphi\partial_\mu\partial_\beta\varphi)\partial_\alpha\partial^\alpha\varphi + 2\partial^\beta\partial_\alpha\varphi\partial^\alpha\partial^\mu\varphi\partial_\mu\partial_\beta\varphi\\ &= -2\partial^\beta\partial^\mu\varphi\partial_\mu\partial_\beta\varphi\partial_\alpha\partial^\alpha\varphi - 2\partial^\mu\varphi\partial_\mu\partial^\beta\partial_\beta\varphi\partial_\alpha\partial^\alpha\varphi + 2\partial^\beta\partial_\alpha\varphi\partial^\alpha\partial^\mu\varphi\partial_\mu\partial_\beta\varphi\\ &\implies -3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi = 2\partial^\mu\varphi\partial_\mu\partial^\beta\partial_\beta\varphi\partial_\alpha\partial^\alpha\varphi - 2\partial^\beta_\alpha\varphi\partial^\alpha\partial^\mu\varphi\partial_\mu\partial_\beta\varphi.\end{aligned} \tag{9.37}$$

And notice that $\partial^\mu\varphi\partial_\mu\partial^\beta\partial_\beta\varphi\partial_\alpha\partial^\alpha\varphi$ can be rewritten as follows:

$$\begin{aligned}\partial^\mu\varphi\partial_\mu\partial^\beta\partial_\beta\varphi\partial_\alpha\partial^\alpha\varphi &= -\partial_\mu(\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi)\partial^\beta\partial_\beta\varphi\\ &= -\partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial^\beta\partial_\beta\varphi - \partial^\mu\varphi\partial_\mu\partial^\alpha\partial_\alpha\varphi\partial_\beta\partial^\beta\varphi\\ &\implies \partial^\mu\varphi\partial_\mu\partial^\beta\partial_\beta\varphi\partial_\alpha\partial^\alpha\varphi = -\frac{1}{2}\partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\partial_\beta\partial^\beta\varphi.\end{aligned} \tag{9.38}$$

Therefore indeed it follows that the term $\propto \bar{G}_{5,X}$ vanishes since:

$$-3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi = -\partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial_\beta\partial^\beta\varphi - 2\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\alpha\varphi\partial^\alpha\partial_\mu\varphi. \tag{9.39}$$

Such that indeed equation (7.47) is found.

9.4.2 $\varphi\varphi\varphi\varphi$-vertex

The $\varphi\varphi\varphi\varphi$-vertex coming from expanding equation (7.24) to quartic order in the field φ is given by:

$$
\begin{aligned}
\mathcal{L}_{\varphi\varphi\varphi\varphi} &= \frac{\Lambda_2^4}{4!}\bar{G}_{2,\phi\phi\phi\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)^4 + \frac{\Lambda_2^4}{2}\bar{G}_{2,X\phi\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)^2 \\
&+ \frac{1}{2}\Lambda_2^4\bar{G}_{2,XX}Y^2 + \frac{\Lambda_2^4}{6\Lambda_3^3}\bar{G}_{3,\phi\phi\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)^3(\partial^\mu\partial_\mu\varphi) \\
&+ \frac{\Lambda_2^4}{\Lambda_3^3}\bar{G}_{3,X\phi}Y\Big(\frac{\varphi}{M_{\rm pl}}\Big)(\partial_\mu\partial^\mu\varphi) + \frac{\Lambda_2^4}{\Lambda_3^6}\bar{G}_{4,XX}Y(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi) \\
&+ \frac{\Lambda_2^4}{2\Lambda_3^6}\bar{G}_{4,X\phi\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)^2(\partial_\mu\partial^\mu\varphi\partial^\nu\partial_\nu\varphi - \partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi) \\
&- \frac{\Lambda_2^4}{6\Lambda_3^9}\bar{G}_{5,X\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)\Big[\partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial_\beta\partial^\beta\varphi + 2\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\alpha\varphi\partial^\alpha\partial_\mu\varphi - 3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi\Big] \\
&= \frac{\Lambda_3^3}{4!M_{\rm pl}^3}\bar{G}_{2,\phi\phi\phi\phi}\varphi^4 - \frac{1}{4M_{\rm pl}^2}\bar{G}_{2,X\phi\phi}\varphi^2\partial_\mu\varphi\partial^\mu\varphi \\
&+ \frac{1}{8\Lambda_3^3M_{\rm pl}}\bar{G}_{2,XX}\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi + \frac{1}{6M_{\rm pl}^2}\bar{G}_{3,\phi\phi\phi}\varphi^3\partial^\mu\partial_\mu\varphi - \frac{1}{2M_{\rm pl}\Lambda_3^3}\bar{G}_{3,X\phi}\varphi\partial_\nu\varphi\partial^\nu\varphi\partial^\mu\partial_\mu\varphi \\
&- \frac{1}{2\Lambda_3^6}\bar{G}_{4,XX}\partial_\alpha\varphi\partial^\alpha\varphi(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi) \\
&+ \frac{1}{2M_{\rm pl}\Lambda_3^3}\bar{G}_{4,X\phi\phi}\varphi^2(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi) \\
&- \frac{1}{6\Lambda_3^6}\bar{G}_{5,X\phi}\varphi\Big[\partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial_\beta\partial^\beta\varphi + 2\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\alpha\varphi\partial^\alpha\partial_\mu\varphi - 3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi\Big].
\end{aligned}
\tag{9.40}
$$

The terms of lower order than $1/M_{\rm pl}$ will be neglected. Notice that the last term can be written in a compact form: $\partial_\mu\partial^\mu\varphi\partial_\alpha\partial^\alpha\varphi\partial_\beta\partial^\beta\varphi + 2\partial^\mu\partial_\nu\varphi\partial^\nu\partial_\alpha\varphi\partial^\alpha\partial_\mu\varphi - 3\partial_\mu\partial^\mu\varphi\partial^\alpha\partial_\beta\varphi\partial^\beta\partial_\alpha\varphi = \delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi\partial^\gamma\partial_\rho\varphi$ and $\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial^\mu\partial_\nu\varphi\partial^\nu\partial_\mu\varphi = \delta^{\mu\nu}_{\alpha\beta}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi$. Thus at the relevant order we are working, the Lagrangian becomes:

$$
\begin{aligned}
\mathcal{L}_{\varphi\varphi\varphi\varphi} &= \frac{1}{8\Lambda_3^3M_{\rm pl}}\bar{G}_{2,XX}\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi - \frac{1}{2M_{\rm pl}\Lambda_3^3}\bar{G}_{3,X\phi}\varphi\partial_\nu\varphi\partial^\nu\varphi\partial^\mu\partial_\mu\varphi \\
&+ \frac{1}{2M_{\rm pl}\Lambda_3^3}\bar{G}_{4,X\phi\phi}\varphi^2\delta^{\mu\nu}_{\alpha\beta}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi - \frac{1}{2\Lambda_3^6}\bar{G}_{4,XX}\partial_\alpha\varphi\partial^\alpha\varphi(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial^\mu\partial_\nu\varphi) \\
&- \frac{1}{6\Lambda_3^6}\bar{G}_{5,X\phi}\varphi\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi\partial^\gamma\partial_\rho\varphi.
\end{aligned}
\tag{9.41}
$$

Let us simplify the terms which are proportional to $\bar{G}_{3,X\phi}$ and $\bar{G}_{4,XX}$.

$$
\begin{aligned}
\varphi\partial_\nu\varphi\partial^\nu\varphi\partial^\mu\partial_\mu\varphi &= -\partial^\mu(\varphi\partial_\nu\varphi\partial^\nu\varphi)\partial_\mu\varphi \\
&= -\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial_\mu\varphi - 2\varphi\partial^\mu\partial^\nu\varphi\partial_\nu\varphi\partial_\mu\varphi \\
&= -\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial_\mu\varphi - \partial^\mu\partial^\nu\varphi\partial_\nu\varphi^2\partial_\mu\varphi \\
&= -\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial_\mu\varphi + \varphi^2\partial_\nu(\partial^\mu\partial^\nu\varphi\partial_\mu\varphi) \\
&= -\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial_\mu\varphi + \varphi^2\partial_\nu\partial^\mu\partial^\nu\varphi\partial_\mu\varphi + \varphi^2\partial^\mu\partial^\nu\varphi\partial_\nu\partial_\mu\varphi \\
&= -\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial_\mu\varphi - \partial^\mu(\varphi^2\partial_\mu\varphi)\partial_\nu\partial^\nu\varphi + \varphi^2\partial^\mu\partial^\nu\varphi\partial_\nu\partial_\mu\varphi \\
&= -\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial_\mu\varphi - 2\varphi\partial^\mu\varphi\partial_\nu\partial^\nu\varphi\partial_\mu\varphi - \varphi^2\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi + \varphi^2\partial^\mu\partial^\nu\varphi\partial_\mu\partial_\nu\varphi \\
&\implies 3\varphi\partial^\mu\varphi\partial_\mu\varphi\partial_\nu\partial^\nu\varphi = -\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial_\mu\varphi - \varphi^2\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi + \varphi^2\partial^\mu\partial^\nu\varphi\partial_\mu\partial_\nu\varphi \\
&\implies \varphi\partial^\mu\varphi\partial_\nu\partial^\nu\varphi\partial_\mu\varphi = -\frac{1}{3}\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi\partial^\mu\varphi - \frac{1}{3}\varphi^2\delta^{\mu\nu}_{\alpha\beta}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi.
\end{aligned}
\tag{9.42}
$$

The term proportional to $\bar{G}_{4,XX}$ can be simplified as follows:

$$
\begin{aligned}
\partial_\alpha\varphi\partial^\alpha\varphi(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi) &= \partial_\alpha\varphi\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial^\nu\partial_\nu\varphi - \partial_\alpha\varphi\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial^\mu\partial_\nu\varphi \\
&= -\varphi\partial_\alpha(\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi) + \varphi\partial_\alpha(\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi) \\
&= -\varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - 2\varphi\partial^\alpha\varphi\partial_\mu\partial^\mu\partial_\alpha\varphi\partial_\nu\partial^\nu\varphi \\
&+ \varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi + 2\varphi\partial^\alpha\varphi\partial_\alpha\partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi \\
&= -\varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi + 2\partial_\mu(\varphi\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi)\partial^\mu\partial_\alpha\varphi \\
&+ \varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial^\mu\partial_\nu\varphi - 2\partial^\nu(\varphi\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi)\partial_\alpha\partial_\mu\varphi \\
&= -\varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi + 2\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi \\
&+ 2\varphi\partial_\mu\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi + 2\varphi\partial^\alpha\varphi\partial_\mu\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi \\
&+ \varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi - 2\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi \\
&- 2\varphi\partial^\nu\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi\partial_\alpha\partial_\mu\varphi - 2\varphi\partial^\alpha\varphi\partial^\nu\partial_\nu\partial^\mu\varphi\partial_\alpha\partial_\mu\varphi \\
&= -\varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi + 2\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi \\
&+ 2\varphi\partial_\mu\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi + \varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi \\
&- 2\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi - 2\varphi\partial^\nu\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi\partial_\alpha\partial_\mu\varphi \\
&= -\varphi\delta^{\mu\beta\rho}_{\alpha\sigma\kappa}\partial_\mu\partial^\alpha\varphi\partial_\beta\partial^\sigma\varphi\partial_\rho\partial^\kappa\varphi + 2\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi \\
&- 2\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi.
\end{aligned}
\tag{9.43}
$$

The expression can be simplified further:

$$
\begin{aligned}
&\partial_\alpha\varphi\partial^\alpha\varphi(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi) \\
&= -\varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - 2\partial^\mu(\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi)\partial_\alpha\varphi + 2\varphi\partial_\mu\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi \\
&+ \varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi + 2\partial^\mu(\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi)\partial_\nu\varphi - 2\varphi\partial^\nu\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi\partial_\alpha\partial_\mu\varphi \\
&= -\varphi\partial^\alpha\partial_\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - 2\partial^\mu\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial_\alpha\varphi - 2\partial_\mu\varphi\partial^\mu\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial_\alpha\varphi - 2\partial_\mu\varphi\partial^\alpha\varphi\partial^\mu\partial_\nu\partial^\nu\varphi\partial_\alpha\varphi \\
&+ 2\varphi\partial_\mu\partial_\alpha\varphi\partial_\nu\partial^\nu\varphi\partial^\mu\partial_\alpha\varphi + \varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\nu\varphi\partial_\nu\partial^\mu\varphi + 2\partial^\mu\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\varphi + 2\partial^\nu\varphi\partial^\mu\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\varphi \\
&+ 2\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\mu\partial^\alpha\varphi\partial_\nu\varphi - 2\varphi\partial^\nu\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi\partial_\alpha\partial_\mu\varphi \\
&\implies 3\partial_\alpha\varphi\partial^\alpha\varphi(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial^\mu\partial_\nu\varphi) = -\varphi\partial_\alpha\partial^\alpha\varphi\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi + 3\varphi\partial_\mu\partial^\nu\varphi\partial^\mu\partial_\nu\varphi\partial_\alpha\partial^\alpha\varphi \\
&+ 2\partial^\mu\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\varphi - 2\partial_\mu\varphi\partial^\mu\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial_\alpha\varphi - 2\varphi\partial^\nu\partial^\alpha\varphi\partial_\nu\partial^\mu\varphi\partial_\alpha\partial_\mu\varphi \\
&= -\varphi\delta^{\mu\beta\rho}_{\alpha\sigma\kappa}\partial_\mu\partial^\alpha\varphi\partial_\beta\partial^\sigma\varphi\partial_\rho\partial^\kappa\varphi + 2\partial^\mu\partial^\nu\varphi\partial_\alpha\partial_\mu\varphi\partial^\alpha\varphi\partial_\nu\varphi - 2\partial_\mu\varphi\partial^\mu\partial^\alpha\varphi\partial_\nu\partial^\nu\varphi\partial_\alpha\varphi.
\end{aligned}
\tag{9.44}
$$

Combining equation (9.43) and (9.44) easily gives:

$$\partial_\alpha\varphi\partial^\alpha\varphi(\partial_\mu\partial^\mu\varphi\partial_\nu\partial^\nu\varphi - \partial_\mu\partial^\nu\varphi\partial^\mu\partial_\nu\varphi) = -\frac{1}{2}\varphi\delta^{\mu\beta\rho}_{\alpha\sigma\kappa}\partial_\mu\partial^\alpha\varphi\partial_\beta\partial^\sigma\varphi\partial_\rho\partial^\kappa\varphi. \tag{9.45}$$

Therefore the Lagrangian $\mathcal{L}_{\varphi\varphi\varphi\varphi}$ can be written as:

$$\begin{aligned}\mathcal{L}_{\varphi\varphi\varphi\varphi} &= \frac{3\bar{G}_{4,XX} - 2\bar{G}_{5,\phi X}}{12\Lambda_3^6}\varphi\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi\partial^\gamma\partial_\rho\varphi \\ &+ \frac{3\bar{G}_{2,XX} + 4\bar{G}_{3,X\phi}}{24M_{\text{pl}}\Lambda_3^3}\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi + \frac{\bar{G}_{3,X\phi} + 3\bar{G}_{4,X\phi\phi}}{6M_{\text{pl}}\Lambda_3^3}\varphi^2\delta^{\mu\nu}_{\alpha\beta}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi.\end{aligned} \tag{9.46}$$

9.4.3 $h\varphi\varphi$-vertex

The relevant Lagrangian containing $h\varphi\varphi$ is found by expanding the action to second order in φ and first order in h:

$$\begin{aligned}\mathcal{L}_{h\varphi\varphi} &= \Lambda_2^4\bar{G}_{2,X}Y(1+\delta_1\sqrt{g}) + \frac{\Lambda_2^4}{2}\bar{G}_{2,\phi\phi}\Big(\frac{\varphi}{M_{\text{pl}}}\Big)^2\delta_1\sqrt{-g} \\ &+ \Lambda_2^4\bar{G}_{3,\phi}(1+\delta_1\sqrt{-g})\Big(\frac{\varphi}{M_{\text{pl}}}\Big)\frac{\nabla^\mu\nabla_\mu\varphi}{\Lambda_3^3} + \bar{G}_{4,X}\sqrt{-g}\Big(M_{\text{pl}}^2YR + \frac{\Lambda_2^4}{\Lambda_3^6}\Big((\nabla^\alpha\nabla_\alpha\varphi)^2 - (\nabla^\alpha\nabla_\beta\varphi)^2\Big)\Big) \\ &+ \frac{M_{\text{pl}}^2}{2}\bar{G}_{4,\phi\phi}\Big(\frac{\varphi}{M_{\text{pl}}}\Big)^2\delta_1 R \\ &+ M_{\text{pl}}^2\bar{G}_{5,\phi}\delta_1 G_{\mu\nu}\Big(\frac{\varphi}{M_{\text{pl}}}\Big)\Big(\frac{\nabla^\mu\nabla^\nu\varphi}{\Lambda_3^3}\Big).\end{aligned} \tag{9.47}$$

In the fourth term we included the full R and $\sqrt{-g}$ (which are present if you would expand the action to infinite order). The reason for this is that this will simplify our expression for $\mathcal{L}_{h\varphi\varphi}$ a lot, since we can neglect some boundary terms which are not present in the full action.

Notice that the second term goes as $\Lambda_3^3/M_{\text{pl}}^2$ and can therefore be neglected at order $1/M_{\text{pl}}$. Recall that $\delta_1\sqrt{-g} = \frac{1}{2M_{\text{pl}}}h_{\mu\nu}\eta^{\mu\nu}$. And we can estimate Y to first order in h:

$$Y = -\frac{1}{2\Lambda_2^4}\nabla^\mu\varphi\nabla_\mu\varphi = -\frac{1}{2\Lambda_2^4}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \approx -\frac{1}{2\Lambda_2^4}\Big(\eta^{\mu\nu} - \frac{1}{M_{\text{pl}}}h^{\mu\nu}\Big)\partial_\mu\varphi\partial_\nu\varphi. \tag{9.48}$$

And the term proportional to $\bar{G}_{3,\phi}$ can also be simplified. Consider the following relevant term for the $h\varphi\varphi$-vertex:

$$
\begin{aligned}
&\varphi\Big(1+\frac{h}{2M_{\rm pl}}\Big)\Big(\eta^{\mu\nu}-\frac{h^{\mu\nu}}{M_{\rm pl}}\Big)\Big(\partial_\mu\partial_\nu\varphi-\delta_1\Gamma^{\alpha}_{\mu\nu}\partial_\alpha\varphi\Big)\\
&=\varphi\Big(1+\frac{h}{2M_{\rm pl}}\Big)\Big(\eta^{\mu\nu}-\frac{h^{\mu\nu}}{M_{\rm pl}}\Big)\Big(\partial_\mu\partial_\nu\varphi-\frac{1}{2M_{\rm pl}}\partial_\alpha\varphi\eta^{\alpha\lambda}(\partial_\nu h_{\mu\lambda}+\partial_\mu h_{\nu\lambda}-\partial_\lambda h_{\mu\nu})\Big)\\
&=\varphi\Big(\eta^{\mu\nu}-\frac{h^{\mu\nu}}{M_{\rm pl}}+\frac{\eta^{\mu\nu}h}{2M_{\rm pl}}-\frac{hh^{\mu\nu}}{2M_{\rm pl}^2}\Big)\Big(\partial_\mu\partial_\nu\varphi-\frac{1}{2M_{\rm pl}}\partial_\alpha\varphi\eta^{\alpha\lambda}(\partial_\nu h_{\mu\lambda}+\partial_\mu h_{\nu\lambda}-\partial_\lambda h_{\mu\nu})\Big)\\
&\approx\varphi\Big(\eta^{\mu\nu}-\frac{h^{\mu\nu}}{M_{\rm pl}}+\frac{\eta^{\mu\nu}h}{2M_{\rm pl}}\Big)\Big(\partial_\mu\partial_\nu\varphi-\frac{1}{2M_{\rm pl}}\partial_\alpha\varphi\eta^{\alpha\lambda}(\partial_\nu h_{\mu\lambda}+\partial_\mu h_{\nu\lambda}-\partial_\lambda h_{\mu\nu})\Big)\\
&\approx\varphi\partial^\mu\partial_\mu\varphi-\frac{\varphi}{2M_{\rm pl}}\partial^\lambda\varphi(2\partial^\mu h_{\mu\lambda}-\partial_\lambda h)-\frac{1}{M_{\rm pl}}\varphi h^{\mu\nu}\partial_\mu\partial_\nu\varphi+\frac{\varphi h}{2M_{\rm pl}}\partial^\mu\partial_\mu\varphi.
\end{aligned}
\tag{9.49}
$$

The first term in the final expression can be omitted since it describes the free part of φ and not the $h\varphi\varphi$-vertex. Using integration by parts this expression can be simplified further, by noticing that:

$$
\begin{aligned}
&-\frac{1}{2}\varphi(\partial^\lambda\varphi)(2\partial^\mu h_{\mu\lambda}-\partial_\lambda h)-\varphi h^{\mu\nu}\partial_\mu\partial_\nu\varphi+\frac{\varphi h}{2}\partial^\mu\partial_\mu\varphi\\
&=-\varphi\partial^\lambda\varphi\partial^\mu h_{\mu\lambda}+\frac{1}{2}\varphi\partial^\lambda\varphi\partial_\lambda h-\varphi h^{\mu\nu}\partial_\mu\partial_\nu\varphi+\frac{1}{2}\varphi h\partial^\mu\partial_\mu\varphi\\
&=\partial^\mu(\varphi\partial^\lambda\varphi)h_{\mu\lambda}-\frac{1}{2}h\partial_\lambda(\varphi\partial^\lambda\varphi)-\varphi h^{\mu\nu}\partial_\mu\partial_\nu\varphi+\frac{1}{2}\varphi h\partial^\mu\partial_\mu\varphi\\
&=h_{\mu\nu}\Big(\partial^\mu\varphi\partial^\nu\varphi-\frac{1}{2}\eta^{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi\Big).
\end{aligned}
\tag{9.50}
$$

Therefore the Lagrangian for the $h\varphi\varphi$-vertex can be written as:

$$
\begin{aligned}
\mathcal{L}_{h\varphi\varphi}=&\frac{\bar G_{2,X}+2\bar G_{3,\phi}}{2M_{\rm pl}}h_{\mu\nu}\Big(\partial^\mu\varphi\partial^\nu\varphi-\frac{1}{2}\eta^{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi\Big)\\
&+\bar G_{4,X}\sqrt{-g}\Big(M_{\rm pl}^2YR+\frac{\Lambda_2^4}{\Lambda_3^6}\Big((\nabla^\alpha\nabla_\alpha\varphi)^2-(\nabla^\alpha\nabla_\beta\varphi)^2\Big)\Big)\\
&+\frac{M_{\rm pl}^2}{2}\bar G_{4,\phi\phi}\Big(\frac{\varphi}{M_{\rm pl}}\Big)^2\delta_1R\\
&+M_{\rm pl}^2\bar G_{5,\phi}\delta_1G_{\mu\nu}\Big(\frac{\varphi}{M_{\rm pl}}\Big)\Big(\frac{\nabla^\mu\nabla^\nu\varphi}{\Lambda_3^3}\Big).
\end{aligned}
\tag{9.51}
$$

Let us first simplify the terms proportional to $\bar G_{4,\phi\phi}$ and $\bar G_{5,\phi}$. From the expression (9.14) we have that:

$$
\begin{aligned}
&\delta_1R_{\sigma\nu}=\frac{1}{2M_{\rm pl}}\Big[\partial^\mu\partial_\sigma h_{\nu\mu}-\partial^\mu\partial_\mu h_{\nu\sigma}-\partial_\nu\partial_\sigma h+\partial^\mu\partial_\nu h_{\mu\sigma}\Big]\\
&\delta_1G_{\sigma\nu}=\delta_1R_{\sigma\nu}-\frac{1}{2}\eta_{\sigma\nu}\delta_1R=\frac{1}{2M_{\rm pl}}\Big[\partial^\mu\partial_\sigma h_{\nu\mu}-\partial^\mu\partial_\mu h_{\nu\sigma}-\partial_\nu\partial_\sigma h+\partial^\mu\partial_\nu h_{\mu\sigma}-\eta_{\sigma\nu}\partial^\alpha\partial^\beta h_{\alpha\beta}+\eta_{\sigma\nu}\partial^\alpha\partial_\alpha h\Big].
\end{aligned}
\tag{9.52}
$$

So therefore it follows that:

$$
\begin{aligned}
\delta_1 G_{\sigma\nu}\varphi\partial^\sigma\partial^\nu\varphi &= \frac{1}{2M_{\rm pl}}\varphi\Big[\partial^\sigma\partial_\nu\varphi\partial^\mu\partial_\sigma h^\nu_\mu - \partial^\sigma\partial_\nu\varphi\partial_\mu\partial^\mu h^\nu_\sigma - \partial^\sigma\partial_\nu\varphi\partial^\nu\partial_\sigma h \\
&+ \partial_\sigma\partial^\nu\varphi\partial^\mu\partial_\nu h^\sigma_\mu - \partial^\nu\partial_\nu\varphi\partial_\alpha\partial^\beta h^\alpha_\beta + \partial^\nu\partial_\nu\varphi\partial^\alpha\partial_\alpha h\Big] \\
&= \frac{1}{2M_{\rm pl}}\varphi\delta^{\mu\rho\alpha}_{\nu\sigma\beta}\partial^\nu\partial_\mu\varphi\partial_\rho\partial^\sigma h^\beta_\alpha.
\end{aligned} \tag{9.53}
$$

From equation (9.15) we also notice that $\varphi^2\delta_1 R = -\varphi^2\delta^{\mu\nu}_{\alpha\beta}\partial^\alpha\partial_\mu h^\beta_\nu$.

Finally, the term proportional to $\bar{G}_{4,X}$ can be simplified by noting that [19]:

$$
-\frac{1}{2}R\nabla_\mu\varphi\nabla^\mu\varphi + (\nabla^\alpha\nabla_\alpha\varphi)^2 - (\nabla^\alpha\nabla_\beta\varphi)^2 = -G_{\mu\nu}\varphi\nabla^\mu\nabla^\nu\varphi + \nabla_\mu(...), \tag{9.54}
$$

where $\nabla_\mu(...)$ indicates a boundary term. The proof of this is quite simple:

$$
\begin{aligned}
(\nabla^\alpha\nabla_\alpha\varphi)^2 - (\nabla^\alpha\nabla_\beta\varphi)^2 &= \nabla_\alpha(\nabla^\alpha\varphi\nabla_\beta\nabla^\beta\varphi - \nabla_\beta\varphi\nabla^\beta\nabla^\alpha\varphi) - \nabla^\alpha\varphi(\nabla_\alpha\nabla_\beta\nabla^\beta\varphi - \nabla_\beta\nabla^\beta\nabla_\alpha\varphi) \\
&= \nabla_\alpha(\nabla^\alpha\varphi\nabla_\beta\nabla^\beta\varphi - \nabla_\beta\varphi\nabla^\beta\nabla^\alpha\varphi) \\
&- \nabla^\alpha\varphi([\nabla_\alpha,\nabla_\beta]\nabla^\beta\varphi - \nabla_\beta\nabla^\beta\nabla_\alpha\varphi + \nabla_\beta\nabla_\alpha\nabla^\beta\varphi) \\
&= \nabla_\alpha(\nabla^\alpha\varphi\nabla_\beta\nabla^\beta\varphi - \nabla_\beta\varphi\nabla^\beta\nabla^\alpha\varphi) - \nabla^\alpha\varphi[\nabla_\alpha,\nabla_\beta]\nabla^\beta\varphi \\
&= \nabla_\alpha(\nabla^\alpha\varphi\nabla_\beta\nabla^\beta\varphi - \nabla_\beta\varphi\nabla^\beta\nabla^\alpha\varphi) - R^\beta_{\ \sigma\alpha\beta}\nabla^\sigma\varphi\nabla^\alpha\varphi \\
&= \nabla_\alpha(\nabla^\alpha\varphi\nabla_\beta\nabla^\beta\varphi - \nabla_\beta\varphi\nabla^\beta\nabla^\alpha\varphi) + R_{\sigma\alpha}\nabla^\alpha\varphi\nabla^\sigma\varphi.
\end{aligned} \tag{9.55}
$$

In the first equal sign and to go from the second to the third line we used that $\nabla_\alpha\nabla^\beta\varphi = \nabla^\beta\nabla_\alpha\varphi$.

By the fact that $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ and the Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$ the result follows directly.

Therefore it follows that:

$$
\bar{G}_{4,X}\sqrt{-g}\Big(M^2_{\rm pl}YR + \frac{\Lambda_2^4}{\Lambda_3^6}\Big((\nabla^\alpha\nabla_\alpha\varphi)^2 - (\nabla^\alpha\nabla_\beta\varphi)^2\Big)\Big) = -\frac{\bar{G}_{4,X}}{H_0^2}\varphi\sqrt{-g}G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi. \tag{9.56}
$$

Focusing on the part relevant for the $h\varphi\varphi$-vertex we find the term:

$$
-\frac{\bar{G}_{4,X}}{H_0^2}\varphi\partial^\mu\partial^\nu\varphi\delta_1 G_{\mu\nu} = -\frac{\bar{G}_{4,X}}{2\Lambda_3^3}\varphi\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu h^\kappa_\rho. \tag{9.57}
$$

Thus the Lagrangian of the $h\varphi\varphi$-vertex becomes:

$$
\begin{aligned}
\mathcal{L}_{h\varphi\varphi} &= -\frac{\bar{G}_{4,X} - \bar{G}_{5,\phi}}{2\Lambda_3^3}\varphi\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu h^\kappa_\rho + \frac{\bar{G}_{2,X} + 2\bar{G}_{3,\phi}}{2M_{\rm pl}}h_{\mu\nu}(\partial^\mu\varphi\partial^\nu\varphi - \frac{1}{2}\eta^{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi) \\
&- \frac{\bar{G}_{4,\phi\phi}}{2M_{\rm pl}}\varphi^2\delta^{\mu\nu}_{\alpha\beta}\partial^\alpha\partial_\mu h^\beta_\nu.
\end{aligned} \tag{9.58}
$$

9.4.4 Other vertices at tree-level

For completeness we will sketch how the hhh-vertex, $hh\varphi$-vertex and $hh\varphi\varphi$-vertex can be derived. These vertices will however not be used to compute the $\varphi\varphi \to \varphi\varphi$ example we are considering. The hhh-vertex is simply the one from GR:

$$\mathcal{L}_{hhh} = \bar{G}_4 M_{\text{pl}}^2 \delta_3(\sqrt{-g}R), \tag{9.59}$$

which is sub-leading order since $\delta_3(\sqrt{-g}R) \propto 1/M_{\text{pl}}^3$. The third order variation δ_3 includes the $1/3!$ from the Taylor series.

The $hhhh$-vertex goes as $1/M_{\text{pl}}^2$ and is therefore not relevant at the order $1/M_{\text{pl}}$ we are working.

The $hh\varphi$-vertex is found by expanding equation (7.24):

$$\begin{aligned}\mathcal{L}_{hh\varphi} &= \Lambda_2^4 \delta_2 \sqrt{-g}\Big(\frac{\varphi}{M_{\text{pl}}}\Big)\bar{G}_{2,\phi} + \Lambda_2^4(1 + \delta_1\sqrt{-g} + \delta_2\sqrt{-g})\bar{G}_3 \frac{\nabla^\mu\nabla_\mu\varphi}{\Lambda_3^3} \\ &+ M_{\text{pl}}^2(1 + \delta_1\sqrt{-g} + \delta_2\sqrt{-g})\bar{G}_{4,\phi}\Big(\frac{\varphi}{M_{\text{pl}}}\Big)(\delta_1 R + \delta_2 R) \\ &+ M_{\text{pl}}^2(1 + \delta_1\sqrt{-g} + \delta_2\sqrt{-g})\bar{G}_5(\delta_1 G_{\mu\nu} + \delta_2 G_{\mu\nu})\frac{\nabla^\mu\nabla^\nu\varphi}{\Lambda_3^3}.\end{aligned} \tag{9.60}$$

Here the δ_2 contains an the factor $1/2!$ from the Taylor series. The term with $\bar{G}_{2,\phi}$ has order $1/M_{\text{pl}}^2$ so can be neglected. It can be seen that the term with $\bar{G}_3$ can be set to zero since $\int d^4x\sqrt{-g}\bar{G}_3\nabla_\mu\nabla^\mu\varphi = 0$ if we neglect boundary terms. The term with $\bar{G}_5$ is zero as well since under integration by parts $\bar{G}_5 G_{\mu\nu}\nabla^\mu\nabla^\nu\varphi = -\bar{G}_5\nabla^\nu G_{\mu\nu}\nabla^\mu\varphi = 0$ by the Bianchi identity, so we may set $\bar{G}_5 = 0$. Therefore only the term with $\bar{G}_{4,\phi}$ contributes.

Eventually in computing the Feynman rules for the vertices the external gravitons will be taken on-shell, i.e. transverse and traceless [18]. In Fourier space the Feynman rules for an external graviton amount to a polarization tensor $\epsilon^{\mu\nu}$ (and complex conjugate if the graviton outgoing) which obeys [20]:

$$\epsilon^{\mu\nu} = \epsilon^{\nu\mu},\ p_\mu\epsilon^{\mu\nu}(p) = 0,\ \eta_{\mu\nu}\epsilon^{\mu\nu} = 0. \tag{9.61}$$

This means that the terms with h and $\partial_\mu h^\mu_\nu$ can be set to zero in the vertex Lagrangian before computing the Feynman rules from it as these will in Fourier space yield terms which will under contraction with these polarization tensors will vanish. This has the advantage of reducing the number of terms by a huge amount [20].

Therefore the relevant on-shell Lagrangian for the $hh\varphi$-vertex can be written as:

$$\mathcal{L}_{hh\varphi} = M_{\text{pl}}\bar{G}_{4,\phi}\varphi\delta_2 R. \tag{9.62}$$

From the computations of the graviton propagator, we recall that $\delta_2 R$ on-shell is given by:

$$\begin{aligned}M_{\text{pl}}^2\delta_2 R &= h^{\sigma\nu}\partial_\mu\partial^\mu h_{\sigma\nu} + \frac{1}{2}(\partial_\nu h^{\mu\lambda})(\partial_\mu h^\nu_\lambda + \partial^\nu h_{\mu\lambda} - \partial_\lambda h^\nu_\mu) \\ &- \frac{1}{4}(\partial^\sigma h^{\mu\alpha} + \partial^\alpha h^{\sigma\mu} - \partial^\mu h^{\sigma\alpha})(\partial_\mu h_{\sigma\alpha} + \partial_\sigma h_{\mu\alpha} - \partial_\alpha h_{\mu\sigma}) \\ &= h^{\sigma\nu}\partial_\mu\partial^\mu h_{\sigma\nu} + \frac{3}{4}\partial_\gamma h_{\alpha\beta}\partial^\gamma h^{\alpha\beta} - \frac{1}{2}\partial^\mu h^{\sigma\alpha}\partial_\alpha h_{\mu\sigma}.\end{aligned} \tag{9.63}$$

Notice that $\varphi h^{\mu\nu}\partial_\mu\partial^\mu h_{\sigma\nu}$ can be simplified:

$$\begin{aligned}
\varphi h^{\mu\nu}\partial_\mu\partial^\mu h_{\sigma\nu} &= -\partial_\mu(\varphi h^{\sigma\nu})\partial^\mu h_{\sigma\nu} \\
&= -h^{\sigma\nu}\partial_\mu\varphi\partial^\mu h_{\sigma\nu} - \varphi\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu} \\
&= \partial^\mu(h^{\sigma\nu}\partial_\mu\varphi)h_{\sigma\nu} - \varphi\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu} \\
&= h^{\sigma\nu}h_{\sigma\nu}\partial^\mu\partial_\mu\varphi + h_{\sigma\nu}\partial_\mu\varphi\partial^\mu h^{\sigma\nu} - \varphi\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu} \\
&= h^{\sigma\nu}h_{\sigma\nu}\partial_\mu\partial^\mu\varphi - h_{\sigma\nu}\varphi\partial^\mu\partial_\mu h^{\sigma\nu} - 2\varphi\partial_\mu h^{\sigma\nu}\partial^\mu h_{\sigma\nu} \\
&\implies \varphi h^{\sigma\nu}\partial_\mu\partial^\mu h_{\sigma\nu} = \frac{1}{2}h^{\sigma\nu}h_{\sigma\nu}\partial^\mu\partial_\mu\varphi - \varphi\partial^\mu h^{\sigma\nu}\partial_\mu h_{\sigma\nu}.
\end{aligned} \tag{9.64}$$

And we can simplify $\varphi\partial^\mu h^{\sigma\alpha}\partial_\alpha h_{\mu\sigma}$:

$$\begin{aligned}
\varphi\partial^\mu h^{\sigma\alpha}\partial_\alpha h_{\mu\sigma} &= -\partial_\alpha(\varphi\partial^\mu h^{\sigma\alpha})h_{\mu\sigma} \\
&= h^{\sigma\alpha}h_{\mu\sigma}\partial_\alpha\partial^\mu\varphi,
\end{aligned} \tag{9.65}$$

where we used the transverse property ($\partial_\alpha h^{\sigma\alpha} = 0$) twice. Hence it follows that $\varphi M_{\text{pl}}^2\delta_2 R$ is of the form:

$$\varphi M_{\text{pl}}^2\delta_2 R = \frac{1}{2}h_{\sigma\nu}\partial_\mu\partial^\mu\varphi - \frac{1}{4}\varphi\partial_\gamma h_{\alpha\beta}\partial^\gamma h^{\alpha\beta} - \frac{1}{2}h^{\sigma\alpha}h_{\mu\sigma}\partial_\alpha\partial^\mu\varphi. \tag{9.66}$$

Thus the $hh\varphi$-vertex is given by:

$$\mathcal{L}_{hh\varphi} = \frac{\bar{G}_{4,\phi}}{4M_{\text{pl}}}(2\delta^{\mu\nu}_{\alpha\beta}\partial^\alpha\partial_\mu\varphi h^\rho_\nu h^\beta_\rho - \varphi\partial_\gamma h_{\alpha\beta}\partial^\gamma h^{\alpha\beta}). \tag{9.67}$$

The $hh\varphi\varphi$-vertex up to order $1/M_{\text{pl}}$ only contains $\bar{G}_{4,X}$ and $\bar{G}_{5,\phi}$ (since other terms are of lower order):

$$\mathcal{L}_{hh\varphi\varphi} = \sqrt{-g}\bar{G}_{4,X}(M_{\text{pl}}^2 YR + \Lambda_2^4([\Phi]^2 - [\Phi^2])) + M_{\text{pl}}^2\bar{G}_{5,\phi}\Big(\frac{\varphi}{M_{\text{pl}}}\Big)\sqrt{-g}\frac{\nabla^\mu\nabla^\nu\varphi}{\Lambda_3^3}(\delta_1 G_{\mu\nu} + \delta_2 G_{\mu\nu}). \tag{9.68}$$

The first term can be simplified since $\Lambda_2^4 YR + (\nabla^\alpha\nabla_\alpha\varphi)^2 - (\nabla^\alpha\nabla_\beta\varphi)^2 = -G_{\mu\nu}\varphi\nabla^\mu\nabla^\nu\varphi$ up to boundary terms. Writing out every term to the right order will yield the $hh\varphi\varphi$-vertex.

9.5 Feynman rules of Horndeski theory on a Minkowski background

In this section we provide detailed calculations of the vertex Feynman rules for Horndeski gravity on a Minkowski background.

9.5.1 $\varphi\varphi\varphi$-vertex Feynman rules

Feynman rules of vertices can be found by computing correlation functions. For the $\varphi\varphi\varphi$-vertex the idea is to compute the three-point correlation function $\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\}|\Omega\rangle$. For the generating function (7.44) we can ignore the graviton part since this is not present in the $\varphi\varphi\varphi$-vertex. Focus on the term with $c^{(m)}_{\varphi\varphi\varphi}$ first. Taking into account the interaction Lagrangian we have that (7.11):

$$Z^{(m)}_{\varphi\varphi\varphi}[J] \approx Z^{(m)}_{\varphi\varphi\varphi,0}[0]\Big(1+i\int d^4x c^{(m)}_{\varphi\varphi\varphi}\Big(-i\frac{\delta}{\delta J(x)}\Big)^2\partial^\mu\partial^\mu\Big(-i\frac{\delta}{\delta J(x)}\Big)\Big)e^{-\frac{1}{2}\int d^4xd^4yJ(x)D(x-y)J(y)}. \quad (9.69)$$

And the correlation function $\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\}|\Omega\rangle^{(m)}_{\varphi\varphi\varphi}$ is given by:

$$\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\}|\Omega\rangle^{(m)}_{\varphi\varphi\varphi} = \frac{1}{Z[0]}\Big(-i\frac{\delta}{\delta J(x_1)}\Big)\Big(-i\frac{\delta}{\delta J(x_2)}\Big)\Big(-i\frac{\delta}{\delta J(x_3)}\Big)Z[J]\Big|_{J=0}. \quad (9.70)$$

As mentioned in [12], we can absorb bubble diagrams involving D^2_{xx} into $Z_0[0]$ to give us $Z[0]$. Also we will ignore other diagrams which involve D_{xx} in the correlation function since we consider tree-level diagrams and we will ignore the free part in the correlation function (only the first-order in the interaction will be considered). We will adopt a notation in which $J(x_i) = J_i$, $D(z-y) = D_{zy}$ and $J(y) = J_y$. We will leave the integration over x implicit and we will adopt the convenient notation $J_zD_{zy}J_y \equiv \int d^4yd^4zJ_zD_{zy}J_y$ in order to make expressions more compact. And the evaluation of the expression at $J = 0$ will also be implicit. Under these notations the correlation function can be written as:

$$i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\}|\Omega\rangle^{(m)}_{\varphi\varphi\varphi} = c^{(m)}_{\varphi\varphi\varphi}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_x}\partial_\mu\partial^\mu\Big(\frac{\delta}{\delta J(x)}\Big)e^{-\frac{1}{2}J_zD_{zy}J_y}. \quad (9.71)$$

Therefore we find that:

$$\begin{aligned}
i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\}|\Omega\rangle^{(m)}_{\varphi\varphi\varphi} &= -c^{(m)}_{\varphi\varphi\varphi}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_x}\frac{\delta}{\delta J_x}e^{-\frac{1}{2}J_zD_{zy}J_y}\partial_\mu\partial_\mu(D_{xz}J_z)\\
&= c^{(m)}_{\varphi\varphi\varphi}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_x}\Big(J_yD_{xy}e^{-\frac{1}{2}J_zD_{zy}J_y}\partial^\mu\partial_\mu(D_{xz}J_z)\Big)\\
&= -c^{(m)}_{\varphi\varphi\varphi}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}(D_{xv}J_vD_{xy}J_y\partial_\mu\partial^\mu(D_{xz}J_z))\\
&= -c^{(m)}_{\varphi\varphi\varphi}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}[2D_{x3}D_{xy}J_y\partial_\mu\partial^\mu(D_{xz}J_z) + D_{xv}D_{xy}J_vJ_y\partial_\mu\partial^\mu D_{x3}]\\
&= -c^{(m)}_{\varphi\varphi\varphi}\frac{\delta}{\delta J_1}[2D_{x3}D_{x2}\partial^\mu\partial_\mu(D_{xz}J_z) + 2D_{x3}D_{x2}\partial_\mu\partial^\mu(D_{xz}J_z)\\
&+ 2D_{x2}D_{xy}J_y\partial_\mu\partial^\mu D_{x3}]\\
&= -2c^{(m)}_{\varphi\varphi\varphi}[D_{x3}D_{x2}\partial_\mu\partial^\mu D_{x1} + D_{x2}D_{x1}\partial_\mu\partial^\mu D_{x3} + D_{x3}D_{x1}\partial_\mu\partial^\mu D_{x2}]\\
&= -2c^{(m)}_{\varphi\varphi\varphi}\int d^4x[D_{x3}D_{x2}\partial_\mu\partial^\mu D_{x1} + D_{x2}D_{x1}\partial_\mu\partial^\mu D_{x3}\\
&+ D_{x3}D_{x1}\partial_\mu\partial^\mu D_{x2}].
\end{aligned} \quad (9.72)$$

In the last line we rewrote the integration over x explicitly. In Fourier space this clearly corresponds to the vertex (incorporating the i in front of the correlation function as well):

$$V^{(m)}_{\varphi\varphi\varphi}(p_1,p_2,p_3) = -2ic^{(m)}_{\varphi\varphi\varphi}[p_1^2+p_2^2+p_3^2], \quad (9.73)$$

where $p_i^2 := p_i \cdot p_i$. It holds that $p_1+p_2+p_3 = 0$ by momentum conservation at the vertex where all momenta are incoming since the integration over x yields a delta function of the form $\delta(p_1+p_2+p_3)$. This definition of the $\varphi\varphi\varphi$-vertex also implies that external scalar fields correspond to 1 and internal scalar fields with momentum p are scalar propagators given by $D(p) = -i/p^2$.

Let us now derive the other vertices with a less explanations since they follow along the same lines. The leading-order vertex for $\varphi\varphi\varphi$ involving $c^{(\Lambda)}_{\varphi\varphi\varphi}$ is found by:

$$\begin{aligned}
i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\}|\Omega\rangle^{(\Lambda)}_{\varphi\varphi\varphi} &= c^{(\Lambda)}_{\varphi\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_x}\Big(\partial_\mu\partial^\alpha\Big(\frac{\delta}{\delta J_x}\Big)\Big)\Big(\partial_\nu\partial^\beta\Big(\frac{\delta}{\delta J_x}\Big)\Big)e^{-\frac{1}{2}J_z D_{zy} J_y} \\
&= -c^{(\Lambda)}_{\varphi\varphi\varphi}[D_{x3}(\partial_\mu\partial^\alpha D_{x2})(\partial_\nu\partial^\beta D_{x1}) + D_{x3}(\partial_\mu\partial^\alpha D_{x1})(\partial_\nu\partial^\beta D_{x2}) \\
&+ D_{x2}(\partial_\mu\partial^\alpha D_{x3})(\partial_\nu\partial^\beta D_{x1}) + D_{x1}(\partial_\mu\partial^\alpha D_{x3})(\partial_\nu\partial^\beta D_{x2}) \\
&+ D_{x2}(\partial_\mu\partial^\alpha D_{x1})(\partial_\nu\partial^\beta D_{x3}) + D_{x1}(\partial_\mu\partial^\alpha D_{x2})(\partial_\nu\partial^\beta D_{x3})].
\end{aligned} \tag{9.74}$$

Therefore the leading-order $\varphi\varphi\varphi$-vertex takes the form:

$$\begin{aligned}
V^{(\Lambda)}_{\varphi\varphi\varphi}(p_1,p_2,p_3) &= ic^{(\Lambda)}_{\varphi\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}[p_{2\mu}p_2^\alpha p_{1\nu}p_1^\beta + p_{1\mu}p_1^\alpha p_{2\nu}p_2^\beta + p_{3\mu}p_3^\alpha p_{1\nu}p_1^\beta \\
&+ p_{3\mu}p_3^\alpha p_{2\nu}p_2^\beta + p_{1\mu}p_1^\alpha p_{3\nu}p_3^\beta + p_{2\mu}p_2^\alpha p_{3\nu}p_3^\beta] \\
&= 2ic^{(\Lambda)}_{\varphi\varphi\varphi}[p_1^2p_2^2 + p_1^2p_3^2 + p_2^2p_3^2 - (p_1\cdot p_2)^2 - (p_1\cdot p_3)^2 - (p_2\cdot p_3)^2].
\end{aligned} \tag{9.75}$$

The momenta in this expression are all ingoing and satisfy $p_1 + p_2 + p_3 = 0$.

9.5.2 $\varphi\varphi\varphi\varphi$-vertex Feynman rules

The Lagrangian for the $\varphi\varphi\varphi\varphi$-vertex can be written as:

$$\mathcal{L}_{\varphi\varphi\varphi\varphi} = c^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\varphi\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi\partial^\gamma\partial_\rho\varphi + c^{(m)}_{\varphi\varphi\varphi\varphi}\partial_\mu\varphi\partial^\mu\varphi\partial_\nu\varphi\partial^\nu\varphi + \tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}\varphi^2\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu\varphi. \tag{9.76}$$

The leading-order term with $c^{(\Lambda)}_{\varphi\varphi\varphi\varphi}$ yields:

$$\begin{aligned}
&-i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|\Omega\rangle^{(\Lambda)}_{\varphi\varphi\varphi\varphi} \\
&= c^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}\frac{\delta}{\delta J_x}\partial^\alpha\partial_\mu\Big(\frac{\delta}{\delta J_x}\Big)\partial^\beta\partial_\nu\Big(\frac{\delta}{\delta J_x}\Big)\partial^\gamma\partial_\rho\Big(\frac{\delta}{\delta J_x}\Big)e^{-\frac{1}{2}J_z D_{zy} J_y} \\
&= c^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\gamma}[D_{x1}\partial^\alpha\partial_\mu D_{x2}\partial^\beta\partial_\nu D_{x3}\partial^\gamma\partial_\rho D_{x4} + \mathrm{perm}],
\end{aligned} \tag{9.77}$$

where perm denotes the sum of all the (other 23) permutations over the indices $\{1,2,3,4\}$ in the expression $D_{x1}\partial^\alpha\partial_\mu D_{x2}\partial^\beta\partial_\nu D_{x3}\partial^\gamma\partial_\rho D_{x4}$.

The corresponding Feynman rule in momentum space is therefore found to be:

$$
\begin{aligned}
V^{(\Lambda)}_{\varphi\varphi\varphi\varphi}(p_1,p_2,p_3,p_4) &= -ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\gamma}[p_{2\mu}p_2^{\alpha}p_3^{\beta}p_{3\nu}p_4^{\gamma}p_{4\rho} + \mathrm{perm}(\partial_\kappa \to p_\kappa)] \\
&= -ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\Big[6p_3^2p_1^2p_2^2 + 6p_4^2p_1^2p_2^2 + 6p_2^2p_3^2p_4^2 + 6p_1^2p_3^2p_4^2 \\
&- 6p_1^2[(p_2\cdot p_3)^2 + (p_2\cdot p_4)^2 + (p_3\cdot p_4)^2] - 6p_2^2[(p_1\cdot p_3)^2 + (p_1\cdot p_4)^2 + (p_3\cdot p_4)^2] \\
&- 6p_3^2[(p_1\cdot p_2)^2 + (p_1\cdot p_4)^2 + (p_2\cdot p_4)^2] - 6p_4^2[(p_1\cdot p_2)^2 + (p_1\cdot p_3)^2 + (p_2\cdot p_3)^2] \\
&+ 12(p_1\cdot p_2)(p_2\cdot p_3)(p_1\cdot p_3) + 12(p_1\cdot p_3)(p_3\cdot p_4)(p_1\cdot p_4) \\
&+ 12(p_2\cdot p_3)(p_3\cdot p_4)(p_2\cdot p_4) + 12(p_1\cdot p_2)(p_2\cdot p_4)(p_1\cdot p_4)\Big] \\
&= -24ic^{(\Lambda)}_{\varphi\varphi\varphi}[-m^6 + m^2[(p_1\cdot p_2)^2 + (p_1\cdot p_3)^2 + (p_1\cdot p_4)^2] \\
&+ 2(p_1\cdot p_2)(p_1\cdot p_3)(p_1\cdot p_4)].
\end{aligned}
\tag{9.78}
$$

In the last line we used that the scalar fields are on-shell, i.e. $p_i^2 = -m^2$ and we used that:

$$
\begin{aligned}
p_1\cdot p_2 &= p_3\cdot p_4 = -s/2 + m^2 \\
p_1\cdot p_3 &= p_2\cdot p_4 = t/2 + m^2 \\
p_1\cdot p_4 &= p_2\cdot p_3 = u/2 + m^2.
\end{aligned}
\tag{9.79}
$$

Again all the momenta are ingoing and satisfy $p_1 + p_2 + p_3 + p_4 = 0$.

Next, consider the sub-leading $\varphi\varphi\varphi\varphi$-vertex with $c^{(m)}_{\varphi\varphi\varphi\varphi}$. This gives:

$$
\begin{aligned}
&- i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|\Omega\rangle^{(m)}_{\varphi\varphi\varphi\varphi} \\
&= c^{(m)}_{\varphi\varphi\varphi\varphi}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}\partial_\mu\Big(\frac{\delta}{\delta J_x}\Big)\partial^\mu\Big(\frac{\delta}{\delta J_x}\Big)\partial_\nu\Big(\frac{\delta}{\delta J_x}\Big)\partial^\nu\Big(\frac{\delta}{\delta J_x}\Big)e^{-\frac{1}{2}J_zD_{zy}J_y} \\
&= 8c^{(m)}_{\varphi\varphi\varphi\varphi}[\partial_\mu D_{x4}\partial^\mu D_{x3}\partial_\nu D_{x2}\partial^\nu D_{x1} + \partial_\mu D_{x4}\partial^\mu D_{x2}\partial_\nu D_{x3}\partial^\nu D_{x1} + \partial_\mu D_{x4}\partial^\mu D_{x1}\partial_\nu D_{x3}\partial^\nu D_{x3}].
\end{aligned}
\tag{9.80}
$$

Henceforth the vertex $V^{(m)}_{\varphi\varphi\varphi\varphi}$ is obtained:

$$
V^{(m)}_{\varphi\varphi\varphi\varphi}(p_1,p_2,p_3,p_4) = 8i[(p_1\cdot p_2)^2 + (p_1\cdot p_3)^2 + (p_1\cdot p_4)^2], \tag{9.81}
$$

where we used equation (9.79). The momenta are all incoming and satisfy $p_1 + p_2 + p_3 + p_4 = 0$.

Finally, let us consider the term with $\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}$:

$$
\begin{aligned}
&- i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|\Omega\rangle^{(\tilde{m})}_{\varphi\varphi\varphi\varphi} = \tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_4}\Big(\frac{\delta}{\delta J_x}\Big)^2 \\
&\partial^\alpha\partial_\mu\Big(\frac{\delta}{\delta J_x}\Big)\partial^\beta\partial_\nu\Big(\frac{\delta}{\delta J_x}\Big)e^{-\frac{1}{2}J_zD_{zy}J_y} \\
&= 2\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}[D_{x1}D_{x2}((\partial^\alpha\partial_\mu D_{x3})(\partial^\beta\partial_\nu D_{x4}) + (\partial^\alpha\partial_\mu D_{x4})(\partial^\beta\partial_\nu D_{x3})) + D_{x1}D_{x3}((\partial^\alpha\partial_\mu D_{x2})(\partial^\beta\partial_\nu D_{x4}) \\
&+ (\partial^\alpha\partial_\mu D_{x4})(\partial^\beta\partial_\nu D_{x2})) + D_{x1}D_{x4}((\partial^\alpha\partial_\mu D_{x3})(\partial^\beta\partial_\nu D_{x2}) + (\partial^\alpha\partial_\mu D_{x2})(\partial^\beta\partial_\nu D_{x3})) \\
&+ D_{x2}D_{x3}((\partial^\alpha\partial_\mu D_{x1})(\partial^\beta\partial_\nu D_{x4}) + (\partial^\alpha\partial_\mu D_{x4})(\partial^\beta\partial_\nu D_{x1})) + D_{x2}D_{x4}((\partial^\alpha\partial_\mu D_{x3})(\partial^\beta\partial_\nu D_{x1}) \\
&+ (\partial^\alpha\partial_\mu D_{x1})(\partial^\beta\partial_\nu D_{x3})) + D_{x3}D_{x4}((\partial^\alpha\partial_\mu D_{x1})(\partial^\beta\partial_\nu D_{x2}) + (\partial^\alpha\partial_\mu D_{x2})(\partial^\beta\partial_\nu D_{x1}))].
\end{aligned}
\tag{9.82}
$$

Therefore the Feynman rule corresponding to this vertex is given by:

$$\begin{aligned}\tilde{V}^{(m)}_{\varphi\varphi\varphi\varphi}(p_1,p_2,p_3,p_4) &= 4i\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}[p_1^2p_2^2+p_1^2p_3^2+p_2^2p_3^2+p_1^2p_4^2+p_2^2p_4^2+p_3^2p_4^2\\ &\quad-(p_1\cdot p_2)^2-(p_1\cdot p_3)^2-(p_1\cdot p_4)^2-(p_2\cdot p_3)^2-(p_2\cdot p_4)^2-(p_3\cdot p_4)^2]\\ &= 8i\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}[3m^4-(p_1\cdot p_2)^2-(p_1\cdot p_3)^2-(p_1\cdot p_4)^2],\end{aligned} \tag{9.83}$$

where we used equation (9.79) and $p_i^2=-m_i^2$ since the scalar fields are on-shell. The momenta in this expression are all incoming and satisfy $p_1+p_2+p_3+p_4=0$.

9.5.3 $h\varphi\varphi$-vertex Feynman rules

The Lagrangian for the $h\varphi\varphi$-vertex can be written as:

$$\mathcal{L}_{h\varphi\varphi}=c^{(\Lambda)}_{h\varphi\varphi}\varphi\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\eta^{\kappa\sigma}\partial^\alpha\partial_\mu\varphi\partial^\beta\partial_\nu h_{\sigma\rho}+c^{(m)}_{h\varphi\varphi}h_{\mu\nu}(\partial^\mu\varphi\partial^\nu\varphi-\frac{1}{2}\eta^{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi)+\tilde{c}^{(m)}_{\varphi\varphi\varphi}\varphi^2\delta^{\mu\nu}_{\alpha\beta}\eta^{\sigma\beta}\partial^\alpha\partial_\mu h_{\sigma\nu}. \tag{9.84}$$

Like in the case of pure scalar field vertices, the correlation function is found as follows (using $J^{\sigma\rho}=J^{\rho\sigma}$ and $D_{\gamma\lambda\iota\delta}=D_{\iota\delta\gamma\lambda}$):

$$\begin{aligned}&i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)h_{\gamma\epsilon}(x_3)\}|\Omega\rangle^{(m)}_{h\varphi\varphi}\\ &=c^{(\Lambda)}_{h\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\eta^{\kappa\sigma}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3}\frac{\delta}{\delta J_x}\partial^\alpha\partial_\mu\Big(\frac{\delta}{\delta J_x}\Big)\partial^\beta\partial_\nu\Big(\frac{\delta}{\delta J_x^{\sigma\rho}}\Big)e^{-\frac{1}{2}J_zD_{zy}J_y}e^{-\frac{1}{2}J_z^{\gamma\lambda}D^{zy}_{\gamma\lambda\iota\delta}J_y^{\iota\delta}}\\ &=-c^{(\Lambda)}_{h\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\eta^{\kappa\sigma}(D_{x1}\partial^\alpha\partial_\mu D_{x2}+D_{x2}\partial^\alpha\partial_\mu D_{x1})\partial^\beta\partial_\nu D^{x3}_{\sigma\rho\gamma\epsilon}.\end{aligned} \tag{9.85}$$

Therefore the Feynman rule for this leading-order vertex is given by:

$$V^{(\Lambda)\rho\sigma}_{h\varphi\varphi}(p_1,p_2,p_3)=ic^{(\Lambda)}_{h\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\eta^{\kappa\sigma}(p_1^\alpha p_{1\mu}+p_2^\alpha p_{2\mu})p_3^\beta p_{3\nu}, \tag{9.86}$$

where p_3 is the momentum of the graviton and p_1,p_2 of the scalar fields. Again the momenta are defined to be all incoming and satisfy the momentum conservation at the vertex $p_1+p_2+p_3=0$.

Next, consider the sub-leading vertex with $c^{(m)}_{h\varphi\varphi}$. We find that:

$$\begin{aligned}&i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)h_{\gamma\epsilon}(x_3)\}|\Omega\rangle^{(m)}_{h\varphi\varphi}\\ &=c^{(m)}_{h\varphi\varphi}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3^{\gamma\epsilon}}\frac{\delta}{\delta J_x^{\mu\nu}}\Big[\partial^\mu\Big(\frac{\delta}{\delta J_x}\Big)\partial^\nu\Big(\frac{\delta}{\delta J_x}\Big)-\frac{1}{2}\eta^{\mu\nu}\partial_\alpha\Big(\frac{\delta}{\delta J_x}\Big)\partial^\alpha\Big(\frac{\delta}{\delta J_x}\Big)\Big]e^{-\frac{1}{2}J_zD_{zy}J_y}e^{-\frac{1}{2}J_z^{\gamma\lambda}D^{zy}_{\gamma\lambda\iota\delta}J_y^{\iota\delta}}\\ &=-c^{(m)}_{h\varphi\varphi}D^{x3}_{\mu\nu\gamma\epsilon}[\partial^\mu D_{x1}\partial^\nu D_{x2}+\partial^\mu D_{x2}\partial^\nu D_{x1}-\eta^{\mu\nu}\partial_\alpha D_{x1}\partial^\alpha D_{x2}].\end{aligned} \tag{9.87}$$

Therefore the Feynman rule for this vertex in momentum space reads:

$$V^{(m)\mu\nu}_{h\varphi\varphi}(p_1,p_2,p_3)=-ic^{(m)}_{h\varphi\varphi}[p_1^\mu p_2^\nu+p_1^\nu p_2^\mu-\eta^{\mu\nu}(p_1\cdot p_2)]. \tag{9.88}$$

Here p_1, p_2 are incoming the momenta of the scalar fields. Again we assume that $p_1 + p_2 + p_3 = 0$ with p_3 the momentum of the graviton.

Finally, the Feynman rule for the part containing $\tilde{c}^{(m)}_{h\varphi\varphi}$ can be found by computing:

$$
\begin{aligned}
& i\langle\Omega|T\{\varphi(x_1)\varphi(x_2)h_{\gamma\epsilon}(x_3)\}|\Omega\rangle^{(\tilde{m})}_{h\varphi\varphi} \\
& = \tilde{c}^{(m)}_{h\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}\eta^{\sigma\beta}\frac{\delta}{\delta J_1}\frac{\delta}{\delta J_2}\frac{\delta}{\delta J_3^{\gamma\epsilon}}\left(\frac{\delta}{\delta J_x}\right)^2\partial^\alpha\partial_\mu\left(\frac{\delta}{\delta J_x^{\sigma\nu}}\right)e^{-\frac{1}{2}J_z D_{zy} J_y}e^{-\frac{1}{2}J_z^{\gamma\lambda}D^{zy}_{\gamma\lambda\iota\delta}J_y^{\iota\delta}} \\
& = -2\tilde{c}^{(m)}_{h\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}\eta^{\sigma\beta}D_{x1}D_{x2}\partial^\alpha\partial_\mu D^{x3}_{\sigma\nu\gamma\epsilon}.
\end{aligned}
\tag{9.89}
$$

Therefore in momentum space the Feynman rule is given by:

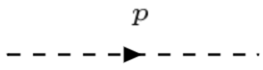
$$
\begin{aligned}
\tilde{V}^{(m)\sigma\nu}_{h\varphi\varphi}(p_1, p_2, p_3) &= -2i\tilde{c}^{(m)}_{h\varphi\varphi}\delta^{\mu\nu}_{\alpha\beta}\eta^{\sigma\beta}p_3^\alpha p_{3\mu} \\
&= 2i\tilde{c}^{(m)}_{h\varphi\varphi}(p_3^\sigma p_3^\nu - \eta^{\sigma\nu}p_3^2).
\end{aligned}
\tag{9.90}
$$

In this expression p_3 is the graviton momentum and the momenta are all ingoing and satisfy $p_1 + p_2 + p_3 = 0$.

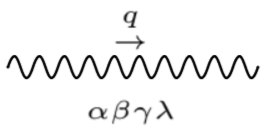
9.5.4 Feynman diagrams for Horndeski gravity

External scalar fields correspond to 1 in momentum space. Internal scalar fields or gravitons with momentum q correspond to the propagators $D(q)$ and $D_{\alpha\beta\gamma\lambda}(q)$ respectively.

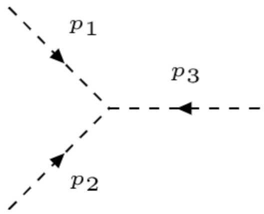
Feynman rules for vertices involving $hh\varphi$, hhh and $hh\varphi\varphi$ can be derived along the above lines. This is useful when considering the scattering processes $h\varphi \to h\varphi$ and $hh \to hh$. As we commented on in section 9.4.4, an external incoming graviton of momentum p is representated by a polarization tensor $\epsilon^{\mu\nu}(p)$ which is symmetric and obeys $\eta_{\mu\nu}\epsilon^{\mu\nu}(p) = p_\mu\epsilon^{\mu\nu}(p) = 0$. Similarly, external outgoing graviton with momentum p is represented by $(\epsilon^{\mu\nu}(p))^\star$.



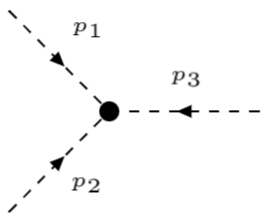
$$= D(p) = -\frac{i}{p^2}$$



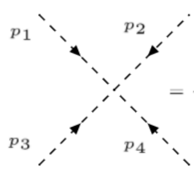
$$= D_{\alpha\beta\gamma\lambda}(q) = -\frac{i}{\bar{G}_4 q^2}(\eta_{\alpha\gamma}\eta_{\beta\lambda} + \eta_{\alpha\lambda}\eta_{\beta\gamma} - \eta_{\alpha\beta}\eta_{\gamma\lambda})$$



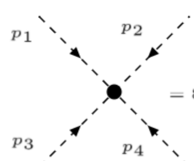
$$= 2ic^{(\Lambda)}_{\varphi\varphi\varphi}[p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2 - (p_1 \cdot p_2)^2 - (p_1 - p_3)^2 - (p_2 \cdot p_3)^2]$$



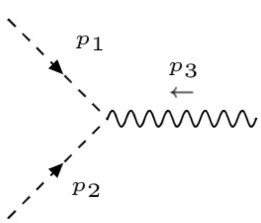
$$= -2ic^{(m)}_{\varphi\varphi\varphi}[p_1^2 + p_2^2 + p_3^2]$$



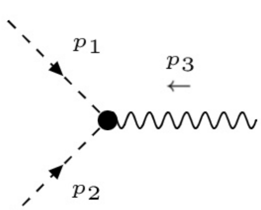
$$= -24ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}[-m^6 + m^2((p_1 \cdot p_2)^2 + (p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2) + 2(p_1 \cdot p_2)(p_1 \cdot p_3)(p_1 \cdot p_4)]$$



$$= 8ic^{(m)}_{\varphi\varphi\varphi\varphi}[(p_1 \cdot p_2)^2 + (p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2] + 8i\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}[3m^4 - (p_1 \cdot p_2)^2 - (p_1 \cdot p_3)^2 - (p_1 \cdot p_4)^2]$$



$$= ic^{(\Lambda)}_{h\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\eta^{\kappa\sigma}(p_1^{\alpha} p_{1\mu} + p_2^{\alpha} p_{2\mu})p_3^{\beta} p_{3\nu}$$



$$= -ic^{(m)}_{h\varphi\varphi}[p_1^{\mu} p_2^{\nu} + p_1^{\nu} p_2^{\mu} - \eta^{\mu\nu}(p_1 \cdot p_2)] + 2i\tilde{c}^{(m)}_{h\varphi\varphi}(p_3^{\mu} p_3^{\nu} - \eta^{\mu\nu} p_3^2)$$

Figure 12: Feynman diagrams relevant for $\varphi\varphi \to \varphi\varphi$ tree-level scattering in Horndeski gravity assuming a flat background. Dots indicate sub-leading vertices ($\mathcal{O}(1/M_{\rm pl})$). These diagrams have been produced with the online tool https://feynman.aivazis.com/.

9.6 Scattering amplitude of Horndeski theory on a Minkowski background

The scattering amplitude for the diagrams in Figure 5 have been computed already. In this part of the Appendix we compute the scattering amplitudes for the other type of diagrams and derive the expression for the tree-level positivity bounds in Horndeski gravity.

The leading-order four-point vertex in Figure 13 has the following scattering amplitude:

$$
\begin{aligned}
V^{(\Lambda)}_{\varphi\varphi\varphi\varphi}(p_1,p_2,-p_3,-p_4) &= -ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}(-24m^6+6m^2((2p_1\cdot p_2)^2+(2p_1\cdot p_3)^2+(2p_1\cdot p_4)^2) \\
&+48(p_1\cdot p_2)(p_1\cdot p_3)(p_1\cdot p_4)) \\
&= -ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\Big(-24m^6+6m^2((s-2m^2)^2+(t-2m^2)^2+(u-2m^2)^2) \\
&+48\Big(-\frac{s}{2}+m^2\Big)\Big(\frac{t}{2}-m^2\Big)\Big(\frac{u}{2}-m^2\Big)\Big) \\
&= -ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\Big[-24m^6+6m^2(12m^4-4m^2(s+t+u)+s^2+u^2+t^2) \\
&+48\Big(-\frac{stu}{8}+\frac{1}{4}m^2(st+su+ts)-\frac{m^4}{2}(s+t+u)+m^6\Big)\Big] \\
&= -ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}\Big[-96m^6+6m^2(u^2+t^2+s^2)-6stu \\
&+12m^2\frac{1}{2}\Big(\Big(u+t+s\Big)^2-s^2-t^2-u^2\Big)\Big] \\
&= 6ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}stu \\
&\approx -6ic^{(\Lambda)}_{\varphi\varphi\varphi\varphi}s^2t \\
&= -\frac{i(3\bar{G}_{4,XX}-2\bar{G}_{5,\phi X})}{2\Lambda_3^6}s^2t.
\end{aligned}
\tag{9.91}
$$

We used that in the forward and high-s limit we have that $stu \approx -s^2t$ because $stu = -16(p^2+m^2)p^4(1-\cos^2(\theta)) \approx -16p^6\theta^2 \approx -s^2t$ for $\theta \ll 1$ and $p^2 \gg m^2$.

The scattering amplitude of the sub-leading four-point vertex (Figure 14) is found by:

$$
\begin{aligned}
&V^{(m)}_{\varphi\varphi\varphi\varphi}(p_1,p_2,-p_3,-p_4)+\tilde{V}^{(m)}_{\varphi\varphi\varphi\varphi}(p_1,p_2,-p_3,-p_4) = 2ic_{\varphi\varphi\varphi\varphi}[(2p_1\cdot p_2)^2+(2p_1\cdot p_3)^2+(2p_1\cdot p_4)^2] \\
&+i\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}[24m^4-2(2p_1\cdot p_2)^2-2(2p_1\cdot p_3)^2-2(2p_1\cdot p_4)^2] \\
&= 2ic^{(m)}_{\varphi\varphi\varphi\varphi}[(s-2m^2)^2+(t-2m^2)^2+(u-2m^2)^2] \\
&+i\tilde{c}_{\varphi\varphi\varphi\varphi}[24m^4-2(s-2m^2)^2-2(t-2m^2)^2-2(u-2m^2)^2] \\
&= 2ic^{(m)}_{\varphi\varphi\varphi\varphi}[s^2+t^2+u^2+12m^4-4m^2(s+t+u)]+i\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}[-2(s^2+t^2+u^2)+8m^2(s+t+u)] \\
&= 2ic^{(m)}_{\varphi\varphi\varphi\varphi}[s^2+t^2+u^2-16m^4]+i\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}[-2(s^2+t^2+u^2)+32m^4] \\
&\approx 4is^2(c^{(m)}_{\varphi\varphi\varphi\varphi}-\tilde{c}^{(m)}_{\varphi\varphi\varphi\varphi}) \\
&= \frac{i(\bar{G}_{2,XX}-4\bar{G}_{4,X\phi\phi})}{2\Lambda_2^4}s^2.
\end{aligned}
\tag{9.92}
$$

Next the scattering amplitude for a diagram consisting of two leading-order three-point vertices (Figure 15) can be computed. Let us focus first on the s-channel diagram. Let $q = p_1+p_2$. The scattering amplitude for the s-channel is:

$$
\begin{aligned}
V^{(\Lambda)}_{\varphi\varphi\varphi}(p_1,p_2,-q)D(q)V^{(\Lambda)}_{\varphi\varphi\varphi}(-p_3,-p_4,q) &= -\frac{4i}{s}c^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[m^4+2m^2s-(p_1\cdot p_2)^2-2(q\cdot p_1)^2\Big]^2 \\
&= -\frac{4i}{s}c^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[m^4+2m^2s-(-s/2+m^2)^2-2(-s/2)^2\Big] \\
&= -\frac{4i}{s}c^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[3m^2s-\frac{3}{4}s^2\Big]^2 \\
&= -\frac{9i}{4s}c^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[4m^2s-s^2\Big]^2 \\
&= -\frac{9i}{4}c^{(\Lambda)2}_{\varphi\varphi\varphi}s(u+t)^2.
\end{aligned}
\tag{9.93}
$$

We used that $(q\cdot p_1)^2=(-m^2+p_1\cdot p_2)^2=(-m^2+(-s/2+m^2))^2=s^2/4$. It is also trivial to see that the t-channel and u-channel are found by $s\leftrightarrow t$ and $s\leftrightarrow u$ (since the vertices and propagator take the same form but with s replaced by t or u when writing $q=p_1-p_3$ or $q=p_1-p_4$). Therefore the total scattering amplitude is easily found by:

$$
\begin{aligned}
&-\frac{9i}{4}c^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[s(u+t)^2+t(s+u)^2+u(s+t)^2\Big] \\
&=-\frac{9i}{4}c^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[4(p^2+m^2)(-4p^2)-2p^2(1-\cos(\theta))(4p^2+4m^2-2p^2(1+\cos(\theta)))^2 \\
&-2p^2(1+\cos(\theta))(4p^2+4m^2-2p^2(1-\cos(\theta)))^2\Big] \\
&=ic^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[144m^4p^2+144m^2p^4\cos^2(\theta)+108p^6\cos^2(\theta)-108p^6\Big] \\
&\approx ic^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[144m^4p^2+144m^2p^4-144m^2p^4\theta^2-108p^6\theta^2\Big] \\
&\approx ic^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[144m^2p^4-108p^6\theta^2\Big] \\
&\approx ic^{(\Lambda)2}_{\varphi\varphi\varphi}\Big[9m^2s^2+\frac{27}{4}s^2t\Big] \\
&=\frac{i(\bar{G}_{3,X}+3\bar{G}_{4,X\phi})^2}{\Lambda_2^4}\frac{m^2}{H_0^2}s^2+\frac{3i(\bar{G}_{3,X}+3\bar{G}_{4,X\phi})^2}{4\Lambda_3^6}s^2t.
\end{aligned}
\tag{9.94}
$$

Let us now consider the diagrams involving a graviton as the propagator. Focus on the diagram consisting of two leading-order vertices (Figure 16) involving the graviton propagator. First note that the vertex $V^{(\Lambda)\rho\sigma}_{h\varphi\varphi}$ can be written as:

$$
\begin{aligned}
V^{(\Lambda)\rho\sigma}_{h\varphi\varphi}(p_1,p_2,q) &= ic^{(\Lambda)}_{h\varphi\varphi}\delta^{\mu\nu\rho}_{\alpha\beta\kappa}\eta^{\kappa\sigma}(p_1^\alpha p_{1\mu}+p_2^\alpha p_{2\mu})q^\beta q_\nu \\
&= ic^{(\Lambda)}_{h\varphi\varphi}[\eta^{\rho\sigma}(p_1^2+p_2^2)q^2-(p_1^2+p_2^2)q^\rho q^\sigma-\eta^{\rho\sigma}((p_1\cdot q)^2+(p_2\cdot q)^2) \\
&+((p_1\cdot q)p_1^\rho+(p_2\cdot q)p_2^\rho)q^\sigma+((p_1\cdot q)p_1^\sigma+(p_2\cdot q)p_2^\sigma)q^\rho-(p_1^\rho p_1^\sigma+p_2^\rho p_2^\sigma)q^2].
\end{aligned}
\tag{9.95}
$$

In the case of the s-channel (in Figure 16) we find the following scattering amplitude (using that p_1,p_2,p_3,p_4 are on-shell, $q=p_1+p_2$, $q^2=-s$ and $p_i\cdot q=-s/2$)

$$
\begin{aligned}
&V^{(\Lambda)\rho\sigma}_{h\varphi\varphi}(p_1,p_2,-q)D_{\rho\sigma\alpha\beta}(q)V^{(\Lambda)\alpha\beta}_{h\varphi\varphi}(-p_3,-p_4,q)\\
&=-\frac{i}{\bar{G}_4 s}c^{(\Lambda)2}_{h\varphi\varphi}[2m^2\eta^{\rho\sigma}s+2m^2q^\rho q^\sigma-2\eta^{\rho\sigma}(p_1\cdot q)^2+2(p_1\cdot q)q^\rho q^\sigma+s(p_1^\rho p_1^\sigma+p_2^\rho p_2^\sigma)](\eta_{\rho\alpha}\eta_{\sigma\beta}+\eta_{\rho\beta}\eta_{\sigma\alpha}\\
&-\eta_{\rho\sigma}\eta_{\alpha\beta})(2m^2\eta^{\alpha\beta}s+2m^2q^\alpha q^\beta-2\eta^{\alpha\beta}(p_1\cdot q)^2+2(p_1\cdot q)q^\alpha q^\beta+s(p_3^\alpha p_3^\beta+p_4^\alpha p_4^\beta))\\
&=-\frac{2i}{\bar{G}_4 s}c^{(\Lambda)2}_{h\varphi\varphi}(2m^2\eta_{\alpha\beta}s+2m^2q_\alpha q_\beta-2\eta_{\alpha\beta}(p_1\cdot q)^2+2(p_1\cdot q)q_\alpha q_\beta+(p_{1\alpha}p_{1\beta}+p_{2\alpha}p_{2\beta})s)\\
&(2m^2\eta^{\alpha\beta}s+2m^2q^\alpha q^\beta-2\eta^{\alpha\beta}(p_1\cdot q)^2+2(p_1\cdot q)q^\alpha q^\beta+(p_3^\alpha p_3^\beta+p_4^\alpha p_4^\beta)s)+\frac{16i}{\bar{G}_4 s}c^{(\Lambda)2}_{h\varphi\varphi}(m^2s-(p_1\cdot q)^2)^2\\
&=-\frac{2i}{\bar{G}_4 s}c^{(\Lambda)2}_{h\varphi\varphi}[4m^4s^2-2m^2s^3+2s^2(p_1\cdot p_3)^2+2s^2(p_1\cdot p_4)^2]+\frac{16i}{\bar{G}_4 s}c^{(\Lambda)2}_{h\varphi\varphi}\Big[m^4s^2-\frac{1}{2}m^2s^3+\frac{1}{16}s^4\Big]\\
&=\frac{i}{\bar{G}_4}c^{(\Lambda)2}_{h\varphi\varphi}[s^3+8m^4s-4m^2s^2-4s(p_1\cdot p_3)^2-4s(p_1\cdot p_4)^2]\\
&=\frac{i}{\bar{G}_4}c^{(\Lambda)2}_{h\varphi\varphi}[s^3+8m^4s-4m^2s^2-s(t-2m^2)^2-s(u-2m^2)^2]\\
&=\frac{i}{\bar{G}_4}c^{(\Lambda)2}_{h\varphi\varphi}[s^3-4m^2s^2-st^2+4m^2st-su^2+4m^2su]\\
&=\frac{2i}{\bar{G}_4}c^{(\Lambda)2}_{h\varphi\varphi}stu.
\end{aligned}
\tag{9.96}
$$

In the last line we used that $4m^2=u+t+s$. In similar fashion we can study the t-channel and u-channel. We will show that the amplitude of the t-channel has the same form but with $s\leftrightarrow t$. The case for the u-channel goes in similar fashion. In the case of the t-channel, define the momentum of the graviton $q=p_1-p_3$. In this case we have that $p_1\cdot q=-p_3\cdot q=-t/2$. The vertex $V^{(\Lambda)\rho\sigma}_{h\varphi\varphi}(p_1,-p_3,-q)$ takes the form (note that we need to flip the sign of p_3 since it is outgoing) [35]

$$
V^{(\Lambda)\rho\sigma}_{h\varphi\varphi}(p_1,-p_3,-q)=ic^{(\Lambda)}_{h\varphi\varphi}[2m^2\eta^{\rho\sigma}t+2m^2q^\rho q^\sigma-2\eta^{\rho\sigma}(p_1\cdot q)^2+2(p_1\cdot q)q^\rho q^\sigma+t(p_1^\rho p_1^\sigma+p_2^\rho p_2^\sigma)].
\tag{9.97}
$$

This illustrates that the total amplitude of the diagrams in Figure 16 is given by:

$$
\begin{aligned}
&\frac{6i}{\bar{G}_4}c^{(\Lambda)2}_{h\varphi\varphi}uts\\
&\approx-\frac{6i}{\bar{G}_4}c^{(\Lambda)2}_{h\varphi\varphi}s^2t\\
&=-\frac{3i}{2\bar{G}_4\Lambda_3^6}(\bar{G}_{4,X}-\bar{G}_{5,\phi})^2s^2t.
\end{aligned}
\tag{9.98}
$$

Finally, the diagrams with the graviton propagator and two vertices of different order are considered (Figure **??**). Focus first on the s-channel of such a diagram with $q=p_1+p_2$. The amplitude of such a diagram is $2V^{(\Lambda)\mu\nu}_{h\varphi\varphi}(p_1,p_2,-q)D_{\mu\nu\sigma\alpha}(q)(V^{(m)\sigma\alpha}_{h\varphi\varphi}(-p_3,-p_4,q)+\tilde{V}^{(m)\sigma\alpha}_{h\varphi\varphi}(-p_3,-p_4,q))$ where the 2 incorporates the symmetry factor coming from exchange of vertices of different order. The product of the first two terms has already been computed, so we can use that result. The amplitude is therefore found by:

[35] Notice that the minus sign from $p_1\cdot q=-p_3\cdot q$ and the minus sign from p_3 being outgoing in the vertex precisely cancel out. Upon defining $q=p_1-p_3$, we recover the vertex like for the s-channel with $s\leftrightarrow t$.

$$
\begin{aligned}
&2V^{(\Lambda)\mu\nu}_{h\varphi\varphi}(p_1,p_2,-q)D_{\mu\nu\sigma\alpha}(q)(V^{(m)\sigma\alpha}_{h\varphi\varphi}(-p_3,-p_4,q)+\tilde{V}^{(m)\sigma\alpha}_{h\varphi\varphi}(-p_3,-p_4,q))\\
&=-\frac{8i}{\bar{G}_4 s}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}(2m^2q_\alpha q_\sigma - s q_\alpha q_\sigma + s(p_{1\sigma}p_{1\alpha}+p_{2\sigma}p_{2\alpha}))(\eta^{\sigma\alpha}s+q^\sigma q^\alpha)\\
&+\frac{4i}{\bar{G}_4 s}c^{(\Lambda)}_{h\varphi\varphi}c^{(m)}_{h\varphi\varphi}(2m^2q_\alpha q_\sigma - s q_\alpha q_\sigma + s(p_{1\sigma}p_{1\alpha}+p_{2\sigma}p_{2\alpha}))(p_3^\sigma p_4^\alpha + p_3^\alpha p_4^\sigma - \eta^{\sigma\alpha}(p_3\cdot p_4))\\
&=-\frac{8i}{\bar{G}_4 s}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}\left(\frac{1}{2}s^3-2m^2s^2\right)+\frac{4i}{\bar{G}_4 s}c^{(\Lambda)}_{h\varphi\varphi}c^{(m)}_{h\varphi\varphi}uts\\
&=-\frac{8i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}\left(\frac{1}{2}s^2-2m^2s\right)+\frac{4i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}c^{(m)}_{h\varphi\varphi}ut.
\end{aligned}
\tag{9.99}
$$

In this computation we used that $p_i\cdot q=-s/2$, $p_1\cdot p_2=-s/2+m^2$, $p_1\cdot p_3=t/2-m^2$ and $p_1\cdot p_4=u/2-m^2$.

The t-channel has the following amplitude (let $q=p_1-p_3$) [36]

$$
\begin{aligned}
&2V^{(\Lambda)\mu\nu}_{h\varphi\varphi}(p_1,-p_3,-q)D_{\mu\nu\sigma\alpha}(q)(V^{(m)\sigma\alpha}_{h\varphi\varphi}(p_2,-p_4,q)+\tilde{V}^{(m)\sigma\alpha}_{h\varphi\varphi}(p_2,-p_4,q))\\
&=-\frac{8i}{\bar{G}_4 t}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}(2m^2q_\alpha q_\sigma - t q_\sigma q_\alpha + t(p_{1\sigma}p_{1\alpha}+p_{3\sigma}p_{3\alpha}))(\eta^{\sigma\alpha}t+q^\sigma q^\alpha)\\
&-\frac{4i}{\bar{G}_4 t}c^{(\Lambda)}_{h\varphi\varphi}c^{(m)}_{h\varphi\varphi}(2m^2q_\alpha q_\sigma - t q_\sigma q_\alpha + t(p_{1\sigma}p_{1\alpha}+p_{3\sigma}p_{3\alpha}))(p_2^\sigma p_4^\alpha + p_2^\alpha p_4^\sigma - \eta^{\sigma\alpha}(p_2\cdot p_4))\\
&=-\frac{8i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}\left(\frac{1}{2}t^2-2m^2t\right)+c^{(\Lambda)}_{h\varphi\varphi}c^{(m)}_{h\varphi\varphi}\frac{4i}{\bar{G}_4}us.
\end{aligned}
\tag{9.100}
$$

The u-channel is completely analogous and gives $t\to u$ for first term in the final expression and $us\to st$. Therefore the total scattering amplitude for diagrams of this type (Figure 17) is given by:

$$
\begin{aligned}
&-\frac{8i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}\Big[\frac{1}{2}(s^2+t^2+u^2)-2m^2(t+u+s)\Big]+\frac{4i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}[ut+us+st]\\
&=-\frac{8i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}\Big[\frac{1}{2}(s^2+t^2+u^2)-8m^4\Big]+\frac{4i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}[ut+us+st]\\
&\approx-\frac{8i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}\tilde{c}^{(m)}_{h\varphi\varphi}s^2-\frac{4i}{\bar{G}_4}c^{(\Lambda)}_{h\varphi\varphi}c^{(m)}_{h\varphi\varphi}s^2\\
&=\frac{i}{\bar{G}_4\Lambda_2^4}(\bar{G}_{5,\phi}-\bar{G}_{4,X})(2\bar{G}_{4,\phi\phi}-\bar{G}_{2,X}-2\bar{G}_{3,\phi})s^2\\
&=\frac{i}{\bar{G}_4\Lambda_2^4}(\bar{G}_{5,\phi}-\bar{G}_{4,X})(2\bar{G}_{4,\phi\phi}-1)s^2.
\end{aligned}
\tag{9.101}
$$

Combining the results indeed yields the coefficients c_{ss}, c_{sst} like in equation (7.55). This completes the proof.

[36] Notice that p_3 and p_4 are outgoing so need to include a minus sign in the vertex.

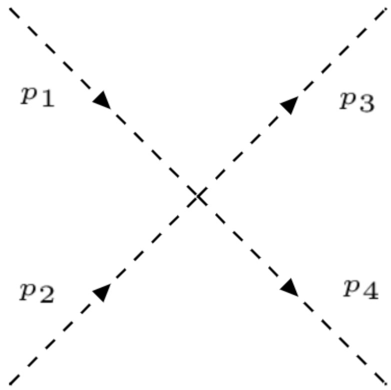


Figure 13: Leading-order four-point scalar vertex.

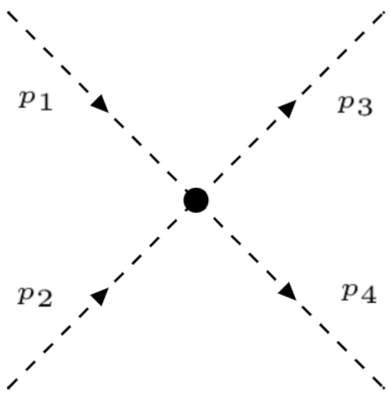


Figure 14: Sub-leading order four-point scalarvertex.

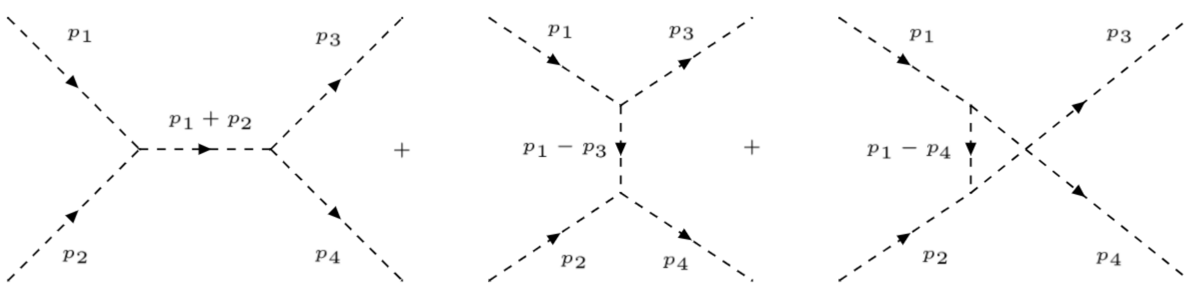


Figure 15: Diagram containing two leading-order three-point scalar vertices.

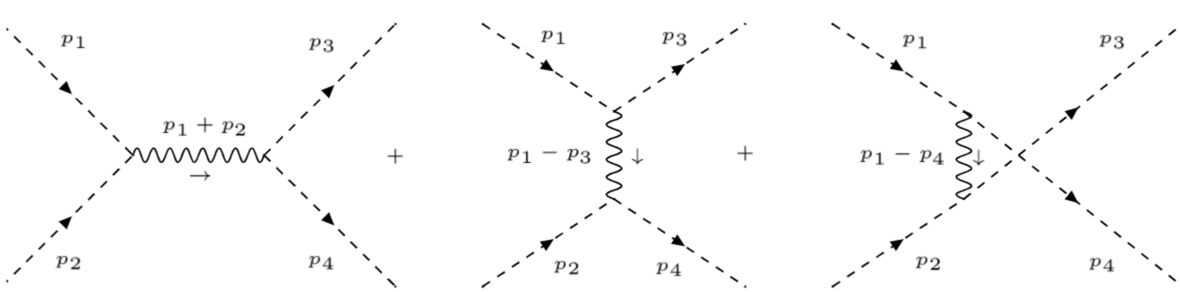


Figure 16: Diagram containing two leading-order three-point vertices with graviton propagator.

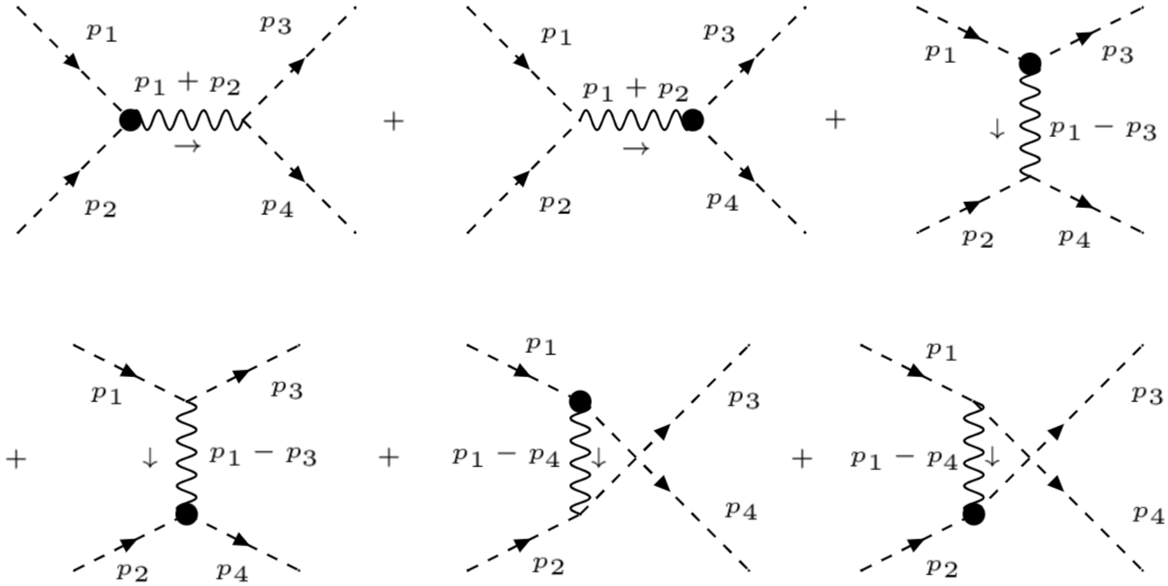


Figure 17: Diagram containing a leading-order three-point vertex and sub-leading order three-point vertex with graviton propagator.

9.7 Example of positivity bounds for shift-symmetric Horndeski theory

In this section we discuss the example of shift-symmetric Horndeski of positivity bounds on cosmological backgrounds in terms of the α-parameters in the ADM formalism for scalar-tensor theories of DE. The α-parameters in these examples follow the convention of [29]. Note that $M_\star$ in that reference is dynamical rather than the fixed $M_\star$ we introduced, to avoid confusion we call it therefore $\tilde{M}$. Notice that the convention for X is slightly different: $X = -\frac{1}{2}(\nabla_\mu\phi)^2$ and the derivatives are again defined differently. We will drop bars on quantities evaluated at the cosmological background.

This particular example has been stated in [17], we will provide the proof of it. Notice that we find the following expressions [29]:

$$\begin{aligned}
\frac{\tilde{M}^2}{M_{\rm pl}^2} &=: M^2 = 2(G_4 - 2XG_{4,X}) \\
M^2\alpha_M &= -\frac{2\dot{X}}{H}(G_{4,X} + 2XG_{4,XX}) \\
M^2\alpha_B &= 8X(G_{4,X} + 2XG_{4,XX}) \\
M^2\alpha_T &= 4XG_{4,X}.
\end{aligned} \tag{9.102}$$

α_K has not been considered, since it turns out it is unconstrained [17]. Notice that the expressions can be combined as [17]:

$$\begin{aligned}
M^2\alpha_B &= 2M^2\alpha_T + 16X^2G_{4,XX} \\
M^2\alpha_M &= -\frac{M^2}{4HX}\dot{X}\alpha_B.
\end{aligned} \tag{9.103}$$

Using the positivity bound $G_{4,XX} \leq -G_{4,X}^2/G_4$ we find that:

$$\begin{aligned}
\alpha_B &= 2\alpha_T + \frac{8X^2}{G_4}(1+\alpha_T)G_{4,XX} \\
&\leq 2\alpha_T - \frac{8X^2}{G_4^2}G_{4,X}^2(1+\alpha_T) \\
&= 2\alpha_T - \frac{M^4\alpha_T^2(1+\alpha_T)}{2G_4^2} \\
&= 2\alpha_T - \frac{2\alpha_T^2}{1+\alpha_T} = \frac{2\alpha_T}{1+\alpha_T},
\end{aligned} \tag{9.104}$$

which confirms the bound found in [17].

9.8 EFTCAMB code notation identities

In this subsection, let $\Omega[\phi]$, $\Lambda[\phi]$, $\Gamma[\phi]$ and $\gamma_i[\phi]$ denote EFT functions in the convention of [26]. The conversion to the EFTCAMB convention [27] can be done using Table 1. The EFT functions in the EFTCAMB convention will be indicated with Ω, Λ, c and γ_i. Recall that $m_0 = M_\star$. Table 1 implies that the relation between these quantities is:

$$\begin{aligned}
\Omega[\phi] &= 1 + \Omega, \\
\Lambda[\phi] &= -\Lambda/m_0^2, \\
\Gamma[\phi] &= 2c/m_0^2, \\
\gamma_1[\phi] &= \gamma_1, \\
\gamma_2[\phi] &= \gamma_2, \\
\gamma_3[\phi] &= \frac{1}{2}\gamma_3.
\end{aligned} \tag{9.105}$$

The EFT functions in the convention of [26] were all derived with respect to ϕ. In code notation however we need to have derivatives of $\mathcal{H}$, c, Λ with respect to the conformal time τ and derivatives of Ω and γ_i with respect to the scale factor a. It is simple to see using the chain rule that:

$$\frac{d}{d\phi} = \frac{1}{m_0^2}\frac{d}{dt} = \frac{aH}{m_0^2}\frac{d}{da} = \frac{\mathcal{H}}{m_0^2}\frac{d}{da}. \tag{9.106}$$

Using this we easily find for an arbitrary EFT function f which depends on ϕ:

$$\begin{aligned}
\frac{df}{d\phi} &= \frac{\mathcal{H}}{m_0^2}\frac{df}{da}, \\
\frac{d^2f}{d\phi^2} &= \frac{1}{m_0^4}\Big(\frac{\dot{\mathcal{H}}}{a}\frac{df}{da} + \mathcal{H}^2\frac{d^2f}{da^2}\Big), \\
\frac{d^3f}{d\phi^3} &= \frac{\mathcal{H}}{m_0^6}\Big[\mathcal{H}^2\frac{d^3f}{da^3} + \frac{3}{a}\dot{\mathcal{H}}\frac{d^2f}{da^2} + \frac{1}{a^2}\Big(\frac{\ddot{\mathcal{H}}}{\mathcal{H}} - \dot{\mathcal{H}}\Big)\frac{df}{da}\Big], \\
\frac{d^4f}{d\phi^4} &= \frac{1}{m_0^8}\Big[\mathcal{H}^4\frac{d^4f}{da^4} + \frac{6\mathcal{H}^2\dot{\mathcal{H}}}{a}\frac{d^3f}{da^3} + \frac{1}{a^2}\Big(-4\mathcal{H}^2\dot{\mathcal{H}} + 3\dot{\mathcal{H}}^2 + 4\mathcal{H}\ddot{\mathcal{H}}\Big)\frac{d^2f}{da^2} + \frac{1}{a^3}\Big(2\mathcal{H}^2\dot{\mathcal{H}} - \dot{\mathcal{H}}^2 - 3\mathcal{H}\ddot{\mathcal{H}} + \dddot{\mathcal{H}}\Big)\frac{df}{da}\Big].
\end{aligned} \tag{9.107}$$

Derivatives of the Hubble parameter should also be converted as follows (where a dot indicates the derivative with respect to τ):

$$\begin{aligned}
\frac{dH}{d\phi} &= \frac{1}{m_0^2 a^2}(\dot{\mathcal{H}} - \mathcal{H}^2), \\
\frac{d^2H}{d\phi^2} &= \frac{1}{m_0^4 a^3}(\ddot{\mathcal{H}} - 4\mathcal{H}\dot{\mathcal{H}} + 2\mathcal{H}^3), \\
\frac{d^3H}{d\phi^3} &= \frac{1}{m_0^6 a^4}(\dddot{\mathcal{H}} - 7\ddot{\mathcal{H}}\mathcal{H} + 18\dot{\mathcal{H}}\mathcal{H}^2 - 4\dot{\mathcal{H}}^2 - 6\mathcal{H}^4).
\end{aligned} \tag{9.108}$$

After converting to the EFTCAMB convention [27] and performing the above derivative conversion equations, it follows that the positivity bounds contain Λ', c', Λ'', where prime will in this convention indicate derivative with respect to a. It is needed to convert these functions as follows:

$$\begin{aligned}
\Lambda' &= \frac{1}{a\mathcal{H}}\dot{\Lambda}, \\
c' &= \frac{1}{a\mathcal{H}}\dot{c}, \\
\Lambda'' &= -\frac{1}{a^2\mathcal{H}}\dot{\Lambda} + \frac{\ddot{\Lambda}}{a^2\mathcal{H}^2} - \frac{\dot{\Lambda}\dot{\mathcal{H}}}{a^2\mathcal{H}^3},
\end{aligned} \tag{9.109}$$

upon which the equations (7.64) and (7.65) follow.

9.9 K-mouflage full model figures

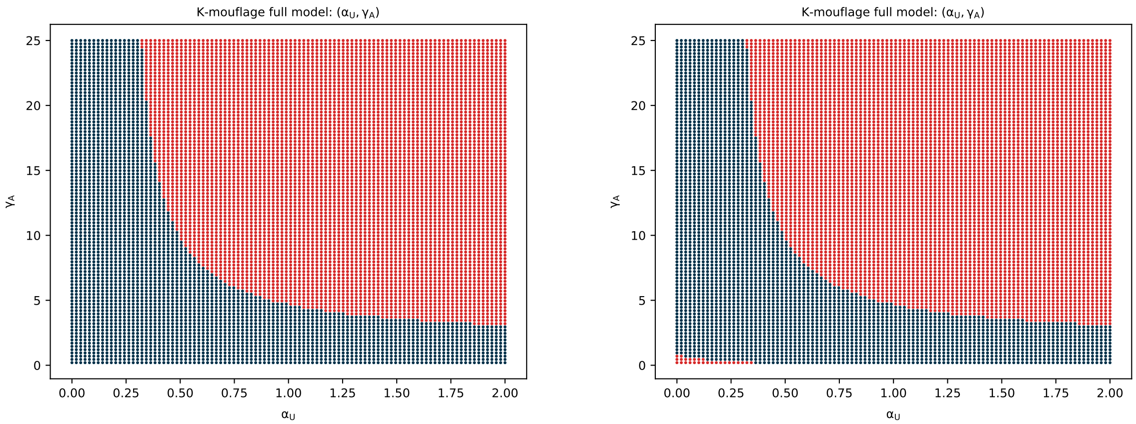


Figure 18: K-mouflage: α_U, γ_A. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\epsilon_{2,0} = -10^{-8}$, $m = 3$ and $\gamma_U = 1$.

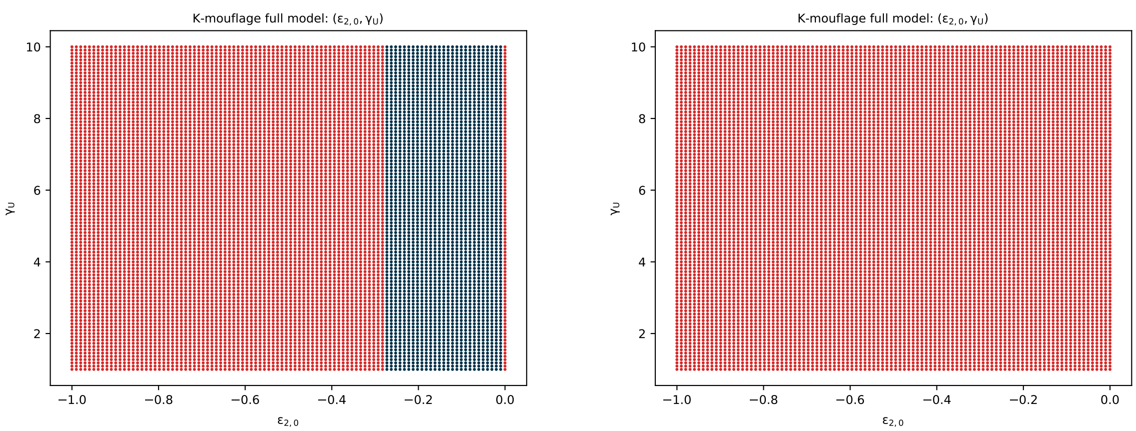


Figure 19: K-mouflage: $\epsilon_{2,0}$, γ_U. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\gamma_A = 0.2$, $m = 3$ and $\alpha_U = 0.1$.

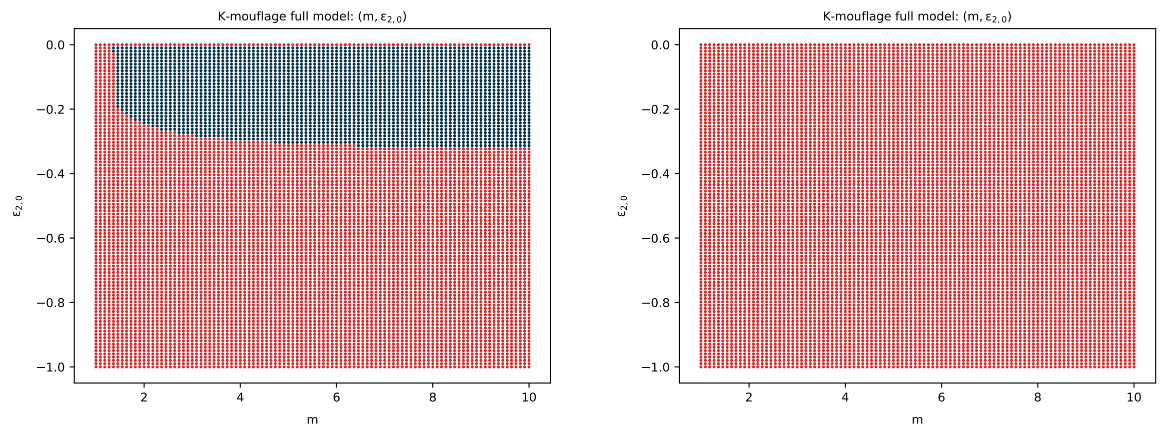


Figure 20: K-mouflage: m, $\epsilon_{2,0}$. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\gamma_A = 0.2$, $\gamma_U = 1$ and $\alpha_U = 0.1$.

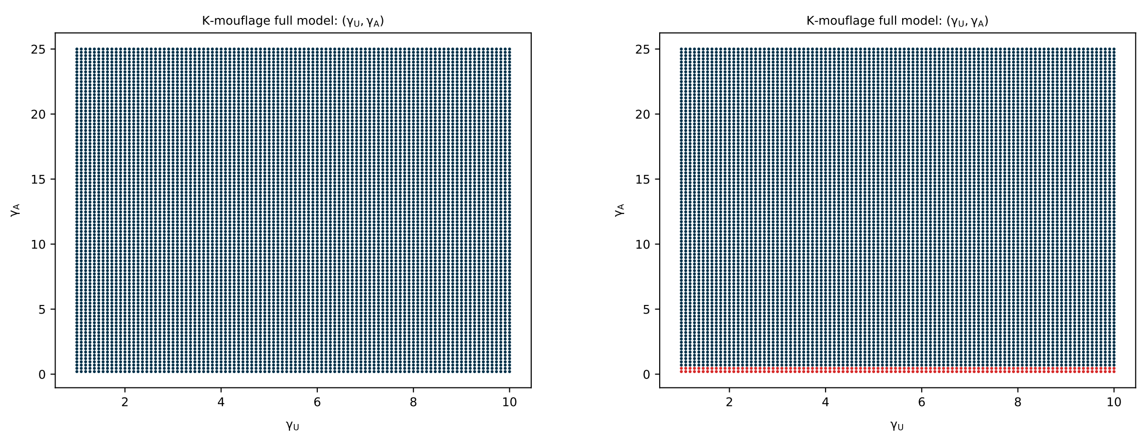


Figure 21: K-mouflage: γ_U, γ_A. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\alpha_U = 0.1$, $m = 3$ and $\epsilon_{2,0} = -10^{-8}$.

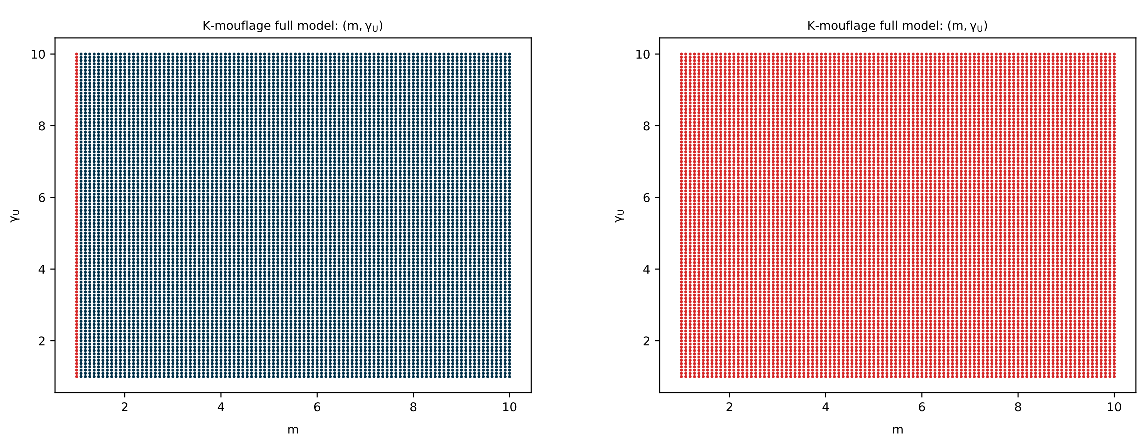


Figure 22: K-mouflage: m, γ_U. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\alpha_U = 0.1$, $\gamma_A = 0.2$ and $\epsilon_{2,0} = -10^{-8}$.

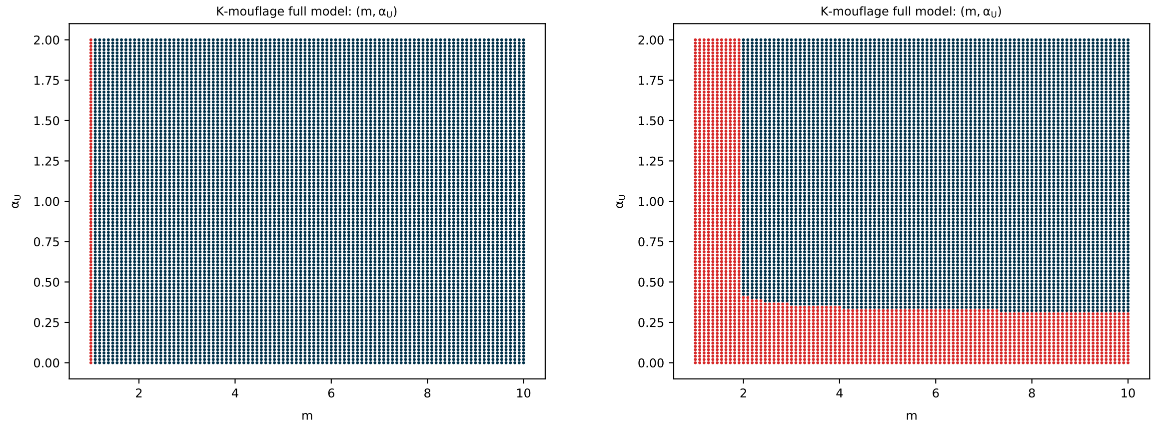


Figure 23: K-mouflage: m, α_U. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\gamma_U = 1$, $\gamma_A = 0.2$ and $\epsilon_{2,0} = -10^{-8}$.

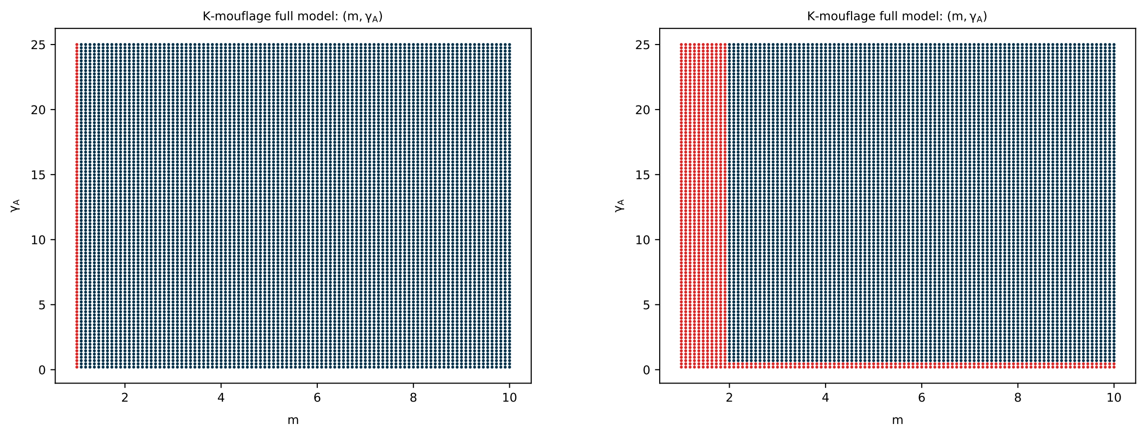


Figure 24: K-mouflage: m, γ_A. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $\epsilon_{2,0} = -10^{-8}$, $\alpha_U = 0.1$ and $\gamma_U = 1$.

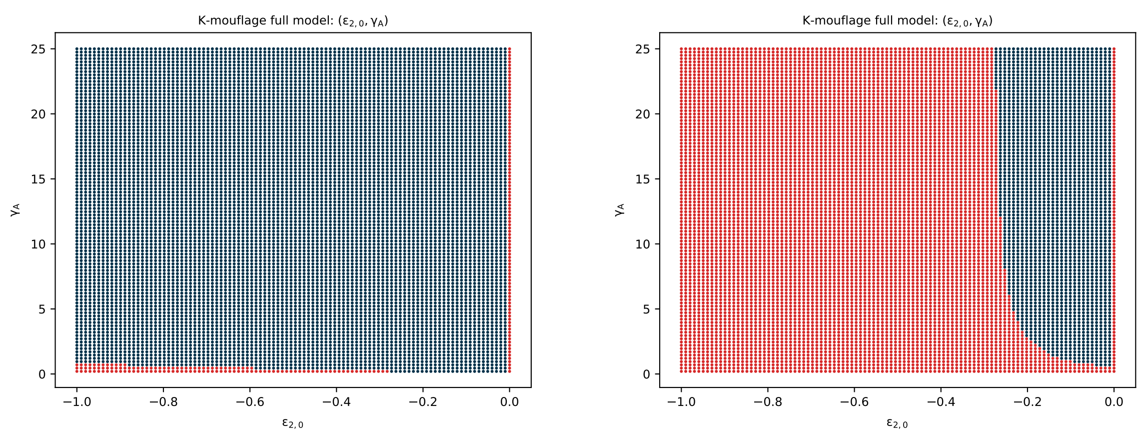


Figure 25: K-mouflage: $\epsilon_{2,0}$, γ_A. Red points indicate unstable models and blue points indicate stable models. On the left the gradient and ghost stability conditions have been imposed and the right includes the positivity condition. In this plot we fixed the other parameters $m = 3$, $\alpha_U = 0.1$ and $\gamma_U = 1$.

9.10 Derivations of Stückelberg tricks up to cubic order

Recall that a general metric in the ADM formalism is given by:

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$

Although $N^i = 0$ in the Newtonian gauge, for the Stückelberg tricks it is important to incorporate how N^i transforms (as it will be non-zero if we are not in the unitary gauge). Of course once we expressed geometrical quantities out of the unitary gauge we can fix the Newtonian gauge for quantities expressed in the unitary gauge. Transformed quantities under $t \mapsto t + \pi(t, \mathbf{x})$ will be indicated with tildes on top. To be precise for the coordinates it holds that $\tilde{t} = t + \pi(t, \mathbf{x})$ and $\tilde{\mathbf{x}} = \mathbf{x}$. As a starting point the inverse metric transforms under a Stückelberg transformation as [35]:

$$g^{\mu\nu} \mapsto \frac{\partial \tilde{x}^\mu}{\partial x^\alpha}\frac{\partial \tilde{x}^\nu}{\partial x^\beta} g^{\alpha\beta} =: \tilde{g}^{\mu\nu}. \tag{9.110}$$

Recall that under a Stückelberg trick it holds that $\tilde{x}^\mu = x^\mu + \delta^\mu_0 \pi$. Therefore we find the transformation for the inverse metric as:

$$\tilde{g}^{\mu\nu} = \frac{\partial(x^\mu + \delta^\mu_0\pi)}{\partial x^\alpha}\frac{\partial(x^\nu + \delta^\nu_0\pi)}{\partial x^\beta} g^{\alpha\beta} = (\delta^\mu_\alpha + \delta^\mu_0\partial_\alpha\pi)(\delta^\nu_\beta + \delta^\nu_0\partial_\beta\pi)g^{\alpha\beta}. \tag{9.111}$$

In particular the following transformation rules are found for the components (using that the metric is diagonal in the Newtonian gauge):

$$\begin{aligned}
\tilde{g}^{00} &= (1+\dot{\pi})^2 g^{00} + g^{ij}\partial_i\pi\partial_j\pi = -\frac{1}{N^2}(1+\dot{\pi})^2 + h^{ij}\partial_i\pi\partial_j\pi, \\
\tilde{g}^{0i} &= h^{ij}\partial_j\pi, \\
\tilde{g}^{ij} &= h^{ij}.
\end{aligned} \tag{9.112}$$

Where we used that $g^{00} = -1/N^2$, $g^{ij} = h^{ij}$ where $N^2 = 1+2\Phi$ and $h_{ij} = a^2(t)(1-2\Psi)(e^\gamma)_{ij}$ in the Newtonian gauge.

In general the metric and inverse metric components in the ADM formalism are easily found to be [35]:

$$\begin{aligned}
&g^{00} = -\frac{1}{N^2}, g^{0i} = \frac{N^i}{N^2}, h^{ij} = g^{ij} + \frac{N^iN^j}{N^2}, \\
&g_{00} = -N^2 + h_{ij}N^iN^j, g_{0i} = N_i = h_{ij}N^j, h_{ij} = g_{ij}.
\end{aligned} \tag{9.113}$$

With these relations we can find the transformations of N, N^i, h_{ij} and N_i up to cubic order in perturbations. We do not need to distinguish between order in metric perturbations and order in π since Φ, Ψ are linear in π and for mixing of γ and π we need at most $\gamma\pi\pi$ which is fully incorporated at cubic order, so no higher order Stückelberg tricks are needed. The expressions derived below are only valid up to cubic order although we explicitly leave quantities like N present in the transformations. In the derivations we will particularly make use of the binomial approximation up to cubic order (for $x \in \mathbb{R}$ with $|x| \ll 1$ and $\alpha \in \mathbb{R}$):

$$(1+x)^\alpha = 1 + \alpha x + \frac{1}{2}\alpha(\alpha-1)x^2 + \frac{1}{6}\alpha(\alpha-1)(\alpha-2)x^3 + \mathcal{O}(x^4). \tag{9.114}$$

Recall that $\tilde{g}^{00} = -1/\tilde{N}^2$, so using the above transformation for the inverse metric we find the transformation for $\tilde{N}$:

$$
\begin{aligned}
\tilde{N}^2 &= \frac{N^2}{(1+\dot{\pi})^2 - N^2 h^{ij}\partial_i\pi\partial_j\pi} \\
\tilde{N} &= N\Big[1 + 2\dot{\pi} + \dot{\pi}^2 - N^2 h^{ij}\partial_i\pi\partial_j\pi\Big]^{-1/2} \\
\tilde{N} &\approx N\Big[1 - \dot{\pi} - \frac{1}{2}\dot{\pi}^2 + \frac{1}{2}N^2 h^{ij}\partial_i\pi\partial_j\pi + \frac{3}{8}\Big(2\dot{\pi} + \dot{\pi}^2 - N^2 h^{ij}\partial_i\pi\partial_j\pi\Big)^2 - \frac{5}{16}\Big(\dot{\pi}^2 + 2\dot{\pi} - N^2 h^{ij}\partial_i\pi\partial_j\pi\Big)^3\Big] \\
\tilde{N} &\approx N\Big(1 - \dot{\pi} + \dot{\pi}^2 + \frac{1}{2}N^2 h^{ij}\partial_i\pi\partial_j\pi - \dot{\pi}^3 - \frac{3}{2}N^2\dot{\pi}h^{ij}\partial_i\pi\partial_j\pi\Big) \\
\tilde{N} &\approx N\Big(1 - \dot{\pi} + \dot{\pi}^2 - \dot{\pi}^3 + \frac{1}{2}N^2 h^{ij}\partial_i\pi\partial_j\pi - \frac{3}{2a^2}\dot{\pi}(\partial_k\pi)^2\Big).
\end{aligned}
\tag{9.115}
$$

In similar fashion it holds that $\tilde{g}^{0i} = \tilde{N}^i/\tilde{N}^2$. Using the transformation rules for $\tilde{g}^{0i}$ and $\tilde{N}^2$ it is possible to find $\tilde{N}^i$:

$$
\begin{aligned}
\tilde{N}^i &= \tilde{N}^2\tilde{g}^{0i} \\
&= \frac{N^2}{(1+\dot{\pi})^2 - N^2 h^{kl}\partial_k\pi\partial_l\pi} h^{ij}\partial_j\pi \\
&\approx N^2\Big[1 - \dot{\pi}^2 - 2\dot{\pi} + N^2 h^{kl}\partial_k\pi\partial_l\pi + (2\dot{\pi} + \dot{\pi}^2 - N^2 h^{kl}\partial_k\pi\partial_l\pi)^2\Big] h^{ij}\partial_j\pi \\
&\approx N^2\Big[1 - 2\dot{\pi} + 3\dot{\pi}^2 + N^2 h^{kl}\partial_k\pi\partial_l\pi\Big] h^{ij}\partial_j\pi.
\end{aligned}
\tag{9.116}
$$

Next, note that $\tilde{h}^{ij}$ can be easily found using the previous results:

$$
\begin{aligned}
h^{ij} &= \tilde{g}^{ij} = \tilde{h}^{ij} + \frac{\tilde{N}^i\tilde{N}^j}{\tilde{N}^2} \\
\tilde{h}^{ij} &= h^{ij} - \frac{\tilde{N}^i\tilde{N}^j}{\tilde{N}^2} \\
\tilde{h}^{ij} &\approx h^{ij} - N^2[(1+\dot{\pi})^2 - N^2 h^{mn}\partial_m\pi\partial_n\pi](1 - 2\dot{\pi} + 3\dot{\pi}^2 + N^2 h^{kl}\partial_k\pi\partial_l\pi)^2 h^{ia}h^{jb}\partial_a\pi\partial_b\pi \\
\tilde{h}^{ij} &\approx h^{ij} - N^2 h^{ia}h^{jb}\partial_a\pi\partial_b\pi + 2N^2\dot{\pi}h^{ia}h^{jb}\partial_a\pi\partial_b\pi.
\end{aligned}
\tag{9.117}
$$

Since $g^{ij} = \tilde{g}^{ij} = h^{ij}$ it follows that:

$$
\begin{aligned}
\tilde{g}^{ik}\tilde{g}^{jl}\tilde{h}_{kl} &= \tilde{g}^{ik}\tilde{g}^{jl}h_{kl} - N^2\tilde{g}^{ik}\tilde{g}^{jl}\partial_k\pi\partial_l\pi + 2N^2\dot{\pi}\tilde{g}^{ik}\tilde{g}^{jl}\partial_k\pi\partial_l\pi \\
\tilde{h}_{kl} &= h_{kl} - N^2\partial_k\pi\partial_l\pi + 2N^2\dot{\pi}\partial_k\pi\partial_l\pi \\
\tilde{h}_{kl} &\approx h_{kl} - N^2\partial_k\pi\partial_l\pi + 2\dot{\pi}\partial_k\pi\partial_l\pi.
\end{aligned}
\tag{9.118}
$$

Recall that $N_j = h_{ij}N^i$. Therefore N_j transforms under a Stückelberg trick as:

$$
\begin{aligned}
\tilde{N}_j &= \tilde{h}_{ij}\tilde{N}^i \\
&\approx N^2(h_{ij} - N^2\partial_i\pi\partial_j\pi + 2\dot{\pi}\partial_i\pi\partial_j\pi)(1 - 2\dot{\pi} + 3\dot{\pi}^2 + N^2h^{kl}\partial_k\pi\partial_l\pi)h^{im}\partial_m\pi \\
&\approx N^2(1 - 2\dot{\pi} + 3\dot{\pi}^2)\partial_j\pi + N^4h^{kl}\partial_k\pi\partial_l\pi\partial_j\pi - N^4h^{im}\partial_i\pi\partial_j\pi\partial_m\pi \\
&\approx N^2(1 - 2\dot{\pi} + 3\dot{\pi}^2)\partial_j\pi.
\end{aligned}
\tag{9.119}
$$

In computing the Stückelberg tricks for geometrical quantities such as the extrinsic curvature tensor it will be useful to know how the partial derivatives ∂_0, ∂_i transform under a Stückelberg transformation. This follows straightforwardly using the chain rule:

$$
\frac{\partial}{\partial\tilde{t}} = \frac{\partial x^\alpha}{\partial\tilde{t}}\frac{\partial}{\partial x^\alpha} = \frac{\partial t}{\partial\tilde{t}}\frac{\partial}{\partial t} = \frac{1}{1+\dot{\pi}}\frac{\partial}{\partial t} \approx (1 - \dot{\pi} + \dot{\pi}^2 - \dot{\pi}^3)\partial_0.
$$

Similarly the derivative with respect to a spatial coordinate transforms as follows:

$$
\begin{aligned}
\frac{\partial}{\partial\tilde{x}^i} &= \frac{\partial x^\alpha}{\partial\tilde{x}^i}\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial x^i} + \frac{\partial t}{\partial\tilde{x}^i}\frac{\partial}{\partial t} \\
&= \frac{\partial}{\partial x^i} + \frac{\partial(\tilde{t} - \pi)}{\partial\tilde{x}^i}\frac{\partial}{\partial t} \\
&= \frac{\partial}{\partial x^i} - \frac{\partial\pi}{\partial\tilde{x}^i}\frac{\partial}{\partial t} \\
&= \partial_i - (\partial_i\pi - \tilde{\partial}_i\pi\partial_0\pi)\partial_0 \\
&= \partial_i - (\partial_i\pi - \partial_i\pi\partial_0\pi + \tilde{\partial}_i\pi\partial_0\pi\partial_0\pi)\partial_0 \\
&\approx \partial_i - (\partial_i\pi - \partial_i\pi\partial_0\pi + \partial_i\pi\partial_0\pi\partial_0\pi)\partial_0 \\
&= \partial_i - (1 - \dot{\pi} + \dot{\pi}^2)\partial_i\pi\partial_0.
\end{aligned}
\tag{9.120}
$$

In this derivation we substituted the result recursively in order to find the transformation up to cubic order.

We are now ready to compute the Stückelberg transformations of various geometrical quantities. Let us start with the transformation of K^i_j. Recall that by definition $K_{\mu\nu} = h^\beta_\mu\nabla_\beta n_\nu = (\delta^\beta_\mu + n^\beta n_\mu)\nabla_\beta n_\nu = \nabla_\mu n_\nu + n^\beta n_\mu\nabla_\alpha n_\nu$. And using the definition of the covariant derivative it follows that $K_{\mu\nu} = \partial_\mu n_\nu - \Gamma^\alpha_{\mu\nu}n_\alpha + n^\alpha n_\mu\partial_\alpha n_\nu - n^\alpha n_\mu\Gamma^\beta_{\alpha\nu}n_\beta$. Although it seems like one could obtain the Stückelberg transformation using the usual gauge transformation rule for tensor fields (like for the inverse metric) this is not possible because 3D hypersurface quantities such as K^μ_ν, K and ${}^{(3)}R$ depend on the foliation (or time slicing). The proper way of finding Stückelberg tricks is to express these quantities in terms of the 4D metric and transform it with the previously derived Stückelberg tricks [37].

Recall that in the unitary gauge we are allowed to pick the normal vector as $n_\mu = -\partial_\mu t/\sqrt{-g^{00}}$ and $n^\mu = g^{\mu\nu}n_\nu$ [30]. Note that $\tilde{\partial}_\mu\tilde{t} = \delta^0_\mu$. Therefore we may write $n_\mu = -\delta^0_\mu/\sqrt{-g^{00}}$ and $n^\mu = -g^{\mu 0}/\sqrt{-g^{00}}$ such that the extrinsic curvature tensor is given by the following expression:

$$
\begin{aligned}
K_{\mu\nu} &= \partial_\mu\Big(-\frac{\delta^0_\nu}{\sqrt{-g^{00}}}\Big) + \Big(\frac{\delta^0_\alpha}{\sqrt{-g^{00}}}\Big)\frac{1}{2}g^{\alpha\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}) \\
&- \frac{\delta^0_\mu g^{\alpha 0}}{-g^{00}}\partial_\alpha\Big(\frac{\delta^0_\nu}{\sqrt{-g^{00}}}\Big) + \frac{g^{\alpha 0}\delta^0_\mu\delta^0_\beta}{2(-g^{00})^{3/2}}g^{\beta\sigma}(\partial_\alpha g_{\nu\sigma} + \partial_\nu g_{\alpha\sigma} - \partial_\sigma g_{\alpha\nu}) \\
&= -\frac{\delta^0_\nu\partial_\mu g^{00}}{2(-g^{00})^{3/2}} + \frac{g^{0\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})}{2(-g^{00})^{1/2}} \\
&- \frac{\delta^0_\mu\delta^0_\nu g^{0\alpha}\partial_\alpha g^{00}}{2(-g^{00})^{5/2}} + \frac{\delta^0_\mu g^{0\alpha}g^{0\sigma}(\partial_\alpha g_{\nu\sigma} + \partial_\nu g_{\alpha\sigma} - \partial_\sigma g_{\alpha\nu})}{2(-g^{00})^{3/2}}.
\end{aligned}
\tag{9.121}
$$

Therefore the component K^i_j is found by:

$$
K^i_j = g^{i\alpha}K_{j\alpha} = -\frac{g^{i0}\partial_j g^{00}}{2(-g^{00})^{3/2}} + \frac{g^{i\alpha}g^{0\sigma}(\partial_j g_{\sigma\alpha} + \partial_\alpha g_{\sigma j} - \partial_\sigma g_{j\alpha})}{2(-g^{00})^{1/2}}. \tag{9.122}
$$

In order to compute the Stückelberg transformation of this quantity we note that first the transformations of g_{00}, g_{0i} and g_{ij} should be known. $\tilde{g}_{0i} = \tilde{N}_i$ and $\tilde{g}_{ij} = \tilde{h}_{ij}$ have already been derived above. The transformation of g_{00} yields:

$$
\begin{aligned}
\tilde{g}_{00} &= -\tilde{N}^2 + \tilde{h}_{ij}\tilde{N}^i\tilde{N}^j \\
&\approx -N^2\Big(1 - \dot{\pi} + \dot{\pi}^2 - \dot{\pi}^3 + \frac{1}{2}N^2h^{ij}\partial_i\pi\partial_j\pi - \frac{3}{2a^2}\dot{\pi}(\partial_k\pi)^2\Big)^2 \\
&+ N^4(h_{ij} - N^2\partial_i\pi\partial_j\pi + 2\dot{\pi}\partial_i\pi\partial_j\pi)(1 - 2\dot{\pi} + 3\dot{\pi}^2 + N^2h^{kl}\partial_k\pi\partial_l\pi)^2h^{im}h^{jn}\partial_m\pi\partial_n\pi \\
&\approx -N^2(1 - 2\dot{\pi} + 3\dot{\pi}^2 - 4\dot{\pi}^3 - \frac{3}{a^2}\dot{\pi}(\partial_k\pi)^2 - N^2\dot{\pi}h^{ij}\partial_i\pi\partial_j\pi + N^2h^{ij}\partial_i\pi\partial_j\pi) \\
&+ N^4h^{mn}\partial_m\pi\partial_n\pi(1 - 4\dot{\pi}) \\
&\approx -N^2(1 - 2\dot{\pi} + 3\dot{\pi}^2 - 4\dot{\pi}^3).
\end{aligned}
\tag{9.123}
$$

To be fully prepared for the computations it is useful to determine $(-\tilde{g}^{00})^{-3/2}$ and $(-\tilde{g}^{00})^{-5/2}$. We easily find that:

$$
\begin{aligned}
\frac{1}{(-\tilde{g}^{00})^{3/2}} &= \tilde{N}^3 \\
&\approx N^3\Big(1 - \dot{\pi} + \dot{\pi}^2 - \dot{\pi}^3 + \frac{1}{2}N^2h^{ij}\partial_i\pi\partial_j\pi - \frac{3}{2a^2}\dot{\pi}(\partial_k\pi)^2\Big)^3 \\
&\approx N^3\Big(1 - 2\dot{\pi} + 3\dot{\pi}^2 - 4\dot{\pi}^3 + N^2h^{ij}\partial_i\pi\partial_j\pi - \frac{4}{a^2}\dot{\pi}(\partial_k\pi)^2\Big)\Big(1 - \dot{\pi} + \dot{\pi}^2 - \dot{\pi}^3 + \frac{1}{2}N^2h^{ij}\partial_i\pi\partial_j\pi \\
&- \frac{3}{2a^2}\dot{\pi}(\partial_k\pi)^2\Big) \\
&\approx N^3\Big(1 - 3\dot{\pi} + 6\dot{\pi}^2 - 10\dot{\pi}^3 + \frac{3}{2}N^2h^{ij}\partial_i\pi\partial_j\pi - \frac{15}{2a^2}\dot{\pi}(\partial_k\pi)^2\Big).
\end{aligned}
\tag{9.124}
$$

$$
\begin{aligned}
\frac{1}{(-\tilde{g}^{00})^{5/2}} &= \tilde{N}^5 \\
&\approx N^5\Big(1 - 3\dot{\pi} + 6\dot{\pi}^2 - 10\dot{\pi}^3 + \frac{3}{2}N^2 h^{ij}\partial_i\pi\partial_j\pi - \frac{15}{2a^2}\dot{\pi}(\partial_k\pi)^2\Big)\Big(1 - 2\dot{\pi} + 3\dot{\pi}^2 - 4\dot{\pi}^3 \\
&+ N^2 h^{ij}\partial_i\pi\partial_j\pi - \frac{4}{a^2}\dot{\pi}(\partial_k\pi)^2\Big) \\
&\approx N^5\Big(1 - 5\dot{\pi} + 15\dot{\pi}^2 - 35\dot{\pi}^3 + \frac{5}{2}N^2 h^{ij}\partial_i\pi\partial_j\pi - \frac{35}{2a^2}\dot{\pi}(\partial_k\pi)^2\Big).
\end{aligned}
\tag{9.125}
$$

Let us now derive the expression for $\tilde{K}^i_j$:

$$
\tilde{K}^i_j = -\frac{\tilde{g}^{i0}\tilde{\partial}_j\tilde{g}^{00}}{2(-\tilde{g}^{00})^{3/2}} + \frac{\tilde{g}^{i\alpha}\tilde{g}^{0\sigma}(\tilde{\partial}_j\tilde{g}_{\sigma\alpha} + \tilde{\partial}_\alpha\tilde{g}_{j\sigma} - \tilde{\partial}_\sigma\tilde{g}_{j\alpha})}{2(-\tilde{g}^{00})^{1/2}}. \tag{9.126}
$$

Combining the above findings it follows that:

$$
\begin{aligned}
\tilde{K}^i_j &= K^i_j - (1 - 2\dot{\pi})h^{ik}\partial_j N\partial_k\pi + Nh^{ik}\partial_k\pi\partial_j\dot{\pi}(1 - 2\dot{\pi}) - \frac{1}{a^4}\partial_i\pi\partial_j\partial_k\pi\partial_k\pi + \frac{1}{a^2}\partial_0 N\partial_i\pi\partial_j\pi \\
&- \frac{\ddot{\pi}}{a^2}\partial_i\pi\partial_j\pi - Nh^{ik}\partial_j\partial_k\pi(1 - \dot{\pi} + \dot{\pi}^2) + Nh^{ik}\partial_j\pi\partial_k\dot{\pi}(1 - 2\dot{\pi}) - \frac{1}{2a^4}(\partial_k\pi)^2\partial_i\partial_j\pi \\
&+ \frac{N}{2}h^{ik}h^{lm}(1 - \dot{\pi})(\partial_m\pi)(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk}) - \frac{N}{2}h^{ik}h^{lm}\dot{h}_{kl}\partial_m\pi\partial_j\pi(1 - 2\dot{\pi}) \\
&- h^{ik}(1 - \dot{\pi})\partial_k N\partial_j\pi + \frac{N}{4}(1 - 2\dot{\pi})h^{ik}h^{mn}\dot{h}_{jk}\partial_m\pi\partial_n\pi,
\end{aligned}
\tag{9.127}
$$

where we used the fact that $K^i_j = \frac{1}{2N}h^{ik}\dot{h}_{kj}$ (since $N_i = 0$ in Newtonian gauge).

As a consistency check, note that the result at second order reduces to:

$$
\begin{aligned}
\tilde{K}^i_j &\approx K^i_j - \frac{1}{a^2}\partial_j N\partial_i\pi + \frac{1}{a^2}\partial_i\pi\partial_j\dot{\pi} - Nh^{ik}(1 - \dot{\pi})\partial_j\partial_k\pi + \frac{1}{a^2}\partial_j\pi\partial_i\dot{\pi} \\
&+ \frac{1}{2a^4}(\partial_m\pi)(-\partial_m h_{ij} + \partial_i h_{jm} + \partial_j h_{im}) - \frac{H}{a^2}\partial_i\pi\partial_j\pi - \frac{1}{a^2}\partial_i N\partial_j\pi + \frac{H}{2a^2}(\partial_k\pi)^2\delta^i_j,
\end{aligned}
\tag{9.128}
$$

which is in accordance with the previous literature [35].

Next, we compute the Stückelberg transformation of $K = g^{\mu\nu}K_{\mu\nu}$. For this notice that $K = g^{\mu\nu}(\delta^\alpha_\mu + n^\alpha n_\mu)\nabla_\alpha n_\nu = \nabla_\alpha n^\alpha$ since the last term is zero by $n_\nu n^\nu = -1$. Namely, we have that $-\nabla_\alpha n^\alpha = \nabla_\alpha(n^\alpha n^\nu n_\nu) = -\nabla_\alpha n^\alpha + 2n^\alpha n_\nu\nabla_\alpha n^\nu$. Therefore we have that $K = \nabla_\alpha n^\alpha = \partial_\alpha n^\alpha + \Gamma^\alpha_{\alpha\mu}n^\mu$. Thus using the definition of the normal vector we find that:

$$
\begin{aligned}
K &= \partial_\alpha\Big(-\frac{g^{\alpha0}}{\sqrt{-g^{00}}}\Big) + \frac{1}{2}\Big(-\frac{g^{\mu0}}{\sqrt{-g^{00}}}\Big)g^{\alpha\sigma}\partial_\mu g_{\alpha\sigma} \\
\tilde{K} &= \tilde{\partial}_0\Big(-\frac{\tilde{g}^{00}}{(-\tilde{g}^{00})^{1/2}}\Big) + \tilde{\partial}_i\Big(-\frac{\tilde{g}^{i0}}{(-\tilde{g}^{00})^{1/2}}\Big) - \frac{\tilde{g}^{\mu0}\tilde{g}^{\alpha\sigma}\tilde{\partial}_\mu\tilde{g}_{\alpha\sigma}}{2(-\tilde{g}^{00})^{1/2}}.
\end{aligned}
\tag{9.129}
$$

Combining the above results yields:

$$
\begin{aligned}
\tilde{K} = {} & K + \frac{1}{a^2}\partial_0 N(\partial_k\pi)^2 + \frac{N}{2}\dot{h}^{ij}\partial_i\pi\partial_j\pi(1-2\dot{\pi}) + 2Nh^{ij}\partial_i\dot{\pi}\partial_j\pi(1-2\dot{\pi}) - \frac{\ddot{\pi}}{a^2}(\partial_k\pi)^2 \\
& - N(1-\dot{\pi})\partial_i h^{ik}\partial_k\pi \\
& - Nh^{ik}\Big(1-\dot{\pi}+\dot{\pi}^2+\frac{3}{2a^2}(\partial_k\pi)^2\Big)\partial_i\partial_k\pi - 2h^{ik}(1-\dot{\pi})\partial_k\pi\partial_i N + \frac{N}{4}h^{ij}h^{kl}\dot{h}_{ij}(1-4\dot{\pi})\partial_k\pi\partial_l\pi \\
& - \frac{N}{2}h^{il}h^{kj}(1-\dot{\pi})\partial_l\pi\partial_i h_{kj}.
\end{aligned}
\tag{9.130}
$$

Note that by $\partial_k\pi\partial_i h^{ik} \approx -\frac{1}{a^4}\partial_k\pi\partial_i h_{ik}$ the result at second order indeed agrees with the literature [35].

Let us now compute the Stückelberg trick of the 3D Ricci scalar ${}^{(3)}R$. For this we use the Gauss-Codazzi equation [30]:

$$
{}^{(3)}R = {}^{(4)}R + 2\,{}^{(4)}R^{\mu\nu}n_\mu n_\nu - K^2 + K^\mu_\nu K^\nu_\mu. \tag{9.131}
$$

The transformation of ${}^{(4)}R$ is trivial: ${}^{(4)}R \mapsto {}^{(4)}R$, because ${}^{(4)}R$ is a 4-scalar and thus diffeomorphism invariant.

Restricting the above results to second order we note that $\tilde{K} = \tilde{K}^i_i$. This means that $\tilde{K}^0_0$ is at least third order. Since $\bar{K}^0_0 = 0$ [37] we know that $K^0_0 K^0_0$ cannot contribute to ${}^{(3)}R$ at cubic order. Next, we will compute $\tilde{K}^0_i$ and see that it vanishes at linear order so it is at least second order. This means by $\bar{K}^i_0 = \bar{K}^0_i = 0$ that only $\delta_1\tilde{K}^i_0\tilde{K}^0_i$ will contribute. In the computations below we will omit the subscript ${}^{(4)}$, e.g. ${}^{(4)}R = R$ for briefness. Let us first detail the computation of the second term. From $n_\mu = -N\partial_\mu t$ we note that:

$$
\begin{aligned}
\tilde{n}_\mu &= -\tilde{N}\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\partial_\nu(t+\pi) \\
&\approx -N\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\Big(1-\dot{\pi}+\dot{\pi}^2-\dot{\pi}^3+\frac{1}{2}N^2h^{ij}\partial_i\pi\partial_j\pi - \frac{3\dot{\pi}}{2a^2}(\partial_k\pi)^2\Big)(\delta^0_\nu+\partial_\nu\pi) \\
&\approx -\frac{\partial x^\nu}{\partial\tilde{x}^\mu}\Big[-n_\nu\Big(1-\dot{\pi}+\dot{\pi}^2-\dot{\pi}^3+\frac{1}{2}N^2h^{ij}\partial_i\pi\partial_j\pi-\frac{3\dot{\pi}}{2a^2}(\partial_k\pi)^2\Big) + N\partial_\nu\pi\Big(1-\dot{\pi}+\dot{\pi}^2+\frac{1}{2a^2}(\partial_k\pi)^2\Big)\Big],
\end{aligned}
\tag{9.132}
$$

and therefore it follows that

[37] Recall that $\bar{K}^i_j = H\delta^i_j$ in a flat cosmological background.

$$
\begin{aligned}
\tilde{R}^{\mu\nu}\tilde{n}_\mu\tilde{n}_\nu \approx R^{\mu\nu}\Big[&- n_\nu\Big(1-\dot{\pi}+\dot{\pi}^2-\dot{\pi}^3+\frac{1}{2}N^2h^{ij}\partial_i\pi\partial_j\pi-\frac{3\dot{\pi}}{2a^2}(\partial_k\pi)^2\Big)+N\partial_\nu\pi\Big(1-\dot{\pi}+\dot{\pi}^2+\frac{1}{2a^2}(\partial_k\pi)^2\Big)\Big] \\
&\cdot\Big[-n_\mu\Big(1-\dot{\pi}+\dot{\pi}^2-\dot{\pi}^3+\frac{1}{2}N^2h^{ij}\partial_i\pi\partial_j\pi-\frac{3\dot{\pi}}{2a^2}(\partial_k\pi)^2\Big)+N\partial_\mu\pi\Big(1-\dot{\pi}+\dot{\pi}^2+\frac{1}{2a^2}(\partial_k\pi)^2\Big)\Big] \\
&\approx R^{\mu\nu}\Big[n_\mu n_\nu\Big(1-2\dot{\pi}+3\dot{\pi}^2-4\dot{\pi}^3+N^2h^{ij}\partial_i\pi\partial_j\pi-\frac{4\dot{\pi}}{a^2}(\partial_k\pi)^2\Big) \\
&-2Nn_\nu\partial_\mu\pi\Big(1-2\dot{\pi}+3\dot{\pi}^2+\frac{1}{a^2}(\partial_k\pi)^2\Big)+N^2(1-2\dot{\pi})\partial_\mu\pi\partial_\nu\pi\Big] \\
&= R^{00}N^2\Big[1+N^2h^{kl}\partial_k\pi\partial_l\pi-\frac{2\dot{\pi}}{a^2}(\partial_k\pi)^2\Big]+2N^2R^{0i}\partial_i\pi\Big[1-2\dot{\pi}+3\dot{\pi}^2+\frac{1}{a^2}(\partial_k\pi)^2\Big] \\
&+N^2R^{ij}(1-2\dot{\pi})\partial_i\pi\partial_j\pi \\
&= R^{\mu\nu}n_\mu n_\nu+N^4R^{00}\Big[h^{kl}\partial_k\pi\partial_l\pi-\frac{2\dot{\pi}}{a^2}(\partial_k\pi)^2\Big]+2R^{0i}N^2\partial_i\pi\Big(1-2\dot{\pi}+3\dot{\pi}^2+\frac{1}{a^2}(\partial_k\pi)^2\Big) \\
&+R^{ij}N^2\partial_i\pi\partial_j\pi(1-2\dot{\pi}),
\end{aligned}
\tag{9.133}
$$

where we used $n_0=-N$ and $n_i=0$ in the last line.

From this we conclude that we need to compute R^{ij}, R^{00} to linear order and R^{0i} to second order. Note that $R^{0i}=g^{00}g^{ij}R_{0j}=-\frac{h^{ij}}{N^2}R_{0j}$, $R^{ij}=h^{ik}h^{jl}R_{kl}$ and $R^{00}=(g^{00})^2R_{00}=\frac{1}{N^4}R_{00}$. Let us compute R_{00}, R_{ij} to linear order and R_{0i} to second order.

$$
R_{00}=\partial_\mu\Gamma^\mu_{00}-\partial_0\Gamma^\mu_{\mu 0}+\Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{00}-\Gamma^\mu_{0\lambda}\Gamma^\lambda_{0\mu}
$$

Note that $\Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{00}\approx\bar{\Gamma}^i_{i0}\Gamma^0_{00}$ at linear order since $\bar{\Gamma}^i_{j0}=H\delta^i_j$ and $\bar{\Gamma}^0_{ij}=a\dot{a}\delta_{ij}$ are the only non-trivial components for a flat cosmological background. Similarly, it follows that $\Gamma^\mu_{0\lambda}\Gamma^\lambda_{0\mu}\approx\Gamma^i_{0j}\Gamma^j_{0i}$ at linear order. Therefore one obtains:

$$
\begin{aligned}
R_{00}\approx&\frac{1}{2}\partial_\mu(g^{\mu\nu}(2\partial_0g_{0\nu}-\partial_\nu g_{00}))-\frac{1}{2}\partial_0(g^{\mu\nu}(\partial_\mu g_{0\nu}+\partial_0g_{\mu\nu}-\partial_\nu g_{0\mu}))+\frac{3H}{2}g^{0\mu}(2\partial_0g_{0\mu}-\partial_\mu g_{00}) \\
&+\frac{1}{4}g^{i\alpha}g^{j\beta}(\partial_0g_{j\alpha}+\partial_jg_{0\alpha}-\partial_\alpha g_{j0})(\partial_0g_{i\beta}+\partial_ig_{0\beta}-\partial_\beta g_{0i}) \\
\approx&\,\partial_0^2N+\frac{1}{a^2}\partial_i^2N-\Big(\partial_0^2N+\frac{1}{2}\dot{h}^{ij}\dot{h}_{ij}+h^{ij}\ddot{h}_{ij}\Big)+3H\partial_0N-\frac{1}{4}h^{ik}h^{jl}\dot{h}_{jk}\dot{h}_{il} \\
=&\,\frac{1}{a^2}\partial_i^2N-\frac{1}{2}(\dot{h}^{ij}\dot{h}_{ij}+h^{ij}\ddot{h}_{ij})+3H\partial_0N-\frac{1}{4}h^{ik}h^{jl}\dot{h}_{jk}\dot{h}_{il}.
\end{aligned}
\tag{9.134}
$$

Next, focus on R_{ij} up to linear order.

$$
R_{ij}=\partial_\mu\Gamma^\mu_{ij}-\partial_j\Gamma^\mu_{\mu i}+\Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{ij}-\Gamma^\mu_{j\lambda}\Gamma^\lambda_{\mu i}.
$$

At linear order we note that $\Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{ij}\approx\bar{\Gamma}^0_{ij}\Gamma^0_{00}+\Gamma^k_{k0}\Gamma^0_{ij}$ and $\Gamma^\mu_{j\lambda}\Gamma^\lambda_{\mu i}\approx\Gamma^0_{jk}\Gamma^k_{0i}+\Gamma^k_{j0}\Gamma^0_{ki}$.

We find the following for R_{ij} at linear order:

$$
\begin{aligned}
R_{ij} &\approx \frac{1}{2}\partial_\mu(g^{\mu\nu}(\partial_i g_{j\nu}+\partial_j g_{i\nu}-\partial_\nu g_{ij}))-\frac{1}{2}\partial_j(g^{\mu\nu}\partial_i g_{\mu\nu})+\frac{\bar{\Gamma}^0_{ij}}{2}g^{0\alpha}(2\partial_0 g_{0\alpha}-\partial_\alpha g_{00})\\
&+\frac{1}{4}g^{k\alpha}g^{0\beta}(\partial_k g_{0\alpha}+\partial_0 g_{k\alpha}-\partial_\alpha g_{k0})(\partial_i g_{j\beta}+\partial_j g_{i\beta}-\partial_\beta g_{ij})\\
&-\frac{1}{4}g^{k\alpha}g^{\beta 0}(\partial_j g_{0\alpha}+\partial_0 g_{j\alpha}-\partial_\alpha g_{j0})(\partial_k g_{i\beta}+\partial_i g_{k\beta}-\partial_\beta g_{ik})-\frac{1}{4}g^{k\alpha}g^{\beta 0}(\partial_i g_{0\alpha}+\partial_0 g_{i\alpha}-\partial_\alpha g_{i0})(\partial_k g_{j\beta}\\
&+\partial_j g_{k\beta}-\partial_\beta g_{jk})\\
&\approx \frac{1}{2a^2}(\partial_i\partial_k h_{jk}+\partial_j\partial_k h_{ik}-\partial_k^2 h_{ij})-2a\dot{a}\delta_{ij}\partial_0 N+\frac{\ddot{h}_{ij}}{2N^2}-\Big(\frac{1}{2a^2}\partial_i\partial_j h_{kk}+\partial_i\partial_j N\Big)+a\dot{a}\delta_{ij}\partial_0 N\\
&+\frac{1}{4N^2}h^{kl}\dot{h}_{kl}\dot{h}_{ij}-\frac{1}{4N^2}\Big(h^{kl}\dot{h}_{jl}\dot{h}_{ik}+h^{kl}\dot{h}_{il}\dot{h}_{jk}\Big)\\
&=\frac{1}{2a^2}(\partial_i\partial_k h_{jk}+\partial_j\partial_k h_{ik}-\partial_k^2 h_{ij}-\partial_i\partial_j h_{kk})-a\dot{a}\delta_{ij}\partial_0 N-\partial_i\partial_j N+\frac{1}{4N^2}\Big(h^{kl}\dot{h}_{kl}\dot{h}_{ij}+2\ddot{h}_{ij}\\
&-h^{kl}\dot{h}_{jl}\dot{h}_{ik}-h^{kl}\dot{h}_{il}\dot{h}_{jk}\Big).
\end{aligned}
\tag{9.135}
$$

Similarly we can compute R_{0i} to second order. Notice first that in the expression of R_{0i} the two terms with Christoffel symbols can be simplified a bit: $\Gamma^\mu_{\mu\lambda}\Gamma^\lambda_{0i}-\Gamma^\mu_{i\lambda}\Gamma^\lambda_{\mu 0}=\Gamma^j_{0j}\Gamma^0_{0i}+\Gamma^j_{jk}\Gamma^k_{0i}-\Gamma^0_{ij}\Gamma^j_{00}-\Gamma^j_{ik}\Gamma^k_{j0}$. The expression for R_{0i} (at second order) follows to be:

$$
\begin{aligned}
R_{0i} &\approx \frac{1}{2}\dot{h}_{ij}\partial_k h^{kj}+\frac{1}{2}h^{kj}\partial_k\dot{h}_{ij}-\frac{1}{2}\dot{h}_{kl}\partial_i h^{kl}-\frac{1}{2}h^{kl}\partial_i\dot{h}_{kl}+\frac{1}{2N}(h^{jk}\dot{h}_{jk}\partial_i N-h^{jk}\dot{h}_{ij}\partial_k N)\\
&+\frac{1}{4}h^{jm}h^{kn}\dot{h}_{in}\partial_k h_{jm}-\frac{1}{4}h^{jm}h^{kn}\dot{h}_{jn}(\partial_i h_{km}+\partial_k h_{im}-\partial_m h_{ik}).
\end{aligned}
\tag{9.136}
$$

Next, let us compute $\tilde{K}^0_i$ at second order. Recall that by definition:

$$
\tilde{K}^0_i=-\frac{\tilde{g}^{00}\tilde{\partial}_i\tilde{g}^{00}}{2(-\tilde{g}^{00})^{1/2}}+\frac{\tilde{g}^{00}\tilde{g}^{0\mu}(\tilde{\partial}_i\tilde{g}_{\mu 0}+\tilde{\partial}_0\tilde{g}_{i\mu}-\tilde{\partial}_\mu\tilde{g}_{i0})}{2(-\tilde{g}^{00})^{1/2}}+\frac{\tilde{g}^{0j}\tilde{g}^{0\mu}(\tilde{\partial}_i\tilde{g}_{j\mu}+\tilde{\partial}_j\tilde{g}_{i\mu}-\tilde{\partial}_\mu\tilde{g}_{ij})}{2(-\tilde{g}^{00})^{1/2}}. \tag{9.137}
$$

Combining the above results in the definition of $\tilde{K}^0_i$ we easily find:

$$
\tilde{K}^0_i\approx\frac{1}{N^4}(1-N^2+2\dot{\pi})(\partial_i N-N\partial_i\dot{\pi}). \tag{9.138}
$$

Notice that this result has indeed no linear order part as we claimed earlier. Thus it suffices to determine $\tilde{K}^i_0$ at linear order for the calculation of ${}^{(3)}R$ up to cubic order. We claim that $\tilde{K}^i_0=\frac{H}{a^2}\partial_i\pi+\mathcal{O}(2)$. Let us illustrate this. Recall that by definition:

$$
\begin{aligned}
\tilde{K}^i_0 &= -\frac{\tilde{g}^{i0}\tilde{\partial}_0\tilde{g}^{00}}{2(-\tilde{g}^{00})^{3/2}}+\frac{\tilde{g}^{i0}\tilde{g}^{0\mu}(2\tilde{\partial}_0\tilde{g}_{0\mu}-\tilde{\partial}_\mu\tilde{g}_{00})}{2(-\tilde{g}^{00})^{1/2}}-\frac{\tilde{g}^{i0}\tilde{g}^{0\alpha}\tilde{\partial}_\alpha\tilde{g}^{00}}{2(-\tilde{g}^{00})^{5/2}}\\
&+\frac{\tilde{g}^{0\kappa}\tilde{g}^{0\alpha}\tilde{g}^{i0}(\tilde{\partial}_\alpha\tilde{g}_{0\kappa}+\tilde{\partial}_0\tilde{g}_{\kappa\alpha}-\tilde{\partial}_\kappa\tilde{g}_{0\alpha})}{2(-\tilde{g}^{00})^{3/2}}-\frac{h^{ij}\tilde{\partial}_j\tilde{g}^{00}}{2(-\tilde{g}^{00})^{3/2}}+\frac{h^{ij}\tilde{g}^{0\mu}(\tilde{\partial}_j\tilde{g}_{0\mu}+\tilde{\partial}_0\tilde{g}_{j\mu}-\tilde{\partial}_\mu\tilde{g}_{j0})}{2(-\tilde{g}^{00})^{1/2}}.
\end{aligned}
\tag{9.139}
$$

Combining the above results indeed yields $\tilde{K}^i_0 \approx \frac{H}{a^2}\partial_i\pi$ at linear order as desired.

Therefore we find that ${}^{(3)}R$ transforms at cubic order as:

$$
\begin{aligned}
{}^{(3)}\tilde{R} = R + 2R^{\mu\nu}n_\mu n_\nu + 2R_{00}\Big[h^{kl}\partial_k\pi\partial_l\pi - \frac{2\dot{\pi}}{a^2}(\partial_k\pi)^2\Big] - 4h^{ij}R_{0j}\partial_i\pi\Big(1 - 2\dot{\pi} + 3\dot{\pi}^2 + \frac{1}{a^2}(\partial_k\pi)^2\Big) \\
+ 2N^2h^{ik}h^{jl}R_{kl}\partial_i\pi\partial_j\pi(1-2\dot{\pi}) - \tilde{K}^2 + \frac{2H}{a^2}\partial_i\pi(1 - N^2 + 2\dot{\pi})(\partial_i N - N\partial_i\dot{\pi}) + \tilde{K}^i_j\tilde{K}^j_i,
\end{aligned}
\tag{9.140}
$$

where $\tilde{K}^i_j$, $\tilde{K}$, R_{00}, R_{ij} and R_{0j} are defined above. We do not write down the terms explicitly and multiply out parentheses since it turns out that the result does not allow for many simplifications.

Finally, we also have to know the Stückelberg trick of the perturbations δK^μ_ν, δK and δg^{00}. Recall that by definition $\delta g^{00} = 1 + g^{00}$. Henceforth it follows that:

$$
\delta\tilde{g}^{00} = 1 + \tilde{g}^{00} = 1 - \frac{1}{N^2}(1+\dot{\pi})^2 + h^{ij}\partial_i\pi\partial_j\pi. \tag{9.141}
$$

Next, it can be seen by definition of δK^μ_ν that $\delta\tilde{K}^\mu_\nu = \tilde{K}^\mu_\nu - H(t+\pi)(\delta^\mu_\nu + \tilde{n}^\mu\tilde{n}_\nu)$. Therefore it follows at cubic order that [38]:

$$
\begin{aligned}
\delta\tilde{K}^i_j &\approx \tilde{K}^i_j - \Big(H + \dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3\Big)\delta^i_j \\
\delta\tilde{K}^0_0 &= \tilde{K}^0_0 \\
\delta\tilde{K}^0_i &= \tilde{K}^0_i \\
\delta\tilde{K}^i_0 &\approx \tilde{K}^i_0 - H(t+\pi)\tilde{n}^i\tilde{n}_0 \\
&\approx \tilde{K}^i_0 + \Big(H + \dot{H}\pi\Big)\frac{\tilde{g}^{i0}}{\tilde{g}^{00}} \\
&\approx \tilde{K}^i_0 - N^2h^{ij}(1-2\dot{\pi})(H+\dot{H}\pi)\partial_j\pi \\
&\approx \tilde{K}^i_0 - N^2H(1-2\dot{\pi})h^{ij}\partial_j\pi - \frac{\dot{H}}{a^2}\pi\partial_i\pi \\
\delta\tilde{K} &= \tilde{K} - 3H(t+\pi) \approx \tilde{K} - 3\Big(H + \dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3\Big).
\end{aligned}
\tag{9.142}
$$

Note that $\delta\tilde{K}^0_i$ has no zeroth and linear order part. Therefore it suffices to expand $\delta\tilde{K}^i_0$ to second order as these quantities will always appear in the form $\delta K^i_0\delta K^0_i$ in the EFT action. Note however that so far we have computed $\tilde{K}^i_0$ only at linear order, whereas we need it to second order. Let us derive the expression for this quantity at second order. Recall the expression (9.139). Combining the above findings in equation (9.139) yields at second order:

$$
\tilde{K}^i_0 \approx \frac{N}{2}(1-2\dot{\pi})h^{ij}h^{kl}\dot{h}_{jk}\partial_l\pi - \frac{1}{a^4}\partial_j\pi\partial_i\partial_j\pi. \tag{9.143}
$$

Note that indeed $\tilde{K}^i_0 \approx \frac{H}{a^2}\partial_i\pi$ at linear order.

The Stückelberg tricks for the perturbations read:

[38] Using the fact that the unit vector transforms as: $\tilde{n}_\mu = -\tilde{\partial}_\mu\tilde{t}\tilde{N} = -\delta^0_\mu\tilde{N}$ and $\tilde{n}^\mu = (1/\tilde{N}, -\tilde{g}^{i0}\tilde{N})$

$$\delta\tilde{K}^i_j = \delta K^i_j - \Big(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3\Big)\delta^i_j - (1-2\dot{\pi})h^{ik}\partial_j N\partial_k\pi + Nh^{ik}\partial_k\pi\partial_j\dot{\pi}(1-2\dot{\pi}) - \frac{1}{a^4}\partial_i\pi\partial_j\partial_k\pi\partial_k\pi$$
$$+ \frac{1}{a^2}\partial_0 N\partial_i\pi\partial_j\pi - \frac{\ddot{\pi}}{a^2}\partial_i\pi\partial_j\pi - Nh^{ik}\partial_j\partial_k\pi(1-\dot{\pi}+\dot{\pi}^2) + Nh^{ik}\partial_j\pi\partial_k\dot{\pi}(1-2\dot{\pi}) - \frac{1}{2a^4}(\partial_k\pi)^2\partial_i\partial_j\pi$$
$$+ \frac{N}{2}h^{ik}h^{lm}(1-\dot{\pi})(\partial_m\pi)(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk}) - \frac{N}{2}h^{ik}h^{lm}\dot{h}_{kl}\partial_m\pi\partial_j\pi(1-2\dot{\pi})$$
$$- h^{ik}(1-\dot{\pi})\partial_k N\partial_j\pi + \frac{N}{4}(1-2\dot{\pi})h^{ik}h^{mn}\dot{h}_{jk}\partial_m\pi\partial_n\pi + \mathcal{O}(4), \tag{9.144}$$

$$\delta\tilde{K} = \delta K - 3\Big(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3\Big) + \frac{1}{a^2}\partial_0 N(\partial_k\pi)^2 + \frac{N}{2}\dot{h}^{ij}\partial_i\pi\partial_j\pi(1-2\dot{\pi}) + 2Nh^{ij}\partial_i\dot{\pi}\partial_j\pi(1-2\dot{\pi})$$
$$- \frac{\ddot{\pi}}{a^2}(\partial_k\pi)^2 - N(1-\dot{\pi})\partial_i h^{ik}\partial_k\pi - Nh^{ik}\Big(1-\dot{\pi}+\dot{\pi}^2 + \frac{3}{2a^2}(\partial_k\pi)^2\Big)\partial_i\partial_k\pi$$
$$- 2h^{ik}(1-\dot{\pi})\partial_k\pi\partial_i N + \frac{N}{4}h^{ij}h^{kl}\dot{h}_{ij}(1-4\dot{\pi})\partial_k\pi\partial_l\pi - \frac{N}{2}h^{il}h^{kj}(1-\dot{\pi})\partial_l\pi\partial_i h_{kj} + \mathcal{O}(4), \tag{9.145}$$

$$\delta\tilde{K}^i_0 = \frac{N}{2}h^{ij}h^{kl}\dot{h}_{jk}\partial_l\pi - \frac{1}{a^4}\partial_j\pi\partial_i\partial_j\pi - N^2 H h^{ij}\partial_j\pi - \frac{\dot{H}}{a^2}\pi\partial_i\pi + \mathcal{O}(3), \tag{9.146}$$

$$\delta\tilde{K}^0_i = \frac{1}{N^4}(1-N^2+2\dot{\pi})(\partial_i N - N\partial_i\dot{\pi}) + \mathcal{O}(3). \tag{9.147}$$

9.11 Expansion of the Horndeski EFT action

The idea of this section is to provide some details of how the EFT action (7.74) has been expanded up to fourth order in the perturbations. We will adopt notation in which $(\partial_k^2\pi)(\partial_k\pi)^2$ means $(\partial_i\partial_i\pi)(\partial_j\pi\partial_j\pi)$ and so on. We will expand to fourth order in π and first order in γ. Furthermore, we will ignore terms of the type $\pi\pi\pi\gamma$ since we are interested in the process $\pi\pi\to\pi\pi$. Recall that $d^4x\sqrt{-g}$ and ${}^{(4)}R$ (we will omit ${}^{(4)}$ in the following) are invariant under the Stückelberg trick of the EFT action. The other transformations we derived before. Note that $\sqrt{-g} = N\sqrt{h}$ where $h = \det(h_{ij})$. By definition of h_{ij} we easily see that:

$$\begin{aligned} h &= a^6(t)(1-2\Psi)^3\det(e^\gamma) \\ &= a^6(t)(1-2\Psi)^3 e^{\mathrm{Tr}(\gamma)} \\ &= a^6(t)(1-2\Psi)^3, \end{aligned} \tag{9.148}$$

where we used that $\mathrm{Tr}(\gamma) = \gamma_{ii} = 0$. Therefore it follows that:

$$\begin{aligned} \sqrt{-g} &= \sqrt{1+2\Phi}\,a^3(t)(1-2\Psi)^{3/2} \\ &\approx a^3(t)\Big[1 - 3\Psi + \frac{3}{2}\Psi^2 + \frac{1}{2}\Psi^3 + \frac{3}{8}\Psi^4 + \Phi - 3\Phi\Psi + \frac{3}{2}\Psi^2\Phi \\ &+ \frac{1}{2}\Phi\Psi^3 - \frac{1}{2}\Phi^2 + \frac{3}{2}\Phi^2\Psi - \frac{3}{4}\Phi^2\Psi^2 + \frac{1}{2}\Phi^3 - \frac{3}{2}\Phi^3\Psi - \frac{5}{8}\Phi^4\Big]. \end{aligned} \tag{9.149}$$

The Ricci scalar $R = g^{\mu\nu}R_{\mu\nu} = g^{00}R_{00} + g^{ij}R_{ij}$ is straightforward but tedious to compute using their definitions. Up to fourth order in π and second order in γ we find the following:

$$
\begin{aligned}
R_{00} \approx\ & \frac{1}{a^2}(1+4\Psi+12\Psi^2)(\delta_{ij}-\gamma_{ij})\partial_i\Phi\partial_j\Psi + \frac{1}{a^2}(\partial_k^2\Phi)(1+2\Psi+4\Psi^2+8\Psi^3) - \frac{1}{a^2}(1+2\Psi)\gamma_{ij}\partial_i\partial_j\Phi \\
& + 3\dot{\Psi}^2(1+4\Psi+12\Psi^2) - 3\frac{\ddot{a}}{a} + 3\ddot{\Psi}(1+2\Psi+4\Psi^2+8\Psi^3) + 3H\dot{\Phi}(1-2\Phi+4\Phi^2-8\Phi^3) \\
& - 3\dot{\Psi}\dot{\Phi}(1+2\Psi-2\Phi+4\Phi^2+4\Psi^2-4\Phi\Psi) - \frac{1}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\Phi\partial_j\Phi(1+2\Psi-2\Phi+4\Phi^2+4\Psi^2-4\Phi\Psi) \\
& + 6H\dot{\Psi}(1+2\Psi+4\Psi^2+8\Psi^3), \\
g^{00}R_{00} \approx\ & -(1-2\Phi+4\Phi^2-8\Phi^3+16\Phi^4)R_{00} \\
h^{ij}R_{ij} \approx\ & \frac{1}{a^2}(\delta_{ij}-\gamma_{ij})[2(1+6\Psi+24\Psi^2)\partial_i\Psi\partial_j\Psi + (1+4\Psi+12\Psi^2+32\Psi^3)\partial_i\partial_j\Psi] \\
& + (1-2\Phi+4\Phi^2-8\Phi^3+16\Phi^4)(3H^2+3\ddot{a}/a) \\
& - 3(4H\dot{\Psi}+2\ddot{\Psi})(1-2\Phi+2\Psi+4\Phi^2+4\Psi^2-4\Phi\Psi-8\Phi^3+8\Psi^3-8\Phi\Psi^2+8\Psi\Phi^2) \\
& - 6H\dot{\Phi}(1-4\Phi+12\Phi^2-32\Phi^3) + 6\dot{\Phi}\dot{\Psi}(1-4\Phi+2\Psi-8\Psi\Phi+12\Phi^2+4\Psi^2) \\
& + \frac{2}{a^2}(\delta_{ij}-\gamma_{ij})[(1+2\Psi-4\Phi-8\Phi\Psi+4\Psi^2+12\Phi^2)\partial_i\Phi\partial_j\Phi + 3(1+6\Psi+24\Psi^2)\partial_i\Psi\partial_j\Psi] \\
& + 3H\dot{\Phi}(1-4\Phi+12\Phi^2-32\Phi^3) - 3\dot{\Psi}\dot{\Phi}(1+2\Psi-4\Phi-8\Psi\Phi+12\Phi^2+4\Psi^2) \\
& + \frac{1}{a^2}(1-2\Phi+4\Psi-8\Phi\Psi+12\Psi^2+4\Phi^2)(\delta_{ij}-\gamma_{ij})\partial_i\Psi\partial_j\Phi + 9H^2(1-2\Phi+4\Phi^2-8\Phi^3+16\Phi^4) \\
& - 18H\dot{\Psi}(1+2\Psi-2\Phi-4\Phi\Psi+4\Psi^2+4\Phi^2-8\Phi\Psi^2+8\Psi\Phi^2+8\Psi^3-8\Phi^3) \\
& + 9\dot{\Psi}^2(1-2\Phi+4\Psi-8\Phi\Psi+4\Phi^2+12\Psi^2) \\
& - \frac{3}{a^2}(1+6\Psi+24\Psi^2)(\delta_{ij}-\gamma_{ij})\partial_i\Psi\partial_j\Psi - \frac{1}{a^2}(1+2\Psi-4\Phi-8\Phi\Psi+4\Psi^2+12\Phi^2)(\delta_{ij}-\gamma_{ij})\partial_i\Phi\partial_j\Phi \\
& - 6H^2(1-2\Phi+4\Phi^2-8\Phi^3+16\Phi^4) \\
& + 12H\dot{\Psi}(1-2\Phi+2\Psi-4\Phi\Psi+4\Phi^2+4\Psi^2-8\Psi^2\Phi+8\Phi^2\Psi-8\Phi^3+8\Psi^3) \\
& - 6\dot{\Psi}^2(1-2\Phi+4\Psi-8\Phi\Psi+4\Phi^2+12\Psi^2) + \frac{1}{a^2}(1+6\Psi+24\Psi^2)(\delta_{ij}-\gamma_{ij})\partial_i\Psi\partial_j\Psi \\
& - \frac{1}{a^2}(1+2\Psi+4\Psi^2+8\Psi^3-2\Phi+4\Phi^2-8\Phi^3-4\Phi\Psi+8\Psi\Phi^2-8\Psi^2\Phi)(\delta_{ij}-\gamma_{ij})\partial_i\partial_j\Phi \\
& + \frac{3}{a^2}(1+4\Psi+12\Psi^2+32\Psi^3)(\delta_{ij}-\gamma_{ij})\partial_i\partial_j\Psi \\
R \approx\ & -\frac{2}{a^2}(1-2\Phi+2\Psi+4\Phi^2+4\Psi^2-4\Phi\Psi-8\Phi^3+8\Psi^3-8\Psi^2\Phi+8\Phi^2\Psi)(\delta_{ij}-\gamma_{ij})\partial_i\partial_j\Phi \\
& + 6(H^2+\ddot{a}/a)(1-2\Phi+4\Phi^2-8\Phi^3+16\Phi^4) \\
& - 9\ddot{\Psi}(1-2\Phi+2\Psi+4\Phi^2+4\Psi^2-4\Phi\Psi-8\Phi^3+8\Psi^3+8\Phi^2\Psi-8\Psi^2\Phi) \\
& - 6H\dot{\Phi}(1-4\Phi+12\Phi^2-32\Phi^3) + 6\dot{\Psi}\dot{\Phi}(1+2\Psi-4\Phi-8\Phi\Psi+12\Phi^2+4\Psi^2) \\
& + \frac{2}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\Phi\partial_j\Phi(1+2\Psi-4\Phi-8\Phi\Psi+12\Phi^2+4\Psi^2) - 24H\dot{\Psi}(1-2\Phi+2\Psi-4\Phi\Psi \\
& + 4\Phi^2+4\Psi^2-8\Phi\Psi^2+8\Psi\Phi^2-8\Phi^3+8\Psi^3) + \frac{6}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\Psi\partial_j\Psi(1+6\Psi+24\Psi^2) \\
& + \frac{4}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\partial_j\Psi(1+4\Psi+12\Psi^2+32\Psi^3).
\end{aligned}
\tag{9.150}
$$

Next, we have to compute K^μ_ν and K. Using the definition $K^\mu_\nu = (g^{\mu\beta} + n^\mu n^\beta)\nabla_\beta n_\nu$ we easily find that $K^0_0 = K^0_i = K^i_0 = 0$ using the expressions for n^μ and the metric. And it follows that $K^i_j = \frac{1}{2N}h^{il}\partial_0 h_{lj}$. Plugging in the metric gives that:

$$
\begin{aligned}
K^i_j \approx{}& H(1-\Phi+\frac{3}{2}\Phi^2-\frac{5}{2}\Phi^3+\frac{35}{8}\Phi^4)(\delta^i_j+\delta^{il}\gamma_{lj}+\gamma^{il}\delta_{lj})\\
&-\dot\Psi(1-\Phi+2\Psi+4\Psi^2+\frac{3}{2}\Phi^2-2\Psi\Phi-\frac{5}{2}\Phi^3+8\Psi^3+3\Psi\Phi^2-4\Psi^2\Phi)(\delta^i_j+\gamma^{il}\delta_{lj}+\delta^{il}\gamma_{lj})\\
&+\delta^{il}\dot\gamma_{lj}(1-\Phi+\frac{3}{2}\Phi^2),\\
K=K^\mu_\mu \approx{}& 3H(1-\Phi+\frac{3}{2}\Phi^2-\frac{5}{2}\Phi^3+\frac{35}{8}\Phi^4)-3\dot\Psi(1-\Phi+2\Psi+4\Psi^2+\frac{3}{2}\Phi^2\\
&-2\Psi\Phi-\frac{5}{2}\Phi^3+8\Psi^3+3\Psi\Phi^2-4\Psi^2\Phi).
\end{aligned}
\tag{9.151}
$$

From this we trivially find the perturbations $\delta K^i_j = K^i_j - H\delta^i_j$ and $\delta K = K - 3H$:

$$
\begin{aligned}
\delta K^i_j \approx{}& H(1-\Phi+\frac{3}{2}\Phi^2-\frac{5}{2}\Phi^3+\frac{35}{8}\Phi^4)(\delta^{il}\gamma_{lj}+\gamma^{il}\delta_{lj})\\
&+H(-\Phi+\frac{3}{2}\Phi^2-\frac{5}{2}\Phi^3+\frac{35}{8}\Phi^4)\delta^i_j\\
&-\dot\Psi(1-\Phi+2\Psi+4\Psi^2+\frac{3}{2}\Phi^2-2\Psi\Phi-\frac{5}{2}\Phi^3+8\Psi^3+3\Psi\Phi^2-4\Psi^2\Phi)(\delta^i_j+\gamma^{il}\delta_{lj}+\delta^{il}\gamma_{lj})\\
&+\delta^{il}\dot\gamma_{lj}(1-\Phi+\frac{3}{2}\Phi^2),\\
\delta K \approx{}& 3H(-\Phi+\frac{3}{2}\Phi^2-\frac{5}{2}\Phi^3+\frac{35}{8}\Phi^4)-3\dot\Psi(1-\Phi+2\Psi+4\Psi^2\\
&+\frac{3}{2}\Phi^2-2\Psi\Phi-\frac{5}{2}\Phi^3+8\Psi^3+3\Psi\Phi^2-4\Psi^2\Phi).
\end{aligned}
\tag{9.152}
$$

Next, we can compute some quantities after performing the Stückelberg trick. Consider $\tilde g^{00}$, $\delta\tilde g^{00}$ and $(\delta\tilde g^{00})^2$. Using the Stückelberg tricks we derived earlier and the metric we find that:

$$
\begin{aligned}
\tilde g^{00} \approx{}& -(1-2\Phi+4\Phi^2-8\Phi^3+16\Phi^4+2\dot\pi-4\dot\pi\Phi-2\Phi\dot\pi^2+\dot\pi^2+8\dot\pi\Phi^2+4\dot\pi^2\Phi^2-16\dot\pi\Phi^3)\\
&+\frac{1}{a^2}(\delta_{ij}-\gamma_{ij})(1+2\Psi+4\Psi^2)\partial_i\pi\partial_j\pi,\\
\delta\tilde g^{00} \approx{}& -(-2\Phi+4\Phi^2-8\Phi^3+16\Phi^4+2\dot\pi-4\dot\pi\Phi-2\Phi\dot\pi^2+\dot\pi^2+8\dot\pi\Phi^2+4\dot\pi^2\Phi^2-16\dot\pi\Phi^3)\\
&+\frac{1}{a^2}(\delta_{ij}-\gamma_{ij})(1+2\Psi+4\Psi^2)\partial_i\pi\partial_j\pi.\\
(\delta\tilde g^{00})^2 \approx{}& 4\dot\pi^2-8\Phi\dot\pi-20\dot\pi^2\Phi-16\Phi\dot\pi^3+4\Phi^2+32\Phi^2\dot\pi+64\Phi^2\dot\pi^2-16\Phi^3-96\Phi^3\dot\pi\\
&+48\Phi^4+4\dot\pi^3+\dot\pi^4+\frac{1}{a^4}\partial_i\pi\partial_j\pi\partial_i\pi\partial_j\pi\\
&-\frac{2}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\pi\partial_j\pi(-2\Phi+4\Phi^2+2\dot\pi-4\dot\pi\Phi-4\Phi\Psi+4\dot\pi\Psi+\dot\pi^2).
\end{aligned}
\tag{9.153}
$$

The Stückelberg trick of δK yields:

$$
\begin{aligned}
\delta\tilde{K} \approx\ & 3H(-\Phi + \frac{3}{2}\Phi^2 - \frac{5}{2}\Phi^3 + \frac{35}{8}\Phi^4) - 3\dot{\Psi}(1 - \Phi + 2\Psi + 4\Psi^2 + \frac{3}{2}\Phi^2 - 2\Psi\Phi \\
& - \frac{5}{2}\Phi^3 + 8\Psi^3 + 3\Psi\Phi^2 - 4\Psi^2\Phi) - 3(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3) + \frac{1}{a^2}\dot{\Phi}(1-\Phi)(\partial_k\pi)^2 \\
& + \frac{H}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\pi\partial_j\pi(\frac{1}{2} + \Psi + \frac{1}{2}\Phi - 4\dot{\pi} - 8\Psi\dot{\pi} + \Phi\Psi - 4\dot{\pi}\Phi - \frac{1}{4}\Phi^2 + 2\Psi^2) \\
& + \frac{\dot{\Psi}}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\pi\partial_j\pi(-\frac{1}{2} - 2\Psi - 6\Psi^2 + 4\dot{\pi} + 16\dot{\pi}\Psi - \frac{1}{2}\Phi - 2\Phi\Psi + 4\dot{\pi}\Phi + \frac{1}{4}\Phi^2) \\
& - \frac{1}{2a^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi + \frac{2}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\dot{\pi}\partial_j\pi(1 + \Phi + 2\Psi + 2\Psi\Phi \\
& - \frac{1}{2}\Phi^2 + 4\Psi^2 - 2\dot{\pi} - 2\dot{\pi}\Phi - 4\dot{\pi}\Psi) \\
& - \frac{\ddot{\pi}}{a^2}(\partial_k\pi)^2 + \frac{1}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\Psi\partial_j\pi(5 + 5\Phi - 5\dot{\pi} - 5\dot{\pi}\Phi - \frac{5}{2}\Phi^2 \\
& + 12\Psi + 12\Phi\Psi + 36\Psi^2 - 12\dot{\pi}\Psi) \\
& - \frac{1}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(1 + 2\Psi + \Phi - \dot{\pi} - \frac{1}{2}\Phi^2 + 4\Psi^2 - 2\Psi\dot{\pi} - \Phi\dot{\pi} + \dot{\pi}^2 \\
& + \frac{3}{2a^2}(\partial_k\pi)^2 + 2\Phi\Psi + \frac{1}{2}\Phi^3 + 8\Psi^3 + 4\Psi^2\Phi \\
& - \Phi^2\Psi + \frac{1}{2}\Phi^2\dot{\pi} - 4\Psi^2\dot{\pi} - 2\Psi\Phi\dot{\pi} + 2\Psi\dot{\pi}^2 + \Phi\dot{\pi}^2 + \frac{3}{a^2}\Psi(\partial_k\pi)^2 + \frac{3}{2a^2}\Phi(\partial_k\pi)^2) \\
& - \frac{2}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\Phi\partial_j\pi(1 - \Phi + 2\Psi - \dot{\pi} + \frac{3}{2}\Phi^2 + \dot{\pi}\Phi - 2\Psi\Phi - 2\dot{\pi}\Psi + 4\Psi^2) \qquad (9.154)
\end{aligned}
$$

$$
\begin{aligned}
\delta\tilde{g}^{00}\delta\tilde{K} \approx\ & \frac{3H}{a^2}(\partial_k\pi)^2(-\Phi-2\Phi\Psi+\frac{3}{2}\Phi^2)+3H(2\Phi\dot{\pi}-2\Phi^2-7\Phi^2\dot{\pi}-2\Phi^2\dot{\pi}^2+7\Phi^3+19\Phi^3\dot{\pi}-19\Phi^4) \\
& -\frac{3\dot{\Psi}}{a^2}(\partial_k\pi)^2(1-\Phi+4\Psi)-3\dot{\Psi}(-2\dot{\pi}-4\dot{\pi}\Psi-8\dot{\pi}\Psi^2+2\Phi+4\Phi\Psi+8\Phi\Psi^2+6\dot{\pi}\Phi+12\dot{\pi}\Phi\Psi+2\Phi\dot{\pi}^2 \\
& -6\Phi^2-12\Phi^2\Psi-15\dot{\pi}\Phi^2+15\Phi^3)-\frac{3}{a^2}(\dot{H}\pi+\frac{1}{2}\ddot{H}\pi^2+2\dot{H}\pi\Psi)(\partial_k\pi)^2 \\
& -3\dot{H}\pi(2\Phi-4\Phi^2+8\Phi^3-2\dot{\pi}+4\dot{\pi}\Phi+2\Phi\dot{\pi}^2-8\dot{\pi}\Phi^2)-\frac{3}{2}\ddot{H}\pi^2(2\Phi-4\Phi^2-2\dot{\pi}+4\dot{\pi}\Phi) \\
& -\dddot{H}\pi^3(\Phi-\dot{\pi})+\frac{2}{a^2}\dot{\Phi}(\Phi-\dot{\pi})(\partial_k\pi)^2+\frac{H}{2a^4}(\partial_k\pi)^4 \\
& +\frac{H}{a^2}(-\dot{\pi}-2\dot{\pi}\Psi+8\dot{\pi}^2+\Phi+2\Phi\Psi-7\dot{\pi}\Phi-\Phi^2)(\partial_k\pi)^2-\frac{\dot{\Psi}(\Phi-\dot{\pi})}{a^2}(\partial_k\pi)^2 \\
& +\frac{2}{a^4}\partial_i\dot{\pi}\partial_i\pi(\partial_k\pi)^2+\frac{2}{a^2}\partial_i\dot{\pi}\partial_i\pi(-2\dot{\pi}-4\dot{\pi}\Psi+2\Phi+4\Phi\Psi-2\Phi\dot{\pi}-2\Phi^2+4\dot{\pi}^2) \\
& -\frac{2}{a^2}\ddot{\pi}(\Phi-\dot{\pi})(\partial_k\pi)^2+\frac{5}{a^4}\partial_i\Psi\partial_i\pi(\partial_k\pi)^2 \\
& +\frac{1}{a^2}\partial_i\Psi\partial_i\pi(-10\dot{\pi}-24\dot{\pi}\Psi+10\dot{\pi}^2+10\Phi+24\Phi\Psi-10\Phi^2) \\
& -\frac{1}{a^4}(\partial_k^2\pi)(\partial_k\pi)^2(1+4\Psi+\Phi-\dot{\pi}) \\
& -\frac{1}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\partial_j\pi(3\Phi^3-\Phi^2\dot{\pi}-4\Phi^2\Psi-2\Phi^2+2\Phi\dot{\pi}^2+8\Phi\Psi^2+4\Phi\Psi \\
& +2\Phi-2\dot{\pi}^3+4\dot{\pi}^2\Psi+2\dot{\pi}^2-8\dot{\pi}\Psi^2-4\dot{\pi}\Psi-2\dot{\pi}+\frac{3}{a^2}(\Phi-\dot{\pi})(\partial_k\pi)^2)-\frac{2}{a^4}\partial_i\Phi\partial_i\pi(\partial_k\pi)^2 \\
& -\frac{2}{a^2}\partial_i\pi\partial_i\Phi(-6\Phi^2+4\dot{\pi}\Phi+4\Phi\Psi+2\Phi+2\dot{\pi}^2-4\Psi\dot{\pi}-2\dot{\pi})-3H\dot{\pi}^2(-\Phi+\frac{3}{2}\Phi^2) \\
& +3\dot{\pi}^2\dot{\Psi}(1-\Phi+2\Psi)+3\dot{\pi}^2(\dot{H}\pi+\frac{1}{2}\ddot{H}\pi^2)-\frac{H}{2a^2}\dot{\pi}^2(\partial_k\pi)^2-\frac{2}{a^2}\dot{\pi}^2\partial_i\dot{\pi}\partial_i\pi-\frac{5}{a^2}\dot{\pi}^2\partial_i\Psi\partial_i\pi \\
& +\frac{1}{a^2}\dot{\pi}^2(\partial_k^2\pi)(1+2\Psi+\Phi-\dot{\pi})+\frac{2}{a^2}\dot{\pi}^2\partial_i\Phi\partial_i\pi
\end{aligned} \tag{9.155}
$$

$$
\begin{aligned}
(\delta\tilde{K})^2 \approx\; & 9H^2(\Phi^2 - 3\Phi^3 + \frac{29}{4}\Phi^4) - 18H\dot{\Psi}(-\Phi - 2\Phi\Psi - 4\Phi\Psi^2 + \frac{5}{2}\Phi^2 + 5\Phi^2\Psi - \frac{11}{2}\Phi^3) \\
& - 18H(\dot{H}(-\Phi + \frac{3}{2}\Phi^2 - \frac{5}{2}\Phi^3) + \frac{1}{2}\ddot{H}\pi^2(-\Phi + \frac{3}{2}\Phi^2) - \frac{1}{6}\dddot{H}\pi^3\Phi) - \frac{6H}{a^2}\dot{\Phi}\Phi(\partial_k\pi)^2 \\
& + \frac{6H^2}{a^2}(\partial_k\pi)^2(-\frac{1}{2}\Phi - \Phi\Psi + 4\Phi\dot{\pi} + \frac{1}{4}\Phi^2) + \frac{3H}{a^2}\Phi\dot{\Psi}(\partial_k\pi)^2 \\
& + \frac{12H}{a^2}\partial_i\dot{\pi}\partial_i\pi(-\Phi + \frac{1}{2}\Phi^2 - 2\Phi\Psi + 2\Phi\dot{\pi}) + \frac{6H}{a^2}\ddot{\pi}\Phi(\partial_k\pi)^2 \\
& + \frac{6H}{a^2}\partial_i\Psi\partial_i\pi(-5\Phi - 12\Phi\Psi + 5\Phi\dot{\pi} + \frac{5}{2}\Phi^2) - \frac{6H}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(-\Phi - 2\Phi\Psi - 4\Phi\Psi^2 + \Phi\dot{\pi} + 2\dot{\pi}\Psi\Phi \\
& - \Phi\dot{\pi}^2 + \frac{1}{2}\Phi^2 + \Phi^2\Psi - \frac{1}{2}\Phi^2\dot{\pi} - \frac{1}{2}\Phi^3 - \frac{3}{2a^2}\Phi(\partial_k\pi)^2) - \frac{12H}{a^2}\partial_i\Phi\partial_i\pi(-\Phi - 2\Phi\Psi + \Phi\dot{\pi} + \frac{5}{2}\Phi^2) \\
& + 9\dot{\Psi}^2(1 + 4\Psi + 12\Psi^2 - 2\Phi - 8\Phi\Psi) + 18\dot{\Psi}(\dot{H}\pi(1 - \Phi + 2\Psi + 4\Psi^2 + \frac{3}{2}\Phi^2 - 2\Phi\Psi) \\
& + \frac{1}{2}\ddot{H}\pi^2(1 - \Phi + 2\Psi) + \frac{1}{6}\dddot{H}\pi^3) \\
& - \frac{6}{a^2}\dot{\Psi}\dot{\Phi}(\partial_k\pi)^2 - \frac{6H}{a^2}\dot{\Psi}(\partial_k\pi)^2(\frac{1}{2} + 2\Psi - 4\dot{\pi}) + \frac{3}{a^2}\dot{\Psi}^2(\partial_k\pi)^2 \\
& - \frac{12\dot{\Psi}}{a^2}\partial_i\dot{\pi}\partial_i\pi(1 + 4\Psi - 2\dot{\pi}) - \frac{6\dot{\Psi}}{a^2}\partial_i\Psi\partial_i\pi(5 + 22\Psi - 5\dot{\pi}) + \frac{6\ddot{\pi}\dot{\Psi}}{a^2}(\partial_k\pi)^2 \\
& + \frac{6\dot{\Psi}}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(1 + 4\Psi + 12\Psi^2 - \dot{\pi} - 4\dot{\pi}\Psi + \dot{\pi}^2 + \frac{3}{2a^2}(\partial_k\pi)^2) + \frac{12\dot{\Psi}}{a^2}\partial_i\Phi\partial_i\pi(1 - 2\Phi - \dot{\pi} + 4\Psi) \\
& + 9(\dot{H}^2\pi^2 + \dot{H}\ddot{H}\pi^3 + \frac{1}{2}\dot{H}\dddot{H}\pi^4 + \frac{1}{4}\ddot{H}^2\pi^4) - \frac{6\dot{H}}{a^2}\dot{\Phi}\pi(\partial_k\pi)^2 - \frac{12}{a^2}\dot{H}\pi\partial_i\dot{\pi}\partial_i\pi(1 + \Phi + 2\Psi - \dot{\pi}) \\
& + \frac{3\dot{H}\dot{\Psi}}{a^2}\pi(\partial_k\pi)^2 - \frac{6H\dot{H}}{a^2}\pi(\frac{1}{2} + \Psi + \frac{1}{2}\Phi - 4\dot{\pi})(\partial_k\pi)^2 - \frac{3H\ddot{H}}{2a^2}\pi^2(\partial_k\pi)^2 - \frac{6\ddot{H}}{a^2}\pi^2\partial_i\dot{\pi}\partial_i\pi + \frac{6\dot{H}\ddot{\pi}}{a^2}\pi(\partial_k\pi)^2 \\
& - \frac{6\dot{H}}{a^2}\pi\partial_i\Psi\partial_i\pi(5 + 5\Phi - 5\dot{\pi} + 12\Psi) + \frac{6}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(\frac{1}{2}\ddot{H}\pi^2 + \dot{H}\pi + 2\dot{H}\pi\Psi + \dot{H}\pi\Phi - \dot{H}\pi\dot{\pi} \\
& + \ddot{H}\Psi\pi^2 + \frac{1}{2}\ddot{H}\pi^2\Phi \\
& - \frac{1}{2}\ddot{H}\pi^2\dot{\pi} + \frac{1}{6}\dddot{H}\pi^3 - \frac{1}{2}\dot{H}\pi\Phi^2 + 4\dot{H}\pi\Psi^2 - 2\dot{H}\pi\Psi\dot{\pi} - \dot{H}\pi\Phi\dot{\pi} + H\pi\dot{\pi}^2 + 2\dot{H}\pi\Phi\Psi + \frac{3\dot{H}}{2a^2}\pi(\partial_k\pi)^2) \\
& - \frac{15\ddot{H}}{a^2}\pi^2\partial_i\Psi\partial_i\pi + \frac{6\ddot{H}}{a^2}\pi^2\partial_i\Phi\partial_i\pi + \frac{12\dot{H}}{a^2}\pi\partial_i\Phi\partial_i\pi(1 - \Phi + 2\Psi - \dot{\pi}) - \frac{2}{a^4}\dot{\Phi}(\partial_k\pi)^2(\partial_k^2\pi) + \frac{H^2}{4a^4}(\partial_k\pi)^4 \\
& + \frac{2H}{a^4}(\partial_k\pi)^2\partial_i\dot{\pi}\partial_i\pi + \frac{5H}{a^4}(\partial_k\pi)^2\partial_i\Psi\partial_i\pi - \frac{2H}{a^4}(\partial_k\pi)^2\partial_k^2\pi(\frac{1}{2} + 2\Psi + \Phi - \frac{9}{2}\dot{\pi}) - \frac{2H}{a^4}(\partial_k\pi)^2\partial_i\Phi\partial_i\pi \\
& + \frac{\dot{\Psi}}{a^4}(\partial_k\pi)^2(\partial_k^2\pi) + \frac{4}{a^4}(\partial_i\dot{\pi}\partial_i\pi)^2 + \frac{20}{a^4}\partial_i\dot{\pi}\partial_i\pi\partial_j\Psi\partial_j\pi \\
& - \frac{4}{a^4}\partial_i\dot{\pi}\partial_i\pi\partial_k^2\pi(1 + 4\Psi + 2\Phi - 3\dot{\pi}) - \frac{8}{a^4}\partial_i\dot{\pi}\partial_i\pi\partial_j\Phi\partial_j\pi + \frac{2\ddot{\pi}}{a^4}(\partial_k\pi)^2(\partial_k^2\pi) \\
& + \frac{25}{a^4}(\partial_i\Psi\partial_i\pi)^2 - \frac{2}{a^4}\partial_i\Psi\partial_i\pi\partial_k^2\pi(5 + 22\Psi + 6\Phi - 6\dot{\pi}) - \frac{20}{a^4}\partial_i\Psi\partial_i\pi\partial_j\Phi\partial_j\pi \\
& + \frac{1}{a^4}(\delta_{ij}\delta_{kl} - \gamma_{ij}\delta_{kl} - \delta_{ij}\gamma_{kl})\partial_i\partial_j\pi\partial_k\partial_l\pi(\frac{3}{a^2}(\partial_k\pi)^2 + 1 + 4\Psi + 12\Psi^2 - 2\dot{\pi} - 8\dot{\pi}\Psi + 3\dot{\pi}^2 + 2\Phi \\
& + 8\Phi\Psi - 4\dot{\pi}\Phi) \\
& + \frac{4}{a^4}\partial_k^2\pi\partial_i\Phi\partial_i\pi(1 + 4\Psi - 2\dot{\pi}) + \frac{4}{a^4}(\partial_i\Phi\partial_i\pi)^2.
\end{aligned}
\tag{9.156}
$$

We can also compute $\delta\tilde{K}^\mu_\nu\delta\tilde{K}^\nu_\mu$. Recall that $\delta\tilde{K}^0_0\tilde{K}^0_0$ can be ignored at fourth order. Therefore we compute $\delta\tilde{K}^0_i$, $\delta\tilde{K}^i_0$ and $\delta\tilde{K}^i_j$ using the Stückelberg tricks we derived earlier. Recall that $\delta\tilde{K}^0_i$ always appears multiplied by $\delta\tilde{K}^i_0$, hence it suffices to consider the quantities at third order (as their background value is zero). At third order we find that:

$$\begin{aligned}
\delta\tilde{K}_i^0 &\approx 2\partial_i\Phi(\dot{\pi}-\Phi-5\dot{\pi}\Phi+5\Phi^2)-2\partial_i\dot{\pi}(\dot{\pi}-\Phi-3\dot{\pi}\Phi+3\Phi^2),\\
\delta\tilde{K}_0^i &\approx \frac{H}{a^2}\partial_l\pi(1+2\Psi+4\Psi^2+\Phi+2\Phi\Psi-\frac{1}{2}\Phi^2)(\delta_{il}-\delta^{ij}\delta^{kl}\gamma_{jk})-\dot{\Psi}\partial_l\pi(1+4\Psi+\Phi)(\delta_{il}-\delta^{ij}\delta^{kl}\gamma_{jk})\\
&\quad-\frac{1}{a^4}\partial_j\pi\partial_i\partial_j\pi-\frac{H}{a^2}(1+2\Psi+4\Psi^2+2\Phi+4\Phi\Psi)(\delta_{ij}-\gamma_{ij})\partial_j\pi-\frac{\dot{H}}{a^2}\pi\partial_i\pi\\
&\quad+\frac{1}{2a^2}(1+\Phi+2\Psi)\delta^{ij}\dot{\gamma}_{jl}\partial_l\pi.
\end{aligned} \tag{9.157}$$

From which we find $\delta\tilde{K}_0^i\delta\tilde{K}_i^0$ up to quartic order:

$$\begin{aligned}
\delta\tilde{K}_0^i\delta\tilde{K}_i^0 &\approx \frac{2H}{a^2}\partial_i\Phi\partial_i\pi(-\dot{\pi}\Phi+\Phi^2)-2\dot{\Psi}\partial_i\Phi\partial_i\pi(\dot{\pi}-\Phi)-\frac{2}{a^4}\partial_i\Phi\partial_j\pi\partial_i\partial_j\pi(\dot{\pi}-\Phi)\\
&\quad-\frac{2\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)(\partial_k\pi)^2-\frac{2H}{a^2}(-\dot{\pi}\Phi+\Phi^2)\partial_i\dot{\pi}\partial_i\pi+2\dot{\Psi}\partial_i\dot{\pi}\partial_i\pi(\dot{\pi}-\Phi)\\
&\quad+\frac{2}{a^4}\partial_i\dot{\pi}\partial_j\pi\partial_i\partial_j\pi(\dot{\pi}-\Phi)+\frac{2\dot{H}}{a^2}(\dot{\pi}-\Phi)\pi\partial_i\dot{\pi}\partial_i\pi.
\end{aligned} \tag{9.158}$$

The Stückelberg trick for $\delta\tilde{K}_j^i$ gives up to cubic order (which again suffices as it is always contracted with a perturbation $\delta\tilde{K}_i^j$):

$$\begin{aligned}
\delta\tilde{K}_j^i &\approx \delta K_j^i-(\dot{H}\pi+\frac{1}{2}\ddot{H}\pi^2+\frac{1}{6}\dddot{H}\pi^3)\delta_j^i-\frac{1}{a^2}\partial_j\Phi\partial_k\pi(\delta^{ik}+\gamma^{ik})(1+2\Psi-2\dot{\pi}-\Phi)\\
&\quad+\frac{1}{a^2}\partial_k\pi\partial_j\dot{\pi}(\delta^{ik}+\gamma^{ik})(1+2\Psi-2\dot{\pi}-\Phi)\\
&\quad-\frac{1}{a^4}\partial_i\pi\partial_j\partial_k\pi\partial_k\pi+\frac{1}{a^2}\dot{\Phi}\partial_i\pi\partial_j\pi-\frac{\ddot{\pi}}{a^2}\partial_i\pi\partial_j\pi\\
&\quad-\frac{1}{a^2}(\delta^{ik}+\gamma^{ik})\partial_j\partial_k\pi(1+2\Psi+4\Psi^2-\dot{\pi}-2\dot{\pi}\Psi+\dot{\pi}^2+\Phi+2\Phi\Psi\\
&\quad-\dot{\pi}\Phi-\frac{1}{2}\Phi^2)+\frac{1}{a^2}\partial_j\pi\partial_k\dot{\pi}(\delta^{ik}+\gamma^{ik})(1+2\Psi-2\dot{\pi}+\Phi)-\frac{1}{2a^4}(\partial_k\pi)^2\partial_i\partial_j\pi\\
&\quad-\frac{1}{a^2}\partial_m\pi\partial_j\Psi(\delta^{im}-\delta^{ik}\delta^{lm}\gamma_{kl})(1+4\Psi-\dot{\pi}+\Phi)\\
&\quad-\frac{1}{a^2}\partial_m\pi\partial_k\Psi(\delta^{ik}\delta_j^m+\delta^{ik}\delta^{lm}\gamma_{jl}+\gamma^{ik}\delta^{lm}\delta_{jl}+\delta^{ik}\gamma^{lm}\delta_{jl})(1+4\Psi-\dot{\pi}+\Phi)\\
&\quad+\frac{1}{a^2}\partial_m\pi\partial_l\Psi(\delta^{lm}\delta_j^i+\delta^{ik}\delta^{lm}\gamma_{jk}+\delta^{ik}\gamma^{lm}\delta_{jk}+\gamma^{ik}\delta^{lm}\delta_{jk})(1+4\Psi-\dot{\pi}+\Phi)\\
&\quad+\frac{1}{2a^2}(1+\Phi+2\Psi-\dot{\pi})(\partial_l\pi\partial_j\gamma_{il}+\partial_l\pi\partial_i\gamma_{jl}-\partial_l\pi\partial_l\gamma_{ij})\\
&\quad-\frac{H}{a^2}(\delta^{im}-\delta^{ik}\delta^{lm}\gamma_{kl})\partial_m\pi\partial_j\pi(1+2\Psi-2\dot{\pi}+\Phi)+\frac{1}{a^2}\dot{\Psi}\partial_i\pi\partial_j\pi\\
&\quad-\frac{1}{2a^2}\partial_l\pi\partial_j\pi\delta^{ik}\dot{\gamma}_{kl}-\frac{1}{a^2}\partial_k\Phi\partial_j\pi(\delta^{ik}+\gamma^{ik})(1+2\Psi-\Phi-\dot{\pi})\\
&\quad+\frac{H}{2a^2}(\delta_j^i\delta^{mn}+\delta^{ik}\delta^{mn}\gamma_{jk}+\gamma^{ik}\delta^{mn}\delta_{jk}+\delta^{ik}\gamma^{mn}\delta_{jk})\partial_m\pi\partial_n\pi(1+\Phi+2\Psi-2\dot{\pi})\\
&\quad-\frac{\dot{\Psi}}{2a^2}(\partial_k\pi)^2\delta_j^i+\frac{1}{4a^2}(\partial_k\pi)^2\delta^{ik}\dot{\gamma}_{jk}.
\end{aligned} \tag{9.159}$$

Therefore it follows that $\delta\tilde{K}^i_j\delta\tilde{K}^j_i$ to quartic order is found by:

$$
\begin{aligned}
\delta\tilde{K}^i_j\delta\tilde{K}^j_i \approx\, & 3H^2(\Phi^2 - 3\Phi^3 + \frac{29}{4}\Phi^4) - 6\dot{\Psi}H(-\Phi - 2\pi\Psi - 4\Phi\Psi^2 + \frac{5}{2}\Phi^2 + 5\Phi^2\Psi - \frac{11}{2}\Phi^3) \\
& + 3\dot{\Psi}^2(1 + 4\Psi + 12\Psi^2 - 2\Phi - 8\Phi\Psi + 4\Phi^2) - 6H(\dot{H}\pi(-\Phi + \frac{3}{2}\Phi^2 - \frac{5}{2}\Phi^3) \\
& + \frac{1}{2}\ddot{H}\pi^2(-\Phi + \frac{3}{2}\Phi^2) - \frac{1}{6}\dddot{H}\pi^3\Phi) \\
& - \frac{2H}{a^2}\partial_i\Phi\partial_i\pi(-\Phi - 2\Phi\Psi + 2\Phi\dot{\pi} + \frac{5}{2}\Phi^2) + 6\dot{\Psi}(\dot{H}\pi(1 - \Phi + 2\Psi + 4\Psi^2 + \frac{3}{2}\Phi^2 - 2\Psi\Phi) \\
& + \frac{1}{2}\ddot{H}\pi^2(1 - \Phi + 2\Psi) \\
& + \frac{1}{6}\dddot{H}\pi^3) + \frac{2}{a^2}\dot{\Psi}\partial_i\Phi\partial_i\pi(1 + 4\Psi - 2\Phi - 2\dot{\pi}) - \frac{2}{a^2}\dot{\gamma}_{ij}\partial_i\Phi\partial_j\pi \\
& + \frac{4H}{a^2}\partial_i\dot{\pi}\partial_i\pi(-\Phi - 2\Phi\Psi + 2\dot{\pi}\Phi + \frac{1}{2}\Phi^2) - \frac{4}{a^2}\dot{\Psi}\partial_i\dot{\pi}\partial_i\pi(1 + 4\Psi - 2\dot{\pi}) \\
& + \frac{4}{a^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\dot{\pi} + \frac{2H}{a^4}\Phi\partial_i\pi\partial_i\partial_k\pi\partial_k\pi \\
& + \frac{2\dot{\Psi}}{a^4}\partial_i\pi\partial_i\partial_k\pi\partial_k\pi - \frac{2H}{a^2}\Phi\dot{\Phi}(\partial_k\pi)^2 + \frac{2}{a^2}\dot{\Psi}\dot{\Phi}(\partial_k\pi)^2 + \frac{2H}{a^2}\Phi\ddot{\pi}(\partial_k\pi)^2 + \frac{2\ddot{\pi}\dot{\Psi}}{a^2}(\partial_k\pi)^2 \\
& - \frac{2H}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(-\Phi - 2\Phi\Psi - 4\Phi\Psi^2 + \Phi\dot{\pi} + 2\dot{\pi}\Phi\Psi - \Phi\dot{\pi}^2 + \frac{1}{2}\Phi^2 + \Phi^2\Psi - \frac{1}{2}\Phi^2\dot{\pi} - \frac{1}{2}\Phi^3) \\
& + \frac{2\dot{\Psi}}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(1 + 4\Psi + 12\Psi^2 - \dot{\pi} - 4\dot{\pi}\Psi + \dot{\pi}^2) - \frac{2}{a^2}\dot{\gamma}_{ij}\partial_i\partial_j\pi(1 + 2\Psi - \dot{\pi}) + \frac{H}{a^4}\Phi(\partial_k\pi)^2(\partial_k^2\pi) \\
& + \frac{\dot{\Psi}}{a^4}(\partial_k\pi)^2(\partial_k^2\pi) - \frac{2H}{a^2}\partial_i\Psi\partial_i\pi(-\Phi + \frac{1}{2}\Phi^2 - 4\Phi\Psi + \Phi\dot{\pi}) + \frac{2\dot{\Psi}}{a^2}\partial_i\pi\partial_i\Psi(1 + 6\Psi - \dot{\pi}) - \frac{2}{a^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\Psi \\
& - \frac{2H}{a^2}\partial_i\Psi\partial_i\pi(-\Phi - 4\Phi\Psi + \Phi\dot{\pi} + \frac{1}{2}\Phi^2) + \frac{2\dot{\Psi}}{a^2}\partial_i\Psi\partial_i\pi(1 - \dot{\pi} + 6\Psi) - \frac{2}{a^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\Psi \\
& + \frac{6H}{a^2}(\partial_k\pi)^2(-\Phi + \frac{1}{2}\Phi^2 - 4\Phi\Psi + \dot{\pi}\Phi) - \frac{6\dot{\Psi}}{a^2}(\partial_k\pi)^2(1 + 6\Psi - \dot{\pi}) - \frac{2H^2}{a^2}(\partial_k\pi)^2(-\Phi - 2\Phi\Psi + 2\Phi\dot{\pi} + \frac{1}{2}\Phi^2) \\
& + \frac{2H}{a^2}\dot{\Psi}(\partial_k\pi)^2(1 + 4\Psi - 2\dot{\pi}) - \frac{2H}{a^2}\dot{\gamma}_{ij}\partial_i\pi\partial_j\pi - \frac{2H}{a^2}\dot{\Psi}\Phi(\partial_k\pi)^2 - \frac{2\dot{\Psi}^2}{a^2}(\partial_k\pi)^2 \\
& - \frac{2H}{a^2}\partial_i\Phi\partial_i\pi(-\Phi - 2\Phi\Psi + \frac{5}{2}\Phi^2 + \dot{\pi}\Phi) + \frac{2\dot{\Psi}}{a^2}\partial_i\Phi\partial_i\pi(1 + 4\Psi - \dot{\pi} - 2\Phi) \\
& - \frac{2}{a^2}\partial_i\Phi\partial_j\pi\dot{\gamma}_{ij} + \frac{3H^2}{a^2}(\partial_k\pi)^2(-\Phi - 2\Phi\Psi + 2\dot{\pi}\Phi + \frac{1}{2}\Phi^2) - \frac{3H}{a^2}\dot{\Psi}(\partial_k\pi)^2(1 + 4\Psi - 2\dot{\pi}) + \frac{3H}{a^2}\dot{\Psi}\Phi(\partial_k\pi)^2 \\
& + \frac{3\dot{\Psi}^2}{a^2}(\partial_k\pi)^2 + 3\dot{H}\pi(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3) + \frac{3}{2}\ddot{H}\pi^2(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2) + \frac{1}{2}\dddot{H}\dot{H}\pi^4 \\
& + \frac{2}{a^2}\partial_i\Phi\partial_i\pi(\dot{H}\pi(1 + 2\Psi - 2\dot{\pi} - \Phi) + \frac{1}{2}\ddot{H}\pi^2) - \frac{4}{a^2}\partial_i\dot{\pi}\partial_i\pi(\dot{H}\pi(1 + 2\Psi - 2\dot{\pi} - \Phi) + \frac{1}{2}\ddot{H}\pi^2) \\
& + \frac{2}{a^4}\dot{H}\pi\partial_i\pi\partial_i\partial_j\pi\partial_j\pi \\
& - \frac{2\dot{H}}{a^2}\dot{\Phi}\pi(\partial_k\pi)^2 + \frac{2}{a^2}\dot{H}\ddot{\pi}\pi(\partial_k\pi)^2 \\
& + \frac{2}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(\dot{H}\pi(1 + 2\Psi + 4\Psi^2 - \dot{\pi} - 2\dot{\pi}\Psi + \dot{\pi}^2 + \Phi + 2\Phi\Psi - \dot{\pi}\Phi - \frac{1}{2}\Phi^2) \\
& + \frac{1}{2}\ddot{H}\pi^2(1 + 2\Psi - \dot{\pi} + \Phi) + \frac{1}{6}\dddot{H}\pi^3) + \frac{\dot{H}}{a^4}\pi(\partial_k\pi)^2(\partial_k^2\pi) + \frac{2}{a^2}\partial_i\Psi\partial_i\pi(\dot{H}\pi(1 + 4\Psi - \dot{\pi} + \Phi) + \frac{1}{2}\ddot{H}\pi^2) \\
& + \frac{2}{a^2}\partial_i\Psi\partial_i\pi(\dot{H}\pi(1 + 4\Psi - \dot{\pi} + \Phi) + \frac{1}{2}\ddot{H}\pi^2) - \frac{6}{a^2}\dot{H}\pi(\partial_k\pi)^2(1 + 4\Psi - \dot{\pi} + \Phi) - \frac{3\ddot{H}}{a^2}\pi^2(\partial_k\pi)^2 \\
& + \frac{2H}{a^2}(\partial_k\pi)^2(\dot{H}\pi(1 + 2\Psi - 2\dot{\pi} + \Phi) + \frac{1}{2}\ddot{H}\pi^2) - \frac{2\dot{H}}{a^2}\dot{\Psi}\pi(\partial_k\pi)^2 \\
& + \frac{2}{a^2}\partial_i\Phi\partial_i\pi(\dot{H}\pi(1 + 2\Psi - \Phi - \dot{\pi}) + \frac{1}{2}\ddot{H}\pi^2)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{3H\dot{H}}{a^2}(\partial_k\pi)^2(1+\Phi+2\Psi-2\dot{\pi})-\frac{3H\ddot{H}}{2a^2}\pi^2(\partial_k\pi)^2+\frac{3\dot{H}\dot{\Psi}}{a^2}\pi(\partial_k\pi)^2+\frac{1}{a^4}(\partial_i\Phi\partial_i\pi)^2-\frac{2}{a^4}\partial_i\dot{\pi}\partial_i\pi\partial_j\Phi\partial_j\pi\\
&+\frac{2}{a^4}(1+4\Psi-3\dot{\pi})\partial_i\Phi\partial_j\pi\partial_i\partial_j\pi-\frac{2}{a^4}\partial_i\Phi\partial_i\dot{\pi}(\partial_k\pi)^2+\frac{2}{a^4}\partial_i\Phi\partial_i\pi\partial_j\Psi\partial_j\pi+\frac{2}{a^4}\partial_i\Phi\partial_i\Psi(\partial_k\pi)^2\\
&-\frac{2}{a^4}\partial_i\Psi\partial_i\pi\partial_j\Phi\partial_j\pi+\frac{2H}{a^4}\partial_i\Phi\partial_i\pi(\partial_k\pi)^2+\frac{2}{a^4}(\partial_k\Phi)^2(\partial_k\pi)^2-\frac{H}{a^4}\partial_i\Phi\partial_i\pi(\partial_k\pi)^2\\
&+\frac{1}{a^4}(\partial_i\dot{\pi}\partial_i\pi)^2-\frac{2}{a^4}\partial_i\pi\partial_j\dot{\pi}\partial_i\partial_j\pi(1+4\Psi-3\dot{\pi}+2\Phi)+\frac{2}{a^4}(\partial_k\pi)^2(\partial_k\dot{\pi})^2-\frac{2}{a^4}\partial_i\dot{\pi}\partial_i\pi\partial_j\Psi\partial_j\pi\\
&-\frac{2}{a^4}\partial_i\Psi\partial_i\dot{\pi}(\partial_k\pi)^2+\frac{2}{a^4}\partial_i\Psi\partial_i\pi\partial_j\dot{\pi}\partial_j\pi-\frac{2H}{a^4}\partial_i\dot{\pi}\partial_i\pi(\partial_k\pi)^2-\frac{2}{a^4}\partial_i\Phi\partial_i\dot{\pi}(\partial_k\pi)^2\\
&+\frac{H}{a^4}\partial_i\dot{\pi}\partial_i\pi(\partial_k\pi)^2+\frac{2}{a^6}\partial_j\pi\partial_i\partial_k\pi\partial_k\pi\partial_i\partial_j\pi-\frac{2}{a^4}\dot{\Phi}\partial_i\pi\partial_j\pi\partial_i\partial_j\pi+\frac{2\ddot{\pi}}{a^4}\partial_i\pi\partial_j\pi\partial_i\partial_j\pi\\
&+\frac{1}{a^4}(\partial_i\partial_j\pi)^2(1+4\Psi+12\Psi^2-2\dot{\pi}-10\dot{\pi}\Psi+3\dot{\pi}^2+2\Phi+8\Phi\Psi-2\dot{\pi}\Phi)-\frac{2}{a^4}\gamma_{jl}\partial_i\partial_j\pi\partial_i\partial_l\pi\\
&-\frac{2}{a^4}\partial_i\dot{\pi}\partial_j\pi\partial_i\partial_j\pi(1+4\Psi-3\dot{\pi}+2\Phi)+\frac{1}{a^6}(\partial_k\pi)^2(\partial_i\partial_j\pi)^2+\frac{2}{a^4}\partial_i\pi\partial_j\Psi\partial_i\partial_j\pi(1+6\Psi-2\dot{\pi}+2\Phi)\\
&+\frac{2}{a^4}\partial_i\Psi\partial_j\pi\partial_i\partial_j\pi(1+6\Psi-2\dot{\pi}+2\Phi)-\frac{2}{a^4}\partial_i\Psi\partial_i\pi(\partial_k^2\pi)(1+6\Psi-2\dot{\pi}+2\Phi)\\
&-\frac{2}{a^4}\partial_i\partial_j\pi\partial_l\pi\partial_j\gamma_{il}+\frac{1}{a^4}\partial_l\pi\partial_l\gamma_{ij}\partial_i\partial_j\pi+\frac{2H}{a^4}\partial_i\pi\partial_j\pi\partial_i\partial_j\pi(1+4\Psi-3\dot{\pi}+2\Phi)\\
&-\frac{2\dot{\Psi}}{a^4}\partial_i\pi\partial_j\pi\partial_i\partial_j\pi+\frac{2}{a^4}\partial_i\Phi\partial_j\pi\partial_i\partial_j\pi-\frac{H}{a^4}(\partial_k\pi)^2(\partial_k^2\pi)(1+4\Psi-2\dot{\pi})+\frac{\dot{\Psi}}{a^4}(\partial_k\pi)^2(\partial_k^2\pi)\\
&+\frac{1}{a^4}(\partial_i\dot{\pi}\partial_i\pi)^2-\frac{2}{a^4}\partial_i\Psi\partial_i\dot{\pi}(\partial_k\pi)^2-\frac{2}{a^4}\partial_i\Psi\partial_i\pi\partial_j\dot{\pi}\partial_j\pi+\frac{2}{a^4}\partial_i\Psi\partial_i\pi\partial_j\dot{\pi}\partial_j\pi-\frac{2H}{a^4}(\partial_k\pi)^2\partial_i\dot{\pi}\partial_i\pi\\
&-\frac{2}{a^4}\partial_i\Phi\partial_i\pi\partial_j\dot{\pi}\partial_j\pi+\frac{H}{a^4}(\partial_k\pi)^2\partial_i\dot{\pi}\partial_i\pi+\frac{1}{a^4}(\partial_i\Psi\partial_i\pi)^2+\frac{2}{a^4}(\partial_k\Psi)^2(\partial_k\pi)^2-\frac{2}{a^4}(\partial_k\pi)^2\partial_i\Psi\partial_i\pi\\
&+\frac{2H}{a^4}\partial_i\Psi\partial_i\pi(\partial_k\pi)^2+\frac{2}{a^4}\partial_i\Psi\partial_i\Phi(\partial_k\pi)^2-\frac{H}{a^4}(\partial_k\pi)^2\partial_i\Psi\partial_i\pi+\frac{1}{a^4}(\partial_i\Psi\partial_i\pi)^2-\frac{2}{a^4}(\partial_i\Psi\partial_i\pi)^2\\
&+\frac{2H}{a^4}(\partial_k\pi)^2\partial_j\Psi\partial_j\pi+\frac{2}{a^4}\partial_i\Phi\partial_i\pi\partial_j\Psi\partial_j\pi-\frac{H}{a^4}(\partial_k\pi)^2\partial_i\Psi\partial_i\pi+\frac{3}{a^4}(\partial_i\Psi\partial_i\pi)^2-\frac{2H}{a^4}\partial_i\Psi\partial_i\pi(\partial_k\pi)^2\\
&-\frac{2}{a^4}\partial_i\Psi\partial_i\pi\partial_j\Phi\partial_j\pi+\frac{3H}{a^4}\partial_i\Psi\partial_i\pi(\partial_k\pi)^2+\frac{H^2}{a^4}(\partial_k\pi)^4+\frac{2H}{a^4}\partial_i\Phi\partial_i\pi(\partial_k\pi)^2\\
&-\frac{H^2}{a^4}(\partial_k\pi)^4+\frac{1}{a^4}(\partial_i\Phi\partial_i\pi)^2-\frac{H}{a^4}\partial_i\Phi\partial_i\pi(\partial_k\pi)^2+\frac{3H^2}{4a^4}(\partial_k\pi)^4.
\end{aligned}
\tag{9.160}
$$

Next, we need the expressions for $\delta\tilde{g}^{00}(\delta\tilde{K})^2$ and $\delta\tilde{g}^{00}(\delta\tilde{K}^i_j\delta\tilde{K}^j_i)$ for the term $\sim m_5^2$. Note also that $\delta\tilde{g}^{00}\delta\tilde{K}^i_0\tilde{K}^0_i \propto \mathcal{O}(5)$, so this term can be ignored in the action. We find the following results:

$$
\begin{aligned}
\delta\tilde{g}^{00}(\delta\tilde{K})^2 \approx & -9H^2(2\Phi^2\dot{\pi}+\Phi^2\dot{\pi}^2-2\Phi^3-10\dot{\pi}\Phi^3+10\Phi^4)+\frac{9H^2}{a^2}\Phi^2(\partial_k\pi)^2 \\
& +18H\dot{\Psi}(-2\Phi\dot{\pi}-4\dot{\pi}\Phi\Psi-\Phi\dot{\pi}^2+2\Phi^2+4\Phi^2\Psi+9\Phi^2\dot{\pi}-9\Phi^3)+\frac{18H}{a^2}\dot{\Psi}\Phi(\partial_k\pi)^2 \\
& +18H(\dot{H}\pi(-2\Phi\dot{\pi}-\Phi\dot{\pi}^2+2\Phi^2+7\Phi^2\dot{\pi}-7\Phi^3)-\ddot{H}\pi^2\Phi(\dot{\pi}-\Phi))+\frac{18H\dot{H}}{a^2}\Phi\pi(\partial_k\pi)^2 \\
& +\frac{6H^2}{a^2}\Phi(\dot{\pi}-\Phi)(\partial_k\pi)^2+\frac{24H}{a^2}\Phi(\dot{\pi}-\Phi)\partial_i\dot{\pi}\partial_i\pi+\frac{60H}{a^2}\Phi(\dot{\pi}-\Phi)\partial_i\Psi\partial_i\pi \\
& +\frac{6H}{a^2}(\partial_k^2\pi)(-2\dot{\pi}\Phi-4\dot{\pi}\Phi\Psi+\Phi\dot{\pi}^2+2\Phi^2+4\Phi^2\Psi+3\Phi^2\dot{\pi}-5\Phi^3) \\
& +\frac{6H}{a^4}(\partial_k^2\pi)(\partial_k\pi)^2\Phi+\frac{24H}{a^2}\Phi(\Phi-\dot{\pi})\partial_i\Phi\partial_i\pi \\
& -9\dot{\Psi}^2(2\dot{\pi}+8\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-8\Phi\Psi-8\Phi\dot{\pi}+8\Phi^2)+\frac{9\dot{\Psi}^2}{a^2}(\partial_k\pi)^2+\frac{6H\dot{\Psi}}{a^2}(\dot{\pi}-\Phi)(\partial_k\pi)^2 \\
& -18\dot{\Psi}(\dot{H}\pi(2\dot{\pi}+4\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-4\Phi\Psi-6\dot{\pi}\Phi+6\Phi^2)+\ddot{H}\pi^2(\dot{\pi}-\Phi))+\frac{18\dot{\Psi}\dot{H}}{a^2}\pi(\partial_k\pi)^2 \\
& +\frac{24\dot{\Psi}}{a^2}(\dot{\pi}-\Phi)\partial_i\dot{\pi}\partial_i\pi+\frac{60\dot{\Psi}}{a^2}(\dot{\pi}-\Phi)(\partial_k\pi)^2 \\
& -\frac{6\dot{\Psi}}{a^2}(\partial_k^2\Psi)(2\dot{\pi}+8\dot{\pi}\Psi-\dot{\pi}^2-2\Phi-8\Phi\Psi-2\Phi\dot{\pi}+4\Phi^2)+\frac{6\dot{\Psi}}{a^4}(\partial_k^2\pi)(\partial_k\pi)^2-\frac{24\dot{\Psi}}{a^2}(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi \\
& +\frac{6H\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)(\partial_k\pi)^2+\frac{9\dot{H}^2}{a^2}\pi^2(\partial_k\pi)^2 \\
& -9\dot{H}^2\pi^2(-2\Phi+4\Phi^2+2\dot{\pi}-4\Phi\dot{\pi}+\dot{\pi}^2)+18\dot{H}^3\ddot{H}\pi^3(\dot{\pi}-\Phi) \\
& +\frac{24\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)\partial_i\dot{\pi}\partial_i\pi+\frac{60\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)\partial_i\Psi\partial_i\pi-\frac{6\dot{H}}{a^2}\pi(\partial_k^2\pi)(-2\Phi+4\Phi^2+2\dot{\pi}-4\Phi\dot{\pi}+\dot{\pi}^2) \\
& -\frac{12}{a^2}(\partial_k^2\pi)(\dot{\pi}-\Phi)(\frac{1}{2}\ddot{H}\pi^2+2\dot{H}\pi\Psi+\dot{H}\pi\Phi-\dot{H}\pi\dot{\pi})+\frac{6\dot{H}}{a^4}\pi(\partial_k^2\pi)(\partial_k\pi)^2-\frac{12\dot{H}}{a^2}\ddot{\pi}(\dot{\pi}-\Phi)(\partial_k\pi)^2 \\
& -\frac{24\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi+\frac{2H}{a^4}(\dot{\pi}-\Phi)(\partial_k^2\pi)(\partial_k\pi)^2 \\
& +\frac{8}{a^4}(\dot{\pi}-\Phi)(\partial_k^2\pi)\partial_i\dot{\pi}\partial_i\pi-\frac{20}{a^4}(\dot{\pi}-\Phi)(\partial_k^2\pi)\partial_i\Psi\partial_i\pi \\
& +\frac{1}{a^6}(\partial_k^2\pi)^2(\partial_k\pi)^2-\frac{1}{a^4}(\partial_k^2\pi)^2(2\dot{\pi}+8\dot{\pi}\Psi-3\dot{\pi}^2-2\Phi-8\Phi\Psi+4\Phi\dot{\pi})-\frac{8}{a^4}(\dot{\pi}-\Phi)(\partial_k^2\pi)\partial_i\Phi\partial_i\pi,
\end{aligned}
\tag{9.161}
$$

$$
\begin{aligned}
\delta\tilde{g}^{00}\delta\tilde{K}^i_j\delta\tilde{K}^j_i \approx & -3H^2(2\Phi^2\dot{\pi}+\Phi^2\dot{\pi}^2-2\Phi^3-10\Phi^3\dot{\pi}+10\Phi^4)+\frac{6H}{a^2}\dot{\Psi}\Phi(\partial_k\pi)^2\\
& -6H\dot{\Psi}(-2\dot{\pi}\Phi-4\dot{\pi}\Phi\Psi-\Phi\dot{\pi}^2+2\Phi^2+4\Phi^2\Psi+9\Phi^2\dot{\pi}-9\Phi^3)+\frac{3H^2}{a^2}\Phi^2(\partial_k\pi)^2\\
& -3\dot{\Psi}^2(2\dot{\pi}+8\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-8\Phi\Psi-8\Phi\dot{\pi}+8\Phi^2)\\
& +\frac{3\dot{\Psi}^2}{a^2}(\partial_k\pi)^2+6H\dot{H}\pi(-2\Phi\dot{\pi}-\Phi\dot{\pi}^2+2\Phi^2+7\Phi^2\dot{\pi}-7\Phi^3)+\frac{6\dot{H}}{a^2}\pi\dot{\Psi}(\partial_k\pi)^2\\
& -6\dot{H}\dot{\Psi}\pi(2\dot{\pi}+4\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-4\Phi\Psi-6\Phi\dot{\pi}+6\Phi^2)-6\ddot{H}\pi^2\dot{\Psi}(\dot{\pi}-\Phi)+\frac{6}{a^2}H\dot{H}\pi\Phi(\partial_k\pi)^2\\
& -6H\ddot{H}\pi^2\Phi(\dot{\pi}-\Phi)-\frac{4H}{a^2}\Phi(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi-\frac{4}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi\\
& +\frac{8H}{a^2}\Phi(\dot{\pi}-\Phi)\partial_i\dot{\pi}\partial_i\pi+\frac{8}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)\partial_i\dot{\pi}\partial_i\pi\\
& +\frac{2H}{a^2}(\partial_k^2\pi)(-2\dot{\pi}\Phi-4\dot{\pi}\Phi\Psi+\Phi\dot{\pi}^2+2\Phi^2+4\Phi^2\Psi-5\Phi^3+3\Phi^2\dot{\pi})\\
& +\frac{2H}{a^4}\Phi(\partial_k^2\pi)(\partial_k\pi)^2-\frac{2}{a^2}\dot{\Psi}(\partial_k^2\pi)(2\dot{\pi}+8\dot{\pi}\Psi-\dot{\pi}^2-2\Phi-8\Phi\Psi-2\Phi\dot{\pi}+4\Phi^2)+\frac{2}{a^4}\dot{\Psi}(\partial_k^2\pi)(\partial_k\pi)^2\\
& +\frac{4}{a^2}\dot{\gamma}_{ij}\partial_i\partial_j\pi(1+2\Psi-\Phi)-\frac{4H}{a^2}\Phi(\dot{\pi}-\Phi)\partial_i\Psi\partial_i\pi+\frac{4}{a^2}\dot{\Psi}(\Phi-\dot{\pi})\partial_i\Psi\partial_i\pi-\frac{4H}{a^2}\Phi(\dot{\pi}-\Phi)\partial_i\Psi\partial_i\pi\\
& -\frac{4}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)\partial_i\Psi\partial_i\pi+\frac{12H}{a^2}\Phi(\dot{\pi}-\Phi)(\partial_k\pi)^2+\frac{12}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)(\partial_k\pi)^2-\frac{4H^2}{a^2}\Phi(\dot{\pi}-\Phi)(\partial_k\pi)^2\\
& -\frac{4H}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)(\partial_k\pi)^2-\frac{4H}{a^2}\Phi(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi-\frac{4}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi+\frac{6H^2}{a^2}\Phi(\dot{\pi}-\Phi)(\partial_k\pi)^2\\
& +\frac{6H}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)(\partial_k\pi)^2+\frac{3\dot{H}^2}{a^2}\pi^2(\partial_k\pi)^2-3\dot{H}^2\pi^2(-2\Phi+4\Phi^2+2\dot{\pi}-4\Phi\dot{\pi}+\dot{\pi}^2)-6\dot{H}\ddot{H}\pi^3(\dot{\pi}-\Phi)\\
& -\frac{4\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi+\frac{8\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)\partial_i\dot{\pi}\partial_i\pi-\frac{2\dot{H}}{a^2}\pi(\partial_k^2\pi)(2\dot{\pi}+4\dot{\pi}\Psi-\dot{\pi}^2-2\Phi-4\Phi\Psi+2\Phi^2)\\
& +\frac{2\dot{H}}{a^4}\pi(\partial_k^2\pi)(\partial_k\pi)^2-\frac{2\ddot{H}}{a^2}\pi^2(\dot{\pi}-\Phi)(\partial_k^2\pi)-\frac{8\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)\partial_i\Psi\partial_i\pi+\frac{12\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)(\partial_k\pi)^2\\
& -\frac{4H\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)(\partial_k\pi)^2-\frac{4\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi+\frac{6\dot{H}}{a^2}\pi(\dot{\pi}-\Phi)(\partial_k\pi)^2-\frac{4}{a^4}(\dot{\pi}-\Phi)\partial_i\Phi\partial_j\pi\partial_i\partial_j\pi\\
& +\frac{4}{a^4}(\dot{\pi}-\Phi)\partial_i\pi\partial_j\dot{\pi}\partial_i\partial_j\pi+\frac{1}{a^6}(\partial_i\partial_j\pi)^2(\partial_k\pi)^2-\frac{1}{a^4}(\partial_i\partial_j\pi)^2(2\dot{\pi}+8\dot{\pi}\Psi-3\dot{\pi}^2-2\Phi-8\Phi\Psi+4\Phi\dot{\pi})\\
& +\frac{4}{a^4}(\dot{\pi}-\Phi)\partial_i\dot{\pi}\partial_j\pi\partial_i\partial_j\pi-\frac{4}{a^4}(\dot{\pi}-\Phi)\partial_i\pi\partial_j\Psi\partial_i\partial_j\pi-\frac{4}{a^4}(\dot{\pi}-\Phi)\partial_i\Psi\partial_j\pi\partial_i\partial_j\Psi\\
& +\frac{4}{a^4}(\dot{\pi}-\Phi)(\partial_k^2\pi)\partial_i\Psi\partial_i\pi-\frac{4H}{a^4}(\dot{\pi}-\Phi)\partial_i\pi\partial_j\pi\partial_i\partial_j\pi\\
& -\frac{4}{a^4}(\dot{\pi}-\Phi)\partial_i\Phi\partial_j\pi\partial_i\partial_j\pi+\frac{2H}{a^4}(\dot{\pi}-\Phi)(\partial_k^2\pi)(\partial_k\pi)^2.
\end{aligned}
\tag{9.162}
$$

Finally, we need to spell out the different parts appearing in ${}^{(3)}\tilde{R}\delta\tilde{g}^{00}$. $\delta\tilde{g}^{00}R$ is easily found using the derived results:

$$
\begin{aligned}
\delta\tilde{g}^{00}R \approx\, & \frac{2}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\partial_j\Phi(2\dot{\pi}+4\dot{\pi}\Psi+8\dot{\pi}\Psi^2+\dot{\pi}^2+2\dot{\pi}^2\Psi-2\Phi-4\Phi\Psi \\
& -8\Phi\Psi^2-8\dot{\pi}\Phi-16\dot{\pi}\Phi\Psi-4\Phi\dot{\pi}^2+8\Phi^2+16\Phi^2\Psi+24\Phi^2\dot{\pi}-24\Phi^3) \\
& -6(H^2+\ddot{a}/a)(2\dot{\pi}+\dot{\pi}^2-2\Phi-8\dot{\pi}\Phi-4\Phi\dot{\pi}^2+8\Phi^2+24\Phi^2\dot{\pi}+12\Phi^2\dot{\pi}^2-24\Phi^3-64\Phi^3\dot{\pi}+64\Phi^4) \\
& +9\ddot{\Psi}(2\dot{\pi}+4\dot{\pi}\Psi+8\dot{\pi}\Psi^2+\dot{\pi}^2+2\dot{\pi}^2\Psi-2\Phi-4\Phi\Psi-8\Phi\Psi^2-8\dot{\pi}\Phi \\
& -16\dot{\pi}\Phi\Psi-4\Phi\dot{\pi}^2+8\Phi^2+16\Phi^2\Psi+24\Phi^2\dot{\pi}-24\Phi^3) \\
& +6H\dot{\Phi}(2\dot{\pi}+\dot{\pi}^2-2\Phi-12\dot{\pi}\Phi-6\dot{\pi}^2\Phi+12\Phi^2+48\dot{\pi}\Phi^2-48\Phi^3) \\
& -6\dot{\Psi}\dot{\Phi}(2\dot{\pi}+4\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-4\Phi\Psi-12\dot{\pi}\Phi+12\Phi^2) \\
& -\frac{2}{a^2}(\partial_k\Phi)^2(2\dot{\pi}+4\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-4\Phi\Psi-12\Phi\dot{\pi}+12\Phi^2) \\
& +24H\dot{\Psi}(2\dot{\pi}+4\dot{\pi}\Psi+8\dot{\pi}\Psi^2+\dot{\pi}^2+2\dot{\pi}^2\Psi-2\Phi-4\Phi\Psi-8\Phi\Psi^2 \\
& -8\dot{\pi}\Phi-16\dot{\pi}\Phi\Psi-4\Phi\dot{\pi}^2+8\Phi^2+16\Phi^2\Psi-24\Phi^3+24\Phi^2\dot{\pi}) \\
& -\frac{6}{a^2}(\partial_k\Psi)^2(2\dot{\pi}+12\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-12\Phi\Psi-4\dot{\pi}\Phi+4\Phi^2) \\
& -\frac{4}{a^2}(\delta_{ij}-\gamma_{ij})\partial_i\partial_j\Psi(2\dot{\pi}+8\dot{\pi}\Psi+24\dot{\pi}\Psi^2+\dot{\pi}^2+4\dot{\pi}^2\Psi-2\Phi \\
& -8\Phi\Psi-24\Phi\Psi^2-4\dot{\pi}\Phi-16\dot{\pi}\Phi\Psi-2\Phi\dot{\pi}^2+4\Phi^2+16\Phi^2\Psi+8\dot{\pi}\Phi^2-8\Phi^3) \\
& -\frac{2}{a^4}(\partial_k^2\pi)(\partial_k\pi)^2(1-2\Phi+4\Psi)+\frac{6}{a^2}(H^2+\ddot{a}/a)(\delta_{ij}-\gamma_{ij})\partial_i\pi\partial_j\pi(1+2\Psi+4\Psi^2-2\Phi-4\Phi\Psi+4\Phi^2) \\
& -\frac{9}{a^2}\ddot{\Psi}(\partial_k\pi)^2(1+4\Psi-2\Phi)-\frac{6H}{a^2}\dot{\Phi}(1+2\Psi-4\Phi)(\partial_k\pi)^2+\frac{6}{a^2}\dot{\Psi}\dot{\Phi}(\partial_k\pi)^2-\frac{24H}{a^2}\dot{\Psi}(1+4\Psi-2\Phi)(\partial_k\pi)^2 \\
& +\frac{6}{a^4}(\partial_k\pi)^2(\partial_k\Psi)^2+\frac{4}{a^4}(\partial_k^2\pi)(\partial_k\pi)^2(1+6\Psi)+\frac{2}{a^4}(\partial_k\pi)^2(\partial_k\Phi)^2.
\end{aligned}
\tag{9.163}
$$

Notice that $2R^{\mu\nu}n_\mu n_\nu = \frac{2}{N^2}R_{00} = \frac{2}{1+2\Phi}R_{00}$. Hence we find using the expression for R_{00}:

$$
\begin{aligned}
2\delta\tilde{g}^{00}R^{\mu\nu}n_\mu n_\nu \approx & -\frac{2}{a^2}\partial_i\Phi\partial_i\Psi(2\dot{\pi}+8\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-8\Phi\Psi-8\dot{\pi}\Phi+8\Phi^2) \\
& -\frac{2}{a^2}(\partial_k^2\Phi)(2\dot{\pi}+4\dot{\pi}\Psi+8\dot{\pi}\Psi^2+\dot{\pi}^2+2\dot{\pi}^2\Psi-2\Phi-4\Phi\Psi-8\Phi\Psi^2 \\
& -8\dot{\pi}\Phi-16\dot{\pi}\Phi\Psi-4\Phi\dot{\pi}^2+8\Phi^2+16\Phi^2\Psi+24\Phi^2\dot{\pi}-24\Phi^3) \\
& +\frac{4}{a^2}\gamma_{ij}(\dot{\pi}-\Phi)\partial_i\partial_j\Phi-6\dot{\Psi}^2(2\dot{\pi}+8\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-8\Phi\Psi-8\Phi\dot{\pi}+8\Phi^2) \\
& +\frac{6\ddot{a}}{a}(2\dot{\pi}+\dot{\pi}^2-2\Phi-8\Phi\dot{\pi}-4\Phi\dot{\pi}^2+8\Phi^2+24\dot{\pi}\Phi^2+12\dot{\pi}^2\Phi^2-24\Phi^3-64\dot{\pi}\Phi^3+64\Phi^4) \\
& -6\ddot{\Psi}(2\dot{\pi}+4\dot{\pi}\Psi+8\dot{\pi}\Psi^2+\dot{\pi}^2+2\dot{\pi}^2\Psi-2\Phi-4\Phi\Psi-8\Phi\Psi^2-8\dot{\pi}\Phi-16\dot{\pi}\Phi\Psi \\
& -4\Phi\dot{\pi}^2+8\Phi^2+16\Phi^2\Psi+24\dot{\pi}\Phi^2-24\Phi^3) \\
& -6H\dot{\Phi}(2\dot{\pi}+\dot{\pi}^2-2\Phi-12\dot{\pi}\Phi-6\Phi\dot{\pi}^2+12\Phi^2+48\dot{\pi}\Phi^2-48\Phi^3) \\
& +6\dot{\Psi}\dot{\Phi}(2\dot{\pi}+4\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-4\Phi\Psi-12\dot{\pi}\Psi+12\Phi^2) \\
& +\frac{2}{a^2}(\partial_k\Phi)^2(2\dot{\pi}+4\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-4\Phi\Psi-12\dot{\pi}\Phi+12\Phi^2) \\
& -12H\dot{\Psi}(2\dot{\pi}+4\dot{\pi}\Psi+8\dot{\pi}\Psi^2+\dot{\pi}^2+2\dot{\pi}^2\Psi-2\Phi-4\Phi\Psi-8\Phi\Psi^2 \\
& -8\dot{\pi}\Phi-16\dot{\pi}\Phi\Psi-4\Phi\dot{\pi}^2+8\Phi^2+16\Phi^2\Psi+24\Phi^2\dot{\pi}-24\Phi^3)+\frac{2}{a^4}(\partial_k\pi)^2\partial_i\Psi\partial_i\Phi \\
& +\frac{2}{a^4}(\partial_k\pi)^2(\partial_k^2\Phi)(1+4\Psi-2\Phi)+\frac{6\dot{\Psi}^2}{a^2}(\partial_k\pi)^2 \\
& -\frac{6\ddot{a}}{a^3}(\delta_{ij}-\gamma_{ij})\partial_i\pi\partial_j\pi(1+2\Psi+4\Psi^2-2\Phi-4\Phi\Psi+4\Phi^2) \\
& +\frac{6}{a^2}\ddot{\Psi}(\partial_k\pi)^2(1+4\Psi-2\Phi)+\frac{6H}{a^2}\dot{\Phi}(1-4\Phi+2\Psi)(\partial_k\pi)^2-\frac{6}{a^2}\dot{\Psi}\dot{\Phi}(\partial_k\pi)^2 \\
& -\frac{2}{a^4}(\partial_k\pi)^2(\partial_k\Phi)^2+\frac{12H}{a^2}\dot{\Psi}(1+4\Psi-2\Phi)(\partial_k\pi)^2.
\end{aligned}
\tag{9.164}
$$

And we find using R_{00} also the following:

$$
\begin{aligned}
2\delta\tilde{g}^{00}R_{00}[h^{kl}\partial_k\pi\partial_l\pi-\frac{2\dot{\pi}}{a^2}(\partial_k\pi)^2] \approx & \frac{4}{a^4}(\Phi-\dot{\pi})(\partial_k^2\Phi)(\partial_k\pi)^2 \\
& +\frac{6\ddot{a}}{a^3}(\partial_k\pi)^2(2\dot{\pi}+4\dot{\pi}\Psi+\dot{\pi}^2-2\Phi-4\Phi\Psi-4\dot{\pi}\Phi) \\
& -\frac{6\ddot{a}}{a^5}(\partial_k\pi)^4+\frac{24\ddot{a}}{a^3}\dot{\pi}(\Phi-\dot{\pi})(\partial_k\pi)^2+\frac{12}{a^2}\ddot{\Psi}(\Phi-\dot{\pi})(\partial_k\pi)^2 \\
& +\frac{12H}{a^2}\dot{\Phi}(\Phi-\dot{\pi})(\partial_k\pi)^2+\frac{24H}{a^2}\dot{\Psi}(\Phi-\dot{\pi})(\partial_k\pi)^2.
\end{aligned}
\tag{9.165}
$$

From equation (7.91) we find $h^{ij}R_{0i}$ to second order:

$$
(\delta^{ij}+\gamma^{ij})R_{0i} \approx 4\dot{\Psi}\partial_j\Psi+2(1+2\Psi)\partial_j\dot{\Psi}+2H(1-2\Phi)\partial_j\Phi-2\dot{\Psi}\partial_j\Phi. \tag{9.166}
$$

Therefore it follows easily that:

$$
\begin{aligned}
-4\tilde{g}^{00}h^{ij}R_{0i}\partial_j\pi(1-2\dot{\pi}+3\dot{\pi}^2+\frac{1}{a^2}(\partial_k\pi)^2) \approx\ & \frac{8}{a^2}\partial_i\dot{\Psi}\partial_i\pi(-2\Phi+4\Phi^2+2\dot{\pi}-4\dot{\pi}\Phi+\dot{\pi}^2) \\
&+\frac{8H}{a^2}\partial_i\Phi\partial_i\pi(-2\Phi+4\Phi^2+2\dot{\pi}-4\dot{\pi}\Phi+\dot{\pi}^2) \\
&+\frac{32}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)\partial_i\Psi\partial_i\pi-\frac{32H}{a^2}(\Phi+\Psi)(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi \\
&-\frac{16}{a^2}\dot{\Psi}(\dot{\pi}-\Phi)\partial_i\Phi\partial_i\pi \\
&-\frac{8}{a^4}(\partial_k\pi)^2\partial_i\pi(\partial_i\dot{\Psi}+H\partial_i\Phi)-\frac{32}{a^2}\dot{\pi}(\dot{\pi}-\Phi)\partial_i\pi(\partial_i\dot{\pi}+H\partial_i\Phi).
\end{aligned}
\tag{9.167}
$$

From equation (7.91) we find that R_{ij} at first order (ignoring γ as it will lead to $\pi\pi\pi\gamma$ in the term we are considering):

$$
\begin{aligned}
R_{ij} \approx\ & \partial_i\partial_j\Psi+\delta_{ij}\partial_k^2\Psi-a\dot{a}\delta_{ij}\dot{\Phi}-\partial_i\partial_j\Phi+\dot{a}^2\delta_{ij}(1-2\Psi+2\Phi) \\
&-2\dot{a}a\delta_{ij}\dot{\Psi}+\ddot{a}a\delta_{ij}(1-2\Psi-2\Phi)+\dot{a}^2\delta_{ij}(1-2\Psi-2\Phi)-2\dot{a}a\dot{\Psi}\delta_{ij}-a^2\ddot{\Psi}\delta_{ij}.
\end{aligned}
\tag{9.168}
$$

Hence it follows that:

$$
\begin{aligned}
2N^2h^{ik}h^{jl}R_{ij}\partial_k\pi\partial_l\pi(1-2\dot{\pi})\delta\tilde{g}^{00} \approx\ & \frac{2}{a^4}(2H^2+\ddot{a}/a)(\partial_k\pi)^4-\frac{4}{a^2}\dot{\pi}\partial_i\pi\partial_j\pi(\partial_i\partial_j\Psi+\delta_{ij}\partial_k^2\Psi-a\dot{a}\dot{\Phi}\delta_{ij} \\
&-\partial_i\partial_j\Phi+2\dot{a}^2\delta_{ij}(1-2\Psi-2\Phi)-4\dot{a}a\delta_{ij}\dot{\Psi} \\
&+a\ddot{a}\delta_{ij}(1-2\Psi-2\Phi)-a^2\ddot{\Psi}\delta_{ij}) \\
&-16\dot{\pi}\Psi(\partial_k\pi)^2(2H^2+\ddot{a}/a)+6\dot{\pi}^2(\partial_k\pi)^2(2H^2+\ddot{a}/a) \\
&+\frac{4}{a^2}\Phi\partial_i\pi\partial_j\pi(\partial_i\partial_j\Psi+\delta_{ij}\partial_k^2\Psi \\
&-a\dot{a}\dot{\Phi}\delta_{ij}-\partial_i\partial_j\Phi+2\dot{a}^2(1-2\Psi-2\Phi)\delta_{ij}-4\dot{a}a\dot{\Psi}\delta_{ij} \\
&+a\ddot{a}(1-2\Psi-2\Phi)\delta_{ij}-a^2\ddot{\Psi}\delta_{ij}) \\
&+16\Phi\Psi(\partial_k\pi)^2(2H^2+\ddot{a}/a)-8\Phi\dot{\pi}(\partial_k\pi)^2(2H^2+\ddot{a}/a).
\end{aligned}
\tag{9.169}
$$

Observe that:

$$
\frac{2H}{a^2}\partial_i\pi(1-N^2+2\dot{\pi})(\partial_iN-N\partial_i\dot{\pi})\delta\tilde{g}^{00}=2\tilde{K}_i^0\tilde{K}_0^i\delta\tilde{g}^{00}. \tag{9.170}
$$

Therefore we compute $\tilde{K}_i^0\tilde{K}_0^i\tilde{g}^{00}$:

$$
\delta\tilde{g}^{00}\tilde{K}_0^i\tilde{K}_i^0 \approx -\frac{8H}{a^2}\partial_i\Phi\partial_i\pi(\dot{\pi}^2-2\dot{\pi}\Phi+\Phi^2)+\frac{8H}{a^2}\partial_i\dot{\pi}\partial_i\pi(\dot{\pi}^2-2\dot{\pi}\Phi+\Phi^2). \tag{9.171}
$$

For the remaining parts, notice that we can write the expressions as follows:

$$
\begin{aligned}
\tilde{K}^2 &= (\delta\tilde{K})^2 + 6H\delta\tilde{K} + 6\delta\tilde{K}(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3) + 9H^2 + 18H(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3) \\
&+ 9(\dot{H}^2\pi^2 + \frac{1}{4}\ddot{H}^2\pi^2 + \frac{1}{3}\dot{H}\dddot{H}\pi^4 + \dot{H}\ddot{H}\pi^3) \\
\tilde{K}^i_j\tilde{K}^j_i &= \delta\tilde{K}^i_j\delta\tilde{K}^j_i + 2H\delta\tilde{K}^i_i + 2\delta\tilde{K}^i_i(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2) + 3H^2 + 3H(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3) \\
&+ 3(\dot{H}^2\pi^2 + \dot{H}\ddot{H}\pi^3).
\end{aligned}
\tag{9.172}
$$

The ingredients we need for $\tilde{K}^2$ are:

$$
\begin{aligned}
\delta\tilde{g}^{00}\delta\tilde{K}(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3) &\approx \frac{3H}{a^2}(\partial_k\pi)^2(-\dot{H}\Phi\pi + \frac{1}{2}\ddot{H}\pi^2) + 3H(\dot{H}\pi(2\Phi\dot{\pi} - 2\Phi^2 - 7\Phi^2\dot{\pi} + 7\Phi^3) \\
&+ \frac{1}{2}\ddot{H}\pi^2(2\Phi\dot{\pi} - 2\Phi^2)) - \frac{3\dot{H}}{a^2}\dot{\Psi}\pi(\partial_k\pi)^2 \\
&- 3\dot{\Psi}(\dot{H}\pi(-2\dot{\pi} - 4\dot{\pi}\Psi + 2\Phi + 4\Phi\Psi + 6\Phi\dot{\pi} - 6\Phi^2) \\
&+ \ddot{H}\pi^2(\Phi - \dot{\pi})) - \frac{3\dot{H}^2}{a^2}\pi^2(\partial_k\pi)^2 - 3\dot{H}\pi(\dot{H}\pi(2\Phi - 4\Phi^2 - 2\dot{\pi} - 4\Phi\dot{\pi}) \\
&+ \ddot{H}\pi^2(\Phi - \dot{\pi})) - 3\ddot{H}\dot{H}\pi^3(\Phi - \dot{\pi}) + \frac{H\dot{H}}{a^2}\pi(\dot{\pi} - \Phi)(\partial_k\pi)^2 \\
&+ \frac{4\dot{H}}{a^2}\pi(\Phi - \dot{\pi})\partial_i\dot{\pi}\partial_i\pi \\
&+ \frac{10\dot{H}}{a^2}\pi(\Phi - \dot{\pi})\partial_i\Psi\partial_i\pi - \frac{\dot{H}}{a^2}\pi(\partial_k^2\pi)(\partial_k\pi)^2 \\
&- \frac{1}{a^2}(\partial_k^2\pi)(\dot{H}\pi(-2\Phi^2 + 4\Phi\Psi + 2\Phi + 2\dot{\pi}^2 \\
&- 4\dot{\pi}\Psi - 2\dot{\pi}) + \ddot{H}\pi^2(\Phi - \dot{\pi})) - \frac{4\dot{H}}{a^2}\pi(\Phi - \dot{\pi})\partial_i\Phi\partial_i\pi \\
&- \dot{H}\dot{\pi}^2\pi(-3H\Phi - 3\dot{\Psi} - 3\dot{H}\pi - \frac{1}{a^2}\partial_k^2\pi) \\
H\delta\tilde{g}^{00}(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2 + \frac{1}{6}\dddot{H}\pi^3) &\approx -H\dot{H}\pi(-2\Phi + 4\Phi^2 - 8\Phi^3 + 2\dot{\pi} - 4\Phi\dot{\pi} - 2\Phi\dot{\pi}^2 + \dot{\pi}^2 + 8\dot{\pi}\Phi^2) \\
&- \frac{H\ddot{H}}{2}\pi^2(-2\Phi + 4\Phi^2 + 2\dot{\pi} - 4\dot{\pi}\Phi + \dot{\pi}^2) - \frac{H\dddot{H}}{3}\pi^3(\dot{\pi} - \Phi) \\
&+ \frac{H\dot{H}}{a^2}\pi(1 + 2\Psi)(\partial_k\pi)^2 + \frac{H\ddot{H}}{2a^2}\pi^2(\partial_k\pi)^2, \\
(\dot{H}^2\pi^2 + \frac{1}{4}\ddot{H}^2\pi^2 + \frac{1}{3}\dot{H}\dddot{H}\pi^4 + \dot{H}\ddot{H}\pi^3)\delta\tilde{g}^{00} &\approx -\dot{H}^2\pi^2(-2\Phi + 4\Phi^2 + 2\dot{\pi} - 4\Phi\dot{\pi} + \dot{\pi}^2) + \frac{\dot{H}^2}{a^2}\pi^2(\partial_k\pi)^2 \\
&- \frac{1}{4}\ddot{H}^2\pi^2(-2\Phi + 4\Phi^2 + 2\dot{\pi} - 4\dot{\pi}\Phi + \dot{\pi}^2) + \frac{\ddot{H}^2}{4a^2}\pi^2(\partial_k\pi)^2 + 2\dot{H}\ddot{H}\pi^3(\Phi - \dot{\pi}),
\end{aligned}
\tag{9.173}
$$

For $\tilde{K}^i_j\tilde{K}^j_i$ note that $\delta\tilde{K}^i_i(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2)\delta\tilde{g}^{00} \approx \delta\tilde{K}(\dot{H}\pi + \frac{1}{2}\ddot{H}\pi^2)\delta\tilde{g}^{00}$, hence this term has already been computed above. The terms we therefore still need to spell out are $3(\dot{H}^2\pi^2 + \dot{H}\ddot{H}\pi^3)\delta\tilde{g}^{00}$ and $2H\delta\tilde{K}^i_i\delta\tilde{g}^{00}$:

$$
\begin{aligned}
3(\dot{H}^2\pi^2 + \dot{H}\ddot{H}\pi^3)\delta\tilde{g}^{00} &\approx -3\dot{H}^2\pi^2(-2\Phi + 4\Phi^2 + 2\dot{\pi} - 4\dot{\pi}\Phi + \dot{\pi}^2) + \frac{3\dot{H}^2}{a^2}\pi^2(\partial_k\pi)^2 - 6\dot{H}\ddot{H}\pi^3(\dot{\pi} - \Phi), \\
2H\delta\tilde{K}^i_i\delta\tilde{g}^{00} &\approx -6H^2(-2\dot{\pi}\Phi - \Phi\dot{\pi}^2 + 2\Phi^2 + 7\Phi^2\dot{\pi} + \frac{7}{2}\Phi^2\dot{\pi}^2 - 7\Phi^3 - 19\Phi^3\dot{\pi} + 19\Phi^4) \\
&+ \frac{6H^2}{a^2}(-\Phi + \frac{3}{2}\Phi^2)(\partial_k\pi)^2 + 6H\dot{\Psi}(2\dot{\pi} + 4\dot{\pi}\Psi + 8\dot{\pi}\Psi^2 + \dot{\pi}^2 + 2\dot{\pi}^2\Psi - 2\Phi - 4\Phi\Psi \\
&- 8\Phi\Psi^2 - 6\Phi\dot{\pi} - 12\dot{\pi}\Phi\Psi - 3\Phi\dot{\pi}^2 + 6\Phi^2 + 12\Phi^2\Psi + 15\Phi^2\dot{\pi}^2 - 15\Phi^3) \\
&- \frac{6H}{a^2}\dot{\Psi}(1 - \Phi + 2\Psi)(\partial_k\pi)^2 + 6H\dot{H}\pi(-2\Phi + 4\Phi^2 - 8\Phi^3 + 2\dot{\pi} - 4\dot{\pi}\Phi - 2\Phi\dot{\pi}^2 + 8\dot{\pi}\Phi^2 + \dot{\pi}^2) \\
&+ 3H\ddot{H}\pi^2(-2\Phi + 2\dot{\pi} + 4\Phi^2 - 4\Phi\dot{\pi} + \dot{\pi}^2) + 2H\dddot{H}\pi^3(\dot{\pi} - \Phi) \\
&- \frac{6H}{a^2}(\partial_k\pi)^2(\dot{H}\pi(1 + 2\Psi) + \frac{1}{2}\ddot{H}\pi^2) \\
&+ \frac{2H}{a^2}\partial_i\Phi\partial_i\pi(2\dot{\pi} + 4\dot{\pi}\Psi - 3\dot{\pi}^2 - 2\Phi - 4\Phi\Psi - 2\Phi\dot{\pi} + 6\Phi^2) - \frac{2H}{a^4}(\partial_k\pi)^2\partial_i\Phi\partial_i\pi \\
&- \frac{2H}{a^2}\partial_i\dot{\pi}\partial_i\pi(2\dot{\pi} + 4\dot{\pi}\Psi - 3\dot{\pi}^2 - 2\Phi - 4\Phi\Psi + 2\Phi\dot{\pi} + 2\Phi^2) + \frac{2H}{a^4}\partial_i\dot{\pi}\partial_i\pi(\partial_k\pi)^2 \\
&+ \frac{4H}{a^4}(\dot{\pi} - \Phi)\partial_i\pi\partial_i\partial_j\pi\partial_j\pi - \frac{4H}{a^2}\dot{\Phi}(\dot{\pi} - \Phi)(\partial_k\pi)^2 - \frac{4H}{a^2}\ddot{\pi}(\dot{\pi} - \Phi)(\partial_k\pi)^2 \\
&+ \frac{2H}{a^2}(\delta_{ij} - \gamma_{ij})\partial_i\partial_j\pi(2\dot{\pi} + 4\dot{\pi}\Psi + 8\dot{\pi}\Psi^2 - \dot{\pi}^2 + \dot{\pi}^3 - 2\Phi - 4\Phi\Psi \\
&- 8\Phi\Psi^2 - 2\dot{\pi}^2\Psi - \Phi\dot{\pi}^2 + 2\Phi^2 + 4\Phi^2\Psi + \Phi^2\dot{\pi} - 3\Phi^3) \\
&- \frac{2H}{a^4}(\partial_k^2\pi)(\partial_k\pi)^2(1 - \dot{\pi} + \Phi + 4\Psi) - \frac{2H}{a^2}\partial_i\dot{\pi}\partial_i\pi(2\dot{\pi} + 4\dot{\pi}\Psi - 3\dot{\pi}^2 - 2\Phi - 4\Phi\Psi + 2\Phi\dot{\pi} + 2\Phi^2) \\
&+ \frac{2H}{a^4}\partial_i\dot{\pi}\partial_i\pi(\partial_k\pi)^2 + \frac{2H}{a^4}(\dot{\pi} - \Phi)(\partial_k^2\pi)(\partial_k\pi)^2 \\
&- \frac{2H}{a^2}\partial_i\Psi\partial_i\pi(2\dot{\pi} + 8\dot{\pi}\Psi - \dot{\pi}^2 - 2\Phi - 8\Phi\Psi + 2\Phi^2) \\
&+ \frac{2H}{a^4}\partial_i\Psi\partial_i\pi(\partial_k\pi)^2 + \frac{2H^2}{a^2}(\partial_k\pi)^2(2\dot{\pi} + 4\dot{\pi}\Psi - 3\dot{\pi}^2 - 2\Phi - 4\Phi\Psi + 2\Phi\dot{\pi} + 2\Phi^2) \\
&- \frac{2H^2}{a^4}(\partial_k\pi)^4 \\
&- \frac{4H}{a^2}\dot{\Psi}(\dot{\pi} - \Phi)(\partial_k\pi)^2 + \frac{2H}{a^2}\partial_i\Phi\partial_i\pi(2\dot{\pi} + 4\dot{\pi}\Psi - \dot{\pi}^2 - 2\Phi - 4\Phi\Psi - 4\Phi\dot{\pi} + 6\Phi^2) \\
&- \frac{2H}{a^4}\partial_i\Phi\partial_i\pi(\partial_k\pi)^2 \\
&- \frac{3H^2}{a^2}(\partial_k\pi)^2(2\dot{\pi} + 4\dot{\pi}\Psi - 3\dot{\pi}^2 - 2\Phi - 4\Phi\Psi - 2\Phi\dot{\pi} + 2\Phi^2) \\
&+ \frac{3H^2}{a^4}(\partial_k\pi)^4 + \frac{6H}{a^2}\dot{\Psi}(\dot{\pi} - \Phi)(\partial_k\pi)^2.
\end{aligned}
\tag{9.174}
$$

9.12 Expressions of EFT coefficients for Horndeski theory

Combining the results of the previous section for the expansion of the EFT action, multiplying by $\sqrt{-g}$ (as given in the previous section) and performing $x^i \mapsto x^i/a$ yields the interaction Lagrangian given in equation (7.92). Let $\Psi =: c_\Psi\pi$ and $\Phi =: c_\Phi\pi$. The coefficients appearing in this expression are given by:

$$
\begin{aligned}
\tilde{\alpha}_1 &= \frac{\sqrt{2}m_4^2}{c_s^3 U M c_T^{3/2}} \\
\tilde{\alpha}_2 &= \frac{m_4^2}{\sqrt{2}c_s^3 U M c_T^{3/2}} \\
\tilde{\beta}_1 &= \frac{1}{2\sqrt{2}c_s^{9/2}U^{3/2}}(2m_2^4 - \frac{3}{2}m_3^3 c_\Psi - 3Hm_4^2 c_\Psi - \frac{33}{2}m_4^2 c_\Psi^2 - \frac{63}{2}m_4^2 c_\Psi c_\Phi) \\
\tilde{\beta}_2 &= \frac{1}{2\sqrt{2}c_s^{9/2}U^{3/2}}(-3m_3^3 - 16Hm_4^2 + 7m_4^2 c_\Psi + 2m_4^2 c_\Phi - 8m_5^2 c_\Psi) \\
\tilde{\beta}_3 &= \frac{1}{2\sqrt{2}c_s^{9/2}U^{3/2}}(3m_4^2 + m_5^2) =: \beta \\
\tilde{\beta}_4 &= \frac{\sqrt{2}m_4^2}{c_s^{9/2}U^{3/2}} =: \alpha \\
\tilde{\beta}_5 &= -\tilde{\beta}_4 \\
\tilde{\beta}_6 &= -\tilde{\beta}_3 \\
\tilde{\beta}_7 &= \frac{3m_4^2 c_\Psi}{4\sqrt{2}c_s^{9/2}U^{3/2}} \\
\tilde{\lambda}_1 &= \frac{1}{4c_s^6 U^2}(\frac{m_2^4}{2} + \frac{3}{2}m_4^2 c_\Psi + \frac{9}{2}m_5^2 c_\Psi^2) \\
\tilde{\lambda}_2 &= \frac{1}{4c_s^6 U^2}(\frac{15}{2}m_3^3 - 30m_4^2 c_\Psi + 16Hm_4^2) \\
\tilde{\lambda}_3 &= -\frac{1}{c_s^6 U^2}(\frac{9}{2}m_4^2 + m_5^2) =: -9\kappa - 2\sigma \\
\tilde{\lambda}_4 &= \tilde{\lambda}_5 = -\frac{m_4^2}{2c_s^6 U^2} =: -\kappa \\
\tilde{\lambda}_6 &= -\frac{1}{4c_s^6 U^2}(\frac{7}{2}m_4^2 + \frac{1}{2}m_5^2) =: -\frac{7}{4}\kappa - \frac{1}{4}\sigma \\
\tilde{\lambda}_7 &= -\frac{1}{4c_s^6 U^2}(\frac{9}{2}m_4^2 + \frac{3}{2}m_5^2) =: -\frac{9}{4}\kappa - \frac{3}{4}\sigma \\
\tilde{\lambda}_8 &= \frac{1}{4c_s^6 U^2}(10m_4^2 + 4m_5^2) =: 5\kappa + 2\sigma \\
\tilde{\lambda}_9 &= \tilde{\lambda}_{10} = \frac{m_4^2}{2c_s^6 U^2} =: \kappa \\
\tilde{\lambda}_{11} &= -\tilde{\lambda}_7 \\
\tilde{\lambda}_{12} &= \frac{1}{4c_s^6 U^2}(\frac{3}{2}m_4^2 + \frac{1}{2}m_5^2) =: \frac{3}{4}\kappa + \frac{1}{4}\sigma,
\end{aligned}
\tag{9.175}
$$

where we defined some new EFT functions $\alpha, \beta, \sigma, \kappa$ which will be useful in the calculation of the scattering amplitudes.

9.13 Scattering amplitudes Horndeski theory on FRW background

Let us detail the computation of the scattering amplitude (7.103). Recall that in Fourier space ∂_μ becomes ip_μ with $p_\mu = (\omega, \mathbf{k})$, and that in our convention outgoing excitations have $\omega < 0$ (such that in a Feynman diagram they move inward). The reason for this convention is that in computing Feynman diagrams one has the condition that $\sum_i p_i = 0$ at a vertex, which is now immediate for 4-point diagrams (for 3-point diagrams you still need to flip the momentum of the virtual particle at one of the vertices). In 4-point diagram vertices we get a factor of $i(-i)^6 = -i$

where i comes from what we had previously on Minkowski space and $(-i)^6$ from that 4-point diagrams contain six derivatives. A 3-point diagram vertex comes with a factor $(-i)(-i)^5 = -1$, where $-i$ comes from the previous derivations and $(-i)^5$ from five derivatives. 3-point diagrams come in two types, with two identical vertices or two different vertices. Diagrams with 2 different vertices come with a symmetry factor of 2, which is easily checked by that $(p_1, p_2) \leftrightarrow (p_3, p_4), q \leftrightarrow -q$ leaves such a scattering amplitude invariant. It turns out that the t-channel diagrams all vanish in the forward limit, which we will illustrate below, therefore it suffices to only consider the s-channel amplitude (the u-channel amplitude can be found and added by $u \leftrightarrow s$ crossing symmetry).

The vertices and scattering amplitudes will be labelled by the EFT coefficients, so e.g. $\mathcal{A}_{\beta_3\beta_4}$ means the amplitude with one vertex with $\tilde{\beta}_3$ and one with $\tilde{\beta}_4$. Before we compute the amplitudes of each diagram, we note that the following identities come in handy:

$$
\begin{aligned}
\omega_1 &= \frac{1}{2}(\omega_s + \omega_u + \omega_t) \\
\omega_2 &= \frac{1}{2}(\omega_s - \omega_u - \omega_t) \\
\omega_3 &= \frac{1}{2}(\omega_t - \omega_u - \omega_s) \\
\omega_4 &= \frac{1}{2}(\omega_u - \omega_s - \omega_t) \\
\mathbf{k}_1 \cdot \mathbf{k}_2 = \mathbf{k}_3 \cdot \mathbf{k}_4 &= -\frac{1}{2}(s - 2\omega_1\omega_2) \\
\mathbf{k}_1 \cdot \mathbf{k}_3 = \mathbf{k}_2 \cdot \mathbf{k}_4 &= -\frac{1}{2}(t - 2\omega_1\omega_3) \\
\mathbf{k}_1 \cdot \mathbf{k}_4 = \mathbf{k}_2 \cdot \mathbf{k}_3 &= -\frac{1}{2}(u - 2\omega_1\omega_4) \\
\omega_s &= M\gamma + \frac{s-u}{8M} \\
\omega_u &= M\gamma + \frac{u-s}{8M} \\
\omega_s^2 - \omega_u^2 &= \frac{\gamma}{2}(s-u).
\end{aligned}
\tag{9.176}
$$

Let us first detail the derivations of the 4-point vertices. The results will be modulo a function which (and its derivatives) vanish in the forward limit. We find the following by using the methods like in the section of scattering amplitudes on Minkowski space:

$$
\begin{aligned}
V_{\lambda_3} &= -i\tilde{\lambda}_3[\omega_1\omega_2^2\omega_3(\mathbf{k}_3 \cdot \mathbf{k}_4) + \text{perm}] \\
&= -i\tilde{\lambda}_3[-(\omega_1\omega_2 + \omega_3\omega_4)(\omega_1 + \omega_2)^2(\mathbf{k}_1 \cdot \mathbf{k}_2) - (\omega_1\omega_3 + \omega_2\omega_4)(\omega_1 + \omega_3)^2(\mathbf{k}_1 \cdot \mathbf{k}_3) \\
&- (\omega_1\omega_4 + \omega_2\omega_3)(\omega_1 + \omega_4)^2(\mathbf{k}_1 \cdot \mathbf{k}_4)] \\
&= \frac{i\tilde{\lambda}_3}{8}[\omega_s^4(\omega_s^2 - \omega_u^2) + \omega_u^4(\omega_u^2 - \omega_s^2) - 2s\omega_s^4 - 2u\omega_u^4 + 2(s+u)\omega_u^2\omega_s^2] \\
&= \frac{i\tilde{\lambda}_3}{4}\gamma(\gamma - 2)s^2\Big(M^2\gamma^2 + \frac{s^2}{16M^2}\Big)
\end{aligned}
\tag{9.177}
$$

$$
\begin{aligned}
V_{\lambda_4} &= -i\tilde{\lambda}_4[\omega_1\omega_3(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}_4)+\text{perm}] \\
&= 2i\tilde{\lambda}_4[\omega_s^2(\mathbf{k}_1\cdot\mathbf{k}_2)^2+\omega_u^2(\mathbf{k}_1\cdot\mathbf{k}_4)^2+\omega_t^2(\mathbf{k}_1\cdot\mathbf{k}_3)^2] \\
&= \frac{i\tilde{\lambda}_4}{2}[s^2(\omega_s^2+\omega_u^2)+s(\omega_u^4-\omega_s^4)+\frac{1}{4}\omega_s^4(\omega_s^2-\omega_u^2)-\frac{1}{4}\omega_u^4(\omega_s^2-\omega_u^2)] \\
&= \frac{i\tilde{\lambda}_4}{2}s^2\Big(2+\frac{1}{2}\gamma^2-2\gamma\Big)\Big(M^2\gamma^2+\frac{s^2}{16M^2}\Big)
\end{aligned}
\tag{9.178}
$$

$$
\begin{aligned}
V_{\lambda_5} &= -4i\tilde{\lambda}_5[\omega_1^2\omega_2^2(\mathbf{k}_3\cdot\mathbf{k}_4)+\omega_1^2\omega_3^2(\mathbf{k}_2\cdot\mathbf{k}_4)+\omega_1^2\omega_4^2(\mathbf{k}_2\cdot\mathbf{k}_3) \\
&\quad+\omega_2^2\omega_3^2(\mathbf{k}_1\cdot\mathbf{k}_4)+\omega_2^2\omega_4^2(\mathbf{k}_1\cdot\mathbf{k}_3)+\omega_3^2\omega_4^2(\mathbf{k}_1\cdot\mathbf{k}_2)] \\
&= \frac{i\tilde{\lambda}_5}{8}[\omega_s^6+\omega_u^6+2\omega_s\omega_u^5+2\omega_u\omega_s^5+7\omega_s^2\omega_u^4+7\omega_s^4\omega_u^2+12\omega_s^3\omega_u^3] \\
&= \frac{i\tilde{\lambda}_5}{8}\Big[32M^6\gamma^6+\frac{\gamma^2 s^4}{8M^2}\Big]
\end{aligned}
\tag{9.179}
$$

$$
\begin{aligned}
V_{\lambda_6} &= -4i\tilde{\lambda}_6[\omega_1^2\omega_2^2(\mathbf{k}_3\cdot\mathbf{k}_4)+\omega_1^2\omega_3^2(\mathbf{k}_2\cdot\mathbf{k}_4)+\omega_1^2\omega_4^2(\mathbf{k}_2\cdot\mathbf{k}_3) \\
&\quad+\omega_2^2\omega_3^2(\mathbf{k}_1\cdot\mathbf{k}_4)+\omega_2^2\omega_4^2(\mathbf{k}_1\cdot\mathbf{k}_3)+\omega_3^2\omega_4^2(\mathbf{k}_1\cdot\mathbf{k}_2)] \\
&= \frac{i}{8}\tilde{\lambda}_6\Big[32M^6\gamma^6+\frac{\gamma^2 s^4}{8M^2}\Big]
\end{aligned}
\tag{9.180}
$$

$$
\begin{aligned}
V_{\lambda_7} &= -4i\tilde{\lambda}_7[\omega_1\omega_2\omega_3^2\omega_4^2+\omega_1\omega_3\omega_2^2\omega_4^2+\omega_1\omega_4\omega_2^2\omega_3^2+\omega_2\omega_3\omega_1^2\omega_4^2+\omega_2\omega_4\omega_1^2\omega_3^2+\omega_3\omega_4\omega_1^2\omega_2^2] \\
&= -\frac{i}{16}\tilde{\lambda}_7[-2\omega_s^6-2\omega_u^6+2\omega_s^2\omega_u^4+2\omega_s^4\omega_u^2] \\
&= \frac{i}{4}\tilde{\lambda}_7\gamma^2 s^2\Big(M^2\gamma^2+\frac{s^2}{16M^2}\Big)
\end{aligned}
\tag{9.181}
$$

$$
\begin{aligned}
V_{\tilde{\lambda}_8} &= -i\tilde{\lambda}_8[\omega_1\omega_3(\mathbf{k}_2\cdot\mathbf{k}_4)(\mathbf{k}_3\cdot\mathbf{k}_4)+\text{perm}] \\
&= 2i\tilde{\lambda}_8[\omega_t^2(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{k}_4)+\omega_s^2(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_1\cdot\mathbf{k}_4)+\omega_u^2(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_1\cdot\mathbf{k}_3)] \\
&= -i\tilde{\lambda}_8(\omega_s^2-\omega_u^2)(\omega_s+\omega_u)^2\Big(s-\frac{1}{2}(\omega_s^2-\omega_u^2)\Big) \\
&= -2i\tilde{\lambda}_8 M^2\gamma^3(2-\gamma)s^2
\end{aligned}
\tag{9.182}
$$

$$
\begin{aligned}
V_{\lambda_9} &= -i\tilde{\lambda}_9[(\mathbf{k}_1\cdot\mathbf{k}_4)(\mathbf{k}_2\cdot\mathbf{k}_4)(\mathbf{k}_2\cdot\mathbf{k}_3)+\text{perm}] \\
&= \frac{i\tilde{\lambda}_9}{2}\Big[\frac{1}{4}(\omega_s^6+\omega_u^6)+s^2(\omega_s^2+\omega_u^2)+2s^2\omega_s\omega_u+s(\omega_u^4-\omega_s^4)+2s\omega_s\omega_u(\omega_u^2-\omega_s^2) \\
&\quad+\frac{1}{2}\omega_u\omega_s(\omega_s^4+\omega_u^4)-\frac{1}{4}\omega_s^2\omega_u^2(\omega_s^2+\omega_u^2)-\omega_s^3\omega_u^3\Big] \\
&= \frac{i\tilde{\lambda}_9}{2}[M^2\gamma^4 s^2-4M^2\gamma^3 s^2+4M^2\gamma^2 s^2]
\end{aligned}
\tag{9.183}
$$

$$
\begin{aligned}
V_{\lambda_{10}} &= -i\tilde{\lambda}_{10}[\omega_1^2(\mathbf{k}_2\cdot\mathbf{k}_4)(\mathbf{k}_3\cdot\mathbf{k}_4)+\text{perm}] \\
&= \frac{i\tilde{\lambda}_{10}}{2}\Big[\frac{1}{4}(\omega_u^6+\omega_s^6)+s(\omega_u^4-\omega_s^4)+s^2(\omega_s^2+\omega_u^2)-\frac{1}{4}\omega_s^2\omega_u^2(\omega_s^2+\omega_u^2)\Big] \\
&= \frac{i\tilde{\lambda}_{10}(\gamma-2)^2(16M^4\gamma^2+s^2)s^2}{64M^2}
\end{aligned}
\tag{9.184}
$$

$$
\begin{aligned}
V_{\lambda_{11}} &= -4i\tilde{\lambda}_{11}[\omega_1\omega_2(\mathbf{k}_3\cdot\mathbf{k}_4)^2+\omega_1\omega_3(\mathbf{k}_2\cdot\mathbf{k}_4)^2+\omega_1\omega_4(\mathbf{k}_2\cdot\mathbf{k}_3)^2 \\
&\quad+\omega_2\omega_3(\mathbf{k}_1\cdot\mathbf{k}_4)^2+\omega_2\omega_4(\mathbf{k}_1\cdot\mathbf{k}_3)^2+\omega_3\omega_4(\mathbf{k}_1\cdot\mathbf{k}_2)^2] \\
&= -\frac{i\tilde{\lambda}_{11}}{4}\Big[-\frac{1}{2}(\omega_s^6+\omega_u^6)-2\omega_u\omega_s(\omega_s^4+\omega_u^4)-4\omega_s^3\omega_u^3-\frac{7}{2}\omega_s^2\omega_u^2(\omega_u^2+\omega_s^2)\Big] \\
&= \frac{i\tilde{\lambda}_{11}}{4}[16M^6\gamma^6+M^2\gamma^4s^2]
\end{aligned}
\tag{9.185}
$$

$$
\begin{aligned}
V_{\lambda_{12}} &= -8i\tilde{\lambda}_{12}[(\mathbf{k}_1\cdot\mathbf{k}_2)^2+(\mathbf{k}_1\cdot\mathbf{k}_3)^2+(\mathbf{k}_1\cdot\mathbf{k}_4)^2] \\
&= i\tilde{\lambda}_{12}\Big[\frac{1}{8}(\omega_s^6+\omega_u^6)+\frac{3}{4}\omega_s\omega_u(\omega_s^4+\omega_u^4)+\frac{15}{8}\omega_s^2\omega_u^2(\omega_u^2+\omega_s^2)+\frac{5}{2}\omega_s^3\omega_u^3\Big] \\
&= 8i\gamma^6M^6.
\end{aligned}
\tag{9.186}
$$

Recall that the amplitude is related to the above vertices by $\mathcal{A}_{\lambda_j}=-iV_{\lambda_j}$. Let us now focus on the 3-point vertices. We first illustrate that t-channel contributions vanish in the forward limit. Let $\mathbf{q}=\mathbf{k}_1+\mathbf{k}_3$. Then we have that the vertex $\dot{\pi}(\partial_k^2\pi)^2$ yields $\omega_1\omega_3^2\mathbf{q}^2+\omega_3\omega_1^2\mathbf{q}^2+\omega_t\omega_1^2\omega_3^2\propto\omega_t$ since $\omega_t=\omega_1+\omega_3$. The vertex $\partial_k^2\pi\partial_i\dot{\pi}\partial_i\pi$ gives $\omega_1^2\omega_3(\mathbf{k}_3\cdot\mathbf{q})+\omega_1^2\omega_t(\mathbf{k}_3\cdot\mathbf{q})+\omega_3^2\omega_1(\mathbf{k}_1\cdot\mathbf{q})+\omega_t\omega_3^2(\mathbf{k}_1\cdot\mathbf{q})+\mathbf{q}^2\omega_3(\mathbf{k}_1\cdot\mathbf{k}_3)+\mathbf{q}^2\omega_1(\mathbf{k}_1\cdot\mathbf{k}_3)$, which is again $\propto\omega_t$. The reason is that it easily follows that $\omega_1^2\omega_3(\mathbf{k}_3\cdot\mathbf{q})+\omega_3^2\omega_1(\mathbf{k}_1\cdot\mathbf{q})\propto\omega_t$ using that $\mathbf{q}=\mathbf{k}_1+\mathbf{k}_3$. Similarly, the vertex $\partial_i\dot{\pi}\partial_j\pi\partial_i\partial_j\pi$ will yield $\omega_1(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_3\cdot\mathbf{q})+\omega_1(\mathbf{k}_1\cdot\mathbf{q})(\mathbf{k}_3\cdot\mathbf{q})+\omega_t(\mathbf{k}_1\cdot\mathbf{q})(\mathbf{k}_1\cdot\mathbf{k}_3)+\omega_t(\mathbf{k}_3\cdot\mathbf{q})(\mathbf{k}_1\cdot\mathbf{k}_3)+\omega_3(\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{q}\cdot\mathbf{k}_1)+\omega_3(\mathbf{q}\cdot\mathbf{k}_3)(\mathbf{q}\cdot\mathbf{k}_1)$ which is again $\propto\omega_t$ since $\omega_1(\mathbf{k}_3\cdot\mathbf{q})+\omega_3(\mathbf{q}\cdot\mathbf{k}_1)=\omega_1\omega_3\omega_t+\omega_t(\mathbf{k}_1\cdot\mathbf{k}_3)$. And the vertex $\dot{\pi}(\partial_i\partial_j\pi)^2$ gives $\omega_1(\mathbf{k}_3\cdot\mathbf{q})^2+\omega_t(\mathbf{k}_1\cdot\mathbf{k}_3)^2+\omega_3(\mathbf{k}_1\cdot\mathbf{q})$ which contains (beyond other terms which vanish in the forward limit) a non-trivial term $\omega_1^3\omega_3+\omega_3^3\omega_1$ which also vanishes in the forward limit. Hence indeed the scattering amplitudes in the t-channel will vanish in the forward limit.

Let us thus focus on the 3-point vertices for the s-channel and the u-channel amplitudes are found by crossing symmetry. We find the following vertices (let all momenta point inward, i.e. let $q=-p_1-p_2$):

$$
\begin{aligned}
V_{\beta_3}(p_1,p_2,q) &= -2\tilde{\beta}_3[\omega_1\omega_2^2\mathbf{q}^2+\omega_2\omega_1^2\mathbf{q}^2+\omega_q\omega_1^2\omega_2^2] \\
&= -2\tilde{\beta}_3[-\omega_q\omega_1\omega_2(\omega_1^2+\omega_2^2+2\mathbf{k}_1\cdot\mathbf{k}_2)+\omega_q\omega_1^2\omega_2^2]
\end{aligned}
\tag{9.187}
$$

$$
\begin{aligned}
V_{\beta_4}(p_1,p_2,q) &= -\tilde{\beta}_4[\omega_1^2\omega_2(\mathbf{k}_2\cdot\mathbf{q})+\omega_1^2\omega_q(\mathbf{q}\cdot\mathbf{k}_2)+\omega_2^2\omega_1(\mathbf{k}_1\cdot\mathbf{q}) \\
&\quad+\omega_2^2\omega_q(\mathbf{k}_1\cdot\mathbf{q})+\omega_q^2\omega_1(\mathbf{k}_1\cdot\mathbf{k}_2)+\omega_q^2\omega_2(\mathbf{k}_1\cdot\mathbf{k}_2)] \\
&= -\tilde{\beta}_4[-\omega_1^2\omega_2^2\omega_q+\omega_q(-\omega_1^2-\omega_2^2+\omega_1\omega_2-\omega_q^2)(\mathbf{k}_1\cdot\mathbf{k}_2)]
\end{aligned}
\tag{9.188}
$$

$$
\begin{aligned}
V_{\beta_5}(p_1,p_2,q) = & -\tilde{\beta}_5[\omega_1(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{q}\cdot\mathbf{k}_2)+\omega_1(\mathbf{k}_1\cdot\mathbf{q})(\mathbf{k}_2\cdot\mathbf{q})+\omega_q(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{q}\cdot\mathbf{k}_2)+\omega_q(\mathbf{k}_1\cdot\mathbf{q})(\mathbf{k}_1\cdot\mathbf{k}_2) \\
& +\omega_2(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{q}\cdot\mathbf{k}_1)+\omega_2(\mathbf{k}_2\cdot\mathbf{q})(\mathbf{k}_1\cdot\mathbf{q})] \\
= & -\tilde{\beta}_5[-2\omega_q(\mathbf{k}_1\cdot\mathbf{k}_2)^2-2\omega_q(\omega_1^2+\omega_2^2)(\mathbf{k}_1\cdot\mathbf{k}_2)+\omega_1\omega_2\omega_q(\mathbf{k}_1\cdot\mathbf{k}_2)-\omega_q\omega_1^2\omega_2^2]
\end{aligned}
\tag{9.189}
$$

$$
\begin{aligned}
V_{\beta_6} &= -2\tilde{\beta}_6[\omega_1(\mathbf{k}_2\cdot\mathbf{q})^2+\omega_q(\mathbf{k}_1\cdot\mathbf{k}_2)^2+\omega_2(\mathbf{k}_1\cdot\mathbf{q})^2] \\
&= -2\tilde{\beta}_6[\omega_1\omega_2(\omega_1^3+\omega_2^3)-2\omega_1\omega_2\omega_q(\mathbf{k}_1\cdot\mathbf{k}_2)].
\end{aligned}
\tag{9.190}
$$

These vertices are then used to compute scattering amplitudes corresponding to all combinations of vertices in Feynman diagrams. Note that the diagrams with vertices of two different types come with a symmetry factor of 2. We find the following amplitudes modulo a function which (and whose derivatives) vanishes in the forward limit:

$$
\begin{aligned}
i\mathcal{A}_{\beta_3^2} &= V_{\beta_3}(p_1,p_2,-q)(-i/s)V_{\beta_3}(p_3,p_4,q)+(u\leftrightarrow s) \\
&= \frac{i\tilde{\beta}_3^2\omega_s^2}{64s}[16s^2\omega_s^4-32s^2\omega_s^2\omega_u^2+16s^2\omega_u^4-24s\omega_s^6+40s\omega_s^4\omega_u^2 \\
&-8s\omega_s^2\omega_u^4-8s\omega_u^6+9\omega_s^8-12\omega_s^6\omega_u^2-2\omega_s^4\omega_u^4+4\omega_s^2\omega_u^6+\omega_u^8]+(u\leftrightarrow s) \\
&= \frac{i\tilde{\beta}_3^2}{64}\Big[\frac{\omega_s^2}{s}\Big(-32\gamma^4M^2s^3-8\gamma^3s^4-\frac{2\gamma^2}{M^2}s^5+16\gamma^2s^4+16\gamma^6M^4s^2 \\
&+8\gamma^5M^2s^3+3\gamma^4s^4+\frac{\gamma^3}{2M^2}s^5+\frac{\gamma^2}{16M^4}s^6\Big)\Big]+(u\leftrightarrow s)
\end{aligned}
\tag{9.191}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_3\beta_4} &= V_{\beta_3}(p_1,p_2,-q)(-2i/s)V_{\beta_4}(p_3,p_4,q)+(u\leftrightarrow s) \\
&= \frac{i\tilde{\beta}_3\tilde{\beta}_4}{64s}\Big[2\omega_u^8+5\omega_s^2\omega_u^6-7\omega_s^4\omega_u^4-9\omega_s^6\omega_u^2+9\omega_s^8-2s\omega_u^6 \\
&+20s\omega_s^2\omega_u^4+6s\omega_s^4\omega_u^2-24s\omega_s^6-24s^2\omega_u^4+8s^2\omega_s^2\omega_u^2+16s^2\omega_s^4\Big]+(u\leftrightarrow s) \\
&= \frac{i\tilde{\beta}_3\tilde{\beta}_4}{64s}\Big[\omega_s^2\Big(20\gamma^6M^4s^2+7\gamma^5M^2s^3+3\gamma^4s^4+\frac{7}{16M^2}\gamma^3s^5+\frac{5}{64M^4}\gamma^2s^6-40\gamma^5M^4s^2 \\
&-26\gamma^4M^2s^3-6\gamma^3s^4-\frac{13}{8M^2}\gamma^2s^5-\frac{5\gamma s^6}{32M^4}+40\gamma^3M^2s^3-4\gamma^2s^4+\frac{5}{2M^2}\gamma s^5\Big)\Big]+(u\leftrightarrow s)
\end{aligned}
\tag{9.192}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_3\beta_5} &= V_{\beta_3}(p_1,p_2,-q)(-2i/s)V_{\beta_5}(p_3,p_4,q)+(u\leftrightarrow s) \\
&= \frac{i\tilde{\beta}_3\tilde{\beta}_5}{64s}\omega_s^2(-32s^3\omega_s^2+32s^3\omega_u^2+80s^2\omega_s^4-64s^2\omega_s^2\omega_u^2-16s^2\omega_u^4-66s\omega_s^6 \\
&+62s\omega_s^4\omega_u^2+10s\omega_s^2\omega_u^4-6s\omega_u^6+18\omega_s^8-24\omega_s^6\omega_u^2-4\omega_s^4\omega_u^4+8\omega_s^2\omega_u^6+2\omega_u^8)+(u\leftrightarrow s) \\
&= \frac{i\tilde{\beta}_3\tilde{\beta}_5}{64s}\omega_s^2\Big(32M^4\gamma^6s^2+16\gamma^5M^2s^3+6\gamma^4s^4+\frac{\gamma^3}{M^2}s^5+\frac{\gamma^2}{8M^4}s^6-64M^4\gamma^5s^2-72\gamma^4M^2s^3 \\
&-22\gamma^3s^4-\frac{9\gamma^2}{2M^2}s^5-\frac{\gamma}{4M^4}s^6+96M^2\gamma^3s^3+32\gamma^2s^4+\frac{6\gamma}{M^2}s^5-32\gamma s^4\Big)+(u\leftrightarrow s)
\end{aligned}
\tag{9.193}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_3\beta_6} &= V_{\beta_3}(p_1,p_2,-q)(-2i/s)V_{\beta_6}(p_3,p_4,q) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_3\tilde{\beta}_6}{64s}\omega_s^2(2\omega_u^8 + 8\omega_s^2\omega_u^6 - 4\omega_s^4\omega_u^4 - 24\omega_s^6\omega_u^2 + 18\omega_s^8 - 16s\omega_u^6 - 16s\omega_s^2\omega_u^4\\
&+ 80s\omega_s^4\omega_u^2 - 48s\omega_s^6 + 32s^2\omega_u^4 - 64s^2\omega_s^2\omega_u^2 + 32s^2\omega_s^4) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_3\tilde{\beta}_6}{64s}\omega_s^2\Big(32\gamma^6M^4s^2 + 16\gamma^5M^2s^3 + 6\gamma^4s^4 + \frac{\gamma^3}{M^2}s^5 + \frac{\gamma^2}{8M^4}s^6\\
&- 64\gamma^4M^2s^3 - 16\gamma^3s^4 - \frac{4\gamma^2}{M^2}s^5 + 32\gamma^2s^4\Big) + (u\leftrightarrow s)
\end{aligned}
\tag{9.194}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_4^2} &= V_{\beta_4}(p_1,p_2,-q)(-i/s)V_{\beta_4}(p_3,p_4,q) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_4^2}{256s}\omega_s^2(4\omega_u^8 + 4\omega_s^2\omega_u^6 - 11\omega_s^4\omega_u^4 - 6\omega_s^6\omega_u^2 + 9\omega_s^8 + 24s\omega_u^6\\
&+ 28s\omega_s^2\omega_u^4 - 28s\omega_s^4\omega_u^2 - 24s\omega_s^6 + 36s^2\omega_u^4 + 48s^2\omega_u^2\omega_s^2 + 16s^2\omega_s^4) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_4^2\omega_s^2}{256s}\Big(25\gamma^6M^4s^2 + 5\gamma^5M^2s^3 + \frac{27}{8}\gamma^4s^4 + \frac{5}{16M^2}\gamma^3s^5 + \frac{25\gamma^2}{256M^4}s^6 - 100\gamma^5M^4s^2\\
&- \frac{23}{2}\gamma^3s^4 - \frac{25}{64M^4}\gamma s^6 + 100\gamma^4M^4s^2 - 20\gamma^3M^2s^3 + \frac{27}{2}\gamma^2s^4 - \frac{5\gamma}{4M^2}s^5 + \frac{25}{64M^4}s^6\Big) + (u\leftrightarrow s)
\end{aligned}
\tag{9.195}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_4\beta_5} &= V_{\beta_4}(p_1,p_2,-q)(-2i/s)V_{\beta_5}(p_3,p_4,q) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_4\tilde{\beta}_5\omega_s^2}{128s}(-66s\omega_s^6 - 12s\omega_s^4\omega_u^2 + 62s\omega_s^2\omega_u^4 + 16s\omega_u^6 + 18\omega_s^8 - 18\omega_s^6\omega_u^2\\
&+ 10\omega_s^2\omega_u^6 + 4\omega_u^8 - 14\omega_s^4\omega_u^4 - 32s^3\omega_s^2 - 48s^3\omega_u^2 + 80s^2\omega_s^4 + 84s^2\omega_u^2\omega_s^2 - 4s^2\omega_u^4) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_4\tilde{\beta}_5\omega_s^2}{128s}\Big(40\gamma^6M^4s^2 + 14\gamma^5M^2s^3 + 6\gamma^4s^4 + \frac{7\gamma^3}{8M^2}s^5 + \frac{5}{32M^4}\gamma^2s^6 - 160\gamma^5M^4s^2\\
&- 50\gamma^4M^2s^3 - 21\gamma^3s^4 - \frac{25}{8M^2}\gamma^2s^5\\
&- \frac{5\gamma}{8M^4}s^6 + 160\gamma^4M^4s^2 + 84\gamma^3M^2s^3 + 18\gamma^2s^4 + \frac{21}{4M^2}\gamma s^5\\
&+ \frac{5}{8M^4}s^6 - 80\gamma^2M^2s^3 + 8\gamma s^4 - \frac{5}{M^2}s^5\Big) + (u\leftrightarrow s)
\end{aligned}
\tag{9.196}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_4\beta_6} &= V_{\beta_4}(p_1,p_2,-q)(-2i/s)V_{\beta_6}(p_3,p_4,q) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_4\tilde{\beta}_6\omega_s^2}{128s}(4\omega_u^8 + 10\omega_s^2\omega_u^6 - 14\omega_s^4\omega_u^4 - 18\omega_s^6\omega_u^2 + 18\omega_s^8 - 4s\omega_u^6\\
&+ 40s\omega_s^2\omega_u^4 + 12s\omega_s^4\omega_u^2 - 48s\omega_s^6 - 48s^2\omega_u^4 + 16s^2\omega_s^2\omega_u^2 + 32s^2\omega + s^4) + (u\leftrightarrow s)\\
&= \frac{i\tilde{\beta}_4\tilde{\beta}_6}{128s}\omega_s^2\Big(40\gamma^6M^4s^2 + 14\gamma^5M^2s^3 + 6\gamma^4s^4 + \frac{7\gamma^3}{8M^2}s^5 + \frac{5\gamma^2}{32M^4}s^6 - 80\gamma^5M^4s^2\\
&- 52\gamma^4M^2s^3 - 12\gamma^3s^4 - \frac{13\gamma^2}{4M^2}s^5 - \frac{5\gamma}{16M^4}s^6 + 80\gamma^3M^2s^3 - 8\gamma^2s^4 + \frac{5\gamma}{M^2}s^5\Big) + (u\leftrightarrow s)
\end{aligned}
\tag{9.197}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_5^2} &= V_{\beta_5}(p_1,p_2,-q)(-i/s)V_{\beta_5}(p_3,p_4,q) + (u \leftrightarrow s) \\
&= \frac{i\tilde{\beta}_5^2}{256s}\omega_s^2\Big(4\omega_u^8 + 16\omega_s^2\omega_u^6 - 8\omega_u^4\omega_s^4 - 48\omega_s^6\omega_u^2 + 36\omega_s^8 + 8s\omega_u^6 + 72s\omega_s^2\omega_u^4 + 88s\omega_s^4\omega_u^2 \\
&- 168s\omega_s^6 - 28s^2\omega_u^4 - 8s^2\omega_s^2\omega_u^2 + 292s^2\omega_s^4 - 32s^3\omega_u^2 - 224s^3\omega_s^2 + 64s^4\Big) + (u \leftrightarrow s) \\
&= \frac{i\tilde{\beta}_5^2}{256s}\omega_s^2\Big(64\gamma^6M^4s^2 + 32\gamma^5M^2s^3 + 12\gamma^4s^4 + \frac{2\gamma^3}{M^2}s^5 \\
&+ \frac{\gamma^2}{4M^4}s^6 - 256\gamma^5M^4s^2 - 160\gamma^4M^2s^3 - 56\gamma^3s^4 \\
&- \frac{10\gamma^2}{M^2}s^5 - \frac{\gamma}{M^4}s^6 + 256\gamma^4M^4s^2 + 320\gamma^3M^2s^3 + 100\gamma^2s^4 + \frac{20\gamma}{M^2}s^5 \\
&+ \frac{s^6}{M^4} + 64s^4 - 256\gamma^2M^2s^3 - 96\gamma s^4 - \frac{16}{M^2}s^5\Big) + (u \leftrightarrow s)
\end{aligned} \tag{9.198}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_5\beta_6} &= V_{\beta_5}(p_1,p_2,-q)(-2i/s)V_{\beta_6}(p_3,p_4,q) + (u \leftrightarrow s) \\
&= \frac{i\tilde{\beta}_5\tilde{\beta}_6}{128s}\omega_s^2\Big(4\omega_u^8 + 16\omega_s^2\omega_u^6 - 8\omega_s^4\omega_u^4 - 48\omega_s^6\omega_u^2 + 36\omega_s^8 - 12s\omega_u^6 + 20s\omega_s^2\omega_u^4 + 124\omega_s^4\omega_u^2 \\
&- 132s\omega_s^6 - 32s^2\omega_u^4 - 128s^2\omega_s^2\omega_u^2 + 160s^2\omega_s^4 + 64s^3\omega_u^2 - 64s^3\omega_s^2\Big) + (u \leftrightarrow s) \\
&= \frac{i\tilde{\beta}_5\tilde{\beta}_6}{128s}\omega_s^2\Big(64\gamma^6M^4s^2 + 32\gamma^5M^2s^3 + 12\gamma^4s^4 + \frac{2\gamma^3}{M^2}s^5 + \frac{\gamma^2}{4M^4}s^6 - 128\gamma^5M^4s^2 \\
&- 144\gamma^4M^2s^3 - 44\gamma^3s^4 - \frac{9\gamma^2}{M^2}s^5 - \frac{\gamma}{2M^4}s^6 + 192\gamma^3M^2s^3 + 64\gamma^2s^4 + \frac{12\gamma}{M^2}s^5 - 64\gamma s^4\Big) \\
&+ (u \leftrightarrow s)
\end{aligned} \tag{9.199}
$$

$$
\begin{aligned}
i\mathcal{A}_{\beta_6^2} &= V_{\beta_6}(p_1,p_2,-q)(-i/s)V_{\beta_6}(p_3,p_4,q) + (u \leftrightarrow s) \\
&= -\frac{i\tilde{\beta}_6^2}{128s}\omega_s^2(-64s^2\omega_s^4 + 128s^2\omega_s^2\omega_u^2 - 64s^2\omega_u^4 + 64s\omega_s^6 - 192s\omega_s^4\omega_u^2 + 192s\omega_s^2\omega_u^4 \\
&- 64s\omega_u^6 - 12\omega_s^8 + 80\omega_s^6\omega_u^2 - 104\omega_s^4\omega_u^4 + 16\omega_s^2\omega_u^6 + 20\omega_u^8) + (u \leftrightarrow s) \\
&= -\frac{i\tilde{\beta}_6^2}{128s}\omega_s^2\Big[64\gamma^6M^4s^2 - 32\gamma^5M^2s^3 - 4\gamma^4s^4 - \frac{2\gamma^3}{M^2}s^5 + \frac{\gamma^2}{4M^4}s^6 + 64\gamma^3s^4 - 64\gamma^2s^4\Big] + (u \leftrightarrow s).
\end{aligned} \tag{9.200}
$$

Adding the above computed amplitudes gives the total scattering amplitude $\mathcal{A} = \sum_i \mathcal{A}_{\lambda_i} + \sum_{i,j} \mathcal{A}_{\beta_i\beta_j}$. The amplitude takes the form (7.103) with coefficients given by:

$$
\begin{aligned}
\chi_1 &= \frac{M^4}{256}[\gamma^6(-256\beta^2+9\alpha^2)-36\gamma^5\alpha^2+36\gamma^4\alpha^2]\\
\chi_2 &= \frac{M^2}{256}[\gamma^5(32\beta^2+9\alpha^2)+\gamma^4(256\beta^2+100\alpha^2)+132\gamma^3\alpha^2-96\gamma^2\alpha^2]\\
\chi_3 &= \frac{1}{256}\Big[\gamma^4\Big(-4\beta^2+\frac{27}{8}\alpha^2\Big)+\gamma^3\Big(-96\beta^2-\frac{51}{2}\alpha^2\Big)+\gamma^2\Big(\frac{155}{2}\alpha^2+64\beta^2\Big)-112\gamma\alpha^2+64\alpha^2\Big]\\
\chi_4 &= \frac{1}{256M^2}\Big[\gamma^3\Big(2\beta^2+\frac{9}{4}\alpha^2\Big)+\gamma^2\Big(8\beta^2+\frac{25}{4}\alpha^2\Big)+\frac{33\gamma}{2}\alpha^2-6\alpha^2\Big]\\
\chi_5 &= \frac{1}{256M^4}\Big[\gamma^2\Big(\frac{9}{256}\alpha^2-\frac{3}{4}\beta^2\Big)-\frac{9\gamma}{64}\alpha^2+9\alpha^2\Big]\\
f_2 &= M^2\Big[\kappa\Big(-\frac{7\gamma^4}{4}+\frac{5\gamma^3}{2}+2\gamma^2+10\gamma-20\Big)+\sigma\Big(-\frac{\gamma^4}{2}+\gamma^3+4\gamma-8\Big)\Big]\\
f_4 &= \frac{\gamma}{64M^2}[(4-3\gamma)\sigma-2(7\gamma-9)\kappa].
\end{aligned}
\tag{9.201}
$$

And the constant and the function $f(t)$ which appear in the scattering amplitude will not be spelled out explicitly since they are not important for the positivity bounds we computed.

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