Neural Networks
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Bachelorscriptie

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1 Background

In neuroscience, a *neural network* is a set of *neurons* that are interconnected by synapses. Together they form the *Central Nervous System* in animals, which can be divided in the brain and the spinal chord. Every neuron can either be in an active state, or in an inactive state. In the brain, whichever state a neuron is in, depends on the state of the neurons with which it is connected. Whenever two connected neurons are simultaneously active, the synapse between them, and thus their connection, is strengthened. This means that it is more likely that if one of them is activated, the other one also gets activated. This way, certain patterns or clusters of connected neurons can arise. These so called cell assemblies can represent all kinds of concepts. To illustrate, whenever one thinks about a fork, it is likely that a spoon also comes to mind, because both concepts have been associated with each other a lot, their cell assemblies are strongly connected.

In machine learning and cognitive science, *artificial neural networks* are statistical learning models that are accurately based on their biological counterparts. An artificial neural network consists of a set of interconnected neurons that can be either active or inactive, respectively represented by it's boolean value 1 or 0. The state of a neuron can be affected by the input of connected neurons and their connection can be strengthened in a similar way as in a biological neural network and certain cell assemblies can arise here, too.

These artificial neural networks are a key element in the development of artificial intelligence. An example of this is *Deep Learning*. This considers neural models that consist of multiple layers. These layers each represent a different stage of information processing and thus the information will travel from the first layer to the last, step-by-step. For example, the first layer represents the retina of the eye, processing the visual input, and deeper layers represent the semantic translation of this information. These deep networks learn by means of the *backpropagation* algorithm, which compares the output of the model with the desired output and minimizes the difference by adjusting the synaptic weights.

The initial synaptic weights influence the behaviour of a neural network. If a network starts off with a cluster of strongly connected neurons, then it is likely that these neurons stay connected. In other words, this gives the network a bias. In this article we consider networks with equal synaptic weights, in other words, unbiased neural networks. These will then evolve randomly, based on the *Pólya urn model*. We will examine the limiting behaviour of these models on the basis of two questions. *What are the equilibria of these models? Are these equilibria stable?*

2 A mathematical model

A graph based model for (artificial) neural networks

To mathematically model the learning process of a neural network, we will use a finite graph.

Definition 2.1 (Finite graph)**.** A *finite graph* G is a pair (V, E), with V a non-empty, finite set of points, called vertices and $E \subset \{\{v_i, v_j\} : v_i, v_j \in V, v_i \neq v_j\}$ a finite set of connections between vertices, called edges. Let $n = |V|$ and $m = |E|$.

Let $G = (V, E)$ be a finite graph. Every vertex $v \in V$ represents a neuron and every edge $e \in E$ between two vertices represents a synapse between two neurons. To represent the strength of the synapses in the model, let $\Delta_m = \{\vec{w} \in \mathbb{R}^m : w_e \geq 0, \sum_{e \in E} w_e = 1\}$. The vectors $\vec{w} \in \Delta_m$ represent the possible ratios of the weights of the edges, relative to the total weight.

For the network to 'learn' is to strengthen particular synapses. In the model this translates to increasing the weight of particular edges relative to others. This process is as follows:

A vertex is picked uniformly at random, say v'. Consider all vertices that are connected to v', that is all $v \in V : \exists\ e = \{v, v'\} \in E$. Consider also the corresponding edges, that is all $e \in E : v' \in e$. Each edge e has a weight w_e. Following a probability distribution, which we will introduce later on, that is based on the weights of the edges, one vertex v that is connected to v' and their corresponding edge e will be picked. The weight of this edge will be increased and thus the weight ratio will change. These steps are then repeated.

Note that because the initial pick is uniformly distributed, the probability of vertex v being picked equals $\frac{1}{|V|} = \frac{1}{n}$ for all $v \in V$.

We will base the second probability function on the Pólya urn model.

Pólya urn model

Consider an urn with coloured balls. In the basic Pólya urn model one ball is randomly drawn from the urn and it's colour is observed, then it and an additional ball of the same colour are placed in the urn. This increases the number and thus the ratio of the balls of the observed colour. Therefore the probability that a ball of the same colour is drawn again also increases. One could say that it is a *"the rich get richer"*-model, because increasing the number of balls of a certain colour will increase the probability that this number will keep increasing.

The basic way of applying the Pólya urn model is as follows. For every vertex we have an urn and for every edge we have a unique colour. If two vertices v_1 and v_2 share an edge e, then the two corresponding urns will contain balls of the colour of e. Note that there are no other urns that contain balls of this colour. Suppose v_1 is picked and a ball of colour e is drawn from the urn of v_1. It's colour is observed, then the ball will be replaced and an additional ball of the same colour will be placed in v_1 and v_2. So the probability of drawing a ball of the same colour increases for both urns. In other words, the synapse between two neurons has been strengthened.

The probability distribution

Consider the finite graph mentioned earlier. Suppose vertex v has been picked uniformly at random at time t in the learning process. Consider the edges that are incident to v and suppose e is one of them. Given a weightfunction $g : \mathbb{R} \to \mathbb{R}_+$, let $\mathbb{P}(e)$, the probability that edge e will be picked and it's weight will be increased, be

$$\mathbb{P}(e) = \frac{1}{n} \cdot \sum_{v \in e} \frac{g(w_e(t))}{\sum\limits_{f \in E : f \ni v} g(w_f(t))}.$$

Assigning this probability to every edge e gives a probability measure. Clearly, $\mathbb{P}(e)$ is non-negative for every $e \in E$. It also holds that

$$\begin{aligned}
\sum_{e\,\in E} \mathbb{P}(e) &= \sum_{e \in E} \frac{1}{n} \cdot \sum_{v \in e} \frac{g(w_e(t))}{\sum\limits_{f \in E : f \ni v} g(w_f(t))} \\
&= \sum_{v \in V} \frac{1}{n} \cdot \sum_{e \in E : e \ni v} \frac{g(w_e(t))}{\sum\limits_{f \in E : f \ni v} g(w_f(t))} \\
&= \sum_{v \in V} \frac{1}{n} \cdot \frac{\sum\limits_{e \in E : e \ni v} g(w_e(t))}{\sum\limits_{f \in E : f \ni v} g(w_f(t))} \\
&= \sum_{v \in V} \frac{1}{n} \\
&= n \cdot \frac{1}{n} \\
&= 1,
\end{aligned} \tag{2.1}$$

which shows that it is indeed a probability measure.

Equilibria of the model

Let the function $F : \Delta_m \to \mathbb{R}^m$ be defined as follows:

$$F_e(\vec{w}) = -w_e + \lim_{t \to \infty} \frac{1}{n} \cdot \sum_{v \in e} \frac{g(w_e t)}{\sum\limits_{f \in E : f \ni v} g(w_f t)}, \tag{2.2}$$

with g a weightfunction. Here we use $w_e t$ in stead of $w_e(t)$, because we are looking for equilibria in the long run and w_e can be seen as the mean $\frac{w_e(t)}{t}$, thus equivalently $w_e t$ as $w_e(t)$.

Let $w_e(t)$ be the weight of edge e at time t, so after t iterations. Lets assume this function is linear in t. If $F_e(\vec{w}(t))$ is positive, then the probability of e being picked is larger then the weight ratio of e and if $F_e(\vec{w}(t))$ is negative, then the probability is smaller then the weight ratio. This tells us something about the expected behaviour of the model. This makes $F_e(\vec{w}(t))$ a derivative of $w_e((t))$. Thus if $F_e(\vec{w}(t)) = 0$, we have a critical point. This is a maximum if the Hessian is negative definite. This is equivalent with the eigenvalues being negative.

Definition 2.2 (Equilibrium)**.** Let F be as defined above and let m be given. A vector $\vec{w} \in \Delta_m$ is an *equilibrium distribution* of the model if $F(\vec{w}) = \vec{0}$.

So an equilibrium is a special case in which the relative weight of edge e equals the probability that the weight of e increases, for all $e \in E$.

In the sequel we consider weightfunctions g of the form $g(w_e) = w_e^{\alpha} t^{\alpha}$, $\alpha \in \mathbb{R}_{>1}$. Note that for $\alpha = 1$ this would give precisely the probability measure that corresponds with the basic Pólya urn model. By defining g like this, it follows that $F_e(\vec{w}) = 0$ can be reduced to the following:

$$
\begin{aligned}
w_e &= \lim_{t\to\infty} \frac{1}{n} \cdot \sum_{v\in e} \frac{g(w_e t)}{\sum\limits_{f\in E: f\ni v} g(w_f t)} \\
&= \lim_{t\to\infty} \frac{1}{n} \cdot \sum_{v\in e} \frac{w_e^\alpha t^\alpha}{\sum\limits_{f\in E: f\ni v} w_f^\alpha t^\alpha} \\
&= \lim_{t\to\infty} \frac{1}{n} \cdot \sum_{v\in e} \frac{w_e^\alpha}{\sum\limits_{f\in E: f\ni v} w_f^\alpha} \cdot \frac{t^\alpha}{t^\alpha} \\
&= \frac{1}{n} \cdot \sum_{v\in e} \frac{w_e^\alpha}{\sum\limits_{f\in E: f\ni v} w_f^\alpha}
\end{aligned}
\tag{2.3}
$$

Let the partial derivatives of $F_e(\vec{w})$ be denoted by $D_{e,d}(\vec{w}) = \frac{\delta F_e(\vec{w})}{\delta w_d}$, for edges e and d that are adjacent or equal. A quick calculation shows that, these partial derivatives are as follows:

$$
D_{e,e}(\vec{w}) = -1 + \alpha w_e^{\alpha-1} \cdot \frac{1}{n} \cdot \sum_{v\in e} \frac{\sum\limits_{f\in E: f\ni v} w_f^\alpha - w_e^\alpha}{\left(\sum\limits_{f\in E: f\ni v} w_f^\alpha\right)^2},
$$

and for $e \neq d$:

$$
D_{e,d}(\vec{w}) = -\alpha w_d^{\alpha-1} w_e^\alpha \cdot \frac{1}{n} \cdot \sum_{v\in e\cap d} \frac{1}{\left(\sum\limits_{f\in E: f\ni v} w_f^\alpha\right)^2}.
$$

If e and d are not adjacent, then this value is 0, indeed.

Let $\mathbf{D}(\vec{w})$ denote the matrix with (e, d) entry $D_{e,d}(\vec{w})$. Note that the initial weights are all equal, thus $w_e = w_d$ for all $e, d \in E$, and thus the same holds for $D_{e,d}(\vec{w})$ and $D_{d,e}(\vec{w})$. It follows that $\mathbf{D}(\vec{w})$ is a symmetric matrix, therefore all eigenvalues are real.

Definition 2.3 (Stable equilibrium)**.** An equilibrium distribution $\vec{w}$ is a *(linearly) stable equilibrium* if all eigenvalues of $\mathbf{D}(\vec{w})$ have negative real parts. If there exists an eigenvalue of $\mathbf{D}(\vec{w})$ that has a positive real part, then $\vec{w}$ is a (linearly) unstable equilibrium. Otherwise, it is critical.

Definition 2.4 (Adjacency matrix)**.** Let $G = (V, E)$ be a graph with n vertices and m edges. The *adjacency matrix* of a graph G is an $(m \times m)$-matrix $\mathbf{A}$, such that

$$
\mathbf{A}_{i,j} = \begin{cases} 1 & \text{if } e_i \cap e_j \neq 0 \text{ and } i \neq j \\ 0 & \text{otherwise} \end{cases}.
$$

Note that $\mathbf{D}(\vec{w})$ can be written as $D_{i,i}(\vec{w})\,\mathbf{I} - D_{i,j}(\vec{w})\,\mathbf{A}$.

From this point we will assume that $\alpha > 1$.

3 Preliminary Considerations

3.1 Step-by-step plan

In each of the following three chapters, a specific kind of graph will be covered. These will be the circlegraph, the stargraph and the complete graph. Let us denote the uniform vector, that is the vector with all entries having the same value w_e, as $(\vec{w_e})$. In each case, the weightvector that we shall consider is $\left(\frac{\vec{1}}{m}\right)$, i.e. the total weight is evenly distributed. Though each of these cases has it's own results, we can use one general method to find those results for each case. The following is a summary of this method:

First
: The first step is to prove that $\left(\frac{\vec{1}}{m}\right)$ is indeed an equilibrium. This is done, simply by proving that $F(\frac{\vec{1}}{m}) = \vec{0}$, (see definition 2.2).

Second
: Then we will calculate both $D_{i,i}(\frac{\vec{1}}{m})$ and $D_{i,j}(\frac{\vec{1}}{m})$. Also we will calculate the adjecency matrix $\mathbf{A}$ of the graph. These values are necessary for the construction of $\mathbf{D}\left(\frac{\vec{1}}{m}\right)$.

Third
: With the information gained in step 2, $\mathbf{D}\left(\frac{\vec{1}}{m}\right)$ will be constructed, as $\mathbf{D}\left(\frac{\vec{1}}{m}\right) = D_{i,i}\left(\frac{\vec{1}}{m}\right)\mathbf{I} - D_{i,j}\left(\frac{\vec{1}}{m}\right)\mathbf{A}$.

Fourth
: Next the eigenvalues of $\mathbf{A}$ will be calculated, and therewith the eigenvalues of $\mathbf{D}\left(\frac{\vec{1}}{m}\right)$, using a lemma that will be introduced in the following subsection.

Fifth
: Finally, the eigenvalues of $\mathbf{D}\left(\frac{\vec{1}}{m}\right)$ will be used to determine if, or under which criterion, the equilibrium is stable.

3.2 Lemmas

In preparation of the following chapters, we state and prove two lemmas.

Lemma 3.1. *Let $p, q \in \mathbb{R}$. Let $\boldsymbol{M}$ and $\boldsymbol{M'}$ be $n \times n$ matrices such that $\boldsymbol{M'} = p\boldsymbol{Id} + q\boldsymbol{M}$. If $\boldsymbol{M}$ has an eigenvalue λ, then $\boldsymbol{M'}$ has a corresponding eigenvalue $\lambda' = p + q\lambda$.*

Proof. Let v be the eigenvector of $\mathbf{M}$ corresponding to the eigenvalue λ. It holds that $\mathbf{M}v = \lambda v$. It follows that:

$$\mathbf{M}'v = (p\mathbf{Id} + q\mathbf{M})v = p\mathbf{Id}v + q\mathbf{M}v = pv + q\lambda v = (p + q\lambda)v$$

Thus $\lambda' = p + q\lambda$ is an eigenvalue of $\mathbf{M}'$. □

Using the eigenvalues of $\mathbf{A}$, lemma 3.1 will be used in chapters 4 and 6 to calculate the eigenvalues of $\mathbf{D}$, which relate to the eigenvalues of $\mathbf{A}$ in a such a way as described in lemma 3.1.

Lemma 3.2. *Let $\boldsymbol{A}$ be an $(n \times m)$-matrix. Consider $\boldsymbol{AA}^T$, which is an $(n \times n)$-matrix, and $\boldsymbol{A}^T\boldsymbol{A}$, which is an $(m \times m)$-matrix. Then the following holds:*

$$\boldsymbol{AA}^T \text{ has an eigenvalue } \lambda \neq 0 \Longleftrightarrow \boldsymbol{A}^T\boldsymbol{A} \text{ has an eigenvalue } \lambda \neq 0$$

Proof. Let λ be an eigenvalue of $\mathbf{AA}^T$. Then there is a vector $v \neq 0$ such that $\mathbf{AA}^T v = \lambda v$.

$$\mathbf{AA}^T v = \lambda v \iff \mathbf{A}^T \mathbf{AA}^T v = \mathbf{A}^T \lambda v \iff \mathbf{A}^T \mathbf{A}(\mathbf{A}^T v) = \lambda(\mathbf{A}^T v)$$

.

Thus, if $(\mathbf{A}^T v) \neq 0$, then λ is an eigenvalue of $\mathbf{A}^T \mathbf{A}$ and $\mathbf{A}^T v$ is the associated/corresponding eigenvector.

Claim: $(\mathbf{A}^T v) \neq 0$

Proof: Suppose $(\mathbf{A}^T v) = 0$. Then $\mathbf{AA}^T v = 0$. Given that $\mathbf{AA}^T v = \lambda v$, it follows that $\lambda v = 0$. However, it holds that $\lambda \neq 0$ and $v \neq 0$, so $\lambda v \neq 0$. This gives a contradiction. Thus $(\mathbf{A}^T v) \neq 0$.

Following this fact, the lemma has been proven.

□

Lemma 3.2 will be of use in section 6, which covers the complete graph.

4 Stargraph results

Definition 4.1 (Stargraph)**.** A simple graph $G = (V, E)$ is a *stargraph* if $\exists v \in V$ such that $v \in e$ $\forall e \in E$.

Note that in this case $m = n - 1$ holds.

Theorem 4.2 (Equilibrium and stability for stargraphs)**.** *Let G be a stargraph with $n \in \mathbb{N}_{\geq 3}$ vertices and $m = n-1$ edges. Then $\left(\frac{\vec{1}}{m}\right) = \left(\frac{\vec{1}}{n-1}\right)$ is an equilibrium for G and it is stable if and only if $\alpha < n$.*

Proof. First we check that $\left(\frac{\vec{1}}{m}\right) = \left(\frac{\vec{1}}{n-1}\right)$ is indeed an equilibrium. To this end, we calculate $F_i\left(\frac{\vec{1}}{n-1}\right)$, according to formula (2.2):

$$\begin{aligned}
F_i\left(\frac{\vec{1}}{n-1}\right) &= -\frac{1}{n-1} + \frac{1}{n} \cdot \left(\frac{(n-1)^{-\alpha}}{(n-1)(n-1)^{-\alpha}} + \frac{(n-1)^{-\alpha}}{(n-1)^{-\alpha}}\right) \\
&= -\frac{1}{n-1} + \frac{1}{n(n-1)} + \frac{1}{n} \\
&= -\frac{n}{n(n-1)} + \frac{1}{n(n-1)} + \frac{n-1}{n(n-1)} \\
&= \frac{-n+1+n-1}{n(n-1)} \\
&= \frac{0}{n(n-1)} \\
&= 0,
\end{aligned} \tag{4.1}$$

which shows that $\left(\frac{\vec{1}}{m}\right)$ is indeed an equilibrium.

Next we will calculate $D_{i,i}\left(\frac{\vec{2}}{n-1}\right)$ and $D_{i,k}\left(\frac{\vec{2}}{n-1}\right)$.

$$\begin{aligned}
D_{i,i}\left(\frac{\vec{1}}{n-1}\right) &= -1 + \alpha\left(\frac{1}{n-1}\right)^{\alpha-1} \cdot \frac{1}{n} \cdot \left(\frac{\frac{1}{n-1}^{\alpha} - \frac{1}{n-1}^{\alpha}}{\frac{1}{n-1}^{2\alpha}} + \frac{(n-1)\frac{1}{(n-1)^{\alpha}} - \frac{1}{(n-1)^{\alpha}}}{(n-1)^2\frac{1}{(n-1)^{2\alpha}}}\right) \\
&= -1 + \alpha\left(\frac{1}{n-1}\right)^{\alpha-1} \cdot \frac{1}{n} \cdot \frac{(n-1)\frac{1}{(n-1)^{\alpha}} - \frac{1}{(n-1)^{\alpha}}}{(n-1)^2\frac{1}{(n-1)^{2\alpha}}} \\
&= -1 + \alpha\left(\frac{1}{n-1}\right)^{\alpha-1} \cdot \frac{1}{n} \cdot \frac{(n-2)\frac{1}{(n-1)^{\alpha}}}{(n-1)^2\frac{1}{(n-1)^{2\alpha}}} \\
&= -1 + \alpha\left(\frac{1}{n-1}\right)^{\alpha-1} \cdot \frac{1}{n} \cdot \frac{(n-2)}{(n-1)^2\frac{1}{(n-1)^{\alpha}}} \\
&= -1 + \alpha\left(\frac{1}{n-1}\right)^{-1} \cdot \frac{1}{n} \cdot \frac{(n-2)}{(n-1)^2} \\
&= -1 + \frac{\alpha}{n} \cdot \frac{(n-2)}{(n-1)}.
\end{aligned} \tag{4.2}$$

For $i \neq k$,

$$\begin{aligned}
D_{i,k}\left(\tfrac{1}{n-1}\right) &= -\alpha \cdot \left(\frac{1}{n-1}\right)^{\alpha-1} \cdot \left(\frac{1}{n-1}\right)^{\alpha} \cdot \frac{1}{n} \cdot \frac{1}{(n-1)^2 \left(\tfrac{1}{n-1}\right)^{2\alpha}} \\
&= -\alpha \cdot \left(\frac{1}{n-1}\right)^{2\alpha-1} \cdot \frac{1}{n} \cdot \frac{1}{(n-1)^2} \cdot \left(\frac{1}{n-1}\right)^{-2\alpha} \\
&= -\frac{\alpha}{n} \cdot \left(\frac{1}{n-1}\right)^{-1} \cdot \left(\frac{1}{n-1}\right)^{2} \\
&= -\frac{\alpha}{n} \cdot \frac{1}{n-1} \\
&= -\frac{\alpha}{n(n-1)}.
\end{aligned} \tag{4.3}$$

For every edge in a stargraph it holds that it is adjacent to every other edge in the graph, because there exists a unique vertex to which all edges are incident. Therefore we know that the adjacency matrix $\mathbf{A}$ of a stargraph of n vertices is of the following form:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}.$$

Here the number of rows and columns of $\mathbf{A}$ both equal $n-1$.

With $D_{i,i}\left(\frac{\vec{1}}{n-1}\right), D_{i,j}\left(\frac{\vec{1}}{n-1}\right)$ and $\mathbf{A}$ as given above, we can write the partial derivative matrix $\mathbf{D}$ of $F\left(\frac{\vec{1}}{n}\right)$ as

$$\mathbf{D}\left(\frac{1}{n-1}\right) = \left(-1 + \frac{\alpha}{n} \cdot \frac{(n-2)}{(n-1)}\right)\mathbf{Id} - \left(\frac{\alpha}{n(n-1)}\right)\mathbf{A}.$$

We will calculate the eigenvalues of $\mathbf{A}$ by rewriting it as $\mathbf{A} = \mathbb{1} - \mathbf{Id}$, with $\mathbb{1}$ being the matrix consisting of ones, namely

$$\mathbb{1} := \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

First we will determine the eigenvalues and eigenvectors of $\mathbb{1}$. For every eigenvalue and it's respective eigenvector(s) it must hold that

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + \dots + x_n \\ x_1 + x_2 + \dots + x_n \\ \vdots \\ x_1 + x_2 + \dots + x_n \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{pmatrix}.$$

One obvious solution for this equation is the following

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1 = x_2 = ... = x_{n-1} = x_n \right\}, \lambda = n,$$

which has dimension 1, since fixing x_i for a certain i will fix all others too.

The other solutions are

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : x_1 + x_2 + ... + x_{n-1} + x_n = 0 \right\}, \lambda = 0.$$

Fixing $n-1$ elements of the vector will fix the last one too, this last one will be the negative sum of the first $n-1$ elements. Which means that this solution has dimension $n-1$.

Now consider the adjacency matrix of the complete graph.

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & \dots & 1 & 1 \\ 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 0 & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix} = \mathbb{1} - \mathbf{Id} = -1 \cdot \mathbf{Id} + 1 \cdot \mathbb{1}.$$

Given that $\mathbb{1}$ has an eigenvalue $\lambda = n$ with multiplicity 1, it now follows from lemma 3.1 that $\mathbf{A}$ has an eigenvalue $\lambda' = n-1$, with multiplicity 1.

Similarly, given that $\mathbb{1}$ has eigenvalue $\mu = 0$ with multiplicity $n-1$, it follows that $\mathbf{A}$ has eigenvalue $\mu' = -1$ with multiplicity $n-1$.

Again using lemma 3.1 we can calculate the eigenvalues of $\mathbf{D}\left(\frac{1}{n-1}\right)$. One eigenvalue is

$$\begin{aligned} \lambda_{\mathbf{D}} &= \left(-1 + \frac{\alpha}{n} \cdot \frac{(n-2)}{(n-1)}\right) - \left(\frac{\alpha}{n(n-1)}\right)(n-1) \\ &= -1 + \frac{\alpha(n-2)}{n(n-1)} - \frac{\alpha(n-1)}{n(n-1)} \\ &= -1 + \frac{\alpha(n-2) - \alpha(n-1)}{n(n-1)} \\ &= -1 + \frac{\alpha(n-2-n+1)}{n(n-1)} \\ &= -1 - \frac{\alpha}{n(n-1)}, \end{aligned} \tag{4.4}$$

with multiplicity 1. The other eigenvalue is

$$\begin{aligned} \lambda_{\mathbf{D}} &= \left(-1 + \frac{\alpha}{n} \cdot \frac{(n-2)}{(n-1)}\right) - \left(\frac{\alpha}{n(n-1)}\right) \cdot -1 \\ &= -1 + \frac{\alpha(n-2) + \alpha}{n(n-1)} \\ &= \frac{\alpha(n-1)}{n(n-1)} \\ &= -1 + \frac{\alpha}{n}, \end{aligned} \tag{4.5}$$

with multiplicity $n-1$.

Therefore the stability criterion is as follows:

$$-1-\frac{\alpha}{n(n-1)}<0 \text{ and } -1+\frac{\alpha}{n}<0.$$

A quick calculation shows that the first part of the criterion is always true:

$$-1-\frac{\alpha}{n(n-1)}<0 \Longleftrightarrow -1<\frac{\alpha}{n(n-1)} \Longleftrightarrow -n(n-1)<\alpha.$$

This is always true because $\alpha>1$ and $n \in \mathbb{N}_{\geq 3}$.

The second part of the criterion gives the following:

$$-1+\frac{\alpha}{n}<0 \Longleftrightarrow \frac{\alpha}{n}<1 \Longleftrightarrow \alpha<n.$$

Thus, $\left(\frac{\vec{1}}{n-1}\right)$ is a stable equilibrium for the stargraph of n vectors and $n-1$ edges if $\alpha<n$, otherwise it is not stable.

$\square$

5 Circlegraph results

Definition 5.1 (Circlegraph)**.** Let $G = (V, E)$ be a simple graph. Order the vertices of G, i.e. $v_0, v_1, \ldots, v_{n-1}$. G is a *circlegraph* if there exits an order of the edges of G such that $e_i = \{v_i, v_{i+1}\}$ $\forall i \in \{0, \ldots, n-2\}$ and $e_{n-1} = \{v_{n-1}, v_0\}$. Note that $m = n$.

Theorem 5.2 (Equilibrium and stability for circlegraphs)**.** *Let G be a circlegraph with $n \in \mathbb{N}_{\geq 3}$ vertices and $m = n$ edges. Then $\left(\frac{\vec{1}}{m}\right) = \left(\frac{\vec{1}}{n}\right)$ is an equilibrium for G and it is stable if and only if n is odd and $\alpha < \frac{2}{1+\cos\left(\frac{\pi}{n}\right)}$.*

Proof. First we check that $\left(\frac{\vec{1}}{m}\right) = \left(\frac{\vec{1}}{n}\right)$ is indeed an equilibrium. To this end, we calculate $F_i\left(\frac{\vec{1}}{n}\right)$.

$$\begin{aligned}
F\left(\frac{\vec{1}}{n}\right)_i &= -\frac{1}{n} + \frac{1}{n} \cdot \frac{\left(\frac{1}{n}\right)^\alpha}{\left(\frac{1}{n}\right)^\alpha + \left(\frac{1}{n}\right)^\alpha} + \frac{1}{n} \cdot \frac{\left(\frac{1}{n}\right)^\alpha}{\left(\frac{1}{n}\right)^\alpha + \left(\frac{1}{n}\right)^\alpha} \\
&= -\frac{1}{n} + \frac{2}{n} \cdot \frac{\left(\frac{1}{n}\right)^\alpha}{2 \cdot \left(\frac{1}{n}\right)^\alpha} \\
&= -\frac{1}{n} + \frac{1}{n} \\
&= 0,
\end{aligned} \tag{5.1}$$

which shows that $\left(\frac{\vec{1}}{m}\right)$ is indeed an equilibrium.

Next we will calculate $D_{i,i}\left(\frac{1}{n}\right)$ and $D_{i,j}\left(\frac{1}{n}\right)$. Note that, because G is a circlegraph, edge e_i can be written as $e_i = \{v_i, v_{i+1}\}$ $\forall i \in \{0, \ldots, n-2\}$ and e_{n-1} can be written as $e_{n-1} = \{v_{n-1}, v_0\}$. Therefore it suffices to calculate $D_{i,i}\left(\frac{1}{n}\right)$ and, using the symmetry of the circle graph, $D_{i,i+1}\left(\frac{1}{n}\right) = D_{i,i-1}\left(\frac{1}{n}\right)$.

$$\begin{aligned}
D_{i,i}\left(\frac{\vec{1}}{n}\right) &= -1 + \alpha\left(\frac{1}{n}\right)^{\alpha-1} \cdot \frac{1}{n} \cdot \left(\frac{2 \cdot \left(\frac{1}{n}\right)^\alpha - \left(\frac{1}{n}\right)^\alpha}{\left(2 \cdot \left(\frac{1}{n}\right)^\alpha\right)^2} + \frac{2 \cdot \left(\frac{1}{n}\right)^\alpha - \left(\frac{1}{n}\right)^\alpha}{\left(2 \cdot \left(\frac{1}{n}\right)^\alpha\right)^2}\right) \\
&= -1 + \alpha\left(\frac{1}{n}\right)^{\alpha-1} \cdot \frac{2}{n} \cdot \left(\frac{2 \cdot \left(\frac{1}{n}\right)^\alpha - \left(\frac{1}{n}\right)^\alpha}{\left(2 \cdot \left(\frac{1}{n}\right)^\alpha\right)^2}\right) \\
&= -1 + \alpha\left(\frac{1}{n}\right)^{\alpha} \cdot \frac{2 \cdot \left(\frac{1}{n}\right)^\alpha}{4 \cdot \left(\frac{1}{n}\right)^{2\alpha}} \\
&= -1 + \alpha\left(\frac{1}{n}\right)^{\alpha} \cdot \frac{1}{2 \cdot \left(\frac{1}{n}\right)^\alpha} \\
&= -1 + \frac{\alpha}{2}.
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
D_{i,i+1}\left(\frac{\vec{1}}{n}\right) &= -\alpha\left(\frac{1}{n}\right)^{\alpha-1} \cdot \left(\frac{1}{n}\right)^{\alpha} \cdot \frac{1}{n} \cdot \frac{1}{\left(2 \cdot \left(\frac{1}{n}\right)^\alpha\right)^2} \\
&= -\alpha\left(\frac{1}{n}\right)^{2\alpha} \cdot \frac{1}{4 \cdot \left(\frac{1}{n}\right)^{2\alpha}} \\
&= -\frac{\alpha}{4}.
\end{aligned} \tag{5.3}$$

We know that edge e_i is adjacent to edges e_{i-1} and e_{i+1} $\forall i \in \{1, \ldots, n-2\}$. Edge e_{n-1} is adjacent to edges e_{n-2} and e_0, and edge e_0 is adjacent to edges e_{n-1} and e_1. From this we can easily derive the adjacency matrix of the circle graph, namely

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ddots & & & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & \ldots & \ddots & 0 & 1 \\ 1 & 0 & \ldots & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

With $D_{i,i}\left(\frac{\vec{1}}{n}\right), D_{i,i+1}\left(\frac{\vec{1}}{n}\right) = D_{i,i-1}\left(\frac{\vec{1}}{n}\right)$ and $\mathbf{A}$ as given above, we can write the partial derivative matrix $\mathbf{D}$ of $F\left(\frac{\vec{1}}{n}\right)$ as

$$\mathbf{D}\left(\frac{\vec{1}}{n}\right) = \left(-1 + \frac{\alpha}{2}\right)\mathbf{Id} - \frac{\alpha}{4}\mathbf{A}.$$

Definition 5.3 (Circulant matrix)**.** An $(n \times n)$ matrix C is a *circulant matrix* if there exist $c_0, \ldots, c_{n-1}$ such that the following holds:

$$C = \begin{pmatrix} c_0 & c_1 & \ldots & \ldots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \ldots & \ldots & c_{n-2} \\ \vdots & c_{n-1} & c_0 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ c_2 & & \ldots & \ddots & \ddots & c_1 \\ c_1 & c_2 & \ldots & \ldots & c_{n-1} & c_0 \end{pmatrix}.$$

Properties of circulant matrix

Let $\phi_j = e^{\frac{2\pi j}{n} i}$. The eigenvalues of a circulant matrix are as follows:

$$\lambda_j = c_0 + c_1\phi_j + c_2\phi_j^2 + \cdots + c_{n-1}\phi^{n-1},\ j = 0, 1, \ldots, n-1,$$

according to [1, Section 3.1].

The adjacency matrix of a circle graph of n vertices is a circulant matrix with the following values:

$c_1 = c_{n-1} = 1$ and $c_2 = c_3 = \ldots c_{n-2} = c_0 = 0$

It follows that

$$\mathbf{D} = \left(-1+\frac{\alpha}{2}\right)\mathbf{Id} - \frac{\alpha}{4}\mathbf{A} = \begin{pmatrix} -1+\frac{\alpha}{2} & -\frac{\alpha}{4} & \dots & \dots & 0 & -\frac{\alpha}{4} \\ -\frac{\alpha}{4} & -1+\frac{\alpha}{2} & -\frac{\alpha}{4} & \dots & \dots & 0 \\ \vdots & -\frac{\alpha}{4} & -1+\frac{\alpha}{2} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \dots & \ddots & \ddots & -\frac{\alpha}{4} \\ -\frac{\alpha}{4} & 0 & \dots & \dots & -\frac{\alpha}{4} & -1+\frac{\alpha}{2} \end{pmatrix}.$$

So $\mathbf{D}$ is a circulant matrix with
$c_0 = -1 + \frac{\alpha}{2}$,
$c_1 = c_{n-1} = -\frac{\alpha}{4}$,
and $c_2 = c_3 = \ldots = c_{n-2} = 0$.
Therefore the eigenvalues of $\mathbf{D}$, with $j \in \mathbb{Z}$, are

$$\begin{aligned}\lambda_j &= -1 + \frac{\alpha}{2} - \frac{\alpha}{4}e^{\frac{2\pi j}{n}i} - \frac{\alpha}{4}e^{\frac{2\pi j}{n}i\cdot(n-1)} \\ &= -1 + \frac{\alpha}{2} - \frac{\alpha}{4}e^{\frac{2\pi j}{n}i} - \frac{\alpha}{4}e^{2\pi j\cdot i - \frac{2\pi j}{n}\cdot i} \\ &= -1 + \frac{\alpha}{2} - \frac{\alpha}{4}e^{\frac{2\pi j}{n}i} - \frac{\alpha}{4}e^{-\frac{2\pi j}{n}\cdot i} \\ &= -1 + \frac{\alpha}{2} - \frac{\alpha}{4}\left(e^{\frac{2\pi j}{n}i} + e^{-\frac{2\pi j}{n}\cdot i}\right) \\ &= -1 + \frac{\alpha}{2} - \frac{\alpha}{4}\left(\cos\left(\frac{2\pi j}{n}\right) + i\sin\left(\frac{2\pi j}{n}\right) + \cos\left(-\frac{2\pi j}{n}\right) + i\sin\left(-\frac{2\pi j}{n}\right)\right) \\ &= -1 + \frac{\alpha}{2} - \frac{\alpha}{4}\left(2\cdot\cos\left(\frac{2\pi j}{n}\right)\right) \\ &= -1 + \frac{\alpha}{2} - \frac{\alpha}{2}\left(\cos\left(\frac{2\pi j}{n}\right)\right) \\ &= -1 + \frac{\alpha}{2}\left(1 - \cos\left(\frac{2\pi j}{n}\right)\right).\end{aligned} \tag{5.4}$$

The equilibrium is stable if and only if all eigenvalues are negative, so if
$\alpha < \dfrac{2}{1-\cos\left(\frac{2\pi j}{n}\right)}$.
The right hand of the inequality attains it's minimum precisely when $\cos\left(\frac{2\pi j}{n}\right)$ attains it's minimum. As the cosine function attains it's minimum at $\pi, 3\pi, \ldots$, it follows that the minimum of $\cos\left(\frac{2\pi j}{n}\right)$ is attained when $j = \frac{n}{2}$. However, because $j \in \mathbb{Z}$ this only holds for n even. This minimum is -1, therefore $\alpha < 1$. By assumption it holds that $\alpha > 1$, so the equilibrium $\left(\frac{\vec{1}}{n}\right)$ is not a stable one, for even n in the model. For odd n, the minimum of $\cos\left(\frac{2\pi j}{n}\right)$ is attained when $j = \frac{n+1}{2}$. The equilibrium is then stable if and only if the following holds:
$\alpha < \dfrac{2}{1-\cos\left(\frac{\pi(n+1)}{n}\right)} = \dfrac{2}{1-\cos\left(\pi+\frac{\pi}{n}\right)} = \dfrac{2}{1+\cos\left(\frac{\pi}{n}\right)}$.

It holds that $\cos\left(\frac{\pi}{n}\right) < 1$, for all $n \in \mathbb{N}$, so this upper bound for α is greater than 1. Therefore the equilibrium $\left(\frac{\vec{1}}{n}\right)$ is stable if and only if α satisfies $1 < \alpha < \frac{2}{1+\cos\left(\frac{\pi}{n}\right)}$.

□

6 Complete graph results

Definition 6.1 (Complete graph)**.** A simple graph $G = (V, E)$ with n vertices is a *complete graph* if $\{v_i, v_k\} \in E \ \forall i, k \in \{1, \ldots, n\}$, $i \neq j$. Note that $m = \frac{n(n-1)}{2}$.

Theorem 6.2 (Equilibrium and stability for complete graphs)**.** *Let G be a complete graph with $n \in \mathbb{N}_{\geq 3}$ vertices and $m = \frac{(n-1)n}{2}$ edges. Then $\left(\frac{\vec{1}}{m}\right) = \left(\frac{\vec{2}}{(n-1)n}\right)$ is an equilibrium for G, but it is not stable.*

Proof. First we check that $\left(\frac{\vec{1}}{m}\right) = \left(\frac{\vec{2}}{n(n-1)}\right)$ is indeed an equilibrium. To this end, we calculate $F_i\left(\frac{\vec{2}}{n(n-1)}\right)$.

$$\begin{aligned} F_i\left(\frac{\vec{2}}{n(n-1)}\right) &= -\frac{2}{n(n-1)} + \frac{2}{n} \cdot \left(\frac{\frac{2}{n(n-1)}^{\alpha}}{(n-1)\frac{2}{n(n-1)}^{\alpha}}\right) \\ &= -\frac{2}{n(n-1)} + \frac{2}{n} \cdot \frac{1}{n-1} \\ &= -\frac{2}{n(n-1)} + \frac{2}{n(n-1)} \\ &= 0, \end{aligned} \tag{6.1}$$

which shows that $\left(\frac{\vec{1}}{m}\right)$ is indeed an equilibrium.

Next we will calculate $D_{i,i}\left(\frac{\vec{2}}{n(n-1)}\right)$ and $D_{i,k}\left(\frac{\vec{2}}{n(n-1)}\right)$.

$$\begin{aligned} D_{i,i}\left(\frac{\vec{2}}{n\,(n-1)}\right) &= -1 + \alpha\frac{2}{n\,(n-1)}^{\alpha-1} \cdot \frac{2}{n} \cdot \frac{(n-2)\left(\frac{2}{n(n-1)}\right)^{\alpha}}{(n-1)^2\left(\frac{2}{n(n-1)}\right)^{2\alpha}} \\ &= -1 + \alpha\left(\frac{2}{n\,(n-1)}\right)^{-1} \cdot \frac{2}{n} \cdot \frac{n-2}{(n-1)^2} \\ &= -1 + \alpha\frac{n\,(n-1)}{2} \cdot \frac{2}{n} \cdot \frac{n-2}{(n-1)^2} \\ &= -1 + \alpha\frac{n-2}{n-1}. \end{aligned} \tag{6.2}$$

$$\begin{aligned} D_{i,j}\left(\frac{\vec{2}}{n\,(n-1)}\right) &= -\alpha\left(\frac{2}{n\,(n-1)}\right)^{\alpha-1} \cdot \left(\frac{2}{n\,(n-1)}\right)^{\alpha} \cdot \frac{1}{n} \cdot \frac{1}{(n-1)^2\left(\frac{2}{n(n-1)}\right)^{2\alpha}} \\ &= -\alpha\left(\frac{2}{n\,(n-1)}\right)^{2\alpha-1} \cdot \frac{1}{n} \cdot \frac{1}{(n-1)^2} \cdot \frac{1}{\left(\frac{2}{n(n-1)}\right)^{2\alpha}} \\ &= -\alpha\left(\frac{2}{n\,(n-1)}\right)^{-1} \cdot \frac{1}{n} \cdot \frac{1}{(n-1)^2} \\ &= -\alpha\frac{n\,(n-1)}{2} \cdot \frac{1}{n} \cdot \frac{1}{(n-1)^2} \\ &= -\frac{\alpha}{2\,(n-1)}. \end{aligned} \tag{6.3}$$

As opposed to the last two chapters, we will not be calculating the adjacency matrix $\mathbf{A}$ of the complete graph of n vertices. We will instead find an eigenvalue of $\mathbf{A}$ by arguing.

So far we can write the partial derivative matrix $\mathbf{D}$ of $F\left(\frac{\vec{2}}{n(n-1)}\right)$ as

$$\mathbf{D}\left(\frac{\vec{2}}{n\,(n-1)}\right) = \left(-1+\alpha-\frac{\alpha}{n-1}\right)\mathbf{Id} - \frac{\alpha}{2(n-1)}\mathbf{A}.$$

We will find an eigenvalue of $\mathbf{A}$ using the so called *incidence matrix*.

Definition 6.3 (Incidence matrix)**.** Let $G = (V, E)$ be a graph with n vertices and m edges. The *incidence matrix* of a graph G is an $(n \times m)$-matrix $\mathbf{N}$, such that

$$\mathbf{N}_{i,j} = \begin{cases} 1 & \text{if } v_i \in e_j \\ 0 & \text{if } v_i \notin e_j \end{cases}.$$

So $\mathbf{N}$ has a row for each vertex and a column for each edge and an element is equal to 1 if and only if the corresponding vertex and edge are incident. Otherwise the value of the element is 0.

Consider the $(n \times n)$-matrix $\mathbf{N}\mathbf{N}^T$, this is the matrix that arises by multiplying the rows of $\mathbf{N}$ with themselves. Thus, the value of $\mathbf{N}\mathbf{N}^T_{i,j}$ equals the number of edges that v_i and v_j share. Note that when $i = j$, this equals the number of edges to which v_i is adjacent.

Now consider the $(m \times m)$-matrix $\mathbf{N}^T\mathbf{N}$, this is the matrix that arises by multiplying the columns of $\mathbf{N}$ with themselves. Thus, the value of $\mathbf{N}^T\mathbf{N}_{i,j}$ equals the number of vertices that e_i and e_j share. Note that for a simple graph, this is 1 when e_i and e_j are adjacent and 0 when they are not. When $i = j$, it equals the number of edges to which e_i is adjacent, which is always equal to 2. Also note that this matrix is very similar to the adjacency matrix, the only difference being that the elements of the main diagonal equal 2. Thus it holds that $\mathbf{A} = \mathbf{N}^T\mathbf{N} - 2\mathbf{Id}$.

$\mathbf{N}\mathbf{N}^T$ has n eigenvalues, of which at most n are non-zero eigenvalues. It follows from Lemma 3.2 that $\mathbf{N}\mathbf{N}^T$ and $\mathbf{N}^T\mathbf{N}$ have the exact same non-zero eigenvalues, therefore $\mathbf{N}^T\mathbf{N}$ too has at most n non-zero eigenvalues. However, $\mathbf{N}^T\mathbf{N}$ has $m > n$ eigenvalues, therefore $m - n > 0$ of those are equal to 0. Thus we can conclude that $\mathbf{N}^T\mathbf{N}$ has an eigenvalue $\lambda = 0$.

Using lemma 3.1 and the equation $\mathbf{A} = -2\mathbf{Id} + \mathbf{N}^T\mathbf{N}$, we can find an eigenvalue λ_A of $\mathbf{A}$:

$$\begin{aligned} \lambda_A &= -2 + 1\cdot\lambda \\ &= -2 + 1\cdot 0 \\ &= -2. \end{aligned} \tag{6.4}$$

Using lemma 3.1 and the equation $\mathbf{D}\left(\frac{\vec{2}}{n(n-1)}\right) = \left(-1+\alpha-\frac{\alpha}{n-1}\right)\mathbf{Id} - \frac{\alpha}{2(n-1)}\mathbf{A}$, we can find an eigenvalue λ_D of $\mathbf{D}\left(\frac{\vec{2}}{n(n-1)}\right)$:

$$\begin{aligned} \lambda_D &= \left(-1+\alpha-\frac{\alpha}{n-1}\right) - \frac{\alpha}{2(n-1)}\cdot\lambda_A \\ &= \left(-1+\alpha-\frac{\alpha}{n-1}\right) - \frac{\alpha}{2(n-1)}\cdot -2 \\ &= -1+\alpha-\frac{\alpha}{n-1} + \frac{\alpha}{(n-1)} \\ &= -1+\alpha. \end{aligned} \tag{6.5}$$

By assumption it holds that $\alpha > 1$, therefore it follows that $-1 + \alpha > 0$.

Thus $\mathbf{D}\left(\frac{\vec{2}}{n(n-1)}\right)$ has an eigenvalue that is greater than 0, therefore the equilibrium is not stable.

$\square$

7 Discussion

We have considered three different types of graphs; stargraphs, circlegraphs and complete graphs. In all three cases we found that $\begin{pmatrix} \vec{1} \\ n \end{pmatrix}$ is an equilibrium. We have shown that for stargraphs this equilibrium is stable if and only if $\alpha < n$. For circlegraphs it is stable if and only if n is odd and if $\alpha < \frac{2}{1+\cos\left(\frac{pi}{n}\right)}$. Lastly for complete graphs we showed that it is never stable.

There are many problems in this model that remain unsolved. The most obvious next step is looking for more (stable) equilibria in the graphs that we studied.

Also, there are many other types of graphs that can be studied, such as bipartite graphs and path graphs. Graphs with a spatial structure would be especially interesting. For example, a two dimensional lattice-shaped graph could be linked to the retina of the eye, and a three dimensional one could be linked to a variety of brain areas.

References

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[2] R. van der Hofstad, M. Holmes, A. Kuznetsov, and W. Ruszel. Strongly reinforced Pólya urns with graph-based competition, Preprint arXiv 1406.0449, 2014.