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Abstract

In this thesis, we will investigate the transformation of electromagnetic fields under conformal maps. When a conformal map is applied to such a field, the resulting field is again a valid electromagnetic field. Even when the conformal map is complex, i.e. it mixes real and complex points of space, the resulting field is valid. To better understand complex conformal maps, we introduce Dirac spinors and Twistor space. Using these concepts, we find a nicer expression for a — possibly complex — conformal transformation. This could ease the calculation of the transformed electromagnetic field.

Contents

Chapter 1

Introduction

In 1989 Rañada published a model of electromagnetism in which a electromagnetic field at time t was associated with a map from the 3-dimensional sphere S^3 to the 2-dimensional sphere S^2. Mathematically, maps from S^2 to S^3 are topologically quantized (lemma 141). In the context of electromagnetic fields, this quantization can be interpreted as the amount of linking between two field lines at a given time t. For fields that are *null*, i.e. $\mathbf{E} \cdot \mathbf{B} = 0$ on all of spacetime, the structure of the field lines is preserved under time evolution. Hence, for these fields the topological quantization can be unambiguously assigned to an electromagnetic field.

However, this quantization is based on the assumption that these fields can indeed be constructed from a map from S^3 to S^2. We would like to verify that this is the case for most electromagnetic null fields. Of course, when we use the formalism of maps from S^3 to S^2, this is trivially the case. Thus we look at a different formalism that can also give similar electromagnetic fields.

In [1] it is shown that the most simple nontrivial field of Rañada, the so-called Hopfion (definition 144), can also be constructed by a complex conformal transformation of an initial field that is constant in all of spacetime. Hence, we will use a formalism in which complex conformal transformations are well understood.

Conformal transformations of complex spacetime occur naturally in the formalism of Twistors, introduced in [2] and more accessably explained in [3]. The Twistor formalism comes with a notion of complexified spacetime and compactified spacetime (see figure 1.1) as well as an action of the unitary group $SU(\mathbb{S}, \Sigma)$ of Twistor space that is translated to an action of the conformal group on (real) Minkowski space. (see figure 1.2).

An important result is theorem 128: we see that the more general group

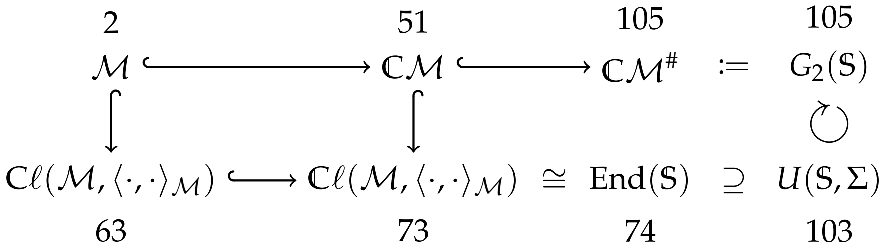

Figure 1.1: *A diagram of the spaces that will be used in this thesis. The numbers refer to the definitions/theorems in which they are defined. $\mathcal{M}$ is standard Minkowski space, $\mathbb{C}\mathcal{M}$ is complexified Minkowski space, $\mathbb{C}\mathcal{M}^\#$ is compactified complexified Minkowski space, which is equal to the Grasmannian $G_2(\mathbb{S})$. The unitary group $U(\mathbb{S})$ acts on $G_2(\mathbb{S})$, where $\mathbb{S}$ is the Dirac spinor space of $\mathcal{M}$, which is defined using the complexified Clifford algebra $\mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ of $\mathcal{M}$, which is the complexification of the Clifford algebra $\mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ of $\mathcal{M}$.*

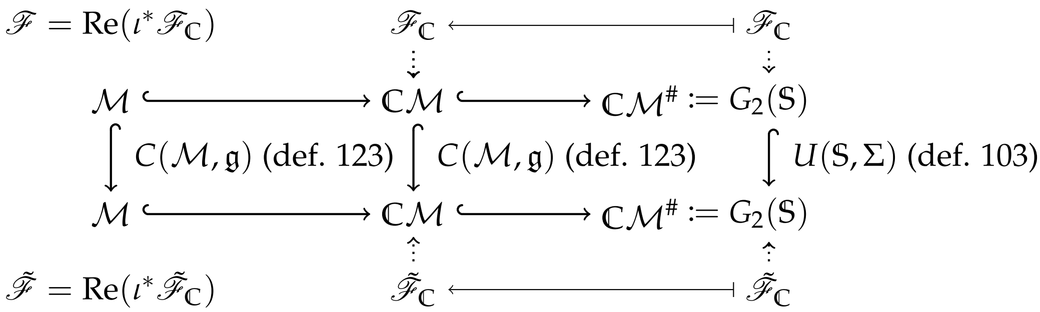

Figure 1.2: *A diagram of the actions between the spaces used in this thesis. The numbers refer to the definitions/theorems in which they are defined. A map from the unitary $U(\mathbb{S},\Sigma)$ is translated to a conformal map of Minkowski space $\mathcal{M}$. These actions can then in turn be applied to an electromagnetic field to obtain a different electromagnetic field.*

$GL(\mathbb{S})$ also gives a conformal mapping of *complexified* spacetime. When such a map is not unitary, it does not leave real Minkowski space invariant, but instead some points of real Minkowski space are mapped to points that originally only existed in complexified Minkowski space and vice-versa.

Furthermore, formulas 113 and 119 give explicit expressions (in terms of Dirac-spinors) of the corresponding translations of points in Minkowski space and tangent vectors of Minkowski space respectively. In further research, these formulas can be applied to ease the calculation of fields that result from conformal transformations. For example, the fields described in section 4.4 could be expressed in these formulas and then investigated further.

Chapter 2

Preliminaries

This thesis was supposed to study several interesting solutions of Maxwell's equations in flat Minkowski space. Therefore, we first introduce Maxwell's equations. For this we can use several different formalisms. Throughout this thesis, the speed of light c is set to 1. Furthermore, we adopt Einstein's summation convention.

Notation 1. *Einstein summation convention means that whenever a letter appears as both a subscript and a superscript in an expression, summation is implied, i.e.* $v^\mu \omega_\mu := \sum_\mu v^\mu \omega_\mu$. *We will use this convention from now on.*

2.1 Maxwell's equations in standard Minkowski space

The first formalism for Maxwell's equations is the oldest and simplest one. First, we introduce Minkowski space.

Definition 2. *Minkowski space* $\mathcal{M}$ *is a 4-dimensional real vector space. The standard basis is referred to as* (e_0, e_1, e_2, e_3), *and vectors in this basis are written as* (t, x, y, z) *or* (x^0, x^1, x^2, x^3) *or* $x^\mu e_\mu$ *or just as* x^μ.

On Minkowski space, we define a Lorentzian inner product using terminology from chapter 8 of [4] (One can compare this to the definition of the Lorentzian metric, definition 29)

Definition 3. *The inner product on* $\mathcal{M}$ *is a non-degenerate symmetric bilinear form* $\langle \cdot, \cdot \rangle_{\mathcal{M}} \colon \mathcal{M}^2 \to \mathbb{R}$ *of rank 4 and signature -2. On the standard basis of* $\mathcal{M}$, *it is represented by the matrix* $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Now, we introduce the electromagnetic fields,

Definition 4. *The electric field* $\mathbf{E}$ *and magnetic field* $\mathbf{B}$ *are two infinitely differentiable functions* $\mathbf{E}, \mathbf{B} \colon \mathcal{M} \to \mathbb{R}^3$.

These field have the following physical interpretation: When a point-charge with charge q is moving with velocity $\mathbf{v}$, the electromagnetic fields exert a force on this particle given by

$$\mathbf{F}(t,x,y,z) = q(\mathbf{E}(t,x,y,z) + \mathbf{v} \times \mathbf{B}(t,x,y,z))$$

This is the well-known Lorentz-force. In 1865, Maxwell enlisted the following equations that these fields obey:

Definition 5. *Maxwell's equations are the 4 equations*

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{2.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

Where ε_0, μ_0 and c are constants introduced for dimensionality purposes. ε_0 is called *permittivity of free space* or *electric constant*, μ_0 is called *permeability of free space* or *magnetic constant* and c is the speed of light, which we set to 1 (we could achieve this by saying we measure distances in units of light-seconds, and time-spans in units of seconds). Furthermore ρ is the charge-density and $\mathbf{J}$ is the current-density. In vacuum, those last two are 0, thus Maxwell's equations reduce to

Definition 6. *Maxwell's equations in vacuum (with c=1) are the 4 equations*

$$\nabla \cdot \mathbf{E} = 0 \tag{2.5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.6}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.7}$$

$$\nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} \tag{2.8}$$

All electromagnetic fields in this thesis satisfy these equations.

2.2 The Riemann-Silberstein vector and Maxwell's equations

Following Bateman [5], we can write Maxwell's equations in terms of the Riemann-Silberstein vector

Definition 7. *The Riemann-Silberstein vector* $\mathbf{F}$ *is an infinitely differentiable function* $\mathbf{F}\colon \mathcal{M} \to \mathbb{C}^3$*. It is related to* $\mathbf{E}$ *and* $\mathbf{B}$ *via* $\mathbf{F} = \mathbf{E} + i\mathbf{B}$

Using this vector, Maxwell's equations in vacuum reduce to 2 equations

Theorem 8. *Maxwell's equations in vacuum (definition 6) are equivalent to the two equations*

$$\nabla \cdot \mathbf{F} = 0 \tag{2.9}$$

$$\nabla \times \mathbf{F} = i\frac{\partial \mathbf{F}}{\partial t} \tag{2.10}$$

Proof. It is clear from definition 7 that $\nabla \cdot \mathbf{F} = 0 \Leftrightarrow \nabla \cdot \mathbf{E} + i\nabla \cdot \mathbf{B} = 0 \Leftrightarrow \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$ and similarly $\nabla \times \mathbf{F} = i\frac{\partial \mathbf{F}}{\partial t} \Leftrightarrow \nabla \times \mathbf{E} + i\nabla \times \mathbf{B} = i\left(\frac{\partial E}{\partial t} + i\frac{\partial B}{\partial t}\right) \Leftrightarrow \left(\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \text{ and } \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}\right)$ $\square$

2.3 Tensors, manifolds and Maxwell's equations

A very frequently used formalism of Maxwell's equations is using the electromagnetic tensor field $\mathbb{F}_{\mu\nu}$. To introduce this, we first need the notion of a tensor field on a manifold, definition 24. A good treatise on this matter, including more intrinsic definitions and subjects here omitted such as maximal atlases and general vector bundles, can be found in [6]. For this thesis, the following definitions will suffice.

Definition 9. *A real differentiable n-manifold is a set* Y *and a covering* $(U_i)_{i\in I}$ *with for each* $i \in I$ *an injective map* $\phi_i\colon U_i \to \mathbb{R}^n$ *such that for any* $p, q \in Y$*, either there exists* U_i *with* $p, q \in U_i$ *or there exist* U_i *and* U_j *with* $U_i \cap U_j = \varnothing$ *and* $p \in U_i$*,* $q \in U_j$*, and there exists a countable subset* $S \subseteq I$ *with* $\cup_{i\in S} U_i = Y$*, and finally for all* $i, j \in I$*,* $\phi_i(U_i \cap U_j)$ *is open and either* $U_i \cap U_j = \varnothing$ *or the map* $\phi_j \circ \phi_i^{-1}\colon \phi_i\left(U_i \cap U_j\right) \to \mathbb{R}^n$ *is infinitely differentiable.*
The topology on Y *is defined to be the topology induced by the maps* ϕ_i*.*
A tuple (U_i, ϕ_i) *is called a chart.*

Notation 10. *Although formally a real differentiable n-manifold thus consists of the tuple* $(Y, (U_i, \phi_i)_{i\in I})$*, it is commonly just written* Y*.*

One can compare this definition to Lemma 1.35 in [6]. Note that $\mathcal{M}$ can be considered a real differentiable 4-manifold when we choose $I = \{1\}$, $U_1 = \mathcal{M}$ and $\phi_1 = \mathrm{id}_{\mathcal{M}}$.

Definition 11. *The tangent space T_pY to an n-dimensional real manifold Y at $p \in Y$ is an n-dimensional real vector space of the form $\{p\} \times \mathbb{R}^n$. Given a chart (U_i, ϕ_i) with $p \in U_i$, the defining basis $(e_1, \ldots, e_n)$ for the codomain of ϕ_i induces a basis for T_pY, written $(\partial_1, \ldots, \partial_n)$ or $(\partial^i_1, \ldots, \partial^i_n)$ or $(\partial_1|_p, \ldots, \partial_n|_p)$ or $(\partial^i_1|_p, \ldots, \partial^i_n|_p)$. Given $(p, v) \in T_pY$, we write $v^\mu \partial^i_\mu$ to express v in terms of the basis $(\partial^i_1, \ldots, \partial^i_n)$.*

Although in the previous definition we wrote $\mathbb{R}^n$ for an n-dimensional real vector space, we would like to stress the fact that, unlike for the codomains of the ϕ_i in definition 9, a basis has not been chosen. Furthermore, any basis for T_xY is x-dependent. The bases induced by a chart give slightly less local bases for each tangent space, and it is these which we will use to define a topology on the tangent bundle. However, first we should know how the bases induced by two different charts are related, which is by their Jacobian.

Definition 12. *Given two charts (U_i, ϕ_i) and (U_j, ϕ_j) of a real differentiable n-manifold Y, and a common point $p \in U_i \cap U_j$, the induced bases $(\partial^i_1, \ldots, \partial^i_n)$ and $(\partial^j_1, \ldots, \partial^j_n)$ of T_pY are related via $\partial^i_\mu = J^\nu_\mu \partial^j_\nu$, where J^ν_μ is the Jacobian of the map $\phi_j \circ \phi_i^{-1}$, i.e. when we write $\phi_j \circ \phi_i^{-1} \colon \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$,
$\begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \mapsto \begin{pmatrix} (\phi_j \circ \phi_i^{-1})^1(v^1, \ldots, v^n) \\ \vdots \\ (\phi_j \circ \phi_i^{-1})^n(v^1, \ldots, v^n) \end{pmatrix}$, we get $J^\nu_\mu = \frac{\partial(\phi_j \circ \phi_i^{-1})^\nu(v^1, \ldots, v^\mu, \ldots, v^n)}{\partial v^\mu}\Big|_{\phi_i(x)}$.*

This can be compared to page 63 in [6]

Definition 13. *The tangent bundle TY of a real differentiable n-manifold Y is a real differentiable $2n$-manifold given by the set $\sqcup_{p \in Y} T_pY = Y \times \mathbb{R}^n$. For each chart (U_i, ϕ_i) of Y, there is a corresponding chart on TY given by $\tilde{U}_i = \sqcup_{p \in U_i} T_pY = U_i \times \mathbb{R}^n$ and $\tilde{\phi}_i \colon \tilde{U}_i \to \mathbb{R}^{2n}$, $(p, v^\mu \partial^i_\mu|_p) \mapsto (\phi_i(p), (v^\mu)^n_{\mu=1})$.*

This can be compared to Prop. 3.18 in [6]. Note that the topology on TY is the one which is induced by the maps $\tilde{\phi}_i$, which in general is different from the product topology on $Y \times \mathbb{R}^n$, as illustrated by the following examples.

Consider the Möbius strip: Let $I = \{1, 2\}$, $\phi_1(U_1) = (-1, 1) \times (-\pi \times \pi)$, $\phi_2(U_2) = (-1, 1) \times (0, 2\pi)$ and consider for $i \in I$ the maps

$$\phi_i^{-1} \colon \phi_i(U_i) \to \mathbb{R}^3,\ (x, y) \mapsto \begin{pmatrix} (2 + x\cos(y/2))\cos(y) \\ (2 + x\cos(y/2))\sin(y) \\ x\sin(y/2) \end{pmatrix}$$

When taking $Y = U_1 \cup U_2$, we get a manifold known as the Möbius strip. The map $\phi_2 \circ \phi_1^{-1}\colon (-1,1) \times \big((-\pi,0) \cup (0,\pi)\big) \to (-1,1) \times \big((0,\pi) \cup (\pi,2\pi)\big)$ is given by $(x,y) \mapsto \begin{cases} (x,y) & \text{if } y \in (0,\pi), \\ (-x, y+2\pi) & \text{if } y \in (-\pi,0), \end{cases}$ and has Jacobian $\begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } y \in (0,\pi), \\ \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) & \text{if } y \in (-\pi,0). \end{cases}$ One sees that around $\phi_1^{-1}\big((-1,1) \times \{0\}\big)$, the basis of the tangent space induced by ϕ_2 gets flipped, as can be understood when looking at a picture of a Möbius strip.

Now consider $S^2 = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ with $I = \{1,2\}$, $U_1 = S^2 \setminus \{(0,0,1)\}$, $\phi_1(x,y,z) = \big(\frac{x}{1-z}, \frac{y}{1-z}\big)$ and $U_2 = S^2 \setminus \{(0,0,-1)\}$, $\phi_2(x,y,z) = (\frac{x}{1+z}, \frac{y}{1+z})$. It can be checked that $\phi_1^{-1}(a,b) = (\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1})$, and thus $\phi_2 \circ \phi_1^{-1}(a,b) = (\frac{a}{a^2+b^2}, \frac{b}{a^2+b^2})$. The corresponding Jacobian is then given by $J = \begin{pmatrix} \frac{b^2-a^2}{(a^2+b^2)^2} & \frac{-2ab}{(a^2+b^2)^2} \\ \frac{-2ab}{(a^2+b^2)^2} & \frac{a^2-b^2}{(a^2+b^2)^2} \end{pmatrix}$, which is an orthogonal matrix with determinant $\det(J) = \frac{-1}{(a^2+b^2)^2}$. The substitution $(a,b) = (r\cos(\vartheta), r\sin(\vartheta))$ then gives $J/(\det(J))^2 = \begin{pmatrix} -\cos(2\vartheta) & -\sin(2\vartheta) \\ -\sin(2\vartheta) & \cos(2\vartheta) \end{pmatrix}$, which is a matrix for a rotation over 2ϑ combined with a reflection. Thus we see the basis induced by ϕ_2 gets rotated over 4π when walking a full circle around the point $(0,0,-1) \in S^2$.

Now before we can define tensors, we first need the notion of a cotangent bundle.

Definition 14. *The cotangent space T_p^*Y to an n-dimensional real manifold Y at $p \in Y$ is an n-dimensional real vector space of the form $\{p\} \times \mathbb{R}^n$, usually identified with the dual of T_pY. Given a chart (U_i, ϕ_i) with $p \in U_i$, the basis $(e^1, \ldots, e^n)$ dual to the defining basis of the codomain of ϕ_i induces a basis for T_p^*Y, written $(\mathrm{d}x^1, \ldots, \mathrm{d}x^n)$ or $(\mathrm{d}x_i^1, \ldots, \mathrm{d}x_i^n)$ or $(\mathrm{d}x^1|_p, \ldots, \mathrm{d}x^n|_p)$ or $(\mathrm{d}x_i^1|_p, \ldots, \mathrm{d}x_i^n|_p)$. Given $(p,\omega) \in T_p^*Y$, we write $\omega_\mu \mathrm{d}x_i^\mu$ to express ω in the basis $(\mathrm{d}x_i^1, \ldots, \mathrm{d}x_i^n)$*

Lemma 15. *Given two charts (U_i, ϕ_i) and (U_j, ϕ_j) of a real differentiable n-manifold Y, and a common point $p \in U_i \cap U_j$, the induced bases $(\mathrm{d}x_i^1, \ldots, \mathrm{d}x_i^n)$ and $(\mathrm{d}x_j^1, \ldots, \mathrm{d}x_j^n)$ of T_p^*Y are related via $\mathrm{d}x_i^\mu = (J^{-1})^\mu_\nu \, \mathrm{d}x_j^\nu$, where J is the Jacobian of $\phi_j \circ \phi_i^{-1}$ as in definition 12.*

Proof. As $T^*_{\phi_i(p)}\phi_i(U_i)$ and $T^*_{\phi_j(p)}\phi_j(U_j)$ are vector spaces, the map between them is a linear one, thus it is given by some matrix A. On the induced bases, we can express A as $A(\mathrm{d}x_i^\mu) = A^\mu_\nu \mathrm{d}x_j^\nu$, which we simply write as $\mathrm{d}x_i^\mu = A^\mu_\nu \mathrm{d}x_j^\nu$. Then, as $\partial^i_\sigma = J^\omega_\sigma \partial^j_\omega$ by definition 12, and for $k \in \{i, j\}$, $\mathrm{d}x_k^\mu(\partial^k_\sigma) = \delta^\mu_\sigma := \begin{cases} 1 & \text{if } \mu = \sigma, \\ 0 & \text{if } \mu \neq \sigma, \end{cases}$ as $\mathrm{d}x_k^\mu$ is a basis dual to ∂^k_μ, we get

$$\delta^\mu_\sigma = \mathrm{d}x_i^\mu(\partial^i_\sigma) = \mathrm{d}x_i^\mu(J^\omega_\sigma \partial^j_\omega) = J^\omega_\sigma \mathrm{d}x_i^\mu(\partial^j_\omega) = J^\omega_\sigma A^\mu_\nu \mathrm{d}x_j^\nu(\partial^j_\omega) = J^\omega_\sigma A^\mu_\nu \delta^\nu_\omega = J^\nu_\sigma A^\mu_\nu,$$

thus $I = J \cdot A$, thus $A = J^{-1}$. (Also compare formula 11.5 of [6]) □

Definition 16. *The cotangent bundle T^*Y of a real differentiable n-manifold Y is a $2n$-manifold given by the set $\sqcup_{p \in Y} T^*_p Y = Y \times \mathbb{R}^n$. For each chart (U_i, ϕ_i) of Y, there is a corresponding chart on T^*Y given by $\hat{U}_i = \sqcup_{p \in U_i} T^*_p Y = U_i \times \mathbb{R}^n$ and $\hat{\phi}_i \colon \hat{U}_i \to \mathbb{R}^{2n}$, $(p, \omega_\mu \mathrm{d}x_i^\mu|_p) \mapsto (\phi_i(p), (\omega_\mu)^n_{\mu=1})$.*

As with the tangent bundle, the cotangent bundle has a topology induced by the maps $\hat{\phi}_i$, which in general is different from the product topology on $Y \times \mathbb{R}^n$. Now that we have tangent and cotangent bundles, we would like to introduce tensor bundles. Recall the definition of a tensor product, (Found e.g. in chapter 12 of either [4] or [6])

Definition 17. *The tensor product between two vector spaces V and W of dimensions respectively n and m, is an nm-dimensional vector space $V \otimes W$ together with a bilinear map $\iota \colon V \times W \to V \otimes W$ such that for any bilinear map $h \colon V \times W \to Z$ to a real vector space Z, there exists a unique linear map $\tilde{h} \colon V \otimes W \to Z$ such that $\tilde{h} \circ \iota = h$. For $\iota(v, w)$, we write $v \otimes w$. Given bases $(e^V_1, \dots, e^V_n)$ and $(e^W_1, \dots, e^W_m)$ for V and W respectively, $(e^V_i \otimes e^W_j)_{(i,j) \in \mathbb{N}_{\leq n} \times \mathbb{N}_{\leq m}}$ forms a basis for $V \otimes W$.*

Lemma 18. *Given three vector spaces V_1, V_2 and V_3, the spaces $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes (V_2 \otimes V_3)$ are canonically isomorphic, and written as $V_1 \otimes V_2 \otimes V_3$.*

Proof. See the proof of note 12.8 of [4]. □

Notation 19. *The k-fold product $\underbrace{V \otimes \cdots \otimes V}_{k}$ is written as either $\bigotimes_k V$ or $V^{\otimes k}$.*

Definition 20. *A type (k, l)-tensor over an n-dimensional vector space V is an element of the n^{k+l}-dimensional vector space $T^k_l(V) := (\bigotimes_k V) \otimes (\bigotimes_l V^*)$, where V^* is the dual of V.*

Definition 21. *The type (k, l)-tensor space $T^k_l(T_p Y)$ to a real differentiable n-manifold Y at $p \in Y$ is an n^{k+l}-dimensional real vector space of the form*

$(\bigotimes_k T_pY) \otimes (\bigotimes_l T_p^*Y)$. Given a chart (U_i, ϕ_i), the induced bases $(\partial_1^i, \ldots, \partial_n^i)$ and $(\mathrm{d}x_i^1, \ldots, \mathrm{d}x_i^n)$ of T_pY and T_p^*Y respectively, induce a basis on $T_l^k(T_pY)$ of the form $(\partial_{\mu_1}^i \otimes \cdots \otimes \partial_{\mu_k}^i \otimes \mathrm{d}x_i^{\nu_1} \otimes \cdots \otimes \mathrm{d}x_i^{\nu_l})_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_l=1}^n$.
For $T \in T_l^k(T_pY)$, we write $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}(\partial_{\mu_1}^i \otimes \cdots \otimes \partial_{\mu_k}^i \otimes \mathrm{d}x_i^{\nu_1} \otimes \cdots \otimes \mathrm{d}x_i^{\nu_l})$ or simply $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}$ or $(T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l})_i$ or even $T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}$ or $\left(T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l}\right)_i$ to express T in the basis induced by (U_i, ϕ_i).

As with tangent and cotangent bundles, the only thing we need before introducing the tensor bundle is the transition between different induced bases.

Lemma 22. *Given two charts (U_i, ϕ_i) and (U_j, ϕ_j) of a real differentiable n-manifold Y, and a common point $p \in U_i \cap U_j$, the bases of $T_l^k(TY)$ $(\partial_{\mu_1}^i \otimes \cdots \otimes \partial_{\mu_k}^i \otimes \mathrm{d}x_i^{\nu_1} \otimes \cdots \otimes \mathrm{d}x_i^{\nu_l})_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_l=1}^n$ induced by ϕ_i and $(\partial_{\mu_1}^j \otimes \cdots \otimes \partial_{\mu_k}^j \otimes \mathrm{d}x_j^{\nu_1} \otimes \cdots \otimes \mathrm{d}x_j^{\nu_l})_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_l=1}^n$ induced by ϕ_j are related via*

$$\partial_{\mu_1}^i \otimes \cdots \otimes \partial_{\mu_k}^i \otimes \mathrm{d}x_i^{\nu_1} \otimes \cdots \otimes \mathrm{d}x_i^{\nu_l} =$$
$$J_{\mu_1}^{\rho_1} \ldots J_{\mu_k}^{\rho_k} (J^{-1})_{\sigma_1}^{\nu_1} \ldots (J^{-1})_{\sigma_l}^{\nu_l} \partial_{\rho_1}^j \otimes \cdots \otimes \partial_{\rho_k}^i \otimes \mathrm{d}x_i^{\sigma_1} \otimes \cdots \otimes \mathrm{d}x_i^{\sigma_l}$$

Proof. From definition 12 and lemma 15 we get for $p \in \mathbb{N}_{\leq k}$ and $q \in \mathbb{N}_{\leq l}$ that $\partial_{\mu_p}^i = J_{\mu_p}^{\rho_p} \partial_{\rho_p}^j$ and $\mathrm{d}x_i^{\nu_q} = (J^{-1})_{\sigma_q}^{\nu_q} \mathrm{d}x_j^{\sigma_q}$. As $\otimes$ is multilinear by definition 17, we get

$$\partial_{\mu_1}^i \otimes \cdots \otimes \partial_{\mu_k}^i \otimes \mathrm{d}x_i^{\nu_1} \otimes \cdots \otimes \mathrm{d}x_i^{\nu_l} =$$
$$(J_{\mu_1}^{\rho_1} \partial_{\rho_1}^j) \otimes \cdots \otimes (J_{\mu_k}^{\rho_k} \partial_{\rho_k}^i) \otimes ((J^{-1})_{\sigma_1}^{\nu_1} \mathrm{d}x_i^{\sigma_1}) \otimes \cdots \otimes ((J^{-1})_{\sigma_l}^{\nu_l} \mathrm{d}x_i^{\sigma_l}) =$$
$$J_{\mu_1}^{\rho_1} \ldots J_{\mu_k}^{\rho_k} (J^{-1})_{\sigma_1}^{\nu_1} \ldots (J^{-1})_{\sigma_l}^{\nu_l} \partial_{\rho_1}^j \otimes \cdots \otimes \partial_{\rho_k}^i \otimes \mathrm{d}x_i^{\sigma_1} \otimes \cdots \otimes \mathrm{d}x_i^{\sigma_l} \qquad \square$$

Definition 23. *The rank (k, l) tensor bundle $T_l^k(TY)$ to a real differentiable n-manifold Y is a real differentiable $(n + n^{k+l})$-manifold given by the set $\sqcup_{p\in Y} T_l^k(T_pY) = Y \times \mathbb{R}^{n^{k+l}}$. For each chart (U_i, ϕ_i) on Y, there is a corresponding chart on $T_l^k(TY)$ given by $\check{U}_i = \sqcup_{p\in U_i} T_l^k(T_pY) = U_i \times \mathbb{R}^{n^{k+l}}$ and $\check{\phi}_i \colon \check{U}_i \to \mathbb{R}^{n+n^{k+l}}$, $(p, (T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l})_i) \mapsto \left(\phi_i(p), (T^{\mu_1\ldots\mu_k}_{\nu_1\ldots\nu_l})_{\mu_1,\ldots,\mu_n,\nu_1,\ldots,\nu_k=1}^n\right)$.*

Again, the topology on $T_l^k(TY)$ is the one induced by the maps $\check{\phi}_i$, which is not necessarily the product topology on $Y \times \mathbb{R}^{n^{k+l}}$. However, for Minkowski space $\mathcal{M}$ we can just give a single chart $(U_1, \phi_1) = (\mathcal{M}, \mathrm{id}_{\mathbb{R}^4})$ and thus the map $\check{\phi}_1$ does identify $T_l^k(T\mathcal{M})$ with $\mathbb{R}^{4+4^{k+l}}$. Furthermore, we have $T_0^1(TY) = TY$ and $T_1^0(TY) = T^*Y$, and we choose $T_0^0(TY) := Y \times \mathbb{R}$.

Definition 24. *A type (k,l) tensor field $\mathbb{T}$ on a real differentiable n-manifold Y is a map $\mathbb{T}\colon Y \to T^k_l(TY)$ such that $\forall p \in Y$, $\mathbb{T}(p) \in T^k_l(T_pY)$ and for any chart (U_i, ϕ_i) of Y, the map $\check{\phi}_i \circ \mathbb{T} \circ \phi_i^{-1}\colon \phi_i(U_i) \to \mathbb{R}^{n+n^{k+l}}$ is infinitely differentiable.*

The condition that $\mathbb{T}(p) \in T^k_l(T_pY)$ is based of our construction of $T^k_l(TY)$ as a disjoint union $\sqcup_{p\in Y} T^k_l(T_pY)$. Another frequently used way to formalize this is to introduce a map $\pi\colon T^k_l(TY) \to Y$ and define $T^k_l(T_pY)$ to be $\pi^{-1}(p)$ endowed with the structure of an n^{k+l}-dimensional vector space. Then this condition becomes $\pi \circ \mathbb{T} = \mathrm{id}_Y$. More on this approach can be found in chapter 10 of [6].

Tensor field of type $(1,0)$ are called *vector fields*, and those of type $(0,1)$ are called *covector fields*. Sometimes it is not possible to define a certain tensor field on the whole manifold, but you can define it almost everywhere.

Definition 25. *A type (k,l) tensor field $\mathbb{T}$ defined almost everywhere on a real differentiable n-mainfold Y is a map $\mathbb{T}\colon U \to T^k_l(T\mathrm{Y})$ such that $\forall p \in U$, $\mathbb{T}(p) \in T^k_l(T_pY)$, U is topologically dense in Y and for any chart (U_i, ϕ_i) of Y, the map $\check{\phi}_i \circ \mathbb{T} \circ \phi_i^{-1}\colon \phi_i(U_i \cap U) \to \mathbb{R}^{n+n^{k+l}}$ is infinitely differentiable.*

Now that we have defined tensor fields, we still need several definitions before we can address Maxwell's equations.

Notation 26. *Given a chart (U_i, ϕ_i) of a manifold Y and a tensor field $\mathbb{T}\colon \mathrm{Y} \to T^k_l(TY)$, the tensor field $\mathbb{T}|_{U_i}$ is usually written using the notations of definition 21, so for example as $(\mathbb{T}^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l})_i$, where the components are considered infinitely differentiable functions $\mathbb{T}^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}\colon U_i \to \mathbb{R}$.*

Lemma 27. *There is a canonical isomorphism $\tilde{\psi}\colon T^k_l(V) \xrightarrow{\sim} \mathrm{L}\left((V^*)^k \times V^l; \mathbb{R}\right)$, where $\mathrm{L}\left((V^*)^k \times V^l; \mathbb{R}\right)$ are the multilinear functions from $(V^*)^k \times V^l$ to $\mathbb{R}$*

Proof. Consider the map $\psi\colon V^k \times (V^*)^l \to \mathrm{L}\big((V^*)^k \times V^l; \mathbb{R}\big)$ such that for any $(v_1, \ldots, v_k, \omega_1, \ldots, \omega_l) \in V^k \times (V^*)^l$ and $(\sigma_1, \ldots, \sigma_k, x_1, \ldots, x_l) \in (V^*)^k \times V^l$ we have $\big(\psi(v_1, \ldots, v_k, \omega_1, \ldots, \omega_l)\big)(\sigma_1, \ldots, \sigma_k, x_1, \ldots, x_l) = \sigma_1(v_1)\cdots\sigma_k(v_k)\cdot\omega_1(x_1)\cdots\omega_l(x_l)$. It is easilly verified that the image $\psi(v_1, \ldots, v_k, \omega_1, \ldots, \omega_l)$ as well as ψ itself are multilinear, so ψ is a well-defined multilinear function, hence it uniquely extends to a linear function $\tilde{\psi}$ by definition 17. Bijectivity of $\tilde{\psi}$ follows from the observation that the image of a basis of $T^k_l(V)$ forms a basis of $L\left((V^*)^k \times V^l; \mathbb{R}\right)$, as in proposition 12.10 in [6]. □

Definition 28. *A tensor field $\mathbb{T}$ on a manifold Υ is called respectively* symmetric, antisymmetric *or* non-degenerate *when the multilinear map $\tilde{\psi}(\mathbb{T}(p))$ is respectively symmetric, antisymmetric or non-degenerate for all $p \in \Upsilon$.*

Definition 29. *The metric $\mathfrak{g}_{\mu\nu}$ of Minkowski space is a symmetric non-degenerate tensor field of type $(0,2)$. In the standard basis, it is given by*

$$\mathfrak{g}_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \mathrm{d}x^0 \otimes \mathrm{d}x^0 - \mathrm{d}x^1 \otimes \mathrm{d}x^1 - \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3.$$

Lemma 30. *The metric $\mathfrak{g}_{\mu\nu}$ induces a canonical isomorphism $\tilde{\mathfrak{g}}_p \colon T_p\mathcal{M} \xrightarrow{\sim} T_p^*\mathcal{M}$, and hence for every $j \in \mathbb{N}_{\leq k}$ an isomorphism $\tilde{\mathfrak{g}}^j \colon T_l^k(T\mathcal{M}) \to T_{l+1}^{k-1}(T\mathcal{M})$.*

Proof. Consider $\tilde{\mathfrak{g}}_p \colon T_p\mathcal{M} \to T_p^*\mathcal{M}$, $v \mapsto \big(w \mapsto \tilde{\psi}(\mathfrak{g}(p))(v,w)\big)$. It is bijective as $\dim(T_p\mathcal{M}) = \dim(T_p^*\mathcal{M})$ and furthermore $\tilde{\mathfrak{g}}_p(v_1) = \tilde{\mathfrak{g}}_p(v_2) \Leftrightarrow \tilde{\mathfrak{g}}_p(v_1 - v_2) = 0 \Leftrightarrow v_1 - v_2 = 0$ as $\mathfrak{g}$ is non-degenerate. Given coordinates, we have for $v = v^\mu \partial_\mu|_p$ that $\tilde{\mathfrak{g}}_p(v) = \mathfrak{g}_{\mu\nu} v^\mu \mathrm{d}x^\nu|_p$. Thus, for $\mathbb{T} \in T_l^k(T\mathcal{M})$, we can let $\tilde{\mathfrak{g}}$ act on the j-th space of $T_l^k(T\mathcal{M})$, i.e.

$$\tilde{\mathfrak{g}}^j(\mathbb{T}^{\mu_1\dots\mu_j\dots\mu_k}{}_{\nu_1\dots\nu_l})^{\mu_1\dots}{}_\sigma{}^{\dots\mu_k}{}_{\nu_1\dots\nu_l} = \mathfrak{g}_{\mu_j\sigma}\mathbb{T}^{\mu_1\dots\mu_j\dots\mu_k}{}_{\nu_1\dots\nu_l}. \qquad \square$$

Lemma 30 allows us to raise and lower indices of tensor fields, given these fields are expressed in coordinates (otherwise they do not even have indices). Whenever this happens, it is important to keep the construction as explained in the proof in mind. We can now introduce the electromagnetic tensor.

Definition 31. *The electromagnetic tensor $\mathbb{F}_{\mu\nu} \in T_2^0(T\mathcal{M})$ is a type $(0,2)$ antisymmetric tensor field on $\mathcal{M}$. The electromagnetic fields $\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} \colon \mathcal{M} \to \mathbb{R}^3$ are related to $\mathbb{F}_{\mu\nu}$ via*

$$\begin{gathered}\mathbb{F}_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = \\ E_1(\mathrm{d}x^0\otimes\mathrm{d}x^1 - \mathrm{d}x^1\otimes\mathrm{d}x^0) + E_2(\mathrm{d}x^0\otimes\mathrm{d}x^2 - \mathrm{d}x^2\otimes\mathrm{d}x^0) + E_3(\mathrm{d}x^0\otimes\mathrm{d}x^3 - \mathrm{d}x^3\otimes\mathrm{d}x^0) + \\ B_1(\mathrm{d}x^3\otimes\mathrm{d}x^2 - \mathrm{d}x^2\otimes\mathrm{d}x^3) + B_2(\mathrm{d}x^1\otimes\mathrm{d}x^3 - \mathrm{d}x^3\otimes\mathrm{d}x^1) + B_3(\mathrm{d}x^2\otimes\mathrm{d}x^1 - \mathrm{d}x^1\otimes\mathrm{d}x^2)\end{gathered}$$

Definition 32. *The Levi-Civita symbol is a function $\varepsilon \colon (\mathbb{Z}_{\geq 0, <n})^n \to \{-1, 0, 1\}$ that assigns to a tuple $(a_1, \dots, a_n)$ the sign of the permutation $(0, \dots, n) \mapsto (a_1, \dots, a_n)$, or 0 when there are distinct i, j such that $a_i = a_j$. $\varepsilon(a_1, \dots, a_n)$ is usually written $\varepsilon^{a_1\dots a_n}$ or $\varepsilon_{a_1\dots a_n}$. One can then also write $\varepsilon^{a_1\dots a_n} = \det(\delta_j^{a_i})_{ij} = \det\begin{pmatrix} \delta_1^{a_1} & \dots & \delta_1^{a_n} \\ \vdots & \ddots & \vdots \\ \delta_n^{a_1} & \dots & \delta_n^{a_n} \end{pmatrix}$.*

Before we can write Maxwell's equations, we first need the notion of a derivative on a manifold. A more natural way to treat this is in the formalism of 2-forms, definition 41. As that will be our main formalism, we give a coordinate-dependent notion of the derivative here. A coordinate-independent notion would use the Levi-Civita connection, which is expressed in coordinates with Christoffel symbols. However, in $\mathcal{M}$ with the standard basis, the Christoffel symbols are all 0, and the Levi-Civita connection is very similar to the following definition.

Definition 33. *The partial derivative $\partial_\nu \mathbb{T}^{\mu_1\cdots\mu_k}$ of a type $(k,0)$ tensor field on an n-manifold Y with respect to a basis $(\partial_1,\dots,\partial_n)$ of TY induced by a chart (U,ϕ) is defined via the representation of $\mathbb{T}$ induced by ϕ, $\check{\mathbb{T}} = \tilde{\pi}\circ\check{\phi}\circ\mathbb{T}\circ\phi^{-1}\colon \phi(U)\to\mathbb{R}^{n^k}$, where $\tilde{\pi}\colon\phi(U)\times\mathbb{R}^{n^k}\to\mathbb{R}^{n^k}$ is the projection, and $\check{\phi}$ is as in definition 23. Then $\partial_\nu \mathbb{T}^{\mu_1\cdots\mu_k}$ is just the partial derivative of the $\mathbb{T}^{\mu_1\cdots\mu_k}$-component of $\check{\mathbb{T}}$ with respect to the νth coordinate of $\phi(U)$.*

Lemma 34. *Maxwell's equations in vacuum (definition 6) in the standard basis for $\mathcal{M}$ are equivalent to the set of equations*

$$\partial_\mu \mathbb{F}^{\mu\nu} = 0 \tag{2.11}$$

$$\partial_\mu(\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathbb{F}_{\rho\sigma}) = 0 \tag{2.12}$$

where $\nu\in\{0,1,2,3\}$

Proof. Using definition 31, formula (2.11) gives for $\nu=0$ that

$$\partial_\mu \mathbb{F}^{\mu 0} = 0 \Leftrightarrow -\partial_1 E_1 - \partial_2 E_2 - \partial_3 E_3 = -\nabla\cdot\mathbf{E} = 0,$$

(Note the extra minus signs, because $\mathbb{F}^{\mu\nu} = \mathfrak{g}^{\mu\rho}\mathfrak{g}^{\nu\sigma}\mathbb{F}_{\rho\sigma}$, which gives a minus sign when one of μ,ν is 0, see lemma 30) and formula (2.12) gives

$$\partial_\mu(\tfrac{1}{2}\varepsilon^{\mu 0\rho\sigma}\mathbb{F}_{\rho\sigma}) = 0 \Leftrightarrow \partial_1(\tfrac{\mathbb{F}_{32}-\mathbb{F}_{23}}{2}) + \partial_2(\tfrac{\mathbb{F}_{13}-\mathbb{F}_{31}}{2}) + \partial_3(\tfrac{\mathbb{F}_{21}-\mathbb{F}_{12}}{2}) = \nabla\cdot\mathbf{B} = 0,$$

while with $\nu = \begin{pmatrix}1\\2\\3\end{pmatrix}$ formula (2.11) gives

$$\begin{pmatrix}\partial_\mu \mathbb{F}^{\mu 1}\\ \partial_\mu \mathbb{F}^{\mu 2}\\ \partial_\mu \mathbb{F}^{\mu 3}\end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix}-\partial_0 E_1 + \partial_2 B_3 - \partial_3 B_2\\ -\partial_0 E_2 - \partial_1 B_3 + \partial_3 B_2\\ -\partial_0 E_3 + \partial_1 B_2 - \partial_2 B_1\end{pmatrix} = -\frac{\partial\mathbf{E}}{\partial t} + \nabla\times\mathbf{B} = 0$$

and formula (2.12) gives

$$\begin{pmatrix}\partial_\mu(\tfrac{1}{2}\varepsilon^{\mu 1\rho\sigma}\mathbb{F}_{\rho\sigma})\\ \partial_\mu(\tfrac{1}{2}\varepsilon^{\mu 2\rho\sigma}\mathbb{F}_{\rho\sigma})\\ \partial_\mu(\tfrac{1}{2}\varepsilon^{\mu 3\rho\sigma}\mathbb{F}_{\rho\sigma})\end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix}-\partial_0\frac{\mathbb{F}_{32}-\mathbb{F}_{23}}{2} - \partial_2\frac{\mathbb{F}_{03}-\mathbb{F}_{30}}{2} + \partial_3\frac{\mathbb{F}_{02}-\mathbb{F}_{20}}{2}\\ -\partial_0\frac{\mathbb{F}_{13}-\mathbb{F}_{31}}{2} + \partial_1\frac{\mathbb{F}_{03}-\mathbb{F}_{30}}{2} - \partial_3\frac{\mathbb{F}_{01}-\mathbb{F}_{10}}{2}\\ -\partial_0\frac{\mathbb{F}_{21}-\mathbb{F}_{12}}{2} - \partial_1\frac{\mathbb{F}_{02}-\mathbb{F}_{20}}{2} + \partial_2\frac{\mathbb{F}_{01}-\mathbb{F}_{10}}{2}\end{pmatrix} =$$

$$\begin{pmatrix}-\partial_0 B_1 - \partial_2 E_3 + \partial_3 E_2\\ -\partial_0 B_2 + \partial_1 E_3 - \partial_3 E_1\\ -\partial_0 B_3 - \partial_1 E_2 + \partial_2 E_1\end{pmatrix} = -\frac{\partial\mathbf{B}}{\partial t} - \nabla\times\mathbf{E} = 0. \qquad \square$$

2.3.1 Intermezzo for physicists

In the previous sections, we have given a general outline of the theory of classical electromagnetism in terms of ordinary differential equations (definition 5), in terms of the Riemann-Silberstein vector (theorem 8) and in terms of the electromagnetic tensor (lemma 34).

The first and the last are very standard, as e.g. in [7]. The Riemann-Silberstein vector introduces complex numbers into the Maxwell's equations. It should be pointed out that this primarily simplifies the mathematics, and there is no clear physical meaning behind this construction. A more natural framework is the formalism of 2-forms, lemma 46. This formalism is only a slight modification of the tensor formalism. When this construction is extended to complexified Minkowski space in lemma 56, one obtains a representation that is again similar to the Riemann-Silberstein vector. But again, only the real part of complexified Minkowski space can unambiguously be given a physical interpretation.

2.4 2-forms and Maxwell's equations

A more natural way to express Maxwell's equations is in the formalism of differential forms. A differential k-form (see definition 37) is just an alternating (see definition 28) tensor field of type $(0,k)$ (see definition 21), but to be able to speak of the space of k-forms, we have to follow the same steps as in definitions 21 up to 24.

Definition 35. *The k-th exterior power of a vector space V, written $\bigwedge^k V$, is the subspace of $T^0_k(V)$ consisting of all alternating tensors of type $(0,k)$ on V.*
There is a natural linear map $\xi\colon T^0_k(V) \to \bigwedge^k V$ that is the identity on $\bigwedge^k(V) \subseteq T^0_k(V)$ given by $\xi(T_{\mu_1\ldots\mu_k}) = \frac{1}{k!}\sum_{\sigma\in S_k}\operatorname{sgn}(\sigma)T_{\mu_{\sigma(1)}\cdots\mu_{\sigma(k)}} = \frac{1}{k!}\varepsilon^{\nu_1\ldots\nu_k}T_{\nu_1\ldots\nu_k}\varepsilon_{\mu_1\ldots\mu_k}$.
Given a basis $(\mathrm{d}x^{\mu_1}\otimes\cdots\otimes\mathrm{d}x^{\mu_k})^n_{\mu_1,\ldots,\mu_k=1}$ of $T^0_k(V)$, its image under ξ forms a basis of $\bigwedge^k V$ written as $(\mathrm{d}x_i^{\mu_1}\wedge\cdots\wedge\mathrm{d}x_i^{\mu_k})_{1\le\mu_1<\cdots<\mu_k\le n}$

For $\alpha \in \bigwedge^k V$ and $\beta \in \bigwedge^l V$, we can construct $\alpha\wedge\beta \in \bigwedge^{k+l} V$ as $\xi(\iota(\alpha,\beta))$, where $\iota\colon (\bigotimes_k V)\times(\bigotimes_l V) \to \bigotimes_{k+l} V$ is as in definition 17 and ξ as in definition 35.

Definition 36. *The k-th exterior power bundle $\bigwedge^k(T^*Y)$ is a real differentiable $\left(n+\binom{n}{k}\right)$-manifold given by the set $\sqcup_{p\in Y}\bigwedge^k(T^*_pY) = Y\times\mathbb{R}^{\binom{n}{k}}$. For each chart (U_i,ϕ_i) on Y, there is a corresponding chart on $\bigwedge^k(T^*Y)$ given by $\grave{U}_i = \sqcup_{p\in U_i}\bigwedge^k(T^*_pY) = U_i\times\mathbb{R}^{\binom{n}{k}}$ and $\grave{\phi}_i\colon \grave{U}_i \to \mathbb{R}^{n+\binom{n}{k}}$, $(p, T_{\mu_1\ldots\mu_k}) \mapsto (\phi_i(p), (T_{\mu_1\ldots\mu_k})_{1\le\mu_1<\cdots<\mu_k\le n})$.*

Definition 37. *A differential k-form* $\omega \in \Omega^k(\Upsilon)$ *on an n-manifold* Υ *is an alternating tensor field of type* $(0,k)$*, i.e. a function* $\omega\colon \Upsilon \to \bigwedge^k(T^*\Upsilon)$ *such that* $\omega(p) \in \bigwedge^k(T_p^*\Upsilon)$ *for all* $p \in \Upsilon$*, and for any chart* (U_i, ϕ_i) *the map* $\dot{\phi}_i \circ \omega \circ \phi_i^{-1}\colon \phi_i(U_i) \to \mathbb{R}^{n+\binom{n}{k}}$ *is infinitely differentiable.*

Note that the space $\Omega^k(\Upsilon)$ of k-forms on a manifold Υ itself can be considered a vector space when addition and scalar multiplication are defined pointwise, i.e. $(\lambda\omega + \mu\eta)(p) = \lambda\omega(p) + \mu\eta(p)$ for $p \in \Upsilon$, $\lambda, \mu \in \mathbb{R}$ and $\omega, \eta \in \Omega^k(\Upsilon)$. Furhermore, it is worth noting that $\Omega^0(\Upsilon)$ is the space of all functions $f\colon \Upsilon \to \Upsilon \times \mathbb{R}$ for which $f(p) = (p, \tilde{f}(p))$ for some differentiable $\tilde{f}\colon \Upsilon \to \mathbb{R}$. Thus $\Omega^0(\Upsilon)$ can be identified with the space of all differentiable functions $\tilde{f}\colon \Upsilon \to \mathbb{R}$. The electromagnetic 2-form is exactly the same as the electromagnetic tensor.

Definition 38. *The electromagnetic 2-form* $\mathscr{F} \in \Omega^2(\mathcal{M})$ *is the electromagnetic tensor (definition 31) viewed as a 2-form. Thus* $\mathscr{F} = \mathbb{F}_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$*.*

Now, we need the notions of the exterior derivative and the Hodge dual. We will first give the exterior derivative, definition 41. For this we need the notion of a differential, which is closely related to the notion of pullbacks.

Definition 39. *The differential* $\mathrm{d}f\colon TM \to TN$ *of a function* $f\colon M \to N$ *is a map such that* $\forall p \in M$*,* $\forall v \in T_pM$*,* $\mathrm{d}f(v) \in T_{f(p)}N$ *and* $\mathrm{d}f|_{T_pM}\colon T_pM \to T_{f(p)}N$ *is linear. Furthermore, for charts* (U_i, ϕ_i)*,* (V_j, ψ_j) *of* M *and* N *respectively, with* $p \in U_i$ *such that* $f(p) \in V_j$*, the map* $\mathrm{d}f|_{T_pM}$ *is given on the induced bases by the Jacobian of* $\psi_j \circ f \circ \phi_i^{-1}$*. (See also definition 12)*

Definition 40. *The pullback* $f^*\colon T^*N \to T^*M$ *of a function* $f\colon M \to N$ *is a map such that* $\forall p \in M$*,* $\forall v \in T^*_{f(p)}N$*,* $f^*(v) \in T_p^*M$ *and* $f^*|_{T^*_{f(p)}N}\colon T^*_{f(p)}N \to T_p^*M$ *is linear.*
Furthermore, for charts (U_i, ϕ_i)*,* (V_i, ψ_i) *of* M *and* N *respectively, with* $p \in U_i$ *such that* $f(p) \in V_i$*, the map* $f^*|_{T^*_{f(p)}N}$ *is given on the induced bases by the transpose of the Jacobian of* $\psi_j \circ f \circ \phi_i^{-1}$*.*

The pullback naturally extends to a map $f_p^*\colon T_k^0(T_{f(p)}N) \to T_k^0(T_pM)$ and hence to a map $f^*\colon \Omega^k(N) \to \Omega^k(M)$: for $p \in M$, we have an induced map $(f^*)^k\colon (T^*_{f(p)}N)^k \to (T_p^*M)^k$, which combined with ι_M from definition 17 gives a map $(T^*_{f(p)}N)^k \overset{(f^*)^k}{\to} (T_p^*M)^k \overset{\iota_M}{\to} T_k^0(T_pM)$.
As this map is multilineair, definition 17 extends it to a lineair map

$f_p^*\colon T_k^0(T_{f(p)}N) \to T_k^0(T_pM)$, which can be restricted to $\bigwedge^k T_{f(p)}^*N \subseteq T_k^0(T_{f(p)}N)$ and composed with $\xi_{T_p^*M}$ from definition 35 to give a map $\xi_{T_p^*M} \circ f_p^*|_{\bigwedge^k T_{f(p)}^*N}\colon \bigwedge^k T_{f(p)}^*N \to \bigwedge^k T_p^*M$. This map can be applied pointwise to a k-form $\mathscr{K} \in \Omega^k(N)$, so $f^*(\mathscr{K})(p) = \xi_{T_p^*M} \circ f_p^*|_{\bigwedge^k T_{f(p)}^*N}(\mathscr{K}(f(p)))$.

Note that with these definitions, the relation between two induced bases as given in definition 12 and lemma 15 can be interpreted as the differential respectively the pullback of the identity $\mathrm{id}_Y\colon Y \to Y$ with respect to two different charts. Thus the differential and pullback are defined such that $\mathrm{d}(\mathrm{id}_Y) = \mathrm{id}_{TY}$ and $(\mathrm{id}_Y)^* = \mathrm{id}_{T^*Y}$. A case of particular interest is when $N = \mathbb{R}$, as the differential of a function $f\colon Y \to \mathbb{R}$ can then be considered a 1-form. By definition 39, the differential $\mathrm{d}f\colon TY \to T\mathbb{R}$ is a function such that $\mathrm{d}f|_{T_pY}(p,v) = (f(p), \widetilde{\mathrm{d}f}_p(v))$. Now the function $\widetilde{\mathrm{d}f}_p$ is a function from T_pY to $\mathbb{R}$, i.e. an element of T_p^*Y. Thus by definition 24, the function $\widetilde{\mathrm{d}f}$ defined by $\widetilde{\mathrm{d}f}\colon Y \to T^*Y$, $p \mapsto \widetilde{\mathrm{d}f}_p$ is a type $(0,1)$ tensor field of Y, i.e. a 1-form. This 1-form is usually written as $\mathrm{d}f$, and this means d can be considered a function from $\Omega^0(Y)$ to $\Omega^1(Y)$. Now we can define the exterior derivative.

Definition 41. *The exterior derivative on k-forms* $\mathrm{d}_k\colon \Omega^k(Y) \to \Omega^{k+1}(Y)$ *is the unique extension of the differential* $\mathrm{d} = \mathrm{d}_0\colon \Omega^0(Y) \to \Omega^1(Y)$ *such that* $\mathrm{d}_{k+1} \circ \mathrm{d}_k = 0$, *and for* $\alpha \in \Omega^k(Y), \beta \in \Omega^l(Y)$ *we have that* $\mathrm{d}_{k+l}(\alpha \wedge \beta) = \mathrm{d}_k(\alpha) \wedge \beta + (-1)^k(\alpha \wedge \mathrm{d}_l(\beta))$.
Given a chart (U_i, ϕ_i) *of* Y, *and* $\alpha \in \Omega^k(Y)$, *on the induced basis* $\mathrm{d}_k(\alpha)|_{U_i}$ *is given by* $\frac{\partial \alpha_{\mu_1 \cdots \mu_k}}{\partial x^{\mu_0}} \mathrm{d}x^{\mu_0} \wedge \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}$, *where for* $p \in Y$, $\frac{\partial \alpha_{\mu_1 \cdots \mu_k}}{\partial x^{\mu_0}}(p)$ *is just the* $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k}$*-component of the partial derivative of* $\grave{\phi}_i \circ \alpha \circ \phi_i^{-1}\colon \phi_i(U_i) \to \mathbb{R}^{n+\binom{n}{k}}$ *with respect to the* μ_0*th coordinate, evaluated in* p.

Now we need the notion of the Hodge dual, definition 45. For this we need the notion of a volume form.

Definition 42. *A manifold* Y *is called orientable when the manifold* $\bigwedge^n(T^*\mathrm{Y})$ *is isomorphic to* $\mathrm{Y} \times \mathbb{R}$ *(either as a topological space or as a differentiable manifold). We then say that the bundle* $\bigwedge^n(T^*\mathrm{Y})$ *is trivial.*

Definition 43. *A volume form* $\omega \in \Omega^n(Y)$ *on an orientable differentiable n-manifold* Y *with respect to a non-degenerate symmetric type* $(0,2)$ *tensor field* $\mathfrak{g}_{\mu\nu}$ *is an n-form such that for a chart* (U_i, ϕ_i), $\omega|_{U_i}$ *is equal to* $\pm\sqrt{\left|\det\begin{pmatrix} (\mathfrak{g}_{11})_i & \cdots & (\mathfrak{g}_{1n})_i \\ \vdots & \ddots & \vdots \\ (\mathfrak{g}_{n1})_i & \cdots & (\mathfrak{g}_{nn})_i \end{pmatrix}\right|}\, \mathrm{d}x_i^1 \wedge \cdots \wedge \mathrm{d}x_i^n$, *where the* $\pm$ *are to be chosen such*

that ω is consistently defined.
On $\mathcal{M}$, we define the volume form as $\omega = \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$, which can then also be given as $\omega = \frac{1}{k!}\varepsilon_{\mu_0\mu_1\mu_2\mu_3}\mathrm{d}x^{\mu_0} \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \mathrm{d}x^{\mu_3}$

Definition 44. *The Hodge star at p, $\star\colon \bigwedge^k(T_p^*\Upsilon) \to \bigwedge^{n-k}(T_p^*\Upsilon)$ is the unique linear map such that for all $\alpha, \beta \in \bigwedge^k(T_p^*\Upsilon)$ we have that $\alpha \wedge (\star\beta) = \langle\alpha, \beta\rangle\omega(p)$, where $\langle\alpha, \beta\rangle = \alpha_{\mu_1\ldots\mu_k}\beta^{\mu_1\cdots\mu_k} = \mathfrak{g}^{\mu_1\nu_1}\ldots\mathfrak{g}^{\mu_k\nu_k}\alpha_{\mu_1\ldots\mu_k}\beta_{\nu_1\ldots\nu_k}$.*
Given a chart (U_i, ϕ_i), we have that $(\star\alpha)_{\nu_1\ldots\nu_{n-k}} = \omega_{\mu_1\ldots\mu_k\nu_1\ldots\nu_{n-k}}\alpha^{\mu_1\cdots\mu_k}$.

Definition 45. *The Hodge dual $\star\alpha \in \Omega^{n-k}(\Upsilon)$ of a k-form $\alpha \in \Omega^k(\Upsilon)$ on a differentiable n-manifold Υ with respect to a symmetric non-degenerate type (0,2) tensor $\mathfrak{g}_{\mu\nu}$ is given by $(\star\alpha)(p) = \star(\alpha(p))$, where $\star(\alpha(p))$ is as in definition 44.*

Now we can write Maxwell's equations in this formalism.

Lemma 46. *Maxwell's equations in vacuum (definition 6) are equivalent to the set of equations*

$$\mathrm{d}_2\mathscr{F} = 0 \tag{2.13}$$

$$\mathrm{d}_2\star\mathscr{F} = 0 \tag{2.14}$$

Proof. We will show these equations are equivalent to equations (2.11) and (2.12). From definition 41 we get $\mathrm{d}_2\mathscr{F} = \partial_\alpha\mathbb{F}_{\beta\gamma}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$. By antisymmety, we then get $\mathrm{d}_2\mathscr{F} = 0 \Leftrightarrow (\forall\delta \in \{0,1,2,3\},\ \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\mathbb{F}_{\beta\gamma} = 0)$, which is equivalent to (2.12). Similarly, using definitions 43, 45 and 41, we get $\mathrm{d}_2\star\mathscr{F} = \partial_\alpha(\frac{1}{2!}\varepsilon_{\mu\nu\beta\gamma}\mathbb{F}^{\mu\nu})\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma$, thus again by antisymmetry we get $\mathrm{d}_2\star\mathscr{F} = 0 \Leftrightarrow (\forall\delta \in \{0,1,2,3\},\ \varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha(\frac{1}{2}\varepsilon_{\mu\nu\beta\gamma}\mathbb{F}^{\mu\nu}) = 0)$. As $\varepsilon^{\alpha\beta\gamma\delta}\varepsilon_{\mu\nu\beta\gamma} = \delta^\delta_\mu\delta^\alpha_\nu - \delta^\delta_\nu\delta^\alpha_\mu$, this is thus equivalent to $\frac{\partial_\alpha\delta^\delta_\mu\delta^\alpha_\nu\mathbb{F}^{\mu\nu} - \partial_\alpha\delta^\delta_\nu\delta^\alpha_\mu\mathbb{F}^{\mu\nu}}{2} = \frac{\partial_\nu\mathbb{F}^{\delta\nu} - \partial_\mu\mathbb{F}^{\mu\delta}}{2} = -\partial_\mu\mathbb{F}^{\mu\delta} = 0$, which is equation (2.11). □

2.5 Maxwell's equations on complex manifolds

Sometimes, Maxwell's equations are considered on complex manifolds. There are several formalisms that can be used for this. We will look into (anti)-self-dual forms and touch upon the Spinor formalism. First, we need to modify the definitions as given in 9 up to 37 to apply to complex manifolds. For definitions 9 up to 37, one can handle exactly the same definitions after changing the word "real" to "complex" and "infinitely differentiable" to "holomorphic". A formal treatise on this can be found in [8]. As an example, we will give the equivalent of definition 9. Equivalents of defin ition 29 and 43 are given in definition 52 and 53 respectively.

Definition 47. *A complex holomorphic n-manifold is a set* Y *and a covering* $(U_i)_{i\in I}$ *with for each* $i \in I$ *an injective map* $\phi_i \colon U_i \to \mathbb{C}^n$ *such that for any* $p, q \in Y$*, either there exists* U_i *with* $p, q \in U_i$ *or there exist* U_i *and* U_j *with* $U_i \cap U_j = \varnothing$ *and* $p \in U_i, q \in U_j$*, and there exists a countable subset* $S \subseteq I$ *with* $\cup_{i\in S} U_i = Y$*, and finally for all* $i, j \in I$*,* $\phi_i(U_i \cap U_j)$ *is open and either* $\phi_j \circ \phi_j^{-1} \colon \phi_i(U_i \cap U_j) \to \mathbb{C}^n$ *is holomorphic or* $U_i \cap U_j = \varnothing$*. The topology on* Y *is defined to be the topology induced by the maps* ϕ_i*. A tuple* (U_i, ϕ_i) *is called a chart.*

We now give some way to relate real manifolds to complex manifolds. More about relating real manifolds to complex manifolds could include almost-complex structures and the Newlander-Nirenberg Theorem, which explains how a real $2n$-manifold can be made into a complex n-manifold. However, we do not include this in this thesis.

Definition 48. *The complexification* $\mathbb{C}V$ *of a real vector space* V *is the tensor product between the real vector spaces* V *and* $\mathbb{C}$*, where for* $\mathbb{C}$ *we choose the basis* $\{1, i\}$ *if necessary. The inclusion* $V \hookrightarrow \mathbb{C}V$ *is given by* $v \mapsto v \otimes 1$ *and complex scalar multiplication is defined by* $\lambda(v \otimes \alpha) = v \otimes (\lambda\alpha)$ *for* $\lambda \in \mathbb{C}, v \otimes \alpha \in \mathbb{C}V$*.*

Definition 49. *The conjugate space* $\overline{\mathbb{S}}$ *of a complex vector space* $(\mathbb{S}, +, \cdot)$ *is a vector space* $(\overline{\mathbb{S}}, +, \bar{\cdot})$ *together with a map* $\mathrm{id} \colon \mathbb{S} \to \overline{\mathbb{S}}$ *such that* id *is a group isomorphism between* $(\mathbb{S}, +)$ *and* $(\overline{\mathbb{S}}, +)$*, and furthermore* $\mathrm{id}(\lambda \cdot v) = \overline{\lambda} \,\bar{\cdot}\, \mathrm{id}(v)$ *for all* $v \in \mathbb{S}$*,* $\lambda \in \mathbb{C}$*.*

Remark 50. *The map* id *in definition 49 is also written as* $^-$*, so* $\mathrm{id}(v) = \overline{v}$*. Given a complexification* $\mathbb{C}V$ *of a vector space* V*, we can identify* $\mathbb{C}V$ *with* $\overline{\mathbb{C}V}$ *via* $v \otimes \lambda \mapsto v \otimes \overline{\lambda}$*.*

Definition 51. *A complexification* $\mathbb{C}Y$ *of a real differentiable n-manifold* Y *is a complex holomorphic n-manifold* $\mathbb{C}Y$ *with an inclusion* $\iota \colon Y \hookrightarrow \mathbb{C}Y$ *such that for every* $x \in Y$ *there is a chart* (U_i, ϕ_i) *on* Y *with* $x \in U_i$ *and a corresponding chart* $(U_{i\mathbb{C}}, \phi_{i\mathbb{C}})$ *on* $\mathbb{C}Y$ *with* $\iota(U_i) \subseteq U_{i\mathbb{C}}$ *and* $\tilde{\iota} = \phi_{i\mathbb{C}} \circ \iota \circ \phi_i^{-1}$*, where* $\tilde{\iota} \colon \mathbb{R}^n \hookrightarrow \mathbb{C}^n$ *is the map that sends* $(x_1, \ldots, x_n) \in \mathbb{R}^n$ *to* $(x_1, \ldots, x_n) \in \mathbb{C}^n$*.*
Complexified Minkowski space $\mathbb{C}\mathcal{M}$ *is the complexification of* $\mathcal{M}$ *as in definition 48, possibly viewed as an manifold via* $(U_i, \phi_i)_{i\in I} = (\mathbb{C}\mathcal{M}, \mathrm{id}_{\mathbb{C}\mathcal{M}})_{i\in\{1\}}$*.*

Note that complexifications of manifolds are not necessarily unique. We give another complexification of $\mathcal{M}$ in definition 105. Note furthermore that any chart (U_i, ϕ_i) with a corresponding chart $(U_{i\mathbb{C}}, \phi_{i\mathbb{C}})$ has another chart $(U_{i\mathbb{C}}, \overline{\phi_{i\mathbb{C}}})$ given by $\overline{\phi_{i\mathbb{C}}}(x) = \overline{\phi_{i\mathbb{C}}(x)}$. One can then create a real manifold $Y^\# = \{x \in \mathbb{C}Y \,|\, \text{for all charts } (U_i, \phi_i), \phi_{i\mathbb{C}}(x) = \overline{\phi_{i\mathbb{C}}}(x)\}$.

Given a complexification of a manifold $Y \overset{\iota}{\hookrightarrow} \mathbb{C}Y$, and a complex k-form $\mathscr{F}_\mathbb{C} \in \Omega^k(\mathbb{C}Y)$, we would like to have some way to construct a real k-form $\mathrm{Re}(\iota^*\mathscr{F}_\mathbb{C})$ on Y. Preferably this goes via the pullback as explained below definition 40. However, an element of $\Omega^k(\mathbb{C}Y)$ has complex coefficients when expressed in a basis, whereas a k-form in $\Omega^k(Y)$ has real coefficients. Formally, one could see $\iota^*\mathscr{F}$ as an element of the space $\Omega^k(Y) \otimes_{\Omega^0(Y)} \mathbb{C}$, which in coordinates goes as follows: Given a point $p \in Y$, by definition 51 we have charts (U_i, ϕ_i) on Y and $(U_{i\mathbb{C}}, \phi_{i\mathbb{C}})$ on $\mathbb{C}Y$ such that $\tilde{\iota} = \phi_{i\mathbb{C}} \circ \iota \circ \phi_i^{-1}$ can be seen as the identity on $\mathbb{R}^n \subseteq \mathbb{C}^n$. Thus the bases $(\mathrm{d}x_{i\mathbb{C}}^{\mu_1} \wedge \cdots \wedge \mathrm{d}x_{i\mathbb{C}}^{\mu_k})_{1\leq\mu_1<\cdots<\mu_k\leq n}$ and $(\mathrm{d}x_i^{\mu_1} \wedge \cdots \wedge \mathrm{d}x_i^{\mu_k})_{1\leq\mu_1<\cdots<\mu_k\leq n}$ induced by $\phi_{i\mathbb{C}}$ and ϕ_i respectively are identified via ι because of definition 40, i.e. $\iota^*(\mathrm{d}x_{i\mathbb{C}}^{\mu_1} \wedge \cdots \wedge \mathrm{d}x_{i\mathbb{C}}^{\mu_k}) = \mathrm{d}x_i^{\mu_1} \wedge \cdots \wedge \mathrm{d}x_i^{\mu_k}$. Thus when we express $\mathscr{F}_\mathbb{C}(p)$ on the basis induced by $\phi_{i\mathbb{C}}$ as $(\mathscr{F}_\mathbb{C}(p))_{\mu_1\ldots\mu_k}\mathrm{d}x_{i\mathbb{C}}^{\mu_1} \wedge \cdots \wedge \mathrm{d}x_{i\mathbb{C}}^{\mu_k}$, where $(\mathscr{F}_\mathbb{C}(p))_{\mu_1\ldots\mu_k} \in \mathbb{C}$, we can write $(\mathscr{F}_\mathbb{C}(p))_{\mu_1\ldots\mu_k} = \mathrm{Re}((\mathscr{F}_\mathbb{C}(p))_{\mu_1\ldots\mu_k}) + i\mathrm{Im}((\mathscr{F}_\mathbb{C}(p))_{\mu_1\ldots\mu_k})$. Now we thus have $\mathrm{Re}(\iota^*\mathscr{F}_\mathbb{C})(p) = \mathrm{Re}((\mathscr{F}_\mathbb{C}(p))_{\mu_1\ldots\mu_k})\mathrm{d}x_i^{\mu_1} \wedge \cdots \wedge \mathrm{d}x_i^{\mu_k}$, where $(\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_k})_{1\leq\mu_1<\cdots<\mu_k\leq n}$ is the basis of $\bigwedge^k T_p^*Y$ with respect to ϕ_i. We would like to be able to take the Hodge dual on complexifications, for which we need the following definitions. As holomorphic fields are difficult to construct globally, we only ask our fields to be defined almost everywhere (definition 25).

Definition 52. *A metric $\mathfrak{g}_\mathbb{C}$ on a complexification of Minkowski space $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}^*$ is a type $(0,2)$ symmetric non-degenerate holomorphic tensor field defined almost everywhere such that for each $p \in \mathcal{M}$ there are charts (U_i, ϕ_i) and $(U_{i\mathbb{C}}, \phi_{i\mathbb{C}})$ as in definition 51 such that when $\mathfrak{g}_\mathbb{C} = \mathfrak{g}_{\mathbb{C}\mu\nu}\mathrm{d}x_{i\mathbb{C}}^\mu \wedge \mathrm{d}x_{i\mathbb{C}}^\nu$ with respect to $\phi_{i\mathbb{C}}$, we have that the standard metric on $\mathcal{M}$ from definition 29 with respect to ϕ_i satisfies $\mathfrak{g} = \mathfrak{g}_{\mu\nu}\mathrm{d}x_i^\mu \wedge \mathrm{d}x_i^\nu = \mathfrak{g}_{\mathbb{C}\mu\nu}\mathrm{d}x_i^\mu \wedge \mathrm{d}x_i^\nu$. Thus $\mathfrak{g} = \iota^*\mathfrak{g}_\mathbb{C}$.*

Definition 53. *A volume form $\omega_\mathbb{C} \in \Omega^n(\mathbb{C}\mathcal{M}^*)$ on a complexification of Minkowski space $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}^*$ with respect to a metric $\mathfrak{g}_\mathbb{C}$ is an n-form defined almost everywhere such that for a chart (V_i, ψ_i) of $\mathbb{C}\mathcal{M}^*$, $\omega_\mathbb{C}|_{V_i}$ is equal to*

$$e^{i\theta}\left(\det\begin{pmatrix}(\mathfrak{g}_{\mathbb{C}11})_i & \cdots & (\mathfrak{g}_{\mathbb{C}1n})_i \\ \vdots & \ddots & \vdots \\ (\mathfrak{g}_{\mathbb{C}n1})_i & \cdots & (\mathfrak{g}_{\mathbb{C}nn})_i\end{pmatrix}\right)^{\frac{1}{2}} \mathrm{d}x_i^1 \wedge \cdots \wedge \mathrm{d}x_i^n,$$

where $\theta \in [0, 2\pi)$ should be chosen such that $\omega_\mathbb{C}$ is consistently defined and $\iota^*\omega_\mathbb{C}$ coincides with a real volume form on $\mathcal{M}$.

The Hodge dual is defined completely analogously to definition 44 and 45. The following definition is motivated by the observation that for a 2-

form $\mathscr{F}_{\mathbb{C}}$ we have that $\star(\star\mathscr{F}_{\mathbb{C}}) = -\mathscr{F}_{\mathbb{C}}$, which means on $\Omega^2(\mathbb{C}\mathrm{Y})$, $\star$ has eigenspaces with eigenvalues $\pm i$.

Definition 54. *A complex holomorphic 2-form $\mathscr{F}_{\mathbb{C}} \in \Omega^2(\mathrm{Y})$ on a complex holomorphic 4-manifold* Y *is called* self-dual *respectively* anti-self-dual *when the Hodge dual $\star\mathscr{F}_{\mathbb{C}}$ satisfies $\star\mathscr{F}_{\mathbb{C}} = i\mathscr{F}_{\mathbb{C}}$ respectively $\star\mathscr{F}_{\mathbb{C}} = -i\mathscr{F}_{\mathbb{C}}$.*

The following lemma is relevant for theorem 130. The proof as such can also be used for a real manifolds $\mathcal{M}$ and $\Omega\colon \mathcal{M} \to \mathbb{R}^*$.

Lemma 55. *Let $\mathscr{F}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M})$ be a 2-form on a complex 4-manifold $\mathbb{C}\mathcal{M}$, let $\mathfrak{g}_{\mathbb{C}}\colon \mathbb{C}\mathcal{M} \to T^0_2(T\mathbb{C}\mathcal{M})$ be a metric following definition 52, and let $\Omega\colon \mathcal{M} \to \mathbb{C}^*$ be a holomorphic function. Then $\Omega\mathfrak{g}_{\mathbb{C}}\colon \mathbb{C}\mathcal{M} \to T^0_2(T\mathbb{C}\mathcal{M})$, $x \mapsto \Omega(x)\mathfrak{g}_{\mathbb{C}}(x)$ is another possible metric on $\mathbb{C}\mathcal{M}$. We have that $\star_{\mathfrak{g}_{\mathbb{C}}}\mathscr{F}_{\mathbb{C}} = \pm\star_{\Omega\mathfrak{g}_{\mathbb{C}}}\mathscr{F}$, i.e. the Hodge dual of $\mathscr{F}_{\mathbb{C}}$ with respect to $\mathfrak{g}_{\mathbb{C}}$ is up to sign equal to the Hodge dual of $\mathscr{F}_{\mathbb{C}}$ with respect to $\Omega\mathfrak{g}_{\mathbb{C}}$.*

Proof. Using definition 43 up to 45, we have that $(\star_{\Omega\mathfrak{g}_{\mathbb{C}}}\mathscr{F}_{\mathbb{C}})_{\mu\nu} = \omega^{\Omega\mathfrak{g}_{\mathbb{C}}}_{\alpha\beta\mu\nu}(\Omega\mathfrak{g}_{\mathbb{C}})^{\alpha\rho}(\Omega\mathfrak{g}_{\mathbb{C}})^{\beta\sigma}\mathscr{F}_{\mathbb{C}\rho\sigma} = \frac{e^{i\theta}}{4!}(\det(\Omega\mathfrak{g}_{\mathbb{C}ij}))^{\frac{1}{2}}\varepsilon_{\alpha\beta\mu\nu}\Omega^{-2}\mathfrak{g}_{\mathbb{C}}^{\alpha\rho}\mathfrak{g}_{\mathbb{C}}^{\beta\sigma}\mathscr{F}_{\mathbb{C}\rho\sigma} = \frac{e^{i\theta}}{4!}(\Omega^4\det(\mathfrak{g}_{\mathbb{C}ij}))^{\frac{1}{2}}\varepsilon_{\alpha\beta\mu\nu}\Omega^{-2}\mathfrak{g}_{\mathbb{C}}^{\alpha\rho}\mathfrak{g}_{\mathbb{C}}^{\beta\sigma}\mathscr{F}_{\mathbb{C}\rho\sigma} = \pm\frac{e^{i\theta}}{4!}(\det(\mathfrak{g}_{\mathbb{C}ij}))^{\frac{1}{2}}\varepsilon_{\alpha\beta\mu\nu}\mathfrak{g}_{\mathbb{C}}^{\alpha\rho}\mathfrak{g}_{\mathbb{C}}^{\beta\sigma}\mathscr{F}_{\mathbb{C}\rho\sigma} = \pm(\star_{\mathfrak{g}_{\mathbb{C}}}\mathscr{F}_{\mathbb{C}})_{\mu\nu}$, where $(\Omega\mathfrak{g}_{\mathbb{C}})^{\alpha\beta} = \Omega^{-1}\mathfrak{g}_{\mathbb{C}}^{\alpha\beta}$ because $(\Omega\mathfrak{g}_{\mathbb{C}})^{\alpha\beta}$ is the $\alpha\beta$-component of the inverse of the map induced by $(\Omega\mathfrak{g}_{\mathbb{C}})_{\alpha\beta}$ via lemma 30. Conversely, $(\Omega\mathfrak{g}_{\mathbb{C}})_{\alpha\beta} = \Omega\mathfrak{g}_{\mathbb{C}\alpha\beta}$ by definition of $\Omega\mathfrak{g}_{\mathbb{C}}$. □

Lemma 56. *Let $\mathscr{F}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M})$ be a (anti-)self-dual 2-form on a complexification of Minkowski space $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}$ that satisfies $\mathrm{d}_2\mathscr{F}_{\mathbb{C}} = 0$. The 2-form $\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})$ then satisfies Maxwell's equations as in lemma 46.*

Proof. We have that $\iota^*(\mathrm{d}_2\mathscr{F}_{\mathbb{C}}) = \mathrm{d}_2(\iota^*\mathscr{F}_{\mathbb{C}}) = \mathrm{d}_2(\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}}) + i\mathrm{Im}(\iota^*\mathscr{F}_{\mathbb{C}})) = \mathrm{d}_2(\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})) + i\mathrm{d}_2(\mathrm{Im}(\iota^*\mathscr{F}_{\mathbb{C}}))$, thus $\mathrm{d}_2\mathscr{F}_{\mathbb{C}} = 0 \Rightarrow \iota^*(\mathrm{d}_2\mathscr{F}_{\mathbb{C}}) = 0 \Leftrightarrow (\mathrm{d}_2(\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})) = 0$ and $\mathrm{d}_2(\mathrm{Im}(\iota^*\mathscr{F}_{\mathbb{C}})) = 0)$, but also $\mathrm{d}_2\mathscr{F}_{\mathbb{C}} = 0 \Rightarrow \mathrm{d}_2 \pm i\mathscr{F}_{\mathbb{C}} = \mathrm{d}_2\star\mathscr{F}_{\mathbb{C}} = 0 \Rightarrow (\mathrm{d}_2(\mathrm{Re}(\iota^*\star\mathscr{F}_{\mathbb{C}})) = 0$ and $\mathrm{d}_2(\mathrm{Im}(\iota^*\star\mathscr{F}_{\mathbb{C}})) = 0)$. As $(\star\mathscr{F}_{\mathbb{C}})_{\mu\nu} = (\omega_{\mathbb{C}})_{\mu\nu\alpha\beta}(\mathscr{F}_{\mathbb{C}})_{\gamma\delta}\mathfrak{g}_{\mathbb{C}}^{\alpha\gamma}\mathfrak{g}_{\mathbb{C}}^{\beta\delta}$ and $\iota^*\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}$ and $\iota^*\omega_{\mathbb{C}} = \omega$, it follows that $\mathrm{Re}(\iota^*\star\mathscr{F}_{\mathbb{C}}) = \star\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})$, thus $\mathrm{d}_2\mathscr{F}_{\mathbb{C}} = 0 \Rightarrow (\mathrm{d}_2\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}}) = 0$ and $\mathrm{d}_2\star\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}}) = 0)$, which is lemma 46 for $\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})$. □

Given a complexification $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}$, it is important to note how the set of real Maxwell forms $\mathscr{F} \in \Omega^2(\mathcal{M})$ is related to the set of complex self-dual or anti-self-dual forms $\mathscr{F}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M})$. Notably, the map $\mathscr{F}_{\mathbb{C}} \mapsto \mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})$ is injective but not surjective, as follows from the identity theorem.

Theorem 57. *The identity theorem on holomorphic functions states that for a path-connected open subset $D \subseteq \mathbb{C}$ and two functions $f, g\colon D \to \mathbb{C}$ either $f = g$ on all of D or the set $\{x \in D | f(x) = g(x)\}$ is discrete in D.*

Lemma 58. *Given two self-dual complex Maxwell forms $\mathscr{F}_{\mathbb{C}}, \mathscr{K}_{\mathbb{C}} \in \Omega(\mathbb{C}\mathcal{M})$ on a path-connected complexification $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}$, either $\mathscr{F}_{\mathbb{C}} = \mathscr{K}_{\mathbb{C}}$ or the set $\{p \in \mathcal{M} | \mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})(p) = \mathrm{Re}(\iota^*\mathscr{K}_{\mathbb{C}})(p)\}$ is discrete in $\mathcal{M}$.*

Proof. Let $\mathscr{F}_{\mathbb{C}}, \mathscr{K}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M})$ be two self-dual Maxwell fields on a complexification $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}$, and consider the set $S = \{p \in \mathcal{M} | \mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})(p) = \mathrm{Re}(\iota^*\mathscr{K}_{\mathbb{C}})(p)\}$. Suppose S has an accumulation point x in $\mathcal{M}$, and let (U_i, ϕ_i) and $(U_{i\mathbb{C}}, \phi_{i\mathbb{C}})$ be charts of $\mathcal{M}$ and $\mathbb{C}\mathcal{M}$ respectively as in definition 51, such that $x \in U_i$ and $U_{i\mathbb{C}}$ is path-connected. As in the proof of lemma 56, we have that $\star\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}}) = \mathrm{Re}(\iota^*\star\mathscr{F}_{\mathbb{C}})$ and $\star\mathrm{Re}(\iota^*\mathscr{K}_{\mathbb{C}}) = \mathrm{Re}(\iota^*\star\mathscr{K}_{\mathbb{C}})$. As $\star\mathscr{F}_{\mathbb{C}} = i\mathscr{F}_{\mathbb{C}}$ and $\star\mathscr{K}_{\mathbb{C}} = i\mathscr{K}_{\mathbb{C}}$, it follows that $\mathrm{Re}(\iota^*\star\mathscr{F}_{\mathbb{C}}) = \mathrm{Re}(i\iota^*\mathscr{F}_{\mathbb{C}}) = \mathrm{Re}(i\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}}) - \mathrm{Im}(\iota^*\mathscr{F}_{\mathbb{C}})) = -\mathrm{Im}(\iota^*\mathscr{F}_{\mathbb{C}})$ and similarly $\mathrm{Re}(\iota^*\star\mathscr{K}_{\mathbb{C}}) = -\mathrm{Im}(\iota^*\mathscr{K}_{\mathbb{C}})$.
For $a \in S$, we thus have $(\iota^*\mathscr{F}_{\mathbb{C}})(a) = \mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})(a) + i\mathrm{Im}(\iota^*\mathscr{F}_{\mathbb{C}})(a) = \mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})(a) - i\star\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})(a) = \mathrm{Re}(\iota^*\mathscr{K}_{\mathbb{C}})(a) - i\star\mathrm{Re}(\iota^*\mathscr{K}_{\mathbb{C}})(a) = \mathrm{Re}(\iota^*\mathscr{K}_{\mathbb{C}})(a) + i\mathrm{Im}(\iota^*\mathscr{K}_{\mathbb{C}})(a) = (\iota^*\mathscr{K}_{\mathbb{C}})(a)$. Using $\grave{\phi}_{i\mathbb{C}}\colon \bigwedge^2(T^*\mathbb{C}\mathcal{M}) \to \mathbb{C}^{10}$ from definition 36 and $\tilde{\iota}$ from definition 51, we thus have for $a \in S$ that $(\grave{\phi}_{i\mathbb{C}} \circ \mathscr{F}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1} \circ \tilde{\iota} \circ \phi_i)(a) = (\grave{\phi}_{i\mathbb{C}} \circ \mathscr{K}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1} \circ \tilde{\iota} \circ \phi_i)(a)$, and thus for $b \in (\tilde{\iota} \circ \phi_i)(S \cap U_i)$ we have that $(\grave{\phi}_{i\mathbb{C}} \circ \mathscr{F}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1})(b) = (\grave{\phi}_{i\mathbb{C}} \circ \mathscr{K}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1})(b)$. As $(\tilde{\iota} \circ \phi_i)(x)$ is an accumulation point in $(\tilde{\iota} \circ \phi_i)(S \cap U_i)$ for every accumulation point $x \in S$, and $\grave{\phi}_{i\mathbb{C}} \circ \mathscr{F}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1}$ and $\grave{\phi}_{i\mathbb{C}} \circ \mathscr{K}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1}\colon \phi_{i\mathbb{C}}(U_{i\mathbb{C}}) \to \mathbb{C}^{10}$ are holomorphic, it follows from the identity theorem that $(\grave{\phi}_{i\mathbb{C}} \circ \mathscr{F}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1})(q) = (\grave{\phi}_{i\mathbb{C}} \circ \mathscr{K}_{\mathbb{C}} \circ \phi_{i\mathbb{C}}^{-1})(q)$ for every $q \in \phi_{i\mathbb{C}}(U_{i\mathbb{C}})$, and thus $\mathscr{F}_{\mathbb{C}}|_{U_{i\mathbb{C}}} = \mathscr{K}_{\mathbb{C}}|_{U_{i\mathbb{C}}}$. As $\mathbb{C}\mathcal{M}$ is path-connected, it follows that $\mathscr{F}_{\mathbb{C}} = \mathscr{K}_{\mathbb{C}}$. □

Lemma 59. *Given a complexification of Minkowski space $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}$, there exist real Maxwell forms $\mathscr{F} \in \Omega^2(\mathcal{M})$ that do not arise as $\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})$ for any self-dual complex Maxwell form $\mathscr{F}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M})$.*

Proof. Consider the field $\mathscr{F} \in \Omega^2(\mathcal{M})$ given on the standard basis by

$$\mathscr{F}(x^0, x^1, x^2, x^3) = \begin{cases} e^{\frac{1}{(x^0+x^3)^2-1}}(\mathrm{d}x^0 \wedge \mathrm{d}x^1 - \mathrm{d}x^1 \wedge \mathrm{d}x^3) & \text{if } |x^0 + x^3| < 1, \\ 0 & \text{else.} \end{cases}$$

It can easily be checked that this indeed is an infinitely differentiable field that satisfies $\mathrm{d}_2\mathscr{F} = \mathrm{d}_2\star\mathscr{F} = 0$. Now let $\mathcal{M} \overset{\iota}{\hookrightarrow} \mathbb{C}\mathcal{M}$ be a complexification of $\mathcal{M}$, and suppose there exists a self-dual $\mathscr{F}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M})$ such that

$\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}}) = \mathscr{F}$. Let $\mathbb{C}\mathcal{M}^\bullet = \{(x^0, x^1, x^2, x^3) \in \mathbb{C}\mathcal{M} | (x^0 + x^3)^2 \neq 1\}$, and consider $\mathscr{K}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M}^\bullet)$ given by $\mathscr{K}_{\mathbb{C}}(x^0, x^1, x^2, x^3) = e^{\frac{1}{(x^0+x^3)^2-1}}(\mathrm{d}x^0 \wedge \mathrm{d}x^1 - \mathrm{d}x^1 \wedge \mathrm{d}x^3 + i\mathrm{d}x^0 \wedge \mathrm{d}x^2 - i\mathrm{d}x^2 \wedge \mathrm{d}x^3)$. As $\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})(\mathbf{x}) = \mathscr{F}(\mathbf{x}) = \mathrm{Re}(\iota^*\mathscr{K}_{\mathbb{C}})(\mathbf{x})$ for $\mathbf{x}$ in the non-discrete set $\{(x^0, x^1, x^2, x^3) \in \mathcal{M} | (x^0 + x^3)^2 < 1\}$, by lemma 58 we have that $\mathscr{K}_{\mathbb{C}} = \mathscr{F}_{\mathbb{C}}$ on $\mathbb{C}\mathcal{M}^\bullet$. However, as for $0 \in \Omega^2(\mathbb{C}\mathcal{M})$ we have that $\mathrm{Re}(\iota^*\mathscr{F}_{\mathbb{C}})(\mathbf{x}) = \mathscr{F}(\mathbf{x}) = \mathrm{Re}(\iota^* 0)(\mathbf{x})$ for $\mathbf{x} \in \{(x^0, x^1, x^2, x^3) \in \mathcal{M} | (x^0 + x^3)^2 \geq 1\}$, lemma 58 also gives that $\mathscr{F}_{\mathbb{C}} = 0$, so we conclude that $\mathscr{K}_{\mathbb{C}} = 0$. This is a contradiction, so we can conclude that such an $\mathscr{F}_{\mathbb{C}}$ does not exist. □

2.5.1 Intermezzo for physicists

In the previous sections, we have seen the 2-form definition of Maxwell's equations (lemma 46), and the complex analog thereof (lemma 56).

The (real) 2-form formalism is very similar to the tensor formalism from [7], where the metric-dependent partial derivatives are replaced by a metric-independent exterior derivative d, and a metric-dependent Hodge-dual $\star$.

The complex version hereof, lemma 56, is related to lemma 46 in a similar way that theorem 8 is related to definition 6. More precisely, in the standard basis of Minkowski space, the components of the complex 2-form of lemma 46 are exactly (up to sign) the components of the Riemann-Silberstein vector. The formulation in terms of 2-forms has the additional advantage of being solely dependent of the metric, i.e. independent of choice of coordinates.

By changing from a real manifold to a complex manifold, the functions on this manifold have changed from being real differentiable into being complex differentiable, which means that these functions are globally determined when defined locally, and several fields that are allowed in the real case are no longer allowed in the complex case (see 57 up to 59, although it should be noted that most fields that are considered by physicists are extendable to the complex case). Furthermore, there is again no unambiguous physical meaning for the complex direction of the coordinates, and on non-real points, there is no clear distinction between the electric and magnetic fields.

Chapter 3

Dirac spinors and twistors

3.1 The Spinor formalism

The spinor formalism is a very important formalism in both physics and mathematics. Spinors were originally introduced to model intrinsic angular momentum in a quantum mechanical particle, incorporated in the Weyl or Dirac equations in the case of spin-$\frac{1}{2}$ (See also chapter eleven of [9], or chapter 3 of [10] for a more thorough treatment of the relation between Quantum equations and observables). Unfortunately, the formal intrinsic definitions of spinors are quite laborious. Our definitions are based on [11], which starts off with the Clifford algebra.

Definition 60. *A unital associative algebra over a field F is a set A together with an addition $+\colon A \times A \to A$, a multiplication $*\colon A \times A \to A$ and a scalar multiplication $\cdot\colon F \times A \to A$ such that $(A, +, \cdot)$ is a vector space and $(A, +, *)$ is a ring with unity, and for $\lambda \in F$ and $v, w \in A$ we have that $(\lambda \cdot v) * w = \lambda \cdot (v * w) = v * (\lambda \cdot w)$.*

Definition 61. *A quadratic form Q on a vector space V over a field F is a map $Q\colon V \to F$ such that for all $\lambda \in F$ and all $v \in V$ we have $Q(\lambda v) = \lambda^2 Q(v)$, and furthermore $(v, w) \mapsto Q(v + w) - Q(v) - Q(w)$ is a bilinear form.*
Given a bilineair form $\langle \cdot, \cdot \rangle\colon V \times V \to F$, the map $v \mapsto \langle v, v \rangle$ is a quadratic form.

Notation 62. *Given a quadratic form Q on a vector space V over a field F of characteristic not 2, the bilinear form $(v, w) \mapsto \frac{1}{2}(Q(v + w) - Q(v) - Q(w))$ is written as $\langle v, w \rangle_Q$. It satisfies $\langle v, v \rangle_Q = Q(v)$*

Definition 63. *The Clifford algebra* $\mathrm{C}\ell(V,Q)$ *of a vector space* V *over a field* F *with respect to a quadratic form* $Q\colon V\times V\to F$ *is a unital associative algebra over* F*, together with an embedding* $\iota\colon V\hookrightarrow \mathrm{C}\ell(V,Q)$ *such that for any* $v\in V$ *we have that* $\iota(v)*\iota(v) = -Q(v)\cdot 1$*, where* $1\in\mathrm{C}\ell(V,Q)$ *is the unity,* $*\colon \mathrm{C}\ell(V,Q)\times\mathrm{C}\ell(V,Q)\to\mathrm{C}\ell(V,Q)$ *is the algebra multiplication and* $\cdot\colon F\times\mathrm{C}\ell(V,Q)\to\mathrm{C}\ell(V,Q)$ *is scalar multiplication. Furthermore,* $\mathrm{C}\ell(V,Q)$ *satisfies the universal property that for any associative unital algebra* A *with an embedding* $j\colon V\hookrightarrow A$ *that satisfies* $\forall v\in V,\ j(v)*j(v) = -Q(v)\cdot 1_A$ *there exists a unique algebra homomorphism* $\tilde{\jmath}\colon \mathrm{C}\ell(V,Q)\to A$ *such that* $\tilde{\jmath}\circ\iota = j$*.*
Given a basis $(e^1,\dots,e^n)$ *for* V*, a basis for* $\mathrm{C}\ell(V,Q)$ *is given by* $(1,(e^{\mu_1})_{1\le\mu_1\le n},(e^{\mu_1}*e^{\mu_2})_{1\le\mu_1<\mu_2\le n},\dots,(e^{\mu_1}*\dots*e^{\mu_n})_{1\le\mu_1<\dots<\mu_n\le n})$*.*

Notation 64. *For* $v\in V$ *we just write* $v\in\mathrm{C}\ell(V,Q)$ *instead of* $\iota(v)\in\mathrm{C}\ell(V,Q)$*.*

Remark 65. *In other sources (notably [9]), the Clifford algebra may be defined using* $\iota(v)*\iota(v) = Q(v)\cdot 1$ *instead of* $\iota(v)*\iota(v) = -Q(v)1$*.*

The previous remark does not pose any problems, as $v\mapsto -Q(v)$ is anther quadratic form that would give a clifford algebra with the other convention. Note that for $v,w\in V$ we have that $v*w+w*v = (v+w)*(v+w)-v*v-w*w = (-Q(v+w)+Q(v)+Q(w))\cdot 1 = -2\langle v,w\rangle_Q\cdot 1$, which allows one to express any product $x_1*\dots*x_n$ in the basis given in definition 63. On the Clifford algebra, there is a canonical automorphism α and two canonical anti-automorphisms $^\flat$ and †.

Definition 66. $\alpha\colon\mathrm{C}\ell(V,Q)\to\mathrm{C}\ell(V,Q)$ *is the unique extension of the map* $j\colon V\to\mathrm{C}\ell(V,Q), v\mapsto -\iota(v)$ *using definition 63.*

Definition 67. *The opposite algebra* A^{op} *of an algebra* $(A,+,*,\cdot)$ *is an algebra* $(A^{\mathrm{op}},+,\tilde{*},\cdot)$ *together with a map* $\mathrm{id}\colon A\to A^{\mathrm{op}}$ *such that* id *is a vector space isomorphism between* $(A,+,\cdot)$ *and* $(A^{\mathrm{op}},+,\cdot)$*, and furthermore* $\mathrm{id}(v*w) = \mathrm{id}(w)\,\tilde{*}\,\mathrm{id}(v)$ *for all* $v,w\in A$*.*

Definition 68. $^\flat\colon\mathrm{C}\ell(V,Q)\to\mathrm{C}\ell(V,Q)$ *is the map given by* $\mathrm{id}^{-1}\circ\widetilde{(\mathrm{id}\circ\iota)}$*, where* $\mathrm{id}\colon\mathrm{C}\ell(V,Q)\to\mathrm{C}\ell(V,Q)^{\mathrm{op}}$ *is as in definition 67, and* $\widetilde{(\mathrm{id}\circ\iota)}\colon\mathrm{C}\ell(V,Q)\to\mathrm{C}\ell(V,Q)^{\mathrm{op}}$ *is the unique extension of* $\mathrm{id}\circ\iota\colon V\hookrightarrow\mathrm{C}\ell(V,Q)^{\mathrm{op}}$ *using definition 63.*

Definition 69. $^\dagger\colon\mathrm{C}\ell(V,Q)\to\mathrm{C}\ell(V,Q)$ *is the composition of* α *and* $^\flat$*, so* $^\dagger = \alpha\circ^\flat = {}^\flat\circ\alpha$*.*

For example, when we have $u,v,w\in V$, we can derive $(u*v*w)^\dagger*(u*v*w) = (\alpha(u*v*w))^\flat*(u*v*w) = \mathrm{id}^{-1}(\mathrm{id}(-\iota(u))\,\tilde{*}\,\mathrm{id}(-\iota(v))\,\tilde{*}$

$\mathrm{id}(-\iota(w))) * (u * v * w) = (-w) * (-v) * (-u) * u * v * w = Q(u)Q(v)Q(w)1$, and similarly $v^\dagger * v = \alpha(v)^\flat * v = -v * v = Q(v)$. Thus $x \mapsto x^\dagger x$ behaves as the quadratic form of V. The automorphism α induces a grading on $C\ell(V, Q)$, which we will use later.

Definition 70. *A unital associative algebra $(A, +, *, \cdot)$ is called S-graded with respect to a moinoid* S when it is a direct sum of subspaces $A = \bigoplus_{s \in S} A_s$ such that $(A_s, +, \cdot)$ are vector spaces and $a_i * a_j \in A_{i \bullet j}$ when $a_i \in A_i$ and $a_j \in A_j$.*

Lemma 71. *The eigenspaces $C\ell^1(V, Q)$ and $C\ell^{-1}(V, Q)$ of the automorphism $\alpha\colon C\ell(V, Q) \to C\ell(V, Q)$ induce a $\{1, -1\}$-grading on $C\ell(V, Q)$.*

Proof. Note that $\alpha \circ \alpha = \mathrm{id}_{C\ell(V,Q)}$, thus α has eigenvalues ± 1. As $\alpha \circ \iota = -\mathrm{id}_V$, we have $C\ell^{-1}(V, Q) \neq \{0\}$. As α is an algebra homomorphism, we find for $v, w \in C\ell(V, Q)$ with $\alpha(v) = \pm_v v$ and $\alpha(w) = \pm_w w$ that $\alpha(v * w) = \alpha(v) * \alpha(w) = \pm_v \pm_w v * w$, thus $C\ell(V, Q)$ is $\{1, -1\}$-graded. □

Remark 72. *In other sources, the grading monoid of $C\ell(V, Q)$ is additively written as $\{0, 1\}$ instead of $\{1, -1\}$. Furthermore, $C\ell^1(V, Q)$ is a subalgebra of $C\ell(V, Q)$, usually referred to as $C\ell^{even}(V, Q)$.*

We now restrict ourselves to the case that V is a real vector space (in this case $C\ell(V, Q)$ is also known as the Geometric algebra), and wish to complexify the Clifford algebra $C\ell(V, Q)$ to $\mathbb{C}\ell(V, Q) = C\ell(V, Q) \otimes \mathbb{C}$.

Definition 73. *The complexified Clifford algebra $\mathbb{C}\ell(V, Q)$ over a real vector space V is the complexification of $C\ell(V, Q)$ as in definition 48.*
By remark 50 we thus have a map $^-\colon \mathbb{C}\ell(V, Q) \to \mathbb{C}\ell(V, Q)$.
By the inclusion $\iota\colon \mathbb{C}V \hookrightarrow \mathbb{C}\ell(V, Q)$, $v \otimes \lambda \mapsto \iota(v) \otimes \lambda$, it can be identified with $C\ell(\mathbb{C}V, Q_\mathbb{C})$ via the universal property in definition 63, which means we also have maps $\alpha, {}^\flat, {}^\dagger\colon \mathbb{C}\ell(V, Q) \to \mathbb{C}\ell(V, Q)$.

Note that for $v \otimes \lambda \in \mathbb{C}V$ we have that $\overline{(v \otimes \lambda)}^\dagger * (v \otimes \lambda) = (v^\dagger * v) \otimes (\bar{\lambda}\lambda) = |\lambda|^2 Q(v)1$, thus for $v \in \mathbb{C}V, v \mapsto \overline{v}^\dagger * v$ behaves similar to $v \mapsto \langle v, v\rangle$ for a sesquilinear form $\langle \cdot, \cdot \rangle\colon \mathbb{C}V \times \mathbb{C}V \to \mathbb{C}V$. We now arrive at the most important theorem for the construction of spinors, which is proposition 11.1.19 in [11].

Theorem 74. *Given a $2k$-dimensional complex vector space V and a non-degenerate quadratic form $Q\colon V \times V \to V$, there exists a 2^k-dimensional complex vector space $\mathbb{S}$ with an isomorphism of algebras $\rho\colon C\ell(V, Q) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$.*

*A monoid is a set S with an associative binary operation $\bullet$ and an identity 1. It can thus be seen as a group without the axiom of extistence of inverses.

Proof. The proof of this can be found in [11]. Note that the isomorphism is not canonical. □

Definition 75. *The Dirac-spinor space of a real even-dimensional vector space V is a complex vector space $\mathbb{S}$ such that $\mathbb{C}\ell(V,Q) \cong \mathrm{End}(\mathbb{S})$ as in theorem 74.*

Now, we would like to introduce Weyl-spinors, for which we need the groups $SO(V,Q)$ and $SO^+(V,Q)$.

Definition 76. *The orthogonal group $O(V,Q)$ of a vector space V with respect to a non-degenerate quadratic form Q is the subgroup $\{f \in \mathrm{End}(V) | Q \circ f = Q\}$.*

Definition 77. *The special orthogonal group $SO(V,Q)$ of a vector space V with a non-degenerate quadratic form Q is the subgroup $\{f \in O(V,Q) | \det(f) = 1\}$.*

Definition 78. *A function $f \in O(V,Q)$ is said to preserve complete orientation if there exists a function $\Gamma_f \colon [0,1] \times V \to V$ such that for any $t \in [0,1]$ we have that $(x \mapsto \Gamma_f(t,x)) \in O(V,Q)$ and for any $x \in V$ we have that $(t \mapsto \Gamma_f(t,x))$ is continuous and $\Gamma_f(0,x) = x$ and $\Gamma_f(1,x) = f(x)$.*

Definition 79. *The identity component of the orthogonal group, $SO^+(V,Q)$, is the subgroup $\{f \in O(V,Q) | f$ preserves complete orientation$\}$.*

The group $SO^+(V,Q)$ is a subgroup of $SO(V,Q)$, as the map $t \mapsto \det(x \mapsto \Gamma_f(t,x))$ is a continuous function with codomain $\{1,-1\}$ and thus the determinant of f is equal to $\det(\mathrm{id}_V) = 1$.

When $\langle\cdot,\cdot\rangle_Q$ and $-\langle\cdot,\cdot\rangle_Q$ are both not positive definite, for example with Minkovski space, the group $SO^+(V,Q)$ is a strict subgroup of $SO(V,Q)$. For example, the map $f \colon \mathcal{M} \to \mathcal{M}, (t,x,y,z) \mapsto (-t,-x,y,z)$ is an element of $SO(\mathcal{M},\langle\cdot,\cdot\rangle_{\mathcal{M}})$ because $\det(f) = (-1)^2 = 1$, but it is not an element of $SO^+(\mathcal{M}\langle\cdot,\cdot\rangle_{\mathcal{M}})$, as for any $\tilde{f} \in O(\mathcal{M},\langle\cdot,\cdot\rangle_{\mathcal{M}})$ we have that $Q(\tilde{f}(1,0,0,0)) > 0$ and thus the time coordinate of $\tilde{f}(1,0,0,0)$ can not be 0. This means the time coordinate of $\Gamma_f(s,(1,0,0,0))$ cannot change sign when varying s, thus f is not in $SO^+(\mathcal{M},\langle\cdot,\cdot\rangle_{\mathcal{M}})$. Sometimes such an f is said not to be *time orientation preserving*.

Before we turn to Weyl-spinors, we give the definitions of the Clifford group and the Spin group.

Definition 80. *The Clifford group $\Gamma(V,Q)$ respectively the complex Clifford group $\Gamma_{\mathbb{C}}(V,Q)$ of a real vector space V with a non-degenerate form Q is the set $\{s \in C\ell(V,Q) | \forall v \in V, s^{-1}vs \in V\}$ resp. $\{s \in \mathbb{C}\ell(V,Q) | \forall v \in V, s^{-1}vs \in V\}$. The group operation is the algebra multiplication of $C\ell(V,Q)$ resp. $\mathbb{C}\ell(V,Q)$.*

Definition 81. *The spin group* $\mathrm{Spin}(V, Q)$ *respectively complex spin group* $\mathrm{Spin}_{\mathbb{C}}(V, Q)$ *of a real vector space V with respect to a non-degenerate quadratic form Q is the subset* $\{s \in \Gamma(V, Q) | s^\dagger * s = 1 \text{ and } \alpha(s) = s\}$ *respectively* $\{s \in \Gamma_{\mathbb{C}}(V, Q) | \bar{s}^\dagger * s = 1 \text{ and } \alpha(s) = s\}$.

The following theorems are very well-known and important.

Theorem 82. *The map* $F\colon \Gamma(V, Q) \to O(V, Q)$, $s \mapsto (v \mapsto s^{-1} v s)$ *respectively* $F_{\mathbb{C}}\colon \Gamma_{\mathbb{C}}(V, Q) \to O(V, Q)$ *is a surjective group homomorphism, with kernel* $\mathbb{R}^*$ *respectively* $\mathbb{C}^*$.

Proof. The real theorem can be proven similar to the proof of theorem 11.1.38 in [11]. The complex theorem can then be proven analogously, as in [11], page 526. □

Theorem 83. *The map* $F|_{\mathrm{Spin}(V,Q)}\colon \mathrm{Spin}(V, Q) \to SO^+(V, Q)$ *is surjective with kernel* $\{1, -1\}$, *and the map* $F_{\mathbb{C}}|_{\mathrm{Spin}_{\mathbb{C}}(V,Q)}\colon \mathrm{Spin}_{\mathbb{C}}(V, Q) \to SO^+(V, Q)$ *is surjective with kernel* $\{z \in \mathbb{C} | |z| = 1\} = S^1$.

Proof. This is proven in [11]. □

We now continue with defining Weyl-spinors. For this we need the following group action of $SO(V, Q)$.

Lemma 84. *Given a vector space V and a non-degenerate quadratic form Q, the group action of* $O(V, Q)$ *respectively* $SO(V, Q)$ *on V induces a natural group action of* $O(V, Q)$ *respectively* $SO(V, Q)$ *on* $\mathbb{C}\ell(V, Q)$.

Proof. Given $f \in O(V, Q)$ and the inclusion $\iota\colon V \hookrightarrow \mathbb{C}\ell(V, Q)$, we have that $\iota \circ f\colon V \hookrightarrow \mathbb{C}\ell(V, Q)$ is an embedding that satisfies $\forall v \in V$,

$$(\iota \circ f)(v) * (\iota \circ f)(v) = \iota(f(v)) * \iota(f(v)) = -Q(f(v)) \cdot 1 = -Q(v) \cdot 1,$$

thus the universal property of Clifford algebras, as in definition 63, gives an algebra homomorphism $\tilde{f}\colon \mathbb{C}\ell(V, Q) \to \mathbb{C}\ell(V, Q)$ such that $\tilde{f}(v) = f(v)$ for all $v \in V \subseteq \mathbb{C}\ell(V, Q)$. We can now take a complexification as $\tilde{f}_{\mathbb{C}}\colon \mathbb{C}\ell(V, Q) \to \mathbb{C}\ell(V, Q)$, $v \otimes \lambda \mapsto \tilde{f}(v) \otimes \lambda$, thus $f \mapsto \tilde{f}_{\mathbb{C}}$ is an action of $O(V, Q)$ on $\mathbb{C}\ell(V, Q)$.
As $SO(V, Q) \subseteq O(V, Q)$, we get an action of $SO(V, Q)$ on $\mathbb{C}\ell(V, Q)$. □

Definition 85. $\mathbb{C}\ell(V, Q)^{SO(V,Q)}$ *is the set of fixed points of* $\mathbb{C}\ell(V, Q)$ *under the action of* $SO(V, Q)$ *on* $\mathbb{C}\ell(V, Q)$, *thus*

$$\mathbb{C}\ell(V, Q)^{SO(V,Q)} = \{x \in \mathbb{C}\ell(V, Q) | \forall f \in SO(V, Q), \tilde{f}_{\mathbb{C}}(x) = x\}.$$

Lemma 86. *$\mathbb{C}\ell(V,Q)^{SO(V,Q)}$ is a linear subspace of $\mathbb{C}\ell(V,Q)$ spanned by 1 and Γ for some $\Gamma \in \mathbb{C}\ell(V,Q)$ with $\Gamma * \Gamma = 1$.*
Furthermore, $\{s \in \mathbb{C}\ell(V,Q)^{SO(V,Q)} | s * s = 1\} = \{1, -1, \Gamma, -\Gamma\}$.
Given a basis $(e^1, \ldots, e^n)$ of V that satisfies $\langle e^i, e^j \rangle_Q = 0$ for all $i \neq j$, we can write Γ as $\Gamma = \pm \eta e^1 * \cdots * e^n$, where $\eta \in \mathbb{C}$ satisfies

$$\eta^2 = \begin{cases} \frac{-1}{Q(e^1)\cdots Q(e^n)} & \text{if } n \equiv 1 \text{ or } 2 \mod 4, \\ \frac{1}{Q(e^1)\cdots Q(e^n)} & \text{if } n \equiv 3 \text{ or } 4 \mod 4. \end{cases}$$

The following proof is lengthy and tedious. It ends at page 39.

Proof. By theorem 8.26 of [4], there exists a basis $(e^1, \ldots, e^n)$ of V such that $\langle e^i, e^j \rangle_Q = 0$ for all $i \neq j$.
Define $\Gamma' = \frac{1}{n!}\varepsilon_{\mu_1 \ldots \mu_n} e^{\mu_1} * \cdots * e^{\mu_n}$. We will show that $\Gamma' \in \mathbb{C}\ell(V,Q)^{SO(V,Q)}$. Let $f \in SO(V,Q)$ be arbitrary and express it in the given basis of V as $f(v_\mu e^\mu) = v_\mu f^\mu_\nu e^\nu$. Note that $\tilde{f}_{\mathbb{C}}(\Gamma') = \frac{1}{n!}\varepsilon_{\mu_1 \ldots \mu_n} f(e^{\mu_1}) * \cdots * f(e^{\mu_n}) = \frac{1}{n!}\varepsilon_{\mu_1 \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_n}_{\nu_n} e^{\nu_1} * \cdots * e^{\nu_n}$. We will now show that $\varepsilon_{\mu_1 \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_n}_{\nu_n} = \varepsilon_{\nu_1 \ldots \nu_n} \det(f)$ for any choise of $\nu_1 \ldots \nu_n$. Note that $\nu \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, $i \mapsto \nu_i$ can be considered a function. If $\nu_i = \nu_j$ for some $i < j$, we have that

$$\begin{aligned} &\varepsilon_{\mu_1 \ldots \mu_i \ldots \mu_j \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_i}_{\nu_i} \cdots f^{\mu_j}_{\nu_j} \cdots f^{\mu_n}_{\nu_n} \\ &= \varepsilon_{\mu_1 \ldots \mu_i \ldots \mu_j \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_j}_{\nu_i} \cdots f^{\mu_i}_{\nu_j} \cdots f^{\mu_n}_{\nu_n} \\ &= \varepsilon_{\mu_1 \ldots \mu_j \ldots \mu_i \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_i}_{\nu_i} \cdots f^{\mu_j}_{\nu_j} \cdots f^{\mu_n}_{\nu_n} \\ &= -\varepsilon_{\mu_1 \ldots \mu_i \ldots \mu_j \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_i}_{\nu_i} \cdots f^{\mu_j}_{\nu_j} \cdots f^{\mu_n}_{\nu_n} \end{aligned}$$

and thus we find $\varepsilon_{\mu_1 \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_n}_{\nu_n} = 0 = \varepsilon_{\nu_1 \ldots \nu_n} \det(f)$ in this case. In the other case, ν is bijective and is thus an element of S_n. We then have

$$\begin{aligned} \varepsilon_{\mu_1 \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_n}_{\nu_n} &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) f^{\sigma(1)}_{\nu_1} \cdots f^{\sigma(l)}_{\nu_n} \\ = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) f^{\sigma(\nu^{-1}(\nu_1))}_{\nu_1} \cdots f^{\sigma(\nu^{-1}(\nu_n))}_{\nu_n} &= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) f^{\sigma \circ \nu^{-1}(1)}_1 \cdots f^{\sigma \circ \nu^{-1}(n)}_n \\ = \sum_{\varsigma \in S^n} \operatorname{sgn}(\varsigma \circ \nu) f^{\varsigma(1)}_1 \cdots f^{\varsigma(n)}_n &= \operatorname{sgn}(\nu) \sum_{\varsigma \in S_n} \operatorname{sgn}(\varsigma) f^{\varsigma(1)}_1 \cdots f^{\varsigma(n)}_n \\ &= \operatorname{sgn}(\nu) \det(f) = \varepsilon_{\nu_1 \ldots \nu_n} \det(f). \end{aligned}$$

Thus in general we find that $\varepsilon_{\mu_1 \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_n}_{\nu_n} = \varepsilon_{\nu_1 \ldots \nu_n} \det(f)$.
We thus obtain that $\tilde{f}_{\mathbb{C}}(\Gamma') = \frac{1}{n!}\varepsilon_{\mu_1 \ldots \mu_n} f^{\mu_1}_{\nu_1} \cdots f^{\mu_n}_{\nu_n} e^{\nu_1} * \cdots * e^{\nu_n} =$

$\frac{\det(f)}{n!}\varepsilon_{\nu_1\ldots\nu_n}e^{\nu_1}*\cdots*e^{\nu_n}=\det(f)\Gamma'$. As $f\in SO(V,Q)$ we have $\det(f)=1$, so $\Gamma'\in\mathbb{C}\ell(V,Q)^{SO(V,Q)}$. It thus follows that $\tilde{\Gamma}:=\frac{1}{\sqrt{\Gamma'*\Gamma'}}\Gamma'\in\mathbb{C}\ell(V,Q)^{SO(V,Q)}$ satisfies $\tilde{\Gamma}*\tilde{\Gamma}=1$, thus we would like $\tilde{\Gamma}=\Gamma$ with Γ as in the lemma.
Note that for $i\neq j$ we have that $0=-2\langle e^i,e^j\rangle_Q=e^i*e^j+e^j*e^i$ as explained below remark 65, thus for $\sigma\in S_n$ we have $e^{\sigma(1)}*\cdots*e^{\sigma(n)}=\operatorname{sgn}(\sigma)e^1*\cdots*e^n$. It follows that

$$\Gamma'=\frac{1}{n!}\varepsilon_{\mu_1\ldots\mu_n}e^{\mu_1}*\cdots*e^{\mu_n}=\frac{1}{n!}\sum_{\sigma\in S_n}\operatorname{sgn}(\sigma)e^{\sigma(1)}*\cdots*e^{\sigma(n)}$$
$$=\frac{1}{n!}\sum_{\sigma\in S_n}(\operatorname{sgn}(\sigma))^2e^1*\cdots*e^n=e^1*\cdots*e^n,$$

and thus $\tilde{\Gamma}=\frac{\Gamma'}{\sqrt{\Gamma'*\Gamma'}}=\frac{\Gamma'}{\sqrt{(e^1*\cdots*e^n)*(e^1*\cdots*e^n)}}=\frac{\Gamma'}{\sqrt{(-1)^{\lfloor\frac{n}{2}\rfloor}(e^1*\cdots*e^n)*(e^n*\cdots*e^1)}}=\frac{\Gamma'}{\sqrt{(-1)^{\lfloor\frac{n}{2}\rfloor+n}Q(e^1)\cdots*Q(e^n)}}=\frac{\pm\sqrt{(-1)^{\lfloor\frac{n}{2}\rfloor+n}}}{\sqrt{Q(e^1)\cdots Q(e^n)}}\Gamma'=\Gamma.$
We thus have $\{1,-1,\Gamma,-\Gamma\}\subseteq\{s\in\mathbb{C}\ell(V,Q)^{SO(V,Q)}|s*s=1\}$. To prove the other inclusion, we assume that our basis is normalized such that $|Q(e^i)|=1$. Let $a,b\in\{1,\ldots,n\}$ be arbitrary, and let $\varpi\in SO(V,Q)$ be a function that satisfies $\varpi(e^i)=e^i$ for $i\notin\{a,b\}$.
We write $\varpi(e^a)=\varpi^a_a e^a+\varpi^a_b e^b$ and $\varpi(e^b)=\varpi^b_a e^a+\varpi^b_b e^b$ (without Einstein summation implied). Let $\theta\in\mathbb{R}$ be arbitrary. If $Q(e^a)=Q(e^b)$, ϖ could be given by

$$\varpi\colon V\to V,\ e^i\mapsto\begin{cases}e^i & \text{if } i\notin\{a,b\}\\ \cos(\theta)e^a+\sin(\theta)e^b & \text{if } i=a,\\ -\sin(\theta)e^a+\cos(\theta)e^b & \text{if } i=b.\end{cases}$$

because we then have $\det(\varpi)=\cos^2(\theta)+\sin^2(\theta)=1$ and

$$Q(\varpi(v_\mu e^\mu))=\langle v_\mu\varpi(e^\mu),v_\nu\varpi(e^\nu)\rangle_Q=$$
$$\langle v_a\varpi(e^a)+v_b\varpi(e^b),v_a\varpi(e^a)+v_b\varpi(e^b)\rangle_Q+\sum_{\mu\notin\{a,b\}}(v_\mu)^2\langle e^\mu,e^\mu\rangle_Q=$$
$$(v_a)^2(\cos^2(\theta)Q(e^a)+\sin^2(\theta)Q(e^b))+2v_av_b\cos(\theta)\sin(\theta)(Q(e^b)-Q(e^a))$$
$$+(v_b)^2(\sin^2(\theta)Q(e^a)+\cos^2(\theta)Q(e^b))+\sum_{\mu\notin\{a,b\}}(v_\mu)^2Q(e^\mu)=$$
$$(v_a)^2Q(e^a)+(v_b)^2Q(e^b)+\sum_{\mu\notin\{a,b\}}(v_\mu)^2Q(e^\mu)=Q(v_\mu e^\mu),$$

whereas when $Q(e^a) = -Q(e^b)$ we can take ϖ as

$$\varpi\colon V \to V,\, e^i \mapsto \begin{cases} e^i & \text{if } i \notin \{a,b\}, \\ \cosh(\theta)e^a + \sinh(\theta)e^b & \text{if } i = a, \\ \sinh(\theta)e^a + \cosh(\theta)e^b & \text{if } i = b. \end{cases}$$

as then again we have $\det(\varpi) = \cosh^2(\theta) - \sinh^2(\theta) = 1$ and

$$\begin{gathered} Q(\varpi(v_\mu e^\mu)) = \langle v_\mu \varpi(e^\mu), v_\nu \varpi(e^\nu)\rangle_Q = \\ \langle v_a\varpi(e^a) + v_b\varpi(e^b), v_a\varpi(e^a) + v_b\varpi(e^b)\rangle_Q + \sum_{\mu\notin\{a,b\}} (v_\mu)^2\langle e^\mu, e^\mu\rangle_Q \\ = (v_a)^2(\cosh^2(\theta)Q(e^a) + \sinh^2(\theta)Q(e^b)) \\ +2v_a v_b \cosh(\theta)\sinh(\theta)(Q(e^a) + Q(e^b)) \\ +(v_b)^2(\sinh^2(\theta)Q(e^a) + \cosh^2(\theta)Q(e^b)) + \sum_{\mu\notin\{a,b\}} (v_\mu)^2 Q(e^\mu) \\ = (v_a)^2 Q(e^a) + (v_b)^2 Q(e^b) + \sum_{\mu\notin\{a,b\}} (v_\mu)^2 Q(e^\mu) = Q(v_\mu e^\mu). \end{gathered}$$

We would like to show that if $v \in \mathbb{C}\ell(V,Q)$ is not an element of $\mathrm{Span}\{1,\Gamma\}$, we can choose a, b and θ such that $\widetilde{\varpi}_{\mathbb{C}}(v) \neq v$.
As claimed in definition 63 and explained below remark 65, a $\mathbb{C}-$basis of $\mathbb{C}\ell(V,Q)$ is given by $(1, (e^{\mu_1})_{1\leq\mu_1\leq n}, \ldots, (e^{\mu_1} * \cdots * e^{\mu_n})_{1\leq\mu_1<\cdots<\mu_n\leq n})$. Now let $v \in \mathbb{C}\ell(V,Q)^{SO(V,Q)}$. When we write

$$v = \sum_{S\subseteq\{1,\ldots,n\}} v_S\, e^{S_1} * \cdots * e^{S_{|S|}}$$

where $S_1 < \cdots < S_{|S|}$, we have expressed v in the given basis. It is then easily checked that $v \in \mathrm{Span}\{1,\Gamma\}$ when $v_S = 0$ for all $S \notin \{\varnothing, \{1,\ldots,n\}\}$.
Let $S \subseteq \{1,\ldots,n\}$ be arbitrary and suppose $a, b \notin S$. Then $\varpi(e^{S_i}) = e^{S_i}$ so $\widetilde{\varpi}_{\mathbb{C}}(e^{S_1} * \cdots * e^{S_{|S|}}) = e^{S_1} * \cdots * e^{S_{|S|}}$.
Now suppose $a \in S$ and $b \notin S$. We then have $\widetilde{\varpi}_{\mathbb{C}}(e^{S_1} * \cdots * e^a * \cdots * e^{S_{|S|}}) = \varpi^a_a e^{S_1} * \cdots * e^a * \cdots * e^{S_{|S|}} + \varpi^a_b e^{S_1} * \cdots * e^b * \cdots * e^{S_{|S|}}$.
If $a \notin S$ and $b \in S$ the result is similar, so now we take $a, b \in S$. Similar to how we proved that $\Gamma' \in \mathbb{C}\ell(V,Q)^{SO(V,Q)}$, one can show that

$$\begin{gathered} \widetilde{\varpi}_{\mathbb{C}}(e^{S_1} * \cdots * e^a * \cdots * e^b * \cdots * e^{S_{|S|}}) \\ = \tfrac{1}{2}\widetilde{\varpi}_{\mathbb{C}}(e^{S_1} * \cdots * e^a * \cdots * e^b * \cdots * e^{S_{|S|}} - e^{S_1} * \cdots * e^b * \cdots * e^a * \cdots * e^{S_{|S|}}) \\ = \frac{\varpi^a_a\varpi^b_b - \varpi^a_b\varpi^b_a}{2}(e^{S_1} * \cdots * e^a * \cdots * e^b * \cdots * e^{S_{|S|}} - e^{S_1} * \cdots * e^b * \cdots * e^a * \cdots * e^{S_{|S|}}) \\ = e^{S_1} * \cdots * e^a * \cdots * e^b * \cdots * e^{S_{|S|}} \end{gathered}$$

Thus for $H \subseteq \{1, \ldots, n\}$ with $b \notin H \ni a$, the $e^{H_1} * \cdots * e^a * \cdots * e^{H_{|H|}}$ component of $\widetilde{\omega}_{\mathbb{C}}(v)$ depends only on the $e^{H_1} * \cdots * e^a * \cdots * e^{H_{|H|}}$ and $e^{H_1} * \cdots * e^b * \cdots * e^{H_{|H|}}$ components of v. Define $\tilde{H} = \{b\} \cup H \setminus \{a\}$ such that the $e^{H_1} * \cdots * e^b * \cdots * e^{H_{|H|}}$ component of v is given by $\pm v_{\tilde{H}}$. As $v \in \mathbb{C}\ell(V, Q)^{SO(V,Q)}$, we thus have

$$
\begin{aligned}
&v_H e^{H_1} * \cdots * e^a * \cdots * e^{H_{|H|}} \pm v_{\tilde{H}} e^{\tilde{H}_1} * \cdots * e^b * \cdots * e^{\tilde{H}_{|H|}} \\
&= \widetilde{\omega}_{\mathbb{C}}(v_H e^{H_1} * \cdots * e^a * \cdots * e^{H_{|H|}} \pm v_{\tilde{H}} e^{\tilde{H}_1} * \cdots * e^b * \cdots * e^{\tilde{H}_{|H|}}) \\
&= (\omega_a^a v_H \pm \omega_a^b v_{\tilde{H}}) e^{H_1} * \cdots * e^a * \cdots * e^{H_{|H|}} + (\omega_b^a v_H \pm \omega_b^b v_{\tilde{H}}) e^{H_1} * \cdots * v^b * \cdots * e^{H_{|H|}}
\end{aligned}
$$

which then gives $v_H = \omega_a^a v_H \pm \omega_a^b v_{\tilde{H}}$ and $v_{\tilde{H}} = \omega_b^b v_{\tilde{H}} \pm \omega_b^a v_H$. We then obtain $(1 - \omega_a^a)(1 - \omega_b^b) v_H = \pm \omega_a^b (1 - \omega_b^b) v_{\tilde{H}} = \omega_b^a \omega_a^b v_H$, so either $v_H = 0$ or $(1 - \omega_a^a)(1 - \omega_b^b) = \omega_b^a \omega_a^b$. The last expression gives $\omega_a^a \omega_b^b - \omega_b^a \omega_a^b = \omega_a^a + \omega_b^b - 1$. As $\det(\omega) = 1$ we thus have $\omega_a^a + \omega_b^b = 2$ or $v_H = v_{\tilde{H}} = 0$. Clearly $\omega_a^a + \omega_b^b = 2$ is not met for all possible ω, thus we conclude that $v_H = v_{\tilde{H}} = 0$ for all $H \notin \{\varnothing, \{1, \ldots, n\}\}$, and thus $\mathbb{C}\ell(V, Q)^{SO(V,Q)} = \mathrm{Span}\{1, \Gamma\}$.
Furthermore, $(\lambda + \mu\Gamma) * (\lambda + \mu\Gamma) = 1$ gives $\lambda^2 + \mu^2 + 2\lambda\mu\Gamma = 1$, thus $\lambda\mu = 0$ and $\lambda^2 + \mu^2 = 1$, which gives $(\lambda, \mu) \in \{(1, 0), (-1, 0), (0, 1), (0, -1)\}$. Thus $\{s \in \mathbb{C}\ell(V, Q)^{SO(V,Q)} | s * s = 1\} = \{1, -1, \Gamma, -\Gamma\}$. □

Note that Γ is not fixed by $O(V, Q)$, as any map $f \in O(V, Q)$ with $\det(f) = -1$ sends Γ to $-\Gamma$. This motivates the following nomenclature.

Definition 87. *The orientation operator Γ of a real vector space V with a non-degenerate quadratic form Q is an element of $\mathbb{C}\ell(V, Q)$ that satisfies $\Gamma * \Gamma = 1$ and which is fixed by $SO(V, Q)$, but not by $O(V, Q)$.*

When V is even-dimensional, given an isomorphism $\rho \colon \mathbb{C}\ell(V, Q) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$ the space $\mathbb{S}$ will split into eigenspaces of $\rho(\Gamma)$, which will then make $\mathrm{End}(\mathbb{S})$ into a $\{1, -1\}$-graded algebra, which coincides with the grading of $\mathbb{C}\ell(V, Q)$. This will follow from the following lemma.

Lemma 88. *The orientation operator $\Gamma \in \mathbb{C}\ell(V, Q)$ corresponding to an even-dimensional vector space V satisfies $\Gamma * x = \alpha(x) * \Gamma$.*

Proof. Express Γ in a suitable basis (see lemma 86) of V as $\eta e^1 * \cdots * e^n$ and let $e^i \in V$ be an arbitrary basis vector. We then have

$$
\begin{aligned}
\Gamma * e^i &= \eta e^1 * \cdots * e^i * \cdots * e^n * e^i = (-1)^{n-i} \eta e^1 * \cdots * e^i * e^i * \cdots * e^n \\
&= (-1)^{n-i} (-1)^{i-1} \eta e^i * e^1 * \cdots * e^i * \cdots * e^n = (-1)^{n-1} e^i * \Gamma.
\end{aligned}
$$

As V is even-dimensional, we have $(-1)^{n-1} = -1$, and thus $\Gamma * e^i = -e^i * \Gamma = \alpha(e^i) * \Gamma$. Now let $e^{\mu_1} * \cdots * e^{\mu_k}$ be an arbitrary basis vector of $\mathbb{C}\ell(V,Q)$. By induction on k, we obtain

$$\Gamma * e^{\mu_1} * \cdots * e^{\mu_k} = \alpha(e^{\mu_1} * \cdots * e^{\mu_{k-1}}) * \Gamma * e^{\mu_k}$$

$$= \alpha(e^{\mu_1} * \cdots * e^{\mu_{k-1}}) * \alpha(e^{\mu_k}) * \Gamma = \alpha(e^{\mu_1} * \cdots * e^{\mu_k}) * \Gamma,$$

and the general result then follows as α is linear. □

Definition 89. *A Weyl-spinor $s \in \mathbb{S}$ of an even-dimensional real vector space V with respect to an isomorphism $\rho\colon \mathbb{C}\ell(V,Q) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$ is a Dirac-spinor that is an eigenvector of $\rho(\Gamma)$.*
The spaces $\mathbb{S}^+ = \{s \in \mathbb{S} | \, \rho(\Gamma)s = s\}$ and $\mathbb{S}^- = \{s \in \mathbb{S} | \, \rho(\Gamma)s = -s\}$ are called the spaces of right-handed respectively left-handed Weyl spinors.

Notation 90. *The projection operators $\frac{1}{2}\rho(1+\Gamma)$ and $\frac{1}{2}\rho(1-\Gamma)$ are written as Γ_+ and Γ_- respectively. The corresponding elements $\frac{1+\Gamma}{2}$ and $\frac{1-\Gamma}{2}$ in $\mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ are also written as Γ_+ and Γ_- respectively. Note that $s \in \mathbb{S}$ can be written as $s = \Gamma_+ s + \Gamma_- s$ with $\Gamma_+ s \in \mathbb{S}^+$ and $\Gamma_- s \in \mathbb{S}^-$.*

Definition 91. *The $\{1,-1\}$-grading on $\mathrm{End}(\mathbb{S})$ is given by $\mathrm{End}^1(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}^+, \mathbb{S}^+) \oplus \mathrm{Hom}(\mathbb{S}^-, \mathbb{S}^-)$ and $\mathrm{End}^{-1}(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}^+, \mathbb{S}^-) \oplus \mathrm{Hom}(\mathbb{S}^-, \mathbb{S}^+)$, such that $\mathrm{End}^1(\mathbb{S}) \oplus \mathrm{End}^{-1}(\mathbb{S}) = \mathrm{Hom}(\mathbb{S}^+ \oplus \mathbb{S}^-, \mathbb{S}^+ \oplus \mathbb{S}^-) = \mathrm{End}(\mathbb{S})$.*

Notation 92. *Given an even-dimensional vector space V and an isomorphism $\rho\colon \mathbb{C}\ell(V,Q) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$, for $a, b \in \{+,-\}$ we write $\chi^a_b\colon \mathbb{C}\ell(V,Q) \twoheadrightarrow \mathrm{Hom}(\mathbb{S}^b, \mathbb{S}^a)$ for the map $v \mapsto \Gamma_a \rho(v) \Gamma_b$, so for example $\chi^+_-(v) = \Gamma_+ \rho(v) \Gamma_-$.*

Lemma 93. *Given an even-dimensional vector space V and an isomorphism $\rho\colon \mathbb{C}\ell(V,Q) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$ we have that $\rho(\mathbb{C}\ell^{\pm 1}(V,Q)) = \mathrm{End}^{\pm 1}(\mathbb{S})$, thus the grading on $\mathrm{End}(\mathbb{S})$ coincides with the grading on $\mathbb{C}\ell(V,Q)$ via ρ.*

Proof. This follows from lemma 88. For $x \in \mathbb{C}\ell^1(V,Q)$ we have that $\alpha(x) = x$, so for $s \in \mathbb{S}^\pm$ we have that $\rho(\Gamma)(s) = \pm s$, thus $\rho(\Gamma)(\rho(x)s) = \rho(\Gamma * x)(s) = \rho(\alpha(x) * \Gamma)(s) = \rho(x)(\rho(\Gamma)(s)) = \pm\rho(x)s$, thus $\rho(x)s \in \mathbb{S}^\pm$ and thus $\rho(x) \in \mathrm{End}^1(\mathbb{S})$, while for $x \in \mathbb{C}\ell^{-1}(V,Q)$ we similarly have that $\alpha(x) = -x$ and thus $\rho(\Gamma)\rho(x)s = \rho(\alpha(x))\rho(\Gamma)s = -\rho(x) \pm s = \mp\rho(x)s$, thus $\rho(x)s \in \mathbb{S}^\mp$ and thus $\rho(x) \in \mathrm{End}^{-1}(\mathbb{S})$. □

In quite some physics literature, Weyl spinors are defined without reference to Dirac spinors. We will call these spinors "basic Weyl spinors" to distuinguish them from the ones defined in definition 89.

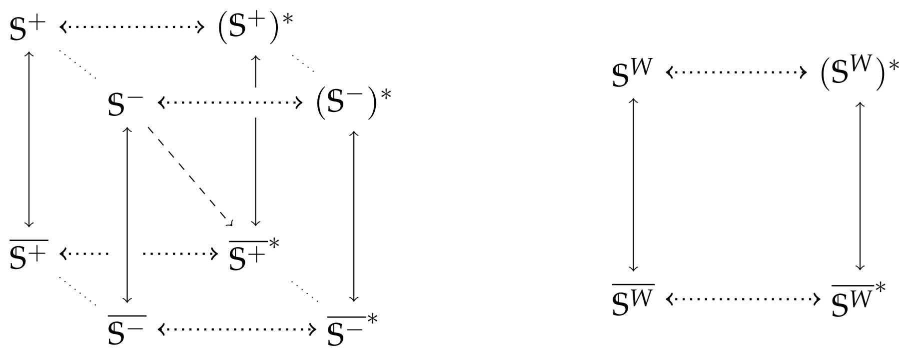

Figure 3.1: A diagram showing the spaces related to the Weyl spinor space of Minkowski space as in definition 89 (left) respectively related to basic Weyl spinors of Minkowski space as in definition 94 (right). The vertical arrows are canonical conjugate-linear maps, and the horizontal dotted arrows are non-canonical linear maps. The dotted lines connecting $\mathbb{S}^+$ *with* $\mathbb{S}^-$ *can be given by any map of* $\mathrm{End}^{-1}(\mathbb{S})$ *(definition 91). The dashed arrow is a canonical conjugate-linear mapping* $\check{\Sigma}$ *(lemma 99), which gives the identification needed to relate the left diagram to the right one.*

Definition 94. *A basic Weyl spinor* $\mathfrak{s} \in \mathbb{S}^W$ *of a* 4*-dimensional real vector space* V *with a basis* (e^0, e^1, e^2, e^3) *and a quadratic form* $x_\mu e^\mu \mapsto (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ *is an element of a* 2*-dimensional complex vector space* $\mathbb{S}^W$*, expressed on a basis* (ι_A, o_A)*.*
We have an inclusion $\zeta\colon V \hookrightarrow \mathbb{S}^W \otimes \overline{\mathbb{S}^W}$ *given by* $x_\mu e^\mu \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} x^0+x^3 & x^1+ix^2 \\ x^1-ix^2 & x^0-x^3 \end{pmatrix}$*.*

We would like to identify the notions of Weyl-spinors as in definition 89 and 94. As can be seen in figure 3.1, this can be done if any of the spaces $\mathbb{S}^+$, $(\mathbb{S}^+)^*$, $\overline{\mathbb{S}^+}$ or $\overline{\mathbb{S}^+}^*$ can (preferably canonically) be identified with $\mathbb{S}^-$. Proposition 11.1.27 in [11] gives, in the case of a *Euclidean* quadratic form Q, a sesquilinear form on $\mathbb{S}$ that is canonically defined up to a multiplicative constant. We will find that a similar construction also works in the case of $\mathcal{M}$. For it's construction, the following lemma is quite important.

Lemma 95. *Let* $\langle\cdot,\cdot\rangle$ *and* $[\cdot,\cdot]$ *be two non-degenerate sesquilinear forms on the Dirac Spinor space* $\mathbb{S}$ *of an even-dimensional real vector space* V *with respect to a non-degenerate quadratic form* Q *and an isomorphism* $\rho\colon \mathbb{C}\ell(V,Q) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$*, that both satisfy* $\forall x \in \mathbb{C}\ell(V,Q), \forall \varsigma, \sigma \in \mathbb{S}$, $\langle \rho(x)\varsigma, \sigma\rangle = \langle \varsigma, \rho(\overline{x}^\dagger)\sigma\rangle$ *and* $\forall x \in \mathbb{C}\ell(V,Q)$,
$\forall \varsigma, \sigma \in \mathbb{S}$, $[\rho(x)\varsigma, \sigma] = [\varsigma, \rho(\overline{x}^\dagger)\sigma]$*. Then* $\langle\cdot,\cdot\rangle = \lambda[\cdot,\cdot]$ *for some* $\lambda \in \mathbb{C}$*.*

Proof. Let $\eta\colon \mathbb{S} \to \mathbb{S}$ be such that $\forall x, y \in \mathbb{S}$, $\langle y, x\rangle = [y, \eta(x)]$. Then for

$v \in \mathbb{C}\ell(V,Q)$ we have that $[y, \eta(\rho(v)x)] = \langle y, \rho(v)x\rangle = \langle \rho(\overline{v}^\dagger)y, x\rangle = [\rho(\overline{v}^\dagger)y, \eta(x)] = [y, \rho(v)\eta(x)]$, thus $\eta(\rho(v)x) = \rho(v)\eta(x)$. This means we can apply Schur's lemma (See chapter 4.5 of [12]), which gives $\eta(x) = \lambda x$ for some $\lambda \in \mathbb{C}$, and thus $\langle y, x\rangle = \overline{\lambda}[y,x]$. □

We thus only need to proof the existence of a sesquilinear form with the given property. For this we first need the following notions (As in the proof of lemma 11.1.27 of [11].)

Definition 96. *The group $G_{e^\mu} \subseteq \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ corresponding to a basis $(e^\mu)_{\mu=0}^3$ of $\mathcal{M}$ is the group generated by $(e^\mu)_{\mu=0}^3$. It is given by the 32 elements $\{\pm 1, (\pm e^\mu)_{0\le\mu\le 3}, (\pm e^\mu * e^\nu)_{0\le\mu<\nu\le 3}, (\pm e^\mu * e^\nu * e^\xi)_{0\le\mu<\nu<\xi\le 3}, \pm\Gamma\}$, and can be written as $G_{e^\mu} = \cup_{S\subseteq\{0,\ldots,3\}}\{v^{S_1} * \cdots * v^{S_{|S|}}, -v^{S_1} * \cdots * v^{S_{|S|}}\}$*

Theorem 97. *There exists a non-degenerate sesquilinear form Σ on $\mathbb{S}$, the Dirac-spinor space of $\mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$, such that $\forall \sigma, \varsigma \in \mathbb{S}, \forall x \in \mathbb{C}\ell(V,Q)$, $\Sigma(\varsigma, \rho(x)\sigma) = \Sigma(\rho(\overline{x}^\dagger)\varsigma, \sigma)$.*

Proof. Let $(e^\mu)_{\mu=0}^3$ be a basis of $\mathcal{M}$ that is orthonormal with respect to $\langle\cdot,\cdot\rangle_{\mathcal{M}}$ (for example the standard basis of $\mathcal{M}$), and consider the sesquilinear form $\Sigma\colon \mathbb{S}\times\mathbb{S}\to\mathbb{C}$ given by $\Sigma(v,w) = \sum_{g\in G_{e^\mu}}\langle\rho(g)v, \rho(g*e^0)w\rangle$ for any sesquilinear inner product $\langle\cdot,\cdot\rangle$ on $\mathbb{S}$.
Now let $h = e^{S_1} * \cdots * e^{S_{|S|}} \in G_{e^\mu}$ be given. It follows from $e^0 * e^a = \begin{cases} e^a * e^0 & \text{if } a = 0 \\ -e^a * e^0 & \text{if } a \neq 0\end{cases}$ that $e^0 * h = \begin{cases} (-1)^{|S|-1} h * e^0 & \text{if } 0 \in S \\ (-1)^{|S|} h * e^0 & \text{if } 0 \notin S\end{cases} = (-1)^{|S|-\mathbf{1}_{0\in S}} h * e^0$. As $\Sigma(\rho(h)v, \rho(h)w) = \sum_{g\in G_{e^\mu}}\langle\rho(g*h)v, \rho(g*e^0*h)w\rangle = (-1)^{|S|-\mathbf{1}_{0\in S}}\sum_{g\in G_{e^\mu}}\langle\rho(g*h)v, \rho(g*h*e^0)w\rangle = (-1)^{|S|-\mathbf{1}_{0\in S}}\sum_{r\in G_{e^\mu}}\langle\rho(r)v, \rho(r*e^0)w\rangle = (-1)^{|S|-\mathbf{1}_{0\in S}}\Sigma(v,w)$, it follows that $\Sigma(v, \rho(h)w) = \Sigma(\rho(h)\rho(h)^{-1}v, \rho(h)w) = \Sigma((-1)^{|S|-\mathbf{1}_{0\in S}}\rho(h)^{-1}v, w) = \Sigma(\rho(h^\dagger * h)\rho(h)^{-1}v, w) = \Sigma(\rho(h^\dagger)v, w)$. As G_{e^μ} includes a basis of $\mathbb{C}\ell(V,Q)$ and $\langle\cdot,\cdot\rangle$ is sesquilinear, it follows that $\Sigma(v, \rho(x)w) = \Sigma(\rho(\overline{x}^\dagger)v, w)$ for any $x \in \mathbb{C}\ell(V,Q)$. □

Lemma 98. *For either $v, w \in \mathbb{S}^+$ or $v, w \in \mathbb{S}^-$, we have that $\Sigma(v,w) = 0$. Furthermore Σ is non-degenerate of signature 0.*

Proof. Let $v, w \in \mathbb{S}^\pm$. Then using lemma 88, we get $\Sigma(v,w) = \sum_{g\in G_{e^\mu}}\langle\rho(g)v, \rho(g*e^0)w\rangle = \frac{1}{2}\sum_{g\in G_{e^\mu}}\langle\rho(g)v, \rho(g*e^0)w\rangle + \langle\rho(g*\Gamma)v, \rho(g*\Gamma*e^0)w\rangle = \frac{1}{2}\left(\Sigma(v,w) + \sum_{g\in G_{e^\mu}}\langle\rho(g*\Gamma)v, \rho(g*\alpha(e^0)*\Gamma)w\rangle\right) = \frac{1}{2}\left(\Sigma(v,w) - \sum_{g\in G_{e^\mu}}\langle\rho(g)\rho(\Gamma)v, \rho(g*e^0)\rho(\Gamma)w\rangle\right) = \frac{1}{2}\left(\Sigma(v,w) - \sum_{g\in G_{e^\mu}}\langle\pm\rho(g)v, \pm\rho(g*e^0)w\rangle\right) = \frac{1}{2}(\Sigma(v,w) - \Sigma(v,w)) = 0.$

Clearly, Σ is non-degenerate as $(v,w) \mapsto \sum_{g\in G_{e^\mu}} \langle \rho(g)v, \rho(g)w\rangle$ is positive-definite and $\rho(e^0)$ is invertible.
Now let (s^1, s^2) be a basis of $\mathbb{S}^+$ and let (s^3, s^4) be a basis of $\mathbb{S}^-$.
The signature of Σ is $\Sigma(s^1,s^1) + \Sigma(s^2,s^2) + \Sigma(s^3,s^3) + \Sigma(s^4,s^4) = 0$. □

The previous lemma is important, as it enables us to make the identification needed to connect the notions of spinors developed thus far as seen in figure 3.1. In particular, we can identify $\mathbb{S}^-$ with $\overline{\mathbb{S}^+}^*$.

Lemma 99. *The sesquilinear form Σ induces a linear map $\check{\Sigma}\colon \mathbb{S} \xrightarrow{\sim} \overline{\mathbb{S}}^*$ given by $s \mapsto (v \mapsto \Sigma(s,v))$, that is also an isomorphism $\check{\Sigma}|_{\mathbb{S}^-}\colon \mathbb{S}^- \xrightarrow{\sim} \overline{\mathbb{S}^+}^*$ when restricted to $\mathbb{S}^-$.*

Proof. The first part is well-known, see for example [4]. We will show that $\ker(\check{\Sigma}|_{\mathbb{S}^-}) = \{0\}$. Suppose $v \in \ker(\check{\Sigma}|_{\mathbb{S}^-})$, so $\forall w \in \overline{\mathbb{S}^+}$, $\Sigma(v,w) = 0$. Then for $s \in \mathbb{S}$ we obtain that $\Sigma(v,s) = \Sigma(v,\Gamma_+ s) + \Sigma(v, \Gamma_- s) = 0$, so we have for any $s \in \mathbb{S}$ that $\Sigma(v,s) = 0$, so $v = 0$ as Σ is non-degenerate. □

Finally, the map ζ of definition 94 can be interpreted in terms of the map χ_-^+ of notation 92. In particular, when $\zeta(v) \in \mathbb{S}^W \otimes \overline{\mathbb{S}^W}$ is interpreted as a map $\zeta(v)\colon \mathbb{S}^+ \to \overline{\mathbb{S}^+}^*$, we get that $\zeta(v) = \check{\Sigma}|_{\mathbb{S}^-} \circ \chi_+^-(v)$, when a suitable basis on $\mathbb{S}$ is chosen.

3.1.1 Intermezzo for physicists

In the previous section, we have defined the machinery that is needed to mathematically define spinors. In physics, there are two important notions of spinors, notably that of Dirac spinors (definition 75) and of Weyl spinors (definition 89 and 94).

In physics, Dirac spinor space is introduced via the 4 complex 4×4 gamma-matrices $(\gamma^0, \gamma^1, \gamma^2, \gamma^3)$, that satisfy $\gamma^\mu \cdot \gamma^\nu + \gamma^\nu \cdot \gamma^\mu = 2\mathfrak{g}_{\mu\nu} I$. The choice of these matrices has some freedom. As we will make extensive use of notation 92 in the next chapter, the so-called Weyl-basis for the gamma matrices is preferable. It is given by

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \ \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \ \gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \ \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$

where σ_μ are the Pauli-matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Any vector $v = v_\mu e^\mu \in \mathbb{C}\mathcal{M}$ can then be associated to the matrix $v_\mu \gamma^\mu$. The 4-dimensional complex vector space on which this matrix operates is the space of Dirac spinors, $\mathbb{S}$.

Furthermore, one has the orientation operator $\Gamma = i\gamma^0\gamma^1\gamma^2\gamma^3 = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$, which is independent of the choice of coordinates on $\mathbb{C}\mathcal{M}$ (lemma 86).

The subspaces of $\mathbb{S}$ that are eigenspaces of this operator are the spaces of Weyl spinors (definition 89). Physicists usually write x^A for a Weyl-spinor, where A ranges from 0 to 1. As seen in figure 3.1, Weyl spinors come in 4 different kinds, which are then written as $x^A, x^{A'}, x_A$ and $x_{A'}$ for spinors in $\mathbb{S}^W, \overline{\mathbb{S}^W}, (\mathbb{S}^W)^*$ and $\overline{\mathbb{S}^W}^*$ respectively. Note the resemblance of the right upper part of $v_\mu \gamma^\mu$ in the Weyl basis and the definition of Weyl spinors in definition 94.

More material on the physical interpretation of spinors can be found in e.g. [3].

3.2 Twistors

Now that we know what a spinor is, we can do two different things. We could try to construct a spinor bundle $\mathbb{S}\mathcal{M}$ on Minkowski space and write Maxwell's equations in terms of spinor fields on $\mathcal{M}$, or we could define twistors without the definition of spinor fields. (This is a huge reduction of the full concept of twistors, as a twistor *should* be defined as a spinor field that can be parametrized with an element of the space in definition 100, see chapter 7 of [3]) We will do the latter.

The twistor formalism was once introduced by Roger Penrose as a possible formalism to enhance the understanding of the interaction between general relativity and particle behaviour. Twistor theory includes the construction of compactified complexified Minkowski space and the (identity component of the) conformal group, which we will use in this thesis. A good book on the matter is [13]. We start with the definition of twistor space as in [3].

Definition 100. *Twistor space $\mathbb{T}$ is the space $\mathbb{S}^W \oplus \overline{\mathbb{S}^W}^*$, where $\mathbb{S}^W$ is the space of basic Weyl vectors corresponding to the vector space $\mathcal{M}$. Twistor space is endowed with a hermitian form $\Sigma_\mathbb{T}\colon \mathbb{T} \times \overline{\mathbb{T}} \to \mathbb{C}$ given by $\Sigma_\mathbb{T}\big((s, \overline{\omega}), (\overline{r}, \sigma)\big) = \sigma(s) + \overline{\omega}(\overline{r})$.*

Lemma 101. *One can identify $\mathbb{T}$ with the Dirac-spinor space $\mathbb{S}$ corresponding to $\mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})$ with $\langle\cdot,\cdot\rangle_\mathcal{M}$ as in definition 3. Using this identification, the hermitian form Σ of theorem 97 coincides with the hermitian form $\Sigma_\mathbb{T}$ of $\mathbb{T}$.*

Proof. Setting $\mathbb{S}^- = \mathbb{S}^W$, lemma 99 gives us a direct means of identifying $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ with $\mathbb{S}^+ \oplus \overline{\mathbb{S}^+}^*$. Furthermore, for $s^+, r^+ \in \mathbb{S}^+$ and $s^-, r^- \in \mathbb{S}^-$ we have that $\Sigma(s^+ + s^-, r^+ + r^-) = \Sigma(s^+, r^-) + \Sigma(s^-, r^+)$, thus after the identification we have that $\Sigma(s^+ + s^-, r^+ + r^-) = \overline{\Sigma(r^-, s^+)} + \Sigma(s^-, r^+) = \overline{\check{\Sigma}|_{\mathbb{S}^-}(r^-)(s^+)} + \check{\Sigma}|_{\mathbb{S}^-}(s^-)(\overline{r^+}) = \Sigma_{\mathbb{T}}\big((s^+, \check{\Sigma}|_{\mathbb{S}^-}(s^-)), (\overline{r^+}, \overline{\check{\Sigma}|_{\mathbb{S}^-}(r^-)})\big)$ □

Remark 102. *As stated in [14], there is a* physical *difference between Dirac spinors and Twistors, in that Dirac spinors are usually made into spinor fields, i.e. every point of sapce-time can have a different Dirac spinor, whereas each Twistor is on its own a spinor field. As we do not use this aspect of twistors, we are allowed to make this identification.*

From now on, we will use $\mathbb{S}$ instead of $\mathbb{T}$. We now define the unitary group $U(\mathbb{S}, \Sigma)$ and complexified compactified Minkowski space $\mathbb{C}\mathcal{M}^\#$.

Definition 103. *The unitary group $U(V, \langle\cdot,\cdot\rangle)$ of a vector space V with respect to a non-degenerate hermitian form $\langle\cdot,\cdot\rangle$ is the subgroup of* $\mathrm{End}(V)$ *given by* $\{f \in \mathrm{End}(V) | \forall v, w \in V, \langle f(v), f(w)\rangle = \langle v, w\rangle\}$.

Lemma 104. *We have* $U(\mathbb{S}, \Sigma) = \{\rho(v) | v \in \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}}), \overline{v}^\dagger * v = 1\}$.

Proof. Clearly, $U(\mathbb{S}, \Sigma) \subseteq \mathrm{End}(\mathbb{S})$, so given $\rho\colon \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}}) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$ we have that $\forall f \in U(\mathbb{S}, \Sigma) \exists v_f \in \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ such that $\rho(v_f) = f$. Now theorem 97 gives that $\Sigma(v, w) = \Sigma(f(v), f(w))$ if and only if $\Sigma(v, w) = \Sigma(\rho(\overline{v_f}^\dagger)\rho(v_f)v, w) = \rho(\overline{v_f}^\dagger * v_f)\Sigma(v, w)$, which is true for all $v, w \in \mathbb{S}$ if and only if $\overline{v_f}^\dagger * v_f = 1$. □

Definition 105. *Complexified compactified Minkowski space $\mathbb{C}\mathcal{M}^\#$ is the space $\{U \subset \mathbb{S} | U$ is a linear subspace of $\mathbb{C}$-dimension $2\}$, also known as the Grassmannian $G_2(\mathbb{S})$. Given a isomorphism $\rho\colon \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}}) \xrightarrow{\sim} \mathrm{End}(\mathbb{S})$, for any $f \in GL(\mathbb{S})$ there is a corresponding chart (U_f, ϕ_f) where ϕ_f is such that*

$$\phi_f^{-1}\colon \mathbb{C}\mathcal{M} \hookrightarrow \mathbb{C}\mathcal{M}^\#, v \mapsto \{s \in \mathbb{S} | \Gamma_+(I + \rho(v)\Gamma_-)f(s) = 0\}.$$

Here Γ_+ and Γ_- are given as in notation 90.

A straightforward but somewhat lengthy calculation show that these maps are injective. Of special interest are the charts (U_f, ϕ_f) with $f \in U(\mathbb{S}, \Sigma)$, because of the following lemma.

Lemma 106. *For $f \in U(\mathbb{S}, \Sigma)$ we have that $\phi_f^{-1}(v)^\perp = \phi_f^{-1}(\overline{v})$, and thus $\phi_f^{-1}(v)^\perp = \phi_f^{-1}(v)$ if and only if $v \in \mathcal{M}$*

Proof. Let $v \in \mathbb{C}\mathcal{M}$ be given, and take $s_1 \in \phi_f^{-1}(v)$ and $s_2 \in \phi_f^{-1}(\overline{v})$. From lemma 98 and because f is unitary we get that

$$\Sigma(s_1, s_2) = \Sigma(f(s_1), f(s_2)) = \Sigma((\Gamma_+ + \Gamma_-)f(s_1), (\Gamma_+ + \Gamma_-)f(s_2)) =$$
$$\Sigma(\Gamma_+ f(s_1), \Gamma_- f(s_2)) + \Sigma(\Gamma_- f(s_1), \Gamma_+ f(s_2)).$$

Because we have that

$$\Gamma_+ f(s_1) = -\Gamma_+ \rho(v)\Gamma_- f(s_1) \text{ and } \Gamma_+ f(s_2) = -\Gamma_+ \rho(\overline{v})\Gamma_- f(s_1),$$

we get

$$\Sigma(s_1, s_2) = \Sigma(-\Gamma_+ \rho(v)\Gamma_- f(s_1), \Gamma_- f(s_2)) + \Sigma(\Gamma_- f(s_1), -\Gamma_+ \rho(\overline{v})\Gamma_- f(s_2)).$$

Again using lemma 98, we see that

$$\Sigma(\Gamma_- \rho(v)\Gamma_- f(s_1), \Gamma_- f(s_2)) = 0 = \Sigma(\Gamma_- s_1, \Gamma_- \rho(\overline{v})\Gamma_- f(s_2)),$$

thus

$$\Sigma(s_1, s_2) = -\Sigma(\rho(v)\Gamma_- f(s_1), \Gamma_- f(s_2)) - \Sigma(\Gamma_- f(s_1), \rho(\overline{v})\Gamma_- f(s_2)),$$

and thus

$$\Sigma(s_1, s_2) = -\Sigma(\Gamma_- f(s_1), \rho(\overline{v}^\dagger)\Gamma_- f(s_2)) - \Sigma(\Gamma_- f(s_1), \rho(\overline{v})\Gamma_- f(s_2)) =$$
$$\Sigma(\Gamma_- f(s_1), \rho(\overline{v} - \overline{v})\Gamma_- f(s_2)) = 0.$$

Thus $\Sigma(s_1, s_2) = 0$ for all $s_1 \in \phi_f^{-1}(v)$ and $s_2 \in \phi_f^{-1}(\overline{v})$.
Because $\dim(\mathbb{S}) = 4$, $\dim(\phi_f^{-1}(v)) = \dim(\phi_f^{-1}(\overline{v})) = 2$ and Σ is non-degenerate, we obtain $\phi_f^{-1}(v)^\perp = \phi_f^{-1}(\overline{v})$.

The last statement of the lemma follows from injectivity of ϕ_f^{-1}. □

Furthermore, the following lemma inspires most people to look at $SU(\mathbb{S}, \Sigma) \subseteq U(\mathbb{S}, \Sigma)$ (the subset of maps with determinant 1) instead of $U(\mathbb{S}, \Sigma)$.

Lemma 107. *For $f \in U(\mathbb{S}, \Sigma)$ and $\lambda \in \mathbb{C}^*$, if we also have that $\lambda f \in U(\mathbb{S}, \Sigma)$, then the charts ϕ_f and $\phi_{\lambda f}$ are equal.*

Proof. Clearly,

$$\phi_f^{-1}(v) = \{s \in \mathbb{S} | \Gamma_+(I + \rho(v)\Gamma_-)f(s) = 0\}$$
$$= \{s \in \mathbb{S} | \Gamma_+(I + \rho(v)\Gamma_-)\lambda f(s) = 0\} = \phi_{\lambda f}^{-1}(v).$$

□

Our definition of complexified compactified Minkowski space is based on our definition of a manifold (definition 47). In particular we have given several charts, whereas other sources usually only give $\phi_{\mathrm{id}_\mathbb{S}}$. In that case, for $f \in U(\mathbb{S}, \Sigma)$, the map $\phi_f \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1}$ from $\phi_f^{-1}(\phi_{\mathrm{id}_\mathbb{S}}(\mathbb{C}\mathcal{M})) \subseteq \mathbb{C}\mathcal{M}$ to $\mathbb{C}\mathcal{M}$ is considered a transformation of $\mathbb{C}\mathcal{M}$.

Notation 108. *We will write δ resp. $\tilde{\delta}$ for the maps $\delta \colon GL(\mathbb{S}) \to Map(\mathbb{C}\mathcal{M})$, $f \mapsto \phi_f \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1}$ resp. $\tilde{\delta} \colon GL(\mathbb{S}) \to Map(\mathbb{C}\mathcal{M})$, $f \mapsto \phi_{\mathrm{id}_\mathbb{S}} \circ \phi_f^{-1}$*

Notation 109. *For $f \in GL(\mathbb{S})$, we write $\tilde{f}$ for the induced map $\tilde{f} \colon G_2(\mathbb{S}) \to \mathbb{S}$, $p \mapsto f(p)$.*

We now try to investigate the relation between $f \in U(\mathbb{S}, \Sigma)$ and the maps $\phi_{\mathrm{id}_\mathbb{S}} \circ \phi_f^{-1}$ and $\phi_f \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1}$.

Lemma 110. *For $f \in GL(\mathbb{S})$, we have that the map $\tilde{\delta}(f) \colon U \to \mathbb{C}\mathcal{M}$, where $U \subseteq \mathbb{C}\mathcal{M}$, is equal to $\phi_{\mathrm{id}_\mathbb{S}} \circ \widetilde{f^{-1}} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1} \colon U \to \mathbb{C}\mathcal{M}$.*
Similarly, $\delta(f) \colon V \to \mathbb{C}\mathcal{M}$ is equal to $\phi_{\mathrm{id}_\mathbb{S}} \circ \tilde{f} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1} \colon U \to \mathbb{C}\mathcal{M}$.
Hence, δ is a group homomorphism.

Proof. Clearly, $\phi_{\mathrm{id}_\mathbb{S}} \circ \widetilde{f^{-1}} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1}(v) = \phi_{\mathrm{id}_\mathbb{S}}(\{f^{-1}(s) \in \mathbb{S} | \Gamma_+(1+\rho(v)\Gamma_-)s = 0\}) = \phi_{\mathrm{id}_\mathbb{S}}(\{s \in \mathbb{S} | \Gamma_+(1+\rho(v)\Gamma_-)f(s) = 0\}) = \phi_{\mathrm{id}_\mathbb{S}} \circ \phi_f^{-1}(v) = \tilde{\delta}(f)(v)$.
A similar argument gives $\delta(f) = \phi_{\mathrm{id}_\mathbb{S}} \circ \tilde{f} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1}$.
Therefore, $\delta(f \circ g) = \phi_{\mathrm{id}_\mathbb{S}} \circ \tilde{f} \circ \tilde{g} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1} = \phi_{\mathrm{id}_\mathbb{S}} \circ \tilde{f} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1} \circ \phi_{\mathrm{id}_\mathbb{S}} \circ \tilde{g} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1} = \delta(f) \circ \delta(g)$. □

Using notation 92, for any map $\rho(z) \in U(\mathbb{S}, \Sigma)$ we can write $\rho(z) = \chi_+^+(z) + \chi_-^+(z) + \chi_+^-(z) + \chi_-^-(z) = \begin{pmatrix} \chi_+^+(z) & \chi_-^+(z) \\ \chi_+^-(z) & \chi_-^-(z) \end{pmatrix}$. We now investigate the maps $\chi_\mp^\pm$ a bit more.

Lemma 111. *The map $\chi_-^+ \colon \mathbb{C}\mathcal{M} \hookrightarrow \mathrm{Hom}(\mathbb{S}^-, \mathbb{S}^+)$ is bijective, and thus admits an inverse $(\chi_-^+)^{-1} \colon \mathrm{Hom}(\mathbb{S}^-, \mathbb{S}^+) \twoheadrightarrow \mathbb{C}\mathcal{M}$.*
Similarly, $\chi_+^- \colon \mathbb{C}\mathcal{M} \hookrightarrow \mathrm{Hom}(\mathbb{S}^+, \mathbb{S}^-)$ has an inverse $(\chi_+^-)^{-1}$

Proof. We will show $\ker(\chi_-^+) = 0$. Let $v \in \ker(\chi_-^+)$ be arbitrary. From $\chi_-^+(v) = \Gamma_+\rho(v)\Gamma_-$ we get $\chi_-^+(v) = 0 \Rightarrow \Gamma_+ v \Gamma_- = 0 \Rightarrow \Gamma_+\Gamma_+ v = \frac{1+\Gamma}{2} v = 0 \Rightarrow \Gamma v \in \mathbb{C}\mathcal{M}$. But this means that $-\Gamma v = (\Gamma v)^\dagger = v^\dagger \Gamma^\dagger = -v\Gamma = -\Gamma\alpha(v) = \Gamma v$, thus $\Gamma v = 0$, and thus $v = \Gamma\Gamma v = 0$.
Completely analogously, $\ker(\chi_+^-) = 0$. The surjectivity follows as $\dim(\mathbb{C}\mathcal{M}) = \dim(\mathrm{Hom}(\mathbb{S}^-, \mathbb{S}^+)) = \dim(\mathrm{Hom}(\mathbb{S}^+, \mathbb{S}^-))$. □

Lemma 112. *For $v \in \mathbb{C}\ell^-(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})$, we have that $\chi^+_-(v) \in \mathrm{Hom}(\mathbb{S}^-, \mathbb{S}^+)$ has an inverse $(\chi^+_-(v))^{-1} \in \mathrm{Hom}(\mathbb{S}^+, \mathbb{S}^-)$ that is given by $\chi^-_+(v^{-1})$.*

Proof. We can derive that $\begin{pmatrix} \mathrm{Id}_{\mathbb{S}^+} & 0 \\ 0 & \mathrm{Id}_{\mathbb{S}^-} \end{pmatrix} = \rho(v^{-1} * v) = \rho(v^{-1})\rho(v) = \begin{pmatrix} 0 & \chi^+_-(v^{-1}) \\ \chi^-_+(v^{-1}) & 0 \end{pmatrix} \begin{pmatrix} 0 & \chi^+_-(v) \\ \chi^-_+(v) & 0 \end{pmatrix} = \begin{pmatrix} \chi^+_-(v^{-1})\chi^-_+(v) & 0 \\ 0 & \chi^-_+(v^{-1})\chi^+_-(v) \end{pmatrix}$ and similarly $\begin{pmatrix} \mathrm{Id}_{\mathbb{S}^+} & 0 \\ 0 & \mathrm{Id}_{\mathbb{S}^-} \end{pmatrix} = \begin{pmatrix} \chi^+_-(v)\chi^-_+(v^{-1}) & 0 \\ 0 & \chi^-_+(v)\chi^+_-(v^{-1}) \end{pmatrix}$ thus $(\chi^+_-(v))^{-1} = \chi^-_+(v^{-1})$. □

Theorem 113. *For $v \in \mathbb{C}\mathcal{M}$ and $\rho(z) \in GL(\mathbb{S})$ we have that*

$$\tilde{\delta}(z)(v) = (\chi^+_-)^{-1}((\chi^+_+(z) + \rho(v)\chi^-_+(z))^{-1}(\rho(v)\chi^-_-(z) + \chi^+_-(z)))$$

Proof. We have $\phi_{\mathrm{id}_\mathbb{S}} \circ \phi^{-1}_{\rho(z)}(v) = \phi_{\mathrm{id}(\mathbb{S})}\{s \in \mathbb{S} | \Gamma_+(I + \rho(v)\Gamma_-)\rho(z)s = 0\} =$
$\phi_{\mathrm{id}(\mathbb{S})}\{s \in \mathbb{S} | \Gamma_+(\Gamma_+\rho(z) + \rho(v)\Gamma_-\rho(z))(\Gamma_+ + \Gamma_-)s = 0\} =$
$\phi_{\mathrm{id}(\mathbb{S})}\{s \in \mathbb{S} | \Gamma_+ \left((\chi^+_+(z) + \rho(v)\chi^-_+(z)) + (\rho(v)\chi^-_-(z) + \chi^+_-(z))\right) s = 0\} =$
$\phi_{\mathrm{id}(\mathbb{S})}\{s \in \mathbb{S} | \Gamma_+ \left(I + (\chi^+_+(z) + \rho(v)\chi^-_+(z))^{-1}(\rho(v)\chi^-_-(z) + \chi^+_-(z))\right) s = 0\}$
$= (\chi^+_-)^{-1}((\chi^+_+(z) + \rho(v)\chi^-_+(z))^{-1}(\rho(v)\chi^-_-(z) + \chi^+_-(z)))$ □

Lemma 114. *The following are true:*

- *For $r \in \mathbb{C}\mathcal{M}$, $\tilde{\delta}(\rho(1 + \Gamma_+ r\Gamma_-))$ is a translation by r. Furthermore, $\rho(1 + \Gamma_+ r\Gamma_-) \in U(\mathbb{S}, \Sigma)$ if and only if $r \in \mathcal{M}$.*
- *For $\alpha, \beta \in \mathbb{C}^*$, $\tilde{\delta}(\alpha\Gamma_+ + \beta\Gamma_-)$ is a dilation by $\frac{\beta}{\alpha}$. Furthermore, $\alpha\Gamma_+ + \beta\Gamma_- \in U(\mathbb{S}, \Sigma)$ if and only if $\frac{\beta}{\alpha} \in \mathbb{R}^*$.*
- *For $z \in \mathrm{Spin}_\mathbb{C}(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})$, $\tilde{\delta}(\rho(z))$ is an element of $SO^+(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})$.*
- *For $z \in \Gamma_\mathbb{C}(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})$ with $\alpha(z) = -z$, $\tilde{\delta}(\rho(z))$ is a conformal inversion composed with an element of $O(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})$ such that it preserves complete orientation.*

Proof. Using theorem 113, we can derive the following:

- If $z = 1 + \Gamma_+ r\Gamma_-$, then we obtain that $\tilde{\delta}(z)(v) = (\chi^+_-)^{-1}((\chi^+_+(1) + \rho(v)\Gamma_-\Gamma_+\rho(r)\Gamma_-\Gamma_+)^{-1}(\rho(v)\chi^-_-(1) + \chi^+_-(r))) =$
$(\chi^+_-)^{-1}(\Gamma_+\rho(v + r)\Gamma_-) = v + r$, thus then $\tilde{\delta}(z)$ is a translation by r. The unitary condition becomes $\bar{z}^\dagger * z = \overline{(1 + \Gamma_+ r\Gamma_-)}^\dagger(1 + \Gamma_+ r\Gamma_-) =$
$(1 + \Gamma_-\bar{r}^\dagger\Gamma_+)(1 + \Gamma_+ r\Gamma_-) = (1 + \Gamma_+\bar{r}^\dagger\Gamma_-)(1 + \Gamma_+ r\Gamma_-) =$
$1 + \Gamma_+(\bar{r}^\dagger + r)\Gamma_- = 1$, thus $r + \bar{r}^\dagger = r - \bar{r} = 0$ thus $r \in \mathcal{M}$.

- If $z = \alpha\Gamma_+ + \beta\Gamma_-$, we obtain that $\tilde{\delta}(z)(v) = (\chi^{\pm}_{-})^{-1}(\alpha^{-1}\beta\chi^{\pm}_{-}(v)) = \frac{\beta}{\alpha}v$. This is a dilation by $\frac{\beta}{\alpha}$. The unitary condition gives $1 = \overline{z}^\dagger * z = \overline{\alpha\Gamma_+ + \beta\Gamma_-}^\dagger(\alpha\Gamma_+ + \beta\Gamma_-) = (\overline{\alpha}\Gamma_- + \overline{\beta}\Gamma_+)(\alpha\Gamma_+ + \beta\Gamma_-) = \overline{\alpha}\beta\Gamma_+ + \overline{\beta}\alpha\Gamma_-$ and thus $\overline{\alpha}\beta = \overline{\beta}\alpha = 1$. Hence, $\overline{\frac{\beta}{\alpha}} = \frac{\overline{\beta}}{\overline{\alpha}} = \frac{\alpha^{-1}}{\beta^{-1}} = \frac{\beta}{\alpha}$, thus $\frac{\beta}{\alpha} \in \mathbb{R}$.

- If $z \in \mathrm{Spin}_{\mathbb{C}}(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ we see $\tilde{\delta}(z)(v) = (\chi^{\pm}_{-})^{-1}((\chi^{\pm}_{+}(z))^{-1}\chi^{\pm}_{-}(v)\chi^{-}_{-}(z)) = (\chi^{\pm}_{-})^{-1}(\chi^{\pm}_{+}(z^{-1})\chi^{\pm}_{-}(v)\chi^{-}_{-}(z)) = (\chi^{\pm}_{-})^{-1}(\chi^{\pm}_{-}(z^{-1} * v * z))$. As $z^{-1} * v * z \in \mathbb{C}\mathcal{M}$ by the definition of $\mathrm{Spin}_{\mathbb{C}}(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$, this equals $z^{-1} * v * z = F|_{\mathrm{Spin}_{\mathbb{C}}(\mathcal{M},\langle\cdot,\cdot\rangle_{\mathcal{M}})}(z)(v)$, with $F|_{\mathrm{Spin}_{\mathbb{C}}(\mathcal{M},\langle\cdot,\cdot\rangle_{\mathcal{M}})}$ as in theorem 83. We thus see any element of $SO^+(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ can be given as $\tilde{\delta}(z)$ for a suitable choice of z.

- If $z \in \Gamma_{\mathbb{C}}(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ with $\alpha(z) = -z$ and $\overline{z}^\dagger * z = 1$, we have that $\tilde{\delta}(z)(v) = (\chi^{\pm}_{-})^{-1}\left((\rho(v)\chi^{-}_{+}(z))^{-1}(\chi^{\pm}_{-}(z))\right) = (\chi^{\pm}_{-})^{-1}(\chi^{\pm}_{-}(z^{-1})\rho(v^{-1})\chi^{\pm}_{-}(z)) = (\chi^{\pm}_{-})^{-1}(\chi^{\pm}_{-}(\frac{-1}{Q(v)}z^{-1}vz)) = \frac{-1}{Q(v)}(\chi^{\pm}_{-})^{-1}(\chi^{\pm}_{-}(z^{-1}vz))$. Because $z^{-1}vz \in \mathbb{C}\mathcal{M}$ by the definition of $\Gamma_{\mathbb{C}}(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$, we get that $\tilde{\delta}(z)(v) = \frac{-1}{Q(v)}z^{-1}vz = \frac{-1}{Q(v)}F_{\mathbb{C}}(z)(v)$, with $F_{\mathbb{C}} \colon \Gamma_{\mathbb{C}}(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}}) \to O(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ as in theorem 82. A slightly more careful examination of the proof of theorem 82 shows that any map in $O(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ that does preserve time-orientation can be given by $v \mapsto z^{-1} * v * z$ for some z with $\overline{z}^\dagger * z = 1$, and that from $\alpha(z) = -z$ it follows that space-orientation is flipped. The map $v \mapsto -z^{-1} * v * z$ therefore preserves space-orientation and flips time-orientation. The map $v \mapsto \frac{v}{Q(v)}$ is known as a conformal inversion, and with our convention (definition 3) it flips time-orientation. Hence, $\tilde{\delta}(z)$ can be seen as an orthogonal transformation that flips time-orientation composed with a conformal inversion that flips time-orientation. As space-orientation is preserved and time-orientation is flipped twice, $\tilde{\delta}(z)$ preserves complete orientation.

□

Now, we want to determine what $\tilde{\delta}(z)$ does on a tangent vector. We use the following definition of the tangent space of the grasmannian, which is in line with the definition given in the introduction of chapter 2 of [15].

Definition 115. *For $p \in G_2(\mathbb{S})$, so $p \subseteq \mathbb{S}$, the space of lineair maps $T_pG_2(\mathbb{S}) = \{f \colon p \to \mathbb{S}/p\}$ can be identified with the tangent space of $G_2(\mathbb{S})$ at p. A tangent vector corresponding to $f \in T_pG_2(\mathbb{S})$ is then given by the abstract expression $\frac{\partial}{\partial t}(1 + tf)p|_{t=0}$*

Lemma 116. *For v with $Q(v) \neq 0$, the differential* $\mathrm{d}\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}\colon T\mathbb{C}\mathcal{M} \to TG_2(\mathbb{S})$ *can be given as* $T_v\mathbb{C}\mathcal{M} \ni r \mapsto \chi^-_-(\frac{v*r}{Q(v)}) \in T_{\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v)}G_2(\mathbb{S})$.

Proof. Let $r \in T_v\mathbb{C}\mathcal{M}$ be given, and consider the map $\gamma\colon (-1,1) \to \mathbb{C}\mathcal{M}$, $t \mapsto v + tr$. As in e.g. [16], $\mathrm{d}\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(r) = \frac{\partial}{\partial t}\gamma(t)|_{t=0}$. Let $s \in \phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v)$ and assume $\Gamma_+(1+\rho(v+tr)\Gamma_-)(s-\lambda) = 0$. Then

$$0 = \Gamma_+(1+\rho(v)\Gamma_-)s + \Gamma_+\rho(tr)\Gamma_-s - \Gamma_+(1+\rho(v+tr)\Gamma_-)\lambda =$$

$$0 + \Gamma_+\rho(tr)\Gamma_-s - \Gamma_+(1+\rho(v+tr)\Gamma_-)\lambda,$$

thus $\Gamma_+\rho(tr)\Gamma_-s = \Gamma_+(1+\rho(v+tr)\Gamma_-)\lambda$. Assuming $Q(v+tr) \neq 0$ for suitable t, we see $\lambda_0 = \Gamma_-\rho((v+tr)^{-1} * tr)\Gamma_-s$ satisfies this equation, and for any $\tilde{s} \in \phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v+tr)$, $\lambda_0 + \tilde{s}$ does so as well.
Thus $\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v+tr) = (1 - \chi^-_-((v+tr)^{-1} * tr))\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v)$.

As $(v+tr)^{-1} * v = \frac{-1}{Q(v+tr)}(v+tr) * tr = \frac{-tv*r - t^2r*r}{Q(v+tr)} = \frac{-t}{Q(v+tr)}v * r + \frac{t^2}{Q(v+tr)}$, we thus get $\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v+tr)(1 + \frac{t}{Q(v+t)}\chi^-_-(v*r) - \frac{t^2}{Q(v+tr)})\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v)$.
Taking the lineair part of $\frac{t}{Q(v+tr)}\chi^-_-(v*r) - \frac{t^2}{Q(v+tr)}$, we get

$$\frac{\partial}{\partial t}\frac{t}{Q(v+tr)}\chi^-_-(v*r) - \frac{t^2}{Q(v+tr)}|_{t=0} =$$

$$(\frac{1}{Q(v+tr)} + t\frac{\partial}{\partial t}\frac{1}{Q(v+tr)})\chi^-_-(v*r) - \frac{2t}{Q(v+tr)} - t^2\frac{\partial}{\partial t}\frac{1}{Q(v+tr)}\bigg|_{t=0} =$$

$$\frac{1}{Q(v)}\chi^-_-(v*r).$$

And thus, up to first order in t, $\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v+tr) \approx (1 + t\frac{1}{Q(v)}\chi^-_-(v*r))\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v)$, thus $\mathrm{d}\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(r) = \chi^-_-(\frac{v*r}{Q(v)})$ by definition 115. □

Clearly, something odd happens when $Q(v) = 0$. In particular, the assumption that we can choose λ_0 with $\Gamma_+\lambda_0 = 0$ is then not entirely true, as $\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(v) \cap \mathbb{S}^- \supsetneq \{0\}$. As the subset $\{v \in \mathbb{C}\mathcal{M} | Q(v) = 0\}$ is sparse in $\mathbb{C}\mathcal{M}$, we can just work on the subset $\{v \in \mathbb{C}\mathcal{M} | Q(v) \neq 0\}$. The general statements then follow for the points with $Q(v) = 0$ from continuity.

Lemma 117. *Given a map $f \in GL(\mathbb{S})$ and a tangent vector $(r\colon p \to \mathbb{S}/p) \in T_pG_2(\mathbb{S})$, the differential $\mathrm{d}\tilde{f}$ of the action $\tilde{f}\colon G_2(\mathbb{S}) \to G_2(\mathbb{S})$, $p \mapsto f(p)$ sends t to $f \circ t \circ f^{-1}\colon f(p) \to \mathbb{S}/f(p)$.*

Proof. For $\frac{\partial}{\partial t}(1+tr)p|_{t=0}$, we have that $\frac{\partial}{\partial t}f((1+tr)p)|_{t=0} = \frac{\partial}{\partial t}(1+tf\circ r\circ f^{-1})f(p)|_{t=0}$. By definition 115, we thus have $\mathrm{d}\tilde{f}(r) = f\circ r\circ f^{-1}$ □

Lemma 118. *For p with $Q(\phi_{\mathrm{id}_\mathbb{S}}(p)) \neq 0$, the differential $\mathrm{d}\phi_{\mathrm{id}_\mathbb{S}}\colon TG_2(\mathbb{S}) \to T\mathbb{C}\mathcal{M}$ of a tangent vector $(f\colon p\to \mathbb{S}/p) \in T_pG_2(\mathbb{S})$ is given by* $(\chi_-^+)^{-1}(-\rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\circ\Gamma_-\circ(1+\rho(\phi_{\mathrm{id}_\mathbb{S}}(p))^{-1})\circ f\circ(\Gamma_- - \rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-))$.

Proof. Note that for $s\in\mathbb{S}^-$ we have $(\Gamma_- - \rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-)s \in p$, because $\Gamma_+(1+\rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-)(\Gamma_- - \rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-)s = 0$.
Furthermore, $\Gamma_-(\Gamma_- - \rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-)s = \Gamma_-s = s$, thus we see Γ_- and $(\Gamma_- - \rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-)$ are inverses of each other.

Besides, one can verify that for any $v\in\mathbb{C}\mathcal{M}$ with $Q(v)\neq 0$, any $s\in\mathbb{S}$ can be written as $s = \Gamma_-(1+\rho(v^{-1}))s + (1-\rho(v^{-1}))\Gamma_+s$, where $\Gamma_-(1+\rho(v^{-1}))s\in\mathbb{S}^-$ and $(1-\rho(v^{-1}))\Gamma_+s \in \phi_{\mathrm{id}_\mathbb{S}}^{-1}(v)$. As $\dim(\phi_{\mathrm{id}_\mathbb{S}}^{-1}(v)) + \dim(\mathbb{S}^-) = \dim(\mathbb{S})$, i.e. $\mathbb{S} = \phi_{\mathrm{id}_\mathbb{S}}^{-1}(v)\oplus\mathbb{S}^-$, these terms are unique. Therefore, for $f\in T_{\phi_{\mathrm{id}_\mathbb{S}}^{-1}(v)}G_2(\mathbb{S})$, the function $\Gamma_-(1+\rho(v^{-1}))f$ is well defined, i.e. for any $r\in p$, when one writes $f(r)\in\mathbb{S}/p$ as $\tilde{f}(r)+p$, this function is independent of the particular point in p we choose.

Hence, $-\rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\circ\Gamma_-\circ(1+\rho(\phi_{\mathrm{id}_\mathbb{S}}(p))^{-1})\circ f\circ(\Gamma_- - \rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-)$ is a well-defined function from $\mathbb{S}^-$ to $\mathbb{S}^+$. By lemma 111, the map in the theorem is thus well-defined.

Lastly, if $f = \mathrm{d}\phi_{\mathrm{id}_\mathbb{S}}^{-1}(r) = \chi_-^-(\frac{v*r}{v*v}) \in T_pG_2(\mathbb{S})$ for $r\in T_v\mathbb{C}\mathcal{M}$, we get $(\chi_-^+)^{-1}(-\rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\circ\Gamma_-\circ(1+\rho(\phi_{\mathrm{id}_\mathbb{S}}(p))^{-1})\circ f\circ(\Gamma_- - \rho(\phi_{\mathrm{id}_\mathbb{S}}(p))\Gamma_-)) = (\chi_-^+)^{-1}(-\rho(v)\circ\Gamma_-\circ(1+\rho(v^{-1}))\circ\chi_-^-(\frac{v*r}{Q(v)})\circ(\Gamma_- - \rho(v)\Gamma_-)) = (\chi_-^+)^{-1}(-\rho(v)\circ(\Gamma_-\Gamma_- + \Gamma_-\rho(v^{-1})\Gamma_-)\circ\rho(\frac{v*r}{Q(v)})\circ(\Gamma_-\Gamma_- - \Gamma_-\rho(v)\Gamma_-)) = (\chi_-^+)^{-1}(-\rho(v)\circ\Gamma_-\circ\rho(\frac{v*r}{Q(v)})\circ\Gamma_-) = (\chi_-^+)^{-1}(\Gamma_+\circ\rho(r)\circ\Gamma_-) = r$, where in the last step we use $r\in\mathbb{C}\mathcal{M}$. Thus the map in the theorem is indeed the inverse of $\mathrm{d}\phi_{\mathrm{id}_\mathbb{S}}^{-1}\colon T_v\mathbb{C}\mathcal{M}\to T_pG_2(\mathbb{S})$ for any $v\in\mathbb{C}\mathcal{M}$ and $p = \phi_{\mathrm{id}_\mathbb{S}}^{-1}(v)$. □

Theorem 119. *For $r\in T_v\mathbb{C}\mathcal{M}$ and $\rho(z)\in GL(\mathbb{S})$, the differential $\mathrm{d}\tilde{\delta}(\rho(z))(r)$ is given by*

$$(\chi_-^+)^{-1}\left(X_+^{+\,-1}\rho(r)(\chi_-^-(z) - \chi_+^-(z)X_+^{+\,-1}X_-^+)\right)$$

where $X_+^+ = \chi_+^+(z) + \rho(v)\chi_+^-(z)$ and $X_-^+ = \chi_+^+(z) + \rho(v)\chi_+^-(z)$.

Proof. Note that $\mathrm{d}\tilde{\delta}(\rho(z))(r) = \mathrm{d}\phi_{\mathrm{id}_\mathbb{S}}\circ\mathrm{d}\widetilde{\rho(z^{-1})}\circ\mathrm{d}\phi_{\mathrm{id}_\mathbb{S}}^{-1}(r)$ by lemma 110. By lemmata 116 and 117, we have $\mathrm{d}\widetilde{\rho(z)}\circ\mathrm{d}\phi_{\mathrm{id}_\mathbb{S}}^{-1}(r) = \mathrm{d}\widetilde{\rho(z^{-1})}(\chi_-^-(\frac{v*r}{Q(v)})) = \rho(z^{-1}*\Gamma_-*\frac{v*r}{Q(v)}*\Gamma_-*z)$.

For convenience, write $\xi = \rho(z^{-1} * \Gamma_- * \frac{v*r}{Q(v)} * \Gamma_- * z)$.
By theorem 113, we have $\phi_{\mathrm{id}_{\mathbb{S}}}(\phi^{-1}_{\rho(z)}(v)) = (\chi^+_-)^{-1}(X^{+\,-1}_+ X^+_-)$.

Now, lemma 118 gives us that $\mathrm{d}\phi_{\mathrm{id}_{\mathbb{S}}} \circ \mathrm{d}\rho(\tilde{z^{-1}}) \circ \mathrm{d}\phi^{-1}_{\mathrm{id}_{\mathbb{S}}}(r) =$
$(\chi^+_-)^{-1}\left(-X^{+\,-1}_+ X^+_- \Gamma_-(1 + X^{+\,-1}_- X^+_+)\xi(1 - X^{+\,-1}_+ X^+_-)\Gamma_-\right) =$
$(\chi^+_-)^{-1}\left(-X^{+\,-1}_+ \Gamma_+(X^+_- + X^+_+)\xi(1 - X^{+\,-1}_+ X^+_-)\Gamma_-\right)$.

As in the proof of lemma 113, we have $X^+_- + X^+_+ = \Gamma_+(1+\rho(v)\Gamma_-)\rho(z)$.
Thus we get $(\chi^+_-)^{-1}\left(-X^{+\,-1}_+ \Gamma_+(1+\rho(v)\Gamma_-)\rho(z)\xi(1 - X^{+\,-1}_+ X^+_-)\Gamma_-\right) =$
$(\chi^+_-)^{-1}\left(-X^{+\,-1}_+ \Gamma_+(1+\rho(v)\Gamma_-)\Gamma_-\rho(\frac{v*r}{Q(v)})\Gamma_-\rho(z)(1 - X^{+\,-1}_+ X^+_-)\Gamma_-\right) =$
$(\chi^+_-)^{-1}\left(-X^{+\,-1}_+ \Gamma_+\rho(v)\rho(\frac{v*r}{Q(v)})\Gamma_-\rho(z)(1 - X^{+\,-1}_+ X^+_-)\Gamma_-\right) =$
$(\chi^+_-)^{-1}\left(X^{+\,-1}_+ \rho(r)\Gamma_-\rho(z)(1 - X^{+\,-1}_+ X^+_-)\Gamma_-\right) =$
$(\chi^+_-)^{-1}\left(X^{+\,-1}_+ \rho(r)(\chi^-_-(z) - \chi^-_+(z)X^{+\,-1}_+ X^+_-)\right)$. □

Furthermore, we have the following nice result:

Lemma 120. *The kernel of the map δ is given by $\mathbb{C}^*$.*

Proof. Let $z \in \rho^{-1}(\ker(\delta)) = \rho^{-1}(\delta^{-1}(\mathrm{id}_{\mathcal{M}}))$ be given.
Then $\tilde{\delta}(\rho(z)) = \delta(\rho(z))^{-1} = (\mathrm{id}_{\mathbb{C}\mathcal{M}})^{-1} = \mathrm{id}_{\mathbb{C}\mathcal{M}}$. Thus we can derive $(\chi^+_-)^{-1}((\chi^+_+(z) + \rho(v)\chi^-_+(z))^{-1}(\rho(v)\chi^-_-(z) + \chi^+_-(z))) = (\chi^+_-)^{-1}(\chi^+_-(v))$ for all $v \in \mathbb{C}\mathcal{M}$.

We then get that $\forall s \in \mathbb{S}$, $(\chi^+_+(z) + \rho(v)\chi^-_+(z))^{-1}(\rho(v)\chi^-_-(z) + \chi^+_-(z))s = \chi^+_-(v)s$, thus $(\rho(v)\chi^-_-(z) + \chi^+_-(z))s = (\chi^+_+(z) + \rho(v)\chi^-_+(z))\chi^+_-(v)s$, and thus $\chi^+_-(v\Gamma_- z + z)s = \chi^+_-(z\Gamma_+ v + v\Gamma_- z\Gamma_+ v)s$ and thus
$\chi^+_-(v\Gamma_- z + z - z\Gamma_+ v - v\Gamma_- z\Gamma_+ v)s = 0$ for all $s \in \mathbb{S}$. We thus get that $0 = \Gamma_+(v\Gamma_- z + z - z\Gamma_+ v - v\Gamma_- z\Gamma_+ v)\Gamma_- = \Gamma_+(vz + z - zv - vzv)\Gamma_- = \Gamma_+(1+v)z(1-v)\Gamma_-$.

Now we have to proof that this only holds if $z \in \mathbb{C} \subseteq \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$. Let $(e^\mu)^3_{mu=0}$ be the standard basis of $\mathbb{C}\mathcal{M}$, and write $z = \lambda + r_\mu e^\mu + s_{\rho\sigma}e^\rho e^\sigma + \check{r}_\nu e^\nu \Gamma + \gamma\Gamma$. Note that $e^\nu\Gamma = \pm i e^\phi * e^\psi * e^\xi$ for ν, ϕ, ψ, ξ all distinct. From $e^\mu * \Gamma_- = \Gamma_+ * e^\mu$ and $\Gamma_+ * \Gamma_- = 0$ we get $\Gamma_+(1+v)z(1-v)\Gamma_- \in \mathrm{Span}((\Gamma_+ e^\mu \Gamma_-)^4_{\mu=0})$. As the subspace $\Gamma_+\mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})\Gamma_-$ is 4-dimensional, this is a basis. Now choose $v = \alpha e^\eta$ for some $0 \leq \eta \leq 4$. The $\Gamma_+ e^\eta \Gamma_-$ component of $\Gamma_+(1+v)z(1-v)\Gamma_-$ then is (without implied summation) $\Gamma_+(r_\eta e^\eta - r_\eta v * e^\eta * v + \check{r}_\eta e^\eta \Gamma - \check{r}_\eta v * e^\eta * \Gamma * v + 2\gamma v * \Gamma)\Gamma_- = (r_\eta - r_\eta \alpha^2 Q(e^\eta) - \check{r}_\eta - \check{r}_\eta \alpha^2 Q(e^\eta) - 2\gamma\alpha)\Gamma_+ e^\eta \Gamma_-$. This should be equal to 0 for all α, so $r_\eta - \check{r}_\eta = -r_\eta Q(e^\eta) - \check{r}_\eta Q(e^\eta) = -2\gamma = 0$, so $r_\eta = \check{r}_\eta = \gamma = 0$, for any η.

We thus get $z = \lambda + s_{\rho\sigma}e^\rho * e^\sigma$. Again take $v = \alpha e^\eta$, take $e^\mu \neq e^\eta$. Because $s_{\rho\sigma}\alpha e^\rho * e^\sigma * e^\eta - s_{\rho\sigma}\alpha e^\eta * e^\rho * e^\sigma = 0$ when η, ρ and σ are distinct, we get that the $\Gamma_+ e^\sigma \Gamma_-$ component of $\Gamma_+(1+v)z(1-v)\Gamma_-$ is equal to $\Gamma_+((s_{\eta\sigma} - s_{\sigma\eta})\alpha e^\eta * e^\eta * e^\sigma - (s_{\eta\sigma} - s_{\sigma\eta})\alpha e^\eta * e^\sigma * e^\eta)\Gamma_- =$ $2(s_{\eta\sigma} - s_{\sigma\eta})\alpha q(e^\eta)\Gamma_+ e^\sigma \Gamma_-$. Thus we find $(s_{\eta\sigma} - s_{\sigma\eta}) = 0$, which gives $z = \lambda$, so $z \in \mathbb{C}$. As $0 \notin GL(\mathbb{S})$, $z \in \mathbb{C}^*$.

Furthermore, for $z \in \mathbb{C}^*$, we have that $\tilde{\delta}(z)(v) =$ $(\chi^+_-)^{-1}((\chi^+_+(z) + \rho(v)\chi^-_+(z))^{-1}(\rho(v)\chi^-_-(z) + \chi^+_-(z))) =$ $(\chi^+_-)^{-1}(\chi^+_+(\frac{1}{z})\rho(v)\chi^-_-(z)) = v$.

And thus $\delta(z) = \tilde{\delta}(z)^{-1} = \mathrm{id}_{\mathbb{C}\mathcal{M}}$, so $\ker(\delta) = \mathbb{C}^*$ □

Now that we have a complexification $\mathbb{C}\mathcal{M}^\#$ of $\mathcal{M}$, we would like to introduce a metric $\mathfrak{g}_\mathbb{C}$ as in definition 52. As we will see, such a metric will diverge on the real manifold $\mathcal{M}^\#$ that is described below definition 51: Let $\mathfrak{g}_\mathbb{C}$ be a metric on $\mathbb{C}\mathcal{M}^\#$ as in definition 52, thus $\mathfrak{g}_\mathbb{C}|_{\iota(\mathcal{M})} = \mathrm{d}x^0_{\mathrm{id}_\mathbb{S}} \otimes \mathrm{d}x^0_{\mathrm{id}_\mathbb{S}} - \mathrm{d}x^1_{\mathrm{id}_\mathbb{S}} \otimes \mathrm{d}x^1_{\mathrm{id}_\mathbb{S}} - \mathrm{d}x^2_{\mathrm{id}_\mathbb{S}} \otimes \mathrm{d}x^2_{\mathrm{id}_\mathbb{S}} - \mathrm{d}x^3_{\mathrm{id}_\mathbb{S}} \otimes \mathrm{d}x^3_{\mathrm{id}_\mathbb{S}}$, using the standard basis (x^0, x^1, x^2, x^3) of $\mathbb{C}\mathcal{M}$. Let $z = e^0$ such that $\overline{z}^\dagger * z = -e^0 * e^0 = 1$ and let $v = (t, x, y, z)$, such that $\tilde{\delta}(z)(v) = \frac{-1}{Q(v)}z^{-1}vz = \frac{-1}{Q(v)}z^{-1}(vz+zv-zv) = \frac{-1}{Q(v)}(\frac{2\langle z,v\rangle_\mathcal{M}}{\langle z,z\rangle_\mathcal{M}}z - v) = \frac{(-t,x,y,z)}{t^2-x^2-y^2-z^2}$. By lemma 22, we can express $\mathfrak{g}_\mathbb{C}$ on the basis induced by $\phi_{\rho(z)}$ using the Jacobian J^μ_ν of $\phi_{\mathrm{id}_\mathbb{S}} \circ \phi^{-1}_{\rho(z)}$.

As $J^\mu_\nu = \frac{-1}{(t^2-x^2-y^2-z^2)^2}\begin{pmatrix} -t^2-x^2-y^2-z^2 & -2tx & -2ty & -2tz \\ -2tx & t^2+x^2-y^2-z^2 & 2xy & 2xz \\ -2ty & 2xy & t^2-x^2+y^2-z^2 & 2yz \\ -2tz & 2xz & 2yz & t^2-x^2-y^2+z^2 \end{pmatrix}$ and $\mathfrak{g}_{\mu\nu}\mathrm{d}x^\mu_{\mathrm{id}_\mathbb{S}} \otimes \mathrm{d}x^\nu_{\mathrm{id}_\mathbb{S}} = J^\rho_\mu J^\sigma_\nu \mathfrak{g}_{\rho\sigma}\,\mathrm{d}x^\mu_{\rho(z)} \otimes \mathrm{d}x^\nu_{\rho(z)}$, this is just a tedious calculation. It turns out that for any choice of μ, ν, we get $\mathfrak{g}_{\mu\nu}\mathrm{d}x^\mu_{\mathrm{id}_\mathbb{S}} \otimes \mathrm{d}x^\nu_{\mathrm{id}_\mathbb{S}} = \frac{1}{(t^2-x^2-y^2-z^2)^2}\mathfrak{g}_{\mu\nu}\mathrm{d}x^\mu_{\rho(z)} \otimes \mathrm{d}x^\nu_{\rho(z)}$. For example, when $\mu = \nu = 1$ we get $J^\rho_1 J^\sigma_1 \mathfrak{g}_{\rho\sigma} = \frac{(-2tx)^2-(t^2+x^2-y^2-z^2)^2-(2xy)^2-(2xz)^2}{(t^2-x^2-y^2-z^2)^4} = \frac{-(t^2+x^2-y^2-z^2)^2+4x^2(t^2-y^2-z^2)}{(t^2-x^2-y^2-z^2)^4} = \frac{-(t^2-y^2-z^2)^2-2x^2(t^2-y^2-z^2)-x^4+4x^2(t^2-y^2-z^2)}{(t^2-x^2-y^2-z^2)^4} = \frac{-(t^2-x^2-y^2-z^2)^2}{(t^2-x^2-y^2-z^2)^4} = \frac{\mathfrak{g}_{11}}{(t^2-x^2-y^2-z^2)^2}$, and $\mu = 0, \nu = 1$ gives $J^\rho_0 J^\sigma_1 \mathfrak{g}_{\rho\sigma} = J^0_0 J^0_1 \mathfrak{g}_{00} + J^1_0 J^1_1 \mathfrak{g}_{11} + J^2_0 J^2_1 \mathfrak{g}_{22} + J^3_0 J^3_1 \mathfrak{g}_{33} = \frac{(-t^2-x^2-y^2-z^2)(-2tx)+2tx(t^2+x^2-y^2-z^2)+2ty(2xy)+2tz(2xz)}{(t^2-x^2-y^2-z^2)^4} = \frac{4tx(-y^2-z^2+y^2+z^2)}{(t^2-x^2-y^2-z^2)^4} = 0 = \frac{\mathfrak{g}_{01}}{(t^2-x^2-y^2-z^2)}$. Thus in the points $v_0 \in \mathbb{C}\mathcal{M}^\#$ with $v_0 = \phi^{-1}_{\rho(z)}(t, x, y, z)$ where $t^2 - x^2 - y^2 - z^2 = 0$, $\mathfrak{g}_\mathbb{C}(v_0)$ cannot be defined, as the limit $\lim_{v \to v_0}(\check{\phi}_{\rho(z)} \circ \mathfrak{g})(v)$, with $\check{\phi}_{\rho(z)}$ as in definition 23, diverges.

However, when we define a metric $\tilde{\mathfrak{g}}_\mathbb{C}$ on $\mathbb{C}\mathcal{M}^\#$ that on the basis $\mathrm{d}x^\mu_{\mathrm{id}_\mathbb{S}} \otimes \mathrm{d}x^\nu_{\mathrm{id}_\mathbb{S}}$ takes the form $(\tilde{\mathfrak{g}}_\mathbb{C})_{\mu\nu}(t, x, y, z) = \frac{2(\mathfrak{g}_\mathbb{C})_{\mu\nu}}{(1+t^2-x^2-y^2-z^2)^2+4(x^2+y^2+z^2)}$

as in chapter 5 of [3], converting this to the basis $\mathrm{d}x^{\mu}_{\rho(z)} \otimes \mathrm{d}x^{\nu}_{\rho(z)}$ gives $(\tilde{\mathfrak{g}}_{\mathbb{C}})_{\mu\nu}\mathrm{d}x^{\mu}_{\rho(z)}\otimes\mathrm{d}x^{\nu}_{\rho(z)}=\frac{2J^{\rho}_{\mu}J^{\sigma}_{\nu}\,\mathfrak{g}_{\rho\sigma}\mathrm{d}x^{\mu}_{\rho(z)}\otimes\mathrm{d}x^{\nu}_{\rho(z)}}{\left(1+\frac{t^2-x^2-y^2-z^2}{(t^2-x^2-y^2-z^2)^2}\right)^2+4\left(\frac{x^2+y^2+z^2}{(t^2-x^2-y^2-z^2)^2}\right)}=\frac{2\mathfrak{g}_{\mu\nu}\mathrm{d}x^{\mu}_{\rho(z)}\otimes\mathrm{d}x^{\nu}_{\rho(z)}}{(1+t^2+x^2+y^2+z^2)^2-4t^2(x^2+y^2+z^2)}$, which only diverges in the points where $(t \pm i)^2 - x^2 - y^2 - z^2 = 0$, which does not happen for $(t, x, y, z) \in \mathcal{M}$. Thus now we have a metric $\tilde{\mathfrak{g}}_{\mathbb{C}}$ that is well-defined on the real manifold $\mathcal{M}^{\#}$, and related to the standard metric $\mathfrak{g}$ on $\mathcal{M}$ via $\tilde{\mathfrak{g}} = \Omega\mathfrak{g}$ for some continuous function $\Omega\colon \mathcal{M} \to \mathbb{R}^* = \mathbb{R} \setminus \{0\}$. (Note that when one extends Ω to $\mathcal{M}^{\#}$, it is 0 in the points where $\mathfrak{g}$ is infinite, resulting in the non-zero value of $\tilde{\mathfrak{g}} = \Omega\mathfrak{g}$ in those points). This motivates the construction of a conformal structure.

Definition 121. *A symmetric non-degenerate type (0,2) tensor field* $\mathfrak{g}$ *on a manifold* $\mathcal{M}$ *is conformally related to a non-degenerate type (0,2) tensor field* $\tilde{\mathfrak{g}}$ *if there exists a infinitely differentiable function* $\Omega\colon \mathcal{M} \to \mathbb{R}^*$ *such that* $\tilde{\mathfrak{g}}(x) = \Omega(x)\mathfrak{g}(x)$ *for all* $x \in \mathcal{M}$.
Similarly, a symmetric non-degenerate type (0,2) tensor field $\mathfrak{g}_{\mathbb{C}}$ *defined almost everywhere on a complex manifold* $\mathbb{C}\mathcal{M}$ *is conformally related to a non-degenerate type (0,2) tensor field* $\tilde{\mathfrak{g}}_{\mathbb{C}}$ *defined almost everywhere if there exists a holomorphic function* $\Omega\colon \mathbb{C}\mathcal{M} \supset U \to \mathbb{C}^*$ *defined almost everywhere such that* $\tilde{\mathfrak{g}}_{\mathbb{C}}(x) = \Omega(x)\mathfrak{g}_{\mathbb{C}}(x)$ *on the points where all three quantities are defined.*

Definition 122. *The conformal structure* $C_{\mathfrak{g}}(\mathcal{M})$ *on a real manifold* $\mathcal{M}$ *corresponding to a symmetric non-degenerate type* $(0,2)$ *tensor field* $\mathfrak{g}$ *is the set* $\{\tilde{\mathfrak{g}}\colon \mathcal{M} \to T^0_2(T\mathcal{M}) | \tilde{\mathfrak{g}} \text{ is a non-degenerate tensor field conformally related to } \mathfrak{g}\}$.
Equivalently, one can see $C_{\mathfrak{g}}(\mathcal{M})$ *as the set of infinitely differentiable sections* $\tilde{\mathfrak{g}}$ *of the bundle* $\bigsqcup_{p\in\mathcal{M}} \mathfrak{g}(p)\mathbb{R} \subseteq T^0_2(T\mathcal{M})$ *with* $\tilde{\mathfrak{g}}(p) \neq 0$ *for all* $p \in \mathcal{M}$.
Similarly, the complex conformal structure $\mathbb{C}_{\mathfrak{g}_{\mathbb{C}}}(\mathbb{C}\mathcal{M})$ *on a complex manifold* $\mathbb{C}\mathcal{M}$ *related to* $\mathfrak{g}_{\mathbb{C}}$ *is the quotient space of the set of holomorphic sections* $\tilde{\mathfrak{g}}_{\mathbb{C}}$ *defined almost everywhere of the bundle* $\bigsqcup_{p\in\mathbb{C}\mathcal{M}} \mathfrak{g}_{\mathbb{C}}(p)\mathbb{C} \subseteq T^0_2(T\mathbb{C}\mathcal{M})$, *under the equivalence relation that two sections are equivalent if they agree on the intersection of their domains.*

A neater definition would replace the word "holomorphic" with the word "meromorphic", and thus the vector bundle $\bigsqcup_{p\in\mathbb{C}\mathcal{M}} \mathfrak{g}_{\mathbb{C}}(p)\mathbb{C}$ with the *fiber bundle* $\bigsqcup_{p\in\mathbb{C}\mathcal{M}} \mathfrak{g}_{\mathbb{C}}(p)\mathbb{C}^{\infty}$, where $\mathbb{C}^{\infty}$ is the Riemann sphere (definition 137).
This would allow one to speak correctly of extensions $\mathfrak{g}_{\mathbb{C}}$ of the Minkowski

metric $\mathfrak{g}$, without writing "almost everywhere" almost everywhere. However, to do this thoroughly, we would need to compactify $T_2^0(T_p\mathcal{M})$. We do not do this, but instead work with functions defined almost everywhere. We now define the conformal group of Minkowski space.

Definition 123. *The conformal group $C(\mathcal{M}, \mathfrak{g})$ is the quotient space of the space $\{f\colon U \to \mathcal{M} | f^* C_{\mathfrak{g}}(\mathcal{M}) = C_{\mathfrak{g}}(\mathcal{M})$ and $\overline{U} = \mathcal{M}\}$ of functions defined on almost all of $\mathcal{M}$ such that for all $\tilde{\mathfrak{g}} \in C_{\mathfrak{g}}(\mathcal{M})$, $f^*\tilde{\mathfrak{g}} \in C_{\mathfrak{g}}(\mathcal{M})$, under the equivalence relation $f\colon U \to \mathcal{M} \simeq g\colon V \to \mathcal{M} \Leftrightarrow f|_{U\cap V} = g|_{U\cap V}$, with composition as group-operation.*
Similarly, the complex conformal group $C(\mathbb{C}\mathcal{M}^{\#}, \mathfrak{g}_{\mathbb{C}})$ is the quotient space of the group $\{f\colon U \to \mathbb{C}\mathcal{M}^{\#} | f^*\mathbb{C}_{\mathfrak{g}_{\mathbb{C}}}(\mathbb{C}\mathcal{M}^{\#}) = \mathbb{C}_{\mathfrak{g}_{\mathbb{C}}}(\mathbb{C}\mathcal{M}^{\#})$ and $\overline{U} = \mathbb{C}\mathcal{M}\}$ under a completely analogous equivalence relation, with composition as group operation.

Now, we need to define the identity component of the conformal group, as $SU(\mathbb{S}, \Sigma)$ will turn out to map onto the identity component only. We would like to define this completely analagously to definitions 78 and 79 of the identity component of $O(V, Q)$. However, as $C(\mathcal{M}, \mathfrak{g})$ is given in terms of functions defined almost everywhere, the definitions are somewhat more involved.

Definition 124. *We say a subset $S \subset [0,1] \times \mathcal{M}$ is regularly dense when for all $t \in [0,1]$, the set $U_t = \{t\} \times \mathcal{M} \cap S$ is dense in $\mathcal{M}$, and for all $x \in \mathcal{M}$, the set $I_x = [0,1] \times \{x\} \cap S$ is dense in $[0,1]$, and I_t is a union of finitely many open subsets of $[0,1]$.*

Definition 125. *A function $f \in C(\mathcal{M}, \mathfrak{g})$ is said to preserve complete orientation if the following holds: There is a representative function $\tilde{f}\colon U \to \mathcal{M}$ for which there exists a regularly dense subset $S \subseteq [0,1] \times \mathcal{M}$ such that $S \cap \{0\} \times \mathcal{M} = \{0\} \times U$. Then, there should exists a function $\Gamma_{\tilde{f}}\colon S \to \mathcal{M}$ such that $(x \mapsto \Gamma_{\tilde{f}}(0, x)) = \mathrm{id}_{\mathcal{M}}$ and $(x \mapsto \Gamma_{\tilde{f}}(1, x)) = \tilde{f}$. Furthermore, this function should satisfy that for every $t \in [0,1]$, the map $(x \mapsto \Gamma_{\tilde{f}}(t, x))$ is a representative of an element of $C(\mathcal{M}, \mathfrak{g})$, and for every $x \in \mathcal{M}$, the function $(t \mapsto G_{\tilde{f}}(t, x))$ is continuous on all open subsets of $[0,1]$ from definition 124.*

Definition 126. *The identity component of the conformal group, $C^+(\mathcal{M}, \mathfrak{g})$, is given by $\{f \in C(\mathcal{M}, \mathfrak{g}) | f$ preserves complete orientation$\}$*

A well-known theorem, due to Liouville, enables one to give generators for $C^+(V, \mathfrak{m})$, where V is an n-dimensional vector space with an inner product $\langle\cdot,\cdot\rangle$, viewed as a manifold in the standard manner, and a corresponding metric $\mathfrak{m}$ induced by the inner product. This theorem generalises to $C(\mathcal{M}, \mathfrak{g})$ with an indefinite bilineair form, as proven by Haantjes:

Theorem 127. *The group $C^+(\mathcal{M},\mathfrak{g})$ is generated by $SO^+(\mathcal{M},\langle\cdot,\cdot\rangle_{\mathcal{M}})$, the group of translations $\{t_r\colon \mathcal{M}\to\mathcal{M},\, v\mapsto v+r | r\in\mathcal{M}\}$, the group of dilations $\{d_\lambda\colon \mathcal{M}\to\mathcal{M},\, v\mapsto \lambda v|\lambda\in\mathbb{R}_{>0}\}$ and a special conformal transformation $v\mapsto \frac{1}{\langle v,v\rangle_{\mathcal{M}}} f(v)$ with f some element of $O(\mathcal{M},\langle\cdot,\cdot\rangle_{\mathcal{M}})$ with $\det(f)=-1$ that preserves space-orientation.*

Proof. A sketchy proof is given in [3], and a proof based on the Euclidean case is sketched in [17]. According to theorem 2.4.1.1 of [13], in [18] a formal proof is given. However, these proofs use a special conformal transformation instead of a conformal inversion, which is a translation conjugated with a conformal inversion. One motivation for using a special conformal transformation instead of a conformal inversion is that a special conformal transformation is always an element of the identity component of the conformal group. In Minkowski space, the conformal inversion itself, when composed with a map that preserves space-orientation and flips time-orientation, is an element of the identity component of the conformal group. This can easily be seen using the function δ of notation 108: Let $z\in\mathcal{M}$ with $Q(z)=1$ be given, and consider the map $\delta(\rho(\cos(\frac{\theta\pi}{2})+z\sin(\frac{\theta\pi}{2})))$. Clearly, when $\theta=0$ we get $\delta(\rho(1))=\mathrm{id}_{\mathcal{M}}$ and when $\theta=1$ we get $\delta(\rho(z))$ which we have seen is given by $(x\mapsto\frac{-1}{Q(x)}z^{-1}*x*z)$. Furthermore, by theorem 128, this map is always conformal.

Therefore, we see that in our case it is allowed to use a conformal inversion of the form $\delta(\rho(z))$ instead of a special conformal transformation in the proof of this theorem. The rest of the proof is given in the sited sources. □

Theorem 128. *The image of the map $\delta\colon GL(\mathbb{S})\to\mathrm{Map}(\mathbb{C}\mathcal{M})$ is contained in $C(\mathbb{C}\mathcal{M}^{\#},\mathfrak{g}_{\mathbb{C}})$.*

Proof. We will proof that for any $\rho(z)\in GL(\mathbb{S})$ and any $v\in\mathbb{C}\mathcal{M}$ for which $\tilde{\delta}(v)$ exists, and any $r\in T_v\mathbb{C}\mathcal{M}$, $\tilde{\delta}(\rho(z))^*\mathfrak{g}_{\mathbb{C}}(r,r)=\Omega(v,z)\mathfrak{g}_{\mathbb{C}}(r,r)$. Note that for $r\in T_v\mathbb{C}\mathcal{M}$ we have that $r*r=-Q(r)=-\mathfrak{g}_{\mathbb{C}}(r,r)$, and $(\Gamma_+r\Gamma_-)^\flat(\Gamma_+r\Gamma_-)=\Gamma_-r\Gamma_+\Gamma_+r\Gamma_-=-Q(r)\Gamma_-$.

By theorem 119, we have $\mathrm{d}\tilde{\delta}(\rho(z))(r)=$
$(\chi^+_-)^{-1}\left(X^{+}_{+}{}^{-1}\rho(r)(\chi^-_-(z)-\chi^-_+(z)X^{+}_{+}{}^{-1}X^+_-)\right)$, and thus

$$\tilde{\delta}(\rho(z))^*\mathfrak{g}_{\mathbb{C}}(r,r)\Gamma_-=Q(\mathrm{d}\tilde{\delta}(\rho(z))(r))\Gamma_-=$$
$$-\left(X^{+}_{+}{}^{-1}r\Gamma_-(z-z\Gamma_+X^{+}_{+}{}^{-1}X^+_-)\Gamma_-\right)^\flat\left(X^{+}_{+}{}^{-1}r\Gamma_-(z-z\Gamma_+X^{+}_{+}{}^{-1}X^+_-)\Gamma_-\right)=$$
$$-\Gamma_-(z-z\Gamma_+X^{+}_{+}{}^{-1}X^+_-)^\flat\Gamma_-r(X^{+\flat}_{+})^{-1}X^{+}_{+}{}^{-1}r\Gamma_-(z-z\Gamma_+X^{+}_{+}{}^{-1}X^+_-)\Gamma_-=$$
$$-\Gamma_-(z-z\Gamma_+X^{+}_{+}{}^{-1}X^+_-)^\flat\Gamma_-r(X^+_+X^{+\flat}_{+})^{-1}r\Gamma_-(z-z\Gamma_+X^{+}_{+}{}^{-1}X^+_-)\Gamma_-.$$

Now we work out $X_+^+ X_+^{+\flat} = (\Gamma_+ z\Gamma_+ + v\Gamma_- z\Gamma_+)(\Gamma_+ z^\flat \Gamma_+ + \Gamma_+ z^\flat \Gamma_- v) = \Gamma_+(1 + v\Gamma_-)z\Gamma_+(z\Gamma_+)^\flat(1 + \Gamma_- v)\Gamma_+$.

Note that $\mathrm{Span}((\Gamma_- e^\mu \Gamma_+)_{\mu=0}^3) = \Gamma_- \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})\Gamma_+$ and both are 4-dimensional. Similarly we have that $\mathrm{Span}((\Gamma_+ e^\mu * e^0 \Gamma_+)_{\mu=0}^3) = \Gamma_+ \mathbb{C}\ell(\mathcal{M}, \langle\cdot,\cdot\rangle_\mathcal{M})\Gamma_+$ and again both are 4-dimensional.

Hence, we can write $z\Gamma_+$ as $z\Gamma_+ = \Gamma_- \check{r}_\mu e^\mu \Gamma_+ + \Gamma_+ s_\nu e^\nu * e^0 \Gamma_+$, which gives $z\Gamma_+(z\Gamma_+)^\flat = (\check{r}_\mu e^\mu + s_\nu e^\nu * e^0)(\check{r}_\rho e^\rho + s_\eta e^0 * e^\eta) = \check{r}_\mu \check{r}_\rho e^\mu * e^\rho + \check{r}_\mu s_\eta e^\mu * e^0 * e^\eta + \check{r}_\rho s_\nu e^\nu * e^0 * e^\rho + s_\nu s_\eta e^\nu * e^0 * e^0 * e^\eta = -Q(\check{r}_\mu e^\mu) + \check{r}_\mu s_\eta e^\mu * e^0 * e^\eta - \check{r}_\rho s_\nu e^\rho * e^0 * e^\nu + Q(s_\nu e^\nu)Q(e^0) = Q(s_\nu e^\nu) - Q(\check{r}_\mu e^\mu)$. This then gives us $X_+^+ X_+^{+\flat} = \Gamma_+(1 + v\Gamma_-)(Q(s) - Q(\check{r}))(1 + \Gamma_- v)\Gamma_+ = (1 - Q(v))(Q(s) - Q(\check{r}))$, thus

$$\tilde{\delta}(\rho(z))^* \mathfrak{g}_\mathbb{C}(r,r)\Gamma_- =$$

$$-\Gamma_-(z - z\Gamma_+ X_+^{+-1} X_-^+)^\flat \Gamma_- r((1 - Q(v)(Q(s) - Q(\check{r})))^{-1} r\Gamma_-(z - z\Gamma_+ X_+^{+-1} X_-^+)\Gamma_- =$$

$$Q(r)\Big(\Gamma_-(z - z\Gamma_+ X_+^{+-1} X_-^+)^\flat \Gamma_-((1 - Q(v))(Q(s) - Q(\check{r})))^{-1}\Gamma_-(z - z\Gamma_+ X_+^{+-1} X_-^+)\Gamma_-\Big) =$$

$$Q(r)\left(X_+^{+-1}\Gamma_-(z - z\Gamma_+ X_+^{+-1} X_-^+)\Gamma_-\right)^\flat \left(X_+^{+-1}\Gamma_-(z - z\Gamma_+ X_+^{+-1} X_-^+)\Gamma_-\right).$$

This last expression is the multiplication of $Q(r)$ with a slightly complicated expression of the form $(\Gamma_+ \xi \Gamma_-)^\flat (\Gamma_+ \xi \Gamma_-)$ which can be simplified to something of the form $-Q(\xi)\Gamma_- = \Omega(v,z)\Gamma_-$. Therefore, we see that for any $r \in T_v \mathbb{C}\mathcal{M}$, $\tilde{\delta}(\rho(z))^* \mathfrak{g}_\mathbb{C}(r,r) = \Omega(v,z)Q(r) = \Omega(v,z)\mathfrak{g}_\mathbb{C}(r,r)$, and thus $\tilde{\delta}(\rho(z))$ is a conformal map, i.e. $\tilde{\delta}(\rho(z)) \in C(\mathbb{C}\mathcal{M}^\#, \mathfrak{g}_\mathbb{C})$. Thus $\delta(\rho(z)) = \tilde{\delta}(\rho(z))^{-1} \in C(\mathbb{C}\mathcal{M}^\#, \mathfrak{g}_\mathbb{C})$ as well. □

Theorem 129. *The map* $\bar{\delta}\colon SU(\mathbb{S}, \Sigma) \to \mathrm{Map}(\mathcal{M}), \rho(z) \mapsto \phi_{\rho(z)} \circ \phi_{\mathrm{id}_\mathbb{S}}^{-1}|_\mathcal{M}$ *is a covering map with 4 sheets onto* $C^+(\mathcal{M}, \mathfrak{g})$.

Proof. Because of lemma 106, $\phi_{\mathrm{id}_\mathbb{S}}^{-1}|_\mathcal{M}$ maps $\mathcal{M}$ into the set $\{V \in G_2(\mathbb{S}) | V^\perp = V\}$, and by the same lemma, $\phi_{\rho(z)}$ maps the part of this set on which $\phi_{\rho(z)}$ is defined into $\mathcal{M}$. Thus $\bar{\delta}(z)(v) \in \mathcal{M}$ for all $z \in \mathcal{M}$ for which $\delta(z)(v)$ is defined. Thus δ is, in this sense, well-defined. Therefore, because of theorem 128, the image of $\bar{\delta}$ is confined to $C(\mathcal{M}, \mathfrak{g})$. Because $SU(\mathbb{S}, \Sigma)$ is connected, as stated in [19], exercise 6.4.4.5.f, it is confined to the connected component $C^+(\mathcal{M}, \mathfrak{g})$. The derived examples of $\tilde{\delta}(\rho(z))$ in lemma 114, combined with theorem 127 and lemma 110 gives that $\bar{\delta}$ is surjective onto $C^+(\mathcal{M}, \mathfrak{g})$.

Lastly, by lemma 120, the kernel of δ is equal to $\mathbb{C}^*$, and thus the kernel of $\bar{\delta}$ is equal to $\rho(\mathbb{C}^*) \cap SU(\mathbb{S}, \Sigma)$. For $\lambda \in \mathbb{C}^*$ with $\rho(\lambda) \in SU(\mathbb{S}, \Sigma)$ we have that $\det(\rho(\lambda)) = \lambda^4 = 1$, so $\lambda \in \{1, i, -1, -i\} \subseteq SU(\mathbb{S}, \Sigma)$. □

In the context of Maxwell fields, conformal transformations are of special interest, because of the following theorem.

Theorem 130. *Let $U \subseteq \mathcal{M}$ be open, and let $f\colon U \to \mathcal{M}$ be a conformal map. Then we have that for any field $\mathscr{F}$ that satisfies Maxwell's equations (Lemma 46), $f^*\mathscr{F}$ satisfies Maxwell's equations as well.*

Proof. This is proven in [17]. It follows easily in our formalism:
If $f^*\mathfrak{g} = \Omega\mathfrak{g}$, it follows from lemma 55 that $\mathrm{d}_2\star_{\mathfrak{g}} f^*\mathscr{F} = \mathrm{d}_2 \pm \star_{\Omega\mathfrak{g}} f^*\mathscr{F} = \pm\mathrm{d}_2\star_{f^*\mathfrak{g}} f^*\mathscr{F} = \pm\mathrm{d}_2 f^*\star_{\mathfrak{g}}\mathscr{F} = \pm f^*\mathrm{d}_2\star_{\mathfrak{g}}\mathscr{F} = f^*0 = 0$. Furthermore, $\mathrm{d}_2 f^*\mathscr{F} = f^*\mathrm{d}_2\mathscr{F} = f^*0 = 0$ holds independently of the properties of f, as found in [20]. The other implication, that any map that preserves Maxwell is conformal, is not necessarily true, as found in e.g. [21]. □

A similar theorem holds in the complex case, as follows from lemma 55:

Lemma 131. *Let $\mathfrak{g}_{\mathbb{C}}$ be a meromorphic metric defined on almost all of $\mathbb{C}\mathcal{M}$ that coincides with the standard Minkowski metric on $\mathcal{M}$, and let $f \in C^+(\mathbb{C}\mathcal{M}^\#, \mathfrak{g}_{\mathbb{C}})$. Furthermore, let $\mathscr{F}_{\mathbb{C}} \in \Omega^2(\mathbb{C}\mathcal{M})$ be an (anti-)self-dual 2-form that satisfies $\mathrm{d}_2\mathscr{F}_{\mathbb{C}} = 0$. Then $f^*\mathscr{F}_{\mathbb{C}}$ is an (anti-) self dual 2-form that satisfies $\mathrm{d}_2\mathscr{F}_{\mathbb{C}}$ as well.*

Proof. We already have from lemma 55 and definition 122 that $\star_{\mathfrak{g}_{\mathbb{C}}} f^*\mathscr{F}_{\mathbb{C}} = \star_{f^*\mathfrak{g}_{\mathbb{C}}} f^*\mathscr{F}_{\mathbb{C}} = f^*\star_{\mathfrak{g}_{\mathbb{C}}}\mathscr{F}_{\mathbb{C}} = f^* \pm i\mathscr{F}_{\mathbb{C}} = \pm i f^*\mathscr{F}_{\mathbb{C}}$, so $f^*\mathscr{F}_{\mathbb{C}}$ is (anti-)self-dual whenever $\mathscr{F}_{\mathbb{C}}$ is. Again $\mathrm{d}_2 f^*\mathscr{F}_{\mathbb{C}} = f^*\mathrm{d}_2\mathscr{F}_{\mathbb{C}} = 0$ holds independently of the properties of f. □

3.2.1 Intermezzo for physicists

In the previous section we have defined twistors and shown how these can be used to give a nice expression for a conformal map.

A twistor is a spinor field Ω^A on $\mathcal{M}$ that is given by the expression $\Omega^A = \omega^A - ix^{AA'}\pi_{A'}$, where ω^A and $\pi_{A'}$ are constants, and $x^{AA'}$ is the (spinor representation of a) space coordinate. The factor $-i$ is arbitrarily chosen, as stated in [14]. We chose not to use this aspect of twistors, but instead look only at the pair $\omega^A, \pi_{A'}$ without the interpretation of it as a spinor field. Furthermore, we changed the arbitrary $-i$ in a 1.

In this way, the pair $\omega^A, \pi_{A'}$ is (at least mathematically) just a Dirac spinor of Minkowski space.

For a given $x^{AA'}$, we can then look for the twistors $(\omega^A, \pi_{A'})$ where $\Omega(x) = 0$. This is always a 2-dimensional plane in twistor space, which gives us a map $\{$ points in (complexified) Minkowski space $\} \to \{$ 2-dimensional planes in Twistor space$\}$ (definition 105).

A function f that acts on Twistor space, then also sends one 2-dimensional plane (say p) of Twistor space to a different 2-dimensional plane of twistor space (say $f(p)$). When one then takes the point of (complexified) minkowski space that corresponds to $f(p)$, one has a map from complexified Minkowski space to complexified minkowski space. In this section, this map is called δ.

This map turns out to always be *conformal*: The angle between any two tangent vectors of Minkowski space is preserved.[†] When f is furthermore unitary with respect to the twistor inner product, it sends real points of Minkowski space to real points of Minkowski space.

Explicitly, when $Z = \left(\begin{array}{cc|cc} z_{00} & z_{01} & z_{02} & z_{03} \\ z_{10} & z_{11} & z_{12} & z_{13} \\ \hline z_{20} & z_{21} & z_{22} & z_{23} \\ z_{30} & z_{31} & z_{32} & z_{33} \end{array}\right) = \begin{pmatrix} Z_+^+ & Z_-^+ \\ Z_+^- & Z_-^- \end{pmatrix}$ has $\det(Z) \neq 0$, a vector $x \in \mathbb{C}\mathcal{M}$, written as $x^{AA'} = \begin{pmatrix} t+z & x+iy \\ x-iy & t-z \end{pmatrix}$, and $x_{AA'} = \begin{pmatrix} t-z & -x-iy \\ -x+iy & t+z \end{pmatrix}$ is send by this procedure to the vector $x' = \begin{pmatrix} t'+z' & x'+iy' \\ x'-iy' & t'-z' \end{pmatrix}$ given by $(Z_+^+ + x_{AA'}Z_+^-)^{-1}(x^{AA'}Z_-^- + Z_-^+)$, where we just do normal multiplication and inversion of 2×2 matrices (theorem 113). A similar, but more complicated expression exists for the transformation of a tangent vector (theorem 119). When Z is unitary with respect to the twistor inner product Σ (Theorem 97), thus when $Z^\dagger \Sigma Z = \Sigma$, this map sends points of real Minkowski space to points in real Minkowski space.

A nice property of conformal maps — and our main motivation to study them — is that they preserve solutions of Maxwells equations. So when f is a conformal map, and $\mathscr{F}$ is a 2-form that satisfies the conditions of lemma 46, $f^*\mathscr{F}$ also satisfies these conditions, which is the main reason we investigated this formalism.

[†]This follows from definition 121, as the angle θ between two tangent vectors r_μ and s_ν is given by $\cos\theta = \frac{r_\mu s_\nu \mathfrak{g}^{\mu\nu}}{\sqrt{r_\mu r_\nu \mathfrak{g}^{\mu\nu} s_\rho s_\sigma \mathfrak{g}^{\rho\sigma}}}$, which is preserved exactly when definition 121 holds.

Chapter 4

Construction methods of knotted solutions of Maxwell's equations

Knotted solutions of Maxwell's equations can be constructed in various ways, most of which are listed in [22]. Of these, Rañada's is used to create the Hopfion. The construction with Bateman variables can be used to analyse and/or obtain certain Maxwell fields, such as the Hopfion, but possibly also more intricate solutions. The Twistor formalism is a formalism of Minkowski space-time in which knotted solutions occur quite naturally. We will use some tools from twistor theory, as introduced in section 3.2. All these solutions have in common that the electric and magnetic field lines form intricate patterns.

4.1 Field lines

To correctly define field lines, we should formally introduce submanifolds, immersions and foliations, which is done very well in [6]. We will not do this, but we will use some of the terminology. The most classical definition of a field line is as follows.

Definition 132. *An electric field line of a vector field* $E\colon \mathcal{M} \to \mathbb{R}^3$ *at a given time* t_0 *is a 1-dimensional submanifold* $L \subseteq \mathcal{M}$ *that satisfies* $\forall (t,x,y,z) \in L$, $t = t_0$ *and* $T_{(t,x,y,z)}L = \mathrm{Span}(E_1\partial_1 + E_2\partial_2 + E_3\partial_3) \subset T_{(t,x,y,z)}\mathcal{M}$.
A magnetic field line is defined analogously for a vector field $B\colon \mathcal{M} \to \mathbb{R}^3$.

However, this notion is highly dependent on the choice of coordinates on $\mathcal{M}$. Newcomb proved the following theorem

Theorem 133. *Let* $\mathbf{E}, \mathbf{B}\colon \mathcal{M} \to \mathbb{R}^3$ *or alternatively* $\mathscr{F} \in \Omega^2(\mathcal{M})$ *be Maxwell fields. Then one can define a notion of time-independent field lines if and only if* $\mathbf{E} \cdot \mathbf{B} = 0$ *or equivalently* $\mathscr{F}_{\mu\nu}(\star\mathscr{F})^{\mu\nu} = 0$ *for all of* $\mathcal{M}$.

Proof. Chapter VIII of Newcomb's article [23] proves this thoroughly. □

Now for a field that satisfies the condition in theorem 133, we will define a field line similar to [23].

Definition 134. *A covariant field line* $\Sigma \subseteq \mathcal{M}$ *of a non-zero Maxwell field* $\mathscr{F} \in \Omega^2(\mathcal{M})$ *that satisfies* $\mathscr{F}_{\mu\nu}(\star\mathscr{F})^{\mu\nu} = 0$ *is a 2-dimensional manifold immersed into* $\mathcal{M}$ *that satisfies for all* $p \in \mathcal{M}$ *that* $T_p\Sigma = \ker(\tilde{\mathscr{F}}_p) \subseteq T_p\mathcal{M}$, *where* $\tilde{\mathscr{F}}_p$ *is defined similar to* $\tilde{\mathfrak{g}}_p$ *in lemma 30.*

The relation between this definition and definition 132 is as follows: Let $\mathscr{F} \in \Omega^2(\mathcal{M})$ be a Maxwell field that satisfies the properties of theorem 133, and let (e_0, e_1, e_2, e_3) be a basis of $\mathcal{M}$. Let Σ be a field line as in definition 134, and let $t_0 \in \mathbb{R}$ be arbitrary. When we consider $\mathcal{M}_{t_0} = \{(t,x,y,z) \in \mathcal{M} | t = t_0\}$, we will see that $\Sigma \cap \mathcal{M}_{t_0}$ satisfies the properties of a magnetic field line as in definition 132: using definition 31, we see that

$$\ker(\tilde{\mathscr{F}}_v) = \ker\begin{pmatrix} 0 & E_1(v) & E_2(v) & E_3(v) \\ -E_1(v) & 0 & -B_3(v) & B_2(v) \\ -E_2(v) & B_3(v) & 0 & -B_1(v) \\ -E_3(v) & -B_2(v) & B_1(v) & 0 \end{pmatrix} \subseteq \operatorname{Span}\left(\begin{pmatrix} 0 \\ B_1(v) \\ B_2(v) \\ B_3(v) \end{pmatrix}, \begin{pmatrix} B_3(v) \\ E_2(v) \\ -E_1(v) \\ 0 \end{pmatrix}\right).$$

As mentioned in [23], this is an equality when $\mathscr{F} \neq 0$. It then follows that

$$T_v(\Sigma \cap \mathcal{M}_{t_0}) = T_v\Sigma \cap T_v\mathcal{M}_{t_0} =$$

$$\operatorname{Span}\left(\begin{pmatrix} 0 \\ B_1(v) \\ B_2(v) \\ B_3(v) \end{pmatrix}, \begin{pmatrix} B_3(v) \\ E_2(v) \\ -E_1(v) \\ 0 \end{pmatrix}\right) \bigcap \operatorname{Span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = \operatorname{Span}\begin{pmatrix} 0 \\ B_1(v) \\ B_2(v) \\ B_3(v) \end{pmatrix}.$$

The following result says that for real conformal transformations field lines can be simply obtained from the original field lines. Unfortunately in the complex case $\tilde{\mathscr{F}}_{\mathbb{C}}$ has a trivial kernel, and thus complex field lines are not easily defined. Thus under complex conformal transformation field lines are not this simply obtained from the original field.

Lemma 135. *Let* $\mathscr{F} \in \Omega^2(\mathcal{M})$ *be a Maxwell field that satisfies* $\mathscr{F}_{\mu\nu}(\star\mathscr{F})^{\mu\nu} = 0$, *and let* $\Sigma \subseteq \mathcal{M}$ *be a covariant field line of* $\mathscr{F}$. *Furthermore, let* $f \in C(\mathcal{M}, \mathfrak{g})$ *be a conformal map. Then* $f^{-1}(\Sigma)$ *is a covariant field line of* $f^*\mathscr{F}$.

Proof. This follows from a careful examination of definitions 39 and 40: Both $f^*\colon T^*\mathcal{M} \to T^*\mathcal{M}$ and $\mathrm{d}f\colon\ T\mathcal{M} \to T\mathcal{M}$ are bijective for a smooth map $f \in C(\mathcal{M}, \mathfrak{g})$. Thus for any $p \in \mathcal{M}$ we have $v \in \ker(\widetilde{f^*\mathscr{F}_p}) \Leftrightarrow \forall w \in T_p\mathcal{M}, (\widetilde{f^*\mathscr{F}_p}(v))(w) = 0 \Leftrightarrow \forall w \in T_p\mathcal{M}, (\tilde{\mathscr{F}}_p(\mathrm{d}f(v))(\mathrm{d}f(w)) = 0 \Leftrightarrow \forall w \in T_p\mathcal{M}, f^*(\tilde{\mathscr{F}}_p(\mathrm{d}f(v)))(w) = 0 \Leftrightarrow f^*(\tilde{\mathscr{F}}_p(\mathrm{d}f(v))) = 0 \Leftrightarrow \mathrm{d}f(v) \in \ker(f^* \circ \tilde{\mathscr{F}}_p) = \tilde{\mathscr{F}}_p^{-1}(\ker(f^*)) = \tilde{\mathscr{F}}_p^{-1}(\{0\}) = \ker(\tilde{\mathscr{F}}_p) \Leftrightarrow v \in (\mathrm{d}f)^{-1}(\ker(\tilde{\mathscr{F}}_p))$ and thus $\ker(\widetilde{f^*\mathscr{F}_p}) = (\mathrm{d}f)^{-1}(\ker(\tilde{\mathscr{F}}_p))$. It thus follows that $T_p f^{-1}(\Sigma) = (\mathrm{d}f)^{-1}(T_p\Sigma)$ satisfies $T_p f^{-1}(\Sigma) = \ker(\widetilde{f^*\mathscr{F}_p})$ if and only if $(\mathrm{d}f)^{-1}(T_p\Sigma) = (\mathrm{d}f)^{-1}(\ker(\tilde{\mathscr{F}}_p))$, which holds if and only if $T_p\Sigma = \ker(\tilde{\mathscr{F}}_p)$. □

4.2 Constructions from complex scalar fields: Rañada

Rañada's construction, mentioned in [24], makes use of two complex scalar field $\phi, \theta \colon \mathcal{M} \to \mathbb{C}$ on Minkowski space that admit certain compactifications. Under these compactifications, the scalars become maps from $\mathbb{R} \times (\mathbb{R}^3 \cup \{\infty\})$ to $\mathbb{C}^\infty$, which topologically are maps from $\mathbb{R} \times S^3$ to S^2. A good treatise on this matter can be found in [25], we quickly give some definitions to outline the main idea's.

Definition 136. *A one-point compactification $\iota \colon X \hookrightarrow \check{X}$ of a topological space X is a compact topological space $\check{X}$ together with an embedding $\iota \colon X \hookrightarrow \check{X}$ such that $\check{X} \setminus \iota(X)$ consist of one point, usually denoted ∞. We will often write $X \cup \{\infty\}$ instead of $\check{X}$.*

Definition 137. *The Riemann sphere $\mathbb{C}^\infty$ is the one-point compactification of $\mathbb{C}$. It can be seen as a complex manifold with the additional structure that $\mathbb{C}^\infty \setminus \{\infty\}$ is a field.*

Definition 138. *A homotopy Γ between two continuous maps $f_0, f_1 \colon X \to Y$ is a continuous map $\Gamma \colon [0,1] \times X \to Y$ such that $\forall x \in X, \Gamma(0,x) = f_0(x)$ and $\Gamma(1,x) = f_1(x)$.*

Definition 139. *The homotopy class $[f]$ of a function $f \colon X \to Y$ is the set $\{g \colon X \to Y |$ there exists a homotopy between f and $g\}$.*

Lemma 140. *Let $\phi \colon \mathcal{M} \twoheadrightarrow \mathfrak{C}$ be a smooth submersion from $\mathcal{M}$ to a compact real 2-manifold $\mathfrak{C}$, and let $\omega \in \Omega^2(\mathfrak{C})$ be a volume form. Then for any $x \in \mathfrak{C}$, the set $\phi^{-1}(x)$ is a 2-dimensional submanifold of $\mathcal{M}$ that satisfies for any $p \in \phi^{-1}(x)$ that $T_p\phi^{-1}(x) = \ker(\widetilde{\phi^*\omega_p})$, thus $\phi^{-1}(x)$ is a covariant field line of $\phi^*\omega$.*

Proof. As in the proof of lemma 135, we have that $\ker(\widetilde{\phi^*\omega_p}) = (\mathrm{d}\phi)^{-1}(\tilde{\omega}_p^{-1}(\ker(\phi^*)) = (\mathrm{d}\phi)^{-1}(\{0_{T_{\phi(p)}\mathfrak{C}}\}) = \ker(\mathrm{d}\phi|_{T_p\mathcal{M}})$. By the definition of $\mathrm{d}\phi|_{T_p\mathcal{M}}$, we have that $T_p\phi^{-1}(x) = \ker(\mathrm{d}\phi|_{T_p\mathcal{M}})$, as also explained in [16]. □

Note that we also have that $\mathrm{d}_2(\phi^*\omega) = \phi^*\mathrm{d}_2\omega = 0$, so for $\phi^*\omega$ to be a Maxwell field, we only need that $\mathrm{d}_2\star\phi^*\omega = 0$. As in [26], the homotopy class $[\eta]$ of a map $\eta\colon S^3 \to S^2$ is determined by the amount in which the inverse images $\eta^{-1}(x)$ and $\eta^{-1}(y)$ are linked for $x, y \in S^2$ distinct. In particular, Hopf has proven the following result:

Lemma 141. *The set of homotopy classes $\pi_3(S^2) = \{[f] | f\colon S^3 \to S^2 \text{ continuous}\}$ admits an isomorphism to $\mathbb{Z}$ via $\psi\colon \mathbb{Z} \to \pi_3(S_2), n \mapsto [h \circ f_n]$, where $h\colon S^3 \to S^2$ is the Hopf-map and $f_n\colon S^3 \to S^3$ is a map with $\deg(f_n) = n$. For a definition of $\deg(f_n)$, see page 339 of [27], and for a definition of the Hopf-map h see either [26] or [25].*

Proof. See [26] (in German). It shows that for h any two circles $h^{-1}(x)$ and $h^{-1}(y)$ are linked, which means h is not in the same homotopy class as a constant function, and then goes on to proof the rest of the lemma. □

This topological result led Rañada to consider a model of Maxwell's equations where every solution is of the form $\phi^*\omega$ for $\phi\colon \mathbb{R} \times (\mathbb{R}^3 \cup \{\infty\}) \to \mathbb{C}^\infty$ where $\omega \in \Omega^2(\mathbb{C}^\infty)$ is the volume form of $\mathbb{C}^\infty$ that can be obtained by the pull-back of the volume form of $S^2 \subseteq \mathbb{R}^3$ resulting from the euclidean metric on $\mathbb{R}^3$ via the stereographic projection $\pi_s\colon \mathbb{C}^\infty \to S^2 \subseteq \mathbb{R}^3, x + iy \mapsto (\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1})$ (As explained in [25]). The resulting model has the interesting property that all solutions thus obtained can be indexed similarly to lemma 141. In section 4.4, we outline solutions that look like they do not fit into this model, and thus cannot be indexed as such.

4.2.1 The Hopfion

The Hopfion $\mathscr{F}_H = \phi^*\omega$ is a solution of Maxwell's equations that is constructed by taking $\phi\colon \mathbb{R} \times (\mathbb{R}^3 \cup \{\infty\}) \overset{\mathrm{id}_\mathbb{R} \times \pi_s}{\to} \mathbb{R} \times S^3 \overset{\tilde{h}}{\to} S^2 \overset{\pi_s}{\to} \mathbb{C}^\infty$, with $\mathrm{id}_\mathbb{R} \times \pi_s$ a stereographic projection on the last coordinates, π_s a stereographic projection and $\tilde{h}$ a map such that $\star\phi^*\omega = i\phi^*\omega$ and $\tilde{h}(0, r) = h(r)$ with $r \in S^3$ and h the Hopf-map, as described in [28]. The resulting field then has Hopf-index 1, which means any two distinct field lines are linked exactly once, as in figure 4.1. An explicit expression is given in definition 144.

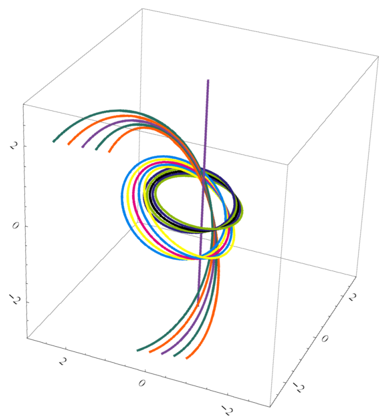

Figure 4.1: *Several field lines of the solution of Maxwell's equation that was described by Rañada in [24]. This solution is called the Hopfion, as it is related to the Hopf map described in [26]. In this solution, any two distinct field lines are linked exactly once.*

4.3 Construction from complex scalar fields: Bateman

In [5], Bateman gave (among many other interesting technical results about Maxwell's equations) a means of constructing a solution of Maxwell's equations from two complex (meromorphic) scalar fields $\alpha, \beta \colon \mathbb{C}\mathcal{M} \to \mathbb{C}^\infty$, now know as Bateman's construction.

Lemma 142. *Let $\alpha, \beta \colon \mathbb{C}\mathcal{M} \to \mathbb{C}^\infty$ be two meromorphic functions. Then the field $\mathscr{F}_\mathbb{C} = \mathrm{d}\alpha \wedge \mathrm{d}\beta \in \Omega^2(\mathbb{C}\mathcal{M})$ satisfies Maxwell's equations if and only if $\star\mathscr{F}_\mathbb{C} = \pm i \mathscr{F}_\mathbb{C}$.*

Proof. $\mathrm{d}_2\mathscr{F}_\mathbb{C} = 0$ directly follows from the construction: we have $(\mathscr{F}_\mathbb{C})_{\mu\nu} = \frac{1}{2}(\partial_\mu\alpha\partial_\nu\beta - \partial_\nu\alpha\partial_\mu\beta)$, thus $(\mathrm{d}_2\mathscr{F}_\mathbb{C})_{\zeta\eta\theta} =$
$\frac{1}{3!}(\partial_\zeta(\partial_\eta\alpha\partial_\theta\beta - \partial_\theta\alpha\partial_\eta\beta) + \partial_\eta(\partial_\theta\alpha\partial_\zeta\beta - \partial_\zeta\alpha\partial_\theta\beta) + \partial_\theta(\partial_\zeta\alpha\partial_\eta\beta - \partial_\eta\alpha\partial_\zeta\beta)) =$
$\frac{1}{6}((\partial_\zeta\partial_\eta\alpha)\partial_\theta\beta - (\partial_\zeta\partial_\eta\alpha)\partial_\theta\beta - (\partial_\zeta\partial_\theta\alpha)\partial_\eta\beta + (\partial_\zeta\partial_\theta\alpha)\partial_\eta\beta + \ldots) = 0.$
Thus indeed $\mathscr{F}_\mathbb{C}$ satisfies the conditions of lemma 56 if and only if it is (anti-)self dual. □

Remark 143. *Most of the time, lemma 142 is written in terms of the Riemann-Silberstein vector $\mathbf{F} = \mathbf{E} + i\mathbf{B}$ instead of a 2-form $\mathscr{F}_\mathbb{C}$. Then it is written as $\mathbf{F} = \nabla\alpha \times \nabla\beta$, with the condition $\star\mathscr{F}_\mathbb{C} = i\mathscr{F}_\mathbb{C}$ being written as $\nabla\alpha \times \nabla\beta = i(\partial_t\alpha\nabla\beta - \partial_t\beta\nabla\alpha)$. Both constructions give rise to the same fields, but for given α and β, the electric field of the one construction has a minus sign with respect to the other construction.*

Using Bateman's construction, the Hopfion can be given by a manageable analytic expression (in contrast to most other ways to describe the Hopfion).

Definition 144. *The Hopfion (or Hopf-field) is the field* $\mathscr{F}_H \in \Omega^2(\mathbb{C}\mathcal{M}^\bullet)$ *given by* $\mathscr{F}_H = \mathrm{d}\alpha \wedge \mathrm{d}\beta$ *where* $\alpha, \beta\colon \mathbb{C}\mathcal{M}^\bullet \to \mathbb{C}$ *are given by* $\alpha(t,x,y,z) = \frac{x+iy}{x^2+y^2+z^2-(t-i)^2}$ *and* $\beta(t,x,y,z) = \frac{i(z+t-i)}{x^2+y^2+z^2-(t-i)^2}$. *In this definition,* $\mathbb{C}\mathcal{M}^\bullet$ *is the set* $\{(t,x,y,z) \in \mathbb{C}\mathcal{M} | x^2+y^2+z^2-(t-i)^2 \neq 0\}$.

Other sources, such as [29] or [22] or [30] all choose the scalars α and β slightly differently. The fields that result from these choices can be obtained from each other by simple coordinate transformations such as reflections and rotations.

An important observation, as made in [1], is the following:

Lemma 145. *Let* $\mathscr{F}_L \in \Omega^2(\mathbb{C}\mathcal{M})$ *be given by* $\mathscr{F}_L = i(\mathrm{d}x^0 \wedge \mathrm{d}x^1 + \mathrm{d}x^1 \wedge \mathrm{d}x^3) - (\mathrm{d}x^2 \wedge \mathrm{d}x^3 + \mathrm{d}x^0 \wedge \mathrm{d}x^3)$, *and let* $\mathfrak{f} \in C(\mathbb{C}\mathcal{M}^\#, \mathfrak{g}_\mathbb{C})$ *be given by the composition* $S \circ T_i \circ S$ *where S is the conformal inversion* $v \mapsto \frac{v}{\langle v,v\rangle_\mathcal{M}}$ *and* T_i *is a translation of* $\mathbb{C}\mathcal{M}$ *by i in the* e^0 *direction. Then* $f^*\mathscr{F}_L = \mathscr{F}_H$.

Proof. For a commplete derivation, see [1]. Note that $\tilde{\alpha}(t,x,y,z) = x+iy$ and $\tilde{\beta}(t,x,y,z) = i(z-t)$ exactly give $\mathscr{F}_L = \mathrm{d}\tilde{\alpha} \wedge \mathrm{d}\tilde{\beta}$, while $\tilde{\alpha} \circ \mathfrak{f}(t,x,y,z) = \frac{x+iy}{x^2+y^2+z^2-(t-i)^2}$ and $\tilde{\beta} \circ \mathfrak{f}(t,x,y,z) = \frac{i(z+t-i)}{x^2+y^2+z^2-(t-i)^2} - 1$, thus $\mathfrak{f}^*\mathrm{d}\tilde{\alpha} \wedge \mathrm{d}\tilde{\beta} = \mathrm{d}\alpha \wedge \mathrm{d}(\beta - 1) = \mathrm{d}\alpha \wedge \mathrm{d}\beta = \mathscr{F}_H$ with α, β as in 144 □

This lemma motivates the construction of a family of solutions $\mathscr{F}_\mathfrak{k}$ of Maxwell's equations, parametrized by a parameter $\mathfrak{k}$, such that $\mathscr{F}_0 = \mathscr{F}_H$ and $\mathscr{F}_\mathfrak{k}$ can not be indexed according to lemma 141 for (some) $\mathfrak{k} \neq 0$. In [29], as well as in [30] several solutions of Maxwell's equations that satisfy $\mathbf{E} \cdot \mathbf{B} = 0$ are given in which most field lines do not close up on themselves, but rather densely fill a 2-dimensional surface. This behaviour is called ergodic behaviour, which has for a long time been known to occur in magnetohydrodynamics (the study of magnetic fields in plasma gasses), as in [31].

4.4 Conformal transformation of a circularly polarized wave

We follow section 3.3 of [30] in constructing the pullback $\mathfrak{f}^*\mathscr{F}_\sim$ of a circularly polarized wave $\mathscr{F}_\sim = \mathrm{d}(x+iy) \wedge \mathrm{d}e^{i(z-t)}$ via the function $\mathfrak{f} \in C(\mathbb{C}\mathcal{M}^\#, \mathfrak{g}_\mathbb{C})$

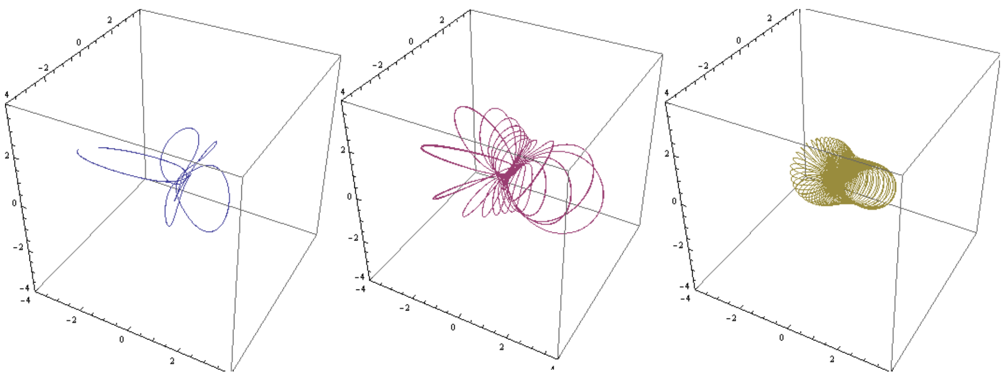

Figure 4.2: *Three field lines of the solution of Maxwell's equation given by $\mathscr{F}_{\mathfrak{k}}$ with $\mathfrak{k} = \pi$ (definition 146). Note how these field lines seem to be lying on nested tori. For bigger $\mathfrak{k}$, the fieldline configurations are even more complicated.*

of lemma 145. We extend this construction to a 1-parameter family of fields $(\mathscr{F}_{\mathfrak{k}})_{\mathfrak{k}\in\mathbb{R}}$ such that $\mathscr{F}_0$ is equal to the Hopfion $\mathscr{F}_H$.

Definition 146. *The generalized Hoyos-field $\mathscr{F}_{\mathfrak{k}}$ is the field obtained by taking the pullback $\mathfrak{f}^*\tilde{\mathscr{F}}_{\mathfrak{k}}$ of the field $\tilde{\mathscr{F}}_{\mathfrak{k}} = \mathrm{d}(x+iy) \wedge \mathrm{d}\frac{e^{\mathfrak{k}i(z-t)}-1}{\mathfrak{k}}$ via the function $\mathfrak{f} \in SC(\mathbb{C}\mathcal{M}^{\#}, \mathfrak{g}_{\mathbb{C}})$ that was defined in lemma 145.*

We can see that $\mathfrak{k} = 0$ gives $\tilde{\mathscr{F}}_0 = \lim_{\mathfrak{k}\to 0} \tilde{\mathscr{F}}_{\mathfrak{k}} = \lim_{\mathfrak{k}\to 0} \mathrm{d}(x+iy) \wedge \mathrm{d}\frac{e^{\mathfrak{k}i(z-t)}-1}{\mathfrak{k}} = \mathrm{d}(x+iy) \wedge \mathrm{d}\left(\lim_{\mathfrak{k}\to 0} \sum_{k=1}^{\infty} \frac{\mathfrak{k}^{k-1}(i(z-t))^k}{k!}\right) = \mathrm{d}(x+iy) \wedge \mathrm{d}i(z-t)$, which is exactly the field of lemma 145, and thus this lemma tells us that $\mathscr{F}_0 = f^*\tilde{\mathscr{F}}_0 = \mathscr{F}_H$. Furthermore, for $\mathfrak{k} > 0$, the field lines of the field $\mathscr{F}_{\mathfrak{k}}$ are quite complicated, as can be seen in figure 4.2. Furthermore it looks like the linking number of two of these field lines is difficult to calculate, possibly even impossible.

Chapter 5

Conclusion

In this thesis, we have seen several formalisms for Maxwells equation, ultimately building up to maxwell fields as a differential form on complexified Minkowski space. Furthermore, we have given an introduction to the Dirac spinor space related to Minkowski space, such that we could study how conformal transformations of Minkowski space can be more easily desribed as elements of the special unitary group of this spinor space. Explicit formulas are given in theorems 113 and 119. Furthermore, we have proven that the *general* linear group of this spinor space gives a conformal transformation of *complexified* Minkowski space. We have then seen how such a conformal transformation can be applied to a simple field to construct the Hopfion. Lastly, we have seen how this conformal transformation applied to a circularly polarised plane wave with wavelength $\frac{1}{\mathfrak{k}}$ creates a family of fields. For $\mathfrak{k} = 0$, this gives us the Hopfion, and for increasingly larger numbers of $\mathfrak{k}$ the field lines form increasingly complex structures.

Chapter 6

Discussion

A few questions have been left unanswered that are interesting to investigate further.

First, it might be interesting to know up to what extend the group $GL(\mathbb{S})$ is mapped to the full group $C(\mathbb{C}\mathcal{M}^{\#}, \mathfrak{g}_{\mathbb{C}})$. In [13] on page 91, it is stated that $\mathbb{C}\ell(\mathcal{M}, \mathfrak{g})$ should be isomorphic to $C(\mathbb{C}\mathcal{M}^{\#}, \mathfrak{g}_{\mathbb{C}})$. By lemma 120, the map $\delta\colon GL(\mathbb{S}) \to C(\mathbb{C}\mathcal{M}^{\#}, \mathfrak{g}_{\mathbb{C}})$ shows that we did not give an isomorphism, and hence we probably did not cover the full group $C(\mathbb{C}\mathcal{M}^{\#}, \mathfrak{g}\mathbb{C})$. Unfortunately, I did not understand how the citation used by [13] proved this isomorphism. Furthermore, if this isomorphism is indeed an isomorphism, I wonder where $0 \in \mathbb{C}\ell(\mathbb{C}\mathcal{M}^{\#}, \mathfrak{g}_{\mathbb{C}})$ should be mapped to. It would be interesting to further look into this.

Second, it would be interesting to investigate if there can be found a relation with the field lines of an electromagnetic field before, and those after a complex conformal transformation. This will be difficult, as definition 134 does not easily generalize to an (anti-)self-dual field, which does not have a kernel as needed in this definition.

Third, it would be interesting to investigate how these fields are related to their singular points in $\mathbb{C}\mathcal{M}$. For example, the Hopfion has a simple pole, and a fairly simple field line structure. The generalised Hoyos field (for $\mathfrak{k} \neq 0$) has an elementary sigularity, and a very complicated field line structure (especially when $\mathfrak{k} \gg 0$). And in [29] field with other singular points are shown to have moderately complicated field line structures. Since the order of this pole seems to be somehow related to the complexity of the field lines, it might be more interesting to look at these singularities instead of the linking of the field lines themselves.

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