Relativistic effects on Gravitational Waves in Scalar-Tensor theories of Gravity

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Abstract

With the development of next-generation gravitational waves detectors, we aim to measure and infer data on a broader range of the energy and redshift spectrum. However, in this range, deviations from theories that predict a non-standard propagation speed for GWs ($C_T^2 \neq 1$) are expected to become non negligible anymore. Moreover, it has been shown that the presence of inhomogeneities and structures in our universe does affect the GWs observables: with the forecasted level of precision of future detectors, such corrections can not be ignored. In this work we set in the frame of quartic scalar-tensor theories of gravity to study such relativistic effects with the presence of an extra scalar degree of freedom. Due to the complexity of the full theory, we opted for a phenomenological approach to describe the dispersion relation and amplitude evolution of the metric perturbation. Using this technique, we evaluated the relativistic corrections to frequency and direction of propagation of the GWs wavevector. On the other hand, it was not possible to analytically calculate the relativistic corrections to the tensor amplitude even using the parametric approach, as a consequence of the elevated number of terms in the amplitude evolution equation. Nevertheless, after selecting from such equation the transverse-traceless modes, we were able to find, using the N-P formalism, which types of term can actually contribute to the evolution of the physical modes.

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Introduction

Gravitational Waves (GWs) are one of the most interesting and important predictions of General Relativity, the theory of gravity proposed by A. Einstein [25]. Their existence was predicted more than a century ago [24], however they have been directly observed only few years ago, thanks to the development of GW interferometers and in particular the LIGO-VIRGO collaboration [1]. This opened up new incredible ways to test a broad range of physical fields, from particle physics (like studying the tidal effects of coalescing binary neutron stars to constraints particle physics models [31]), to astrophysics (studying the population of black holes, or more exotic compact objects), to finally cosmology and multi-messenger astrophysics, which will be the main focus of this project.
This observational successes pushed the scientific world to improve and enhance the world of gravitational waves detectors, with the development and, subsequently, the construction of future new generation interferometers.
In particular, two main collaborations are developing the new, so called, third-generation detectors: the Einstein telescope [50] and LISA detector [6]. While the first one will follow the road of the previous interferometers, as it will be a ground-based observatory with substantial and structural improvements, the second one will be a space-based detector, trailing the Earth in its orbit around the Sun. These two new detectors promise to achieve together a substantial sensitivity improvement in frequency ranges.
Due to the recent observational improvements, new interests has started to sprout regarding the relativistic corrections to GWs observables due to Large-Scale structures, the largest known gravitationally collapsed objects: a GW signal travelling trough astrophysical distances, will be affected by the presence of matter structures and inhomogeneities along its path [9].

This will result in modifications to GWs observables (such as frequency, direction of propagation or luminosity distance) due to relativistic effects like gravitational lensing, ISW effect or peculiar velocities.
Moreover, one of the listed observables above, the luminosity distance, is probably the most important quantity we can measure from gravitational waves in cosmology, since from GWs we can infer a direct measure of distance from their amplitude, while one normally can only work with redshift values of standard electromagnetic observations. This allow us to construct a map between redshift and physical distances, making possible to extract information about the cosmological parameters [53] or test new gravity models beyond GR.
This last topic in particular has been of interest after the discovery of the late-time cosmic accelerated expansion from the study of the Hubble diagram of a type Ia supernovae catalog. Since any known component in the universe would, in fact, create a deceleration in the expansion of the universe, according to GR, a great effort has been put in order to explain such phenomenon and its origin, by proposing several explanations to such unknown component, generally referred to as Dark Energy. In the standard picture, Dark Energy is modeled as a cosmological constant, since the easiest mathematical (as it will be shown in the following chapters) way to obtain an accelerated expansion is adding a constant value to the standard Einstein action. This cosmological constant is one of the main building blocks of the standard model of cosmology, namely the ΛCDM model, since, from an observational point of view, it is a great candidate to solve the Dark Energy problem as it describes quite nicely our data [3]. However, it does bring several theoretical issues. Firstly, this vacuum energy does not have a well-defined and justified origin, as we can not relate it to any known physical interaction. In addition to this, its measured value has a huge discrepancy with the expected vacuum energy-density value one can calculate from Quantum Field Theory, the standard theory of particle physics. The last issue is generally addressed to as the *cosmological constant problem*.
All these reasons brought physicists to find alternative explanations for the accelerated expansion of the universe, which apparently GR does not seem to fully describe. Even though there is no unique road to address this problem (a brief introduction to alternative theories of gravity will be presented in the second chapter), one of the simplest and most straightforward ways to extend and modify our standard theory of gravity can be achieved by adding a classical scalar field ϕ, coupled to the metric $g_{\mu\nu}$. In this frame (generally referred as Scalar-Tensor theories of gravity), the scalar field will characterize the Dark Energy in such a way we can still

find self-accelerated solutions.
The presence of this additional degree of freedom would also result in the modification of cosmological physics like, for example, the growth dynamics of matter perturbations. However, no observational confirmation of modified theories of gravity has been achieved yet, nor evidence of deviations from General Relativity.
The new generation of galaxy surveys, like the Euclid telescope [43], will also aim to constrain GR on large scales and rule out or detect signatures of modifications of it. However, the GW detectors opened another new way to test different cosmologies that, combined with EM observations or other type of signals, would provide powerful results in a multi-messenger physics approach.
The aim of this project is to study the relativistic effects of gravitational waves in scalar-tensor theories, hence in gravitational theories with the presence of an extra scalar field. As a matter of fact, in general scalar-tensor theories the propagation speed of the gravitational waves, usually called c_T (while the propagation of the scalar field will be referred to as c_S), can be different from the speed of light c, which is the standard propagation speed of GWs in General Relativity. This modification implies that GWs will not follow geodesic paths anymore and, moreover, the relativistic effects will affect them in different ways.
From observations we get some pretty tight constraints on the GWs propagation speed, $|c_T/c - 1| \leq 5 \cdot 10^{-16}$ [2], however such measurement comes from a low-redshift, high-energy event and in this limit we expect to recover a standard propagation speed for c_T [19]. Furthermore, next generation detectors are expected to perform measurements in less accessible ranges of redshift and energy, meaning that they will be able to put more accurate and precise constraints on the propagation speed.
Therefore, we will try to apply the Cosmic Rulers formalism [52] on the perturbed equations of motion that one can derive form the (quartic) Horndenski action. From here, one can then evaluate the relativistic corrections to the GWs observables.
However, due to a very large number of terms in the perturbed equations of motion, the exact calculation on the full theory was not to achievable. As a result, we decided to define a parametric formalism and use this approach to solve the problem.

0.1 Notation and conventions

Natural constants as the speed of light c and the (reduced) Planck constant $\hbar$ are usually set to unity, while in some situations they might be written explicitly, but always after clearly stating it.
Greek indices run from 0 to 3, while Latin letters take values from 1 to 3 and are usually denoted for the spatial indices. The convention for the sign of the flat metric is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1), \tag{1}$$

and the space-time coordinates are denoted as

$$x^\mu = (x^0, x'i)\ ,\ \ x^0 = \eta, \tag{2}$$

where η is the conformal time, hence directional partial derivatives are written as $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Covariant derivatives are written as ∇_μ and covariant directional derivatives along a path defined by a parameter λ will be denoted as

$$\frac{\mathcal{D}}{d\lambda} = \frac{dx^\mu}{d\lambda}\nabla_\mu. \tag{3}$$

Indices are generally raised and lowered using the metric, however in this work this is not always straightforward and true (background metrics or different definitions of the metric might be used for raising and lowering). Nevertheless, the procedure for raising and lowering indices will always be clearly stated if the standard metric is not used. Einstein notations is used, hence repeated indices are summed over.
The curved space-time metric is denoted as $g_{\mu\nu}$. We define the Christoffel symbols as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}), \tag{4}$$

while the Riemann tensor will be

$$R^\mu_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\alpha\rho}\Gamma^\alpha_{\nu\sigma} - \Gamma^\mu_{\alpha\sigma}\Gamma^\alpha_{\nu\rho}. \tag{5}$$

From this we can evaluate the Ricci tensor and the Ricci scalar as, respectively

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}, \qquad \text{and} \qquad R = g^{\mu\nu}R_{\mu\nu}. \tag{6}$$

Finally, the Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{7}$$

Symmetrization and anti-symmetrization of the indices of a tensor are denoted respectively as

$$A_{(\alpha\beta)} = \frac{1}{2}(A_{\alpha\beta} + A_{\beta\alpha})\ , \ \ A_{[\alpha\beta]} = \frac{1}{2}(A_{\alpha\beta} - A_{\beta\alpha}). \tag{8}$$

Chapter 1

Gravitational Waves

1.1 Gravitational Waves on a flat spacetime

In this section, we will briefly derive how gravitational waves naturally emerge from the frame of General Relativity, using the so-called geometrical approach. For a more detailed derivation, in particular about the gauge freedom discussion, see Chapter 1 of [44].
In this set-up, the main idea of GWs starts from the concept of being able to split our metric into slowly varying background and a small perturbing part, however the ambiguity of this separation is not a trivial problem and it will addressed in more detailed manner later.
Starting from the gravitational action $S = S_E + S_M$, where S_E is the Einstein's action, namely

$$S_E = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R, \tag{1.1}$$

and S_M is the matter action. Varying the total action with respect to $g_{\mu\nu}$, it will bring us the well known Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}, \tag{1.2}$$

where the left hand-side can be more compactly written using the Einstein tensor: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.
It is important to state that General Relativity is invariant under diffeomorphisms,

$$x^\mu \to x'^\mu(x) = f(x^\mu), \tag{1.3}$$

if the arbitrary function of x^μ, $f(x^\mu)$, is differentiable, invertible with a differentiable inverse and under local Poincaré transformations.
We start the derivation of GWs in a special case, by perturbing the Einstein's equations around a flat Minkowksi metric, $\eta_{\mu\nu}$, that is

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\ , \quad |h_{\mu\nu}| \ll 1. \tag{1.4}$$

We expand the equations of motion up to linear order in $h_{\mu\nu}$, so, starting from the Riemann tensor, we find

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\mu\partial_\sigma h_{\rho\nu} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}). \tag{1.5}$$

It is usually a standard procedure to write the equations of motion in a more efficient way by first defining the trace

$$h = \eta^{\mu\nu} h_{\mu\nu}, \tag{1.6}$$

and

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu} h, \tag{1.7}$$

where the last expression is generally called the trace-reverse form metric perturbation.
Using the fact that $\bar{h} \equiv \eta^{\mu\nu}\bar{h}_{\mu\nu} = -h$, one can find $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}$ and write explicitly the perturbed Einstein equations up to linear order:

$$\Box\bar{h}_{\mu\nu} + \eta_{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\rho\sigma} - 2\partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} = -16\pi G T_{\mu\nu}. \tag{1.8}$$

We point out that in the linearized theory one will raise and lower indices with the background metric, which, in this derivation, will be the flat Minkowski metric.
However, we can notice that under a coordinate transformation like

$$x \to x'^\mu = x^\mu + \xi^\mu(x), \tag{1.9}$$

where ξ^μ are arbitrary functions of the coordinates, and their derivatives $|\partial_\nu \xi_\mu|$ are of the same order of magnitude as $|h_{\mu\nu}|$, the GWs will transform as

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}, \tag{1.10}$$

up to linear order.
Using these gauge freedom, one can choose the functions ξ^μ so that

$$\partial^\nu\bar{h}_{\mu\nu} = 0, \tag{1.11}$$

usually known as the harmonic (or de -Donder) gauge and find the well-known wave equation

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{1.12}$$

To study the propagation of gravitational waves and how they interact with detectors, one will be more interested to study the last equation outside the source, where $T_{\mu\nu} = 0$. Please note that the hypothesis of GWs travelling through the vacuum does not hold for scalar-tensor theories (as instead of a vacuum energy there will be a scalar field) nor cosmological scales: in particular, in this thesis we are interested in how inhomogeneities present in our universe will affect the GWs. However, this problem shall be addressed in following chapters, here we will make the assumption that GWs are propagating in the vacuum. Therefore, setting $T_{\mu\nu} = 0$, we have

$$\Box\bar{h}_{\mu\nu} = 0, \tag{1.13}$$

from which we can infer an important property of Gravitational Waves in GR: since the d'Alembertian in Minkowski space is $\Box = -\frac{1}{c^2}\partial_0 + \nabla^2$ (here the factor of c is kept explicit for didactic reasons) it follows that Gravitational Waves propagate at the speed of light.
From the study of the gauge freedom of the system it is possible to still find some residual gauge freedom: under another coordinate transformation

$$x \to x'^{\mu} = x^{\mu} + \xi^{\mu}(x) \qquad \text{with} \qquad \Box\xi^{\mu} = 0, \tag{1.14}$$

the harmonic gauge $\partial_\nu \bar{h}^{\mu\nu}$ is not spoiled. Hence, we can choose ξ^0 such that $\bar{h} = 0$, hence $\bar{h}_{\mu\nu} = h_{\mu\nu}$. The other functions ξ^i are chosen in order to have $h_{0i} = 0$. Adding these gauge choices to the harmonic gauge with $\mu = 0$, one can find

$$\partial_0 h^{00} + \partial_i h^{i0} = 0. \tag{1.15}$$

Since $h_{0i} = 0$, it follows that

$$\partial_0 h^{00} = 0, \tag{1.16}$$

so the 00-component does not depend on time. However, a time-independent h_{00} term represents the Newtonian potential of the source that genereated the GW. Since the gravitational wave is the propagating time-dependent part of the gravitational interaction, if $\partial_0 h^{00} = 0$, then it follows that $h_{00} = 0$.
As a summary of all the gauge fixing, we have:

$$h_{0\mu} = 0 \;\; h^i_i = 0\,, \;\; \partial^j h_{ij} = 0, \tag{1.17}$$

which is generally called the transverse-traceless gauge (or TT gauge) of the metric perturbation. Eq. (1.13) admits a plain wave solution of the form

$$h_{ij}^{TT}(x) = e_{ij}(\mathbf{k}^i)\exp[ik^\mu x_\mu], \tag{1.18}$$

where $k^\mu = (\nu, \mathbf{k}^i)$ is the wave-vector associated with the GW and e_{ij} is generally called the polarization tensor. If we perform a rotation of the coordinates in order to have the direction of propagation of the plain wave along the z-axis, then, due to the harmonic gauge, we will have perturbation transverse to such direction. Imposing the traceless and symmetric properties, and choosing only the real part from the complex exponential, we will have

$$h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos[\omega(t-z)]. \tag{1.19}$$

$h_+ = h_+(t,z)$ and $h_\times = h_\times(t,z)$ are usually called the amplitudes of the *plus* and *cross* polarization of the wave (see figure 1.1).

1.2 Gravitational Waves on a curved manifold

We now move on to a more realistic setup, where the background spacetime is no longer flat. In the previous case the splitting between the perturbation and the background was clear and unambiguous, but now some problems arise: if we follow the same procedure as before, we split the metric as

$$g_{\mu\nu}(x^\alpha) = \bar{g}_{\mu\nu}(x^\alpha) + h_{\mu\nu}(x^\alpha)\ , \ \ |h_{\mu\nu}| \ll 1, \tag{1.20}$$

where $\bar{g}_{\mu\nu}$ is dynamical background and $h_{\mu\nu}$ is the perturbation. In principle one could always move coordinates dependent terms from the background to the perturbation, as there is no longer an unambiguous splitting (flat vs. curved manifold). The most straightforward way is to restrict ourselves when there is a clear difference in terms of scales in our physical system. That means that the splitting can be performed in systems where the scales involved for spatial variations in background metric $\bar{g}_{\mu\nu}$ are way larger than wavelength of the small superimposed perturbation. Another way of thinking this is to move into Fourier space and consider systems where the background has frequency up to a certain value f_{BG} which is way smaller than the peak frequency of the perturbation f. In this sense, the perturbation $h_{\mu\nu}$ is a high-frequency perturbation of a slowly changing

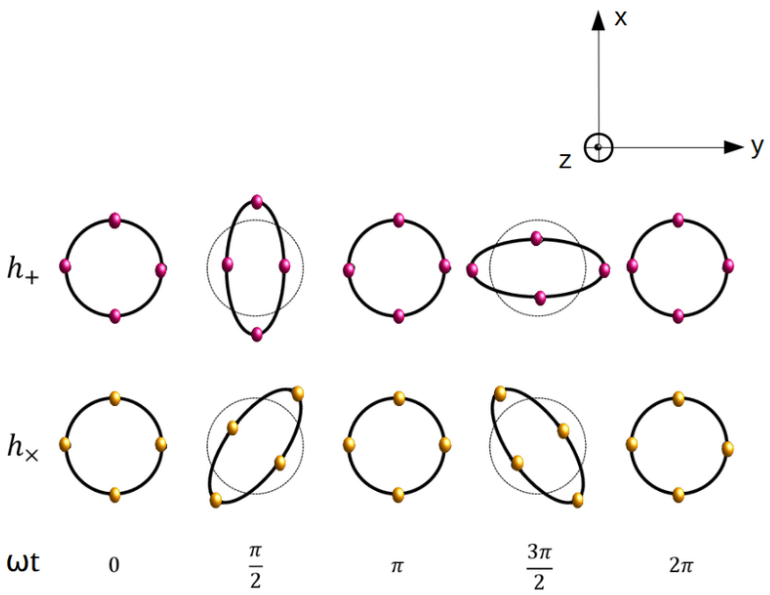

Figure 1.1: *Plus and cross polarizations of a Gravitational Wave propagating in the z-direction. The red and yellow points represent the spatial separations being modified by the presence of a GW.*

background. Therefore $h_{\mu\nu}$ will be characterised by high-frequency modes while on the other hand $\bar{g}_{\mu\nu}$ will be characterised by low-frequency modes. Assuming that we are in a system where eq. (1.20) holds, we can repeat the calculations performed in the previous section under the right adjustments. Starting from the Einstein equations, we can rearrange them in the following form

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T), \tag{1.21}$$

with $T_{\mu\nu}$ as the already introduced energy-momentum tensor and T as its trace. If we expand the Ricci tensor up to $\mathcal{O}(h^2)$, we will have

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}, \tag{1.22}$$

where $R^{(0)}_{\mu\nu}$ is constructed only by the background metric $\bar{g}_{\mu\nu}$, therefore it will be characterized only by low-frequency modes; $R^{(1)}_{\mu\nu}$ contains only linear terms of $h_{\mu\nu}$, hence will be characterized only by low-frequency modes; $R^{(1)}_{\mu\nu}$ is quadratic in $h_{\mu\nu}$, so it can contain both low and high-frequency modes. While the first two statements are pretty straightforward, the last one is not: let us consider a quadratic term, like $h_{\mu\nu}h_{\alpha\beta}$ for instance. $h_{\mu\nu}$ will bring a high wave-vector k^i which might combine with another high wave-vector $q^i \approx -k^i$ from the $h_{\alpha\beta}$ term, ending in a low-frequency mode. As a result, the Einstein equations can be split into two separate equations, one describing the low modes while the other describes the high modes:

$$R^{(0)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^L + 8\pi G\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^L, \tag{1.23}$$

$$R^{(1)}_{\mu\nu} = -\left[R^{(2)}_{\mu\nu}\right]^H + 8\pi G\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^H, \tag{1.24}$$

where the superscript L stands for low-frequency modes and H for the high-frequency ones. The first equation will describe how the GWs affect and curve the background metric, while the second tells us how the GWs propagate in the background spacetime. For the scope of this project we will be focusing on the latter.
We will also restrict to only linear terms in $h_{\mu\nu}$ and neglect higher order terms, meaning that we can set $\left[R^{(2)}_{\mu\nu}\right]^H = 0$. Performing the full algebra on the LHS of eq. (1.24), one shall find

$$R^{(1)}_{\mu\nu} = \bar{g}^{\alpha\beta}\left(\bar{\nabla}_\alpha\bar{\nabla}_\nu h_{\mu\beta} + \bar{\nabla}_\alpha\bar{\nabla}_\mu h_{\nu\beta} - \bar{\nabla}_\mu\bar{\nabla}_\nu h_{\alpha\beta} + \bar{\nabla}_\alpha\bar{\nabla}_\beta h_{\mu\nu}\right), \tag{1.25}$$

with the covariant derivative taken with respect to the background metric. After substituting the trace-reversed form of the metric perturbation, namely

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}h, \tag{1.26}$$

where h is the trace of $h_{\mu\nu}$, and imposing the harmonic gauge

$$\bar{\nabla}_\mu\bar{h}^{\mu\nu} = 0, \tag{1.27}$$

we can further rewrite eq. (1.25) as follows

$$R^{(1)}_{\mu\nu} = \bar{\nabla}_\alpha\bar{\nabla}^\alpha\bar{h}_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\beta}\bar{h}^{\alpha\beta} - \bar{R}_{\mu\alpha}\bar{h}^\alpha_\nu - R_{\nu\alpha}\bar{h}^\alpha_\mu, \tag{1.28}$$

where the Riemann and Ricci tensors $\bar{R}_{\mu\alpha\nu\beta}$ and $\bar{R}_{\mu\alpha}$ are constructed with the background metric. We acknowledge an overlapping in the notation, as $\bar{h}_{\mu\nu}$ is the trace-reversed form of the GW, while the bar symbol on any other objects just represents it is taken with respect to the background metric. However, this is the most used notation in the literature, hence we will continue with it.

Combining this last result with eq. (1.24), we will find the propagation equation for the gravitational waves in the high-energy limit on a generic background:

$$\bar{\nabla}_\alpha \bar{\nabla}^\alpha \bar{h}_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\beta}\bar{h}^{\alpha\beta} - \bar{R}_{\mu\alpha}\bar{h}^\alpha_\nu - R_{\nu\alpha}\bar{h}^\alpha_\mu = 8\pi G \left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^H . \quad (1.29)$$

This result will be the starting point for calculating the relativistic corrections in the following chapters.

1.3 Tetrads formalism

In this section we will introduce the tetrad formalism which shall be used in the following chapters when calculating the relativistic corrections to the gravitational waves. For a more detailed description please look [57] or, for a more rigorous differential geometric approach, [18].
Until now we briefly presented General Relativity and discussed one of its most known applications: gravitational waves. Still, we discussed such phenomenon by itself, independent from an observer or a measurement, although we know, especially from a quantum point of view, that our only way to interface to the physical world is through measurements.
We shall now properly define the concepts of observer or measurement, starting from the statement that a measurement is physically meaningful if and only if it is a coordinate independent quantity, and an observer can interpret it in a coordinate independent way as well. However, in real life, these measurements will be performed by different observers, so they are observer dependent. It is necessary then to define some criteria to relate different observers. Therefore, we define an observer as a collection of measuring devices which must be confined in a sufficiently small space, the laboratory. Moreover, this measurements must be performed over a sufficiently small amount of time: by this, the curvature effects can be neglected and it is possible to locally split space-time into space and time in an unambiguous way.
The key concept of the tetrad formalism is that, for observers, it is natural to perform physical measurements in a Cartesian set of coordinates, since this would provide them an instantaneous inertial frame. Therefore, assuming that a physical phenomenon is described by a generic tensor, the measurement will be interfaced with the invariant projection of such tensor on the chosen Cartesian coordinates.
The main property of Cartesian coordinates is that they are the only orthonormal basis in flat space-time. Hence, we can define a non-holomonic

orthonormal basis (i.e. coordinate independent) of smooth vector fields $e_{\hat{a}}$ with

$$e_{\hat{a}}e_{\hat{b}} = \eta_{\hat{a}\hat{b}}, \tag{1.30}$$

where the Latin letters run over $\hat{a} = 0, 1, 2, 3$. Moreover, $\eta_{\hat{a}\hat{b}}$ defines the scalar product rules for the orthonormal vectors and it has the same symbol of Minkowski metric because the vector are orthonormal. Since spacetime is four dimensional, this set is generally called tetrad.
If we define a generic coordinate basis x_μ on our manifold, from the properties of the vector basis, it follows that

$$e_{\hat{a}} = (e_{\hat{a}})^\mu x_\mu \qquad \text{and} \qquad x_\mu = (e_\mu)^{\hat{a}} e_{\hat{a}}. \tag{1.31}$$

Moreover, from eq. (1.30) and definitions (1.31), one can infer following properties:

$$g^{\mu\nu}(e_\mu)^{\hat{b}}(e_\nu)_{\hat{a}} = \delta^{\hat{b}}_{\hat{a}}, \tag{1.32}$$

$$(e_\mu)^{\hat{a}}\eta_{\hat{a}\hat{b}} = (e_\mu)_{\hat{b}}, \tag{1.33}$$

$$\eta_{\hat{a}\hat{b}}(e_\mu)^{\hat{a}}(e_\nu)^{\hat{b}} = g_{\mu\nu}, \tag{1.34}$$

where $\delta^{\hat{b}}_{\hat{a}}$ is the identity map on vectors and $g^{\mu\nu}$ is the inverse of metric. Therefore indices of the tetrad are raised and lowered with the Minkowski metric $\eta_{\hat{a}\hat{b}}$.
Consider now, for example, a (2,0) rank tensor $U^{\mu\nu}$: one can find that its tetrad components are coordinate independent:

$$U^{\hat{a}\hat{b}} = U^{\mu\nu}(e_\mu)^{\hat{a}}(e_\nu)^{\hat{b}}, \tag{1.35}$$

with inverse relation

$$U^{\mu\nu} = U^{\hat{a}\hat{b}}(e^\mu)_{\hat{a}}(e^\nu)_{\hat{b}}. \tag{1.36}$$

We shall now apply this method to the already introduced phenomenon of gravitational waves. Let us consider a gravitational perturbation $h_{\mu\nu}$, split from the background metric $\bar{g}_{\mu\nu}$, in its trace-reversed metric form (1.26), written as $\bar{h}_{\mu\nu}$ and we select the transverse-traceless gauge. We then perform a 3-D Fourier decomposition to find

$$\bar{h}_{\mu\nu} = \int d^3k \; e^{ik^i \cdot x^i} \bar{h}_{\mu\nu(k)} \,, \tag{1.37}$$

where k^μ is the 4-dimensional wave-vector of the GW (with spatial part k^i) and $\bar{h}_{\mu\nu,k}$ is the Fourier transform. In addition to this, we assumed $\bar{h}_{\mu\nu,k}$

depends only on the modulus of the wave-vector and not on its direction. The choice of the tetrad construction will follow the procedure of [29].
We then start by constructing a new vector basis which will form the legs of our tetrad. Firstly, we shall define a time-like vector

$$e^{\mu}_{\hat{0}} = u^{\mu} = (1, 0, 0, 0), \tag{1.38}$$

then we use the wave-vector defined in eq. (1.37) to define the space-like vector

$$e^{\mu}_{\hat{3}} = \frac{k^{\mu} - k^{0}u^{\mu}}{|k|} = (0, k^{i}), \tag{1.39}$$

where $|k|$ is the pseudo-norm of the wave-vector. We can notice that equations (1.38) and (1.39) are orthogonal to each other and normalized. They shall be, respectively, the first and fourth leg of the tetrad. The remaining two vectors, that is $e^{\mu}_{\hat{1}}$ and $e^{\mu}_{\hat{2}}$, are chosen to be space-like, normalized and orthogonal to $e^{\mu}_{\hat{0}}$ and $e^{\mu}_{\hat{3}}$. Following the formalism introduced by Newman and Penrose [47] we shall perform a rotation on the vectors $e^{\mu}_{\hat{1}}$ and $e^{\mu}_{\hat{2}}$ and use the combinations:

$$m^{\mu} = \frac{e^{\mu}_{\hat{1}} + ie^{\mu}_{\hat{2}}}{\sqrt{2}}, \qquad \tilde{m}^{\mu} = \frac{e^{\mu}_{\hat{1}} - ie^{\mu}_{\hat{2}}}{\sqrt{2}}, \tag{1.40}$$

with scalar products

$$\bar{g}_{\mu\nu}m^{\mu}m^{\nu} = \bar{g}_{\mu\nu}\tilde{m}^{\mu}\tilde{m}^{\nu} = 0, \qquad \bar{g}_{\mu\nu}m^{\mu}\tilde{m}^{\nu} = 1. \tag{1.41}$$

Plugging the vectors together we can define the following basis

$$\{e^{\mu}_{\hat{0}}, m^{\mu}, \tilde{m}^{\mu}, e^{\mu}_{\hat{3}}\} \tag{1.42}$$

One can notice that the vectors m^{μ} and $\tilde{m}^{\mu}$ are related to the circular polarization basis of Gravitational Waves, while e^{μ}_{0} and e^{ν}_{3} are used to define the linear polarization basis.
Using this vectors, we can create a new basis for rank (0,2) symmetric tensors, by defining

$$\Theta^{\mu\nu}_{\hat{a}\hat{b}} = \frac{1}{2}(e^{(\mu}_{\hat{a}}e^{\nu)}_{\hat{b}}), \tag{1.43}$$

where $e^{\mu}_{\hat{a}}$ and $e^{\mu}_{\hat{b}}$ span over (1.42). Then the standard procedure would be decomposing the tensorial amplitude as

$$\bar{h}_{\mu\nu,k} = \sum_{\hat{a},\hat{b}} \bar{h}_{\hat{a}\hat{b},k}\Theta^{\hat{a}\hat{b}}_{\mu\nu}, \qquad \text{with} \qquad \bar{h}_{\hat{a}\hat{b},k} = \bar{h}_{\hat{b}\hat{a},k}. \tag{1.44}$$

The factor of $\frac{1}{2}$ in (1.43) follows from the symmetry of changing the decomposition coefficients a and b.
After imposing the gauge choices, one can find some constraints on the coefficients $h_{\hat{a}\hat{b}}$: starting from transversality condition, namely,

$$u^{\mu}\bar{h}_{\mu\nu,k} = 0, \qquad k^{i}\bar{h}_{i\nu,k} = 0, \tag{1.45}$$

one would find the constraints

$$\bar{h}_{\hat{0}\hat{a},k} = 0, \qquad \bar{h}_{\hat{3}\hat{a},k} = 0. \tag{1.46}$$

The final condition, tracelessness, will define the final constraint: from the trace of $\bar{h}_{\mu\nu,k}$, that is

$$\bar{g}_{\mu\nu}\bar{h}_{\mu\nu,k} = \bar{g}_{\mu\nu}(\bar{h}_{mm,k}\Theta^{mm}_{\mu\nu} + \bar{h}_{\tilde{m}\tilde{m},k}\Theta^{\tilde{m}\tilde{m}}_{\mu\nu} + \bar{h}_{m\tilde{m},k}\Theta^{m\tilde{m}}_{\mu\nu}) = 0, \tag{1.47}$$

after plugging the scalar products (1.41), the last condition simplify as

$$\bar{h}_{m\tilde{m},k} = 0. \tag{1.48}$$

Therefore, in the TT-gauge, the gravitational wave will have the form

$$\bar{h}_{\mu\nu,k} = \bar{h}_{mm,k}\Theta^{mm}_{\mu\nu} + \bar{h}_{\tilde{m}\tilde{m},k}\Theta^{\tilde{m}\tilde{m}}_{\mu\nu}, \tag{1.49}$$

where $\bar{h}_{mm,k}$ and $\bar{h}_{\tilde{m}\tilde{m},k}$ are the left-handed and right-handed helicity modes. Such modes are the physical part of the gravitational waves which can not be gauged away.
We showed how to apply the tetrad formalism to gravitational waves, and we will later use the results from this section to study the polarization of such phenomenon.

Chapter 2

Modified gravity theories

After discussing one of the most important applications of General Relativity, we will switch topic and introduce in this chapter the other main aspect of this thesis, that is modified theories of gravity.
As already said in the introduction, General relativity is our standard gravitational theory. It has been intensively tested for more than a century at this point, always proving to be true with uncanny precise levels, starting from early tests like the perihelion precession of Mercury [13] or the gravitational deflection of a light-ray [22], up to modern day tests, like the direct observation of a Gravitational Wave or a Black hole [4].
Nevertheless, it appears that GR is still not sufficient to comprehensively explain all the gravitationally related phenomena in our universe, failing at very small and very large scales. The first issue is related by the quantization of gravity: three out of four interactions present in our universe can be fully described by the formalism of quantum mechanics (QM) and quantum field theory (QFT), constructing our standard model of particle physics [45], while GR uses the formalism of classical physics to describe gravity. As a result, when the gravitational effects of strong fields are noticeable even at particles scale GR will not be sufficient to describe them. However, after trying to perform a quantization of gravity, one would have to face the non-renormalizability nature of gravity [28]. In order to solve this issue, several alternative theories have been proposed, like the string theory or loop quantum gravity.
Still, for the scope of this thesis, we shall focus on the problems of General Relativity at largest scales: from the Friedmann's solution of GR, verified by the Hubble's observation [34], the whole new scientific field of cosmology was born. Several experimental tests confirmed this model of the universe, such as observation of the Cosmic Microwave Background (CMB),

Large Scale Structures forming from gravitational instabilities and so on. In spite of that, the first observations of a cosmic acceleration, found from the Hubble diagram of type Ia supernovae [51], produced some cracks in our cosmological model. If we consider an expanding universe following the rules of GR and populated by standard matter, this will undergo a decelerated expansion rather than an accelerated one.
Moreover, looking at the composition of the universe, one would also find that this unknown component, usually called Dark Energy (DE) is the main component of our cosmos, roughly constituting 70% of it [3].
This observed cosmic acceleration can be achieved by introducing a cosmological constant Λ in the Einstein action, namely

$$S = \int d^4x \sqrt{-g} \left[R(x) + \Lambda \right], \tag{2.1}$$

where R is the Ricci scalar. By doing so, the resulting Friedmann equations will describe an acceleration in the expansion parameter a at late times. The cosmological constant Λ is one of the building blocks of the scandard model of cosmology, the ΛCDM model. For a complete and detailed derivation of the Friedmann equations with a cosmological constant please look [7].
Such model exquisitely describes our data, still it faces several theoretical issues. Firstly, a cosmological constant does not come from any physical interaction, hence it just represents the vacuum energy density emerging from all quantum fields and it does not bring any further information about our universe: if indeed the cosmic acceleration is caused by Λ, then no deeper understanding is needed and we can not do nothing more besides finding the most possible precise value of it. Therefore, it seems reasonable, from the point of view of theoretical physics, to consider the cosmological constant as the extreme solution, when nothing more can be found.
A more quantitative issue of the Λ can be found if we use Particle Physics to test this model. As a matter of fact, it is possible to calculate, using QFT, the expected contributions to the cosmological constant from quantum fluctuations in vacuum. Then one could compare the expected vacuum energy density from QFT with the observed energy density of Λ. After a rough calculation, the two values have a discrepancy in almost 30 orders of magnitude [54], meaning that one of the two models is not accurate enough. This is generally addressed as the cosmological constant problem.
The Standard Model of physics has been the most successful theory of physics, predicting with extreme accuracy the nature of particles at most

fundamental levels. It seems straightforward then, to consider the cosmological constant as an approximation that matches quite well our observations, but still it is far from being the full theory.
Even though General Relativity has been successfully proven on astronomical scales, we still have to test the nature of gravity at cosmological scales, which can be better achieved by exploring a broad range of options rather than just trying to fit GR.
All these previous reasons (testing gravity at largest scales, the cosmological constant problem), combined with other issues appearing in cosmology (for example the Hubble tension [21]), led scientists to formulate and propose several kinds of alternative theories for gravity. The simplest possible approach would be changing the Einstein action: instead of it being linear to the Ricci scalar R, one could define an action which is a more complicate function of the curvature scalar, namely

$$S = \int d^4x \sqrt{-g} f(R). \tag{2.2}$$

This family of theories is generally referred to as $f(R)$ theories. Other possible roads would be adding extra dimensions (instead of standard 4-D space-time) or extra degrees of freedom. For a full summary of the most important alternative gravitational theory in cosmology, please look at [14]. In this work we choose to work in the latter setup, in particular we will consider Scalar-Tensor theories of gravity. Before doing so, however, we shall also introduce the concept of Galileons and covariant formalism

2.1 Galileons

With the discovery of the accelerated expansion, several families of theories were proposed to explain this phenomenon, each one with dozens of specific models. Still, there is no theoretical *a priori* reason to choose one rather than another given also the difficulties to put cosmological observational constraints. Because of this, a formalism to analyse large classes of modified gravity theories in a model independent way was created [48], called Galileon theory. In this theoretical frame, General Relativity is modified by adding an extra degree of freedom, in this case a single scalar field, the galileon, usually referred to as π, with derivative self-interaction. This family of theories, usually called scalar-tensor theories, is one of the most used and studied MG theory in cosmology. One of the main reasons is because their simple structure (compared to other more complicated theories) can grant us access to analytic solutions for a large number of phys-

ical systems. Moreover, the simple scalar field has been a very successful guinea pig in other context to then further construct and define theories in a more accurate way (like QFT, for example).
Nevertheless, we must keep in mind that this gravity modification should be valid at very large scale, as we still want to recover our standard theory at scales where GR has been extensively proved. In order to do so, the most straightforward way would be to define very weak couplings but other mechanisms (like the Chameleon mechanism) has been proposed in order to maintain valid the Equivalence Principle even at smaller scales [39].
With this in mind we shall present the Galileon formalism. If we neglect the presence of matter, the resulting Lagrangian of vacuum is invariant under the transformation of the galileon field

$$\pi \to \pi + b_\mu x^\mu + c. \tag{2.3}$$

Such invariance is just a generalization of the most common Galilean symmetry, therefore this is the property giving the name to the theory. We now want to construct an action with this new constituent. Considering the decoupling limit in a Minkowski background, such action will have the form

$$S = \int d^4x[\mathcal{L}_{GR} + \mathcal{L}_\pi], \tag{2.4}$$

with $\mathcal{L}_{GR}$ as the linearized Einstein action and

$$\mathcal{L}_\pi = \mathcal{L}_{Gal}(\pi, \partial_\mu\pi, \partial_\mu\partial_\nu\pi) + \pi T, \tag{2.5}$$

where T is the energy-density tensor of the Galileon. We require the vacuum part of $\mathcal{L}_\pi$, namely $\mathcal{L}_{Gal}$, to give second order equations of motion (in order to avoid ghost instabilities) and to be invariant under the Galilean symmetry. The last requirement means that if we perform the transformation (2.3), than the vacuum lagrangian will transform as

$$\mathcal{L}_{Gal} \to \mathcal{L}_{Gal} + \text{total derivatives}. \tag{2.6}$$

Such Lagrangian, on a 4-D manifold, will have the following form [20]

$$\mathcal{L}_{Gal}(\pi, \partial_\mu\pi, \partial_\mu\partial_\nu\pi) = \sum_{i=1}^{5} c_i\mathcal{L}_i(\pi, \partial_\mu\pi, \partial_\mu\partial_\nu\pi), \tag{2.7}$$

where c_i are constants and

$$\begin{aligned}
\mathcal{L}_1 &= \pi && (2.8)\\
\mathcal{L}_2 &= -\frac{1}{2}(\partial_\mu\pi)^2, && (2.9)\\
\mathcal{L}_3 &= -\frac{1}{2}\Box\pi(\partial_\mu\pi)^2, && (2.10)\\
\mathcal{L}_4 &= -\frac{1}{2}\left[(\Box\pi)^2-(\partial_\mu\partial_\nu\pi)^2\right](\partial_\mu\pi)^2, && (2.11)\\
\mathcal{L}_5 &= -\frac{1}{2}\left[(\Box\pi)^3-3\Box\pi(\partial_\mu\partial_\nu\pi)^2+2(\partial_\mu\partial_\nu\pi)^3\right], && (2.12)
\end{aligned}$$

where we define

$$\begin{aligned}
&\Box\pi = \eta^{\mu\nu}\partial_\mu\partial_\nu\pi,\\
&(\partial_\mu\pi)^2 = (\partial_\mu\pi)(\partial^\mu\pi),\\
&(\partial_\mu\partial_\nu\pi)^2 = (\partial_\mu\partial_\nu\pi)(\partial^\mu\partial^\nu\pi).
\end{aligned} \tag{2.13}$$

Such Lagrangian, however, has been defined on a Minkowski background and does not define a full covariant theory, which is what we would be interested for cosmological purposes. A covariant completion of eq. (2.5) has been found [20], whose action is

$$S[g_{\mu\nu},\pi] = \int d^4x\sqrt{-g}\,[R+\mathcal{L}^{cov}_{Gal}] + S_{matter}\left[\hat{g}_{\mu\nu},\psi_i\right], \tag{2.14}$$

where the first part is the standard Einstein action and the matter fields ψ_i are minimally coupled to the conformal metric $\hat{g}_{\mu\nu} = f(\pi)g_{\mu\nu}$ (the conformal factor f depends on the galileon field). It turns out that the covariant completion of $\mathcal{L}_{Gal}$ is given by $\mathcal{L}^{cov}_{Gal} = \sum_i c_i\mathcal{L}^{cov}_i$, with

$$\begin{aligned}
\mathcal{L}^{cov}_2 &= -\frac{1}{2}(\nabla_\mu\pi)^2, && (2.15)\\
\mathcal{L}^{cov}_3 &= -\frac{1}{2}\Box\pi(\nabla_\mu\pi)^2 && (2.16)\\
\mathcal{L}^{cov}_4 &= \frac{1}{2}\left[(\Box\pi)^2-(\nabla_\mu\nabla_\nu\pi)^2-\frac{1}{4}R\right](\nabla_\mu\pi)^2, && (2.17)\\
\mathcal{L}^{cov}_5 &= \frac{1}{2}[(\Box\pi)^3-3\Box\pi(\nabla_\mu\nabla_\nu\pi)^2+2(\nabla_\mu\nabla_\nu\pi)^3+\\
&\quad -6G_{\mu\nu}(\nabla^\mu\pi)(\nabla^\nu\nabla_\alpha\pi)(\nabla^\alpha\pi)](\nabla_\mu\pi)^2, && (2.18)
\end{aligned}$$

where $G_{\mu\nu}$ is the Einstein tensor, ∇_μ the covariant derivative , both defined with respect to the metric $g_{\mu\nu}$, and

$$\begin{aligned}\Box\pi &= g^{\mu\nu}\nabla_\mu\nabla_\nu\pi,\\ (\nabla_\mu\pi)^2 &= g^{\mu\nu}(\nabla_\mu\pi)(\nabla_\nu\pi),\\ (\nabla_\mu\nabla_\nu\pi)^2 &= g^{\mu\alpha}g^{\nu\beta}(\nabla_\mu\nabla_\nu\pi)(\nabla_\alpha\nabla_\beta\pi).\end{aligned} \tag{2.19}$$

$\mathcal{L}_4^{cov}$ and $\mathcal{L}_5^{cov}$ contain non-minimal coupling between gravity and the galileons: if we evaluate the EOMs from the most straightforward covariant completion of (2.7), we would find derivatives of order higher than two. This coupling is added in order to avoid such higher derivatives. However, after finding a covariant theory which is giving at most second order EOMs, we must notice the the Galilean symmetry is now broken [20].

2.2 Horndenski theories

In this section we will try, starting from the Galileon theory and its covariance completion, to extend even further this scalar-tensor to what is now generally accepted to be the most general theory with a single extra scalar field giving us second order equations of motion at most.
In order to be consistent with the mainstream notation, we shall now refer to the extra scalar field with the Greek letter ϕ. This choice will be kept from now on until the end of the thesis.
The following action is the starting point to setup the theory:

$$S[g_{\mu\nu},\phi] = \int d^4x\sqrt{-g}\left[\sum_{i=2}^{5}\mathcal{L}_i[g_{\mu\nu},\phi] + \mathcal{L}_m[g_{\mu\nu},\psi_i]\right], \tag{2.20}$$

where $\mathcal{L}_m$ is the matter Lagrangian described by the metric and the matter fields ψ_i. The Lagrangian densities will have the form

$$\mathcal{L}_2 = G_2(\phi, X), \tag{2.21}$$

$$\mathcal{L}_3 = G_3(\phi, X)\Box\phi, \tag{2.22}$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4,X}(\phi, X)\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)^2\right], \tag{2.23}$$

$$\begin{aligned}\mathcal{L}_5 = {} & G_5(\phi, X)G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \frac{1}{6}G_{5,X}[(\Box\phi)^3\\ & + 2(\nabla_\mu\nabla_\nu\phi)^3 - 3\Box\phi(\nabla_\mu\nabla_\nu\phi)^2],\end{aligned} \tag{2.24}$$

where R is the Ricci scalar, $G_{\mu\nu}$ is the Eisntein tensor, X is the kinetic energy associated with the scalar field,

$$X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \tag{2.25}$$

G_i are generic functions of the scalar field and its kinetic term and, in addition to this, $G_{i,X} = \partial G_i / \partial X$. The covariant derivatives, the Ricci scalar and the Einstein tensor are defined with respect to the metric $g_{\mu\nu}$ and notation used for Laplacian operators or terms like $(\nabla_\mu \nabla_\nu \phi)^2$ is the same as the ones defined in (2.19).
Such theory was firstly modeled by Horndenski in the '70 [33] without using the Galilean formalism, although only relatively recently it has been shown how Horndenski's theory was equivalent with what is going to presented in this section [40]. Therefore, this is the reason why we shall refer to it with the name Horndenski theory and, sometimes, in the literature the functions G_i are called Horndenski functions.
Again the non-minimal coupling in $\mathcal{L}_4$ and $\mathcal{L}_5$ are necessary to avoid terms with derivative order higher than two in the equations of motion. An important feature of this model is that we do not need a separate Lagrangian density for the gravitational part: if we set all the Horndenski functions to zero except for $\mathcal{L}_4 = M_{Pl}/2$, where M_{Pl} is the planck mass, one should recover the standard Einstein action.
In this thesis, we will apply the linear perturbation approach of GW presented in the previous chapter to the Horndenski action, rather than the GR standard one in order to study the evolution of the metric and scalar field perturbations.

Chapter 3

Scalar-Tensor Waves

3.1 Perturbed equations of motion

The starting point of this calculation is the Horndenski action presented in eq. (2.20), the most general 4-D theory of gravity leading to second order equations of motion whose action is constructed by the metric tensor and a scalar field. In order to find the equations of motion, one has to vary the action with respect to the metric tensor and the scalar field, in order to find

$$\delta\left(\sqrt{-g}\sum_{i=2}^{5}\mathcal{L}_i\right)=\sqrt{-g}\left(\sum_{i=2}^{5}\mathcal{G}^i_{\mu\nu}\delta g^{\mu\nu}+\sum_{i=2}^{5}\mathcal{F}_i\delta\phi\right). \tag{3.1}$$

The last equation is true up to total derivative terms, which can always be neglected after the use of Gauss theorem and setting boundary conditions at infinity. The equations of motion of the tensor and scalar field will respectively be

$$\sum_{i=2}^{5}\mathcal{G}^i_{\mu\nu}=0\ ;\ \sum_{i=2}^{5}\mathcal{F}^i=0. \tag{3.2}$$

For the explicit form of the EOMs in Horndenski theories please have a look at [40].
The following step will be the splitting of metric and the scalar field in a slowly changing background and a small perturbative part:

$$\begin{aligned} g_{\mu\nu}&=\bar{g}_{\mu\nu}+\epsilon h_{\mu\nu},\\ \phi&=\varphi+\epsilon\delta\phi, \end{aligned} \tag{3.3}$$

where ϵ is a small parameter.
Please be aware of the misleading notation in eqs. (3.1) and (3.3): in the

first case $\delta\phi$ refers to tha variation of the scalar field, while in the second it represents the perturbative part of ϕ after the splitting. From now on $\delta\phi$ will always have the latter interpretation.
Another useful shortcut notation is the directional derivative of the background scalar field φ, which from now on it will always be written as

$$v_\mu \equiv \partial_\mu \varphi. \tag{3.4}$$

After this splitting, one can find the perturbed equations of motion at the first order in ϵ, which can be formally written as

$$\left[\begin{pmatrix} \mathbb{K}^{(\phi)\alpha\beta}_{(\phi)} & \mathbb{K}^{\rho\sigma\alpha\beta}_{(\phi)} \\ \mathbb{K}^{(\phi)\alpha\beta}_{\mu\nu} & \mathbb{K}^{\rho\sigma\alpha\beta}_{\mu\nu} \end{pmatrix} \bar{\nabla}_\alpha \bar{\nabla}_\beta + \begin{pmatrix} \mathbb{A}^{(\phi)\alpha}_{(\phi)} & \mathbb{A}^{\rho\sigma\alpha}_{(\phi)} \\ \mathbb{A}^{(\phi)\alpha}_{\mu\nu} & \mathbb{A}^{\rho\sigma\alpha}_{\mu\nu} \end{pmatrix} \bar{\nabla}_\alpha + \begin{pmatrix} \mathbb{M}^{(\phi)}_{(\phi)} & \mathbb{M}^{\rho\sigma}_{(\phi)} \\ \mathbb{M}^{(\phi)}_{\mu\nu} & \mathbb{M}^{\rho\sigma}_{\mu\nu} \end{pmatrix}\right] \begin{pmatrix} \delta\phi \\ h_{\rho\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where each symbol $\mathbb{K}$, $\mathbb{A}$ and $\mathbb{M}$ can be seen as a 11×11 matrix (10 independents components from the 4-D symmetric matrices group, i.e. the tensor perturbation, and 1 from the scalar perturbation). The covariant derivatives are taken with respect to the background metric $\bar{g}_{\mu\nu}$, while the subscripts (or superscripts) (ϕ) are not used as the traditional Einstein's notation indices, but they are just to remind whether a matrix is related to the scalar field. The matrices K are usually called kinetic matrices, as they are related to second order derivatives of the fields, while the matrices A are defined as friction (or amplitude) matrices. Finally the matrices M are mass-like terms and they can be neglected in the optical limit, as it will be shown.
All the passages presented up to now can be performed using the Wolfram Mathematica Software [35]. As a first try only quartic theories will be considered (G_5=0) since they will already include non standard propagation speed for the tensor perturbation.
After perturbing the Einstein equation and the EOM of the scalar field, one can find the evolution equation for the GWs and the scalar perturbation. This process was performed thanks the help of the Mathematica suite of packages `xAct` [46] for tensorial computation, with a main use of the package `xPand` [12] for cosmological perturbation theory.
The full expressions consist of almost 400 terms for each equation, hence they will not transposed in this generic form for legibility.
Since in our case $K^{\rho\sigma\alpha\beta}_{(\phi)}$ and $K^{(\phi)\alpha\beta}_{\mu\nu}$ are not zero, we will find second order derivatives of the scalar perturbation in the Einstein perturbed equation of motion, and vice versa. This means that the perturbations defined before are not the true propagating degrees of freedom, hence a "diagonalization" of the EOMs and a redefinition of the variables is required. With diagonalization we mean in this specific situation that we want to move ta

basis where $K^{\rho\sigma\alpha\beta}_{(\phi)}$ and $K^{(\phi)\alpha\beta}_{\mu\nu}$ are zero. Hence the effect of this transformation on the equations of motion will be

$$\left[\begin{pmatrix} K^{\phi\alpha\beta}_{\phi} & 0 \\ 0 & K^{\rho\sigma\alpha\beta}_{\mu\nu} \end{pmatrix} \bar{\nabla}_\alpha \bar{\nabla}_\beta + \begin{pmatrix} A^{\phi\alpha}_{\phi} & A^{\rho\sigma\alpha}_{\phi} \\ A^{\phi\alpha}_{\mu\nu} & A^{\rho\sigma\alpha}_{\mu\nu} \end{pmatrix} \bar{\nabla}_\alpha + \begin{pmatrix} M^{\phi}_{\phi} & M^{\rho\sigma}_{\phi} \\ M^{\phi}_{\mu\nu} & M^{\rho\sigma}_{\mu\nu} \end{pmatrix}\right] \begin{pmatrix} \delta\phi \\ \gamma_{\rho\sigma} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.5}$$

where K, A, M are the resulting matrices from the process of diagonalizing the Equations of motion, $\delta\phi$ and $\gamma_{\mu\nu}$ are the new propagating degrees of freedom and, in particular, $\gamma_{\mu\nu}$ will be a combination of $h_{\mu\nu}$ and $\delta\phi$. For an example of the complete procedure in the case of a simpler theory, the reader can refer to [32].
Again, one can use the gauge freedom due to the invariance of $h^{\mu\nu}$ under a diffeomorphism to choose a certain gauge and simplify the equations. The harmonic (or De Donder) gauge

$$\bar{\nabla}_\mu h^{\mu\nu} = 0, \tag{3.6}$$

(the covariant derivative is taken with respect to the background metric $\bar{g}_{\mu\nu}$) already introduced in (1.11), will be used again. The harmonic gauge will not help too much in general, since it will get rid of ~ 20 terms for each equation, while in the eikonal approximation (formally explained in the following section) things might improve: one can notice that, if the harmonic gauge is imposed, then

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha h^{\mu\nu} = 0, \tag{3.7}$$

up to mass-like terms. Starting from the commutative properties of covariant derivatives, namely

$$\bar{\nabla}_\mu \bar{\nabla}_\alpha h^{\mu\nu} - \bar{\nabla}_\alpha \bar{\nabla}_\mu h^{\mu\nu} = \bar{R}^{\alpha}_{\gamma\alpha\mu} h^{\gamma\beta} + \bar{R}^{\beta}_{\gamma\alpha\mu} h^{\alpha\gamma}, \tag{3.8}$$

it is possible to use the harmonic gauge to eliminate the second term on the left-hand side of the equation, and notice than on the right-hand side there are only negligible (in the high-energy limit) mass-like terms. Therefore eq. (3.7) is true up to mass-like terms and, in principle, we are able to simplify the kinetic matrices.
Nevertheless, even with the use of these computational tricks, the situation does not improve significantly. Different gauge choices were tried in order to simplify more the equations, like for example $v_\mu h^{\mu\nu} = 0$, but with the same outcome.
Due to the incredibly high number of terms, it was not possible to perform

the diagonalization of the kinetic matrix and the redefinition of the variables, therefore another approach is required.
At the cost of loosing some generality, one could decide to choose a certain self-accelerating solution for the Horndenski function in order to simplify the EOMs. One of the simplest ansazt to achieve this family of solution is:

$$G_2 = c_2, \tag{3.9}$$

$$G_3 = c_3, \tag{3.10}$$

$$G_4 = c_4 M_p^2 X^2, \tag{3.11}$$

where c_2, c_3 and c_4 are constants, M_p is the Planck mass.
Still, even after choosing such solution, the EOMs do not simplify enough to perofrm the diagonalization. This is because the majority of terms are proportional to $G_{4,X}$ therefore selecting a quadratic solution for this function does not really reduce the number of terms.
It was decided to take a more phenomenological road, where the new independent degrees of freedom are left unknown as well as the exact form of the kinetic and amplitude matrices (mass-like matrices will not be used for calculations): the evolution of the perturbations will be studied by considering different parametrizations of the matrices.
Such kind of approach has already been developed and used for other purposes in the context of alternative gravitational theories: the presence of extra degrees of freedom might complicate the equations in way that is not possible to work with the full theory anymore. Moreover, working with parametrized functions from the full theory is more interfaceable with experiments, rather than just select a subset of your theory.
However, this road has never been walked in the framework of Gravitational Waves and we claim to do it for the first time since, as we showed in this section, considering the (almost) complete theory does not allow to work out all the steps and find the exact equations.

3.2 High-energy limit and eikonal approximation

Following the work introduced by Isaacson in 1967 [36], we set ourselves in the high-energy limit of the perturbations so it is possible to restrict to the eikonal (or WKB) approximation. This means that the tensor and scalar perturbations can be described by a wave ansatz and they will take the following form:

$$\begin{aligned} \gamma_{\mu\nu} &= \Gamma_{\mu\nu} e^{i\theta/\epsilon} = e_{\mu\nu} \Gamma e^{i\theta/\epsilon}, \\ \delta\phi &= \Xi e^{i\xi/\epsilon}, \end{aligned} \tag{3.12}$$

where $\Gamma_{\mu\nu}$ and Θ describe respectively the tensor and scalar amplitude, while θ and ξ represent the waves' phases and ϵ is a small parameter. The tensor amplitude can be additionally written as the product of a real function Γ and a polarization tensor $e_{\mu\nu}$.
In our set-up, the eikonal approximation means that the wavelengths involved are way smaller than the typical curvature radius of the background space-time.
We then define the wave-vectors of the tensor and scalar perturbations:

$$k_\mu = \partial_\mu \theta \ \ , \ \ q_\mu = \partial_\mu \xi. \tag{3.13}$$

Since ϵ is a small parameter, the phases will evolve much faster than the corresponding amplitudes. It seems straightforward to regroup the terms in the equations of motion in powers of ϵ and then solve those one by one. Terms up to $\mathcal{O}(\epsilon^{-2})$ will describe the dispersion relation of the waves, while terms up to $\mathcal{O}(\epsilon^{-1})$, which are smaller than the previous ones, will represent the amplitudes evolution. Finally, terms up to $\mathcal{O}(\epsilon^{0})$ are mass-like terms, that in the high-energy limit can be neglected.
The following step is evaluating the first and second order derivatives of the perturbations that enters in the equations of motion, and then regroup the terms in power of ϵ (or in powers of k_μ and q_μ). Note that thanks to the eikonal approximation this calculation is extremely simplified.
The first order derivatives take the following form

$$\begin{aligned} \bar{\nabla}_\alpha \gamma_{\mu\nu} &= (i\epsilon^{-1}\Gamma_{\mu\nu}k_\alpha + \bar{\nabla}_\alpha \Gamma_{\mu\nu})e^{i\theta/\epsilon}, \\ \bar{\nabla}_\alpha \delta\phi &= (i\epsilon^{-1}\Xi q_\alpha + \bar{\nabla}_\alpha \Xi)e^{i\xi/\epsilon}. \end{aligned} \tag{3.14}$$

We now evaluate the second order derivatives :

$$\begin{aligned} \bar{\nabla}_\alpha \bar{\nabla}_\beta \gamma_{\mu\nu} &= (-\epsilon^{-2}\Gamma_{\mu\nu}k_\alpha k_\beta + 2i\epsilon^{-1}k_{(\alpha}\bar{\nabla}_{\beta)}\Gamma_{\mu\nu} + \\ &\quad + i\epsilon^{-1}\Gamma_{\mu\nu}\bar{\nabla}_\beta k_\alpha + \bar{\nabla}_\alpha \bar{\nabla}_\beta \Gamma_{\mu\nu})e^{i\theta/\epsilon}, \end{aligned} \tag{3.15}$$

$$\begin{aligned} \bar{\nabla}_\alpha \bar{\nabla}_\beta \delta\phi &= (-\epsilon^{-2}\Xi q_\alpha q_\beta + 2i\epsilon^{-1}q_{(\alpha}\bar{\nabla}_{\beta)}\Xi + \\ &\quad + i\epsilon^{-1}\Xi\bar{\nabla}_\beta q_\alpha + \bar{\nabla}_\alpha \bar{\nabla}_\beta \Xi)e^{i\xi/\epsilon}. \end{aligned} \tag{3.16}$$

It is now possible to split the equations of motion at different orders of ϵ. Starting from the leading order, namely ϵ^{-2}, we can select these terms from equations (3.14) and (3.15) (actually only one term from the scalar and one from the tensor perturbation), then plug them in eq.(3.5). Therefore we will find the dispersion relations of our perturbations:

$$K_{\mu\nu}^{\rho\sigma\alpha\beta}\Gamma_{\rho\sigma}k_\alpha k_\beta = 0, \tag{3.17}$$

$$K_{(\phi)}^{(\phi)\alpha\beta}q_\alpha q_\beta = 0. \tag{3.18}$$

Due to the diagonalization of the kinetic matrices, at level ϵ^{-2} there is no interaction, hence the dispersion relation of the tensor perturbation is not affected by the scalar perturbation, and vice versa.
Moving to the order ϵ^{-1}, we can find:

$$K_{\mu\nu}^{\rho\sigma\alpha\beta}(2k_{(\alpha}\bar{\nabla}_{\beta)}\Gamma_{\rho\sigma}+\Gamma_{\rho\sigma}\bar{\nabla}_{\beta}k_{\alpha})+A_{\mu\nu}^{\rho\sigma\alpha}\Gamma_{\rho\sigma}k_{\alpha}+A_{\mu\nu}^{(\phi)\alpha}\Xi q_{\alpha}e^{i(\xi-\theta)/\epsilon}=0, \tag{3.19}$$

$$K_{(\phi)}^{(\phi)\alpha\beta}(2q_{(\alpha}\bar{\nabla}_{\beta)}\Xi+\Xi\bar{\nabla}_{\beta}q_{\alpha})+A_{(\phi)}^{(\phi)\alpha}\Xi q_{\alpha}+A_{(\phi)}^{\rho\sigma\alpha}\Gamma_{\rho\sigma}k_{\alpha}e^{i(\theta-\xi)/\epsilon}=0. \tag{3.20}$$

Please notice that, since the amplitude matrices are not diagonalized (in the sense explained before) at level ϵ^{-1} there could be in principle interaction between the scalar and tensor perturbation.
Our goal is to solve equations (3.18), (3.18), (3.19) and (3.20) in order to find the expressions for the wave-vectors and amplitudes. However, another problem arise: these equations contain covariant derivatives, taken with respect to the background metric. The cosmological principle, namely that the universe is isotropic and homogeneous (hence we can represent it with a FRW metric), only holds for the largest scales. In reality perturbations propagating to us might experience the effects of inhomogeneities, traditionally called Large-Scale Structures (LSS), departing from the cosmological principle. These are, as the name suggests, the largest gravitationally collapsed objects of our universe and they appear to be dominated by a matter component not represented by the Standard Model of Particle Physics [45], ususally referred to as Dark Matter [56].
As a result, the background metric we use to take covariant derivatives will deviate from the standard FRW metric, due to the presence of the LSS, in a non-trivial way, complicating the EOMs of the scalar and tensor perturbations, even in the high-energy approximation.
In order to solve the problem we shall appeal to perturbative methods that we will introduce and describe in the following chapter, firstly in General Relativity and in the frame of Modified Theories of Gravity.
As we will discuss better in the following chapter, such approach has never been used in the context of gravitational waves. We choose it due to the difficulties of working with the equations of a full theory and it is, in the spirit, similar to parametrization for LSS in modified gravity [49].

Chapter 4

Relativistic corrections

4.1 Relativistic Corrections in General Relativity

As already discussed before, one of the key feature of gravitational waves is that they, unlike Electromagnetic radiation, can provide a direct measurement of luminosity distance which is, at the end of the day, is probably the most important observable that we can infer from GWs for cosmological information.
Due to the smallness of gravitational cross-section, GWs can travel from the source to the observer without any effective absorption or scattering with the matter medium. However, as showed in the first section of this chapter for frequency and direction of propagation, they are still subject to relativistic effects caused by such structures present in our universe. It is crucial then, to study how such phenomena affect the GWs and, subsequently, the magnitude of this corrections. In order to do this, the cosmic rulers formalism [52] shall be used.
Some initial attempts have been performed to investigate the ISW (Integrated Sachs-Wolf) effect on GWs from supermassive black-holes mergers [41] and from stochastic background [15]. Other studies on the effect of lensing [16] or peculiar velocities [11] have also been done. Finally, it has also been tried to generalize all these studies in a formalism which will allow us to evaluate all the relativistic effects and their corrections to GWs observables starting from the cosmological perturbations [9]. While all these works have been performed in the frame of General Relativity, in this chapter we will try to use this formalism in order to extend these results in the frame of modified gravity.
It seems convenient tough to first give a summary of derivations of relativistic corrections on Gravitational Waves propagation in the regime of

GR. The formalism and calculations presented in the following section are the results of the work of Bertacca et al. [9], Laguna et al. [42]. We shall illustrate the key aspects in order to use later the same formalism applied in the context of scalar-tensor theories.

4.1.1 Linearized Einstein equations in high-energy limit

We start again by splitting our metric in a slowly changing background (in this set-up it will describe a FRW metric and first-order perturbations) and a GW metric perturbation:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{4.1}$$

In section 1.2, we derived the propagation equation for the GWs in a curved background: that result will be the starting point the following calculations. We will rewrite eq. (1.29) for the sake of clearness:

$$\bar{\Box}\bar{h}_{\mu\nu} + 2\bar{R}_{\mu\alpha\nu\beta}\bar{h}^{\alpha\beta} - \bar{R}_{\mu\alpha}\bar{h}^{\alpha}_{\nu} - \bar{R}_{\nu\alpha}\bar{h}^{\alpha}_{\mu} = 8\pi G\left[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right]^{H}, \tag{4.2}$$

where $\bar{h}_{\mu\nu}$ is the trace-reversed form of the GW perturbation already presented in eq. (1.26); while $\bar{\nabla}_{\mu}$, $\bar{R}_{\mu\alpha\nu\beta}$ and $\bar{R}_{\mu\nu}$ are respectively the covariant derivative, the Riemann tensor and the Ricci tensor taken with respect to the background metric $\bar{g}_{\mu\nu}$. In addition to this:

$$\bar{\Box} = \bar{g}^{\mu\nu}\bar{\nabla}_{\mu}\bar{\nabla}_{\nu}. \tag{4.3}$$

We acknowledge an overlapping in the notation, as $\bar{h}_{\mu\nu}$ is the trace-reversed form of the GW, while the bar symbol on any other objects just represents it is taken with respect to the background metric. However, this is the most used notation in the literature, hence we will continue with it.
Since we are interested in the propagation of Gravitational Waves on cosmological scales, we can neglect the energy stress tensor contribution and set $T_{\mu\nu} = 0$. This is not because we are in vacuum (on the contrary, we are looking for the effects of matter structures on GWs), but rather because we assume that matter does not have high-frequency perturbation modes (we are neglecting the response of matter background to the presence of GWs)[44]. This extremely simplifies eq. (4.2), and hence we can find

$$\bar{\Box}\bar{h}_{\mu\nu} + 2\bar{R}_{\gamma\mu\lambda\nu}\bar{h}^{\gamma\lambda} - \bar{R}_{\mu\alpha}\bar{h}^{\alpha}_{\nu} - \bar{R}_{\nu\alpha}\bar{h}^{\alpha}_{\mu} = 0. \tag{4.4}$$

Using the notation presented in the previous chapter, we can identify from eq. (4.4) the kinetic matrices (3.5), along with the amplitude and mass-like matrices. Since, of course, there is no scalar field, there will not be any scalar perturbation, hence for the kinetic matrices

$$K^{(\phi)\alpha\beta}_{(\phi)} = 0, \quad K^{\rho\sigma\alpha\beta}_{\mu\nu} = \delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu}\bar{g}^{\alpha\beta}. \tag{4.5}$$

Moving to first derivative order, we can see that there is no friction contribute in GR as all amplitude matrices are zero. Finally, the only non-trivial mass-like matrix is

$$M^{\rho\sigma}_{\mu\nu} = 2\bar{R}^{\rho\sigma}_{\mu\nu} - 2\bar{R}^{\rho}_{(\mu}\delta^{\sigma}_{\nu)}. \tag{4.6}$$

We will now impose the eikonal (or short-wave) approximation firstly introduced in this context by Isaacson in 1968 ([36]-[37]). Under this approximation, we claim that the typical wavelength of the GW perturbation is substantially smaller than the typical length scales involved (i.e. curvature of background spacetime). If we define a ϵ as the ration between the last two mentioned quantities, we will have

$$\epsilon \equiv \frac{\lambda}{L} \ll 1, \tag{4.7}$$

where λ is the typical wavelength and L is the typical length scale. Hence we can treat the problem using the tools of geometrical optics.
Therefore, we can write the GWs as

$$\bar{h}_{\mu\nu} = A_{\mu\nu}e^{i\theta/\epsilon} = e_{\mu\nu}\mathcal{A}e^{i\theta/\epsilon} = e_{\mu\nu}h, \tag{4.8}$$

where θ and $\mathcal{A}$ are real functions of retarded time and describe respectively the phase and the amplitude of the wave, while $e_{\mu\nu}$ is a polarization tensor and ϵ can be now considered a small parameter used to keep track of the order of perturbations. Plugging (4.8) into eq. (4.4), we can find

$$\begin{aligned}\epsilon^{-2}[-k_\alpha k^\alpha A_{\mu\nu}] + i\epsilon^{-1}[2k^\alpha\bar{\nabla}_\alpha A_{\mu\nu} + A_{\mu\nu}\bar{\nabla}_\beta k^\beta]+\\ + [\bar{\Box}A_{\mu\nu} + 2\bar{R}_{\gamma\mu\lambda\nu}A^{\gamma\lambda} - \bar{R}_{\mu\alpha}A^{\alpha}_{\nu} - \bar{R}_{\nu\alpha}A^{\alpha}_{\mu}] = 0,\end{aligned} \tag{4.9}$$

where $k_\mu = \partial_\mu\theta$ is the GW wave vector and we use the background metric to raise and lower indices. In order to be consistent with the notation, we will define:

$$\begin{aligned}\bar{k}_\mu &= k_\mu = \partial_\mu\theta,\\ \bar{k}^\mu &= \bar{g}^{\mu\nu}\bar{k}_\nu.\end{aligned} \tag{4.10}$$

Given the different orders of magnitude of each term, in order for this equation to be satisfied each part proportional to a different power of ϵ must be equal to zero, therefore we can treat and solve each power order independently. Terms proportional to ϵ^{-2} are the dominant ones and they define the dispersion relation of the wave, while the terms proportional to ϵ^{-1} will describe the evolution of the amplitude. Terms proportional to ϵ^0 are called mass-like and under the eikonal approximation can always be neglected as their contribution to the evolution equations will be way smaller than the higher order terms.
In order to simplify calculations, a conformal transformation is performed and hence we define the comoving metric $\hat{g}_{\mu\nu} = \bar{g}_{\mu\nu}/a^2$, where a is the scale factor. The wave-vector will transform as

$$\begin{aligned} \hat{k}_\mu &= k_\mu = \partial_\mu \theta, \\ \hat{k}^\mu &= \hat{g}^{\mu\nu}\bar{k}_\nu = a^2 \bar{k}_\nu. \end{aligned} \tag{4.11}$$

If we consider terms up to $\mathcal{O}(\epsilon^{-2})$, eq. (4.9) will give the dispersion relation

$$\hat{k}_\mu \hat{k}^\mu = 0, \tag{4.12}$$

and, after taking a derivative

$$\hat{\nabla}_\nu (\hat{k}^\mu \hat{k}_\mu) = 2\hat{k}^\mu \hat{\nabla}_\mu \hat{k}_\nu = 0, \tag{4.13}$$

where $\hat{\nabla}_\nu$ is the covariant derivative with respect to the covariant metric $\hat{g}_{\mu\nu}$ and we used that

$$\hat{\nabla}_\nu \hat{k}_\mu = \hat{\nabla}_\nu \partial_\mu \theta = \hat{\nabla}_\mu \partial_\nu \theta = \hat{\nabla}_\mu \hat{k}_\nu. \tag{4.14}$$

This means that, in General Relativity, in the eikonal approximation, GWs can be described effectively as particles propagating along rays which are null geodesics of the background spacetime with curves x^μ defined by null vectors $\hat{k}^\mu = dx^\mu/d\chi$, where χ is an affine parameter. Another property we can find from the dispersion relation is that

$$\hat{k}^\mu \hat{\nabla}_\mu e_{\alpha\beta} = 0, \tag{4.15}$$

i.e. the polarization tensor is parallel transported along null geodesics.
Now, switching considering $\mathcal{O}(\epsilon^{-1})$ terms, we can find

$$\frac{d}{d\chi} \ln(a\mathcal{A}) = -\frac{1}{2}\hat{\nabla}_\mu \hat{k}^\mu \tag{4.16}$$

where we used the following chain rule:

$$\hat{k}^\mu \hat{\nabla}_\mu = \frac{dx^\mu}{d\chi}\hat{\nabla}_\mu = \frac{d}{d\chi}. \tag{4.17}$$

Eq. (4.16) describes the evolution of the GW amplitude, and tells us that such amplitude will decrease as the rays diverge.

4.1.2 Mapping between Real and Redshift-frame

Now, we defined x^μ as the coordinates in real space while χ is the distance from the source to the detector, still in the real space. However, we perform our measurements in another frame, usually called "Redshift-GW" frame. We do so in order to take coordinates that will flatten the past GW-cone, hence the coordinates of a GW geodesic travelling to us from a distant source can be written in this frame as

$$\underline{x}^\mu = (\underline{\eta}, \underline{x}^i) = (\eta_0 - \underline{\chi}, \underline{\chi} n^i), \tag{4.18}$$

where $\underline{\eta}$ is conformal time (and η_0 is the conformal time at observation) while $\underline{n^i}$ is the observed direction of arrival from the sky of the Gravitational Wave or, alternatively, $n^i = \bar{x}^i/\bar{\chi}$. This procedure of the Redshift-frame mapping was already performed for electromagnetic waves in [38] and readapted for gravitational waves in [9].
Due to the presence of perturbations around the FRW Universe, there will be a displacement between the real-space and the redshift-space.
Due to the presence of LSS, the background metric will depart from a FRW description and it will also contain information about the structures of the universe. Assuming this deviations from FRW are small, we write the background standard metric as

$$\bar{g}_{\mu\nu} = a^2[\eta_{\mu\nu} + \delta\bar{g}_{\mu\nu}], \tag{4.19}$$

where a is the scale factor, $\eta_{\mu\nu}$ the Minkowski metric and $\delta\bar{g}_{\mu\nu}$ the perturbations due to the presence of LSS. This equation, after a conformal transformation will be

$$\hat{g}_{\mu\nu} = \eta_{\mu\nu} + \delta\hat{g}_{\mu\nu}. \tag{4.20}$$

It is important to notice that not only there will be a displacement between real and redshift-space in any physical quantity (which is function of space-time) but also in the affine parameter χ, so we can set up a mapping between the two frames in the following way:

$$\chi = \underline{\chi} + \delta\chi. \tag{4.21}$$

From now on, the underline symbol will be used to represent quantities at 0th order perturbation around FRW. Then, moving to spacetime coordinates

$$x^\mu(\chi) = (\underline{x}^\mu + \delta x^\mu)(\underline{\chi} + \delta\chi) = \underline{x}^\mu(\underline{\chi}) + \underline{x}^\mu(\delta\chi) + \delta x^\mu(\underline{\chi}), \tag{4.22}$$

up to second order terms. After Taylor expanding the second term and using

$$\underline{\hat{k}}^\mu = \frac{d\underline{x}^\mu}{d\underline{\chi}}, \tag{4.23}$$

we can write

$$x^\mu(\chi) = \underline{x}^\mu(\underline{\chi}) + \underline{\hat{k}}^\mu\delta\chi + \delta x^\mu(\underline{\chi}) = \underline{x}^\mu(\underline{\chi}) + \Delta x^\mu(\underline{\chi}), \tag{4.24}$$

where we have defined the Δ variation symbol which represent the combination of the displacement of the physical quantity itself and the effect of the displacement of the affine parameter on the physical quantity.

4.1.3 Perturbation of the wave-vector and geodesic equation

Repeating the splitting for $\hat{k}^\mu$, one can find

$$\hat{k}^\mu(\chi) = \frac{dx^\mu(\chi)}{\chi} = \frac{d\underline{\chi}}{d\chi}\frac{d(\underline{x}^\mu + \Delta x^\mu)}{d\underline{\chi}}, \tag{4.25}$$

where a chain rule and the result from eq. (4.24) were used. It follows that

$$\begin{aligned}\hat{k}^\mu(\chi) = \frac{d(\chi - \delta\chi)}{d\chi}\left[\frac{d\underline{x}^\mu}{d\underline{\chi}} + \frac{d(\underline{\hat{k}}^\mu\delta\chi)}{d\underline{\chi}} + \frac{d\delta x^\mu}{d\underline{\chi}}\right] = \\ = \left(1 - \frac{d\delta\chi}{\chi}\right)\left[\underline{\hat{k}}^\mu + \underline{\hat{k}}^\mu\frac{d\delta\chi}{\underline{\chi}} + \frac{d\delta x^\mu}{d\underline{\chi}}\right],\end{aligned} \tag{4.26}$$

since

$$\frac{d(\underline{\hat{k}}^\mu\delta\chi)}{d\underline{\chi}} = \underline{\hat{k}}^\mu\frac{d\delta\chi}{\underline{\chi}} + \frac{d\underline{\hat{k}}^\mu}{d\underline{\chi}}\delta\chi \tag{4.27}$$

and

$$\frac{d\underline{\hat{k}}^\mu}{d\underline{\chi}}\delta\chi = \underline{\hat{k}}^\nu\hat{\nabla}_\nu(\underline{\hat{k}}^\mu)\delta\chi = 0, \tag{4.28}$$

which is the unperturbed form of eq. (4.13).
Taking the first order of eq. (4.26), the wave-vector will become

$$\hat{k}^\mu(\chi) = \underline{\hat{k}}^\mu + \underline{\hat{k}}^\mu \frac{d\delta\chi}{d\chi} + \frac{d\delta x^\mu}{d\underline{\chi}} - \underline{\hat{k}}^\mu \frac{d\delta\chi}{d\underline{\chi}}, \tag{4.29}$$

and finally, after finding that, up to first order,

$$\frac{d\delta\chi}{d\chi} = \frac{d\underline{\chi}}{d\chi}\frac{d\delta\chi}{d\underline{\chi}} = \frac{d(\chi - \delta\chi)}{d\chi}\frac{d\delta\chi}{d\underline{\chi}} = \frac{d\delta\chi}{d\underline{\chi}}, \tag{4.30}$$

the wave-vector will be

$$\hat{k}^\mu(\chi) = \underline{\hat{k}}^\mu(\underline{\chi}) + \frac{d\delta x^\mu(\underline{\chi})}{d\underline{\chi}} = \underline{\hat{k}}^\mu + \delta\hat{k}^\mu, \tag{4.31}$$

where we have defined $\delta\hat{k}^\mu \equiv d\delta x^\mu / d\underline{\chi}$. Therefore, due to the definition of a tangent vector, the perturbed part of $\hat{k}^\mu$ is just $\delta\hat{k}^\mu$ rather than $\Delta\hat{k}^\mu$. This feature is not very straightforward, and creates a subtle difference between perturbing the wave-vector $\hat{k}^\mu$ or another generic physical quantity.
Hence, we define the null unperturbed geodesic vector in the redshift-frame as

$$\underline{\hat{k}}^\mu(\underline{\chi}) = \frac{d\underline{x}^\mu}{d\underline{\chi}} = (-1, n^i), \tag{4.32}$$

and the perturbed vector in the real-space evaluated at $\underline{\chi}$ as

$$\hat{k}^\mu(\underline{\chi}) = \frac{dx^\mu}{d\chi}(\underline{\chi}) = \frac{d}{d\underline{\chi}}(x^{\mu(0)} + \delta x^\mu)(\underline{\chi}) = (-1 + \delta\nu, n^i + \delta n^i)(\underline{\chi}). \tag{4.33}$$

The total displacement in time and space can be calculated by integrating eq. (4.33) over $\underline{\chi}$:

$$\delta x^0(\underline{\chi}) = \int_0^{\underline{\chi}} d\tilde{\chi}\delta\nu(\tilde{\chi})\,, \quad \delta x^i(\underline{\chi}) = \int_0^{\underline{\chi}} d\tilde{\chi}\delta n^i(\tilde{\chi}), \tag{4.34}$$

after setting the integrating constant to zero (or equivalently saying that $\delta x^\mu_{obs} = 0$).
It is useful to define the perpendicular and parallel projections along the observed line-of-sight direction. For a generic spatial tensor T_{ij}, its parallel projection will be

$$T_\parallel = n^i n^j A_{ij}, \tag{4.35}$$

while for a generic spatial vector V^i, its perpendicular projection will be

$$V^i_\perp = \mathcal{P}^{ij} V_j, \quad \mathcal{P}^i_j = \delta^i_j - n^i n_j. \tag{4.36}$$

From these equation it is possible to define the projected directional derivatives:

$$\partial_\parallel = n^i \partial_i, \quad \partial_{\perp i} = \mathcal{P}^j_i \partial_j, \quad \partial_i n^j = \frac{1}{\underline{\chi}} \mathcal{P}^j_i. \tag{4.37}$$

Starting from the null geodesic

$$\frac{d\hat{k}^\mu}{d\chi} + \hat{\Gamma}^\mu_{\alpha\beta} \hat{k}^\alpha \hat{k}^\beta = 0, \tag{4.38}$$

with $\hat{\Gamma}^\mu_{\alpha\beta}$ as the comoving Christoffel symbols, we can perturb it

$$\frac{d(\chi - \delta\chi)}{d\chi} \frac{d(\hat{\underline{k}}^\mu + \delta\hat{k}^\mu)}{d\underline{\chi}} + (\hat{\underline{\Gamma}}^\mu_{\alpha\beta} + \delta\hat{\Gamma}^\mu_{\alpha\beta})(\hat{\underline{k}}^\alpha + \delta\hat{k}^\alpha)(\hat{\underline{k}}^\beta + \delta\hat{k}^\beta) = 0, \tag{4.39}$$

where we used the same chain rule of eq. (4.25) for the first term, while $\hat{\underline{\Gamma}}^{(0)\mu}_{\alpha\beta}$ and $\delta\hat{\Gamma}^\mu_{\alpha\beta}$ are, respectively, the unperturbed and first order perturbed Christoffel symbols of $\hat{g}_{\mu\nu}$. After keeping only first order perturbation terms we find

$$\frac{d\delta\hat{k}^\mu}{d\underline{\chi}} - \frac{d\delta\chi}{d\chi} \frac{d\hat{\underline{k}}^\mu}{d\underline{\chi}} + \delta\hat{\Gamma}^\mu_{\alpha\beta} \hat{\underline{k}}^\alpha \hat{\underline{k}}^\beta + 2\hat{\underline{\Gamma}}^\mu_{\alpha\beta} \delta\hat{k}^\alpha \hat{\underline{k}}^\beta = 0. \tag{4.40}$$

The last term is equal to zero in our setup, since Christoffel symbols are always null in Minkowski metric, while for the second term some manipulation can be done in order to eliminate it. Using the following derivation rule,

$$\frac{d}{d\underline{\chi}} = -\partial_\eta + n^i \partial_i, \tag{4.41}$$

we can write

$$\frac{d\hat{\underline{k}}^\mu}{d\underline{\chi}} = -\partial_\eta \hat{\underline{k}}^\mu + n^i \partial_i \hat{\underline{k}}^\mu = n^i \partial_i n^j, \tag{4.42}$$

since no component of $\hat{\underline{k}}^\mu$ is time-dependent. Now, from the directional derivatives presented in eq. (4.37), we know that $\partial_i n^j = \frac{1}{\underline{\chi}} \mathcal{P}^j_i$, where $\mathcal{P}^j_i$ is the orthogonal projector. Therefore

$$\frac{d\hat{\underline{k}}^\mu}{d\underline{\chi}} = n^i \partial_i n^j = n^i \frac{1}{\underline{\chi}} \mathcal{P}^j_i = \frac{1}{\underline{\chi}} n^i (\delta^j_i - n^i n_j) = \frac{1}{\underline{\chi}} (n^i - n^i) = 0. \tag{4.43}$$

It follows that, up to linear order, we can write the null geodesic equation for the wave-vector as

$$\frac{d\delta\hat{k}^\mu}{d\underline{\chi}} + \delta\hat{\Gamma}^\mu_{\alpha\beta}\underline{\hat{k}}^\alpha\underline{\hat{k}}^\beta = 0, \tag{4.44}$$

4.1.4 Poisson gauge for background spacetime

Choosing the Poisson gauge, the comoving conformal line element will be

$$d\hat{s}^2 = -(1+2\Phi)d\eta^2 + \delta_{ij}(1-2\Psi)dx^i dx^j, \tag{4.45}$$

where Φ and Ψ are the scalar perturbations of the metric. The following step requires plugging all the explicit values in eq. (4.44) (see Appendix B.1), so the corrections to frequency and direction of propagation will be:

$$\frac{d}{d\underline{\chi}}(\delta\nu - 2\Phi) = \Phi' + \Psi' \ ; \ \frac{d}{d\underline{\chi}}(\delta n^i - 2\Psi n^i) = -\partial^i(\Phi+\Psi). \tag{4.46}$$

The total corrections can be found by integrating along the affine parameter and setting the values of $\delta\nu$ and δn^i in the observer frame:

$$\delta\nu = \delta\nu_{obs} + 2\Phi + \int_0^{\underline{\chi}} d\chi(\Phi' + \Psi'), \tag{4.47}$$

$$\delta n^i = \delta n^i_{obs} + 2n^i\Psi - \int_0^{\underline{\chi}} d\chi\partial^i(\Phi+\Psi). \tag{4.48}$$

We can recognize in (4.47) a contribution from the ISW effect, namely

$$I = \int_0^{\underline{\chi}} d\chi(\Phi' + \Psi'), \tag{4.49}$$

while the last term of (4.48) several relativistic effects: starting from it we can split a partial derivative in a parallel component (to direction of arrival) and a perpendicular one

$$\begin{aligned}\int_0^{\underline{\chi}} d\chi\partial^i(\Phi+\Psi) &= \int_0^{\underline{\chi}} d\chi\left[n^i n^j\partial_j + \partial^i_\perp\right](\Phi+\Psi) = \\ &= n^i\int_0^{\underline{\chi}} d\chi(\Phi'+\Psi') + n^i(\Psi+\Phi) + \int_0^{\underline{\chi}} d\chi\partial^i_\perp(\Phi+\Psi),\end{aligned} \tag{4.50}$$

where we used the the properties of directional derivatives (4.37) and $d/d\underline{\chi} = -\partial_\eta + n^i\partial_i$. Again, the first term is an ISW effect contribution, while if we integrate over $\underline{\chi}$ the last two terms (in order to find the total spatial displacement δx^i) we can find, repsectively, a time-delay and lensing contribution:

$$T = \int_0^{\underline{\chi}} d\chi(\Phi+\Psi), \quad \text{and} \quad L = \int_0^{\underline{\chi}} d\tilde{\chi}(\underline{\chi} - \tilde{\chi})\partial_i(\Phi+\Psi) \tag{4.51}$$

4.2 Frequency and direction of propagation in Horndenski theories

4.2.1 Introducing the parametric approach

In order to help the reader with the notation used in this section, we refer to table A.1 in the Appendix.
Moving back to scalar-tensor theories, we aim to extend the previous calculations of relativistic corrections to tensor perturbations with general propagation speed. This calculations have already been performed in the subcase where $c_T^2 = 1$ [30], so the goal is to study the effects that a different propagation might have.
The first effect of such generalization is the non-trivial transformation of the dispersion relation. We start from the tensorial dispersion relation in eq. (3.18) which, in the most general form, can be written as [27]

$$K_{\mu\nu}^{\rho\sigma\alpha\beta} = [K1]_{\mu\nu}^{\rho\sigma}\,\bar{g}^{\alpha\beta} + [K2]_{\mu\nu}^{\rho\sigma} v^\alpha v^\beta + [K3]_{\mu\nu}^{\rho\sigma} \bar{\nabla}^\alpha v^\beta, \tag{4.52}$$

where the form of the tensorial prefactors $K1$, $K2$ and $K3$ will depend on the choices of the Horndenski functions and the covariant derivative in the last term is related to the background metric $\bar{g}_{\mu\nu}$. In general relativity $[K1]_{\mu\nu}^{\rho\sigma} = \delta_\mu^{(\rho}\delta_\nu^{\sigma)}$, while all other prefactors are set to zero. For theories with $c_T^2 = 1$, $K1_{\mu\nu}^{\rho\sigma}$ can have a non-trivial form, however the other prefactors are still set to zero [17]. Finally, for quartic theories, $[K2]_{\mu\nu}^{\rho\sigma} \neq 0$ [10] and for quintic theories $[K3]_{\mu\nu}^{\rho\sigma} \neq 0$ [55].
Before entering into the details of the calculations, we shall recall the reasons that justified the use of a parametric approach. The expressions derived from the full theory appear in a very intricate form and they involve complex combinations of the parameters of the theories, which will not be fixed by experiments probably. For this reasons, it is interesting trying to already understand whether data indicates deviations from the standard theory, without necessarily pointing toward one specific model or another. To this end, parametrized approaches are better suited.
This kind of approach has already been intensively used for studying the LSS in the context of Modified Gravity, with the introduction of the parametric functions Σ and μ [49]. These functions were created with the goal to study the deviations from standard Einstein equations of LSS in GR.
Nevertheless, even though such approach has been used in the context of Gravitational Waves [8](or more correctly scalar-tensor perturbations), we will try to apply it for the first time in the context of relativistic corrections to scalar-tensor waves. Therefore, in this section we aim to use this

formalism to find the relativistic corrections to frequency and direction of propagation for tensor perturbations.
As a first simplified case, we consider only for quartic theories, hence $[K3]^{\rho\sigma}_{\mu\nu} = 0$. Recalling the values of kinetic matrices in General Relativity (see eq. (4.5)) and for standard tensorial perturbations propagation speed [17], it seems consistent to choose $K1^{\rho\sigma}_{\mu\nu}$ and $K2^{\rho\sigma}_{\mu\nu}$ such that they are $\propto \delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu}$. With this choice it is possible to factor out the amplitude tensor $\Gamma_{\rho\sigma}$, hence the dispersion relation will be independent of such quantity. Writing everything explicitly, we have:

$$\begin{aligned}
[K1]^{\rho\sigma}_{\mu\nu} &= A\,\delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu}, \\
[K2]^{\rho\sigma}_{\mu\nu} &= B\,\delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu}, \\
[K3]^{\rho\sigma}_{\mu\nu} &= 0,
\end{aligned} \tag{4.53}$$

where $A = A(\phi, v_\mu)$ and $B = B(\phi, v_\mu)$ are parametric functions that will depend on the scalar field, the kinetic term X and their derivatives, while their form is left generic. The dispersion relation will have the following form:

$$(A\bar{g}^{\alpha\beta} + Bv^\alpha v^\beta)\bar{k}_\alpha\bar{k}_\beta = 0, \tag{4.54}$$

where the wave-vectors $\bar{k}_\alpha$ have been defined in eq. (4.10) and the partial derivatives v_μ in eq. (3.4).
Due to the additional term $v^\alpha v^\beta$ and the generic form of A and B, the tensor perturbation will no longer follow null geodesics, hence in principle it will move on different paths than photons, giving us a powerful tool for multi-messengers observational cosmology, like for example in the physics of gravitational lensing: if the tensor perturbation has a EM counterpart, the lensing due to the presence of LSS could result in different paths for the two signals.

4.2.2 Dispersion relation in a general background metric

The most straightforward approach to find the relativistic corrections to the wave-vector $\hat{k}_\mu$ would be taking a covariant derivative with respect to the standard metric $\tilde{g}_{\mu\nu}$, repeat the cosmic rulers splitting which was shown in the previous section and then solve the differential equations for the corrections on the wave-vector. This method, however, will take to non-trivial differential equations that cannot be solved by simply integrating over the world-line parameter. Therefore we will take a different

approach, as shown later.
Before proceeding, we will perform again the comoving transformation already used in the previous section. Here is the list of all the used quantities after the conformal transformation:

$$\begin{aligned}\hat{g}_{\mu\nu} &= a^{-2}\bar{g}_{\mu\nu},\\ \hat{k}_\mu &= k_\mu = \partial_\mu\theta,\\ \hat{k}^\mu &= \hat{g}^{\mu\nu}\bar{k}_\nu = a^2\bar{k}_\mu,\\ \hat{v}_\mu &= v_\mu = \partial_\mu\varphi,\\ \hat{v}^\mu &= \hat{g}^{\mu\nu}\bar{v}_\nu = a^2\bar{v}_\mu.\end{aligned} \tag{4.55}$$

Under this transformation eq. (4.54) will change as follow:

$$\left(\hat{g}^{\alpha\beta} + \frac{C}{a^2}\hat{v}^\alpha\hat{v}^\beta\right)\hat{k}_\alpha\hat{k}_\beta = 0, \tag{4.56}$$

where we define

$$C = \frac{B}{A}. \tag{4.57}$$

Then, another possible road can be walked if one notices that the wave-vectors are still null vectors with respect to a new "effective metric"

$$\breve{g}_{\mu\nu} \equiv \hat{g}_{\mu\nu} + C\hat{v}_\mu\hat{v}_\nu. \tag{4.58}$$

This operation has the mathematical form of a disformal transformation [5], a generalization of the more common conformal transformation widely used in cosmology. In this context, the function A is the conformal factor, while B is the disformal factor.
A new rule for raising and lowering indices is defined:

$$\begin{aligned}\breve{k}_\mu &= k_\mu = \partial_\mu\theta,\\ \breve{k}^\mu &= \breve{g}^{\mu\nu}\breve{k}_\nu.\end{aligned} \tag{4.59}$$

Moreover,

$$\breve{k}^\mu = \frac{dx^\mu}{d\breve{\chi}}, \tag{4.60}$$

where $\breve{\chi}$ is a another world-line parameter. Thanks to this new effective metric we settled again in a null geodesic, therefore it is possible to repeat all the steps performed in the previous section in the GR frame, with just a redefinition of covariant derivatives. Starting from

$$\breve{g}^{\mu\nu}\breve{k}_\mu\breve{k}_\nu = (\hat{g}_{\mu\nu} + \frac{C}{a^2}\hat{\vartheta}_\mu\hat{\vartheta}_\nu)\breve{k}^\mu\breve{k}^\nu = 0, \tag{4.61}$$

we take a covariant derivative in order to find the trajectory, namely

$$\breve{\nabla}_\nu\left[\breve{k}^\mu\breve{k}_\mu\right] = 0, \tag{4.62}$$

where $\breve{\nabla}_\mu$ is the covariant derivative with the respect to the effective metric. Again, this last equation can be rewritten as

$$\breve{k}^\mu\breve{\nabla}_\mu\breve{k}^\nu = 0, \tag{4.63}$$

since $k_\mu = \partial_\mu\theta$ and derivatives of scalar functions up to second order can commute. We follow the same procedure as before, since $\breve{k}_\mu$ describes a null geodesic with respect to the effective metric, hence we find the following equation:

$$\frac{d\breve{k}^\nu}{d\breve{\chi}} + \breve{\Gamma}^\nu_{\mu\lambda}\breve{k}^\mu\breve{k}^\lambda = 0, \tag{4.64}$$

where we used $\breve{k}^\mu\partial_\mu = \dfrac{d}{d\breve{\chi}}$, and $\breve{\Gamma}^\nu_{\mu\lambda}$ represents the Christoffel symbols taken with respect to the effective metric $\breve{g}_{\mu\nu}$. We can now see that $\breve{k}^\mu$ is null geodesic with respect to the new effective metric with $\breve{\chi}$ as its affine parameter.

A perturbation around an homogeneous and isotropic universe is taken into account as before, therefore we can split all quantities into an unperturbed part and a perturbed one. Starting from the standard background metric, we have

$$\hat{g}_{\alpha\beta} = \underline{\hat{g}}_{\alpha\beta} + \delta g_{\alpha\beta}, \tag{4.65}$$

where $\underline{\hat{g}}_{\alpha\beta}$ is the Minkowski metric and $\delta g_{\alpha\beta}$ is the perturbation of the metric due to the LSS.

Furthermore, also the world-line parameter $\breve{\chi}$ will be perturbed and, accordingly, all functions of space-time will be affected by the displacement between observed and real frame in the same way as it has been shown previously:

$$\breve{k}^{\mu}(\breve{\chi}) = \underline{\breve{k}}^{\mu}(\underline{\breve{\chi}}) + \delta \breve{k}^{\mu}(\underline{\breve{\chi}}) \tag{4.66}$$

$$\hat{v}_{\mu}(x^{\nu}) = \underline{\hat{v}}_{\mu} + \Delta \hat{v}_{\mu}, \tag{4.67}$$

$$C(x^{\nu}) = \underline{C} + \Delta C, \tag{4.68}$$

$$a(x^{\nu}) = \underline{a} + \Delta a, \tag{4.69}$$

$$\breve{\chi} = \underline{\breve{\chi}} + \delta \breve{\chi}. \tag{4.70}$$

where

$$\begin{aligned}
\Delta \hat{v}_{\mu} &= \delta v_{\mu}(\underline{x}^{\nu}) + (\underline{\breve{k}}^{\alpha} \delta \breve{\chi} + \delta x^{\alpha}) \breve{\nabla}_{\alpha} \underline{\hat{v}}_{\mu}, \\
\Delta C &= \delta C(\underline{x}^{\nu}) + (\underline{\breve{k}}^{\alpha} \delta \breve{\chi} + \delta x^{\alpha}) \breve{\nabla}_{\alpha} C, \\
\Delta a &= \delta a(\underline{x}^{\nu}) + (\underline{\breve{k}}^{\alpha} \delta \breve{\chi} + \delta x^{\alpha}) \breve{\nabla}_{\alpha} a.
\end{aligned} \tag{4.71}$$

We can see that the correction of $\breve{k}^{\mu}(\breve{\chi})$ is denoted by the δ symbol, rather than Δ like the other quantities. This is because we can do all the steps previously done in the case of GR.
Similarly as done before, we make explicit the splitting of the wave-vector, as

$$\breve{k}^{\mu} = (-1 + \delta \mathcal{K}^0, n^i + \delta \mathcal{K}^i)(\underline{\breve{\chi}}), \tag{4.72}$$

where the $\delta \mathcal{K}^{\mu}$ are not associated with any physical interpretation, but are just a mathematical tool to find the relativistic corrections.
Finally, perturbing eq. (4.64), we can evaluate the relativistic corrections to the wave-vector, starting from the perturbed geodesic equation, namely

$$\frac{d\delta \mathcal{K}^{\nu}}{d\underline{\breve{\chi}}} + \delta \breve{\Gamma}^{\nu}_{\mu\lambda} \underline{\breve{k}}^{\mu} \underline{\breve{k}}^{\lambda} + 2 \underline{\breve{\Gamma}}^{\nu}_{\mu\lambda} \underline{\breve{k}}^{\mu} \delta \mathcal{K}^{\lambda} = 0. \tag{4.73}$$

4.2.3 Dispersion relation in Poisson Gauge

We will choose again the Poisson gauge, while only considering scalar perturbations to the background metric in the same way we did for the calculations in GR.
Compatibly with the symmetries of the background metric, also the background scalar field will have a profile which is described by a time dependent function plus small corrections of the same order of the LSS, namely

$$\varphi = \underline{\phi}(\eta) + \delta \phi, \tag{4.74}$$

where, for sake of clearness, we shall explicitly define each term: φ is the background scalar field (i.e. it is the background part of the scalar perturbation and it contains the information on the FRW+LSS background), $\underline{\phi}$ is the unperturbed background scalar field (i.e. only at level of a FRW universe) and $\delta\phi$ accounts for the corrections of the scalar field due to the presence of LSS. Please note that only $\underline{\phi}$ is a funtcion of time, while the perturbed background still depends on all space-time coordinates.
Therefore, the gradient of the scalar field, v_μ, takes the form

$$v_\mu = \partial_\mu(\underline{\phi}(\eta) + \delta\phi) = (\underline{v}_0 + \Delta v_0, \Delta v_i)(\underline{\breve{\chi}}), \tag{4.75}$$

and, using (4.55), one can find

$$\hat{v}_\mu = (\hat{\underline{v}}_0 + \Delta\hat{v}_0, \Delta\hat{v}_i)(\underline{\breve{\chi}}). \tag{4.76}$$

It follows that also $\underline{C}$ will not be space dependent, as C is a parametric function constructed with φ and its derivatives.
Considering the effective metric, one shall find in the Poisson gauge the following

$$\breve{g}_{\mu\nu} = \underline{\breve{g}}_{\mu\nu} + \delta\breve{g}_{\mu\nu}, \tag{4.77}$$

where the unperturbed part, after the approximation of eq. (4.76), is

$$\underline{\breve{g}}_{\mu\nu} = \eta_{\mu\nu} + \frac{\underline{C}}{\bar{a}^2}\hat{\underline{v}}_0\hat{\underline{v}}_0\delta_{\mu 0}\delta_{\nu 0} = \begin{pmatrix} -1 + \frac{\underline{C}}{\underline{a}^2}\hat{\underline{v}}_0^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.78}$$

Please notice that the background of the unperturbed metric is not simple Minkowski anymore, but there is an additional term in the 00-th component that will only depend on the time coordinate: all its components, namely $\underline{C}$, $\underline{v}_0$ and $\underline{a}$, are only time-dependent.
Since now the unperturbed metric has the form of a diagonal matrix, its inverse can be easily calculated:

$$\underline{\breve{g}}^{\mu\nu} = \begin{pmatrix} \left(-1 + \frac{\underline{C}}{\underline{a}^2}\hat{\underline{v}}_0^2\right)^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.79}$$

From a mathematical point of view, eq. (4.73) is formally the same kind of differential equation of eq. (4.44) however, in the comoving effective metric, the unperturbed Christoffel symbols are not always equal to zero: the unperturbed background is not Minkowski anymore, due to the presence of the extra time dependent term in the 00-th component. It follows that this new differential equation can not be solved just by integrating over $\breve{\chi}$. Now, looking to the perturbed effective metric, it will have two contributions:

$$\delta \breve{g}_{\mu\nu} = \delta \hat{g}_{\mu\nu} + U_{\mu\nu}, \tag{4.80}$$

where $\delta \hat{g}_{\mu\nu}$ is the perturbation around the FRW universe due to the Large-scale Structures in the conformal frame and

$$U_{\mu\nu} = \Delta \left[\frac{C}{a^2} \hat{v}_\mu \hat{v}_\nu \right], \tag{4.81}$$

is a new defined tensor describing the perturbative part of the disformal transformation. Focusing on the first term, $\delta \hat{g}_{\mu\nu}$, we will again only consider scalar perturbations of the metric, hence in the Poisson gauge it will have the same form used in the context of GR (section B.2 in Appendix). Moving to $U_{\mu\nu}$, we can evaluate the specific components of this tensor, that is

$$\begin{aligned} U_{00} &= \Delta \left(\frac{C}{a^2} v_0 v_0 \right) = \Delta \left(\frac{C}{a^2} \right) \underline{\hat{v}}_0^2 + 2 \, \frac{\underline{C}}{\underline{a}^2} \, \underline{\hat{v}}_0 \, \Delta \left(\hat{v}_0 \right), \\ U_{0i} &= \Delta \left(\frac{C}{a^2} v_0 v_i \right) = \frac{\underline{C}}{\underline{a}^2} \, \underline{\hat{v}}_0 \, \Delta (v_i), \\ U_{ij} &= \Delta \left(\frac{C}{a^2} v_i v_j \right) = 0, \end{aligned} \tag{4.82}$$

where we used eq. (4.76) to further simplify the expressions. Since we started our calculations from ϕ, it seams useful to rewrite the last equations in terms of the background scalar field:

$$\begin{aligned} U_{00} &= \Delta \left(\frac{C}{a^2} \right) (\underline{\phi}')^2 + 2 \frac{\underline{C}}{\underline{a}^2} \underline{\phi}' \Delta \phi', \\ U_{0i} &= \frac{\underline{C}}{\underline{a}^2} \underline{\phi}' \Delta (\partial_i \phi), \\ U_{ij} &= 0, \end{aligned} \tag{4.83}$$

where $\underline{\phi}$ is the background scalar field at 0-th order perturbation around FRW.

Combining equations (4.45) and (4.82) together, one can find the explicit expression of the perturbed effective metric in Poisson gauge:

$$\delta \breve{g}_{\mu\nu} = \begin{pmatrix} U_{00} - 2\Phi & U_{01} & U_{02} & U_{03} \\ U_{10} & -2\Psi & 0 & 0 \\ U_{20} & 0 & -2\Psi & 0 \\ U_{30} & 0 & 0 & -2\Psi \end{pmatrix}. \tag{4.84}$$

The final step requires finding the Christoffel symbols for the effective metric: we can perform the same splitting so we will have

$$\breve{\Gamma}^{\lambda}_{\mu\nu} = \underline{\breve{\Gamma}}^{\lambda}_{\mu\nu} + \delta\breve{\Gamma}^{\lambda}_{\mu\nu}. \tag{4.85}$$

We can notice that in the previous section the unperturbed Christoffel symbols were always zero, but this does not hold anymore. However, due to the simple form of eq. (4.78), the only non-trivial component will be

$$\underline{\breve{\Gamma}}^{0}_{00} = \frac{\underline{D}'}{2(\underline{D}-1)} \qquad \text{with} \qquad \underline{D} \equiv \frac{\underline{C}}{\underline{a}^2}\underline{v}_0^2. \tag{4.86}$$

The function $\underline{D}$ has been introduced to keep expressions shorter and the underline is just implemented to keep track that this object is constructed with just unperturbed background quantities.
For the perturbed effective Christoffel symbols, we can use the following formula to evaluate them:

$$\delta\breve{\Gamma}^{\rho}_{\mu\nu} = \frac{1}{2}\breve{g}^{\rho\lambda}\left(\breve{\nabla}_{\mu}\delta\breve{g}_{\nu\lambda} + \breve{\nabla}_{\nu}\breve{g}_{\mu\lambda} - \breve{\nabla}_{\lambda}\breve{g}_{\mu\nu}\right), \tag{4.87}$$

for the complete derivation have a look at Appendix B.
Plugging the explicit values in eq. (4.73), we find:

$$\frac{d\delta\mathcal{K}^{\nu}}{d\underline{\breve{\chi}}} + \delta\breve{\Gamma}^{\nu}_{\mu\lambda}\underline{\breve{k}}^{\mu}\underline{\breve{k}}^{\lambda} + \frac{\underline{D}'}{(1-\underline{D})}\delta\breve{k}^{\nu}\delta_{\nu 0} = 0, \tag{4.88}$$

resulting in the following system of differential equations:

$$\frac{d\delta\mathcal{K}^{0}}{d\underline{\breve{\chi}}} + \delta\breve{\Gamma}^{0}_{\mu\lambda}\underline{\breve{k}}^{\mu}\underline{\breve{k}}^{\lambda} + \frac{\underline{D}'}{(1-\underline{D})}\delta\mathcal{K}^{0} = 0, \tag{4.89}$$

$$\frac{d\delta\mathcal{K}^{i}}{d\underline{\breve{\chi}}} + \delta\breve{\Gamma}^{i}_{\mu\lambda}\underline{\breve{k}}^{\mu}\underline{\breve{k}}^{\lambda} = 0, \tag{4.90}$$

where the values of $\underline{\breve{k}}^{\mu}$ are defined in eq. (4.72) and the perturbed Christoffel symbols are given in Appendix B.

4.2.4 Expanding around de Sitter universe

The spatial equation can be easily solved by integrating over the affine parameter. However, the first equation takes the form of an inhomogeneous differential equation, which, without knowing the exact form of the sources, gives rise to several difficulties in finding an exact solution.
A possible approach to avoid the mathematical issues is to perturb around a de Sitter Universe, because with this approximation, as we will show, the scalar field is constant so that $\underline{D} = 0$ and the term in the equation (4.90) making it non-homogeneous vanishes.
Starting from our Horndenski action (2.20), we want to recover a pure de Sitter solution: when the action takes the following form, that is

$$S = \int d^4x \sqrt{-g}\, [R + \Lambda] \,, \tag{4.91}$$

where R is the Ricci scalar (also present in the Einstein action of GR) and Λ is a cosmological constant., one can recover a pure de Sitter solution.
This is can be achieved by approximating φ as constant field. Indeed, if we plug this assumptions in (2.20), one shall find

$$S = \int d^4x \sqrt{-g}\, [G_4 R + G_2] \,, \tag{4.92}$$

where now G_2 and G_4 are just constant values and G_2 in particular plays the role of the cosmological constant. Additionally, one would have to fine-tune $G_4 \approx 1$ to completely recover eq. (4.91).
As already mentioned, from the observations of SN1A [51] and Baryon acoustic oscillations (BAO) [26], we know that the universe is in a phase of accelerated expansion which must be close to de Sitter solution. For this reason we decide to to solve our eq. (4.90) in a perturbative way by considering that the homogeneous and isotropic part of the scalar field, $\underline{\phi}$, drives an almost de Sitter expansion. In light of the arguments given before, this means considering $\underline{\phi}$ almost constant. To this end, we introduce the slow-roll parameter α and write the background scalar field around FRW as

$$\underline{\phi}(\eta) = \underline{\phi}|_{deSitter} + \alpha \underline{\phi}^{(1)}(\eta). \tag{4.93}$$

As a result, $\underline{\hat{v}}^0 \sim 0$ and hence $\underline{D} \sim 0$, meaning that we can use perform another perturbative approach. We start by expanding $\underline{D}$ and $\delta\mathcal{K}^0$ around a de Sitter universe, cutting the expansion at first order:

$$\begin{aligned} \underline{D} &= 0 + \alpha \underline{D}_{(1)} + \mathcal{O}(\alpha^2), \\ \delta\mathcal{K}^0 &= \delta\mathcal{K}^0_{(0)} + \alpha \delta\mathcal{K}^0_{(1)} + \mathcal{O}(\alpha^2), \end{aligned} \tag{4.94}$$

The cosmological constant is one of the building blocks of the ΛCDM model, which is at the moment the standard theoretical description of our observations [3], meaning that, an approximate level, our modified theory of gravity must converge to the action (4.91). As a result, at unperturbed order, the background scalar field behaves a constant field and therefore, at a this level, the contribution of $\underline{D}$ to our EOMs can be neglected, meaning that $\underline{D}_{(0)} \approx 0$.
We can notice that, at $\alpha = 0$, eq. (4.90) can be easily solved by integrating over $\breve{\chi}$ to find $\delta\mathcal{K}^0_{(0)}$, then, moving to $\alpha = 1$, we will use $\delta\mathcal{K}^0_{(0)}$ to iteratively solve the differential equation, up to linear order in α.
Using this procedure, we can split eq. (4.90) into a 0-th and first order equations:

$$\alpha \;=\; 0 \qquad \rightarrow \qquad \frac{d\delta\mathcal{K}^0_{(0)}}{d\underline{\breve{\chi}}} + \delta\breve{\Gamma}^0_{\mu\lambda(0)}\underline{\breve{k}}^\mu\,\underline{\breve{k}}^\lambda = 0, \tag{4.95}$$

$$\alpha \;=\; 1 \qquad \rightarrow \qquad \frac{d\delta\mathcal{K}^0_{(1)}}{d\underline{\breve{\chi}}} + \delta\breve{\Gamma}^0_{\mu\lambda(1)}\underline{\breve{k}}^\mu\,\underline{\breve{k}}^\lambda + \underline{D}'_{(1)}\delta\mathcal{K}^0 = 0, \tag{4.96}$$

where we used that

$$\left[\frac{\underline{D}'}{1-\underline{D}}\right]_{(1)} \approx \left[\alpha\underline{D}'_{(1)}(1+\alpha\underline{D}_{(1)})\right] \approx \alpha\underline{D}'_{(1)}, \tag{4.97}$$

up to first order perturbation in α, while the Christoffel symbols are also split around a de Sitter universe, since they contain $\underline{D}$. Therefore $\delta\breve{\Gamma}^0_{\mu\lambda(0)}$ and $\delta\breve{\Gamma}^0_{\mu\lambda(1)}$ are respectively the 0-th order and first order parts of the perturbed Christoffel symbols (around a FRW universe) in the α expansion.
The 0-th order equation can easily be solved by integrating over $\underline{\breve{\chi}}$ from emission to observation and then, after plugging the unperturbed solution into eq. (4.96), the first order perturbed relativistic correction can be found.
Plugging the explicit values of the Christoffel symbols (see Appendix B), it is possible to find the corrections $\delta\mathcal{K}^\mu$. For $\mu = 0$, at order $\alpha = 0$, we start from eq. (4.95) and, after writing the Christoffel symbols (B.14), one will find

$$\frac{d}{d\underline{\breve{\chi}}}[\delta\mathcal{K}^0_{(0)} - 2\Phi + U_{00} - U_{0i}n^i] = \Phi' + \Psi' - \frac{1}{2}U'_{00} + U'_{0i}n^i, \tag{4.98}$$

therefore, after the integration, the correction at 0-th order will be

$$\delta\mathcal{K}^0_{(0)} = 2\Phi - U_{00} + U_{0i}n^i + \int d\bar{\chi}[\Phi' + \Psi' - \frac{1}{2}U'_{00} + U'_{0i}n^i], \tag{4.99}$$

up to an integration constant. Moving to $\alpha = 1$ order, the differential equation (4.96) will have the form

$$\begin{aligned}\frac{d\delta\mathcal{K}^0_{(1)}}{d\breve{\underline{\chi}}} + \underline{D}_{(1)}\frac{d}{d\breve{\underline{\chi}}}[-2\Phi + U_{00} - U_{0i}n^i] = \underline{D}'_{(1)}(-\delta\mathcal{K}^0_{(0)} + \frac{1}{2}U_{00}) + \\ + \underline{D}_{(1)}(\Phi' + \Psi' - \frac{1}{2}U'_{00} + U'_{0i}n^i),\end{aligned} \tag{4.100}$$

where $\delta\mathcal{K}^0_{(0)}$ is the 0-th order solution written in eq. (4.99). However, before integrating over the affine parameter as before, we perform the following integration by part:

$$\begin{aligned}\underline{D}_{(1)}\frac{d}{d\breve{\underline{\chi}}}[-2\Phi + U_{00} - U_{0i}n^i] = \frac{d}{d\breve{\underline{\chi}}}[\underline{D}_{(1)}(-2\Phi + U_{00} - U_{0i}n^i)] + \\ - [-2\Phi + U_{00} - U_{0i}n^i]\frac{d}{d\breve{\underline{\chi}}}\underline{D}_{(1)}\end{aligned} \tag{4.101}$$

now, using $\frac{d}{d\breve{\underline{\chi}}} = -\frac{d}{d\eta} + n^i\partial_i$, the last term can be rearranged since $\underline{D} = \frac{\underline{C}}{\underline{a}^2}\underline{v}_0^2$ is just a function of time ($\underline{v}_0$, $\underline{C}$ and $\underline{a}$ do not have any spacial dependency), hence

$$[-2\Phi + U_{00} - U_{0i}n^i]\frac{d}{d\breve{\underline{\chi}}}\underline{D}_{(1)} = -\underline{D}'_{(1)}[-2\Phi + U_{00} - U_{0i}n^i]. \tag{4.102}$$

Combining everything together and integrating over $\breve{\underline{\chi}}$, one can find

$$\begin{aligned}\delta\mathcal{K}^0_{(1)} = \underline{D}_{(1)}(-2\Phi + U_{00} - U_{0i}n^i) + \int d\breve{\underline{\chi}}(-\delta\hat{k}^0_{(0)} - \frac{1}{2}U_{00} + 2\Phi + \\ + U_{0i}n^i)\underline{D}'_{(1)} + \int d\breve{\underline{\chi}}(\Phi' + \Psi' - \frac{1}{2}U'_{00} + U'_{0i}n^i)\underline{D}_{(1)},\end{aligned} \tag{4.103}$$

up to a integration constant.
Keep in mind that this is an approximate solution around a de Sitter universe, where we ended the expansion at the first order.

Moving to the spatial corrections (4.90), we can solve the differential equation just by integrating over $\underline{\breve{\chi}}$ due to its simpler form. Therefore, there is no need to perturb around de Sitter and an exact and complete solution can be found starting from:

$$\frac{d}{d\underline{\breve{\chi}}}[\delta\mathcal{K}^i - \breve{U}_0^i + 2n^i\Psi] = -\breve{\partial}^i(\Phi + \Psi - U_{00} + U_{0j}n^j) - \frac{1}{2}\frac{\bar{D}'}{1-\bar{D}}\breve{U}_0^i, \quad (4.104)$$

where we used the effective metric to raise indices, hence $\breve{U}^\mu_\nu = \underline{\breve{g}}^{\mu\alpha}U_{\alpha\nu}$ and $\breve{\partial}^i = \underline{\breve{g}}^{i\alpha}\partial_\alpha$, we can easily integrate and find

$$\delta\mathcal{K}^i = \breve{U}_0^i - 2n^i\Psi + \int d\underline{\breve{\chi}}\left[-\breve{\partial}^i(\Phi + \Psi - U_{00} + U_{0j}n^j) - \frac{1}{2}\frac{\bar{D}'}{1-\bar{D}}\breve{U}_0^i\right], \quad (4.105)$$

up to an integration constant.
It is important to remind that the quantities found above are mostly a mathematical tool and their physical interpretation may not be too straightforward. In order to recover the standard corrections to frequency and direction of propagation, one must track back and work out the lowering and raising of the indices.
We recall the comoving wave-vector $\hat{k}^\mu$ defined in the previous section at eq. (4.11), which is the true quantity we are interested into and from that we can infer the physically meangingful relativistic corrections. Therefore, we perform again the same splitting in eq. (4.33), so that now we can write:

$$\delta\mathcal{K}^\mu = \underline{\breve{g}}^{\mu\nu}\delta k_\nu = \underline{\breve{g}}^{\mu\nu}\delta\hat{k}^\alpha\underline{\hat{g}}_{\alpha\nu}, \quad (4.106)$$

which, after writing the explicit entries of the standard and effective metric can give the relations between $\delta\hat{k}^\mu$ and $\delta\mathcal{K}^\mu$.
Considering $\nu = i$ for eq. (4.88), the relation turns out to be just the identity:

$$\delta\mathcal{K}^i = \delta\hat{k}^i = \delta n^i, \quad (4.107)$$

therefore the relativistic corrections to the direction of propagation is simply eq. (4.105). No particular physical meaning should be behind this, rather the effect of the approximation on the background scalar field.
For (4.88), if $\nu = 0$, one can find

$$\delta\mathcal{K}^0 = \delta\hat{k}^0(1 - \underline{D}) = \delta\nu(1 - \underline{D}), \quad (4.108)$$

hence the relativistic correction to the frequency is $\delta\mathcal{K}^0$ adjusted by a factor of $(1 - \underline{D})^{-1}$. In order to be consistent one should be careful with orders of expansion around de Sitter, hence terms that are quadratic, or more, in $\underline{D}$ should be neglected. As a result, the relation will be:

$$\delta\mathcal{K}^0 = \delta\hat{k}^0(1-\underline{D}) = \delta\nu(1-\underline{D}) = (\delta\nu_{(0)} + \alpha\delta\nu_{(1)})(1-\alpha\underline{D}_{(1)}) = \\ = \delta\nu_{(0)} + \alpha(\delta\nu_{(1)} + \delta\nu_{(0)}\underline{D}_{(1)}), \quad (4.109)$$

Hence, for sake of clearness we shall finally write the physical relativistic corrections to the wave-vector of tensorial perturbations: combining (4.107) with (4.105), the correction to the direction of propagation will be:

$$\delta n^i = \breve{U}_0^i - 2n^i\Psi + \int d\underline{\breve{\chi}} \left[-\breve{\partial}^i (\Phi + \Psi - U_{00} + U_{0j}n^j) - \frac{1}{2}\frac{\bar{D}'}{1-\bar{D}}\breve{U}_0^i \right]. \quad (4.110)$$

Using the same calculations of (4.50), one can notice an ISW effect and lensing contributions from the term $\int d\underline{\breve{\chi}}\, \breve{\partial}^i (\Phi+\Psi)$ (after dividing the partial derivative in a parallel component and a perpendicular one), while the last term accounts for perturbations of the scalar field mediated by a function of the disformal factor along the line of sight.
Combining (4.108) with (4.111),the frequency correction, up to terms quadratic around the de Sitter expansion, will be

$$\delta\nu_{(1)} = \underline{D}_{(1)}(-2\Phi + U_{00} - U_{0i}n^i) + \int d\underline{\breve{\chi}}(-\delta\nu_{(0)} - \frac{1}{2}U_{00} + 2\Phi + \\ + U_{0i}n^i)\underline{D}'_{(1)} + \int d\underline{\breve{\chi}}(\Phi' + \Psi' - \frac{1}{2}U'_{00} + U'_{0i}n^i)\underline{D}_{(1)}, \quad (4.111)$$

with

$$\delta\nu_{(0)} = 2\Phi - U_{00} + U_{0i}n^i + \int d\bar{\chi}[\Phi' + \Psi' - \frac{1}{2}U'_{00} + U'_{0i}n^i], \qquad (4.112)$$

where we used equations (4.112) and (4.108) to find the above result. One can see one more time a standard ISW effect in eq. (4.112), while in eq. (4.111) the ISW correction is mediated by the disformal factor.

4.3 Relativistic Corrections in Horndenski Theories: Tensor amplitude

In this section, we will move on to one of the main GW observable for cosmology: the luminosity distance. As, already mentioned, one of the main features of Gravitational Waves, if they are emitted from inspirals of compact objects binaries, is that from their detection we can directly infer the distance, since their amplitude is inversely proportional to the luminosity distance, namely

$$\mathcal{A} = \frac{\mathcal{Q}(1+z)}{\mathcal{D}_L}, \tag{4.113}$$

(where z is the redshift, $\mathcal{Q}$ depends on the source and $\mathcal{D}_L$ is the luminosity distance), while we usually have to deal with measurements of redshift from EM measurements (with some exceptions to this, like Type 1a Supernovae detections).

After evaluating the relativistic corrections to the wave-vector, hence the modifications to frequency and direction of propagation, one can try to the effects of such corrections also for the luminosity distance by studying the the amplitude of the scalar or tensorial perturbation waves (in our case $\delta\phi$ and $\Gamma_{\mu\nu}$).

As showed in section 4.1, in General Relativity, the evolution of the wave-vector for the tensor perturbation does not depend on the amplitude (in the high-energy limit), giving us the dispersion relation in eq. (4.12).

Another important feature of GR is that in the perturbed equations of motion (4.4) there is no contribution from the amplitude matrix in the EOMs (in the harmonic gauge), therefore no first derivatives in the differential equation, but only from the kinetic matrix. It follows that, in the eikonal approximation, the evolution of the amplitude will just be described by the terms linear in the wave-vector that arise from the kinetic matrix, as shown in eq. (3.15). As a result, the standard GWs amplitude (obtained in GR) multiplied by the scale factor, that is $a\mathcal{A}$, will decrease as the wave-vectors diverge, accordingly to eq. (4.16). This relation can also be rewritten as

$$\hat{\nabla}_\mu (a^2 \mathcal{A} \hat{k}^\mu) = 0, \tag{4.114}$$

which means, in a particle physics language, that the comoving number of graviton is conserved.

However, things are quite different if we consider scalar-tensor theories of gravity, as the resulting amplitude matrices are not usually trivial.

For sake of clearness, we report here again the equations at order ϵ^{-1} which describe the evolution of the amplitudes Γ and Ξ already calculated in Chapter 3 (equations (3.20) and (3.19)):

$$K^{\rho\sigma\alpha\beta}_{\mu\nu}(2k_{(\alpha}\bar{\nabla}_{\beta)}\Gamma_{\rho\sigma}+\Gamma_{\rho\sigma}\bar{\nabla}_{\beta}k_{\alpha})+A^{\rho\sigma\alpha}_{\mu\nu}\Gamma_{\rho\sigma}k_{\alpha}+A^{(\phi)\alpha}_{\mu\nu}\Xi q_{\alpha}e^{i(\xi-\theta)/\epsilon}=0, \tag{4.115}$$

$$K^{(\phi)\alpha\beta}_{(\phi)}(2q_{(\alpha}\bar{\nabla}_{\beta)}\Xi+\Xi\bar{\nabla}_{\beta}q_{\alpha})+A^{(\phi)\alpha}_{(\phi)}\Xi q_{\alpha}+A^{\rho\sigma\alpha}_{(\phi)}\Gamma_{\rho\sigma}k_{\alpha}e^{i(\theta-\xi)/\epsilon}=0. \tag{4.116}$$

As a result, the differential equations will have a more complicated form and, most importantly, there might be an interaction between the amplitudes of the scalar and tensor perturbations. As showed in section 3.1, the diagonalization process was performed only on the kinetic matrix, in order to find the true propagating degrees of freedom. However, the amplitude (and mass-like) matrices will not be diagonalized, in the sense that the equations of the tensor and scalar sectors are coupled at order ϵ^{-1}.
We approach the problem with the same parametric formalism presented in the previous section in order to find the corrections to the tensorial amplitude.
In this section we will apply the first rule introduced for raising and lowering indices, hence we will use the background metric $\bar{g}_{\mu\nu}$ and the wave-vector will have the form described in eq. (4.10).
Due to the interactions between scalar and tensor amplitude, we cannot ignore the scalar EOM anymore, therefore one needs to consider both perturbed EOMs where, in the eikonal approximation, contributions will arise from the linear terms (in the wave-vectors) in the kinetic and amplitude matrices.

4.3.1 Phase difference

Due to the phase difference, an additional complication arise when calculating the amplitudes evolution: scalar and tensor perturbations might propagate at different speed, therefore their phases ξ and θ could differ. Since the majority of the literature on scalar-tensor theories restrict themselves to the subclass of models for which $c_T^2=1$, one can then select three different situations for the scalar perturbations speed (see [17] for a complete discussion): luminal regime ($c_S^2=1$), quasiluminal regime ($c_S^2\approx 1$) and nonluminal regime ($c_S^2\neq 1$). Since in our case also the tensor propagation speed is generic, we must consider also the three subcases (luminal, quasiluminal, nonluminal regimes) for the tensor perturbation.
In the case of $c_T^2=1$, [17] provides a complete discussion of all possible

subcases of the scalar propagation speed.
From GW170817, we can infer some quite strict constraints on the tensorial propagation speed [2]:

$$|c_T/c - 1| \leq 5 \cdot 10^{-16}. \tag{4.117}$$

It is important to remind, though, that this is still a low-redshift, high-energy measurement and in this limit we expect to recover $c_T^2 = 1$ [19]: in the context of scalar-tensor theories we expect significant deviations from $c_T^2 = 1$ to be observed at higher redshift and lower energies than GW170817. However, it seems reasonable to restrict ourselves to the subcase $c_T^2 \approx 1$ as a first attempt (which is also in line with the quasi-de Sitter expansion performed in the previous section). With this choice, we can consider the luminal and quasiluminal regimes for scalar propagation speed altogether, as long c_S is close to c_T in the following sense: if we consider a scalar and tensor perturbation emitted simultaneously and travelling through a distance l, their phase difference at the time of observation shall remain much smaller than their period. In this situation $q_\alpha \approx k_\alpha$, as the two perturbations follow almost the same world-line. It has been shown that, in the subcase $c_T^2 = 1$, the scalar-tensor amplitude interaction terms can be neglected (as they will have the same order of magnitude of mass-like terms) if one consider only the transverse-traceless modes of the tensor perturbation $\gamma_{\rho\sigma}$ [17], utterly simplifying the amplitude equations. This does not hold necessarily for quartic and quintic theories, hence in the following subsection we will study how projecting the amplitude equations in the transverse-traceless modes might affect the interaction terms.
If $c_T = c_S$, then the phase difference will be identically zero and the amplitudes evolution will not depend on the phases. However, this is a very special subcase and there is no particular reason to choose this *a priori*.
Finally, if $c_S^2 \neq 1$, the situation will drastically aggravate: equations (4.115) and (4.116) describe only the local evolution of the amplitudes Ξ and $\Gamma_{\mu\nu}$ along their respective wolrdlines, therefore it is not guaranteed that is possible to simultaneously solve both the equations for the amplitude's evolution. Some methods have been tried to solve the problem under certain assumptions [17], but this goes out of the scope of the thesis.

4.3.2 Parametrization of Amplitude matrices

After discussing the matter of the phase difference, we will move on to study the parametrization choice for the kinetic and amplitude matrices in equations (4.115) and (4.116). In order to be consistent, we will choose again to the parametrization of the kinetic matrix chosen in the previous section for quartic theories (4.53), that is

$$K_{\mu\nu}^{\rho\sigma\alpha\beta} = \delta_{\mu}^{(\rho}\delta_{\nu}^{\sigma)}\left(A\bar{g}^{\alpha\beta} + B\bar{v}^{\alpha}\bar{v}^{\beta}\right). \tag{4.118}$$

Therefore, combining the terms of order $\mathcal{O}(\epsilon^{-1})$ in eq. (3.15) with (4.118), one will find for the tensorial amplitude evolution (4.115)

$$\begin{aligned}\delta_{\mu}^{(\rho}\delta_{\nu}^{\sigma)}\breve{g}^{\alpha\beta}[2(\nabla_{(\alpha}\Gamma_{\rho\sigma})k_{\beta)} + \Gamma_{\rho\sigma}\nabla_{\beta}k_{\alpha}] = A[\Gamma_{\mu\nu}\nabla_{\alpha}k^{\alpha} + 2k^{\alpha}\nabla_{\alpha}\Gamma_{\mu\nu}]+\\ + B[\Gamma_{\mu\nu} + v^{\alpha}v^{\beta}\nabla_{\alpha}k_{\beta} + 2(v^{\beta}k_{\beta})v^{\alpha}\nabla_{\alpha}\Gamma_{\mu\nu}],\end{aligned} \tag{4.119}$$

where the standard background metric $\bar{g}_{\mu\nu}$ is used to raise and lower the indices. Notice that it is possible to recover GR by setting $A = 1$ and $B = 0$, or any gravitational theory with $c_T^2 = 1$ just by setting $B = 0$.
The following steps would require to choose a parametrization of the amplitude matrices, and then perform the perturbation splitting around an FRW universe already introduced in section 4.1. Nevertheless, it is not possible to ignore the incredibly high number of terms contributing to the amplitude matrices $A_{\mu\nu}^{\rho\sigma\alpha}$ and $A_{\mu\nu}^{(\phi)\alpha}$, which will make it extremely hard to solve the amplitude equation without loosing too much generality (by neglecting several possible terms).
Therefore, it appears to be better to study first how the amplitude matrices can be constructed and how they can affect the amplitudes evolution, rather than guess a certain parametrization that would just result in a toy model.
We start by studying the possible components of the amplitude matrices, namely:

- the background scalar field φ,
- its gradient $v_{\mu} = \partial_{\mu}\varphi$,
- the kinetic factor of the scalar field X and its gradient $\partial_{\mu}X$,
- the background metric $\bar{g}_{\mu\nu}$,
- Kronecker's delta δ_{ν}^{μ}.

The background scalar field φ and its kinetic term X are scalar functions, therefore we can factor out the dependencies on φ and X into some phenomenological function $F = F(\varphi, X)$, while the remaining part of the amplitudes matrices will be built-up by v_{μ}, $\partial_{\mu}X$ (and their covariant derivatives), $\bar{g}_{\mu\nu}$ and δ_{ν}^{μ} (the same objects we used for constructing the kinetic

matric in the previous section). Moreover, given the fact the amplitude matrices are tensors with an odd number of indices, and all the building blocks are rank-2 tensors (the background metric or the Kronecker delta) besides the background scalar field gradient $\bar{v}_\mu$ and $\partial_\mu X$, we conclude that they must be proportional to either one of such quantities.
Then, the simplest parametrization would be to factor out the index contracted with the covariant derivative using the scalar field gradient v_μ in the following way

$$A^{\rho\sigma\alpha}_{\mu\nu} = [A1]^{\rho\sigma}_{\mu\nu} v^\alpha, \tag{4.120}$$
$$A^{(\phi)\alpha}_{\mu\nu} = [A2]^{(\phi)}_{\mu\nu} v^\alpha, \tag{4.121}$$

since, this way, when the scalar field is just constant, the amplitude matrices vanish. With this assumption, one would easily recover such situation. With this assumption, eq. (4.115) will have the form

$$K^{\rho\sigma\alpha\beta}_{\mu\nu}[2(\nabla_{(\alpha}\Gamma_{\rho\sigma})k_{\beta)} + \Gamma_{\rho\sigma}\nabla_\beta k_\alpha] + [A1]^{\rho\sigma}_{\mu\nu}\bar{v}^\alpha k_\alpha \Gamma_{\rho\sigma} + \\ + [A2]^{(\phi)}_{\mu\nu}\bar{v}^\alpha q_\alpha \Xi e^{i(\xi-\theta)/\epsilon} = 0, \tag{4.122}$$

while the simplified amplitude matrix (4.121), due to its low number of indices and the limited amount of building blocks, can be written in such parametric form:

$$[A2]^{(\phi)}_{\mu\nu} = M\bar{g}_{\mu\nu} + N\bar{v}_\mu\bar{v}_\nu + O\bar{\nabla}_{(\mu}\bar{v}_{\nu)} + \\ + P\bar{\nabla}_\mu X\bar{\nabla}_\nu X + Q\bar{v}_{(\mu}\bar{\nabla}_{\nu)}X + R\bar{\nabla}_{(\mu}\bar{\nabla}_{\nu)}X, \tag{4.123}$$

where M, N, O, P, Q, R and S are functions depending on the scalar field and its kinetic term. Eq. (4.123) is justified by the limited number of objects that can construct amplitude matrices and by the fact that $A2$ is just a (0,2) rank tensor, resulting in this relatively small amount of possible terms.
Such description in principle could be extended to (4.120) but, since we are dealing with a (2,2) rank tensor, one would have to define an utterly elevated number of parametric functions, that is

$$[A1]^{\rho\sigma}_{\mu\nu} = E\delta^{(\rho}_\mu\delta^{\sigma)}_\nu + F\bar{v}_\mu\bar{v}_\nu\bar{v}^\rho\bar{v}^\sigma + G\bar{g}_{\mu\nu}v^\rho v^\sigma + H\bar{v}_\mu\bar{v}_\nu\bar{\nabla}^\rho X\bar{\nabla}^\sigma X + \\ + I\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma} + J\bar{g}_{\mu\nu}\bar{\nabla}^\rho X\bar{\nabla}^\sigma X + K\bar{\nabla}_{(\mu}v_{\nu)}\bar{\nabla}^{(\rho}v^{\sigma)} + \cdots, \tag{4.124}$$

making it impossible to give a generic parametrization.

4.3.3 TT modes projection

In the previous subsection we described which objects can construct the amplitude matrices $A^{\rho\sigma\alpha}_{\mu\nu}$ and $A^{(\phi)\alpha}_{\mu\nu}$, then we defined a first attempt to parametrize those, namely

$$\begin{aligned} A^{\rho\sigma\alpha}_{\mu\nu} &= [A1]^{\rho\sigma}_{\mu\nu}\bar{v}^{\alpha}, \\ A^{(\phi)\alpha}_{\mu\nu} &= [A2]^{(\phi)}_{\mu\nu}\bar{v}^{\alpha}. \end{aligned}$$

Even so, it is not easy to find a consistent and accurate way to choose the parametrization of $[A1]^{\rho\sigma}_{\mu\nu}$ and $[A2]^{(\phi)}_{\mu\nu}$ as we have many objects er can use to construct these matrices ($\bar{v}_{\mu}$, $\bar{g}_{\mu\nu}$, ...). As a result we end up with an elevated number of different possible parametrizations.
In the first chapter of this thesis, we showed how the physical modes of gravitational waves consist in the transverse-traceless modes. Still, after the diagonalization, it is not clear at all which kind of modes are included in the metric perturbation $\gamma_{\rho\sigma}$. The diagonalization is theory dependent, since we are taking a parametrized approach it makes little sense to relate it to a specific theory. In principle $\gamma_{\rho\sigma}$ has every mode inside. Another criterion one could use to reduce this number of possible parametrization would be projecting the amplitude equation (3.19) and then extract only the equations for the TT modes (and still the possible interaction terms with the scalar perturbation).
In order to achieve so, we shall use the tetrad formalism introduced in section 1.3 to define a new vector basis. We first introduce an orthonormal basis, where the first leg of the tetrad is gradient of the background scalar field after a normalization:

$$e^{\mu}_{\hat{0}} = \frac{\bar{v}^{\mu}}{|\bar{v}|} = n^{\mu}, \tag{4.125}$$

where $|\bar{v}|$ is the norm of $\bar{v}^{\mu}$. We will use the wave-vector k^{μ} to define the second leg of the tetrad:

$$e^{\mu}_{\hat{3}} = \frac{k^{\mu} + n^{\alpha}k_{\alpha}n^{\mu}}{|k|}, \tag{4.126}$$

where $|k|$ is the norm of the wave-vector k^{μ}. With this choice, it automatically follows that $e^{\mu}_{\hat{0}}$ and $e^{\mu}_{\hat{3}}$ are orthogonal. One can also notice that, if $\bar{v}^{\mu}$ has only a time component, then $e^{\mu}_{\hat{3}}$ will automatically turn into the spatial part of the tensor wave-vector, namely k^{i}.

The remaining two vectors $e^\mu_{\hat{1}}$ and $e^\mu_{\hat{2}}$ are chosen to be space-like, normalized, and orthogonal to each other and to the rest of the basis. It is then convenient to perform the following rotations on the tetrad legs [23] as already showed in eq. (1.40):

$$m^\mu = \frac{e^\mu_{\hat{1}} + ie^\mu_{\hat{2}}}{\sqrt{2}}, \qquad \tilde{m}^\mu = \frac{e^\mu_{\hat{1}} - ie^\mu_{\hat{2}}}{\sqrt{2}}. \tag{4.127}$$

The new tetrad then it will be

$$\{e^\mu_{\hat{0}}, m^\mu, \tilde{m}^\mu, e^\mu_{\hat{3}}\}, \tag{4.128}$$

with scalar products

$$g_{\mu\nu} m^\mu \tilde{m}^\nu = -g_{\mu\nu} e^\mu_{\hat{0}} e^\nu_{\hat{3}} = 1, \tag{4.129}$$

while all other contractions are equal to zero. From this, it is possible to define a basis for symmetric rank (0,2) tensor objects:

$$\Theta^{\mu\nu}_{\hat{a}\hat{b}} = \frac{1}{4}(e^\mu_{\hat{a}} e^\nu_{\hat{b}} + e^\nu_{\hat{a}} e^\mu_{\hat{b}}), \tag{4.130}$$

so we can decompose the tensorial amplitude as

$$\Gamma_{\mu\nu} = \sum_{\hat{a},\hat{b}} \Gamma_{\hat{a}\hat{b}} \Theta^{\hat{a}\hat{b}}_{\mu\nu}, \qquad \text{with} \qquad \Gamma_{\hat{a}\hat{b}} = \Gamma_{\hat{b}\hat{a}}. \tag{4.131}$$

The factor of $\frac{1}{4}$ in (1.43) again follows from the symmetry of changing the decomposition coefficients $\hat{a}$ and $\hat{b}$ in the tensor perturbation $\Gamma_{\mu\nu}$.
After inserting the explicit contribute from the kinetic matric (4.119), one will have To find the equation of motion of each coefficient $\Gamma_{\hat{a}\hat{b}}$ one can take eq. (4.115) and project with the basis elements (4.130)

$$\begin{aligned}\{2[Ak^\alpha \bar{\nabla}_\alpha \Gamma_{\mu\nu} + B\bar{v}^\alpha k_\alpha \bar{v}^\beta \bar{\nabla}_\beta \Gamma_{\mu\nu}] + \Gamma_{\mu\nu}(A\nabla_\alpha k^\alpha + B\bar{v}^\alpha \bar{v}^\beta \bar{\nabla}_\beta k_\alpha) + \\ + [A1]^{\rho\sigma}_{\mu\nu} \bar{v}^\alpha k_\alpha \Gamma_{\rho\sigma} + [A2]^{(\phi)}_{\mu\nu} \bar{v}^\alpha q_\alpha \Xi e^{i(\xi-\theta)/\epsilon} = 0\} \times \Theta^{\mu\nu}_{ab} = 0.\end{aligned} \tag{4.132}$$

In section 1.3, we pointed out that the circular polarization basis for metric perturbations can be constructed from the vectors m and $\tilde{m}$ in the following way

$$\Gamma_{\rho\sigma} = \Gamma_{\circlearrowleft} m_\rho m_\sigma + \Gamma_{\circlearrowright} \tilde{m}_\rho \tilde{m}_\sigma, \tag{4.133}$$

where $\Gamma_{\circlearrowleft}$ and $\Gamma_{\circlearrowright}$ are, respectively, the left-handed and right-handed helicity modes of the tensor perturbation. This modes are the physical ones, like the more used plus and cross polarization (1.19), which are related to (4.133) as

$$\Gamma_{\circlearrowleft} = \Gamma_{+} - i\Gamma_{\times}, \tag{4.134}$$
$$\Gamma_{\circlearrowright} = \Gamma_{+} + i\Gamma_{\times}, \tag{4.135}$$

where Γ_{+} and $\Gamma_{\times}$ are the plus and cross polarizations. We want then to project the amplitude equation (3.19) along $\Theta^{\mu\nu}_{mm}$ and $\Theta^{\mu\nu}_{\tilde{m}\tilde{m}}$ to gain some insights on the amplitude matrices for the TT modes.
We shall study the projection of each term in eq. (4.132) and we will arbitrarily choose $\Theta^{\mu\nu}_{mm}$ for the projection (all the following calculations and results will be the same along the $\tilde{m}\tilde{m}$ projection): starting from the first one, we will have

$$\Theta^{\mu\nu}_{mm}\{2Ak^{\alpha}\bar{\nabla}_{\alpha}\Gamma_{\mu\nu}\} = 2Ak^{\alpha}\bar{\nabla}_{\alpha}(\sum_{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}}\Theta^{\hat{a}\hat{b}}_{\mu\nu})\Theta^{\mu\nu}_{mm} = \frac{1}{2}Ak^{\alpha}\bar{\nabla}_{\alpha}\Gamma_{\tilde{m}\tilde{m}}, \tag{4.136}$$

where the multiplying factor of $\frac{1}{2}$ comes from the contraction

$$\Theta^{\mu\nu}_{mm}\Theta^{\tilde{m}\tilde{m}}_{\mu\nu} = \frac{1}{4}. \tag{4.137}$$

Accordingly, the second term in (4.132) is evaluated using the same procedure

$$\Theta^{\mu\nu}_{mm}\{2B\bar{v}^{\alpha}k_{\alpha}\bar{v}^{\beta}\bar{\nabla}_{\beta}\Gamma_{\mu\nu}\} = \frac{1}{2}B\bar{v}^{\alpha}k_{\alpha}\bar{v}^{\beta}\bar{\nabla}_{\beta}\Gamma_{\tilde{m}\tilde{m}}, \tag{4.138}$$

where we skipped the algebra since the calculation is the same as eq. (4.136).
Moving to the contributions from the amplitude matrices, we will first have a look at the term proportional to the tensorial amplitude matrix, namely

$$\Theta^{\mu\nu}_{mm}\{\bar{v}^{\alpha}k_{\alpha}\Gamma_{\rho\sigma}[A1]^{\rho\sigma}_{\mu\nu}\}. \tag{4.139}$$

We can see that the TT-modes tensor does not contract with the tensor amplitude Γ, but only with the amplitude matrix $[A1]$. Hence, in principle, there could be a mixing between different modes of the tensor perturbation, depending on the specific form of $[A1]^{\rho\sigma}_{\mu\nu}$ so we are rather interested in studying the contraction $\Theta^{\mu\nu}_{mm}[A1]^{\rho\sigma}_{\mu\nu}$. Using the generic parametrization for $[A1]^{\rho\sigma}_{\mu\nu}$ introduced in (4.124), we can explicitly calculate the contraction

$$\begin{aligned}\Theta^{\mu\nu}_{mm}\Gamma_{\rho\sigma}[A1]^{\rho\sigma}_{\mu\nu} = \Theta^{\mu\nu}_{mm}\Gamma_{\rho\sigma}\{&E\delta^{(\rho}_{\mu}\delta^{\sigma)}_{\nu} + F\bar{v}_{\mu}\bar{v}_{\nu}\bar{v}^{\rho}\bar{v}^{\sigma} + G\bar{g}_{\mu\nu}v^{\rho}v^{\sigma} + \\ + H\bar{v}_{\mu}\bar{v}_{\nu}\bar{\nabla}^{\rho}X\bar{\nabla}^{\sigma}X + I\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma} &+ J\bar{g}_{\mu\nu}\bar{\nabla}^{\rho}X\bar{\nabla}^{\sigma}X + K\bar{\nabla}_{(\mu}v_{\nu)}\bar{\nabla}^{(\rho}v^{\sigma)} + \cdots\}.\end{aligned} \tag{4.140}$$

Now, using the orthogonality properties and the scalar products (4.129), we can see that terms like the second or the third one will not contribute to the evolution of TT modes. Moreover, all terms proportional to $\bar{v}_\mu$ (or $\bar{v}_\nu$) will always vanish when contracted with $\Theta^{\mu\nu}_{mm}$, since $\bar{v}_\mu$ is orthogonal to m_μ (and $\tilde{m}_\mu$). In addition to this, all terms proportional to the background metric $\bar{g}_{\mu\nu}$ will vanish too, as

$$\bar{g}_{\mu\nu} m^\mu m^\nu = \bar{g}_{\mu\nu} \tilde{m}^\mu \tilde{m}^\nu = 0. \tag{4.141}$$

As a result, only the first and last term written in (4.140) will survive:

$$\Theta^{\mu\nu}_{mm} \Gamma_{\rho\sigma} [A1]^{\rho\sigma}_{\mu\nu} = \frac{1}{4} E \Gamma_{\tilde{m}\tilde{m}} + K \Theta^{\mu\nu}_{mm} \Gamma_{\rho\sigma} \bar{\nabla}_{(\mu} v_{\nu)} \bar{\nabla}^{(\rho} v^{\sigma)} + \cdots \tag{4.142}$$

Finally, for the last term in (4.132), will will repeat the procedure and evaluate

$$\Theta^{\mu\nu}_{mm} \{ [A2]^{(\phi)}_{\mu\nu} \bar{v}^\alpha q_\alpha \Xi e^{i(\xi-\theta)/\epsilon} \}, \tag{4.143}$$

where again we are interested in the contraction between $\Theta^{\mu\nu}_{mm}$ and $[A2]^{(\phi)}_{\mu\nu}$. Plugging the parametrization for $[A2]^{(\phi)}_{\mu\nu}$ given in (4.123), one will have

$$\begin{aligned} \Theta^{\mu\nu}_{mm} \{ \bar{v}^\alpha q_\alpha \Xi [A2]^{(\phi)}_{\mu\nu} \} e^{i(\xi-\theta)/\epsilon} &= \bar{v}^\alpha q_\alpha \Xi e^{i(\xi-\theta)/\epsilon} \Theta^{\mu\nu}_{mm} \{ M \bar{g}_{\mu\nu} + N \bar{v}_\mu \bar{v}_\nu + \\ &+ O \bar{\nabla}_{(\mu} \bar{v}_{\nu)} + P \bar{\nabla}_\mu X \bar{\nabla}_\nu X + Q \bar{v}_{(\mu} \bar{\nabla}_{\nu)} X + R \bar{\nabla}_{(\mu} \bar{\nabla}_{\nu)} X \}, \end{aligned} \tag{4.144}$$

where, after repeating the previous considerations on orthogonality and vanishing scalar products, we will find

$$\begin{aligned} \Theta^{\mu\nu}_{mm} \{ \bar{v}^\alpha q_\alpha \Xi [A2]^{(\phi)}_{\mu\nu} \} e^{i(\xi-\theta)/\epsilon} &= \bar{v}^\alpha q_\alpha \Xi e^{i(\xi-\theta)/\epsilon} \Theta^{\mu\nu}_{mm} (O \bar{\nabla}_{(\mu} \bar{v}_{\nu)} + \\ &+ P \bar{\nabla}_\mu X \bar{\nabla}_\nu X + Q \bar{v}_{(\mu} \bar{\nabla}_{\nu)} X + R \bar{\nabla}_{(\mu} \bar{\nabla}_{\nu)} X). \end{aligned} \tag{4.145}$$

However, we are still interested in the perturbations around a FRW Universe. Therefore, we will use again the splitting of the background metric (4.19):

$$\bar{g}_{\mu\nu} = \underline{\bar{g}}_{\mu\nu} + \delta \bar{g}_{\mu\nu},$$

where $\underline{\bar{g}}_{\mu\nu}$ describes a FRW metric and $\delta\bar{g}_{\mu\nu}$ represents the perturbations due to LSS. Unfortunately, if we perturb eq. (4.132) around FRW, the number of terms one has to consider will increase even more, but projecting again along TT modes might give us some insights for the contributions of the amplitude matrices, especially at unperturbed level.
We first need to define how the perturbation around FRW will affect the

legs of the tetrad. Considering the first one, that is $e^{\mu}_{\hat{0}}$, we can easily find its value using (4.76):

$$e^{\mu}_{\hat{0}} = \underline{e}^{\mu}_{\hat{0}} + \Delta e^{\mu}_{\hat{0}} = \frac{\underline{\bar{v}}^{\mu}}{|\underline{\bar{v}}|} + \Delta\left(\frac{\bar{v}^{\mu}}{|\bar{v}|}\right) = (1 + \Delta n^0, \Delta n^i), \tag{4.146}$$

where we defined

$$\underline{n}^{\mu} = \frac{\underline{\bar{v}}^{\mu}}{|\underline{\bar{v}}|} = (1,0,0,0) \quad \text{and} \quad \Delta n^{\mu} = \Delta\left(\frac{\bar{v}^{\mu}}{|\bar{v}|}\right). \tag{4.147}$$

Moving to $e^{\mu}_{\hat{3}}$, if we perturb this quantity around FRW we will have

$$e^{\mu}_{\hat{3}} = \underline{e}^{\mu}_{\hat{3}} + \Delta e^{\mu}_{\hat{3}} = \frac{\underline{k}^{\mu} + \underline{n}^{\alpha}\underline{k}_{\alpha}\underline{n}^{\mu}}{|\underline{k}|} + \Delta\left(\frac{k^{\mu} + n^{\alpha}k_{\alpha}n^{\mu}}{|k|}\right). \tag{4.148}$$

After inserting the values we found after the splitting in (4.147) for n^{μ} we can see that $\underline{e}^{\mu}_{\hat{3}}$ is the vector pointing along the unperturbed direction of propagation of the tensor perturbation.
For the remaining two vectors of the basis, that is m^{μ} and $\tilde{m}^{\mu}$, after the splitting they will be

$$m^{\mu} = \underline{m}^{\mu} + \Delta m^{\mu} \quad \text{and} \quad \tilde{m}^{\mu} = \underline{\tilde{m}}^{\mu} + \Delta \tilde{m}^{\mu}, \tag{4.149}$$

where not much can be said about them, as they are a combination of generic orthonormal vectors. Still, if we perform a rotation of our reference system in order to have the direction of the incoming tensorial signal precisely along our z-axis, we can find, after inserting the results of equations (4.147) and (4.148) and using the orthogonality properties, that

$$\underline{m}^{\mu} = \frac{1}{\sqrt{2}}(0,1,i,0) \quad \text{and} \quad \underline{\tilde{m}}^{\mu} = \frac{1}{\sqrt{2}}(0,1,-i,0). \tag{4.150}$$

After evaluating the effects of the perturbation around FRW for the tetrad legs, we can move back to eq. (4.132). We will start by firstly considering eq. (4.132) at unperturbed level.
One can notice that, in addition to the considerations done previously on the terms actually contributing to the evolution of the TT-modes, terms proportional to $\bar{\nabla}_{\mu}\underline{X}$ (or $\bar{\nabla}_{\nu}\underline{X}$) will not contribute to eq. (4.132) at unperturbed level. This is because in a FRW universe we assume (4.76) and, as a consequence, $\underline{m}^{\mu}$ and $\underline{\tilde{m}}^{\mu}$ will not have a time component, since they must be orthogonal to $\underline{e}^{\mu}_{\hat{0}}$ (see eq. (4.147) for the explicit value). On the other hand, $\bar{\nabla}_{\mu}\underline{X}$ will only have a time component, hence its contraction with

$\underline{m}^\mu$ or $\underline{\tilde{m}}^\mu$ will be zero.
Unfortunately we could not find other meaningful insights of this kind at first perturbation level around FRW, since the perturbed form of the various elements building the tensorial amplitude evolution will be much more complicated.
For sake of clearness we will list, of all the possible terms contained in the amplitude matrices, the ones that are not contributing to the evolution of the TT-modes:

- terms proportional to the gradient of the background scalar field, that is $\bar{v}^\mu$ or $\bar{v}^\nu$ (due to orthogonality of the tetrad legs),
- terms proportional to the background metric $\bar{g}_{\mu\nu}$ (due to scalar product properties of the tetrad legs),
- at FRW level, terms proportional to the gradient of the background scalar field kinetic term, namely $\bar{\nabla}_\mu \underline{X}$ (due to the assumption (4.76)).

We remind that another contribution to the amplitude evolution of the tensor perturbation in principle can come from eq. (4.116), due to the interaction term with the scalar perturbation. One can still repeat the previous procedure of projecting along the TT-modes, finding the same results that we listed above.
We leave as a future goal to develop other techniques in order to get further insights on building blocks of the amplitude evolution.

Chapter 5

Conclusions

In this concluding chapter, we will summarize the main findings of the research conducted in this thesis. In the context of scalar-tensor theories, our goal was to study the relativistic effects on the perturbations of the metric by expanding the ”cosmic rulers” formalism for modified gravity theories that may predict a propagation speed for the tensorial perturbation different from lightspeed.
To do so, we started from a scalar-tensor action in a covariant Galileon formalism, and restricted ourselves to quartic theories and then, using the variation principle, we found the Equations of motion. From this, we split the metric and the scalar field in a slowly changing background and a small perturbative part, in order to then find the EOMs linear in the perturbations. Given the generality of the starting action, the perturbed EOMs will result in an extremely convoluted set of inhomogeneous second-order differential equations. We regroup all the terms proportional to second order derivatives under the mathematical objects called kinetic matrices, while all terms proportional to first order derivatives are collected under the amplitude matrices. The remaining terms are called mass-like terms.
Moreover, since there is a mixing between second order derivatives of the scalar field and metric perturbation, it follows these two are not the true propagating degrees of freedom, therefore a redefinition of the variables is required (such passage is referred to in the thesis as ”diagonalization of the kinetic matrices”). Due to the incredibly number of terms in the perturbed EOMs, such procedure was not possible, even after using the residual gauge freedom or selecting a self-accelerating solution for the Horndenski functions. We decided then to follow a parametric approach to deal with the problem: we leave the explicit form of the independent degrees of freedom unknown, as well as the exact form of kinetic and amplitude

matrices and we will study the evolution of the perturbations by selecting the parametrization of the matrices. However, before starting calculating the relativistic corrections, we assume the eikonal approximation, which is valid if the typical involved wavelengths are much smaller than the typical curvature radius of the background space-time. Under this approximation, the scalar and tensor perturbation can be described by a wave ansatz. Moreover, in this limit, mass-like terms can be neglected and, using an order of magnitude argument, we can collect all the terms linear in the wave-vectors of the tensor and scalar perturbations in one equation (amplitude evolution), repeat the procedure for the terms quadratic in the wave-vectors (dispersion relation) and solve this set of equations.
After briefly introducing and reviewing the formalism and results of the cosmic rulers approach in General Relativity which include the key concept of our scalar-tensor waves travelling through inhomogeneities and structures in our universe (the so called Large Scale Structure), hence another linear perturbation approach is used and we explain how to perturb our quantities around a FRW universe. After this, we aim to extend such calculations to our set-up.
The main feature of considering quartic theories, is that the dispersion relation of the tensor perturbation is no longer the standard one ($\bar{g}_{\mu\nu}k^{\mu}k^{\nu} = 0$, where k^{μ} is the wave-vector of the tensor perturbation and $\bar{g}_{\mu\nu}$ the background metric), but it will be corrected by the addition of a disformal factor $\propto v_{\mu}v_{\nu}$, where v_{μ} is the gradient of the background scalar field. As a result, in the eikonal ansatz, tensor perturbation will no longer necessarily follow null paths, meaning their propagation speed might differ from the speed of light.
We defined an effective metric $\breve{g}_{\mu\nu} = A\bar{g}_{\mu\nu} + Bv_{\mu}v_{\nu}$, where A and B are parametric functions (usually referred to as, respectively, conformal and disformal factors) so that now the dispersion relation reads $\breve{g}_{\mu\nu}k^{\mu}k^{\nu} = 0$, meaning that the wave-vector is a null vector with respect to the effective metric. In this way we can repeat the same procedure presented in GR, meaning we can perturb around a FRW universe and find the equations for the relativistic corrections to frequency and direction of propagation. An important assumption we make is that the unperturbed background scalar field (hence at FRW level) can be described as just a function of time, simplifying our calculations. In this way we were able to find the relativistic corrections to the direction of propagation for our tensor perturbation. Nevertheless, the equation describing the relativistic corrections to the frequency has the form of an inhomogeneous first order differential equation, which given the unknown form of sources would be extremely com-

plicated to solve analytically. Therefore, we performed a third perturbative expansion, this time around a de Sitter solution: from observation we know that our universe can be quite precisely described by the Einstein-Hilbert Lagrangian with the addition of a cosmological constant. The most straightforward way, starting from our scalar-tensor action, to asymptotically find this solution is to approximate the background scalar field as a constant field. Hence we expanded the equation representing the relativistic corrections to the frequency around a de Sitter solution and we solved such equation in a iterative way.
Moving to the amplitude equation, the number of terms appearing in the amplitude matrices was too high to find a parametrization that would allow for an analytical solution without loosing too much generality. Since, in principle, there could be interaction terms between scalar and tensor waves at level of the amplitudes, we firstly discussed the issue of the phase difference appearing in the amplitude equations, where we restricted ourselves to the regime $c_T^2 \approx 1$ and then studied the subcases $c_S^2 \approx 1$ and $c_S^2 \neq 1$ (where c_S is the propagation speed of the scalar perturbation).
After an examination of the building blocks of the amplitude matrices and how one can construct them, we decided to focus on the evolution of trasnverse-traceless modes of the tensor perturbation: the diagonalization process is theory dependent and, since we are using a parametrized approach, it is not clear which modes are included in the tensor perturbation. Therefore we decided to project the amplitude equations and then extract such equations for the TT modes and the scalar field. Therefore, we first constructed a tetrad using the Newman-Penrose formalism and then a basis for symmetric rank-2 tensor. Using this basis we were able to project the amplitude equation along the TT modes so we could get some insights on the evolution of such modes. We found all the terms in the amplitude matrices that cannot contribute to the evolution of the amplitude for TT modes using the properties of the tetrad. We finally performed the perturbation around FRW for the amplitude equation in order to find other insights, however could only further simplify the amplitude equation at unperturbed level using the projection on the TT modes.
A future goal will be finding further simplifications to the amplitude equation in order to solve it also for quartic theories. Moreover, the next generation of detectors might provide us with new data in ranges where deviations from theories that predict $c_T^2 = 1$ could be observed, helping us to get better constraints on our theories and therefore, on the choice of the parametric functions.

Appendices

Appendix A

Summary of notations for the Cosmic Rulers

For sake of clearness, we summarize in a table here all quantities that will be used in the following section:

	Standard Metric	Com. Metric	Eff. Metric
Background metric	$\bar{g}_{\mu\nu} = \underline{\bar{g}}_{\mu\nu} + \delta\bar{g}_{\mu\nu}$	$\hat{g}_{\mu\nu} = \underline{\hat{g}}_{\mu\nu} + \delta\hat{g}_{\mu\nu}$	$\breve{g}_{\mu\nu} = \underline{\breve{g}}_{\mu\nu} + \delta\breve{g}_{\mu\nu}$
Cov. derivative	$\bar{\nabla} = \partial + \underline{\bar{\Gamma}} + \delta\bar{\Gamma}$	$\hat{\nabla} = \partial + \underline{\hat{\Gamma}} + \delta\hat{\Gamma}$	$\breve{\nabla} = \partial + \underline{\breve{\Gamma}} + \delta\breve{\Gamma}$
Wave-vector	$\bar{k}_\mu = \partial_\mu\theta$ $\bar{k}^\mu = \bar{g}^{\mu\nu}\bar{k}_\nu$	$\hat{k}_\mu = \bar{k}_\mu$ $\hat{k}^\mu = \hat{g}^{\mu\nu}\hat{k}_\nu$	$\breve{k}_\mu = \bar{k}_\mu$ $\breve{k}^\mu = \bar{g}^{\mu\nu}\breve{k}_\nu$
W.v. splitting	$\bar{k}_\mu = \underline{\bar{k}}_\mu + \delta\bar{k}_\mu$	$\hat{k}_\mu = \underline{\hat{k}}_\mu + \delta\hat{k}_\mu$	$\breve{k}_\mu = \underline{\breve{k}}_\mu + \delta\breve{\mathcal{K}}_\mu$
Grad. scalar field	$\bar{v}_\mu = \partial_\mu\varphi$ $\hat{v}^\mu = \hat{g}^{\mu\nu}\hat{v}_\nu$	$\hat{v}^\mu = \hat{g}^{\mu\nu}\hat{v}_\nu$ $\hat{v}^\mu = \hat{g}^{\mu\nu}\hat{v}_\nu$	$\breve{v}_\mu = \bar{v}_\mu$ $\breve{v}^\mu = \breve{g}^{\mu\nu}\breve{v}_\nu$

Table A.1

Due to the different rules used for raising and lowering indices, in the table you will find the main mathematical operators and physical objects used for the calculation of relativistic effects in different definitions of the background metric.
The first column represents the standard background metric emerging from the initial linearization of the EOMs. In the second column one can read the same operators and quantities after a comoving transformation of the background metric. Finally, in the last column it is presented the effective metric, used for calculating the relativistic corrections in quar-

tic theories, and its rules for total derivatives and for raising or lowering indices.

Appendix B

Perturbation Terms in Poisson Gauge

B.1 Perturbed Christoffel symbols for generic perturbed background metric

We shall define a background metric and its Christoffel symbols perturbed around a FRW universe, that is

$$g_{\mu\nu} = \underline{g}_{\mu\nu} + \delta g_{\mu\nu} \qquad \text{and} \qquad \Gamma^{\alpha}_{\mu\nu} = \underline{\Gamma}^{\alpha}_{\mu\nu} + \delta\Gamma^{\alpha}_{\mu\nu}. \tag{B.1}$$

Then the perturbed Christoffel symbols $\delta\Gamma^{\alpha}_{\mu\nu}$ can be found using

$$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\underline{g}^{\alpha\rho}\left(\nabla_{\mu}\delta g_{\nu\rho} + \nabla_{\nu}\delta g_{\rho\mu} - \nabla_{\rho}\delta g_{\mu\nu}\right), \tag{B.2}$$

where the covariant derivative are taken with respect to $\underline{g}_{\mu\nu}$. In order to demonstrate this short-cut formula we start by writing the definition of perturbed Christoffel symbol, namely

$$\begin{aligned}\Gamma^{\alpha}_{\mu\nu} = \underline{\Gamma}^{\alpha}_{\mu\nu} + \delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}(\underline{g}^{\alpha\rho} - \delta g^{\alpha\rho})[\partial_{\mu}(\underline{g}_{\nu\rho} + \delta g_{\nu\rho}) +& \\ + \partial_{\nu}(\underline{g}_{\rho\mu} + \delta g_{\rho\mu}) - \partial_{\rho}(\underline{g}_{\mu\nu} + \delta g_{\mu\nu})],&\end{aligned} \tag{B.3}$$

and, after selecting only first order perturbations terms, we will have

$$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}[\underline{g}^{\alpha\rho}(\partial_{\mu}h_{\nu\rho} + \partial_{\nu}h_{\rho\mu} - \partial_{\rho}h_{\mu\nu}) - h^{\alpha\rho}(\partial_{\mu}\underline{g}_{\nu\rho} + \partial_{\nu}\underline{g}_{\rho\mu} - \partial_{\rho}\underline{g}_{\mu\nu})], \tag{B.4}$$

which is the expression we want to recover.
Performing explicitly the algebra on eq. (B.2), we will have

$$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\underline{g}^{\alpha\rho}(\partial_\mu h_{\nu\rho} - \underline{\Gamma}^{\lambda}_{\mu\nu}h_{\lambda\rho} - \underline{\Gamma}^{\lambda}_{\mu\rho}h_{\nu\lambda} + \partial_\nu h_{\mu\rho} - \underline{\Gamma}^{\lambda}_{\rho\nu}h_{\lambda\mu} + \\ - \underline{\Gamma}^{\lambda}_{\mu\nu}h_{\rho\lambda} - \partial_\rho h_{\nu\mu} + \underline{\Gamma}^{\lambda}_{\mu\rho}h_{\lambda\nu} + \underline{\Gamma}^{\lambda}_{\nu\rho}h_{\mu\lambda}), \quad \text{(B.5)}$$

and, after eliminating all terms that cancel out, it will result in

$$\delta\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}\underline{g}^{\alpha\rho}(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu} - 2\underline{\Gamma}^{\lambda}_{\mu\nu}h_{\lambda\rho}) = \frac{1}{2}[\underline{g}^{\alpha\rho}(\partial_\mu h_{\nu\rho} + \\ + \partial_\nu h_{\rho\mu} - \partial_\rho h_{\mu\nu}) - \underline{g}^{\alpha\rho}\underline{g}^{\lambda\tau}h_{\lambda\rho}(\partial_\mu \underline{g}_{\nu\rho} + \partial_\nu \underline{g}_{\rho\mu} - \partial_\rho \underline{g}_{\mu\nu})], \quad \text{(B.6)}$$

which is, after contracting the indices of $\underline{g}^{\alpha\rho}\underline{g}^{\lambda\tau}h_{\lambda\rho}$, the same expression of eq. (B.4). We have thus found an identity, hence the formula (B.2) has been demonstrated.

B.2 Perturbation Terms in General Relativity

At unperturbed level the background comoving metric will have the simple form of Minkowski metric

$$\hat{\underline{g}}_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{(B.7)}$$

while the first order perturbed metric components are

$$\delta\hat{g}_{00} = -2\Phi, \;\; \delta\hat{g}_{0i} = 0, \;\; \delta\hat{g}_{ij} = -2\delta_{ij}\Psi. \quad \text{(B.8)}$$

Because of (B.7), all unperturbed comoving Christoffel symbols will be equal to zero, namely

$$\hat{\underline{\Gamma}}^{\alpha}_{\mu\nu} = 0. \quad \text{(B.9)}$$

Moving to first perturbation order, the comoving Christoffel symbols will be

$$\begin{aligned} \delta\hat{\Gamma}^{0}_{00} &= \Phi', & \delta\hat{\Gamma}^{0}_{0i} &= \partial_i\Phi, \\ \delta\hat{\Gamma}^{0}_{ij} &= -\delta_{ij}\Psi', & \delta\hat{\Gamma}^{i}_{00} &= \partial^i\Phi, \\ \delta\hat{\Gamma}^{i}_{0j} &= -\delta^i_j\Psi', & \delta\hat{\Gamma}^{i}_{jk} &= -\delta^i_j\partial_k\Psi - \delta^i_k\partial_j\Psi + \delta_{jk}\partial^i\Psi. \end{aligned} \quad \text{(B.10)}$$

B.3 Perturbation Terms in Quartic theories

For the comoving effective metric, at unperturbed level we have

$$\hat{\mathcal{G}}_{00} = -1 + \bar{D}, \qquad \hat{\mathcal{G}}_{0i} = 0, \qquad \hat{\mathcal{G}}_{ij} = \delta_{ij}, \tag{B.11}$$

and, at first perturbations order

$$\delta\mathcal{G}_{00} = U_{00} - 2\Phi, \quad \delta\mathcal{G}_{0i} = U_{0i}, \quad \delta\mathcal{G}_{ij} = -2\delta_{ij}\Psi. \tag{B.12}$$

For the Christoffel symbols $\hat{\hat{\Gamma}}^{\alpha}_{\mu\nu} = \hat{\hat{\Gamma}}^{\alpha(0)}_{\mu\nu} + \hat{\hat{\Gamma}}^{\alpha(1)}_{\mu\nu}$ of the comoving effective metric, we have at unperturbed level:

$$\begin{aligned} \hat{\hat{\Gamma}}^{0(0)}_{00} = \frac{\bar{D}'}{2(\bar{D}-1)}; \quad \hat{\hat{\Gamma}}^{0(0)}_{0i} = 0; \quad \hat{\hat{\Gamma}}^{0(0)}_{ij} = 0; \\ \hat{\hat{\Gamma}}^{i(0)}_{00} = 0; \qquad \hat{\hat{\Gamma}}^{i(0)}_{0j} = 0; \quad \hat{\hat{\Gamma}}^{i(0)}_{jk} = 0. \end{aligned} \tag{B.13}$$

And, at first order perturbations:

$$\begin{aligned}
\hat{\hat{\Gamma}}^{0(1)}_{00} &= \frac{1}{2}\frac{1}{\bar{D}-1}\left(-2\Phi' + U'_{00} + \frac{\bar{D}'}{1-\bar{D}}U_{00}\right); \\
\hat{\hat{\Gamma}}^{0(1)}_{0i} &= \frac{1}{2}\frac{1}{\bar{D}-1}\left(-2\partial_i\Phi + \partial_i U_{00}\right); \\
\hat{\hat{\Gamma}}^{0(1)}_{ij} &= \frac{1}{2}\frac{1}{\bar{D}-1}\left(\partial_i U_{0j} + \partial_j U_{0i} + 2\Psi'\delta_i j\right); \\
\hat{\hat{\Gamma}}^{i(1)}_{00} &= \frac{1}{2}\delta^{ij}\left(2U'_{j0} + \frac{\bar{D}'}{1-\bar{D}}U_{0j} - \partial_j U_{00} + 2\partial_j\Phi\right); \\
\hat{\hat{\Gamma}}^{i(1)}_{0j} &= \frac{1}{2}\delta^{jk}\left(-2\delta_{kj}\Psi' + \partial_j U_{0k} - \partial_k U_{0j}\right); \\
\hat{\hat{\Gamma}}^{i(1)}_{jk} &= \frac{1}{2}\delta^{il}\left(-2\delta_{lk}\partial_j\Psi - 2\delta_{jl}\partial_k\Psi + 2\delta_{jk}\partial_l\Psi + \partial_j U_{lk} + \partial_k U_{lj} - \partial_l U_{jk}\right)
\end{aligned} \tag{B.14}$$

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