Norm-induced partially ordered vector spaces
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Contents

Abstract

In this thesis, we will see a criterion for positive operators on a partially ordered vector space induced by a polyhedral cone with linearly independent extreme vectors, as well as for block-diagonal maps on a partially ordered vector space ordered by a norm-induced cone. Finally, we will show that positive operators on a complete partially ordered vector space ordered by a norm-induced cone are continuous.

1 Introduction

First, we will define the major components in this setting.

1.1 Set-up

Definition 1.1. Partially ordered vector space
A real vector space equipped with a partial order $(V, \leq)$ is called a *partially ordered vector space* if it satisfies

- For all $u, v, w \in V$, if $u \leq v$, then $u + w \leq v + w$;
- For all $u, v \in V$, if $u \leq v$, then for any non-negative scalar λ we have $\lambda u \leq \lambda v$.

Dual to a partially ordered vector space is a vector space V equipped with a cone.

Definition 1.2. Cone
We call a subset $K \subset V$ a *cone* if

- For all $x, y \in K$, $x + y \in K$;
- For all $x \in K$ and λ a non-negative scalar, we have $\lambda x \in K$;

- The intersection $K \cap -K$ is the singleton $\{0\}$.

Given a cone K, we can introduce a partial order on V by defining $x \leq y$ if and only if $y - x \in K$ and given a partially ordered vector space $(V, \leq)$, the subset $\{v \in V : 0 \leq v\}$ is a cone according to our definition, and called the *positive cone*. We can define any partially ordered vector space by equipping a real vector space V with a cone[1, p. 3-4], and throughout this thesis, we will consider partially ordered vector spaces within the context of having been induced by a cone K, which is then also the positive cone of the partial order.

Given a partially ordered vector space V, we are interested in structure-preserving maps T, thus linear maps $T : V \to V$ such that $v \leq w \Rightarrow Tv \leq Tw$. Due to linearity, this can be rewritten as $0 \leq w - v \Rightarrow 0 \leq Tw - Tv = T(w - v)$, which leaves us with the criterion that $x \geq 0$ must imply $Tx \geq 0$.

Definition 1.3. Positive linear operator
A linear map $T : V \to V$ such that $T[K] \subset K$ is called a *positive linear operator*, or simply a *positive operator*. We will also describe this property as T being *positive with regard to* K.

1.2 Properties

If two partially ordered vector spaces are isomorphic, we obtain a nice way to consider positive maps on one of the spaces in terms of what we know about maps on the other space.

Fact 1.4. Given a partially ordered vector space V, a linear map T and a V-automorphism A, we find that T is positive with regard to K if and only if ATA^{-1} is positive with regard to AK.

Proof. Assuming $T[K] \subset K$, we consider $ATA^{-1}[AK]$, which is equal to $AT[K]$. Because $T[K] \subset K$, we find that this is contained in AK, so ATA^{-1} is positive with regard to AK. The other implication is simply applying what we have just proven by conjugating with A^{-1}, which is also an automorphism. □

Two concepts about partially ordered vector spaces we will use are the following.

Definition 1.5. Directed
We call a partially ordered vector space V *directed* if any element can be written as the difference of two positive elements.

Definition 1.6. Monotone
If V is a partially ordered vector space equipped with some norm $||\cdot||$, we call this norm *monotone* if $0 \leq u \leq v$ implies $||u|| \leq ||v||$.

2 Polyhedral cones in the finite-dimensional case

In finite dimensional real vector spaces, spaces isomorphic to $\mathbb{R}^d$, we consider a family of cones that can be constructed by drawing lines from the origin through the vertices of a convex polytope and extending to infinity. Letting these lines be the edges of some solid K in $\mathbb{R}^d$, we obtain what is called a polyhedral cone, named after its finitely many faces. To formalise this, we must first introduce the notion of positive linear independence.

Definition 2.1. Positively linearly independent
A finite set of vectors $v_1, ..., v_n$ is called *positively linearly independent* if, for non-negative scalars $\lambda_1, ..., \lambda_n$ the equation $\lambda_1 v_1 + ... + \lambda_n v_n = 0$ implies that all λ_i are 0.

Definition 2.2. Polyhedral cone & edge representation
Let $\{v_1, ..., v_n\}$ be a set of vectors in $\mathbb{R}^d$ such that all the v_i are positively linearly independent. Then the positive linear span of all these vectors, or $\text{Pos}(v_1, .., v_n)$ is called a *polyhedral cone*, and the vectors are an *edge representation* for this cone.

An example of a polyhedral cone in $\mathbb{R}^d$ would be the positive 2^d-tant; simply $\text{Pos}(e_1, ..., e_d)$ where e_i is the i-th standard basis vector, and this cone would induce the standard order on $\mathbb{R}^d$, where $x \leq y$ if and only if for all $1 \leq j \leq d$, we have $x_j \leq y_j$. It is already well-known that a linear map on this partially ordered vector space is positive if and only if all the matrix coefficients are non-negative[2, p. 315].

Theorem 2.3. *If $K = \text{Pos}(v_1, ..., v_n)$ is a cone and $v_1, ..., v_n$ are linearly independent, then $(v_1, ..., v_n)$ is a basis for the linear subspace $U = \text{Span}(v_1, ..., v_n) \subset \mathbb{R}^d$ and a linear map $T : U \to U$ is positive with regard to K if and only if the coefficients of the matrix T with regard to basis $(v_1, ..., v_n)$ are non-negative.*

Proof. We see that K lies in U and defines a partial order on U. We know that positive maps on $\mathbb{R}^d$ are maps such that the matrix coefficients are non-negative, and through the basis transformation A, which maps v_j to e_j, U can be considered as $\mathbb{R}^n$ with the standard basis. The cone is mapped to the cone belonging to the standard order on $\mathbb{R}^n$, and combining this knowledge with Fact 1.4, we obtain the desired equivalence. □

For $n < d$, we are considering the case where the cone K lies within a proper subspace of $\mathbb{R}^d$, and through embedding, we can say something about the positivity of T on all of $\mathbb{R}^d$, as elements in $\mathbb{R}^d$ that lie outside this subspace are not positive to begin with.

For $n = d$, we are considering the case where $U = \mathbb{R}^d$ and we can simplify the statement by forgetting about subspaces and just talk about the entire space.

However, for $n > d$, our vectors $v_1, ..., v_n$ cannot be linearly independent, so Lemma 2.3 cannot be applied. An example of this would be an upside-down pyramid in $\mathbb{R}^3$, which can also be described with the ∞-norm on $\mathbb{R}^2$; this cone can be described as $\{(x, y, z) \in \mathbb{R}^3 : ||(x, y)||_\infty \leq z\}$. The next chapter studies cones induced by norms in a similar fashion.

3 Norm-induced cones

From now on, we will consider a family of cones which will be constructed as follows.

3.1 Construction

Let $(X, ||\cdot||)$ be a real, normed space. Let $Y = X \times \mathbb{R}$ and define $K \subset Y$ to be $\{(x,\alpha) : ||x|| \leq \alpha\}$. Furthermore, make Y a normed space by defining $||(x,\alpha)||_Y = ||x|| + |\alpha|$. An example of such a cone would be the upside-down pyramid we made reference to in the previous chapter, or the famous ice cream cone in $\mathbb{R}^3$. Cones of this type are more commonly studied when defined by an inner product that makes X a Hilbert space, and are sometimes referred to as *Lorentz cones* [3, p. 211, Section 5.1].

If we want to think about linear operators on Y, we can do this in terms of maps relating to X and $\mathbb{R}$. If we have a linear map $T : Y \to Y$, we can look at the images of elements $(x, 0)$ and $(0, \alpha)$ and project, and find that we can consider T as a block matrix in the following way:

$$T = \begin{array}{c} \\ X \\ \mathbb{R} \end{array} \begin{array}{c} \begin{array}{cc} X & \mathbb{R} \end{array} \\ \left(\begin{array}{c|c} f & v \\ \hline \phi & c \end{array}\right) \end{array},$$

where f is a linear map $X \to X$, v a vector in X representing a linear map $\mathbb{R} \to X$, ϕ an element in the dual space X^* and c a scalar in $\mathbb{R}$ representing a linear map $\mathbb{R} \to \mathbb{R}$. We can thus consider $T(x,\alpha) = \begin{pmatrix} f(x) + \alpha v \\ \phi(x) + c\alpha \end{pmatrix}$.

3.2 Block-diagonal maps on Y

We call T *block-diagonal* if it is represented as $\left(\begin{array}{c|c} f & 0 \\ \hline 0 & c \end{array}\right)$, thus $T(x,\alpha) = (f(x), c\alpha)$. For block-diagonal maps, we have a very nice equivalence for positivity. By $||\cdot||_{op}$, we denote the operator norm of X^*, where $||f||_{op}$ is defined to be $\inf\{c \geq 0 : ||f(v)|| \leq c||v||, \text{ for all } v \in X\}$, a notion that only applies to continuous operators.

Theorem 3.1. *A block-diagonal map* $T = \left(\begin{array}{c|c} f & 0 \\ \hline 0 & c \end{array}\right)$ *is positive if and only if* $||f||_{op} \leq c$.

Proof. "⇒" Assuming T is positive, we find that, for $(x,\alpha) \in Y$, $||f(x)|| \leq c\alpha$, given $||x|| \leq \alpha$. We assume $\alpha \neq 0$, as otherwise we would just be looking at the origin which would give no information on the operator norm. We consider $\frac{x}{\alpha}$, which lies in the unit ball in X, as $||x|| \leq \alpha$, and each element in the unit ball can be represented as $\frac{x}{\alpha}$ for some $x \in X$ with $||x|| \leq \alpha$. Note that $||f(\frac{x}{\alpha})|| = \frac{1}{\alpha}||f(x)||$, which is smaller than or equal to c. So for all elements $u \in X$ with $||u|| \leq 1$, we have $||f(u)|| \leq c$, thus $||f||_{op} \leq c$.

"⇐" Assuming $||f||_{op} \leq c$, we take an arbitrary element (x,α) in K, and note that $||f(x)|| \leq ||f||_{op}||x|| \leq c||x|| \leq c\alpha$, thus $(f(x), c\alpha) = T(x,\alpha)$ is contained in K and so T is positive. □

Not all linear maps on Y are block-diagonal, but some are block-diagonalisable; there exists some automorphism A such that ATA^{-1} is block-diagonal, and by Fact 1.4, we can say something about positivity on isomorphic partially ordered vector spaces. Sadly, not all maps are block-diagonalisable. If we take $(X, ||\cdot||) = (\mathbb{R}, |\cdot|)$, then $Y = \mathbb{R}^2$ with the 1-norm, and our notion of block-diagonal collapses into simply diagonal, and not all 2×2 matrices are diagonalisable; take for instance $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which has complex eigenvalues i and $-i$.

3.3 Continuity of positive maps on Y

Block-diagonal maps on Y that are positive are clearly continuous, as their operator norm equals c, which raises the question if all positive maps on Y are continuous.

Theorem 3.2. *Let $(X, ||\cdot||)$ be a Banach space, Y, K and $||\cdot||_Y$ be as constructed before and $T : Y \to Y$ positive. Then T is continuous.*

The rest of this chapter will be dedicated to proving Theorem 3.2, and we assume the conditions of that theorem throughout.

Lemma 3.3. *There is a $k \geq 0$ such that, for all $(x, \alpha) \in K$, $||T(x, \alpha)||_Y \leq k||(x, \alpha)||_Y$.*

Proof. Let $(x, 1) \in K$, so, $||x|| \leq 1$. Note that $T(x, 1)$ again lies in K, so, considering $T(x, 1) = (f(x)+v, \phi(x)+c)$, we find that $||f(x)+v|| \leq \phi(x)+c$. If we now take $(-x, 1)$, we find that $||f(x)+v|| \leq -\phi(x)+c$, so $\phi(x) \leq -||f(x)+v|| + c \leq c$. This holds for every x such that $||x|| \leq 1$, so ϕ is continuous on X.

As the map $x \mapsto \phi(x)$ is continuous, so is $(x, \alpha) \mapsto \phi(x)$, thus $(x, \alpha) \mapsto \phi(x) + \alpha c$ is as well, so there is some $k_1 \geq 0$ such that, for all (x, α), $\phi(x) + c\alpha \leq k_1||(x, \alpha)||_Y$. If (x, α) is in K, then $||f(x) + \alpha v|| \leq \phi(x) + c\alpha \leq k_1||(x, \alpha)||_Y$, and so $||T(x, \alpha)||_Y = ||f(x) + \alpha v|| + \phi(x) + c\alpha \leq 2k_1||(x, \alpha)||_Y$, which proves the lemma. □

We will introduce the notion of a regular norm[4, p. 54, Definition 3.39], as it is vital to the proof.

Definition 3.4. Regular norm
A *regular norm* on a directed partially ordered vector space V is a norm $|| \cdot ||_r$ such that $||v||_r = \inf\{||u||_r : -u \leq v \leq u\}$.

A regular norm has the nice property that for all $v \in V$, for all $\epsilon > 0$, there exists some $u \in V$ with $-u \leq v \leq u$ such that $||u||_r \leq ||v||_r + \epsilon$. This implies a nice decomposition property.

Lemma 3.5. *Let V be a directed partially ordered vector space supplied with a regular norm $|| \cdot ||_r$, then for all $v \in V$, for all $\epsilon > 0$, there exist $x, y \in V$ with $v = x - y$ such that $x, y \geq 0$ and $||x||_r, ||y||_r \leq ||v|| + \epsilon$.*

Proof. Let $v \in V$ be given and let ϵ be greater than 0. Take $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$, where u is such that $-u \leq v \leq u$ and $||u||_r \leq ||v||_r + \epsilon$. The fact that

$-u \leq v$ implies that $u+v$ is positive, and $v \leq u$ implies that $u-v$ is positive, so both x and y are positive. Their difference, $x-y$, is clearly equal to v. Checking the norms, we see that $||x||_r \leq \frac{1}{2}||u||_r + \frac{1}{2}||v||_r \leq \frac{1}{2}(||v||_r + \epsilon) + \frac{1}{2}||v||_r < ||v||_r + \epsilon$. Similarly for y. This proves the existence of the desired x and y. □

Our next step is to show that our Y-norm is equivalent to a regular norm, but for this, we will first need to check that $||\cdot||_Y$ is monotone, that Y is complete and directed, and that K is closed.

Lemma 3.6. $||\cdot||_Y$ *is monotone.*

Proof. Assume that $0 \leq (x, \alpha) \leq (y, \beta)$, so $||y - x|| \leq \beta - \alpha$. Then $||(x, \alpha)||_Y = ||x|| + \alpha \leq ||x - y|| + ||y|| + \alpha \leq \beta - \alpha + \alpha + ||y|| = ||y|| + \beta = ||(y, \beta)||_Y$. □

Remark 3.7. Y is complete.
We assumed X was complete, so $Y = X \times \mathbb{R}$ is complete with regard to the norm $||(x, \alpha)||_Y = ||x|| + |\alpha|$.

Lemma 3.8. Y *is directed.*

Proof. Let $(x, \alpha) \in Y$ be given. For $\alpha \geq 0$, we can write (x, α) as the difference of two elements in K as $\frac{(x, ||x|| + 2\alpha)}{2} - \frac{(-x, ||x||)}{2}$. For $\alpha < 0$, we can write (x, α) as $\frac{(x, ||x||)}{2} - \frac{(-x, ||x|| - 2\alpha)}{2}$, which is the difference of two positive elements. □

Lemma 3.9. K *is closed.*

Proof. Let (x_n, α_n) be a sequence in K that converges to (x, α) in Y. For all n, we have $||x_n|| \leq \alpha_n$, so for the limit we also have $||x|| \leq \alpha$, which means (x, α) is contained in K. □

Having checked that Y is a directed partially ordered vector space with a monotone norm $||.||_Y$ such that K is close, and that Y is norm complete, we find that $||\cdot||_Y$ is equivalent to a regular norm $||\cdot||_r$ on Y [4, p. 58, Corollary 3.48].

Knowing this, we can prove Theorem 3.2: If $(X, ||\cdot||)$ is a Banach space and Y, K and $||\cdot||_Y$ are as constructed in section 3.1 and $T : Y \to Y$ is positive, then T is continuous.

Proof. Let $v \in Y$ and $\epsilon > 0$ be given, and consider $||Tv||_Y$. We choose $x, y \in K$ such that $v = x - y$ and $||x||_r, ||y||_r \leq ||v|| + \epsilon$, which we can do because of Lemma 3.5. Substituting v by $x - y$, we find that $||Tv||_Y$ is bounded from above by $||Tx||_Y + ||Ty||_Y$ which is smaller than or equal to $k||x||_Y + k||y||_Y$ due to Lemma 3.3. Because $||\cdot||_Y$ and $||\cdot||_r$ are equivalent, there exist $m, M > 0$ such that, for all $\chi \in Y$, $m||\chi||_r \leq ||\chi||_Y \leq M||\chi||_Y$. We thus conclude that $||Tv||_Y \leq Mk||x||_r + Mk||y||_r$. This is smaller than or equal to $2Mk(||v||_r + \epsilon)$ due to our choice of x and y. Using equivalence of norms again, we find that this is smaller than or equal to $2kM(m||v||_Y + \epsilon)$.

As this holds for all $\epsilon > 0$, we see that $||Tv||_Y \leq 2kMm||v||_Y$, where $2kMm$ does not depend on v. Thus T is continuous. □

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