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A look at positivity bounds on effective field theories with spontaneously broken Lorentz invariance and how to use them to constrain the cosmological model

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Abstract

Our research is related to testing dark energy/modified gravity theories. We determine the positivity bounds on effective field theories with spontaneously broken Lorentz invariance. We consider all the operators in a low-energy — effective field theory (EFT) approach and gain the conditions for EFT coefficients so that a theory is healthy (without instabilities). These conditions are called the positivity bounds, for which a theory works. These positivity bounds can give us constraints about the cosmological model. We mainly follow the paper *Positivity bounds on effective field theories with spontaneously broken Lorentz invariance* by Paolo Creminelli, Oliver Janssen, and Leonardo Senatore, where the positivity bounds are calculated from the two-point correlation functions of conserved quantities like the Noether current and stress-energy tensor. Then we show how this new mechanism of finding positivity bounds can be used for real cosmological models.

Contents

Chapter 1

Introduction

The context of our thesis is related to the late time universe and effective field theories of dark energy (DE), which address the cosmic acceleration and possibly the cosmological constant problem. In the following lines, we explain why we use the EFT approach and what the cosmological problem means.

The idea of using EFTs is that one can make calculations without knowing the exact theory. We write down the actions in the effective field theory approach. The EFT approach means writing down all the possible operators in the low-energy degrees of freedom (while high-energy degrees of freedom (DoFs) are integrated out). From the symmetries of the theory, we guess which operators to use, and from the *coefficients* of operators, we gain information about the details of the UV theory.

Now, let us explain the case of the cosmological problem. For a massless spin-2 field in General Relativity, we have the following action:

$$S = \int d^4x \sqrt{-g}\frac{M_p^2}{2}[R + \Lambda_0] \tag{1.1}$$

Where g is a determinant of the metric, M_p is the Planck mass, R is the Ricci (curvature) scalar and Λ_0 is the cosmological constant (we will see later that it can get time-dependent corrections). Looking at the energy density of dark energy (vacuum energy) Λ_{vacuum} calculated from the theory including all the fields from the standard model (SM), we will see that it is 120 orders of magnitude higher than the appropriate observed value of it ($\Lambda_{observed}$). From the energy (mass) scale's point of view, it is equivalent to the discrepancy of 30 orders between theory and observation (as energy density scales as $[energy^{-4}]$)[1]. We can write down the following expression.

$$\Lambda_0 + \Lambda_{vacuum} = \Lambda_{measured}, \tag{1.2}$$

where Λ_{vacuum} is the theoretical value of the cosmological constant, calculated from considering standard model fields, $\Lambda_{measured}$ is the value which can be received from the observations, and Λ_0 is the cosmological constant related to the dark energy (the term we have in the action (1.1)). Λ_{vacuum}, as we already mentioned, is 120 degrees higher than $\Lambda_{measured}$, that leaves us with the negative value of Λ_0. The value of $\Lambda_{measured}$ is fixed while the value of Λ_{vacuum} may change with phase transitions as the universe expands. Hence, Λ_0 would here be very fine-tuned in a time-dependent way.

All this suggests the idea that action (1.1) has to be changed. Proposed action for the effective field theory (EFT) of dark energy (DE)/EFT of cosmological perturbations/EFT of Inflation is:

$$S = \int d^4x\sqrt{-g}\frac{M_p^2}{2}[(1+\Omega(t))R + \Lambda_0(t) + ...] \tag{1.3}$$

In the simplest and most suitable way to discuss the ghost and gradient instabilities, the action can be written as follows:

$$S = \int d^4x\sqrt{-g}\frac{M_p^2}{2}[R+\Lambda_0] + \int d^4x\sqrt{-g}\frac{M_p^2}{2}\left[A(t)\frac{\dot{\phi}^2}{2} + B(t)\frac{(\nabla\phi)^2}{2}\right] \tag{1.4}$$

As later can be shown, it turns out that for certain conditions for these EFT coefficients, Lagrangian received from action (1.4) should give us a healthy theory. Under "healthy" we mean the theory that is free of instabilities. In order to be free of instabilities, the theory should satisfy some stability criteria given below:

- to avoid the ghosts, Lagrangian should give a positive energy term (for (1.4) A should be positive);
- to avoid the gradient instabilities, DoFs should be propagating with non-negative speed;
- to avoid the tachyonic and Jeans instabilities, DoF should have a positive mass squared.

Satisfaction of these criteria specifies conditions for EFT coefficients and that conditions are called the positivity bounds.

We mainly follow the paper: *Positivity bounds on effective field theories with spontaneously broken Lorentz invariance* by Paolo Creminelli Oliver Janssen and Leonardo Senatore, re-derive (get each formula and condition) independently and try to relate the given mechanism to the cosmological models [2].

Under some assumptions about UV completion of a theory (Lorentz invariance, locality, unitarity) we can derive inequalities that the low-energy (EFT) coefficients must satisfy. Mainly, there are two ways to get conditions for coefficients. One of them is considering the analytic structure of the S-matrix, where one uses a tree-level scattering amplitude of two particles. The amplitude is written in the EFT coefficients. This matrix gives crucial information: using Cauchy's theorem one can relate an integral (over some physical quantity constructed from scattering amplitude) along a low-energy contour (using the EFT approach with coefficients) to a high-energy one. This can be proven to have a definite sign, so conditions for the EFT coefficients are obtained. We provide an exact way of doing this in chapter 2. Another way is to look at two-point correlation functions of conserved quantities, like the Noether current and stress-energy tensor. Conservation laws of these quantities will help us to get the conditions for the EFT coefficients, so we will get the positivity bounds. This is provided in chapter 4.

Generally, in late cosmology Lorentz invariance is spontaneously broken because we use the FLRW metric which is not invariant under Lorentz transformations (the metric is evolving in time),

$$g \neq \Lambda^{-1} g \Lambda \tag{1.5}$$

where Λ is the Lorentz transformation matrix. Also, contrary to de-Sitter and Minkowski cases, FLRW background does not acquire the local time-like killing vector (so, we cannot find any scale up to which FLRW metric would be Lorentz invariant) [3]. Conclusively, in late-time cosmology, we have the preferred time direction. So, while choosing the relevant theories in cosmology one could reduce the allowed parameter space by knowing which theories of inflation and DE admit a standard UV completion. To deal with the inflation and DE models we need to consider the UV theory to be the Lorentz-breaking theory and we cannot directly expand the "positivity" arguments gained from the Lorentz-invariant cases. It is wrong to think about the EFT in which Lorentz symmetry is broken as a Lorentz-invariant theory around another vacuum.

Consider we are in a Lorentz-breaking theory. Generally, the low and high energy states are connected by the boosts, but in Lorentz-breaking theory we do not have the boost-invariance. This hints that the S-matrix approach which describes the scattering of the EFT degrees of freedom cannot be extended to the UV, as the states are intrinsically in low-energy and without boosts one cannot relate them to states in the UV theory (high energy). So, it is not clear how to **connect the low-energy EFT calculation to the UV** which is quite meaningful when working with positivity logic. So, the S-matrix approach is good for Lorentz-invariant theories and not the best way for our investigation when we have the Lorentz-breaking theory.

In [2] another route is used. They look at the correlation function $\langle J^\mu(-k)J^\nu(k)\rangle$ of a conserved current J^μ (and the stress-energy tensor $T^{\mu\nu}$). From the conservation laws of these two-point correlation functions they gain the conditions for EFT coefficients, so-called positivity bounds. For these derivations we will use the conformal field theory (CFT) approach (from David Tong's lecture notes) [4] because its characteristics match our requirements (for example, the calculation should be valid both in IR and UV) quite well as we will see later. "A conformal field theory is a field theory which is invariant under the *conformal transformations*" defined as follows: Under the change of coordinates: $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$, the metric changes as:

$$g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma) g_{\alpha\beta}(\sigma) \tag{1.6}$$

CFT transformation preserves the angles so the angles are rather important here than the quantities like distance etc. The assumption in paper [2] is that in deep UV the theory reaches a conformal fixed point, which means the correlation function of J^μ is fully fixed by conformal invariance. UV limit together with the analytic properties of the two-point function of J^μ give us a possibility to have an argument similar to the one of the S-matrix and derive positivity bounds on the coefficients of EFT.

In paper[2] they consider the EFT that describes the low-energy excitations of a CFT at finite chemical potential μ for an internal $U(1)$ symmetry. According to this logic, there is only one extra scalar degree of freedom so both UV conformal invariance and Lorentz invariance are broken only by μ. μ can be seen as a cut-off of the effective field theory (the point in near UV, where the EFT approach ceases to be valid). From the beginning, it maybe seems strange to relate CFT formalism to cosmology. Until we explain it more thoroughly, let us provide here a little overview: we know that at low energy cosmological perturbations are not Lorentz-invariant, but in UV they should have Lorentz-invariant completion. Then, we know that CFTs are Lorentz-invariant in UV and they also are connected to the EFT by the large charge section. So, if we use CFTs for our theory, we can have a possibility to find positivity bounds for the low-energy EFT coefficients. These coefficients would dictate when our Lagrangian (for the DE/MG theory, which in low energy limit is Lorentz breaking and in UV has Lorentz invariant completion) gives the healthy theory. In chapter 3 we provide more explicit explanations of why and how we can use CFTs to gain positivity bounds, which are connected to the Lagrangian coefficients and, therefore, to the cosmology.

In our project, we want to find the conditions for the EFT coefficients from two-point correlation functions of conserved quantities like the Noether current and stress-energy tensor. That conditions represent the positivity bounds on the EFT coefficients and give a possibility to predict when DE Lagrangian will give the healthy theory. Then the results can be related to real cosmology models to constrain the cosmological constant values.

Chapter 2

Formalism

Even though we are not going to use the S-matrix method, it is worth explaining how it works, because we will need to compare then this method to the two-point correlation function's approach. For this, we adopt some findings from [5]. S-matrix approach means considering of $2 \to 2$ scattering in forward limit (i.e. $t \to 0$, where s, t, u are Mandelstam variables)[6]. Then there is calculated the tree-level scattering amplitude in low-energy EFT, so the amplitude is written in EFT coefficients. After this, the authors construct a physical quantity from this amplitude which can be used for deriving the positivity bounds for the EFT coefficients. Then they demonstrate that the S-matrix in a given limit has certain properties which make it possible to derive the positivity bounds:

1. S-matrix is a physically well-defined function for all values of the real variable s.

2. S-matrix is field-redefinition independent.

3. The consequences of locality and Lorentz-invariance:

    - S-matrix has an analytic continuation to the upper and lower half complex s-planes, where singularities belong only on the real axis. This includes unitarity cuts for energies $|s| > 4m^2$, where m is a non-zero mass gap in the theory. Here scattering is for 2 particles, and minimal energy for each particle is the rest energy m, $c = 1$, thus $|s| > (m + m)^2$.

    - S-matrix has a crossing symmetry: $M(s)^\star = M(4m^2 - s^\star)$.

4. A consequence of unitarity: the main idea is that the discontinuity across the cut has a definite sign, namely, it is $i\times$ a positive number.

5. A consequence of the minimal requirements to derive the Froissart bound[6]: Scattering amplitude decays as $|(M(s))|/s^2 \to 0$ as $|s| \to \infty$.

Here, $\hat{s}$ is defined as: $\hat{s} = s - 4m^2$ and $\hat{M}(\hat{s}) \equiv M(s)$, then Eq. (5) gives: $\hat{M}(\hat{s})^\star = \hat{M}(-\hat{s}^\star)$.

At the tree level, the scattering amplitude in the low-energy EFT is:

$$\hat{M}(\hat{s}) = c_0 + c_2\frac{\hat{s}^2}{\Lambda^4} + c_4\frac{\hat{s}^4}{\Lambda^8} + ...; \tag{2.1}$$

where c_i-s are real numbers and Λ is the scale suppressing the higher-dimension operators in the EFT.
Now, we provide an example of how one can use scattering amplitude to derive the positivity bounds. To construct a physical quantity that allows applying of the Cauchy theory, the authors [2] use function $\hat{M}(\hat{s})/\hat{s}^3$:

$$\frac{\hat{M}(\hat{s})}{\hat{s}^3} = c_0\frac{1}{\hat{s}^3} + c_2\frac{1}{\hat{s}\Lambda^4} + c_4\frac{\hat{s}}{\Lambda^8} + ... \tag{2.2}$$

Next the integral is computed counterclockwise. Then the following is obtained:

$$\oint d\hat{s}\frac{\hat{M}(\hat{s})}{\hat{s}^3} = 2\pi i\frac{c_2}{\Lambda^4}. \tag{2.3}$$

As we can see from Cauchy's integral theorem, the only pole that does not give residue 0 is one with c_2 term. We can perform this integral around the contour shown in Fig.2.1. Now observe, that $\hat{M}$ decays sufficiently quickly at ∞. Therefore, the integral along the large circle is negligible as $r \to \infty$. Then, integrals along the negative s cut and along the positive s cut are the same. Thus $i \times c_+$ with c_+ non-negative number is obtained. That gives condition $c_2 \geq 0$.

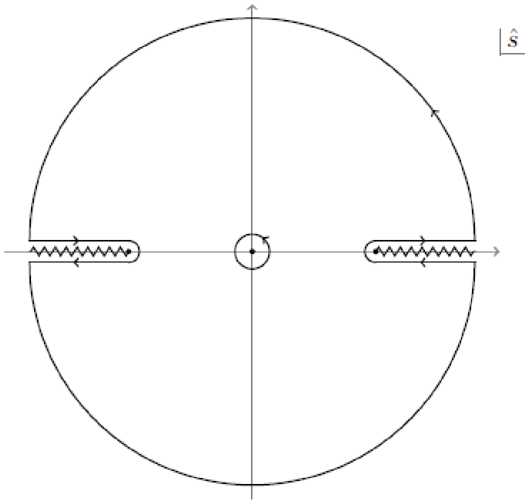

Figure 2.1: *The two contours in the $\hat{s}$-plane for S-matrix argument. The integral around the origin gives the coefficient of the operator in the EFT. It is equivalent to the large contour that reduces to an integral along the cuts since the arcs at infinity vanish. The integral along the cuts is positive definite[2]*

All this logic (S-matrix approach) works well when we have a Lorentz-invariant theory because we know how to relate low-energy states to high-energy ones, but in this thesis, we are focusing on theories for which the Lorentz-invariance is spontaneously broken. Upon Spontaneous breaking of Lorentz invariance, Goldstone bosons appear, which represent degrees of freedom relevant to these kinds of EFTs.

An important obstacle in our reasoning is that when Lorentz-invariance is broken, we have no boost transformation that relates high-energy to low-energy states. Due to this, it is difficult and quite compelling to connect the low and high-energy S-matrices.

Now let us switch to another approach.

2.1 IR/UV control

From the situation explained above we can see that we should search for such a quantity whose behavior can be controlled both in IR and UV. Under the assumption that deep UV completions of our EFTs are Lorentz-invariant, unitary CFT-s[2], good candidates with required characteristics can be the **correlation functions** of conserved quantities, like conserved currents or stress-energy tensor. Conserved currents are **primary** operators (a detailed definition of primary can be seen in section 2.4) for CFT-s and have fixed scaling dimensions. If we consider a d dimensional space-time, the conserved currents will have scaling dimension $d-1$, the stress-energy tensor will have scaling dimension d.

The 2-point correlation function of the conserved currents in Fourier space will be:

$$\langle J^{\mu}(-k)J^{\nu}(k)\rangle = c_J(k^{\mu}k^{\nu} - \eta^{\mu\nu}k^2)k^{d-4} \tag{2.4}$$

Derivation of (2.4) ([7]):

$$\pi^{\mu}_{\alpha}(\mathbf{p}) = \delta^{\mu}_{\alpha} - \frac{p^{\mu}p_{\alpha}}{p^2} \tag{2.5}$$

This operator is a projector onto tensors transverse to $\mathbf{p}$, which means $p_{\mu}\pi^{\mu}_{\alpha}(\mathbf{p}) = 0$ (because of transversality condition). From the transverse Ward identities (identities that arise from the conservation law of currents) it can be shown that the divergence of any 2-point function of conserved currents is proportional to the 1-point functions (derivations can be seen in [7]). Under the assumption that 1-point correlation functions vanish, 2 point correlation function is divergenceless, therefore transverse. It can be expressed as follows:

$$\langle J^{\mu}(\mathbf{p})J^{\nu}(\text{-}\mathbf{p})\rangle = \pi^{\mu\nu}(\mathbf{p})C(p) \tag{2.6}$$

Where $C(p)$ is any function of the magnitude of momentum. From the Dilation Ward identity, it can be shown that the form factors* should be the homogeneous functions† of degree $2\triangle - d$. For the conserved current, scaling dimension $\triangle = d-1$, so $2\triangle - d = d-2$. $\Rightarrow C(p) = \tilde{c}_J p^{d-2}$ ($\tilde{c}_J$ is arbitrary constant here).

So we can see that in our case 4-momentum is k, we have different sign from different ordering, we have $4D$, so $\delta^{\mu}_{\nu} \to \eta^{\mu\nu}$ and $1/k^2$ is out of the parenthesis, so using (2.5) we definitely get (2.4).

Under the assumption that conserved currents are Hermitian, the correlation functions of the Noether currents become field-redefinition independent. This is another property in common with the S matrix approach.

*a function that encapsulates the properties of certain particle interaction without considering whole underlined physics but providing suitable momentum dependence of suitable matrix elements instead

†a function of several variables such that, if all its arguments are multiplied by a scalar, then its value is multiplied by some power of this scalar, called the degree of homogeneity

Focusing on the correlation functions, we should decide which one to study. Locality and Lorentz invariance are very meaningful, thus quite natural decision would be if we use the retarded and advanced Green's functions[7]:

$$\begin{aligned} G_R^{\mu\nu}(x-y) &= i\theta(x^0-y^0)\,\langle 0|[J^\mu(x),J^\nu(y)]|0\rangle \\ G_A^{\mu\nu}(x-y) &= -i\theta(y^0-x^0)\,\langle 0|[J^\mu(x),J^\nu(y)]0|\rangle\,. \end{aligned} \tag{2.7}$$

2.2 Analiticity

Next, we look at analytic properties of $G_R^{\mu\nu}$ and $G_A^{\mu\nu}$. Here we are using logic and expressions from paper [2].
Fourier transform of Green's function is (definition):

$$\tilde{G}^{\mu\nu}_{R,A}(\omega,\mathbf{p}) = \int_{R^d} d^d x e^{-ip\cdot x} G^{\mu\nu}_{R,A}(x). \tag{2.8}$$

Metric convention is $(-,+,...,+)$.

For retarded function one observes:
$G_R^{\mu\nu} = 0$ for $x^0 < 0$ because if $x^0 < 0$, then $\theta = 0$ and so becomes $G_R^{\mu\nu}$;
$G_R^{\mu\nu} = 0$ for $x^2 > 0$ as commutator of the currents will be 0 because quantum fields at space-like separation commute.

From the calculations‡ we have that for $p^{Im} \in$ FLC, $\tilde{G}_R^{\mu\nu}(\omega,\mathbf{p})$ is analytic. Analogously, for $p^{Im} \in$ BLC $\tilde{G}_A^{\mu\nu}(\omega,\mathbf{p})$ is analytic.

Now we will show analicity explicitly. For this we write the momentum in the following way:

$$\boldsymbol{p} = \boldsymbol{k_0} + \omega\boldsymbol{\xi} \tag{2.9}$$

$\boldsymbol{k_0},\boldsymbol{\xi} \in \mathbb{R}^{d-1}$ are constants, $|\boldsymbol{\xi}| \equiv \xi < 1$, $\omega^{Im} > 0$ for $\tilde{G}_R$ and $\omega^{Im} < 0$ for $\tilde{G}_A$.
Then we assume that upon appropriate limits - $\omega^{Im} \to 0^{\pm}$ - both functions $\tilde{G}^{\mu\nu}_{R,A}(\omega)$ can be defined on the real line $\omega \in \mathbb{R}$.

‡Integration of (2.8) - FLC (forward lightcone): $x^0 > 0, x^2 < 0$. Considering the complex values for p and assuming polynomial boundedness (The function f is polynomially bounded means that there exist such polynomials g and h that for all x we have $g(x) \leq f(x) \leq h(x)$) of the real-space correlation functions, (2.8) converges for $Re(-ip\cdot x) < 0$ or $p^{Im}\cdot x < 0$ (negative $i\times$ negative imaginary part gives overall $''-''$) as $|x| \to \infty$ in the FLC. $\Rightarrow p^{Im} \in$ FLC.

For each $\boldsymbol{k_0}$ and $\boldsymbol{\zeta}$ let us define the following function on the whole complex ω-plane:

$$\tilde{G}^{\mu\nu}(\omega) = \begin{cases} \tilde{G}_R^{\mu\nu}(\omega, \mathbf{p}) & if \quad \omega^{Im} \geq 0, \\ \tilde{G}_A^{\mu\nu}(\omega, \mathbf{p}) & if \quad \omega^{Im} < 0. \end{cases} \tag{2.10}$$

It can be shown that $\tilde{G}^{\mu\nu}(\omega)$ is analytic on $\mathbb{C} \setminus \{(-\infty, -m) \cup (m, \infty)\}$, with m positive mass. Let us consider $\omega \in \mathbb{R}$ (or $\omega \pm \epsilon, \epsilon \to 0$). Inserting a unit operator $1 = \sum_n |P_n\rangle\langle P_n|$ and remembering that $J^\mu(x) = e^{-i\hat{P}\cdot x} J^\mu(0) e^{i\hat{P}\cdot x}$ we have:

$$\begin{aligned}
&\lim_{\epsilon\to 0} \left(\tilde{G}^{\mu\nu}(\omega + i\epsilon) - \tilde{G}^{\mu\nu}(\omega - i\epsilon)\right) = i \int_{R^d} d^d x e^{-ip\cdot x} \langle 0|[J^\mu(x), J^\nu(0)]|\rangle \\
&= i \int_{R^d} e^{-ip\cdot x} \langle 0|J^\mu(x) \Big(\sum_n |P_n\rangle\langle P_n| \Big) J^\nu(0)|\rangle - (\mu \leftrightarrow \nu, x \leftrightarrow 0) \\
&= i \int_{R^d} e^{-ip\cdot x} \langle 0|e^{-i\hat{P}\cdot x} J^\mu(0) e^{i\hat{P}\cdot x} \Big(\sum_n |P_n\rangle\langle P_n| \Big) J^\nu(0)|\rangle - (\mu \leftrightarrow \nu, x \leftrightarrow 0) \\
&= i(2\pi)^d \sum_n \{\delta^{(d)}(p - P_n)\langle 0|J^\mu(0)|P_n\rangle\langle P_n|J^\nu(0)|0\rangle - \delta^{(d)}(p + P_n)\langle 0|J^\mu(0)|P_n\rangle\langle P_n|J^\nu(0)|0\rangle\}
\end{aligned} \tag{2.11}$$

Where we have integrated over x to get delta functions and $(2\pi)^d$ and use:
$\langle 0|e^{-i\hat{P}x} = 0, \quad e^{-i\hat{P}x}|0\rangle = -Pe^{-i\hat{P}x}|0\rangle$ and
$e^{i\hat{P}x}|0\rangle = 0, \quad \langle 0|e^{i\hat{P}x} = P\langle 0|e^{i\hat{P}x}$,
For simplicity we will consider theory with a mass gap: so $P_n^0 > m > 0$.
$p^0 = \omega$, thus if $\omega| < m$ arguments of delta functions never vanish and $\tilde{G}^{\mu\nu}$ is continuous as we cross the imaginary axis at this point. So, we can see that for $\omega > m$, only the first term of the last line of Eq. (2.11) contributes, whereas for $\omega < m$ only the second term contributes. So, we can conclude that $\tilde{G}^{\mu\nu}(\omega)$ is analytic on the doubly-cut plane $\mathbb{C} \setminus \{(-\infty, -m) \cup (m, \infty)\}$.

2.3 Positivity of cut contributions

We want to check if the contribution from the discontinuity across the cuts has a definite sign. Let us integrate $\tilde{G}\mu\nu$ around the (m, ∞) cut in a clockwise direction, changing measure with some arbitrary powers of ω. We get a contribution only from the first part of (2.11), from $\delta^{(d)}(p - P_n)$. It is an useful idea to contract $\tilde{G}^{\mu\nu}$ with 2 copies of a constant real vector V_μ:

$$\frac{1}{(2\pi)^d} \int_{(m,\infty)cut} \frac{d\omega}{\omega^l} \tilde{G}^{\mu\nu}(\omega) V_\mu V_\nu = i \int_m^\infty \frac{d\omega}{\omega^l} \sum_n \delta^{(d)}(p - P_n) |\langle P_n | J^\mu(0) V_\mu | 0 \rangle|^2 \tag{2.12}$$

This is true because, V_μ and J^μ are Hermitian, so we can write that $\langle 0|J^\mu(0)|P_n\rangle = \langle P_n|J^m u(0)|0\rangle$ and in summation we can write down $\sum |V_\mu|^2 = \sum V_\mu V_\nu$. This is of the form $i\times$ (positive). (because, integration gives $-\frac{1}{l+1}$ from ∞ to $m \Rightarrow$, we get $-(\frac{1}{\infty} - \frac{1}{m})$ that gives $\frac{1}{m}$ and m is positive).

Integrating over $(-\infty, -m)$, we obtain:

$$\frac{1}{(2\pi)^d} \int_{(m,\infty)cut} \frac{d\omega}{\omega^l} \tilde{G}^{\mu\nu}(\omega) V_\mu V_\nu = -i \int_{-\infty}^{-m} \frac{d\omega}{\omega^l} \sum_n \delta^{(d)}(p + P_n) |\langle P_n | J^\mu(0) V_\mu | 0 \rangle|^2 \tag{2.13}$$

(2.13) is $i\times$ (positive) for odd l and $i\times$ (negative) for even l. We are interested in odd l cases as we want the contribution from both sides to be $i\times$ (positive).

Crossing symmetry and reality properties.

Let us see if the Green's functions $\tilde{G}^{\mu\nu}_{R,A}(p)$ satisfy a crossing symmetry property.
For $p^{Im} \in FLC$ we have:

$$\begin{aligned}
\tilde{G}^{\nu\mu}_A(-p) &= -i \int_{R^d} d^d x e^{ip\cdot x} \theta(-x^0) \langle 0 | [J^\nu(x), J^\mu(0)] | 0 \rangle \\
&= -i \int_{R^d} d^d x e^{-ip\cdot x} \theta(x^0) \langle 0 | [J^\nu(-x), J^\mu(0)] | 0 \rangle \\
&= -i \int_{R^d} d^d x e^{ip\cdot x} \theta(-x^0) \langle 0 | [J^\nu(0), J^\mu(x)] | 0 \rangle = \tilde{G}^{\mu\nu}_R(p)
\end{aligned} \tag{2.14}$$

Where to go from the first line to the second we change the integration variable $(x \to -x)$, to go from the second line to the third we use translation invariance (translate it by x).
So, we can see that the crossing symmetry property is satisfied.
In the case $\mathbf{k_0} = 0$, from Eq. (9) we have:

$$\tilde{G}^{\mu\nu}(\omega) = \tilde{G}^{\nu\mu}(-\omega) \quad when \quad \mathbf{k_0} = 0 \tag{2.15}$$

Let us highlight, that Green's advanced and retarded functions have certain reality properties as well. Due to the fact that $G^{\mu\nu}(x)$ is real for real x, its Fourier transform satisfies:

$$\tilde{G}_R^{\mu\nu}(p) = (\tilde{G}_R^{\mu\nu}(-p^\star))^\star \tag{2.16}$$

where $p^{Im} \in$ FLC. (because $\tilde{G}_R^{\mu\nu}(-p) = \int_{r^d} d^d x e^{ip\cdot x} G_R^{\mu\nu}(x)$ and then taking its conjugate will give $\int_{r^d} d^d x e^{ip\cdot x} G_R^{\mu\nu}(x) = \tilde{G}_R^{\mu\nu}(p)$).

Together Eq.s $(16), (14)$ imply:

$$\tilde{G}_R^{\mu\nu}(p) = (\tilde{G}_A^{\nu\mu}(p^\star))^\star. \tag{2.17}$$

2.4 EFT contact terms and gauging symmetry

To help understand this subsection, we will first introduce the notion of the contact terms. Let us use the definition for the operator product expansion (OPE) [4]. OPE represents the behavior of local operators when they approach each other. Given two local operators O_i and O_j at nearby points z and w on the complex plane (Fig2.2) can be closely approximated by a string of operators at one of the points z or w.

$$O_i(z,\bar{z})O_j(\omega,\bar{\omega}) = \sum_k C_{ij}^k(z-\omega,\bar{z}-\bar{\omega})O_k(\omega,\bar{\omega}) \tag{2.18}$$

$C_{ij}^k(z-\omega,\bar{z}-\bar{\omega})$ is a set of functions. Because of the translation invariance, C_{ij}^k-s depend only on the separation between 2 operators. The operators like (2.18) represent operator insertions inside time-ordered correlations functions. Now we want to define the primary operator. Ward identities give that OPE for T (stress-energy tensor) with any operator O must be in form:

$$T(z)O(\omega,\bar{\omega}) = ... + \frac{\partial O(\omega,\bar{\omega})}{\bar{z}-\bar{\omega}} + ... \tag{2.19}$$

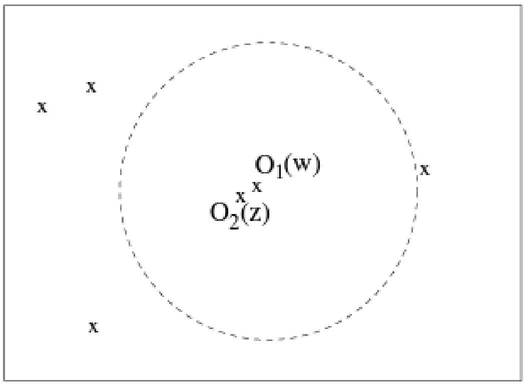

Figure 2.2

A *primary* operator is an operator whose OPE with T ($\bar{T}$) diminishes at order $(z-\omega)^{-2}$ ($(\bar{z}-\bar{\omega})^{-2}$)[4]. So, there are no higher-order singularities:

$$T(z)O(\omega,\bar{\omega}) = h\frac{O(\omega,\bar{\omega})}{(z-\omega)^2} + \frac{\partial O(\omega,\bar{\omega})}{\bar{z}-\bar{\omega}} + non-singular \tag{2.20}$$

The equation of energy conservation in a complex plane looks like [4]:

$$\partial T_{\bar{z}\bar{z}} = -\bar{\partial} T_{zz} \tag{2.21}$$

This leads to OPE:

$$T_{z\bar{z}} T_{\omega\bar{\omega}} = \frac{c\pi}{6}\partial_z \bar{\partial}_{\bar{\omega}} \delta(z-\omega, \bar{z}-\bar{\omega}) \tag{2.22}$$

Eq. (2.22) implies that OPE of $T_{z\bar{z}}$ and $T_{\omega\bar{\omega}}$ almost vanishes, however, we can notice certain singular behavior as $z \to \omega$. This term (a derivative of δ function) is the contact term between operators and is needed to guarantee the energy-momentum conservation.

After presenting the properties of $G^{\mu\nu}$ in the UV let us now discuss the calculation of $\tilde{G}^{\mu\nu}$ in the EFT.
We can understand the EFT as the low-energy expansion of the UV theory where the heavy degrees of freedom are integrated out. Integration of the heavy modes generates the contact terms. An example of such a term generation would be when an expansion of a heavy propagator proportional to $(p^2+m^2)^{-1}$ at low energy returns a polynomial in p^2. This polynomial represents the Fourier transform of a sum of the generated contact terms, which themselves are the derivatives of the delta function (as we

saw above in Eq. (2.22)).

Commonly, coupling a given operator to an external source allows to efficiently calculate the correlation functions. If our operator is the conserved current, then such a field can be a non-dynamical vector potential gauge field A_μ, if the operator is the energy-momentum tensor, then such a field can be a non-dynamical metric $g_{\mu\nu}$.

When $(U(1))$ is spontaneously broken, this gauge symmetry is spontaneously broken as well. Consequently, $U(1)$ symmetry leads to a gauge symmetry for A_μ.
Rather than calculating the correlation functions for J^μ, we can obtain the functional derivatives in the path integral of the gauged theory.

We can express Green's function in terms of the time-ordered product operators:

$$\begin{aligned} G_R^{\mu\nu}(x-y) =& i\theta(x^0-y^0)\langle 0|[J^\mu(x), J^\nu(y)]|0\rangle = \\ &= i\langle 0|T\{J^\mu(x)J^\nu(y)\}|0\rangle - i\langle 0|J^\nu(y)J^\mu(x)|0\rangle. \end{aligned} \tag{2.23}$$

(as θ leaves only the part where $x^0 > y^0$, so if time order is $J^\nu(y)J^\mu(x)$, expression should vanish, as it does).

The contact terms will appear only in the time-ordered product but not in the last term of Eq.(2.23).
We will indeed have:

$$\begin{aligned} & i\int_{R^d} d^d x e^{-ip\cdot x}\langle 0|J^\nu(0)J^\mu(x)|0\rangle = \\ & = i(2\pi)^d \sum_n \delta^{(d)}(p+P_n)\langle 0|J^\nu(0)|P_n\rangle\langle P_n|J^\mu(0)|0\rangle \end{aligned} \tag{2.24}$$

(Active $\omega < -m$ mode, so no heavy-energy contribution).
However, due to the product with θ function in real space, the first term of RHS of (2.23) involves a convolution in Fourier space. Thus, the heavy modes also contribute to the time-ordered correlation function at the low energy.

Useful is to see in path integral approach: The first term is time-ordered, so we can write:

$$\langle 0|T\{J^\mu(x)J^\nu(y)\}|0\rangle = \frac{1}{Z}\int D\phi e^{i\int_{R^d} d^d x L(\phi)} J^\mu(x)J^\nu(y) \tag{2.25}$$

Z - normalization:

$$Z = \int D\phi e^{i\int_{R^d} d^d x L(\phi)} \tag{2.26}$$

ϕ is the whole set of heavy and light dynamical fields in theory. The second term of (2.23) is without any prescription about ordering, but still interesting for the observables different from the flat space S-matrix (in Cosmology, etc). After applying the time-evolution operator: $U(t, t_0)$ from $t_0 \to t$, the correlator in Schrodinger picture becomes:

$$\langle 0|J^\nu(y)J^\mu(x)|0\rangle = \langle 0|U(+\infty, y^0)J^\nu_{(s)}(\mathbf{y})U(y^0, x^0)J^\mu_{(s)}(\mathbf{x})U(x^0, -\infty)|0\rangle \tag{2.27}$$

Let us insert twice the identity element at a fixed time:

$$1 = \int D\phi(\tilde{\mathbf{x}})|\phi(\mathbf{x})\rangle\langle\phi(\mathbf{x})|, \tag{2.28}$$

at the time $t = x^0$ and $t = y^0$. Then let us express each time-evolution operator by path integral with appropriate boundary conditions, we will get:

$$\langle\phi(y^0, \tilde{\mathbf{y}})|U(y^0, x^0)|\phi(x^0, \tilde{\mathbf{x}})\rangle = \int_{\phi(\tilde{\mathbf{x}})}^{\phi(\tilde{\mathbf{y}})} D\phi e^{i\int_{x^0}^{y^0} d^d x L(\phi)}, \tag{2.29}$$

So, we can write

$$\begin{aligned}\langle 0|J^\nu(y), J^\mu(x)|0\rangle =& \frac{1}{Z}\int D\phi(\tilde{\mathbf{x}})\int D\phi(\tilde{\mathbf{y}})J^\nu(\phi(y^0, \mathbf{y}))J^\mu(\phi(x^0, \mathbf{x}))\int_{\phi(\tilde{\mathbf{y}})} D\phi_3 e^{i\int_{y^0}^{+\infty} d^d x L(\phi_3)} \times \\ &\times \int_{\phi(\tilde{\mathbf{x}})}^{\phi(\tilde{\mathbf{y}})} D\phi_2 e^{i\int_{x^0}^{y^0} d^d x L(\phi_2)} \int^{\phi(\tilde{\mathbf{x}})} D\phi_1 e^{i\int_{-\infty}^{x^0} d^d x L(\phi_1)}.\end{aligned} \tag{2.30}$$

It is worth noticing that $\phi(x^0, \mathbf{x})$ is fixed in terms $\phi(\tilde{\mathbf{x}}) : \phi(x^0, \mathbf{x}) = \phi(\tilde{\mathbf{x}})\Big|_{\tilde{\mathbf{x}}=\mathbf{x}}$.

Next the authors[2] introduce the external sources coupled to J^μ and write the correlation of currents as the functional derivatives wrt gauge bosons.

$L(\phi_i) \to L(\phi_i, A^i_\mu)$:

$$G_R^{\mu\nu}(x,y) =$$
$$= \frac{i}{Z}\Bigg(\int D\phi_0 e^{i\int_{R^d}^{d^d x L\left(\phi_0, A_\mu^{(0)}\right)}} J^\mu(\phi_0(x))J^\nu(\phi_0(y))\Big|_{A_\mu^{(0)}=0} +$$
$$\int D\phi(\tilde{\mathbf{x}}) \int D\phi(\tilde{\mathbf{y}}) J^\nu(\phi(y^0,\mathbf{y}))J^\mu(\phi(x^0,\mathbf{x})) \int_{\phi(\tilde{\mathbf{y}})} D\phi_3 e^{i\int_{y^0}^{+\infty} d^d x L(\phi_3, A_\mu^{(3)})} \times$$
$$\int_{\phi(\tilde{\mathbf{x}})}^{\phi(\tilde{\mathbf{y}})} D\phi_2 e^{i\int_{x^0}^{y^0} d^d x L(\phi_2, A_\mu^{(2)})} \int^{\phi(\tilde{\mathbf{x}})} D\phi_1 e^{i\int_{-\infty}^{x^0} d^d x L(\phi_1, A_\mu^{(1)})}\Big|_{A_\mu^{(1,2,30}=0}\Bigg). \quad (2.31)$$

Let us rewrite the currents as derivatives wrt A_μ:

$$G_R^{\mu\nu}(x,y) = \frac{i}{Z}\Bigg(-\frac{\delta^2}{\delta A_\mu^{(0)}(x)\delta A_\nu^{(0)}(y)} \int D\phi_0 e^{i\int_{R^d}^{d^d x L\left(\phi_0, A_\mu^{(0)}\right)}}\Big|_{A_\mu^{(0)}=0} -$$
$$-\frac{\delta^2}{\delta A_\mu^{(1)}(x)\delta A_\nu^{(3)}(y)} \int D\phi(\tilde{\mathbf{x}}) \int D\phi(\tilde{\mathbf{y}}) \int_{\phi(\tilde{\mathbf{y}})} D\phi_3 e^{i\int_{y^0}^{+\infty} d^d x L(\phi_3, A_\mu^{(3)})} \times$$
$$\int_{\phi(\tilde{\mathbf{x}})}^{\phi(\tilde{\mathbf{y}})} D\phi_2 e^{i\int_{x^0}^{y^0} d^d x L(\phi_2, A_\mu^{(2)})} \int^{\phi(\tilde{\mathbf{x}})} D\phi_1 e^{i\int_{-\infty}^{x^0} d^d x L(\phi_1, A_\mu^{(1)})}\Big|_{A_\mu^{(1,2,30}=0}\Bigg). \quad (2.32)$$

This expression is in full UV theory. If we want to gain an appropriate low-energy EFT expression, we should integrate out the heavy fields: splitting the fields in heavy, ϕ_h, and light, ϕ_l, the following expression can be obtained:

$$e^{iS_{EFT}(\phi_l, A_\mu)} = \int D\phi_h e^{iS_{EFT}(\phi_h, \phi_l, A_\mu)}. \quad (2.33)$$

Now resulting S_{EFT} is gauge invariant, therefore it contains all the ”minimal” couplings of ϕ_l to A_μ induced by gauging. Integrating out the heavy fields leads to other operators as well.

Operators **quadratic** in A_μ contribute the **contact terms** to the $\langle JJ \rangle$ (so we see that only time-ordered part gives contribution).

2.5 Contour argument

We now will see the example of how one can gain positivity bounds for theories in which boosts are spontaneously broken. Consider a low-energy EFT with a cutoff Λ. Let us compute $\tilde{G}^{\mu\nu}(\omega)$ in this case. 00 component will look like this:

$$\tilde{G}^{00}(\omega) = \mu^{d-2}\left[c_1 \frac{1}{1-c_s^2\xi^2} + \frac{\omega^2}{\Lambda}\Big(\frac{c_2}{(1-c_s^2\xi^2)^2} + d_1\Big) + O\Big(\frac{\omega^4}{\Lambda^2}\Big)\right] \quad (2.34)$$

The propagators of the low energy degree of freedom (which has a speed of propagation c_s) give us the denominators in Eq. (2.34), μ is overall scale, c_i and d_i are the EFT coefficients.
Similar logic what was provided in the case of the S-matrix can be used here:

$$\oint d\omega \frac{\tilde{G}^{00}(0)}{\omega^3} = 2\pi i \Big(\frac{c_2}{(1-c_s^2\xi^2)^2} + d_1\Big)\frac{\mu^{d-2}}{\Lambda^2} \quad (2.35)$$

From the CFT correlator (2.4) we have an expression for $\tilde{G}^{\mu\nu}$. Then we can do the same trick that was used in (2.3) by applying the Cauchy theorem. If we have $d = 3$, $\tilde{G}^{\mu\nu} \sim \omega$ in the limit $|\omega| \to \infty$ and contribution from the circle at infinity is negligible. Conclusively, one can guess that the integral around the origin is equal to the integral along the cuts, which is $i\times$ (positive). This will give us the following bound:

$$\frac{c_2}{(1-c_s^2\xi^2)^2} + d_1 \geq 0. \quad (2.36)$$

To get the most general bounds (inequalities) ξ should be varied in the interval $0 \leq \xi < 1$ and $\tilde{G}^{\mu\nu}$ should be contracted with a generic vector V^μ.

Getting rid of the mass gap

In the absence of the mass gap, the low energy excitation loops will open a cut in the ω-plane all along the real axis (Fig.2.3). And one can integrate over 2 separate contours getting the same results for positivity bound as from integrating over the contour shown on Fig.2.1.

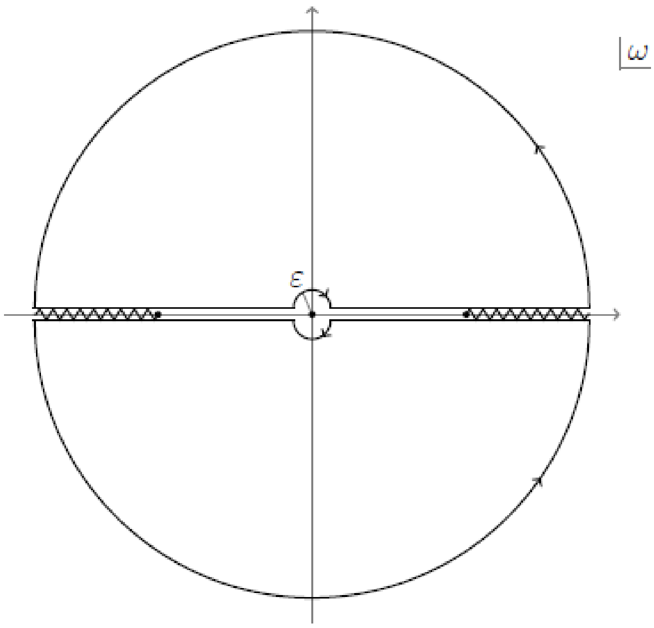

Figure 2.3: *Different contours that give the same result as in Fig. 1. This choice is suitable when a cut is running all along the real axis.*

Chapter 3

Conformal fields and Superfluids

Now we will discuss the reason for using a conformal field theory, explain superfluidity and relate them.

3.1 Superfluids

Generally, the superfluid is called the fluid which flows without friction. A flow can be observed by transporting something through a capillary[8]. For example, for liquid helium superfluidity can be considered as the frictionless transport of a conserved charge. There exists a certain critical temperature, below which helium becomes superfluid and above which it behaves like a normal fluid. So, at the critical temperature, a phase transition happens, leaving the system in a superfluid phase below this temperature. This transition can be described as Bose-Einstein condensation, where helium atoms occupy a single quantum state forming a ”condensate”. This phase transition can be characterized by the symmetries of the system, so one can relate it to the conserved charge: in the superfluid phase of a system, a symmetry of the system associated with a charge is spontaneously broken.

We can now switch from liquid helium to a more microscopic approach for superfluids. To understand the characteristic properties of a superfluid, one can consider the theory for the degrees of freedom forming Bose condensate. The useful general concepts are stated below:

- $U(1)$ *symmetry*: is the simplest continuous symmetry under which the Lagrangian of the model is invariant; $U(1)$ symmetry is characterized by one real parameter. The necessary condition for superfluidity is the spontaneous breaking of Lagrangian invariance under $U(1)$;

- *conserved charge*: is essential for all superfluids due to the charge being transported by a superflow; this charge is a consequence of $U(1)$ symmetry via Noether's theorem
- *Bose-Einstein condensatrion*: is an alternative way to express that $U(1)$ symmetry is spontaneously broken in the context of the bosonic superfluid;
- *spontaneous symmetry breaking*: is the evidence that the ground state (the Bose-Einstein condensate) in a superfluid is not invariant wrt to the original symmetry transformations of the system's Lagrangian;
- *Goldstone mode*: is a massless mode, which arises for all temperatures below the critical temperature if a spontaneously broken symmetry is global [8].

Now we will keep this in mind for some time and explain why are we using a conformal field approach. Then it will be clear that this two can be related.

3.2 Conformal field theory approach

Generally, we are trying to look at the modified gravity by considering an extra scalar degree of freedom. So, we are interested if there exists a "global completion" for dynamics of our scalar degree of freedom [9], that means if the description of π (the scalar field) is still working at large distances. One way to achieve this is to make 4D Lagrangian conformally invariant. The 4D conformal group is $SO(4,2)$. It has the following maximal subgroups: $SO(3,2)$, $4D$ Poincare, and $SO(4,1)$ with their isometry groups of four-dimensional spaces - anti-deSitter, Minkovski, and deSitter, respectively. A single scalar, namely dilaton, can cause a spontaneous breakdown of the conformal group into any of the above subgroups [9].

In [10], the authors consider a limit in which the gravity is decoupled $M_{Pl} \to \infty$. The dynamical gravity significantly changes the picture at large distances, but for local analysis, at distances much smaller than the Hubble scale, the effects of the dynamical gravity can be neglected. Above mentioned AdS, Minkowski, and dS are the three possible maximally symmetric solutions corresponding to different unbroken combinations of the following generators: dilations, infinitesimal special conformal transformations, translations, and boosts (see expressions for these generators in appendix A). For example, consider the Minkowski space case. It corresponds to a trivial (constant) configuration of a scalar degree of freedom. The translations and boosts for this space are unbroken, while dilations and special conformal transformations are spontaneously broken. Now, let us write down conformally invariant Lagrangians. To do so, we write down all possible diff-invariant* Lagrangian terms involving the metric defined as following (consider that matter is minimally coupled to the metric):

$$g_{\mu\nu} = e^{2\pi}\eta_{\mu\nu} \tag{3.1}$$

where π is our scalar degree of freedom. Note that a conformal group is such a subgroup of diffeomorphisms that leaves the metric conformally flat. The simplest term of the Lagrangian will be:

$$\int d^4x\sqrt{-g} = \int d^4x e^{4\pi} \tag{3.2}$$

where $g = det g_{\mu\nu} = e^{4\pi} \cdot det(\eta_{\mu\nu}) = e^{4\pi} \cdot (-1)$. This is conformally invariant, upon changing the integration variable (see appendix A). For the next terms in the derivative expansion, we will have curvature invariants R, R^2, $R_{\mu\nu}R^{\mu\nu}$, $R^{\mu}_{\nu\rho\sigma}R^{\nu\rho\sigma}_{\mu}$. At short distances and for small values of π, the conformal invariance reduces to the Galilean invariance [10].

So we can construct Lagrangian for our theory in the CFT approach where we have spontaneously broken symmetry, but we should be careful that received Lagrangian gave at most second order in equations of motion in order to avoid the ghosts and other instabilities. Because of this, we should add some terms to the Lagrangian that will cancel the terms gained from the curvature that give us higher-order EOMs.

Now as we already said that spontaneous breaking is a necessary condition for superfluidity and we have also the Lagrangian in a CFT approach with spontaneously broken symmetry, we can relate superfluids Lagrangian to conformal field Lagrangian (just use Lagrangian that we gained for CFT for superfluids). From now, we will try to construct the Lagrangian, which will give us the two-point correlation functions for conserved current and stress-energy tensor.

*an invariant for the action of a Lie group on a space that involves the derivatives of graphs of functions in the space

Chapter 4

Conformal superfluids

In this chapter, we will use the conformal field theory approach for superfluids and consider the case where gravity is decoupled. The whole CFT may be quite complicated and subtle to consider but we can consider its separate sections. We are interested in a large charge Q sector as the EFT encodes all the information about that, and also it can be identified with the derivative expansion (as we show in calculations). Also, we are interested in a theory where Lorentz invariance is spontaneously broken because of the preferred time direction, while we know that for conformal superfluids the charge causes the breaking of $U(1)$ symmetry. Thus it will be a wise idea, if we consider our CFT at a finite chemical potential μ as a state evolving in time around $U(1)$ (in this case we are trying to look at our theory in the conformal superfluid's approach) (μ will turn out to be the EFT cut-off of the theory)[2]. The spontaneous symmetry breaking of Lorentz invariance leads to a Goldstone boson — the unique degree of freedom at much lower energies than cut-off μ. This symmetry breaking can be expressed as the charged scalar χ linearly evolving in time

$$\chi(x) = \mu t + \pi(x), \tag{4.1}$$

π is the Goldstone boson here. One can gain the most general action through a coset construction [11] or by using an effective metric (appendix B).

Generally, the CFT correlation functions containing two or more operators with large charge can be obtained using the EFT[12]. The EFT operators' coefficients contain all the specifics about the certain CFT, thus the positivity constraints will specify the region of possible CFT data.

We will work in $d = 3$, however, our arguments are general. Reasons why we will consider $d = 3$ are the following:

- $d = 3$ the CFTs are the main objects in second-order phase transitions;
- in $d = 3$ it is sufficient to consider the operators of the theory at next to linear order (NLO), whereas in $d = 4$ the contour we are going to perform integral around would not converge at infinity and one would have to consider higher order terms.

4.1 $\langle JJ \rangle$ calculation

Considering the terms next to linear order (NLO) in derivatives, the EFT Lagrangian for our scalar field π that is coupled to A_μ in $d = 3$ (relation to superfluid Lagrangian can be seen in appendix B) is:

$$\begin{aligned} L =& \frac{c_1}{6}|\nabla\chi|^3 - 2c_2\frac{(\partial|\nabla\chi|)^2}{|\nabla\chi|} + c_3\Big(2\frac{(\nabla^\mu\chi\partial_\mu|\nabla\chi|)^2}{|\nabla\chi|^3} + \partial_\mu\big(\frac{\nabla^\mu\chi\nabla^\nu\chi}{|\nabla\chi|^2}\big)\partial_\nu|\nabla\chi|\Big) - \\ &- \frac{b}{4}\frac{F_{\mu\nu}F^{\mu\nu}}{|\nabla\chi|} + \frac{d}{2}\frac{F_i^\mu F^{\nu i}}{|\nabla\chi|^3}\nabla_\mu\chi\nabla_\nu\chi, \end{aligned} \tag{4.2}$$

where χ is from Eq.(4.1) and:

$$\begin{aligned} \nabla_\mu\chi &\equiv \partial_\mu\chi - A_\mu \\ |v| &\equiv \sqrt{-v_\mu v^\mu} \end{aligned} \tag{4.3}$$

For obtaining a healthy kinetic term, we need the following condition: $c_1 > 0$.

Now if we expand the Lagrangian up to the second order in π and A and integrate by parts, we will get:

$$\begin{aligned} L =& \frac{c_1\mu^3}{6} + \frac{c_1\mu}{2}\Big[(\dot\pi + A^0)^2 - \frac{1}{2}(\partial_i - A_i)^2 + \mu(\dot\pi + A^0)\Big] + \\ &+ \frac{2c_2}{\mu}\Big[-\pi\Box\ddot\pi + 2A^0\Box\dot\pi + A^0\Box A^0\Big] + \\ &+ \frac{2c_3}{\mu}\Big[-\pi\Box\ddot\pi + 2A^0\Box_{c_s}\dot\pi - A^i\partial_i\ddot\pi + (\dot A^0)^2 + \dot A^0\partial_i A^i\Big] + \\ &+ \frac{(b+d)}{2\mu}\Big[(\partial_i A^0)^2 + (\partial_0 A_i)^2 + 2\dot A^0(\partial_i A_i)\Big] - \frac{b}{4\mu}(\partial_i A_j - \partial_j A_i)^2 \end{aligned} \tag{4.4}$$

where $\Box \equiv \partial_\mu \partial^\mu$ and $\Box_{cs} = -\partial_t^2 + c_s^2 \partial_i \partial^i$, $c_s^2 = \frac{1}{2}$.

Now up to quadratic order, we can write down this expression:

$$L = L(A=0) + A_\mu J_N^\mu + O(A^2) \tag{4.5}$$

Prove that we can do that is provided below:
considering Lagrangian for the Klein-Gordon field;

$$L_{KG} = (\partial_\mu \phi)^\star (\partial^\mu \phi) - m^2 \phi^\star \phi, \tag{4.6}$$

For global symmetry: $\phi \to e^{i\alpha}\phi$;
For local symmetry: $\phi \to e^{i\alpha(x)}\phi$, that needs also to have

$$\begin{aligned} A_\mu &\to A_\mu - \partial_\mu \alpha(x) \\ \partial_\mu &\to D_\mu = \partial_\mu - iqA_\mu \end{aligned} \tag{4.7}$$

Then we would have:

$$\begin{aligned} L &= (D_\mu \phi)^\star (D^\mu \phi) - m^2 \phi^\star \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \\ &= (\partial_\mu \phi)^\star (\partial^\mu \phi) - m^2 \phi^\star \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu. \end{aligned} \tag{4.8}$$

Where the first two terms are the Lagrangian with vector-potential zero, $L(A=0)$, the third one is quadratic in A so we neglect it $\Rightarrow$ we could write down L as:

$$L = L(A=0) + A_\mu J^\mu. \tag{4.9}$$

Let us find the Noether currents (associated to symmetry $\chi \to \chi + c$ for the Lagrangian (4.2)):

$$\begin{aligned} A_\mu J^\mu &= L - L(A=0) \\ &= \frac{\mu c_1}{2}\Big(2\dot{\pi} A^0 + (A^0)^2 - \frac{1}{2}(-2\partial_i \pi A_i + A_i^2) + \mu A^0\Big) + \\ &+ \frac{2c_2}{\mu}(2A^0 \Box \dot{\pi} + A^0 \Box A^0) + \frac{2c_3}{\mu}(2A^0 \Box_{cs} \dot{\pi} - A^i \partial_i \ddot{\pi} + (\dot{A}^0)^2 + \dot{A}^0 \partial_i A^i) + \\ &+ \frac{b+d}{2\mu}((\partial_i A^0)^2 + (\partial_0 A_i)^2 + 2\dot{A}^0(\partial_i A_i)) - \frac{b}{4\mu}(\partial_i A_j - \partial_j A_i)^2 \end{aligned} \tag{4.10}$$

If we neglect the quadratic terms in A and consider only the Noether current without the contract terms (that are b, d) we will be left with:

$$\begin{aligned} A_\mu J^\mu &= L - L(A=0) \\ &= \frac{\mu c_1}{2}\Big(2\dot{\pi}A^0 - \frac{1}{2}(-2\partial_i\pi A_i) + \mu A^0\Big) + \\ &+ \frac{2c_2}{\mu}(2A^0\Box\dot{\pi}) + \frac{2c_3}{\mu}(2A^0\Box_{c_s}\dot{\pi} - A^i\partial_i\ddot{\pi} + \dot{A}^0\partial_i A^i) \end{aligned} \tag{4.11}$$

The Noether currents are expressed in the following way:

$$J_N^0 = -\frac{\mu c_1}{2} - \mu c_1\dot{\pi} - \frac{4c_2}{\mu}\Box\dot{\pi} - \frac{4c_3}{\mu}\Box_{c_s}\dot{\pi} \tag{4.12}$$

$$J_N^i = \frac{\mu c_1}{2}\partial_i\pi - \frac{2c_3}{\mu}\partial_i\ddot{\pi}, \tag{4.13}$$

which are derived from Noether procedure given below:
$L = L(A=0) + A_\mu J_N^\mu + O(A^2)$.

Now we will try to obtain a propagator of π as we will need it to calculate the two-point function J_N^μ in EFT. We get it from the quadratic action, considering $A = 0$. We have:

$$L_{(2)A=0} = \frac{\mu c_1}{2}\Big(\dot{\pi}^2 - \frac{1}{2}(\partial_i\pi)^2\Big) - \pi\Box\ddot{\pi}\Big(\frac{2c_2}{\mu} + \frac{2c_3}{\mu}\Big) \tag{4.14}$$

Let us rewrite (4.14) in a bit different way:
We will use:

$$\begin{aligned} \dot{\pi}^2 - \frac{1}{2}(\partial_i\pi)^2 =& \partial_t(\pi\dot{\pi}) - \pi\ddot{\pi} - \frac{1}{2}\partial_i(\pi\partial^i\pi) + \frac{1}{2}\pi\partial_i\partial^i\pi = \\ &= \partial_\mu(\pi\partial^\mu\pi) - \pi\ddot{\pi} + \frac{1}{2}\pi\partial_i\partial^i\pi; \end{aligned} \tag{4.15}$$

The first term is full derivative so we can imagine: $L \to L - \partial_\mu(\pi\partial^\mu\pi)$ and use the result from (4.15). Then let's have a look at the following:

$$\begin{aligned} \pi\Box\ddot{\pi} &= (\pi\Box\dot{\pi})' - \dot{\pi}\Box\dot{\pi} = (\pi\Box\dot{\pi})' - \dot{\pi}\partial_\mu\partial^\mu\dot{\pi} = \\ &= (\pi\Box\dot{\pi})' - \partial_\mu(\dot{\pi}\partial^\mu\dot{\pi}) + (\partial_\mu\pi)^2 \approx -\pi^2 + (\partial_i\dot{\pi})^2. \end{aligned} \tag{4.16}$$

Here we neglected the first term because it had a III derivative and the second term because it was a full derivative.

So, using (4.15) and (4.16) we can write:

$$\begin{aligned} L_{(2),A=0} =& \frac{\mu c_1}{2}\pi\Box c_s\pi - \frac{2(c_2+c_3)}{\mu}\pi\Box\ddot{\pi} = \\ & \frac{1}{2}\dot{\pi}_c^2 - \frac{1}{4}(\partial_i\pi_c)^2 + \frac{2(c_2+c_3)}{\mu}(\ddot{\pi}_c^2 - (\partial_i\dot{\pi})^2), \end{aligned} \tag{4.17}$$

where $\pi_c \equiv \sqrt{\mu c_1}\pi$.

In Fourier space:

$$L_{(2),A=0}(k) = \frac{1}{2}\tilde{\pi}_c(-k)\left(\omega^2 - c_s^2\boldsymbol{k}^2 + \frac{4(c_2+c_3)}{\mu^2 c_1}\omega^2(\omega^2 - \boldsymbol{k}^2)\right)\tilde{\pi}_c(k) \tag{4.18}$$

Fourier convention:

$$f(x) = (2\pi)^{-3}\int d^3k e^{ik\cdot x}\tilde{f}(k) \tag{4.19}$$

The $\pi\tilde{\pi}$ propagator:

$$\langle\tilde{\pi}_c(-k)\tilde{\pi}_c(k)\rangle = \frac{i}{\omega^2 - c_s^2\boldsymbol{k}^2}\left(1 - \frac{4(c_2+c_3)}{\mu^2 c_1}\omega^2\frac{(\omega^2-\boldsymbol{k}^2)}{\omega^2 - c_s^2\boldsymbol{k}^2}\right) \tag{4.20}$$

The current conservation constrains the structure of the current-current correlator ($\langle J^\mu(-k)J^\nu(k)\rangle$):

$$k_\mu\langle J^\mu(-k)J^\nu(k)\rangle = 0 \tag{4.21}$$

In the absence of the Lorentz invariance, we can express the current-current correlator with 2 possible tensorial quantities, that guarantee the conservation,

$$i\langle J^\mu(-k)J^\nu(k)\rangle = A(k^\mu k^\nu - \eta^{\mu\nu}k^2) + B(k^i k^j - \delta^{ij}\boldsymbol{k}^2), \tag{4.22}$$

here A and B are general functions of ω and $|\boldsymbol{k}|$.

Let us try to calculate A and B. We will do that in the following way:

1. Firstly, we will consider the correlation function of 00 components of the currents, it will give us A (as we will see a bit later);
2. And then we will consider the correlation function of ii component of the currents and with already known A we will find B.

For 00 component for the RHS of the equation (4.22) we will have only A part as following (i and j do not have 0 meanings, so we will not have B part):

$$A(k^0k^0 - \eta^{00}k^2) = A(\omega^2 + (-\omega^2 + \boldsymbol{k}^2)) = A\boldsymbol{k}^2 \tag{4.23}$$

Now let us see what will be LHS of (4.22) considering the 00 component. Firstly, let us write down our current in Fourier space:

$$\tilde{J}^0_N(k) = i\omega + i\omega\frac{4c_2}{\mu}(\omega^2 - \boldsymbol{k}^2) + i\omega\frac{4c_3}{\mu}(\omega^2 - c_s^2\boldsymbol{k}^2) \tag{4.24}$$

Then correlator for 00 component will be:

$$\begin{aligned}
\langle \tilde{J}^0_N(-k)\tilde{J}^0_N(k)\rangle &= \omega^2\mu^2c_1^2\Big(1 + \frac{4c_2}{\mu^2c_1}(\omega^2 - \boldsymbol{k}^2) + \frac{4c_3}{\mu^2c_1}(\omega^2 - c_s^2\boldsymbol{k}^2)\Big)^2\cdot \\
&\cdot\frac{i}{\omega^2 - c_s^2\boldsymbol{k}^2}\Big(1 - \frac{4c_2\omega^2(\omega^2 - \boldsymbol{k}^2)}{\mu^2c_1(\omega^2 - c_s^2\boldsymbol{k}^2)} - \frac{4c_3\omega^2(\omega^2 - \boldsymbol{k}^2)}{\mu^2c_1(\omega^2 - c_s^2\boldsymbol{k}^2)}\Big) = \\
&= \frac{i\omega^2\mu c_1}{\omega^2 - c_s^2\boldsymbol{k}^2}\Big(1 + \frac{8c_2}{\mu^2c_1}(\omega^2 - \boldsymbol{k}^2) + \frac{8c_3}{\mu^2c_1}(\omega^2 - c_s^2\boldsymbol{k}^2)\Big)\cdot \\
&\cdot\frac{i}{\omega^2 - c_s^2\boldsymbol{k}^2}\Big(1 - \frac{4c_2\omega^2(\omega^2 - \boldsymbol{k}^2)}{\mu^2c_1(\omega^2 - c_s^2\boldsymbol{k}^2)} - \frac{4c_3\omega^2(\omega^2 - \boldsymbol{k}^2)}{\mu^2c_1(\omega^2 - c_s^2\boldsymbol{k}^2)}\Big)
\end{aligned} \tag{4.25}$$

Then let us remind RHS of (4.22). For 00 components we will have:

$$\frac{i\langle \tilde{J}^0_N(-k)\tilde{J}^0_N(k)\rangle}{\boldsymbol{k}^2} = A \tag{4.26}$$

Let us calculate A term by term. We will also consider the on-shell condition which means:

$$2\omega^2 = \boldsymbol{k}^2 \tag{4.27}$$

Also, we will need to take into account that $c_s^2 = 1/2$.

1. For the first — c_1 term we have:

$$A_1 = \Big(\frac{i\langle \tilde{J}^0_N(-k)\tilde{J}^0_N(k)\rangle}{\boldsymbol{k}^2}\Big)_1 = -\frac{\mu c_1}{2(\omega^2 - c_s^2\boldsymbol{k}^2)}\frac{2\omega^2}{\boldsymbol{k}^2} = -\frac{\mu c_1}{2(\omega^2 - c_s^2\boldsymbol{k}^2)}. \tag{4.28}$$

2. For the second — c_2 term we have:

$$\begin{aligned}
A_2 &= \Big(\frac{i\langle \tilde{J}^0_N(-k)\tilde{J}^0_N(k)\rangle}{\boldsymbol{k}^2}\Big)_2 = \\
&= -\frac{\omega^2\mu c_1}{\boldsymbol{k}^2(\omega^2 - c_s^2\boldsymbol{k}^2)}\Big(\frac{8c_2}{\mu^2 c_1}(\omega^2 - \boldsymbol{k}^2) - \frac{4c_2}{\mu^2 c_1}\omega^2\frac{\omega^2 - \boldsymbol{k}^2}{\omega^2 - c_s^2\boldsymbol{k}^2}\Big) = \\
&= -\frac{c_2(\omega^2 - \boldsymbol{k}^2)\omega^2}{\boldsymbol{k}^2\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2}(8\omega^2 - 8c_s^2\boldsymbol{k}^2 - 4\omega^2) = \\
&= -\frac{c_2(\omega^2 - \boldsymbol{k}^2)\omega^2}{\boldsymbol{k}^2\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2}4(\omega^2 - \boldsymbol{k}^2) = \frac{c_2(\omega^2 - \boldsymbol{k}^2)\boldsymbol{k}^4}{\boldsymbol{k}^2\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2} = \\
&= \frac{c_2}{\mu}\frac{(\omega^2 - \boldsymbol{k}^2)\boldsymbol{k}^2}{\mu(\omega^2 - c_s^2 - \boldsymbol{k}^2)^2}.
\end{aligned} \tag{4.29}$$

Where we used the following:

$$\begin{aligned}
\omega^2 &= \frac{\boldsymbol{k}}{2}, \\
\omega^2 - \boldsymbol{k}^2 &= -\frac{\boldsymbol{k}^2}{2}
\end{aligned} \tag{4.30}$$

3. For the third — c_3 term we have:

$$\begin{aligned}
A_3 &= \Big(\frac{i\langle \tilde{J}^0_N(-k)\tilde{J}^0_N(k)\rangle}{\boldsymbol{k}^2}\Big)_3 = \\
&= -\frac{\omega^2\mu c_1}{\boldsymbol{k}^2(\omega^2 - c_s^2\boldsymbol{k}^2)}\Big(\frac{8c_3}{\mu^2 c_1}(\omega^2 - c_s^2\boldsymbol{k}^2) - \frac{4c_3\omega^2}{\mu^2 c_1}\frac{\omega^2 - \boldsymbol{k}^2}{\omega^2 - c_s^2\boldsymbol{k}^2}\Big) = \\
&= -\frac{\omega^2\mu c_1}{\boldsymbol{k}^2(\omega^2 - c_s^2\boldsymbol{k}^2)}\Big(\frac{1}{\omega^2 - c_s^2\boldsymbol{k}^2}\Big(-\frac{4c_3}{\mu^2 c_1}\Big)\cdot\frac{\boldsymbol{k}^2}{2}\cdot\Big(\frac{-\boldsymbol{k}^2}{2}\Big)\Big) = \\
&= -\frac{c_3\omega^2\boldsymbol{k}^4}{\boldsymbol{k}^2\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2} = \\
&= -\frac{c_3}{\mu}\frac{\omega^2\boldsymbol{k}^2}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2}.
\end{aligned} \tag{4.31}$$

So, to sum up, our A_N (contribution to A from only Noether currents, without contact terms) will be $A_N = A_1 + A_2 + A_3$:

$$A_N = -\frac{\mu c_1}{\omega^2 - c_s^2\boldsymbol{k}^2} + \frac{c_2}{\mu}\frac{\omega^2 - \boldsymbol{k}^2\boldsymbol{k}^2}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{c_3}{\mu}\frac{\omega^2\boldsymbol{k}^2}{(\omega^2 - c_s^2\boldsymbol{k}^2))} + \frac{b}{\mu} + \frac{d}{\mu} \tag{4.32}$$

Now let us find the contact terms for A and call them A_c. To do so, we will use the following expression written for correlation function[2]:

$$\frac{1}{Z}\int D\phi e^{i\int_R d^3xL(\phi_0,A_\mu^{(0)})}\frac{\delta^2 L(\phi_0,A_\mu^{(0)})}{\delta A_\mu^{(0)}\delta A_\nu^{(0)}}\Big|_{A_\mu^{(0)}=0}. \tag{4.33}$$

Let us look at the Fourier expansion of A^0.

$$A^0=(2\pi)^{-3}\int d^3ke^{ik\cdot x}\tilde{A}^0=\boldsymbol{k}A^0 \tag{4.34}$$

Now, if we take the fourth line of the Lagrangian (4.4) and try to take the 00 component, we will get:

$$\frac{b+d}{2\mu^2}(-\boldsymbol{k}^2A^0-2i\omega ik_iA_i) \tag{4.35}$$

Now if we differentiate it twice wrt A^0 we get:

$$\langle J_c^\mu(-k)J_c^\nu(k)\rangle=-i2\boldsymbol{k}^2 \tag{4.36}$$

$\Rightarrow$

$$i\langle J_c^\mu(-k)J_c^\nu(k)\rangle=i(-i2\boldsymbol{k}^2)=2\boldsymbol{k}^2\frac{b+d}{2\mu}=A_c\boldsymbol{k}^2 \tag{4.37}$$

$\Rightarrow$

$$A_c=\frac{b+d}{\mu} \tag{4.38}$$

So, for full A we have $A=A_N+A_c$:

$$A=-\frac{\mu c_1}{\omega^2-c_s^2\boldsymbol{k}^2}+\frac{c_2}{\mu}\frac{\omega^2-\boldsymbol{k}^2\boldsymbol{k}^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}-\frac{c_3}{\mu}\frac{\omega^2\boldsymbol{k}^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}+\frac{b}{\mu}+\frac{d}{\mu}. \tag{4.39}$$

Now let us derive B. We have the following Fourier expansion:

$$J^i=ik_i\frac{\mu c_1}{2}\tilde{\pi}(k)+i\omega k_i\frac{2c_3}{\mu}\tilde{\pi}(k) \tag{4.40}$$

Let us consider the ii component. Then:

$$\begin{aligned}i\langle \boldsymbol{J}(-k)\boldsymbol{J}(k)\rangle &= i\frac{\boldsymbol{k}^2\mu^2c_1^2}{4}\left(1+\frac{4c_3\omega^2}{\mu^2c_1}\right)^2\langle\tilde{\pi}(-k)\tilde{\pi}(k)\rangle=\\ &=i^2\frac{\boldsymbol{k}^2\mu^2c_1}{4(\omega^2-c_s^2\boldsymbol{k}^2)}\left(1+\frac{8c_3\omega^2}{\mu^2c_1}\right)\left(1-\frac{4c_2\omega^2(\omega^2-\boldsymbol{k}^2)}{\mu^2c_1(\omega^2-c_s^2\boldsymbol{k}^2)}-\frac{4c_3\omega^2(\omega^2-\boldsymbol{k}^2)}{\mu^2c_1(\omega^2-c_s^2\boldsymbol{k}^2)}\right)=\\ &=-\frac{\boldsymbol{k}^2\mu^2c_1}{4(\omega^2-c_s^2\boldsymbol{k}^2)}\left(1-\frac{4c_2\omega^2(\omega^2-\boldsymbol{k}^2)}{\mu^2c_1(\omega^2-c_s^2\boldsymbol{k}^2)}-\frac{4c_3\omega^2(\omega^2-\boldsymbol{k}^2)}{\mu^2c_1(\omega^2-c_s^2\boldsymbol{k}^2)}+\frac{8c_3\omega^2}{\mu^2c_1}\right)\end{aligned} \tag{4.41}$$

Where we used expansion for small $1/\mu^2$. On the other side, from RHS of (4.22) we have:

$$\begin{aligned} i\langle \boldsymbol{J}(-k)\boldsymbol{J}(k)\rangle =& A(\boldsymbol{k}^2 - 3(-\omega^2 + \boldsymbol{k}^2)) + B(\boldsymbol{k}^2 - 3\boldsymbol{k}^2) = \\ &= 3A\omega^2 - 2A\boldsymbol{k}^2 - 2B\boldsymbol{k}^2. \end{aligned} \tag{4.42}$$

So, we have:

$$\begin{aligned} B =& \frac{3}{2}A\frac{\omega^2}{\boldsymbol{k}^2} - A - \frac{i}{2\boldsymbol{k}^2}\langle \boldsymbol{J}(-k)\boldsymbol{J}(k)\rangle = \frac{3}{4}A - A - \frac{i}{2\boldsymbol{k}^2}\langle \boldsymbol{J}(-k)\boldsymbol{J}(k)\rangle = \\ &= -\frac{A}{4} - \frac{i}{2\boldsymbol{k}^2}\langle \boldsymbol{J}(-k)\boldsymbol{J}(k)\rangle. \end{aligned} \tag{4.43}$$

Let us calculate term by term again:

1. For the first — c_1 term we have:

$$\begin{aligned} B_1 =& -\frac{1}{4}\Big(-\frac{\mu c_1}{2(\omega^2 - c_s^2\boldsymbol{k}^2)}\Big) + \frac{\mu c_1}{8(\omega^2 - c_s^2\boldsymbol{k}^2)} = \\ &= \frac{\mu c_1}{8(\omega^2 - c_s^2\boldsymbol{k}^2)} + \frac{\mu c_1}{8(\omega^2 - c_s^2\boldsymbol{k}^2)} = \frac{\mu c_1}{4(\omega^2 - c_s^2\boldsymbol{k}^2)} \end{aligned} \tag{4.44}$$

2. For the second — c_2 term we have:

$$\begin{aligned} B_2 =& -\frac{c_2(\omega^2 - \boldsymbol{k}^2)\boldsymbol{k}^2}{4\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{c_2(\omega^2 - \boldsymbol{k}^2)\omega^2}{2\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2} = \\ &= -\frac{c_2(\omega^2 - \boldsymbol{k}^2)\boldsymbol{k}^2}{4\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{\boldsymbol{k}^2}{2}\frac{c_2(\omega^2 - \boldsymbol{k}^2)\boldsymbol{k}^2}{2\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2} = \\ &= -\frac{\boldsymbol{k}^2}{2}\frac{c_2(\omega^2 - \boldsymbol{k}^2)}{\mu(\omega^2 - c_s^2)^2} = \frac{c_2(\omega^2 - \boldsymbol{k}^2)^2}{\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2} \end{aligned} \tag{4.45}$$

   Here we used that

$$-\frac{\boldsymbol{k}^2}{2} = \omega^2 - \boldsymbol{k}^2. \tag{4.46}$$

3. For the third — c_3 term we have:

$$\begin{aligned} B_3 =& \frac{c_3}{4\mu(\omega^2 - c_s^2\boldsymbol{k}^2)^2}\Big(\omega^2 + 2\omega^2 - \boldsymbol{k}^2\Big) - \frac{c_3}{2\mu}\frac{\omega^2(\omega^2 - \boldsymbol{k}^2)}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} = \\ &= \frac{c_3\omega^2}{2\mu}\frac{1/2\boldsymbol{k}^2}{(\omega^2 - c_s^2\boldsymbol{k}^2)} - \frac{c_3}{2\mu}\frac{\omega^2(\omega^2 - \boldsymbol{k}^2)}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} = \\ &= -\frac{c_3}{\mu}\omega^2\frac{\omega^2(\omega^2 - \boldsymbol{k}^2)}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} \end{aligned} \tag{4.47}$$

So, for B_N we have $B_N = B_1 + B_2 + B_3$:

$$B_N = \frac{\mu c_1}{4(\omega^2 - c_s^2 \boldsymbol{k}^2)} + \frac{c_2}{\mu}\frac{(\omega^2 - \boldsymbol{k}^2)^2}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{c_3}{\mu}\omega^2 \frac{(\omega^2 - \boldsymbol{k}^2)}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2}. \tag{4.48}$$

Now let us find contact terms for B.

$$L_c = \frac{(b+d)}{2\mu}\Big[(\partial_i A^0)^2 + (\partial_0 A_i)^2 + 2\dot{A}^0(\partial_i A_i)\Big] - \frac{b}{4\mu}(\partial_i A_j - \partial_j A_i)^2 \tag{4.49}$$

So with the procedure (4.33), we will have the following derivative (differenciating twice wrt A_i):

$$-3\frac{b+d}{\mu} + \frac{4b}{\mu} \tag{4.50}$$

Here I used Fourier expansion for A and the fact that in Fourier space one can derive that$(\partial_i A_j - \partial_j A_i) = 2\partial_i A_j$. For the correlator we will have:

$$\langle \boldsymbol{J}_c(-k)\boldsymbol{J}_c\rangle = i\Big(-3\frac{b+d}{\mu} + \frac{4b}{\mu}\Big). \tag{4.51}$$

Then from equation (4.43) we have relationship:

$$\begin{aligned} B_c = &- \frac{A}{4} - \frac{i\cdot i}{4\omega^2}(-3\frac{b+d}{\mu} + \frac{4b}{\mu}) = \\ &= -\frac{A}{4} + \frac{1}{4\omega^2}(-3\frac{b+d}{\mu}\omega^2 + \frac{4b}{\mu}) = -\frac{d}{4}. \end{aligned} \tag{4.52}$$

Then for full B we will have $B_N + B_c$:

$$\frac{\mu c_1}{4(\omega^2 - c_s^2 \boldsymbol{k}^2)} + \frac{c_2}{\mu}\frac{(\omega^2 - \boldsymbol{k}^2)^2}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{c_3}{\mu}\omega^2 \frac{(\omega^2 - \boldsymbol{k}^2)}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{d}{\mu} \tag{4.53}$$

So, in our EFT we get:

$$A = -\frac{\mu c_1}{\omega^2 - c_s^2\boldsymbol{k}^2} + \frac{c_2}{\mu}\frac{\omega^2 - \boldsymbol{k}^2\boldsymbol{k^2}}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{c_3}{\mu}\frac{\omega^2\boldsymbol{k^2}}{(\omega^2 - c_s^2\boldsymbol{k}^2))} + \frac{b}{\mu} + \frac{d}{\mu}, \tag{4.54}$$

$$B = \frac{\mu c_1}{4(\omega^2 - c_s^2 \boldsymbol{k}^2)} + \frac{c_2}{\mu}\frac{(\omega^2 - \boldsymbol{k}^2)^2}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{c_3}{\mu}\omega^2 \frac{(\omega^2 - \boldsymbol{k}^2)}{(\omega^2 - c_s^2\boldsymbol{k}^2)^2} - \frac{d}{\mu}. \tag{4.55}$$

4.2 Positivity bounds from $\langle JJ \rangle$

We want to derive the positivity bounds on coefficients c_2, c_3, d. The logic is similar to Chapter 2. We find the conditions from the two-point correlation function using Cauchy integral formula. We take aim at the tree-level approximation in the EFT. Let us consider the function:

$$\tilde{f}(\omega) = \tilde{G}^{\mu\nu}(k)V_\mu(k)V_\nu(k)|_{k=(\omega,\boldsymbol{k_0}+\omega\boldsymbol{\xi})} \tag{4.56}$$

where

$$\tilde{G}^{\mu\nu}(k) = i\,\langle J^\mu(-k)J^\nu(k)\rangle \tag{4.57}$$

and $V_\mu(k = (\omega, \boldsymbol{k_0} + \boldsymbol{\omega\xi})) \equiv V(\omega)$, whose components are the arbitrary polynomials in ω (we use this vector just for contraction with our function). Consider expressions:

$$\begin{aligned} \hat{K} &= \frac{(1,\boldsymbol{\xi})}{\sqrt{1-\xi^2}}, \\ \hat{E} &= \frac{(\xi,\hat{\boldsymbol{\xi}})}{\sqrt{1-\xi^2}} \\ \hat{F} &= (0,\hat{\mathbf{f}}) \end{aligned} \tag{4.58}$$

$$k = (\omega, \boldsymbol{k_0} + \omega\boldsymbol{\xi}). \tag{4.59}$$

Hats denote unit vectors, so we have:

$$\begin{aligned} &\hat{\boldsymbol{\xi}}\cdot\hat{\boldsymbol{\xi}} = 1 = \hat{\boldsymbol{f}}\cdot\hat{\boldsymbol{f}}, \quad \hat{\boldsymbol{\xi}}\cdot\hat{\boldsymbol{f}} = 0, \\ &\hat{K}\cdot\hat{K} = -1, \quad \hat{E}\cdot\hat{E} = 1 = \hat{F}\cdot\hat{F}, \\ &\hat{K}\cdot\hat{E} = \hat{K}\cdot\hat{F} = \hat{E}\cdot\hat{F} = 0. \end{aligned} \tag{4.60}$$

At each ω we can expand $V(\omega)$ in this basis like:

$$V(\omega) = \alpha(\omega)\hat{K} + \beta(\omega)\hat{E} + \gamma(\omega)\hat{F}; \tag{4.61}$$

We consider $\boldsymbol{k_0} = 0$ and notice that $\hat{K}$ mimics k, conclusively, when we multiply it by correlator its contribution will be zero, so we will not take into account the contribution from α. So,

$$k = (\omega, \omega\boldsymbol{\xi}). \tag{4.62}$$

$$V(\omega) = \beta(\omega)\hat{E} + \gamma(\omega)\hat{F} \tag{4.63}$$

For two point correlation function (4.22) we have:

$$i\langle J^\mu(-k)J^\nu(k)\rangle = A(k^\mu k^\nu - \eta^{\mu\nu}k^2) + B(k^ik^j - \delta^{ij}\mathbf{k}^2); \tag{4.64}$$

Together , (4.22), (4.56), (4.57) give:

$$\tilde{f}(\omega) = \big(A(k^\mu k^\nu - \eta^{\mu\nu}k^2) + B(k^ik^j - \delta^{ij}\mathbf{k}^2)\big)V_\mu V_\nu \tag{4.65}$$

Let us do it part by part. Start from A part:

$$A(k^\mu k^\nu - \eta^{\mu\nu}k^2)V_\mu V_\nu = A\big((k^\mu V_\mu)^2 - k^2V^2\big) \tag{4.66}$$

For the first term we have:

$$\begin{aligned}(k^\mu V_\mu)^2 &= (-k_0V_0 + \boldsymbol{k}\cdot\boldsymbol{V})^2 = \\ &= \Big(-\frac{\omega\beta\xi}{\sqrt{1-\xi^2}} + \omega\boldsymbol{\xi}\big(\frac{\hat{\boldsymbol{\xi}}\beta}{\sqrt{1-x^2}} + \gamma\hat{\boldsymbol{f}}\big)\Big) = \\ &= \Big(-\frac{\omega\beta\xi}{\sqrt{1-\xi^2}} + \frac{\omega\beta\xi}{\sqrt{1-\xi^2}}\Big)^2 = 0.\end{aligned} \tag{4.67}$$

For the second term we have:

$$\begin{aligned}-k^2V^2 = &- (-\omega^2 + \omega^2\xi^2)(-V_0^2 + \boldsymbol{V}^2) = \\ &= \omega^2(1-\xi^2)\Big(\frac{-\beta\xi^2}{1-\xi^2} + \Big(\frac{\beta\hat{\boldsymbol{\xi}}}{\sqrt{1-\xi^2}} + \gamma\hat{\boldsymbol{f}}\Big)^2\Big) = \\ &= \omega^2(1-\xi^2)\Big(\frac{-\beta\xi^2}{1-\xi^2} + \frac{\beta^2}{1-\xi^2} + \gamma^2\Big) = \\ &= \omega^2(1-\xi^2)\Big(\frac{\beta^2}{1-\xi^2}(1-\xi^2) + \gamma^2\Big) \\ = &= \omega^2(1-\xi^2)(\beta^2+\gamma^2).\end{aligned} \tag{4.68}$$

So, for our A part we have $A\omega^2(1-\xi^2)(\beta^2+\gamma^2)$.

For the B part we should consider that:

$$\begin{aligned}k^\mu &= \delta^\mu_0 K^0 + \delta^\mu_i k^i; \\ \delta^{ij} &= (\delta^\mu_i \delta^\nu_j); \\ \delta^\mu_i\delta^\nu_j V_\mu V_\nu &= (\delta^\mu_i V_\mu)^2\end{aligned} \tag{4.69}$$

So we have:

$$\begin{aligned}(k^iV^i)^2 - \boldsymbol{k}^2\boldsymbol{V}^2 =& \Big(\omega\boldsymbol{\xi}\Big(\frac{\beta\hat{\boldsymbol{\xi}}}{\sqrt{1-\xi^2}} + \gamma\hat{\boldsymbol{f}}\Big)\Big)^2 - \omega^2\xi^2\Big(\frac{\beta^2}{1-\xi^2} + \gamma^2\Big)\\ =&\frac{\omega^2\xi^2\beta^2}{1-\xi^2} - \frac{\beta^2\omega^2\xi^2}{1-\xi^2} - \gamma^2\omega^2\xi^2 = \\ =& -\gamma^2\omega^2\xi^2\end{aligned} \tag{4.70}$$

So, for the B part we have: $-B\gamma^2\omega^2\xi^2, \Rightarrow$:

$$\tilde{f}(\omega) = A\omega^2(1-\xi^2)(\beta^2+\gamma^2) - B\gamma^2\omega^2\xi^2 \tag{4.71}$$

Now we have expressions for the coefficients A and B. Keeping in mind that : $k^2 = -\omega^2(1-\xi^2)$,
$k = (\omega, \omega\boldsymbol{\xi}), \Rightarrow$

$$\begin{aligned}\tilde{f}(\omega) =& A\omega^2(1-\xi^2)(\beta^2+\gamma^2) - B\gamma^2\omega^2\xi^2\\ &- \frac{\mu c_1}{2(\omega^2 - c_s^2\vec{k}^2)}\omega^2(1-\xi^2)(\beta^2+\gamma^2) + \frac{\mu c_1}{4(\omega^2 - c_s^2\vec{k}^2)}\xi^2\omega^2\gamma^2 +\\ &+ \frac{c_2}{\mu}\frac{\omega^2 - \vec{k}^2}{(\omega^2 - c_s^2\vec{k}^2)^2}\vec{k}^2(1-\xi^2)(\beta^2+\gamma^2) + \frac{c_2}{mu}\frac{(\omega^2-\vec{k}^2)^2}{\omega^2 - c_s^2\vec{k}^2}\xi^2\omega^2\beta^2 -\\ &- \frac{c_3}{\mu}\frac{\omega^2\vec{k}^2}{\omega^2 - c_s^2\vec{k}^2}\omega^2(1-\xi^2)(\beta^2+\gamma^2) - \frac{c_3}{\mu}\frac{\omega^2(\omega^2-\vec{k}^2)}{\omega^2 - c_s^2\vec{k}^2}\xi^2\gamma^2\omega^2 +\\ &+ \frac{b+d}{\mu}\omega^2(1-\xi^2)(\beta^2+\gamma^2) - \frac{d}{\mu}\xi^2\omega^2\gamma^2 \equiv (1)+(2)+(3)+(4)\end{aligned} \tag{4.72}$$

Where $(1), (2)$ etc. are lines' numbers. Let us calculate line by line:
We have:

$$\begin{aligned}(1) \equiv& -\frac{\mu c_1\omega^2}{2\omega^2(1-\frac{\xi^2}{2})}(1-\xi^2)\beta^2 - \frac{\mu c_1}{2(1-\frac{\xi^2}{2})}(1-\xi^2)\gamma^2 + \frac{\mu c_1}{4(1-\frac{\xi^2}{2})}\xi^2\gamma^2 =\\ =& -\frac{\mu c_1}{2(1-\frac{\xi^2}{2})}(1-\xi^2)\beta - \frac{\mu c_1}{2(1-\frac{\xi^2}{2})}\gamma^2(1-\xi^2+\frac{\xi^2}{2}) =\\ =& -\frac{\mu c_1}{2(1-\xi^2/2)}\Big((1-\xi^2)\beta^2 + (1-\xi^2/2)\gamma^2\Big).\end{aligned} \tag{4.73}$$

$$\begin{aligned}(2) \equiv& \frac{c_2}{\mu}\frac{\omega^2(1-\xi^2)}{\omega^2(1-\frac{\xi^2}{2})^2}\omega^2\xi^2(1-\xi^2)(\beta^2+\gamma^2) + \frac{c_2}{\mu}\frac{(1-\xi^2)\xi^2\omega^2\gamma^2}{(1-\frac{\xi^2}{2})^2} =\\ =& \frac{c_2}{\mu}\frac{(1-\xi^2)^2}{(1-\xi^2/2)^2}\omega^2\xi^2\beta^2.\end{aligned} \tag{4.74}$$

$$
\begin{aligned}
(3) \equiv & \Big(\frac{c_3}{\mu}\frac{\omega^2\omega^2\xi^2(1-\xi^2)}{(1-\xi^2/2)^2}(\beta^2+\gamma^2)+\frac{c_3}{\mu}\frac{(1-\xi^2)}{(1-\xi^2/2)^2}\xi^2\gamma^2\omega^2\Big) = \\
& = -\frac{c_3}{\mu}\frac{\omega^2\xi^2(1-\xi^2)}{(1-\xi^2/2)^2}\beta^2.
\end{aligned}
\tag{4.75}
$$

$$
(4) \equiv \frac{b\omega^2}{\mu}(1-\xi^2)(1-\xi^2)(\beta^2+\gamma^2)+\frac{d\omega^2}{\mu}\Big((1-\xi^2)\beta^2+\gamma^2\Big). \tag{4.76}
$$

So, we will have:

$$
\begin{aligned}
\tilde{f}(\omega) = & -\frac{\mu c_1}{2(1-\xi^2/2)}\Big((1-\xi^2)\beta^2+(1-\xi^2/2)\gamma^2\Big)+ \\
& +\frac{c_2}{\mu}\frac{(1-\xi^2)^2}{(1-\xi^2/2)^2}\omega^2\xi^2\beta^2 - -\frac{c_3}{\mu}\frac{\omega^2\xi^2(1-\xi^2)}{(1-\xi^2/2)^2}\beta^2+ \\
& \frac{b\omega^2}{\mu}(1-\xi^2)(1-\xi^2)(\beta^2+\gamma^2)+\frac{d\omega^2}{\mu}\Big((1-\xi^2)\beta^2+\gamma^2\Big).
\end{aligned}
\tag{4.77}
$$

Then using the known trick we have:

$$
\oint d\omega \frac{\tilde{f}(\omega)}{\omega^3} = i\pi \tilde{f}''(0) \tag{4.78}
$$

Then we know that contribution from cuts should $i \times positive \Rightarrow \tilde{f}(\omega)'' \geq 0$, that gives us the following condition:

$$
\begin{aligned}
& 2\frac{c_2}{\mu}\frac{\xi^2(1-\xi^2)^2}{(1-\xi^2/2)^2}\beta^2 - 2\frac{c_3}{\mu}\xi^2\frac{(1-\xi^2)}{(1-\xi^2/2)^2}\beta^2+2\frac{b}{\mu}(1-\xi^2)(\beta^2+\gamma^2)+ \\
& +2\frac{d}{\mu}\Big((1-\xi^2)\beta^2+\gamma^2\Big) \geq 0
\end{aligned}
\tag{4.79}
$$

So,

$$
c_2\frac{\xi^2(1-\xi^2)}{(1-\xi^2/2)^2}\beta^2 - c_3\xi^2\frac{\xi^2}{(1-\xi^2/2)^2}\beta^2+b(\beta^2+\gamma^2)+d\Big(\beta^2+\frac{\gamma^2}{1-\xi^2}\Big) \geq 0. \tag{4.80}
$$

For all $\xi \in [0,1), \beta, \gamma$.

We can find the bounds by taking the following limits:

- $\xi \to 1, \quad \gamma \neq 0, \quad \Rightarrow d \geq 0$
- $\xi \to 0 \quad \Rightarrow b + d \geq 0,$

- $\propto \gamma^2 \to RHS, \quad \Rightarrow$

$$c_2 \frac{\xi^2(1-\xi^2)}{(1-\xi^2/2)^2}\beta^2 - c_3\xi^2 \frac{\xi^2}{(1-\xi^2/2)^2}\beta^2 + b\beta^2 + d\beta^2 \geq -b\gamma^2 - d\frac{\gamma^2}{1-\xi^2} \tag{4.81}$$

So, the most stringent bounds we get when $\gamma = 0$, then,

$$c_2 \frac{\xi^2(1-\xi^2)}{(1-\xi^2/2)^2} - c_3\xi^2 \frac{\xi^2}{(1-\xi^2/2)^2} + b + d \geq 0 \tag{4.82}$$

If we devide everything on $\xi^2(b+d)$ and multiply by $(1-\xi^2/2)^2$, we will get the following constraint:

$$\frac{c_2}{b+d}(1-\xi^2) - \frac{c_3}{b+d} \geq -\frac{(1-\xi^2/2)^2}{\xi^2}. \tag{4.83}$$

These constraints hold for all $\xi \in [0,1)$, and they are plotted in Fig.4.1. The boundary curve for $\tilde{c}_2 \leq -3/4, \tilde{c}_3 \leq 1/4$ is given by

$$\tilde{c}_3 = \tilde{c}_2 - 1 + \sqrt{1-4\tilde{c}_2}, \tag{4.84}$$

where $\tilde{c}_{2,3} \equiv c_{2,3}/(b+d)$, when for $\tilde{c}_2 \geq -3/4$ the boundary curve is just the horizontal line $\tilde{c}_3 = 1/4$ [2].

If $\boldsymbol{k_0} \neq 0$:

$$\begin{aligned} \hat{K} &= k = (\omega, \boldsymbol{k}), \\ \hat{E} &= (\boldsymbol{k}^2, \omega\boldsymbol{k}), \\ \hat{F} &= (0, -\omega k_2, \omega k_1). \end{aligned} \tag{4.85}$$

Then: For the A part we have:

$$\begin{aligned} &A\Big((k^\mu V_\mu)^2 - k^2V^2\Big) = \\ &\quad = A\Big((-k_0V_0 + \boldsymbol{k}\boldsymbol{V})^2 + (\omega^2 - \boldsymbol{k}^2)V^2\Big) = \\ &\quad = A\Big(-\omega\beta\boldsymbol{k}^2 + k_1V_1 + k_2V_2)^2 + (\omega^2 - \boldsymbol{k}^2)V^2\Big) = \\ &\quad = A\Big((-\omega\beta\boldsymbol{k}^2 + \beta k_1^2\omega - \gamma\omega k_1k_2 + \omega\beta k_2^2 + \gamma\omega k_1k_2)^2 + (\omega^2 - \boldsymbol{k}^2)V^2\Big) = \\ &\quad = A(0 + (\omega^2 - \boldsymbol{k}^2)V^2) = \\ &\quad = A(\omega^2 - \boldsymbol{k}^2)(-V_0^2 + V_1V_1 + V_2V_2) = \\ &\quad = A(\omega^2 - \boldsymbol{k}^2)\Big(-\beta^2\boldsymbol{k}^4 + \omega^2\boldsymbol{k}^2(\beta^2 + \gamma^2)\Big) = \\ &\quad = A(\omega^2 - \boldsymbol{k}^2)\boldsymbol{k}^2\Big((\omega^2 - \boldsymbol{k}^2)\beta^2 + \omega^2\gamma^2\Big), \end{aligned} \tag{4.86}$$

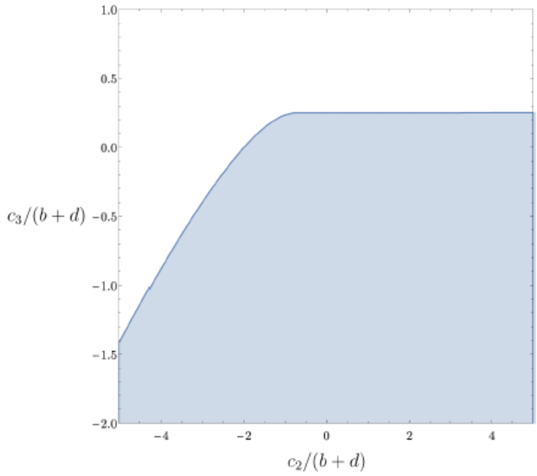

Figure 4.1: *The constraints* (4.83) *(the blue region is allowed). There are also the constraints* $d \geq 0$ *and* $b + d \geq 0$. $c_1 > 0$ *is required to have a healthy kinetic term for fluctuations.*

For the B part we have:

$$\begin{aligned} B\Big(k^iV^i)^2 - \boldsymbol{k}^2\boldsymbol{V}^2\Big) = \\ &= B\Big((k_1V_1 + k_2V_2)^2 - \boldsymbol{k}^2(\beta^2\omega^2\boldsymbol{k}^2 - (\beta^2+\gamma^2)\omega^2\boldsymbol{k}^2)\Big) = \\ &= -B\gamma^2\omega^2\boldsymbol{k}^4 \end{aligned} \tag{4.87}$$

So,

$$\tilde{f}(\omega) = A(\omega^2 - \boldsymbol{k}^2)\boldsymbol{k}^2\Big((\omega^2 - \boldsymbol{k}^2)\beta^2 + \omega^2\gamma^2\Big) - B\gamma^2\omega^2\boldsymbol{k}^4 \tag{4.88}$$

We will have 2 new poles:

$$\omega^2 = c_s^2\boldsymbol{k}^2(\omega) \tag{4.89}$$

We know that $\boldsymbol{k} = \boldsymbol{k_0} + \omega\boldsymbol{\xi}$, so,

$$\begin{aligned}
&\omega^2 = \frac{1}{2}k_0^2 + \frac{2}{2}k_0\omega\xi cos\theta + \frac{\omega^2\xi^2}{2} \\
&\omega^2(1-\xi^2/2) - \omega k_0\xi cos\theta - \frac{1}{2}k_0 = 0 \\
&D = k_0^2\xi^2cos^2\theta + 2(1-\xi^2/2)k_0^2 = k_0^2\sqrt{\xi^2cos^2\theta + 2 - \xi^2} = k_0^2\sqrt{2-\xi^2sin^2\theta} \\
&\omega_\pm = \frac{k_0\xi cos\theta \pm k_0\sqrt{2-\xi^2sin^2\theta}}{2(1-\xi^2/2)} = \\
&= \frac{k_0}{2(1-\xi^2/2)}\Big(\xi cos\theta \pm k_0\sqrt{2-\xi^2\sin^2\theta}\Big).
\end{aligned} \tag{4.90}$$

Here θ is an angle between $\boldsymbol{k_0}$ and $\boldsymbol{\xi}$. Now we have 2 more poles so we consider a contour including all three low-energy poles. The resulting integral is again positive. Summing the contributions from all the three poles together gives us exactly the same inequality Eq. (4.82)[2]. So we can say that non-zero $\boldsymbol{k_0}$ will not give any other different condition, which means our assumption $\boldsymbol{k_0} = 0$ works.

4.3 $\langle TT \rangle$ calculation

We can consider with similar logic the two-point correlation function of the stress-energy tensor. From the dimensional analysis, we guess that the CFT correlator involving a stress-energy tensor will contain two more powers of ω than the correlation function of the currents. So, to have a converging contour at infinity, we should construct the function (to which we apply the Cauchy integral formula) dividing the correlator by two more powers of ω, thus we should consider the EFT operators with two more derivatives[2]. In this case, it is convenient to couple χ field with a non-dynamical metric $g_{\mu\nu}$. Then the most general action non-linearly realizing the conformal symmetry can be written using a Weyl invariant conformal metric $\hat{g}_{\mu\nu} = g_{\mu\nu}|g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi|$. At next to next to linear order (NNLO) in derivatives we gain three new operators (compared to the Lagrangian (4.2)) and we have the following action:

$$S = \int d^3x\sqrt{-\hat{g}}\Big(\frac{c_1}{6} - c_2\hat{R} + c_3\hat{R}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + c_4\hat{R}^2 + c_5\hat{R}_{\mu\nu}\hat{R}^{\mu\nu} + c_6\hat{R}^0_\mu\hat{R}^{\mu 0}\Big) \tag{4.91}$$

where $\hat{R}^0_\mu \equiv \hat{R}^\lambda_\mu \partial_\lambda \chi$ (derivation that the first three terms of this action coincide with an appropriate action for Lagrangian (4.2) and therefore (4.2) is also derived from this logic can be seen in appendix B). The logic and the procedure are similar to what we provided for the current.

For the calculation of the stress-energy tensor, one should use linear perturbations around the flat space, $T^{\mu\nu} = (-g)^{1/2}\delta S/\delta g_{\mu\nu}|_{g=\eta}$. We want to investigate the following correlator:

$$\langle T^{\mu\nu}(-k)T^{\rho\sigma}(k)\rangle \tag{4.92}$$

Like $\langle JJ\rangle$, $\langle TT\rangle$ will also include the contact terms generated by $O(\delta g^2)$ terms in the action[2].

We will discuss the conserved traceless object $\langle T^{\mu\nu}T^{\rho\sigma}\rangle_{subl.}$ (we consider subleading orders i.e. NLO and NNLO) with the relevant symmetries $(\mu \leftrightarrow \nu, \quad \rho \leftrightarrow \sigma, \quad (\mu\nu) \leftrightarrow (\rho\sigma))$. One can express this correlator in $d > 2$ dimensions as the following linear combination:

$$i\langle T^{\mu\nu}(-k)T^{\rho\sigma}(k)\rangle_{subl.} = C(k)\Pi^{\mu\nu\rho\sigma}(k) + D(k)\tilde{\Pi}^{\mu\nu\rho\sigma}(k) \tag{4.93}$$

with

$$\Pi^{\mu\nu\rho\sigma} = \frac{1}{2}(\pi^{\mu\rho}\pi^{\nu\sigma} + \pi^{\mu\sigma}\pi^{\nu\rho}) - \frac{1}{d-1}\pi^{\mu\nu}\pi^{\rho\sigma}, \tag{4.94}$$

$$\tilde{\Pi}^{\mu\nu\rho\sigma} = \frac{1}{4}(\pi^{\mu\rho}\tilde{\pi}^{\nu\sigma} + \pi^{\mu\sigma}\tilde{\pi}^{\mu\rho} + \pi^{\nu\rho}\tilde{\pi}^{\mu\sigma}) + \frac{1}{d-2}\tilde{\pi}^{\mu\nu}\tilde{\pi}^{\rho\sigma}, \tag{4.95}$$

where

$$\pi^{\mu\nu} \equiv \eta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}, \tag{4.96}$$

$$\tilde{\pi}^{\mu\nu} = \delta^{mn} - \frac{k^m k^n}{\boldsymbol{k}^2}. \tag{4.97}$$

With the same logic as we calculated A and B coefficients can be calculated C and D as well, but we should consider the ii components first, that leaves expression for C. For ii component D is multiplied by 0 (as $\tilde{\pi}^{ii} = \delta^{ii} - \frac{k^i k^i}{\boldsymbol{k}^2} = 3 - 3 = 0$ for ii component), so from (4.91) we can calculate C and than use this information to calculate D.

Expressions for C and D are the following:

$$
\begin{aligned}
C = &-\frac{\mu}{2}\frac{\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}(c_2+c_3)+\frac{1}{\mu}\frac{\boldsymbol{k}^4(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_4+\\
&+\frac{1}{2\mu}\frac{(\omega^2-\boldsymbol{k}^2)^2(\omega^2(\omega^2-\boldsymbol{k}^2)+\boldsymbol{k}^4)}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_5+\frac{1}{4\mu}\frac{\boldsymbol{k}^2\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_6-\\
&-\frac{1}{2\mu}\frac{(c_2+c_3)^2}{c_1}\frac{\boldsymbol{k}^4\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^3},
\end{aligned}
\tag{4.98}
$$

$$
\begin{aligned}
D = &-\frac{\mu}{4}\frac{\boldsymbol{k}^4(\omega^2-\boldsymbol{k}^2)}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}(c_2+c_3)-\frac{1}{\mu}\frac{\boldsymbol{k}^4(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}\left(2c_4+\frac{3}{4}c_5\right)+\\
&+\frac{1}{8\mu}\frac{\boldsymbol{k}^6(\omega^2-\boldsymbol{k}^2)}{(\omega^2-c_s^2\boldsymbol{k}^2)^2}c_6+\frac{1}{\mu}\frac{(c_2+c_3)^2}{c_1}\frac{\boldsymbol{k}^2\omega^2(\omega^2-\boldsymbol{k}^2)^2}{(\omega^2-c_s^2\boldsymbol{k}^2)^3}.
\end{aligned}
\tag{4.99}
$$

4.4 Positivity bounds from $\langle TT \rangle$

Using the same logic as for $\langle JJ \rangle$, we contract (4.91) by multiplying it twice with a general symmetric two-tensor A:$\langle T^{\mu\nu}T^{\rho\sigma}\rangle A_{\mu\nu}A_{\rho\sigma}$. Let us take $\boldsymbol{k_0} = 0$ here as well and expand this 2-tensor A with the constant coefficients as follows:

$$
A_{\mu\nu} = \alpha\hat{K}_\mu\hat{K}_\nu+\beta\hat{E}_\mu\hat{E}_\nu+\gamma\hat{F}_\mu\hat{F}_\nu+\tilde{\alpha}(\hat{K}_\mu\hat{E}_\nu+\hat{K}_\nu\hat{E}_\mu)+\tilde{\beta}(\hat{K}_\mu\hat{F}_\nu+\hat{F}_\nu\hat{F}_\mu)+\tilde{\gamma}(\hat{E}_\mu\hat{F}_\nu+\hat{E}_\nu\hat{F}_\mu),
\tag{4.100}
$$

where $\hat{K}$, $\hat{E}$ and $\hat{F}$ are given in Eqns. (4.85). Through the calculation we will use the identities expressed below:

$$
\begin{aligned}
&\pi^{\mu\alpha}\hat{K}_\mu = \tilde{\pi}^{\mu\alpha}\hat{E}_\mu = 0,\\
&\pi^{\mu\alpha}\hat{E}_\mu = \hat{E}^\alpha,\\
&\pi^{\mu\alpha}\hat{F}_\mu = \hat{F}^\alpha,\\
&\tilde{\pi}^{\mu\alpha}\hat{F}_\mu = \hat{F}^\alpha.
\end{aligned}
\tag{4.101}
$$

Now we will start calculating $\langle T^{\mu\nu}T^{\rho\sigma}\rangle A_{\mu\nu}A_{\rho\sigma}$ step by step. Firstly, we will calculate the C part:

$$\begin{aligned}
C\Pi A_{\mu\nu} =& C\Big(\frac{1}{2}(\pi^{\mu\rho}\pi^{\nu\sigma} + \pi^{\mu\sigma}\pi^{\nu\rho}) - \frac{1}{d-1}\pi^{\mu\nu}\pi^{\rho\sigma}\Big)\cdot \\
&\cdot\Big(\alpha\hat{K}_\mu\hat{K}_\nu + \beta\hat{E}_\mu\hat{E}_\nu + \gamma\hat{F}_\mu\hat{F}_\nu + \tilde{\alpha}(\hat{K}_\mu\hat{E}_\nu + \hat{K}_\nu\hat{E}_\mu) + \\
&+ \tilde{\beta}(\hat{K}_\mu\hat{F}_\nu + \hat{F}_\nu\hat{F}_\mu) + \tilde{\gamma}(\hat{E}_\mu\hat{F}_\nu + \hat{E}_\nu\hat{F}_\mu)\Big) = \\
&= C\Big(\frac{1}{2}(\beta\hat{E}^\rho\hat{E}^\sigma + \gamma\hat{F}^\rho\hat{F}^\sigma + \tilde{\gamma}(\hat{E}^\rho\hat{F}^\sigma + \hat{E}^\sigma\hat{F}^\rho)) \\
&+ \frac{1}{2}(\beta\hat{E}^\sigma\hat{E}^\rho + \gamma\hat{F}^\sigma\hat{F}^\rho + \tilde{\gamma}(\hat{E}^\sigma\hat{F}^\rho + \hat{E}^\rho\hat{F}^\sigma)) - \\
&- \frac{1}{2}(\beta\hat{E}^\nu\pi^{\rho\sigma\hat{E}_\nu} + \gamma\hat{F}^\nu\pi^{\rho\sigma}\hat{F}_\nu) + \tilde{\gamma}(\hat{E}^\nu\pi^{\rho\sigma}\hat{F}_\nu - \hat{E}^\mu\pi^{\rho\sigma}\hat{F}_\mu)\Big),
\end{aligned} \tag{4.102}$$

$$\begin{aligned}
C\Pi A_{\mu\nu}A^{\mu\nu} =& \Big(\frac{1}{2}(\beta\hat{E}^\rho\hat{E}^\sigma + \gamma\hat{F}^\rho\hat{F}^\sigma + \tilde{\gamma}(\hat{E}^\rho\hat{F}^\sigma + \hat{E}^\sigma\hat{F}^\rho) \\
&+ \beta\hat{E}^\sigma\hat{E}^\rho + \gamma\hat{F}^\sigma\hat{F}^\rho + \tilde{\gamma}(\hat{E}^\sigma\hat{F}^\rho + \hat{E}^\rho\hat{F}^\sigma) - \\
&- \beta\hat{E}^\nu\pi^{\rho\sigma\hat{E}_\nu} + \gamma\hat{F}^\nu\pi^{\rho\sigma}\hat{F}_\nu) + \tilde{\gamma}(\hat{E}^\nu\pi^{\rho\sigma}\hat{F}_\nu - \hat{E}^\mu\pi^{\rho\sigma}\hat{F}_\mu))\Big)\cdot \\
&\cdot\Big((\alpha\hat{K}_\rho\hat{K}_\sigma + \beta\hat{E}_\rho\hat{E}_\sigma + \gamma\hat{F}_\rho\hat{F}_\sigma + \tilde{\alpha}(\hat{K}_\rho\hat{E}_\sigma + \hat{K}_\sigma\hat{E}_\rho) + \\
&+ \tilde{\beta}(\hat{K}_\rho\hat{F}_\nu + \hat{F}_\nu\hat{F}_\mu) + \tilde{\gamma}(\hat{E}_\rho\hat{F}_\sigma + \hat{E}_\sigma\hat{F}_\rho)\Big) = \\
&= \frac{C}{2}(\beta\hat{E}^2 + \gamma\hat{F}^2 - 2\beta\gamma\tilde{\gamma}^2 + \tilde{\gamma}^2 + \tilde{\gamma}^2 + \tilde{\gamma}^2) = \\
&= \frac{C}{2}[(\beta-\gamma)^2 + 4\tilde{\gamma}^2]
\end{aligned} \tag{4.103}$$

we used here (4.60). Now we calculate the D part:

$$\begin{aligned}
D\Pi A_{\mu\nu} =& D\Big(\frac{1}{4}(\pi^{\mu\rho}\tilde{\pi}^{\nu\sigma} + \pi^{\mu\sigma}\tilde{\pi}^{\nu\rho} + \pi^{\nu\sigma}\tilde{\pi}^{\mu\rho} + \pi^{\nu\rho}\tilde{\pi}^{\mu\sigma}) - \tilde{\pi}^{\mu\nu}\tilde{\pi}^{\rho\sigma}\Big)\cdot \\
&\cdot \Big((\alpha\hat{K}_\mu\hat{K}_\nu + \beta\hat{E}_\mu\hat{E}_\nu + \gamma\hat{F}_\mu\hat{F}_\nu + \tilde{\alpha}(\hat{K}_\mu\hat{E}_\nu + \hat{K}_\nu\hat{E}_\mu) + \\
&+ \tilde{\beta}(\hat{K}_\mu\hat{F}_\nu + \hat{F}_\nu\hat{F}_\mu) + \tilde{\gamma}(\hat{E}_\mu\hat{F}_\nu + \hat{E}_\nu\hat{F}_\mu)\Big) = \\
&= D\Big(\frac{1}{4}(\gamma\hat{F}^\rho\hat{F}\sigma + \tilde{\gamma}\hat{E}^\rho\hat{E}^\sigma + \gamma\hat{F}^\rho\hat{F}\sigma + \tilde{\gamma}\hat{E}^\rho\hat{E}^\sigma + \\
&+ \gamma\hat{F}^\rho\hat{F}_\sigma + \tilde{\gamma}\hat{E}^\rho\hat{E}^\sigma + \gamma\hat{F}^\rho\hat{F}\sigma + \tilde{\gamma}\hat{E}^\rho\hat{E}^\sigma) - \gamma\hat{F}^\nu\hat{F}_\nu\tilde{\pi}^{\rho\sigma}\Big)\cdot \\
&\cdot \Big(\alpha\hat{K}_\rho\hat{K}_\sigma + \beta\hat{E}_\rho\hat{E}_\sigma + \gamma\hat{F}_\rho\hat{F}_\sigma + \tilde{\alpha}(\hat{K}_\rho\hat{E}_\sigma + \hat{K}_\sigma\hat{E}_\rho) + \\
&+ \tilde{\beta}(\hat{K}_\rho\hat{F}_\nu + \hat{F}_\nu\hat{F}_\mu) + \tilde{\gamma}(\hat{E}_\rho\hat{F}_\sigma + \hat{E}_\sigma\hat{F}_\rho)\Big) = \\
&= D\Big(\frac{1}{4}(4\gamma^2 + 4\tilde{\gamma}^2) - \gamma^2\Big) = D\tilde{\gamma}^2
\end{aligned} \tag{4.104}$$

we used (4.60) and (4.100) and we are in $d = 3 \Rightarrow (d-2) = 1$. (4.102) and (4.103) give:

$$i\langle T^{\mu\nu}(-k)T^{\rho\sigma}(k)\rangle_{subl.} = \frac{C}{2}[(\beta-\gamma)^2 + 4\tilde{\gamma}^2] + D\tilde{\gamma}^2. \tag{4.105}$$

Now let find the facilitated expression for C using that $\boldsymbol{k} = \omega\boldsymbol{\xi}$.

$$\begin{aligned}
C =& -\frac{\mu}{2}\omega^2\frac{(\omega^2 - \omega^2\xi^2)^2}{(\omega^2(1-1/2\xi^2))^2}(c_2+c_3) + \\
&+ \frac{\omega^4\xi^4(\omega^2-\omega^2\xi^2)^2}{\mu(\omega^2 - 1/2\omega^2\xi^2)^2}c_4 + \frac{1}{2\mu}\frac{(\omega^2-\omega^2\xi^2)^2(\omega^2(\omega^2-\omega^2\xi^2)+\omega^4\xi^4)}{(\omega^2(1-1/2\xi^2))^2}c_5 + \\
&+ \frac{1}{4\mu}\frac{\omega^4\xi^4(\omega^2-\omega^2\xi^2)^2}{(\omega^2(1-1/2\xi^2))^2}c_6 - \frac{1}{2\mu}\frac{(c_1+c_2)^2}{c_1}\frac{\omega^6\xi^4(\omega^2-\omega^2\xi^2)^2}{(\omega^2(1-1/2\xi^2))^3} = \\
&= -\frac{\mu}{2}\frac{\omega^2(1-\xi^2)^2}{(1-1/2\xi^2)^2}(c_2+c_3) + \omega^4\frac{\xi^4(1-\xi^2)^2}{(1-1/2\xi^2)^2}c_4 + \\
&+ \omega^4\frac{(1-\xi^2)^2(1-\xi^2+\xi^4)}{2\mu(1-1/2\xi^2)^2}c_5 + \omega^4\frac{\xi^2(1-\xi^2)^2}{4\mu(1-1/2\xi^2)^2}c_6 - \omega^4\frac{(c_2+c_3)^2}{c_1}\frac{\xi^4(1-\xi^2)^2}{2\mu(1-1/2\xi^2)^3}.
\end{aligned} \tag{4.106}$$

From Eq.(4.106) we can see that in order to neglect the contour at infinity and construct a physical quantity for which we will use the Cauchy theorem, we should divide (4.105) by at least ω^5. Let us divide (4.105) by

ω^5 and integrate around the origin (in a counterclockwise direction, using Cauchy integral formula (4.78)). This procedure will give us the positivity bounds. (We are going to express RHS of Eq.(4.93). Plugging C from Eq.(4.106) to Eq.(4.102) and plugging the result from Eq. (4.104) and D in Eq.(4.93) gives the following ($\delta \equiv \beta - \gamma$):

$$\frac{1}{2\mu}\frac{(1-\xi^2)}{(1-1/2\xi^2)^2}\xi^4 c_4(\delta^2+4\tilde{\gamma}^2)+\frac{1}{4\mu}\frac{(1-\xi^2)^2}{(1-1/2\xi^2)^2}c_5(1-\xi^2+\xi^4)(\delta^2+4\tilde{\gamma}^2)+$$
$$+\frac{1}{8\mu}\frac{\xi^2(1-\xi^2)^2}{(1-1/2\xi^2)^2}c_6(\delta^2+4\tilde{\gamma}^2)-\frac{1}{4\mu}\frac{(c_2+c_3)^2}{c_1(1-1/2\xi^2)}\frac{(1-\xi^2)^2}{(1-1/2\xi^2)^2}(\delta^2+4\tilde{\gamma}^2)-$$
$$-\frac{1}{\mu}\frac{\xi^4(1-\xi^2)^2}{(1-1/2\xi^2)^2}\Big(2c_4+\frac{3}{4}c_5\Big)\tilde{\gamma}^2+\frac{1}{8\mu}\frac{\xi^6(1-\xi^2)}{(1-1/2\xi^2)^2}c_6\tilde{\gamma}^2+$$
$$+\frac{1}{\mu}\frac{(c_2+c_3)^2}{c_1(1-\xi^2)}\frac{\xi^4(1-\xi^2)^2}{(1-1/2\xi^2)^2}\tilde{\gamma}^2\geq 0. \tag{4.107}$$

Facilitation of (4.107) by dividing everything by $\frac{(1-\xi^2)^2}{(1-1/2\xi^2)^2}$ leads to:

$$\frac{1}{2\mu}\xi^4 c_4\delta^2+\frac{2}{\mu}\xi^4 c_4\tilde{\gamma}^2+\frac{c_5}{4\mu}(1-\xi^2+\xi^4)\delta^2+\frac{c_5}{\mu}(1-\xi^2+\xi^2)\tilde{\gamma}^2+$$
$$+\frac{c_6}{8\mu}\xi^2\delta^2+\frac{c_6}{2\mu}\xi^2\tilde{\gamma}^2-\frac{1}{4\mu}\frac{\xi^4(c_2+c_3)^2}{c_1(1-1/2\xi^2)}\delta^2-\frac{1}{\mu}\frac{(c_2+c_3)^2}{c_1(1-1/2\xi^2)}\tilde{\gamma}^2+$$
$$+\frac{c_6}{8\mu}\frac{\xi^6}{(1-\xi^2)}\tilde{\gamma}^2\geq 0 \tag{4.108}$$

$$c_4\Big(\frac{1}{2}\xi^4\delta^4+\frac{2}{\mu}\xi^4\tilde{\gamma}^2-\frac{2}{\mu}\xi^2\tilde{\gamma}^2\Big)+c_5\Big(\frac{1}{4}(1-\xi^2+\xi^4)\delta^2+(1-\xi^2+\xi^4)\tilde{\gamma}^2-\frac{3}{4}\xi^4\tilde{\gamma}^2\Big)+$$
$$+c_6\Big(\frac{1}{8}\xi^2\delta^2+\frac{1}{2}\xi^2\tilde{\gamma}^2+\frac{1}{8}\frac{\xi^6}{1-\xi^2}\tilde{\gamma}^2\Big)-\frac{(c_2+c_3)^2\xi^4}{4c_1(1-1/2\xi^2)}\delta^2\geq 0 \tag{4.109}$$

$$\begin{aligned}&4c_4\xi^4\delta^2+2c_5\Big((1-\xi^2+\xi^4)\delta^2+(2-\xi^2)^2\tilde{\gamma}^2\Big)+\\&+\xi^2\Big(\frac{\xi^4}{1-\xi^2}\tilde{\gamma}^2+4\tilde{\gamma}^2+\delta^2\Big)c_6\geq\frac{4(c_2+c_3)^2}{c_1(2-\xi^2)}\delta^2\xi^4\end{aligned} \tag{4.110}$$

$$4c_4\xi^4+2\Big((1-\xi^2+\xi^4)\delta^2+(2-\xi^2)^2\tilde{\gamma}^2 c_5\Big)+\xi^2\Big(\frac{(2-\xi^2)^2}{1-\xi^2}\tilde{\gamma}^2+\delta^2\Big)c_6\geq 4\delta^2\xi^4\frac{(c_2+c_3)^2}{c_1(2-\xi^2)} \tag{4.111}$$

Let us try to get stringent bounds. For this reason, let us make several assumptions:

- $\delta = 0$ then

$$2(2-\xi^2)^2\tilde{\gamma}^2 c_5 + \xi^2 \frac{(2-\xi^2)^2}{1-\xi^2}\tilde{\gamma}^2 c_6 \geq 0 \tag{4.112}$$

$$2(1-\xi^2)c_5 + \xi^2 c_6 \geq 0 \tag{4.113}$$

  - ⋆ $\xi = 1 \rightarrow c_6 \geq 0$,
  - ⋆ $\xi = 0 \rightarrow c_5 \geq 0$,

- $\tilde{\gamma} = 0$

$$4\xi^2\delta^2 c_4 + 2(1-\xi^2+x^4)\delta^2 c_5 + \xi^2\delta^2 c_6 \geq \frac{4\xi^4\delta^2}{2-\xi^2}\frac{(c_2+c_3)^2}{c_1} \tag{4.114}$$

  - ⋆ $\xi = 1 \rightarrow$

$$4c_4 + 2c_5 + c_6 \geq 4\frac{(c_2+c_3)^2}{c_1}. \tag{4.115}$$

Chapter 5

Results

To summarize, for the conformal field theory Lagrangians (that we believe can be extended to more general theories for DE/Modified Gravity models) from the two-point correlation functions of conserved current $(5.1)-(5.4)$ and stress-energy tensor $(5.5)-(5.7)$ the following bounds can be obtained[2]:

$$c_1 \geq 0 \quad (for \quad healthy \quad conditions), \tag{5.1}$$

$$\frac{c_2}{b+d}(1-\xi^2) - \frac{c_3}{b+d} \geq -\frac{(1-\xi^2/2)^2}{\xi^2}, \tag{5.2}$$

$$d \geq 0, \tag{5.3}$$

$$b+d \geq 0, \tag{5.4}$$

$$4c_4 + 2c_5 + c_6 \geq 4(c_2+c_3)^2/c_1, \tag{5.5}$$

$$c_5 \geq 0, \tag{5.6}$$

$$c_6 \geq 0. \tag{5.7}$$

The constraints in (5.2) hold for all $\xi \in [0,1)$ and are plotted in figure (2.1). Spontaneous breaking of Lorentz boosts leads to a Goldstone boson π, whose EFT action is given by Eq. (4.91) where $\hat{g}_{\mu\nu} = g_{\mu\nu}|g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi|$ and $\chi = \mu t + \pi$.

5.1 Applying results to cosmology

Until now we presented a quite new method of getting the constraints for EFT coefficients, but now we would like to explain how this method will work in cosmology for real cosmological models, in our case for DE/MG. For that firstly we should write down the most general EFT action for DE/MG models up to second-order perturbations, including the most relevant operators [13].

$$
\begin{aligned}
S = \frac{1}{2}\int d^4x\sqrt{-g}[&M_{pl}^2 f(t)R - 2\Lambda(t) - 2c(t)g^{00} + \\
&+ M_2^4(t)(\delta g^{00})^2 - m_1^3(t)\delta g_{00}K - \bar{M}_2^2(t)\delta K^2 - \\
&- \bar{M}_3^2(t)\delta K_\mu^\nu \delta K_\nu^\mu + \mu_1^2(t)\delta g^{00}\delta R + m_2^2(t)h^{\mu\nu}\partial_\mu g^{00}\partial_\nu g^{00} + +...] \\
&+ S_m[g_{\mu\nu,\chi_m}].
\end{aligned}
\tag{5.8}
$$

M_{pl} is the Plank mass, $g^{00} = -1 + \delta g^{00}$, g — determinant of the metric, δR and $\delta R_{\mu\nu}$ — perturbations of Ricci scalar and tensor, S_m — matter action for all matter fields χ_m.

EFT functions — $f, \quad \Lambda, \quad c, \quad M_i, \quad m_i, \quad \mu_i.$

This model-independent construction of the action can be connected to Horndeski theories as follows:

$$
\bar{M}_2^2 = -\bar{M}_3^2 = 2\mu_1^2, \quad m_2^2 = 0 \tag{5.9}
$$

After writing the general action (5.8) one can find the EFT functions from modified Friedmann equations. But as we see, there are 4 free functions - EFT functions $f, \quad \Lambda, \quad c$ together with the unknown expansion history H, and 2 Friedmann equations. So, in order to find 2 functions from this system of equations, one should fix the other 2. Another interesting fact is that as in the unitary gauge, the extra DoF is hidden in the metric, we cannot get an explicit equation for the scalar field. So, evolution histories of the DoF and metric perturbations cannot be studied separately in the unitary gauge. However, one can make such a field appear explicitly in the action by using the Stuckelberg trick *[11]. For this one needs to force back broken gauge transformation on the field Lagrangian. After this pro-

*restoring the full diffeomorphism invariance cedure, the resulting action will be:

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}[&\frac{M_{pl}^2}{2}f(t+\pi)R - \Lambda(t+\pi) \\
&- c(t+\pi)(-1+\delta g^{00} - 2\dot{\pi} + 2\dot{\pi}\delta g^{00}) + 2\nabla_i\pi g^{0i} - \dot{\pi}^2 + \frac{1}{a^2}\nabla i\pi\nabla_i\pi \\
&+ \frac{M_2^4(t)}{2}(\delta g^{00} - 2\dot{\pi})^2 \\
&- \frac{\overline{m}_1^3(t)}{2}(\delta g^{00} - 2\dot{\pi})(\delta K_\mu^\mu + 3\dot{H}\pi + \frac{\nabla_i\nabla^i\pi}{a^2}) \\
&- \frac{\overline{M}_2^2(t)}{2}(\delta K_\mu^\mu + 3\dot{H}\pi + \frac{\nabla_i\nabla^i\pi}{a^2}) \\
&- \frac{\overline{M}_3^2(t)}{2}[(\delta K_j^i + \dot{H}\pi\delta_j^i + \frac{1}{a^2}\nabla^i\nabla^j\pi)(\delta K_i^j + \dot{H}\pi\delta_i^j \\
&+ \frac{1}{a^2}\nabla^i\nabla^j\pi) + (\delta K_0^0)^2 + 2(\delta K_0^i - \frac{H}{a^2}\nabla^i\pi)(\delta K_i^0 + H\nabla_i\pi)] \\
&+ \frac{\mu_1^2(t)}{2}(\delta g^{00} - 2\dot{\pi})(\delta R + 4H\frac{\nabla_i\nabla^i\pi}{a^2}) \\
&+ \frac{m_2^2(t)}{2}(g^{\mu\nu} + n^\mu n^\nu)\partial_\mu(g^{00} - 2\dot{\pi})\partial_\nu(g^{00} - 2\dot{\pi})] + S_m[g_{\mu\nu}, \chi_m][11].
\end{aligned}
\tag{5.10}
$$

Now we would like to explain (show the explicit steps theoretically) how the equations (4.91) and (5.10) relate to each other. Firstly, in Eq. (4.91) we are using the conformal transformations to express the field and the curvature explicitly (exact calculations are done in appendix B) [14], then we decouple the gravity and get an action similar to the action for Lagrangian (4.2) with some extra terms (extra terms are from the three more terms in (4.91), that give us more bounds). For this derived action we already have obtained the positivity bounds (above in Chap.5). Now, let us compare this gained action with the action (5.10). If we compare the corresponding terms and find the connection between them we will be able to use the same positivity bounds for the functions in (5.10). The latter corresponds to a real cosmological model. According to [15], there exists a connection between the EFT functions in (5.10), also called Wilson functions,[†] and the terms of our action received from (4.91) [15]. However, the coefficients from the action (4.91) are complicated functions of Wilson functions and we could not find the exact relation between them at the

[†]the coefficients of metric perturbations

moment of submission of this thesis because of the lack of time. Also, the CFT model works for 3D, but it is not clearly known yet how it will work for 4D. The needed assumption is that CFT starts to work at a scale lower than the string theory scale. But it is quite a strong assumption that yet needs to be checked carefully. Next, if it works and if a relation between Wilson functions and obtained action's terms is found, we will be able to use the gained positivity bounds for the Wilson functions to constrain the α values of the theory. Consequently, we will obtain the constraints about the cosmological constant.

5.2 Conclusion

In this thesis, we looked at a new mechanism of finding the positivity bounds (in Sec. 4), which would guarantee the proposed Lagrangian/action to give the healthy theory [2]. This new approach could be a useful tool for testing DE/MG theories. So, we tried to apply this mechanism to the real cosmological models in Sec. 5 (with provided assumption, that needs to be carefully checked yet). The mechanism considers finding the positivity bounds from 2-point correlation functions of conserved quantities like the Noether current and the stress-energy tensor. We propose the idea of relating the new mechanism of bound-constraining to the real cosmological models. The idea is the following: firstly, we take the Lagrangian/action of the model—in our case we consider the DE/MG model, which has an extra scalar degree of freedom. Next, we write down the equation of motion (EOM) for the scalar field of our theory, expand the solution of EOM in the low-energy limit, and plug the solution back into the initial Lagrangian/action. As a result, we get an expression similar to (4.91). Finally, we find the positivity bounds following the approach explained in Sec. 5.1 and use them to constrain our model of DE/MG.

Chapter 6

Future plans

As I already mentioned we were not yet able to find the exact connection between the Wilson functions and the coefficients from the (4.91) action, but we enjoyed working on this and we hope we will be able to find them soon. Also, we want to investigate how CFT works in 4D and try to find other constraints from 2-point correlation functions of a linear combination of the conserved current and stress-energy tensor $\langle (J(-k) + \mu T(-k))(J(k) + \mu T(k)) \rangle$ as it is proposed in paper [2].

Appendix A

CFT/superfluids

Generators:

- Dilations

$$D: \quad \pi(x) \to \pi'(x) \equiv \pi(\lambda x) + log\lambda; \tag{A.1}$$

- Infinitesimal conformal transformations

$$K_\mu: \quad \pi(x) \to \pi'(x) \equiv \pi(x + (cx^2 - 2(c \cdot x)x)) - 2c_\mu x^\mu; \tag{A.2}$$

- Translations

$$P_\mu: \quad \pi(x) \to \pi'(x) \equiv \pi(x + a); \tag{A.3}$$

- Boosts

$$M_{\mu\nu}: \quad \pi(x) \to \pi'(x) \equiv \pi(\Lambda \cdot x). \tag{A.4}$$

If we check the following Lagrangian:

$$\int d^4x\sqrt{-g} = \int d^4x e^{4\pi}, \tag{A.5}$$

we will see that it is conformally invariant upon changing the integration variable. For instance, under spacial conformal transformations, Eq.$(A.2)$, we get:

$$\int d^4x e^{4\pi(x)} \to \int d^4x e^{4\pi(x+(cx^2-2(c\cdot x)x))} e^{-8c_\mu x^\mu} = \int d^4x' e^{4\pi(x')}. \tag{A.6}$$

Appendix B

Modified metric & EFT operators

The modified metric can be written as:

$$\hat{g}_{\mu\nu} \equiv g_{\mu\nu}|g^{\alpha\beta}\partial_\alpha\chi\partial_\beta\chi|, \tag{B.1}$$

where $\chi = \mu t + \pi(t, x)$. At leading orders derivative we have:

$$S^{(1)} = \frac{c_1}{6}\int d^3x\sqrt{-\hat{g}} = \frac{c_1}{6}\int d^3x\sqrt{-g}|\partial_\chi|^3 \tag{B.2}$$

For the next order, we have to add the terms:

$$S^{(2)} = \int d^3x\sqrt{-\hat{g}}(-c_2\hat{R} + c_3\hat{R}^{\mu\nu}\hat{\partial}_\mu\chi\hat{\partial}_\nu\chi) \tag{B.3}$$

Now let us consume that the first three terms in action (4.91) give us the same Lagrangian as (4.2) (up to total derivative) Let us consider term by term. First term of the action (4.92) is:

$$\frac{c_1}{6}\int d^3x\sqrt{-\hat{g}} = \frac{c_1}{6}\int d^3x\sqrt{-g}\partial\chi^3 \tag{B.4}$$

So, for the Lagrangian, the first term will be

$$\frac{c_1}{6}\partial\chi^3 \tag{B.5}$$

That perfectly coincides with the c_1 term of the Lagrangian (4.2) (we do not have a covariant derivative here, because it is not coupled to A_μ, not any other difference). For c_2 term in (4.91) we will use the following formula [14]:

$$\hat{R} = \frac{R}{|\partial\chi|^2} - 2(n-1)g^{\alpha\beta}(\partial\chi)^3(\nabla_\alpha\nabla_\beta\partial\chi) - (n-1)(n-4)g^{\alpha\beta}\partial\chi^{-4}\nabla_\alpha\partial\chi\nabla_\beta\partial\chi \tag{B.6}$$

Putting the partial derivative everywhere instead of covariant, $n = 3$, and $g_{\mu\nu} = \eta_{\mu\nu}$ gives:

$$\hat{R} = \frac{R}{|\partial\chi|^2} - 2 * 2(\partial\chi)^{-3}\partial\chi^3 + 2 * \partial\chi^{-4}(\partial\partial\chi)^2 \tag{B.7}$$

Now we consider that the gravity is decoupled and plug the result from $(B.7)$ we get for c_2 term in (4.91):

$$\begin{aligned} &\int d^3x\sqrt{-\hat{g}}(-c_2\hat{R}) = \int d^3x\sqrt{-g}\partial\chi^3(-c_2\hat{R}) = \\ &= \int d^3x\sqrt{-g}(-c_2)\frac{(\partial\partial\chi)^2}{\partial\chi}, \end{aligned} \tag{B.8}$$

where we threw away the first term of $(B.7)$ because of the decoupled gravity and the second term because it gives us a total derivative and would not make a change in EOMs. So, we can see that the second Lagrangian term that we get from the action (4.91) also coincides with the relevant c_2 term of the Lagrangian (4.2).

Now let us look at the (c_3) term. For this one, we also use formula:[15]

$$\begin{aligned} \hat{R}^{\mu\nu} &= \hat{R}_{\sigma\rho}\hat{g}^{\sigma\nu}\hat{g}^{\rho\mu} = |\partial\chi|^{-4}\hat{R}_{\sigma\rho}g^{\sigma\nu}g^{\rho\mu} = \\ &= |\partial\chi|^{-4}g^{\sigma\nu}g^{\rho\mu}\Big(R_{\sigma\rho} - (n-2)(\delta^\alpha_\sigma\delta^\beta_\rho + g_{\sigma\rho}g^{\alpha\beta})|\partial\chi|^{-1}\nabla_\alpha\nabla_\beta|\partial\chi| + \\ &+ 2(n-2)(\delta^\alpha_\sigma\delta^\beta_\rho + g_{\sigma\rho}g^{\alpha\beta})|\partial\chi|^{-2}\nabla_\alpha|\partial\chi|\nabla_\beta|\partial\chi|\Big) = \\ &= |\partial\chi|^{-4}\Big(R^{\mu\nu} - |\partial\chi|^{-1}\nabla_\sigma\nabla\rho|\partial\chi|g_{\sigma\nu}g^{\rho\mu} - |\partial\chi|^{-1}g_{\sigma\rho}g^{\alpha\beta}g^{\sigma\nu}g^{\rho\mu}\nabla_\alpha\nabla_\beta|\partial\chi| + \\ &+ 2|\partial\chi|^{-2}\nabla_\sigma|\partial\chi|\nabla_\rho|\partial\chi|g^{\sigma\nu}g^{\rho\mu} + 2|\partial\chi|^{-2}g_{\rho\sigma}g^{\alpha\beta}g^{\sigma\nu}g^{\rho\mu}\nabla_\alpha|\partial\chi|\nabla_\beta|\partial\chi|\Big), \end{aligned} \tag{B.9}$$

From the action (4.91) we have:

$$\int d^3x\sqrt{-\hat{g}}\hat{R}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = \int d^3x\sqrt{-g}\partial\chi^3\hat{R}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi, \tag{B.10}$$

Then for the Lagrangian term equivalent to this, we will have:

$$\begin{aligned} &\partial\chi^3\hat{R}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = \\ &\qquad \partial\chi^3|\partial\chi|^{-4}\partial_\mu\chi\partial_\nu\chi\Big(R^{\mu\nu} - |\partial\chi|^{-1}\partial^\mu\partial^\nu|\partial\chi| - |\partial\chi|^{-1}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha\partial_\beta|\partial\chi| + \\ &\qquad + 2|\partial\chi|^{-2}\partial^\nu|\partial\chi|\partial^\mu|\partial\chi| + 2|\partial\chi|^{-2}g^{\mu\nu}g^{\alpha\beta}\partial_\alpha\partial\chi\partial_\beta\partial\chi\Big), \end{aligned} \tag{B.11}$$

Considering the gravity decoupling we have:

$$\partial\chi^3\hat{R}^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = -\frac{\partial_\mu\chi\partial_\nu\chi\partial^\mu\partial^\nu|\partial\chi|}{|\partial\chi|^2} - \frac{\partial_\mu\chi\partial^\mu\chi\partial^\alpha\partial_\alpha\partial\chi}{|\partial\chi|^2} + +2\frac{\partial_\mu\chi\partial_\nu\chi\partial^\nu|\partial\chi|\partial^\mu|\partial\chi|}{|\partial\chi|^3} =$$
$$= -\frac{\partial_\mu\chi\partial_\nu\chi\partial^\mu\partial^\nu|\partial\chi|}{|\partial\chi|^2} - \partial^\alpha\partial_\alpha\partial\chi + 2\frac{\partial_\mu\chi\partial_\nu\chi\partial^\nu|\partial\chi|\partial^\mu|\partial\chi|}{|\partial\chi|^3}. \tag{B.12}$$

We can see that the third term of the $(B.12)$ is similar to the $1-st$ part of the c_3 term in (4.2), the second term is the full derivative and we can ignore that as it will not change EOMs. For the first term of $(B.12)$ we have:

$$-\frac{\partial_\mu\chi\partial_\nu\chi\partial^\mu\partial^\nu|\partial\chi|}{|\partial\chi|^2} =$$
$$= \partial_\mu\Big(\frac{\partial^\mu\chi\partial^\nu\chi\partial_\nu\partial\chi}{\partial\chi^2}\Big) + \partial_\mu\Big(\frac{\partial^\mu\chi\partial^\nu\chi}{\partial\chi^2}\Big)\partial_\nu\partial\chi, \tag{B.13}$$

So we can see that from here we could keep only the second term of RHS of $(B.13)$ if we want which coincides with the second part of the C_3 term in (4.2), as the first one is the total derivative. So, we have shown that the action (4.91) definitely leads us to the Lagrangian like (4.2).

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