Parity, Dynamos and Cosmic Magnetogenesis

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Author :	M.J.N. Versnel
Student ID :	s1861115
Supervisor :	Prof.dr. A. Boyarsky
Second corrector :	Dr. E. Sellentin

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M.J.N. Versnel

Huygens-Kamerlingh Onnes Laboratory, Leiden University
P.O. Box 9500, 2300 RA Leiden, The Netherlands

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Abstract

The evolution of magnetic helicity and the alpha effect in galaxy clusters is studied and compared to the amplification of magnetic fields in the early universe by the chiral magnetic effect, which are both systems with violated parity. A dynamical system for the quantities magnetic helicity and alpha is derived using nonlinear dynamo theory. The solutions of this system show that an inverse cascade takes place, on the time scale of millions of years.

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Chapter 1

Introduction

Magnetic fields are ubiquitous in space and play important roles in the generation and behavior of stars, black holes, cosmic rays and galaxies. They are, thus, of great interest in many areas of astrophysical research. Yet much remains unclear about their origin and evolution.

Large-scale magnetic fields of the magnitude of microgauss have been observed in galaxies and galaxy clusters. They are thought to be formed and maintained by a specific kind of interaction between the flows of interstellar plasma and the magnetic fields themselves. Turbulent currents generate magnetic fields, which reinforce these currents. A positive feedback loop ensues, causing the magnetic field to grow rapidly in strength, until dampening effects take over and equilibrium is reached. A closed system in which this interaction occurs is called a dynamo and the study of dynamos is called dynamo theory.* A completely satisfactory theory of dynamos is yet to be developed.

But consideration of a mostly conserved quantity called magnetic helicity has accelerated understanding. It can be thought of as the amount of twisting of magnetic field lines and interlinking of field line loops. Such features tend to 'relax' on the small scale, but since the helicity is mostly conserved, the helicity, with its associated energy, is transported to the large scale. This is known as the inverse cascade.

Dynamos are mathematically described by Maxwell's laws and the Navier-Stokes equation. Unfortunately, this last equation, which describes the motion of fluids, makes an analytical approach very difficult in any system in which it appears. Numerical simulations provide a lot of knowledge and new insights, but they have a flaw. Because of possible positive feedback loops, the start of the dynamo process is unstable. A small discretization error can therefore affect the results tremendously. Here, an analytical model provides a control, as well as intuition.

The formation of a galactic magnetic field by the dynamo mechanism requires an already existing, much weaker magnetic field. Such fields, known as a seed fields, should then be present in intergalactic space. Fields with a strength of at least $3 \cdot 10^{-16}$ gauss have indeed been observed in these regions. [14]

Some magnetic fields were probably created immediately after the Big Bang. In this period, the universe consisted of a very hot plasma. Elementary particles have a quantum property called chirality, which can be either left-handed or right-handed. The primordial plasma is thought to have had a chirality imbalance. In presence of a mag-

*There are also devices called dynamos, which have little to do with this concept.

netic field, currents are generated in such a plasma. This process is called the chiral magnetic effect (CME) and it caused an exponential amplification of the magnetic field. This amplification was dampened by another quantum effect, the chiral anomaly. [5]. Chirality is not a conserved property, so over time the chirality imbalance disappeared and these processes ceased. But the conservation of helicity prevented the magnetic fields from decaying too quickly. In fact, they seem to have survived to the present, thus explaining the provenance of the seed fields.

This amplification of magnetic fields by the CME in the early universe will be called 'CFA' (CME Field Amplification) for short. Remarkable analogies exist between this CFA and dynamos. The aim of this thesis is to compare these two physical systems and to create a model for dynamos based on the CFA process.

Conventions

Three-vector quantities will be denoted by boldface letters ($\boldsymbol{A}$), scalar quantities, four-vectors and higher order tensors by regular letters (A). Unless specified otherwise, a natural unit system is used, where quantities such as Boltzmann's constant k_B and the magnetic constant (μ_0) are unity. A partial derivative such as $\frac{\partial}{\partial t}$ is written as ∂_t. The Einstein summation convention is used. When it is implied by a Latin letter, the three spatial components are meant. A Greek letter indicates the four spacetime dimensions with the Minkowski metric $(1, -1, -1, -1)$.

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Chapter 2

Mathematical Introduction

Many physical systems are described by fields, which are physical quantities consisting of either scalar or vector values at every point in space and time. Barring relativity, they are functions of the form $f : D \to \mathbb{R}$ for scalar fields and $g : D \to \mathbb{R}^3$ for vector fields, to state it mathematically. In special relativity, we most often talk about four-vector fields $h : D \to \mathbb{R}^4$. Here, D is generally a convex subset of $\mathbb{R}^4$. The four components of the domain, or the variables, are associated with time and three spatial dimensions. Such functions f, g and h are assumed to be continuous and smooth. Examples include the scalar temperature and Schrödinger fields, the vectorial velocity and magnetic fields, and the four-vectorial current density and Dirac spinor* fields.

2.1 Partial Differential Equations

These fields must obey certain partial differential equations (PDEs) which characterize the physical system in question. In the case of ordinary differential equations with only one variable, usually interpreted as time, the equations together with the values of the fields at $t = 0$ give a unique solution. For partial differential equations, not only the initial conditions are needed, but also the spatial boundary conditions at every point in time. Fortunately, for most systems, these are quite simple: the fields vanish (have value zero) at the boundaries at all times. The existence and uniqueness of solutions are usually guaranteed for physical reasons. However, it is an interesting mathematical problem, which has never been solved for the Navier-Stokes equation of fluid dynamics. Still, even without considering existence problems and with simple boundary conditions, providing solutions to PDE problems can be very challenging. Analytically it is most often impossible, and numerically, it usually requires a lot of computing power.

2.2 Instability

Differential equations can have exponentially growing solutions. The main equation of this thesis (4.6) has a possible solution [16]

$$\overline{\boldsymbol{B}} = B_0(\cos kz, \sin kz, 0)e^{Ct},$$

*Dirac spinors are not truly four-vectors, but they have four components.

where k is some wave number and $C = -(\lambda + \beta)k^2 \pm \alpha k$. This means that two sets of slightly different initial conditions can provide drastically different solutions. This phenomenon is called instability. Since numerical simulations are always discrete, a small discretization offset can create unphysical artifacts in unstable systems. Therefore, it is important to have a good understanding of the sources of growth and dissipation.

2.3 Dynamical Systems

Physical systems are often simplified to remove any spatial derivatives, for example by only considering global quantities. This will be done in this thesis as well. The result is a dynamical system, which, for the purposes of this thesis, is simply a set of ordinary differential equations. The simplest dynamical systems are linear dynamical systems, which are of the form

$$\boldsymbol{v}' = A\boldsymbol{v},$$

where $\boldsymbol{v} = (v^1(t), v^2(t), \ldots, v^n(t))$ is a vector in phase space, whose components are the physical quantities of interest, and A is a constant matrix. These systems can be solved analytically. However, in most systems, the components of $\boldsymbol{v}$ also depend on each other. In contrast to PDEs, these dynamical systems can be solved numerically rather easily. Instability also occurs in dynamical systems, but in this thesis, the system is simplified for the exact reason to extract the sources of growth and dissipation.

2.4 Solving Dynamical Systems

The Euler method is the most straightforward method of solving dynamical systems. Time is evenly discretized $(t_0, t_1, t_2, \ldots)$, where the difference between two consecutive points is a small quantity of time h. Differentiation approximates a function by a linear function, so the difference $\boldsymbol{v}(t_{n+1}) - \boldsymbol{v}(t_n)$ is approximately $\boldsymbol{v}'(t_n) \cdot h$, for all integers $n \geq 0$. The result is the recursive rule $\boldsymbol{v}(t_{n+1}) = \boldsymbol{v}(t_n) + \boldsymbol{v}'(t_n) \cdot h$. If $\boldsymbol{v}(t_0)$ is known, then from the differential equation, also $\boldsymbol{v}'(t_0)$ is known. The recursive formula provides the rest.

A bit more accurate is the classical Runge-Kutta method. It is related to Simpson's rule

$$\boldsymbol{v}(t+h) - \boldsymbol{v}(t) \approx \frac{h}{6}(\boldsymbol{v}'(t) + 4\boldsymbol{v}'(t+h/2) + \boldsymbol{v}'(t+h)),$$

which itself is based on the Lagrange interpolating polynomial of the points $(t, \boldsymbol{v}(t))$ and $(t+h, \boldsymbol{v}(t+h))$. For clarity, the dynamical system is now written as $\boldsymbol{v}'(t) = f(t, \boldsymbol{v})$. Suppose that $\boldsymbol{v}_n := \boldsymbol{v}(t_n)$ is already known. The classical Runge-Kutta method is the following algorithm.

1. Then, like in the Euler method, $\boldsymbol{v}(t_n + h/2)$ can be estimated by $\boldsymbol{v}_n + \frac{h}{2}k_1$, where $k_1 := f(t_n, \boldsymbol{v}_n)$.

2. Now, $\boldsymbol{v}'(t_n + h/2)$ can be estimated by $k_2 := f(t_n + h/2, \boldsymbol{v}_n + \frac{h}{2}k_1)$.

3. A new estimate for $\boldsymbol{v}(t_n + h/2)$ can be made $\boldsymbol{v}(t_n + h/2) \approx \boldsymbol{v}_n + \frac{h}{2}k_2$.

4. This leads to a new estimate $\boldsymbol{v}'(t_n + h/2) \approx \boldsymbol{k}_3 := f(t_n + h/2, \boldsymbol{v}_n + \frac{h}{2}k_2)$.

5. Then, $\boldsymbol{v}(t + h) \approx \boldsymbol{v}_n + hk_3$.

6. The last derivative we estimate is $\boldsymbol{v}'(t_n + h) \approx f(t_n + h, \boldsymbol{v}_n + hk_3)$.

7. These four derivatives are used to decide $\boldsymbol{v}(t_{n+1})$. From Simpson's rule, we know that the derivatives at $t_n + h/2$ should have a weight of $2/3$. This is divided equally over k_2 and k_3. The other derivatives k_1 and k_4 have weight $1/6$. Hence $\boldsymbol{v}(t_{n+1}) = \boldsymbol{v}_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$.

The error of this method per time step is of the order h^5. [1]

However, the method used for this thesis, integrate.solve_ivp from the Scipy package of Python, uses a more complicated algorithm. This is the Dormand-Prince method [9], which evaluates the derivative seven times between t_n and $t_n + h$. Also, more complicated linear combinations of the $k_1, k_2, \ldots, k_m$ are used to calculate k_{m+1}, optimized to produce the smallest error in fifth order. But most importantly, it is an adaptive method. That means it estimates the error. When it is too high, the time step is decreased and the iteration is repeated, guaranteeing an accurate solution.

Chapter 3

Parity

One might assume that, when a physical system is reflected, its vector quantities are reflected and its scalar quantites stay the same. But this is not true for all physical quantities. For example, imagine a top spinning clockwise, meaning its angular momentum vector points downward. If a vertical mirror is placed next to it, the mirror image will spin counterclockwise, and the angular momentum will point upward. Such vectors, which are not only reflected but also reversed, are known as pseudovectors or axial vectors. Vectors that are only reflected are called polar vectors. For scalars, it is the other way around. True scalars are invariant under reflections, and pseudoscalars change sign.

The cross product of two polar vectors, such as angular momentum, is a pseudovector. This is because the definition of the cross product depends on the coordinate system, whether it is right handed or left handed. The rotation of a polar vector is therefore also a pseudovector and vice versa. The cross product of two pseudovectors is again a pseudovector, and the cross product of a polar vector and a pseudovector is a polar vector. So the parity of the polar vectors decides the outcome, an even number of polar vectors in a cross product produces a pseudovector, an odd number produces a polar vector. Other kinds of products, most importantly scalar products and dot products, do not depend on the coordinate system. Here, the parity of the pseudovectors and pseudoscalars decides the result.

More precisely, a vector quantity is a pseudovector if it is invariant under the parity transformation P. This is defined as a sign flip in the spatial coordinates

$$t' = t \qquad x' = -x \qquad y' = -y \qquad z' = -z,$$

or a reflection through the origin. It is both a Galilean and a Lorentz transformation. It would be natural to assume that our universe is not fundamentally distinguishable from its mirror image. However, this does not seem to be the case. The weak interaction can cause parity violation, so the 'total amount of parity', very loosely defined as all pseudoscalars and pseudovectors combined, can be nonzero.

Chirality is a pseudoscalar. There was a non-zero total chirality in the universe, probably created by weak processes. This chirality imbalance disappeared, but the parity asymmetry did not. Under the chiral anomaly, he chirality was converted into magnetic helicity, another pseudoscalar explored in the next chapter.

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Chapter 4

Dynamo theory

Dynamo theory attempts to explain dynamos, that is the generation of a stable magnetic field by a conductive fluid. This section will derive the main equations of the theory in a mean field approach. It will also explore the concept of magnetic helicity. The first two sections will mainly follow [16].

This thesis will only consider large scale dynamos, which are larger than the typical scale of the turbulent velocity field. They are named in constrast to small scale or turbulent dynamos, see for example [6].

4.1 Mean Field Approach

A region of space containing a galaxy or galaxy cluster with its magnetosphere will be considered. This region V will be the global scale and an average of a quantity A over its volume will be written as

$$\langle A \rangle := \frac{\int_V A(r)\, dr}{\int_V dr}.$$

The region extends so far in space, that the relevant quantities are either negligible at the boundaries, or parallel to them, so that the flux is zero.* This simplifies many global integrals, because divergence terms disappear. This follows from the divergence theorem:

$$\int_V \nabla \cdot \boldsymbol{A}\, dr = \int_{\partial V} \boldsymbol{A} \cdot d\boldsymbol{a} = 0,$$

where ∂V is the boundary. Gradient terms also disappear: for any constant $\boldsymbol{C}$ we have $\nabla \cdot (A\boldsymbol{C}) = (\nabla A) \cdot \boldsymbol{C} + A\nabla \cdot \boldsymbol{C} = (\nabla A) \cdot \boldsymbol{C}$, so

$$\int_V (\nabla A) \cdot \boldsymbol{C}\, dr = \int_V \nabla \cdot (A\boldsymbol{C}) dr = 0.$$

Since this holds for any constant $\boldsymbol{C}$, we also have $\int_V \nabla A\, dr = 0$.

The physical quantities will be separated into two modes: the large scale with a characteristic length of k_2^{-1} and the small, or 'turbulent' scale of k_1^{-1}.† Large scale quantities by a bar $\overline{A}$ and small scale quantities (or fluctuations) by undercase letters

*Assuming periodic boundaries would be equivalent.

†Nota bene: many authors define this the other way around.

a, so $A = \overline{A} + a$. The large scale quantity at a point $\mathbf{r}$ is nothing more than an average of the quantity around a neighborhood of scale k_2^{-1}. Then it can be easily seen that for any quantities A and B, and any constant C, the following hold

$$\overline{A+B} = \overline{A} + \overline{B}, \quad \overline{CA} = C\overline{A}, \quad \overline{\partial_x A} = \partial_x \overline{A}, \quad \overline{a} = 0.$$

The average of a product is, however, not the product of the averages. Since $AB = (\overline{A} + a)(\overline{B} + b) = \overline{A} \cdot \overline{B} + \overline{A}b + a\overline{B} + ab$, we have

$$\begin{aligned}
\overline{AB} &= \overline{\overline{A} \cdot \overline{B}} + \overline{\overline{A}b} + \overline{a\overline{B}} + \overline{ab} \\
&= \overline{A} \cdot \overline{B} + \overline{A} \cdot \overline{b} + \overline{a} \cdot \overline{B} + \overline{ab} \\
&= \overline{A} \cdot \overline{B} + \overline{ab}.
\end{aligned}$$

Large scale quantities could be pulled out of the integrals, because they were constant under these integrations.

4.2 The Induction Equation

From fluid mechanics, we know that large scale flows generally break up into smaller flows. So in a sense, the energy is transported from the large scale to the small scale, eventually being converted into heat. This is known as the energy cascade. For this reason, we can assume that the large scale velocity field is zero, $\overline{\boldsymbol{V}} = 0$ or $\boldsymbol{V} = \boldsymbol{v}$.

In space, charged particles can move freely, so the conductivity σ is high. This prevents the buildup of large electric fields, as the charges will rearrange themselves to lower any electric potential differences. We assume that the conductivity is constant. Ohm's law in a fluid is

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{4.1}$$

where $\boldsymbol{J}$ is the current density, simply called current in this thesis, $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field. It is best to have a differential equation in one quantity only, which in our case will be the magnetic field. It will be derived using the Maxwell equations

$$\nabla \times \boldsymbol{E} = -\partial_t \boldsymbol{B} \quad \nabla \cdot \boldsymbol{B} = 0 \quad \nabla \times \boldsymbol{B} = \boldsymbol{J}.$$

The term $\partial_t \boldsymbol{E}$ that normally appears in the Ampère-Maxwell equation can be neglected, because the current (with a factor σ in Ohm's law) will be much larger. We will use several times that the vector Laplacian of the magnetic field equals

$$\begin{aligned}
\nabla^2 \boldsymbol{B} &= \nabla(\nabla \cdot \boldsymbol{B}) - \nabla \times (\nabla \times \boldsymbol{B}) \\
&= -\nabla \times (\nabla \times \boldsymbol{B}) \\
&= -\nabla \times \boldsymbol{J}.
\end{aligned}$$

Applying the rotation operator on both sides of Ohm's law (4.1) then gives

$$\begin{aligned}
-\nabla^2 \boldsymbol{B} &= \sigma(\nabla \times \boldsymbol{E} + \nabla \times (\boldsymbol{v} \times \boldsymbol{B})) \\
\partial_t \boldsymbol{B} &= \lambda \nabla^2 \boldsymbol{B} + \nabla \times (\boldsymbol{v} \times \boldsymbol{B}),
\end{aligned} \tag{4.2}$$

where the terms were rearranged and Faraday's law was used. This $\lambda = \sigma^{-1}$ is called the magnetic diffusivity.‡ Equation (4.2) is known as the induction equation of magnetohydrodynamics. The large scale version of this equation is

$$\begin{aligned}\partial_t \overline{\boldsymbol{B}} &= \lambda \nabla^2 \overline{\boldsymbol{B}} + \nabla \times \overline{\boldsymbol{v} \times \boldsymbol{B}} \\ &= \lambda \nabla^2 \overline{\boldsymbol{B}} + \nabla \times (\overline{\boldsymbol{v}} \times \overline{\boldsymbol{B}} + \overline{\boldsymbol{v} \times \boldsymbol{b}}) \\ &= \lambda \nabla^2 \overline{\boldsymbol{B}} + \nabla \times \boldsymbol{\mathcal{E}}, \end{aligned} \tag{4.3}$$

where $\boldsymbol{\mathcal{E}} := \overline{\boldsymbol{v} \times \boldsymbol{b}}$ is an electromotive force.

4.3 The alpha Effect

The small scale version of the induction equation is given by the difference of (4.2) and (4.3)

$$\begin{aligned}\partial_t \boldsymbol{b} &= \lambda \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{v} \times \boldsymbol{B} - \overline{\boldsymbol{v} \times \boldsymbol{b}}) \\ &= \lambda \nabla^2 \boldsymbol{b} + \nabla \times (\boldsymbol{v} \times \overline{\boldsymbol{B}} + \boldsymbol{v} \times \boldsymbol{b} - \overline{\boldsymbol{v} \times \boldsymbol{b}}).\end{aligned}$$

After rearranging

$$\partial_t \boldsymbol{b} - \lambda \nabla^2 \boldsymbol{b} - \nabla \times (\boldsymbol{v} \times \boldsymbol{b} - \overline{\boldsymbol{v} \times \boldsymbol{b}}) = \nabla \times (\boldsymbol{v} \times \overline{\boldsymbol{B}}),$$

the left hand side is a linear operator applied to $\boldsymbol{b}$ and the right side is linear in $\boldsymbol{v}$ and $\overline{\boldsymbol{B}}$. Consequently, $\boldsymbol{b}$ can be separated in a part linear in $\overline{\boldsymbol{B}}$ and $\boldsymbol{\mathcal{E}}_{(\mathbf{0})}$, a part only dependent on $\boldsymbol{v}$. The same holds for $\boldsymbol{\mathcal{E}}$. The turbulent quantities $\boldsymbol{b}$ and $\boldsymbol{v}$ are assumed to be distributed homogeneously and isotropically, but importantly, not necessarily axisymmetrically. This means that any large scale quantity depending on them is invariant under translations and rotations of these fields. So the vector $\boldsymbol{\mathcal{E}}_{(\mathbf{0})}$ must be zero.[17] In conclusion, $\boldsymbol{\mathcal{E}}$ can be written as a continuous linear combination

$$\mathcal{E}^i(\boldsymbol{x}, t) = \int_0^\infty \int_V K^i_j(\boldsymbol{x}, t; \boldsymbol{\xi}, \tau) \overline{B}^j(\boldsymbol{x} + \boldsymbol{\xi}, t - \tau) d\xi \, d\tau,$$

where K is a second rank pseudotensor. Here and further on, the Einstein summation convention is used.

The tensor can be separated in $\mathcal{A}$, a part symmetric in $\boldsymbol{\xi}$, and K', a part antisymmetric in $\boldsymbol{\xi}$. The second part can be written as negative the divergence of a third order tensor $K'^i_j = -\partial_k \mathcal{B}^{ik}_j$. This tensor is again symmetric in $\boldsymbol{\xi}$. With the assumed boundary conditions, integration by parts then gives

$$-\int_V \partial_k \mathcal{B}^{ik}_j \overline{B}^j d\xi = \int_V \mathcal{B}^{ik}_j \partial_k \overline{B}^j d\xi.$$

Note that

$$\frac{\partial \overline{\boldsymbol{B}}(\boldsymbol{x} + \boldsymbol{\xi})}{\partial \xi_k} = \frac{\partial \overline{\boldsymbol{B}}(\boldsymbol{x} + \boldsymbol{\xi})}{\partial x_k},$$

‡In SI units, it is $\eta = \frac{1}{\mu_0 \sigma}$.

so the notation ∂_k is unambiguous. This results in

$$\mathcal{E}^i = \int_0^\infty \int_V \mathcal{A}^i_j \overline{B}^j + \mathcal{B}^{ik}_j \partial_k \overline{B}^j \, d\xi \, d\tau.$$

Naturally, it is expected that $\boldsymbol{\mathcal{E}}(\boldsymbol{x}, t)$ depends only values of $\boldsymbol{b}$ and $\boldsymbol{v}$ close to the point $(\boldsymbol{x}, t)$. The behavior of these fluctuations outside the typical length scale k_1^{-1} and time scale is irrelevant, so the tensors should be close to zero when $|\boldsymbol{\xi}| > k_1^{-1}$ and τ is large. But $\overline{\boldsymbol{B}}$, a large scale quantity, varies little on these scales. Assuming ideal scale separation, $\overline{\boldsymbol{B}}$ and its derivatives can be treated as constants and pulled out of the integral. This yields

$$\mathcal{E}^i = a^i_j \overline{B}^j + b^{ik}_j \partial_k \overline{B}^j. \tag{4.4}$$

Here, $a = \int_0^\infty \int_V \mathcal{A} \, d\xi d\tau$ and b, defined equivalently, are pseudotensors. Like $\boldsymbol{\mathcal{E}}_{(\mathbf{0})}$, a and b must also be homogeneous and isotropic. The only tensors with this property are multiples of the Kronecker tensor and the Levi-Civita pseudotensor, so $a^i_j = \alpha \delta^i_j$ and $b^{ik}_j = \beta \epsilon^{ik}_j$. Then (4.4) becomes

$$\boldsymbol{\mathcal{E}} = \alpha \overline{\boldsymbol{B}} - \beta \nabla \times \overline{\boldsymbol{B}}. \tag{4.5}$$

Note that α is a pseudoscalar, but β is an ordinary scalar.

Together with (4.3), the result is

$$\begin{aligned} \partial_t \overline{\boldsymbol{B}} &= \lambda \nabla^2 \overline{\boldsymbol{B}} + \nabla \times (\alpha \overline{\boldsymbol{B}} - \beta \nabla \times \overline{\boldsymbol{B}}) \\ &= (\lambda + \beta) \nabla^2 \overline{\boldsymbol{B}} + \alpha \nabla \times \overline{\boldsymbol{B}}. \end{aligned} \tag{4.6}$$

The large scale version of Ohm's law (4.1) becomes

$$\begin{aligned} \overline{\boldsymbol{J}} &= \sigma(\overline{\boldsymbol{E}} + \boldsymbol{\mathcal{E}}) \\ &= \sigma(\overline{\boldsymbol{E}} + \alpha \overline{\boldsymbol{B}} - \beta \nabla \times \overline{\boldsymbol{B}}) \\ &= \sigma(\overline{\boldsymbol{E}} + \alpha \overline{\boldsymbol{B}}) - \sigma \beta \overline{\boldsymbol{J}} \\ &= \sigma_m (\overline{\boldsymbol{E}} + \alpha \overline{\boldsymbol{B}}), \end{aligned} \tag{4.7}$$

where $\sigma_m = \frac{\sigma}{1 + \sigma\beta}$. If α is large enough, the magnetic field will grow exponentially. This is called the α effect.

4.4 Magnetic Helicity

A very useful quantity is the so called magnetic helicity, defined as $H^M = \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle$, where $\boldsymbol{A}$ is the magnetic potential. With the assumed boundary conditions, the magnetic helicity is independent of the gauge invariance of $\boldsymbol{A}$. If we had chosen another potential $\boldsymbol{A}'$ such that $\boldsymbol{A}' = \boldsymbol{A} + \nabla \phi$ for some scalar field ϕ, then

$$\begin{aligned} \langle \boldsymbol{A}' \cdot \boldsymbol{B} \rangle &= \langle \boldsymbol{A} \cdot \boldsymbol{B} + (\nabla \phi) \cdot \boldsymbol{B} \rangle \\ &= \langle \boldsymbol{A} \cdot \boldsymbol{B} + (\nabla \phi) \cdot \boldsymbol{B} + \phi \nabla \cdot \boldsymbol{B} \rangle \\ &= \langle \boldsymbol{A} \cdot \boldsymbol{B} + \nabla \cdot (\phi \boldsymbol{B}) \rangle \\ &= \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle. \end{aligned}$$

In case of high conductivity, this quantity is mostly conserved. The following vector calculus identity

$$\nabla \cdot (\boldsymbol{A} \times \partial_t \boldsymbol{A}) = \boldsymbol{A} \cdot (\nabla \times \partial_t \boldsymbol{A}) - \partial_t \boldsymbol{A} \cdot (\nabla \times \boldsymbol{A}),$$

implies that

$$\langle \boldsymbol{A} \cdot (\nabla \times \partial_t \boldsymbol{A}) \rangle = \langle \partial_t \boldsymbol{A} \cdot (\nabla \times \boldsymbol{A}) \rangle = \langle \partial_t \boldsymbol{A} \cdot \boldsymbol{B} \rangle. \tag{4.8}$$

Because we will integrate later, we can ignore gauge invariance and remove the rotations from the induction equation (4.2)

$$\partial_t \boldsymbol{A} = \boldsymbol{v} \times \boldsymbol{B} - \lambda \boldsymbol{J}.$$

Then the time derivative of the magnetic helicity is equal to

$$\begin{aligned} \partial_t H^M &= \langle \partial_t \boldsymbol{A} \cdot (\nabla \times \boldsymbol{A}) + \boldsymbol{A} \cdot \partial_t (\nabla \times \boldsymbol{A}) \rangle \\ &= 2 \langle \partial_t \boldsymbol{A} \cdot \boldsymbol{B} \rangle \\ &= 2 \langle (\boldsymbol{v} \times \boldsymbol{B} - \lambda \boldsymbol{J}) \cdot \boldsymbol{B} \rangle \\ &= -2\lambda \langle \boldsymbol{J} \cdot \boldsymbol{B} \rangle, \end{aligned}$$

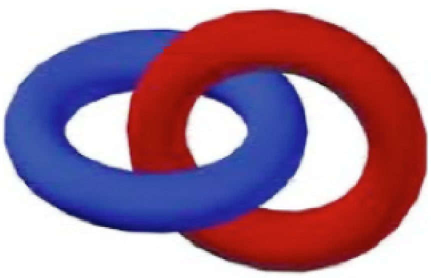
because $\boldsymbol{v} \times \boldsymbol{B}$ is perpendicular to $\boldsymbol{B}$. This is well conserved, because λ is small. This time derivative is also equal to

$$\begin{aligned} \partial_t H^M &= 2 \langle \boldsymbol{A} \cdot \partial_t \boldsymbol{B} \rangle \\ &= 2 \langle \boldsymbol{A} \cdot (-\nabla \times \boldsymbol{E}) \rangle \\ &= -2 \langle (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{E} \rangle \\ &= -2 \langle \boldsymbol{B} \cdot \boldsymbol{E} \rangle, \end{aligned} \tag{4.9}$$

where equation (4.8) was used again.

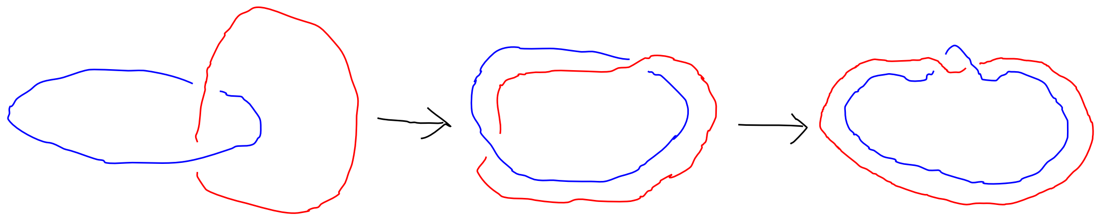
Until now, magnetic helicity has been an abstract concept, but a visualization is certainly possible. Informally, it is a measure of the 'linkage', 'twist' and 'writhe' of the magnetic field lines, using the terms in [2]. Linkage occurs when field lines form loops going through each other like the links of a chain, figure 1.

Figure 4.1: *Linkage. Source: Eric G. Blackman [2].*



As a simple example, presume there are two rotationally symmetric tori of magnetic field lines as in figure 4.1. Let Φ_1 be the magnetic flux ($\Phi_1 = \int \boldsymbol{B}_1 \cdot d\boldsymbol{S}_1$) through one of the tubes and Φ_2 the flux through the other. The magnetic field outside the tori

Figure 4.2: *Linkage to twist in three steps.*



is zero, so the helicity is $\int \boldsymbol{A} \cdot \boldsymbol{B}\, dV = \int \boldsymbol{A}_1 \cdot \boldsymbol{B}_1\, dV + \int \boldsymbol{A}_2 \cdot \boldsymbol{B}_2\, dV$. Since the magnetic field is constant when a path along the torus is followed, we find

$$\int \boldsymbol{A}_1 \cdot \boldsymbol{B}_1\, dV = \iint \boldsymbol{A}_1 \cdot \boldsymbol{B}_1\, d\ell dS = \int \left(\int \boldsymbol{A}_1 \cdot d\boldsymbol{\ell} \right) \boldsymbol{B}_1 \cdot d\boldsymbol{S} = \Phi_2 \int \boldsymbol{B}_1 \cdot d\boldsymbol{S} = \Phi_1 \Phi_2,$$

where $\int \boldsymbol{A}_1 \cdot d\boldsymbol{\ell} = \Phi_2$ because of Stokes' theorem. The other term gives the same result, so the total helicity is $2\Phi_1\Phi_2$. If there were no linkage, then of course $\int \boldsymbol{A}_1 \cdot d\boldsymbol{\ell}$ would be zero, yielding no helicity.

Twist occurs when the field lines coil around a torus, like a Möbius strip. In figure 4.2, a continuous deformation of two field lines is shown, from linkage to twist. The lines in the right image can also be imagined as the one-dimensional edges of a Möbius strip with two twists, showing that these concepts are closely related.

It also illustrates why magnetic helicity is well conserved. The field lines cannot cross and the dissipation of the magnetic field is very slow because of the high conductivity and low viscosity. So the only way to go to configuration of lower energy, is for the lines to unwind and unknot. This magnetic relaxation corresponds to the transport of magnetic helicity from smaller scales to larger scales. This will be explored in detail in the next chapters.

Chapter 5

The Amplification of Magnetic Fields by the CME

In the beginning, the universe was hot, small and dense. Around 10^{-11}s after the Big Bang, when the temperature was between 100 and 200GeV, the weak and the electromagnetic interactions separated. The strong interaction and presumably gravitation had already been separated by then, so the fundamental interactions had their present forms. At about 10^{-5}s or 160MeV, quarks, which had been free until then, condensed into hadrons, mostly protons and neutrons. Atomic nuclei (other than hydrogen-1) only began forming after 1s, corresponding to 1MeV. So between 10^{-11}s and 1s, the universe consisted of elementary particles, protons and neutrons in a hot and dense plasma, where collisions occurred very often. [19]

5.1 The Dirac Equation

The following is based mostly on [15] and [11]. These particles were very fast, so special relativity must be taken into account. There are many fundamental differences between relativistic and nonrelativistic quantum mechanics, but for the purposes of this thesis it is not necessary to dwell on them. What is important is that electrons and positrons, the particles of most interest to this thesis, are described by the Dirac equation. It can be derived, heuristically, as follows. For simplicity, free particles are considered, so potentials can be ignored.

In nonrelativistic quantum mechanics,the Schrödinger equation describes particles. This equation is based on the equation $E = \frac{p^2}{2m}$, that is, for a free particle, the total energy equals the kinetic energy. When the quantities are replaced by operators $E \to i\partial_t$, $\boldsymbol{p} \to -i\nabla$ and when we let them act on a wave function, the Schrödinger equation appears ($i\partial_t\psi = -\frac{1}{2m}\nabla^2\psi$). It would be natural to try something similar for the relativistic case.

In special relativity, there is a similar equation involving energy and momentum, namely $E^2 = p^2 + m^2$. By using the four-momentum $p = (E, \boldsymbol{p})$, it can be written more compactly as $p^\mu p_\mu = m^2$. Doing the same as above, with $p^\mu \to i\partial^\mu$, yields

$$(\partial^\mu\partial_\mu + m^2)\psi = 0.$$

This is the Gordon-Klein equation. This turns out to be the right equation for bosons with spin 0. However, it does not correctly describe electrons. This has to do with the second order time derivative. Paul Dirac tried to solve this problem, by factoring the energy-momentum relation $p^\mu p_\mu - m^2 = (\gamma^\nu p_\nu + m)(\gamma^\lambda p_\lambda - m)$, where γ is to be decided. We must have $p^\mu p_\mu = \gamma^\nu \gamma^\lambda p_\nu p_\lambda$. Expanding the right side gives

$$p^\mu p_\mu = (\gamma^\mu)^2 (p_\mu)^2 + \sum_{\mu \neq \nu} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) p_\mu p_\nu.$$

This implies that $(\gamma^\mu)^2 = \eta^{\mu\mu}$ for every μ and the anticommutators $\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu$ are zero for all $\mu \neq \nu$. Then the components of γ must be noncommutative. They can be represented by 4×4 matrices, known as the gamma matrices or Dirac matrices. In fact, several equivalent matrix representations exist. One of these is the Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where 0 and 1 stand for the 2×2 zero matrix and identity matrix respectively. The σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It will be useful to define another matrix,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This matrix anticommutes with all other gamma matrices, for example

$$\begin{aligned} \gamma^5\gamma^0 &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0 \\ &= -i\gamma^0\gamma^1\gamma^2\gamma^0\gamma^3 \\ &= i\gamma^0\gamma^1\gamma^0\gamma^2\gamma^3 \\ &= -i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 \\ &= -\gamma^0\gamma^5, \end{aligned}$$

because γ^0 anticommutes with the other matrices.

But most importantly, the resulting equation is the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0,$$

which turns out to be correct when describing electrons. Here, ψ is called the Dirac spinor field. It has four components, though it is not a four-vector. Therefore, $\psi^*\psi$, where the asterisk stands for the Hermitian conjugate, is not a Lorentz invariant scalar. But $\psi^*\gamma^0\psi$ is, meaning all measurable quantities are of the form $\overline{\psi} D \psi$, where $\overline{\psi} := \psi^*\gamma^0$ and D is some operator. One of the most important of these is $\overline{\psi}\gamma^\mu\psi$, which is identified with the electric current density four-vector $j^\mu = (\rho, \boldsymbol{J})$, where ρ is the charge density. With the Dirac equation, the divergence can be calculated

$$\begin{aligned} \partial_\mu j^\mu &= \partial_\mu(\overline{\psi})\gamma^\mu\psi + \overline{\psi}\gamma^\mu\partial_\mu\psi \\ &= \overline{-im\psi}\psi - \overline{\psi}im\psi \\ &= im\overline{\psi}\psi - im\overline{\psi}\psi \\ &= 0, \end{aligned}$$

which shows that charge is conserved.

5.2 The Chiral Magnetic Effect

The chirality of a Dirac spinor is given by the operator γ^5. This matrix has only two eigenvalues, ± 1. So when the chirality of an electron is measured, it can only have these values, where $+1$ is a right handed electron, and -1 is a left handed one. This property is Lorentz invariant, but in general not conserved. And, as suggested by the terms 'left' and 'right', it is a pseudoscalar. The parity transformation on a Dirac spinor is given by $P\psi = \gamma^0\psi$. Hence

$$\begin{aligned} P\overline{\psi}\gamma^5\psi P &= \overline{\psi}\gamma^0\gamma^5\gamma^0\psi \\ &= -\overline{\psi}(\gamma^0)^2\gamma^5\psi \\ &= -\overline{\psi}\gamma^5\psi. \end{aligned}$$

In any case, the right and left chiral parts of the Dirac spinor can be separated by the projector operators $P_R = \frac{1}{2}(1+\gamma^5)$ and $P_L = \frac{1}{2}(1-\gamma^5)$ and right and left chiral currents can be defined as $j_R^\mu = \overline{\psi}\gamma^\mu P_R\psi$ and $j_L^\mu = \overline{\psi}\gamma^\mu P_L\psi$. These can be understood as the electric currents of only the right handed electrons or only the left handed electrons. Then naturally, $j_R + j_L = j$. But the difference is also an interesting quantity called the chiral, or axial current $j^{\mu 5} = j_R - j_L$. Its divergence is

$$\begin{aligned} \partial_\mu j^{\mu 5} &= \partial_\mu(\overline{\psi}\gamma^\mu\gamma^5\psi) \\ &= im\overline{\psi}\gamma^5\psi + \overline{\psi}\gamma^\mu\gamma^5\partial_\mu\psi \\ &= im\overline{\psi}\gamma^5\psi - \overline{\psi}\gamma^5\gamma^\mu\partial_\mu\psi \\ &= 2im\overline{\psi}\gamma^5\psi. \end{aligned} \tag{5.1}$$

So chiral charge is generally not conserved.

So far, chirality has been a very abstract property. There is a related quantity which is much more easily visualized, namely the helicity of a particle. This is nothing more than the projection of the spin unto the direction of the momentum. For Dirac spinors, the spin operators are

$$\Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

So the helicity operator is $\frac{1}{|\boldsymbol{p}|}\Sigma^i p_i$. As an electron only has two measurable spin states, a helicity measurement can only give two outcomes. Again, $+1$ corresponds to right handed helicity and -1 to left handed helicity. In the primordial plasma, the electrons had enormous amounts of kinetic energy, so for many purposes, their mass can be neglected. In this approximation, chirality and helicity are the same. Using the momentum space version of the Dirac equation, $\gamma^\mu p_\mu\psi = 0$, we find

$$\begin{aligned} \gamma^0 E\psi &= \gamma^i p_i\psi \\ \gamma^5\gamma^0\gamma^0 E\psi &= \gamma^5\gamma^0\gamma^i p_i\psi \\ \gamma^5\psi &= \pm\frac{1}{|\boldsymbol{p}|}\gamma^5\gamma^0\gamma^i p_i\psi. \end{aligned}$$

In the massless limit, $E^2 = p^2$. The plus corresponds to electrons and the minus to positrons, which have negative energies in relativistic quantum theory.

Notice that

$$
\begin{aligned}
\gamma^5\gamma^0\gamma^i &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \\
&= \Sigma^i.
\end{aligned}
$$

These two results show that $\gamma^5\psi = \pm\frac{1}{|\boldsymbol{p}|}\Sigma^i p_i\psi$, so for electrons, chirality is the same as helicity when mass is ignored. These properties are both Lorentz invariant and conserved, which is also directly seen in equation (5.1) with $m = 0$.

Until now, only free particles have been discussed. But something interesting happens when a magnetic field is present. For simplicity, consider a uniform magnetic field in the upward direction. The spins of the electrons will anti-align with the field, i.e. point downward. And since the helicity is conserved, the vertical parts of the momenta of right handed electrons will point in the same direction as the spins, so they will go downward. Left handed electrons will go upward. If the density of right and left electrons is different, a current is established. Unlike electrons, right handed positrons will go upward and left handed ones go downward. So they go in the opposite direction, but the charge is also opposite. Then, the currents of both particles go in the same direction. The resulting net current is the chiral magnetic effect.[12]

Without the massless approximation, there are types of collisions that can flip the chirality of an electron. However, in the epoch of interest, described in the introduction of this chapter, these are much rarer than chirality preserving collisions. This means that the chiral flipping rate Γ_f was much smaller than the ordinary collision rate Γ_o. Therefore, between Γ_o^{-1} and Γ_f^{-1}, thermal equilibrium, in the form of Fermi-Dirac distributions, is reached for right and left electrons separately, with chemical potentials μ_R and μ_L. The chiral magnetic effect is usually expressed in the difference of these quantities* $\mu^5 := \mu_R - \mu_L$. We will assume that this quantity is constant in space. The complete derivation is somewhat elaborate and can be found in [10]. The result, however, is quite simple

$$
\boldsymbol{J} = \frac{e^2}{4\pi^2}\mu^5\boldsymbol{B} = \frac{\alpha}{\pi}\mu^5\boldsymbol{B}. \tag{5.2}
$$

Here, e is the elementary charge and $\alpha = \frac{e^2}{4\pi}$ is the fine structure constant.. It is similar to equation (4.7) and implies an exponential growth of the magnetic field. However, there is another effect that comes into play.

5.3 The Dynamical System for CFA

Without chirality flipping collisions, one would expect the number densities n_R and n_L of left and right electrons to be conserved and therefore also their chemical potentials. However, there is another way chirality can flip, but only in the presence of both electric and magnetic fields. This is caused by a quantum effect called the chiral, or

*Nota bene: others define this as $\frac{1}{2}(\mu_R - \mu_L)$ or as $\mu_L - \mu_R$.

axial anomaly. Its derivation is beyond the scope of this thesis, but it can be found in chapter 19 of [15]. It is given by [10]

$$\partial_t(n_R - n_L) = \frac{2\alpha}{\pi}\langle \boldsymbol{E} \cdot \boldsymbol{B} \rangle. \tag{5.3}$$

In the high temperature (T) limit, we have $n_R - n_L = \frac{1}{6}\mu^5 T^2$ [10]. Then (5.3) can be rewritten as

$$\begin{aligned}\partial_t \mu^5 &= \frac{12\alpha}{\pi T^2}\langle \boldsymbol{E} \cdot \boldsymbol{B} \rangle \\ &= \frac{6\alpha}{\pi T^2}\partial_t H^M.\end{aligned} \tag{5.4}$$

For the last step, see equation (4.9).

It is possible to construct a dynamical system with the chiral chemical potential μ^5 and the magnetic helicity as in [5]. Fluid dynamical effects will be ignored, so the velocity field is not considered here. So Ohm's law is simply $\boldsymbol{J} = \sigma \boldsymbol{E}$, but because of the chiral magnetic effect (5.2), the total current is $\boldsymbol{J} = \sigma \boldsymbol{E} + \frac{\alpha}{\pi}\mu^5 \boldsymbol{B}$. Then with Ampère's law, the same can be done as in equation (4.2)

$$\partial_t \boldsymbol{B} = \lambda \nabla^2 \boldsymbol{B} + \frac{\alpha\lambda}{\pi}\mu^5 \nabla \times \boldsymbol{B}. \tag{5.5}$$

We would like to see how the magnetic helicity changes on several length scales. This can be done most easily by going to Fourier space. A k in the subscript will make clear a quantity is in Fourier space. Parseval's theorem states that

$$H^M = \frac{1}{V}\int_V \boldsymbol{A} \cdot \boldsymbol{B}^* \, d\boldsymbol{r} = \frac{1}{(2\pi)^3 V}\int_V \boldsymbol{A}_k \cdot \boldsymbol{B}_k^* \, d\boldsymbol{k},$$

where $\boldsymbol{B}^* = \boldsymbol{B}$, because the magnetic field is a real quantity. Then the following can be defined

$$H_k^M = \frac{1}{(2\pi)^3 V}\int_{|\boldsymbol{k}'|=k} \boldsymbol{A}_{k'} \cdot \boldsymbol{B}_{k'}^* \, d\boldsymbol{k}'. \tag{5.6}$$

For the magnetic energy $E^M = \langle \frac{1}{2}B^2 \rangle$, something similar holds

$$E_k^M = \frac{1}{(2\pi)^3 2V}\int_{|\boldsymbol{k}'|=k} |\boldsymbol{B}_{k'}|^2 d\boldsymbol{k}'. \tag{5.7}$$

The dot product of the Fourier version of (5.5) with $\boldsymbol{B}_k^*$ gives

$$\frac{1}{2}\partial_t |\boldsymbol{B}_k|^2 = (\partial_\eta \boldsymbol{B}_k) \cdot \boldsymbol{B}_k^* = -\lambda k^2 \boldsymbol{B}_k \cdot \boldsymbol{B}_k^* + \frac{i\alpha\lambda}{\pi}\mu^5 (\boldsymbol{k} \times \boldsymbol{B}_k) \cdot \boldsymbol{B}_k^*. \tag{5.8}$$

Since $\boldsymbol{B}_k^* = -i\boldsymbol{k} \times \boldsymbol{A}_k^*$ and $\boldsymbol{k} \cdot \boldsymbol{B}_k^* = 0$,

$$\begin{aligned}(\boldsymbol{k} \times \boldsymbol{B}_k) \cdot \boldsymbol{B}_k^* &= -i(\boldsymbol{k} \times \boldsymbol{B}_k) \cdot (\boldsymbol{k} \times \boldsymbol{A}_k^*) \\ &= -ik^2 \boldsymbol{A}_k \cdot \boldsymbol{B}_k^* + i(\boldsymbol{k} \cdot \boldsymbol{B}_k^*)(\boldsymbol{k} \cdot \boldsymbol{A}_k) \\ &= -ik^2 \boldsymbol{A}_k \cdot \boldsymbol{B}_k^* \\ &= -ik^2 \boldsymbol{A}_{-k}^* \cdot \boldsymbol{B}_{-k}.\end{aligned}$$

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Because the domain of integration $|\boldsymbol{k}'| = k$ is symmetric in $\boldsymbol{k}'$, the negative signs will have no effect on the following results. Using this in (5.8) and then integrating as in (5.6) gives

$$\partial_t E_k^M = -2\lambda k^2 E_k^M + \frac{\alpha\lambda}{\pi} k^2 \mu^5 H_k^M. \tag{5.9}$$

Similarly, the dot product of (5.5) with $i\boldsymbol{k} \times \boldsymbol{B}_k^*$ gives

$$\partial_t H_k^M = -2\lambda k^2 H_k^M + \frac{4\alpha\lambda}{\pi} \mu^5 E_k^M. \tag{5.10}$$

The magnetic helicity is subject to the realizability condition $E_k^M \geq \frac{1}{2} k H_k$. In case of equality, the magnetic field is maximally helical. It was shown that magnetic fields in dynamo processes become maximally helical over time [7]. Using this, we find

$$\partial_t H_k^M = -2\lambda k^2 H_k^M + \frac{2\alpha\lambda}{\pi} k \mu^5 H_k^M. \tag{5.11}$$

From (5.6) and the chiral anomaly (5.3), an equation for the evolution of μ^5 can be obtained. However, the weak interaction also has a small effect on the chiral difference, so a constant c, incorporating the factor $\frac{6}{\pi}$, is introduced to account for this. The chiral flipping rate Γ_f is also reintroduced. The result is

$$\partial_t \mu^5 = \frac{c\alpha}{T^2} \int_k \partial_t H_k^M - \Gamma_f \mu^5. \tag{5.12}$$

Equations (5.11) and (5.12) form the dynamical system studied in [5]. In particular, it showed that magnetic helicity was transported from the small scale to the large scale.

In the primordial plasma, space expanded very rapidly. This expansion was not accounted for in the equations of this chapter. Fortunately, all equations pertaining to this thesis stay the same, when time is replaced by a quantity called conformal time and when other replacements such as $\boldsymbol{B} \to a^2 \boldsymbol{B}$ are made, where a is the cosmological scale factor. For simplicity, these replacements were omitted.

Chapter 6

The Inverse Cascade

Equations (4.6) and (5.5) are very similar. The quantities α (not to be confused with the fine structure constant) and β will be assumed constant over space, or, alternatively, they are replaced by their global averages. Then they are not subject to any Fourier transforms. The evolution of the magnetic helicity for the dynamo can be derived the same way, as

$$\partial_t H_k^M = -2(\lambda + \beta)k^2 H_k^M + 2k\alpha H_k^M. \tag{6.1}$$

for all $k \ll k_1$.

6.1 The Time Evolution of Alpha

The quantities α and β can be shown to be equal to [3], [16], with ρ the mass density

$$\begin{aligned}
\alpha &= \alpha_M + \alpha_K \\
\alpha_M &= \frac{1}{3\rho}\tau\langle \boldsymbol{b} \cdot (\nabla \times \boldsymbol{b})\rangle \\
\alpha_K &= -\frac{1}{3}\tau\langle \boldsymbol{v} \cdot (\nabla \times \boldsymbol{v})\rangle \\
\beta &= \frac{1}{3}\tau_\beta\langle v^2\rangle.
\end{aligned}$$

Here, τ is the fluid correlation time. It is reasonable to assume β is constant. Even though the energy of the magnetic field is derived from the kinetic energy of the plasma, the kinetic energy is many orders of magnitude greater, so any decrease in it will be negligible. The evolution of α_M is given by [8]

$$\begin{aligned}
\partial_t \alpha_M &= -\left\langle 2\beta k_1^2 \left(\frac{\boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}}}{B_{eq}^2} - \frac{\alpha_M}{R_m} \right) - \nabla \cdot \boldsymbol{F} \right\rangle \\
&= -2k_1^2 \left(\tau \frac{\langle \boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} \rangle}{3\rho} - \frac{\beta\alpha_M}{R_m} \right),
\end{aligned} \tag{6.2}$$

where B_{eq} is the equipartition field strength, defined so that $B_{eq}^2 = \rho\overline{v^2} \approx \rho\langle v^2\rangle$. And R_m is the magnetic Reynolds number, defined as $R_m = \frac{UL}{\lambda}$, where U is the typical velocity scale of the plasma and L is the typical length scale, in this case k_1.

The evolution of α_K is given by [4]

$$\partial_t \langle \boldsymbol{v} \cdot \boldsymbol{\omega} \rangle \approx 2\rho^{-1} k_1 (k_1 - k_2) \langle \boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} \rangle - 2\nu \langle \partial_i v^j \partial_i \omega^j \rangle,$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ is the vorticity and ν is the kinematic viscosity. Here, the fluid is assumed to be incompressible ($\nabla \times \boldsymbol{v} = 0$) and the magnetic field is assumed to be force free, meaning that $\boldsymbol{B} \times \boldsymbol{J} = 0$. For the last term, we can do a partial integration and a one mode approximation

$$\begin{aligned} \langle \partial_i v^j \partial_i \omega^j \rangle &= -\langle \partial_i^2 (v^j) \omega^j \rangle \\ &= -\langle (\nabla^2 \boldsymbol{v}) \cdot \boldsymbol{\omega} \rangle \\ &\approx k_1^2 \langle \boldsymbol{v} \cdot \boldsymbol{\omega} \rangle. \end{aligned}$$

In terms of α_K, the result is

$$\begin{aligned} \partial_t \alpha_K &= -\frac{2\tau}{3\rho} k_1 (k_1 - k_2) \langle \boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} \rangle - 2\nu k_1^2 \alpha_K \\ &\approx -\frac{2\tau}{3\rho} k_1^2 \langle \boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} \rangle - 2\nu k_1^2 \alpha_K. \end{aligned} \tag{6.3}$$

Using the Maxwell equations and Ohm's law, we find that the electromotive force equals

$$\boldsymbol{\mathcal{E}} = \alpha \overline{\boldsymbol{B}} - \beta \nabla \times \overline{\boldsymbol{B}} = \alpha \overline{\boldsymbol{B}} - \beta \sigma_m (\overline{\boldsymbol{E}} + \boldsymbol{\mathcal{E}})$$

or $\boldsymbol{\mathcal{E}} = C\alpha \overline{\boldsymbol{B}} - C\beta\sigma_m \overline{\boldsymbol{E}}$ where $C = (1 + \beta\sigma_m)^{-1}$. Hence

$$\boldsymbol{\mathcal{E}} \cdot \overline{\boldsymbol{B}} = C\alpha \overline{B}^2 - C\beta\sigma_m \overline{\boldsymbol{E}} \cdot \overline{\boldsymbol{B}}.$$

For more clarity, define $\frac{4\tau}{3\rho} k_1^2 C = C_1$. The result is

$$\begin{aligned} \partial_t \alpha_M &= -C_1 \alpha_M \langle \overline{B}^2 \rangle + C_1 \beta\sigma \langle \overline{\boldsymbol{E}} \cdot \overline{\boldsymbol{B}} \rangle - C_2 \alpha_M \\ \partial_t \alpha_K &= -C_1 \alpha_K \langle \overline{B}^2 \rangle + C_1 \beta\sigma \langle \overline{\boldsymbol{E}} \cdot \overline{\boldsymbol{B}} \rangle - 2k_1^2 \nu \alpha_K, \end{aligned} \tag{6.4}$$

where $C_2 = \frac{2k_1^2\beta}{\alpha_M}$. However, the kinematic viscosity is thought to be very high, $\nu = 10^{24}\mathrm{m}^2\mathrm{s}^{-1}$, compared to $\beta/R_m \approx 10^{-3}\mathrm{m}^2\mathrm{s}^{-1}$ [18]. But the source terms for α_M and α_K are the same. Consequently, α_K will be completely negligible compared with α_M. So henceforth, we have $\alpha = \alpha_M$.

6.2 The Dynamical System

Introducing a new scale $k_3 < k_2 \ll k_1$, the notation H_i will be used for $H^M_{k_i}$. The aim is to make a closed system with the variables α, H_2 and H_3. Similarly as in (4.9), we have $\langle \overline{\boldsymbol{E}} \cdot \overline{\boldsymbol{B}} \rangle = \lambda \langle \overline{\boldsymbol{J}} \cdot \overline{\boldsymbol{B}} \rangle$. Also, $\langle \nabla^2 \boldsymbol{A} \cdot \boldsymbol{B} \rangle = -\langle (\nabla \times (\nabla \times \boldsymbol{A})) \cdot \boldsymbol{B} \rangle = -\langle \boldsymbol{J} \cdot \boldsymbol{B} \rangle$, and the same holds for the large and small scale quantities. So for all $k \ll k_1$,

$$\begin{aligned} \langle \overline{\boldsymbol{E}} \cdot \overline{\boldsymbol{B}} \rangle &= -\lambda \langle \nabla^2 \overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}} \rangle \\ &\approx \lambda k^2 H_k. \end{aligned}$$

For maximally helical fields, $\langle B^2 \rangle = \int_k kH_k \approx k_2H_2 + k_3H_3$. In this two modes approximation, H_1, the helicity of scale k_1, is neglected, see [4].

With equation (6.1), the final system is

$$\begin{aligned}
\partial_t H_2 &= -2(\lambda+\beta)k_2^2 H_2 + 2k_2\alpha H_2 \\
\partial_t H_3 &= -2(\lambda+\beta)k_3^2 H_3 + 2k_3\alpha H_3 \\
\partial_t \alpha &= -C_1\alpha(k_2H_2 + k_3H_3) + C_1\beta\sigma_m\lambda(k_2^2H_2 + k_3^2H_3) - C_2\alpha,
\end{aligned} \tag{6.5}$$

where we assume that $H_2(0)$, the quantity H_2 at $t = 0$, is positive. It is convenient to use dimensionless quantities. Therefore, we replace t by t/τ and define the dimensionless variables $h_2 = H_2/H_2(0)$ and $h_3 = H_3/H_2(0)$, as well as $A = k_2\tau\alpha$. The dimensionless constants are defined as

$$\begin{aligned}
K_2 &= k_2/k_1 \\
K_3 &= k_3/k_1 \\
c_1 &= 2(\lambda+\beta)k_1^2\tau \\
c_2 &= \frac{1}{\mu_0}C_1H_2(0)k_1\tau \\
c_3 &= \beta k_1^2\tau \\
c_4 &= \tau C_2,
\end{aligned}$$

where the magnetic constant μ_0 was reintroduced. The result is

$$\begin{aligned}
\partial_t h_2 &= -c_1K_2^2h_2 + 2K_2Ah_2 \\
\partial_t h_3 &= -c_1K_3^2h_3 + 2K_3Ah_3 \\
\partial_t A &= -c_2A(K_2h_2 + K_3h_3) + c_2c_3(K_2^2h_2 + K_3^2h_3) - c_4A.
\end{aligned} \tag{6.6}$$

This system can be easily generalized for more scales $h_4, h_5, \ldots$.

The value k_1^{-1} can be approximated by $\lambda_{\mathrm{mfp}} = 0.05\mathrm{kpc}$ from table 1 in [18]. Also from this table, $\beta = \frac{1}{3}t_{\mathrm{visc}}U^2 = 3.3\cdot 10^{24}\mathrm{m}^2\mathrm{s}^{-1}$. Compared with this, $\lambda = 0.02\mathrm{m}^2\mathrm{s}^{-1}$ is negligible. Then $c_1 \approx 450$ and $c_3 \approx 225$. For c_4, the value $2\beta k_1^2\tau/R_m$ will be used. Because of the very large magnetic Reynolds number of the order 10^{27}, the result is $c_4 \approx 10^{-25}$, which is negligible even as a dissipation rate, when compared to the lifetime of the universe. The density is around $10^{-23}\mathrm{kgm}^{-3}$ [13]. Since $\mu_0\beta\sigma = \beta/\lambda \gg 1$, we have $\sigma_m = \frac{\sigma}{1+\mu_0\sigma\beta} \approx \frac{1}{\mu_0\beta}$ and $C = (1+\mu_0\beta\sigma_m)^{-1} = \frac{1}{2}$. This gives a value $C_1\tau k_1 = 4.5\cdot 10^{-4}\mathrm{kg}^{-1}\mathrm{s}^2$. Let $K_2 = 0.1$ and $K_3 = 0.01$. Then $H_2(0)/\mu_0 \approx \frac{1}{\mu_0 k_2}B^2 = 1.1\cdot 10^8\mathrm{kgs}^{-2}$ and $c_2 = 25\cdot 10^3$. Keep in mind that most of these figures are based on rough measurements and assumptions.

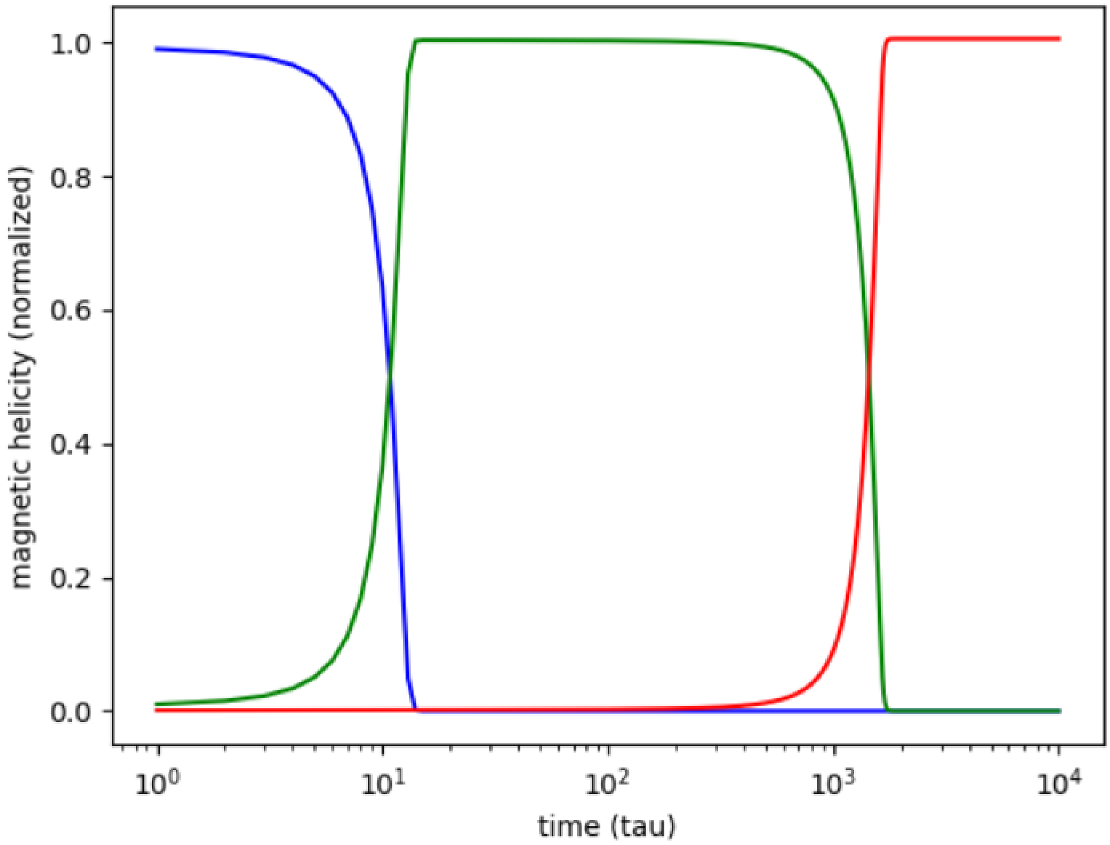
The initial value of h_2 is 1 by definition. We assume a very small positive initial value for h_3, because if $h_3(0) = 0$, then its evolution in (6.6) is zero, so it will stay zero. We also assume that $\partial_t A(0) = 0$, so $A(0)$ is equal to its equilibrium value. Then $0 = -c_2A(0)K_2 + c_2c_3K_2^2$, or $A(0) = c_3K_2$.

6.3 Results

These equations, with one mode $k_4 = 0.001k_1$ added, gave the following solution (figures 6.1 and 6.2) with solve_ivp from the SciPy package of Python, with the default

RK45 method and a time step of 1. In order for the program to work, c_3 had to be lowered to 10,000, from the estimated 25,000.

Figure 6.1: *The evolution of magnetic helicity for three modes. Blue is the mode $k_2 = 0.1k_1$, green is $k_3 = 0.01k_1$ and $k_4 = 0.001k_1$.*



The following (figures 6.3 and 6.4) has the modes closer to each other. Here the DOP853 was used, because the default did not work. The time step is 0.1. In fact, six modes were coded but the last is not visible in the plots.

Figure 6.2: *The evolution of α ($k_1 \tau \alpha$), with three helicity modes, corresponding to figure 6.1.*

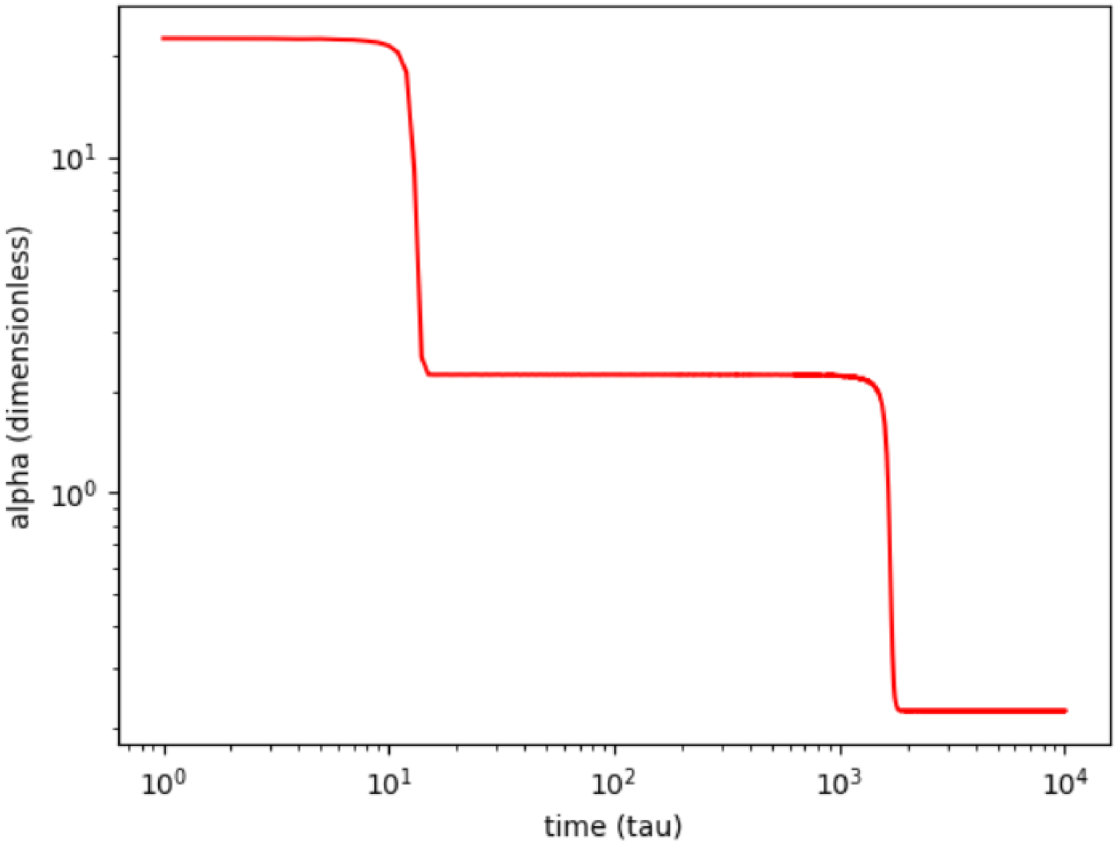


Figure 6.3: *The evolution of magnetic helicity for five modes. Blue is the mode* $k_2 = k_1/3$*, orange is* $k_3 = k_1/3^2$*, green is* $k_4 = k_1/3^3$*, red is* $k_4 = k_1/3^4$ *and purple is* $k_5 = k_1/3^5$*.*

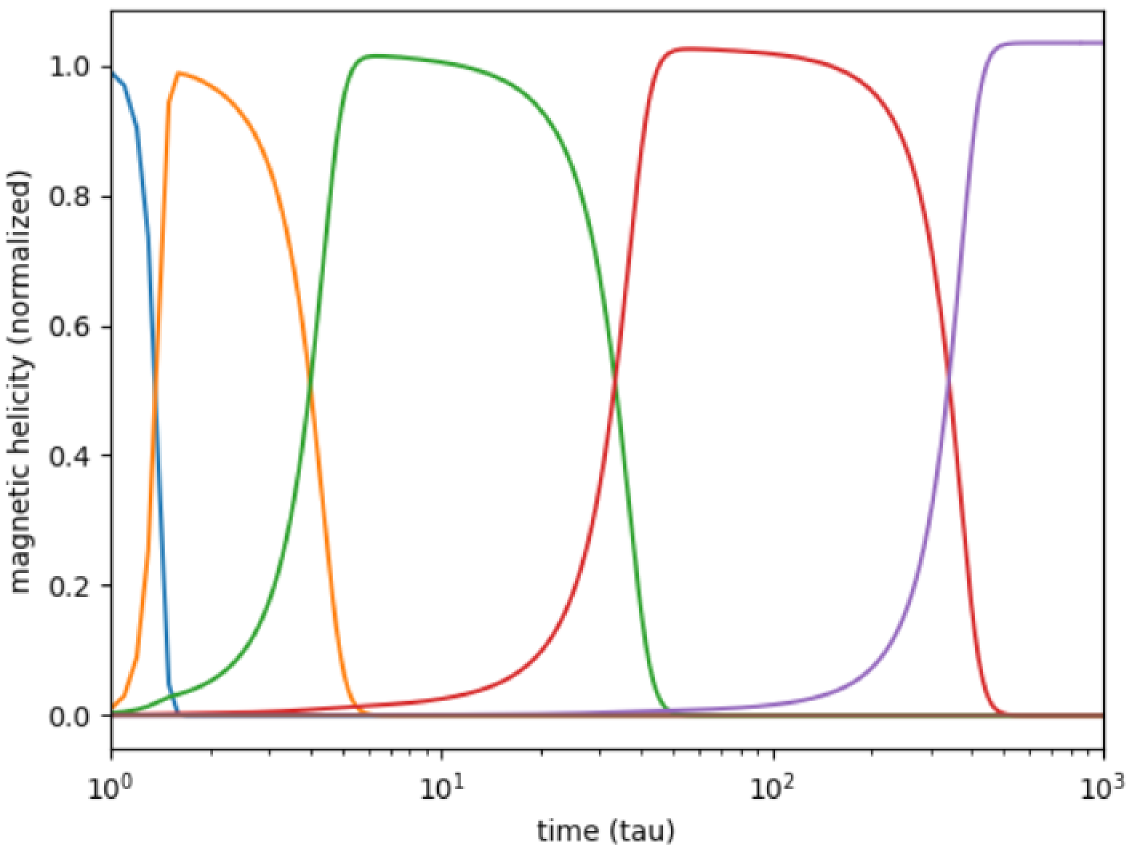
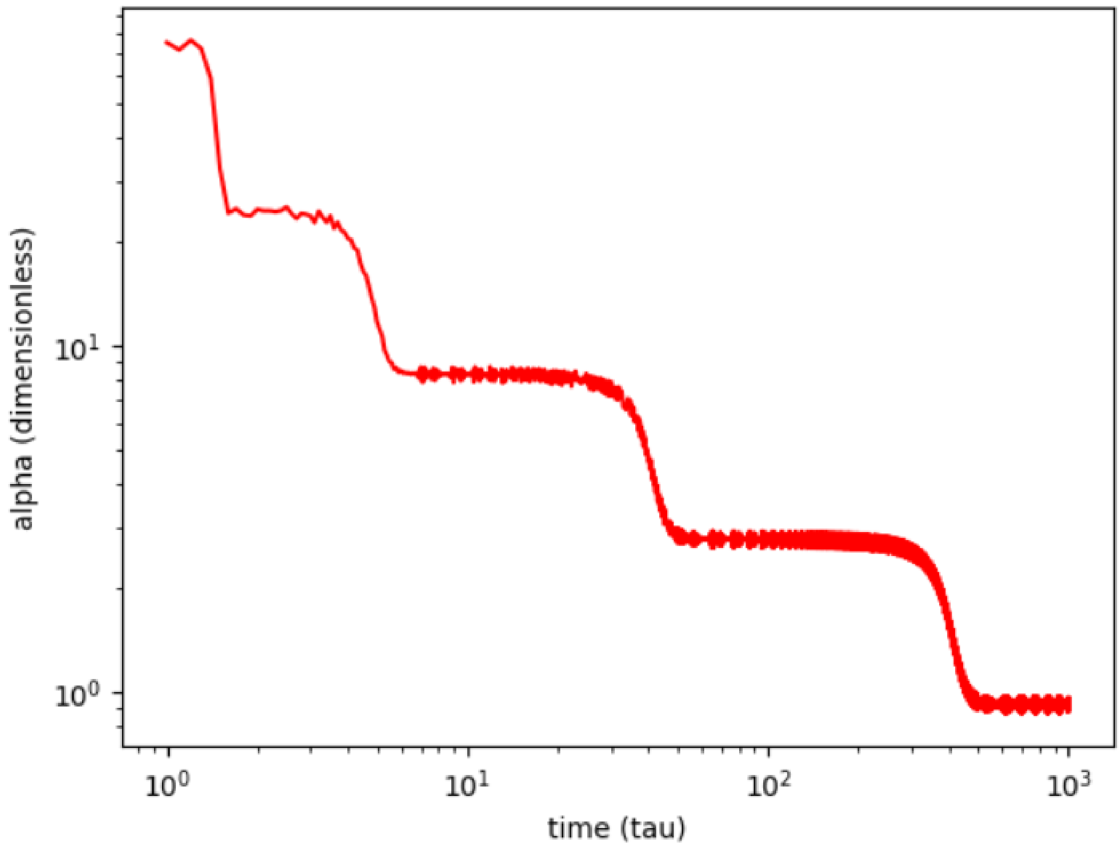


Figure 6.4: *The evolution of α ($k_1 \tau \alpha$), with five helicity modes, corresponding to figure 6.3,*



Chapter 7

Discussion

Figures 6.1 and 6.3 nicely show the inverse cascade. Magnetic helicity is transported to the larger scale. Also shown is the conservation of helicity, as at any time, the sum of all helicity modes is approximately one. At the beginning, $\partial_t h_2 = 0$, so $2A = c_1 K_2$. Then, $2A > c_1 K_3$, which implies that $\partial_t h_3 > 0$. So h_3 starts to grow exponentially, and the dissipative factor $K_2 h_2 + K_3 h_3$ in the time evolution of A grows as well, faster than the source factor, because $K_3^2 \ll K_3$. Then A starts to decrease exponentially, diminishing the source factor in the time evolution of h_2, causes is to decrease exponentially as well. This continues until a new equilibrium is reached, with $2A = c_1 k_3$.

Note that if $\tau = t_{\text{visc}} = 5 \cdot 10^6$ years, then $10^4\tau$ is almost the age of the universe. So if the results are correct, the inverse cascade is a slow process. Galaxy clusters are on the scale of 1Mpc and $k_1^{-1} = 0.05$kpc, but the magnetic helicity scale must not have grown much further than $k_4^{-1} = 50$kpc. Also, figure 6.3 seems to show that the growth slows down, because the peaks are further apart for the larger scales (the modes descend exponentially, but the x-axis is logarithmic). Lastly, the peaks in both figures seem to be slightly higher than the peaks from the previous modes, which is strange, because the total helicity is expected to slightly decrease, not increase. This seems to be caused by the lower value for c_2, because even lower values exacerbate this problem.

The magnetic energy density is $E^M = \int_k kH_k$. Since at their peaks, $h_2 \approx h_3 \approx h_4$, etc., the energy density decreases. This is exactly compensated by its growth in scale.

Of course, the stepwise form of the evolution of α is an artifact from the discrete mode approximation. It seems this would in reality be a straight line, signifying a power function. Each time a helicity mode reaches the maximum value, the other helicity modes are close to zero. Then α has a temporal equilibrium value defined by $0 = \partial_t h_i = -c_1 K_i^2 h_i + 2K_i A h_i$, that is $A = c_1 K_i/2$.

Despite the fact that the physical system is a priori completely different from the one described in chapter 5 and [5], the evolutions are comparable. This can be sourced to the fact that the α effect and the chiral magnetic effect are, coincidentally, very similar. The major difference is that the dissipation rate of α is all but negligible, while the chiral flipping rate is not.

7.1 Conclusions

The evolution of magnetic fields in galaxy clusters experiences an inverse cascade of magnetic helicity by the α effect, similarly to the inverse cascade in the early universe caused by the chirality imbalance μ^5. The time scales, however, are much greater, on the order of millions to billions of years. The inverse cascade also seems to slow down somewhat when the length scale increases.

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Appendix A

Python Code Used for the Results

For figures 6.1 and 6.2.

```
from scipy.integrate import solve_ivp
import numpy as np
import matplotlib.pyplot as plt
begintime=1 # chosen to be 1, not 0, because of the log axis
endtime = 10000
timestep = 1
t = np.arange(begintime,endtime,timestep)
k2 = 0.1
k3 = 0.01
k4 = 0.001
con1 = 450
con2 = 10000
con3 = 225
con4 = 10**(-25)

#dynamicalsystem[0] is h_2, [1] is h_3, [2] is h_4 and [3] is alpha

def diffequations(t, y):
    dynamicalsystem = np.empty(4)
    dynamicalsystem[0] = -con1 * k2**2 * y[0] + 2*k2 * y[0] * y[3]
    dynamicalsystem[1] = -con1 * k3**2 * y[1] + 2*k3 * y[1] * y[3]
    dynamicalsystem[2] = -con1 * k4**2 * y[2] + 2*k4 * y[2] * y[3]
    dynamicalsystem[3] = -con2 * y[3]
        * (k2 * y[0] + k3 * y[1] + k4 * y[2]) + con2 * con3 *
        (k2**2 * y[0] + k3**2 *y[1] + k4**2 * y[2])- con4 * y[3]
    return dynamicalsystem

sol = solve_ivp(diffequations, (begintime,endtime),
    [0.99,0.01, 0.001, con3*k2], t_eval = t)

plt.figure(0)
plt.plot(sol.t, sol.y[0], 'b')
plt.plot(sol.t, sol.y[1], 'g')
```

```
plt.plot(sol.t, sol.y[2], 'r')
plt.xscale("log")
plt.xlabel("time (tau)")
plt.ylabel("magnetic helicity (normalized)")
plt.title("Figure 1")
plt.show()

plt.figure(1)
plt.plot(sol.t, sol.y[3], 'r')
plt.xscale("log")
plt.yscale("log")
plt.xlabel("time (tau)")
plt.ylabel("alpha (dimensionless)")
plt.title("Figure 2")
plt.show()
```

For figures 6.3 and 6.4.

```
from scipy.integrate import solve_ivp
import numpy as np
import matplotlib.pyplot as plt
begintime=1 # chosen to be 1, not 0, because of the log axis
endtime = 10000
timestep = 1
t = np.arange(begintime,endtime,timestep)
k2 = 0.1
k3 = 0.01
k4 = 0.001
con1 = 450
con2 = 10000
con3 = 225
con4 = 10**(-25)

#dynamicalsystem[0] is h_2, [1] is h_3, [2] is h_4 and [3] is alpha

def diffequations(t, y):
    dynamicalsystem = np.empty(4)
    dynamicalsystem[0] = -con1 * k2**2 * y[0] + 2*k2 * y[0] * y[3]
    dynamicalsystem[1] = -con1 * k3**2 * y[1] + 2*k3 * y[1] * y[3]
    dynamicalsystem[2] = -con1 * k4**2 * y[2] + 2*k4 * y[2] * y[3]
    dynamicalsystem[3] = -con2 * y[3]*(k2 * y[0] + k3 * y[1] + k4 * y[2])
        + con2 * con3 * (k2**2 * y[0] + k3**2 *y[1] + k4**2 * y[2])
        - con4 * y[3]
    return dynamicalsystem

sol = solve_ivp(diffequations, (begintime,endtime),
    [0.99,0.01, 0.001, con3*k2], t_eval = t)

plt.figure(0)
```

```
plt.plot(sol.t, sol.y[0], 'b')
plt.plot(sol.t, sol.y[1], 'g')
plt.plot(sol.t, sol.y[2], 'r')
plt.xscale("log")
plt.xlabel("time (tau)")
plt.ylabel("magnetic helicity (normalized)")
plt.title("Figure 1")
plt.show()

plt.figure(1)
plt.plot(sol.t, sol.y[3], 'r')
plt.xscale("log")
plt.yscale("log")
plt.xlabel("time (tau)")
plt.ylabel("alpha (dimensionless)")
plt.title("Figure 2")
plt.show()
```