G-bundles: Cech Cohomology and the Fundamental Group

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Chapter 1

Introduction

If we let $\mathbb{Z}$, the group of integers, act on $\mathbb{R}$, the set of real numbers, by translation, the resulting orbit space is homeomorphic to S^1, a circle. There are (at least) two interesting 'coincidences' to note in this example. The first is a purely topological observation: $\mathbb{R}$ and S^1 locally look the same. In other words, the original space and the orbit space are (in this case) locally homeomorphic. The second is an algebraic observation: it so happens that $\mathbb{Z}$ is isomorphic to the fundamental group of S^1. In other words, the group acting on the original space is (in this case) isomorphic to the fundamental group of the orbit space.

Do these phenomena always occur when we let a group act on a topological space? Certainly not, even if we stipulate that the action of the group be a continuous one. If we replace $\mathbb{Z}$ with $\mathbb{Q}$ in the example above, the orbit space is a uncountable space with trivial topology and certainly not locally homeomorphic to $\mathbb{R}$. But we will see that under a certain reasonable condition, the first observation will hold. The second observation is less robust, but will hold under the additional condition that the original space is simply connected.

The action of $\mathbb{Z}$ on $\mathbb{R}$ and the induced map $\mathbb{R} \to S^1$ is a motivating example for the study of G-bundles. These objects, which will be defined formally in chapter 2, are often mentioned in books on algebraic topology, but usually as examples of more general structures like covering spaces or fiber bundles, and rarely studied as objects per se. In the literature, G-bundles are sometimes known as G-covers, G-coverings or principal bundles, depending on the context.

In this thesis, we will study G-bundles from an algebraic-topological viewpoint, elucidating a connection between G-bundles over a fixed well-behaved topological space on one hand and Čech cohomology and its fundamental groupoid on the other. In the last chapters we specialize to connected base spaces to set up a Galois connection between G-bundles and normal subgroups of the fundamental groups. Besides these results, we also give some applications of this theory to construct for example the orientation bundle over manifolds and the universal cover over well-behaved spaces. In the last chapter, we use the theory of G-bundles to give an alternative proof of the Seifert-van Kampen theorem, due to Grothendieck.

The contents of this thesis are meant to be understandable to an undergraduate mathematics student. In terms of preliminary knowledge, the text assumes familiarity with basic point-set topology and concepts like compactness and local connectedness. From algebraic topology, the concept of the fundamental group will often be used. We will also need some group theory, but this will be limited to some basic facts about actions and normal subgroups. In fact, if one has no problems understanding the first two paragraphs of this introduction, then one will probably have no problem with the rest of this thesis either. The main exception is chapter 4, in which we will consider manifolds. However, the later chapters are largely independent from the results in this chapter, and readers who are not familiar with manifolds may skip this chapter without loss of continuity. Another exception is chapter 5, where we will borrow some language from category theory.

Chapter 2

G-bundles: Definition and First Properties

We begin by defining G-bundles over a given topological space. For this we need the notion of an even action of a group on a space.

All groups are assumed to be equipped with the discrete topology.

Definition 2.1. Let G be a group and Y a topological space. An *even action*[1] on Y is an action of G on Y such that the map $G \times Y \to Y$ given by $(g, y) \mapsto g \cdot y$ is continuous, and such that every point $y \in Y$ has an open neighbourhood $y \in U \subset Y$ such that $U \cap gU = \emptyset$ for every $g \in G$ with $g \neq e$.

Examples 2.2.

1. The action of $\mathbb{Z}$ on $\mathbb{R}$ by translation is even: for every point $y \in \mathbb{R}$ the neighbourhood $U = (y - \frac{1}{2}, y + \frac{1}{2})$ satisfies the above criterium. Likewise, we get an even action of the cyclic group C_n of n elements on $\mathbb{C}^*$ by identifying C_n with the nth roots of unity and multiplying.
2. The action of $\mathbb{Q}$ on $\mathbb{R}$ by translation is not even, since every non-empty open $U \subset \mathbb{R}$ intersects $q+U$ for some $q \in \mathbb{Q}$. Also, the action of C_n on $\mathbb{C}$ by multiplication by nth roots of unity is not even (assuming $n > 1$): since 0 is a fixed point, no neighbourhood of 0 can satisfy the criterium in the definition.

Now that we have defined an even action, we can define the concept of a G-bundle.

Definition 2.3. Let G be a group and X be a topological space. A *G-bundle over X* is a topological space Y, an even action of G on Y and a map $p : Y \to X$ such that there exists a homeomorphism $\varphi : X \xrightarrow{\sim} G \setminus Y$ such that the composition $\varphi \circ p$ maps each $y \in Y$ to its orbit $\overline{y}$. We call X the *base space*, Y the *covering space* and p the *projection map*.

A few examples:

Examples 2.4.

1. A rather trivial example is given by $Y = X \times G$ for any topological space X and any discrete group G. We let G act on Y by $g \cdot (x, h) = (x, gh)$ and define $p : Y \to X$ by $(x, g) \mapsto x$. This example (and any G-bundle isomorphic to it, see definition 2.6) is called a *trivial* G-bundle over X.
2. The prototypical G-cover is the case where $G = \mathbb{Z}$ is the addition group of the integers, $Y = \mathbb{R}$ is the real line, $X = S^1$ is the circle, and the projection map $p : Y \to X$ is given by $x \mapsto \exp(2\pi i \cdot x)$. The reader is invited to check that this does indeed satisfy the definition of a G-bundle.
3. Using the same group, we can extend the previous example to an annulus $X = \{z \in \mathbb{C}$:

[1] The word 'even' was introduced by Fulton in [Ful95], and is used in [Sza09] as well. The traditional term 'properly discontinuous' is also still common.

$\frac{1}{2} \leq |z| \leq 2\}$ if we use

$$Y = \left\{ z \in \mathbb{C} : -\frac{\ln(2)}{2\pi} \leq \Im(z) \leq \frac{\ln(2)}{2\pi} \right\}$$

as covering space and $x \mapsto \exp(2\pi i \cdot x)$ as projection map.

4. In the same way we can construct $\mathbb{Z}$-bundles over various different annuli in $\mathbb{C}$ using the exponent map. But we should note the importance of the role played by the hole in the centre. In fact, we will soon show that there are no non-trivial $\mathbb{Z}$-bundles (or any non-trivial G-bundles for any G) over the disk $D = \{z \in \mathbb{C} : |z| \leq 1\}$, for example.

A few easy consequences follow from this definition:

Lemma 2.5. *Let G be a group and $p : Y \to X$ be a G-bundle over X.*

1. *The projection map p is open (meaning that open sets are mapped to open sets) and surjective.*
2. *Let $A \subset X$ be a subset, and define $B = p^{-1}(A)$. Then the restriction $p|_B : B \to A$ again defines a G-bundle, called the restriction of the bundle to A.*
3. *For each $x \in X$, there exists an open neighbourhood $U \subset X$ of x such that the restriction of $p : Y \to X$ to U is a trivial G-bundle (thus, any G-bundle is locally trivial).*
4. *For each $x \in X$ the action of G on Y restricts to the set $p^{-1}(x)$ (called the fiber over x), and the restricted action is both transitive (meaning that for each $y, y' \in p^{-1}(x)$ there is some $g \in G$ such that $y' = gy$) and free (meaning that $gy \neq hy$ whenever $g \neq h$, for each $y \in p^{-1}(x)$).*

Proof. Property 1 clearly holds for the map $Y \to G \setminus Y$ and therefore also for $Y \to X$. Property 2 follows from noting that $p^{-1}(A)$ is the union of orbits and therefore closed under action of G on Y. Property 3 is a direct consequence of the definition of an even action. In property 4 transitivity follows from the definition of an orbit and freedom follows from evenness of the action. □

Note that property 1 and 3 together imply that the projection is a *local homeomorphism* (i.e. every point $y \in Y$ has an open neighbourhood U such that $p(U)$ is open in X, and that p is an homeomorphism between U and $p(U)$). Therefore X and Y have the same local properties.

Now that we have defined G-bundles, a natural next step is to define morphisms of G-bundles.

Definition 2.6. Let $p : Y \to X$ and $p' : Y' \to X$ be G-bundles over X. A continuous map $\varphi : Y \to Y'$ is called a *morphism of G-bundles over X* if $p = p' \circ \varphi$, and $\varphi(gy) = g\varphi(y)$ for each $g \in G$ and each $y \in Y$. If φ is an homeomorphism, then φ is called an *isomorphism of G-bundles over X*.

Example 2.7. Let $p : Y \to X$ be a G-bundle, and suppose G is abelian. Then the map $\lambda_g : Y \to Y$ given by $y \mapsto g \cdot y$ is a morphism, since it is continuous by definition 2.1 and satisfies $\lambda_g(hy) = ghy = hgy = h\lambda_g(y)$. More generally, when G is not abelian, then λ_g is a morphism if and only if g is an element of the center of G.

This definition turns the collection of G-bundles over a base space X into a category, which we will denote $\mathrm{Bun}(G, X)$. A rather surprising consequence of the definition of the morphisms is that every morphism is an isomorphism:

Proposition 2.8. *Let $\varphi : (Y, p) \to (Y', p')$ be a morphism of G-bundles over X. Then φ is an isomorphism.*

Proof. Let $a, b \in Y$ be such that $\varphi(a) = \varphi(b)$. Then we must have $p(a) = p(b)$, so a and b are in the same fiber of the bundle. Therefore there is some $g \in G$ such that $b = g \cdot a$, and therefore $\varphi(b) = g \cdot \varphi(a)$. Since the action of G on Y' is free, this implies that $g = e$, and therefore $a = b$, so φ is injective. On the other hand, let $y' \in Y'$ be given. Let y be any point in the fiber $p^{-1}(p'(y'))$. Then $\varphi(y)$ and y' must be in the same fiber, so again there is some $g \in G$ such that $y' = g \cdot \varphi(y)$. But then $y' = \varphi(gy)$, so φ is surjective as well.

To show that φ is open, we note that this is clearly the case when Y and Y' are equal to the trivial G-bundle $X \times G$. Since any G-bundle is locally trivial, this means that φ is locally open. But being open is a local property, so φ is open. Therefore, φ is a homeomorphism, and thus a isomorphism of G-bundles. □

We saw in lemma 2.5 that every G-bundle is locally trivial. A reverse of this assertion is also true:

Lemma 2.9. *Let G be a group, Y be a G-space, X be a topological space and $p : Y \to X$ be a continuous map such that every point $x \in X$ has an open neighbourhood $U \subset X$ such that $p^{-1}(U) \to U$ defines a trivial G-bundle over U. Then $p : Y \to X$ is a G-bundle over X.*

Proof. Let $x \in X$ be a point and $U \subset X$ an open neighbourhood of x as above. Then there is some isomorphism $\varphi : U \times G \to p^{-1}(U)$ such that $p \circ \varphi$ is the projection on the first coordinate. Since φ is by assumption an isomorphism of G-bundles, it is a G-map, and therefore we find $\varphi(y, g) = g \cdot \varphi(y, e)$ for every $y \in U$ and $g \in G$.

Assume that there is some $g \in G$ with $gU \cap U \neq \emptyset$. Let $y \in gU \cap U$, we then have $y \in U$ and $g^{-1}y \in U$. But since $p(g^{-1} \circ \varphi(y, e)) = p(\varphi(y, g^{-1})) = y$ it follows that $g \cdot y = y$. But $p^{-1}(U)$ is isomorphic to $U \times G$, and in the latter the action of G is free, so it should be free in $p^{-1}(U)$ as well. Therefore, from $g \cdot y = y$ it follows that $g = e$. Thus we only have $gU \cap U \neq \emptyset$ in the case $g = e$, and since $x \in X$ was arbitrary, this proves that the action of G on Y is even.

Now we define $\varphi : X \to G \setminus Y$ by $x \mapsto p^{-1}(x)$. This map is well-defined since it is locally well-defined: for every $x \in X$ we can find an open neighbourhood U of x as in the lemma and an isomorphism $\varphi : U \times G \to p^{-1}(U)$ as above. Then for every $y \in p^{-1}(x)$ we have $y = \varphi(x, g)$ for some $g \in G$, and therefore, y' is in the same orbit as y if and only if $p(y') = x$, for in that case we have $y' = \varphi(x, h)$ for some $h \in G$. This shows that $p^{-1}(x)$ is exactly the orbit of y, so φ is well-defined. As similar argument shows that φ is both continuous and open, since it locally has those properties. Also, φ is bijective, since it is easily checked that $\bar{y} \mapsto p(y)$ is the inverse of φ. Therefore, φ is a homeomorphism between X and $G \setminus Y$ with $\varphi \circ p$ equal to the canonical projection $Y \to G \setminus Y$. Thus $p : Y \to X$ is an G-bundle. □

Suppose we have a G-bundle $p : Y \to X$. Then by definition, the projection map induces a homeomorphism $G \setminus Y \to X$, but as we have seen in the examples, this does not generally imply that Y is homeomorphic to $G \times X$. This situation is somewhat similar to a short exact sequence in group theory. If we have a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ of groups, then by the isomorphism theorems, this implies that g induces a isomorphism $B/f[A] \to C$, but not in general that B is isomorphic to $A \times C$. However, the short exact sequence of groups is split if and only if g has a right inverse. In the case of G-bundles, the analogous statement is true as well.

Proposition 2.10. *Let G be a group, X be a topological space and $p : Y \to X$ a G-bundle. Then Y is trivial if and only if there exists an continuous map $s : X \to Y$ such that $p \circ s = \mathrm{id}_X$ (such a map is called a* section*).*

Proof. If Y is trivial, then there is a G-bundle isomorphism $\varphi : X \times G \to Y$. The map $s : X \to Y$ defined by $x \mapsto \varphi(x, e)$ is then as we wanted.

Now assume we are given a continuous map $s : X \to Y$ satisfying $p \circ s = \mathrm{id}_X$. Then the map $\varphi : X \times G \to Y$ defined by $(x, g) \mapsto g \cdot s(x)$ is continuous, since it is the composition of the continuous map $(x, g) \mapsto (s(x), g)$ with the continuous map $(y, g) \mapsto g \cdot y$. Also, for each $x \in X$ and $g, h \in G$ we have $g \cdot \varphi(x, h) = gh \cdot s(x) = \varphi(x, gh)$, so φ is G-equivariant. Thus φ is a morphism of G-bundles, and by proposition 2.8 φ is an isomorphism. $\square$

The idea underlying the previous proposition is the following: G acts freely and transitively on each fiber. If we would choose some base point y_0 in some fiber $p^{-1}(x)$, then that would give us a canonical bijection $G \to p^{-1}(x)$ by $g \leftrightarrow g \cdot y_0$. However, this bijection depends on the choice of base point. A section $s : X \to Y$ as in the proposition essentially defines a choice of base point for each fiber. The continuity of s then makes sure that the bijections between G and each fiber are 'aligned', in the sense that we can regard Y as a product of G with X.

Chapter 3

Construction and Classification of G-bundles[1]

Assume we are given a group G and a base space X. We would like to know which spaces Y and which maps $p : Y \to X$ can occur as G-bundles over X (up to isomorphism, of course). In general questions like these are rather hard to answer, but in this case we are lucky, as long as the topology of X is nice enough. In this chapter, we will show a classification of G-bundles using so-called Čech cocycles. This approach uses the fact that the G-bundles over X are easily classified if X happens to be simply connected.

In this chapter and in the rest of the thesis, we take *simply connected* to mean 'non-empty, path-connected, and every loop is homotopic (with fixed endpoints) to a constant loop'. In particular, our notion of simple connectedness assumes path connectedness.

Theorem 3.1. *Let G be a group and X be a simply connected and locally path-connected topological space. Then every G-bundle over X is trivial.*

The proof of this theorem relies on the following lemma on lifting paths and homotopies, which we will quote without proof (lemma 2.3.2 in [Sza09]).

Lemma 3.2. *Let $p : Y \to X$ be a G-bundle, let $y \in Y$ and let $x = p(y)$.*

1. *Given a path $\gamma : [0,1] \to X$ with $\gamma(0) = x$, there is a unique path $\widetilde{\gamma} : [0,1] \to Y$ with $\widetilde{\gamma}(0) = y$ and $p \circ \widetilde{\gamma} = \gamma$ called the* lifting *of γ.*
2. *Homotopic paths in X have homotopic liftings in Y. In particular, the endpoints of the lifting are the same.*

Proof of theorem 3.1. Suppose $p : Y \to X$ is a G-bundle. Let $y_0 \in Y$ be any point, and define $x_0 = p(y_0)$. For each $x \in X$ we let $s(x)$ be the endpoint in Y of the lifting of any path from x_0 to x. The lemma guarantees that $s(x)$ does not depend on the path chosen, since any path from x_0 to x is homotopic to any other path from x_0 to x. This defines a map $s : X \to Y$ which clearly satisfies $p \circ s = \mathrm{id}_X$. Notice that the image of s equals the path-connected component of Y containing y_0: every point in Y that can be connected to y_0 by some path γ is the endpoint of the unique lift of the path $p \circ \gamma$ in X, and a point that can not be connected to y_0 by any path in Y can not be the endpoint of the lifting of a path in X if this lifting starts at y_0. In particular this implies that the image of s is open, since X is locally path-connected. Now, let $V \subset Y$ be any open set. Then we have $s^{-1}(V) = p(V \cap s(X))$. Since $s(X)$ is open, $V \cap s(X)$ is open as well. Since p is an open map, this means that $s^{-1}(V)$ is open, so s is continuous. The claim now follows from proposition 2.10. □

Another ingredient we will need is the following classification of the automorphisms of a trivial bundle.

[1] This chapter follows the lines of [Ful95], paragraph 15

Lemma 3.3. *Let G be a group and X a topological space. Let $Y = X \times G$ be the trivial bundle and let $\varphi : Y \to Y$ be a morphism of G-bundles. Then there is a unique locally constant function $f : X \to G$ such that $\varphi(x, g) = (x, g \cdot f(x))$.*

Proof. It is clear that there is at most one such f, so it suffices to give one. Define $s : X \to Y$ by $x \mapsto (x, e)$ and $\pi : Y \to G$ by $(x, g) \mapsto g$, and let $f = \pi \circ \varphi \circ s$. Since each of these maps is continuous, it follows that f is also continuous, and since G is discrete, f must be locally constant. Now, for each $(x, g) \in Y$ we have

$$\pi(\varphi(x, g)) = \pi(g \cdot \varphi(x, e)) = g \cdot (\pi \circ \varphi \circ s)(x) = g \cdot f(x),$$

and since φ must preserve the first coordinate, we find that $\varphi(x, g) = (x, g \cdot f(x))$. □

The following definition will be instrumental in what follows. Recall that an *open cover* $\mathcal{U}$ of some topological space X is a collection of open subsets of X such that X is the union of these subsets.

Definition 3.4. Let X be a topological space and $\mathcal{U}$ be an open cover of X. Then $\mathcal{U}$ is called *good* if each element of $\mathcal{U}$ is simply connected and locally simply connected. Notice that X has a good cover if and only if it is locally simply connected.

Now, if X is covered by a good cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$, then any G-bundle over X must be trivial over each U_α by theorem 3.1. Therefore, we can think of Y as a union of spaces $G \times U_\alpha$ for $\alpha \in A$, pasted together in some way. This pasting is based on the behaviour of the bundle on the overlaps $U_\alpha \cap U_\beta$, and can be descriped nicely using so-called Čech cocycles.

Definition 3.5. Let X be a locally simply connected space and $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a good cover of X. Then a *Čech cocycle on $\mathcal{U}$ with coefficients in G* is a collection $(c_{\alpha\beta})_{\alpha,\beta \in A}$ of locally constant functions $c_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$ satisfying the so-called *cocycle condition*:

$$c_{\alpha\gamma}(x) = c_{\alpha\beta}(x) \cdot c_{\beta\gamma}(x) \tag{3.1}$$

for every $x \in U_\alpha \cap U_\beta \cap U_\gamma$.

Two Čech cocycles $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$ are called *cohomologous* if for each $\alpha \in A$ there is a locally constant function $h_\alpha : U_\alpha \to G$ such that

$$h_\alpha(x) \cdot d_{\alpha\beta}(x) = c_{\alpha\beta}(x) \cdot h_\beta(x) \tag{3.2}$$

holds for every $x \in U_\alpha \cap U_\beta$ and every $\alpha, \beta \in A$.

Note that 'being cohomologous' defines a equivalence relation on the set of Čech cocycles. The equivalence classes are called Čech cohomology classes and the set of these classes is denoted $H^1(\mathcal{U}; G)$. Also note that from equation 3.1 it follows that $c_{\alpha\alpha}(x) = e$ for each $\alpha \in A$ and each $x \in U_\alpha$, and that $c_{\alpha\beta}(x) = c_{\beta\alpha}(x)^{-1}$ for every $\alpha, \beta \in A$ and every $x \in U_\alpha \cap U_\beta$.

At first, these definitions may seem a bit bewildering and the reader may feel that these cocycles have little to do with G-bundles. However, the opposite is true. As it turns out, these cohomology classes can be used to perfectly describe G-bundles. The rest of this chapter will be devoted to two constructions. First, we show how we can use Čech cocycles as 'pasting data' to glue together a G-bundle, and then we show how to reverse the process.

Construction 3.6 (From Čech cocycle to G-bundle)**.** Let G be a group, let X be a locally simply connected space with a good cover $\{U_\alpha\}_{\alpha \in A}$ and let $(c_{\alpha\beta})$ be a Čech cocycle on $\mathcal{U}$ with coefficients in G. We define the space Y' as

$$Y' = \{(x, g, \alpha) \in X \times G \times A : x \in U_\alpha\},$$

where we give both G and A the trivial topology and Y' the induced topology. Now we define a relation $\sim$ on Y' as follows:

$$(x, g, \alpha) \sim (y, h, \beta) \quad \text{iff} \quad x = y \text{ and } g = h \cdot c_{\beta\alpha}(x).$$

The cocycle condition guarantees that $\sim$ is an equivalence relation. We let $Y = Y'/\sim$, and define $p : Y \to X$ by $\overline{(x, g, \alpha)} \mapsto x$. Furthermore, we let G act on Y by left multiplication on the second coordinate: $g \cdot \overline{(x, h, \alpha)} = \overline{(x, gh, \alpha)}$. It is easily checked that both p and the action of G on Y are well-defined: choosing different representatives in both definitions still results in the same equivalence classes.

We claim that $p : Y \to X$ is a G-bundle. For this, we first note that p is continuous. Now, let $x \in X$ be given, and let $\alpha \in A$ be such that $x \in U_\alpha$. Then the inverse image of U_α under p consists the equivalence classes of points $(y, g, \beta) \in Y'$ for each $y \in U_\alpha$, each $g \in G$ and each β such that $y \in U_\beta$. But such a (y, g, β) is equivalent to the point $(y, g \cdot c_{\beta\alpha}(x), \alpha)$, so every equivalence class in $p^{-1}(U_\alpha)$ contains a point labeled with α as third coordinate. On the other hand, (y, g, α) and (y, h, α) can only be equivalent if $g = h$, so every equivalence class in $p^{-1}(U_\alpha)$ contains exactly one point with α as third coordinate. In particular, we can define a map $\varphi : p^{-1}(U_\alpha) \to U_\alpha \times G$ by mapping the equivalence class of (y, g, α) to (y, g). This map is easily seen to be a homeomorphism, and this shows that the restriction of p to $p^{-1}(U_\alpha)$ gives a trivial G-bundle over U_α. Since $x \in X$ was arbitrary, we can invoke lemma 2.9, and find that $p : Y \to X$ is a G-bundle.

And now for the reverse:

Construction 3.7 (From G-bundle to cohomology class)**.** Let G be a group, let X be a locally simply connected space with a good cover $\{U_\alpha\}_{\alpha\in A}$ and let $p : Y \to X$ be a G-bundle over X. By theorem 3.1, the restriction of the G-bundle to each U_α is trivial, since each U_α is simply connected by assumption. Therefore, we can for each $\alpha \in A$ find a G-bundle isomorphism $\varphi_\alpha : U_\alpha \times G \to p^{-1}(U_\alpha)$. For $\alpha, \beta \in A$ the restrictions of φ_α and φ_β to the overlap $U_\alpha \cap U_\beta$ give rise to a transition isomorphism

$$(U_\alpha \cap U_\beta) \times G \xrightarrow{\varphi_\alpha} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{\varphi_\beta^{-1}} (U_\alpha \cap U_\beta) \times G$$

of trivial G-bundles. Applying lemma 3.3 on this isomorphism gives a unique locally constant function $c_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ such that the transition map can be written as $(x, g) \mapsto (x, g \cdot c_{\alpha\beta}(x))$. Doing this for all $\alpha, \beta \in A$ gives a collection $(c_{\alpha\beta})$ of locally constant functions. The cocycle condition follows from the identity $\varphi_\gamma^{-1} \circ \varphi_\alpha = \varphi_\gamma^{-1} \circ \varphi_\beta \circ \varphi_\beta^{-1} \circ \varphi_\alpha$, which holds on all of $U_\alpha \cap U_\beta \cap U_\gamma$. Therefore, $(c_{\alpha\beta})$ defines a Čech cocycle on $\mathcal{U}$.

But this cocycle is not uniquely determined: it depends on our choice of trivialisations φ_α. If we had chosen another set $\{\varphi'_\alpha\}$ of isomorphisms $\varphi'_\alpha : U_\alpha \times G \to p^{-1}(U_\alpha)$, we would have found other transition maps on the overlaps, and we would have ended up with another cocycle $(d_{\alpha\beta})$. We claim now, however, that $(c_{\alpha\beta})$ is cohomologous to $(d_{\alpha\beta})$. To see this, we note that for each $\alpha \in A$ the composition $\varphi'^{-1}_\alpha \circ \varphi_\alpha$ defines an isomorphism $U_\alpha \times G \to U_\alpha \times G$. Applying lemma 3.3 again gives us a locally constant function $h_\alpha : U_\alpha \to G$ for each $\alpha \in A$, such that ${\varphi'_\alpha}^{-1} \circ \varphi_\alpha$ is given by $(x, g) \mapsto (x, g \cdot h_\alpha(x))$. It is quickly checked that for each $\alpha, \beta \in A$ and each $x \in U_\alpha \cap U_\beta$ these functions satisfy $h_\alpha(x) \cdot d_{\alpha\beta}(x) = c_{\alpha\beta}(x) \cdot h_\beta(x)$, showing that the two cocycles are indeed cohomologous.

We now know how to build a G-bundle from a Čech cocycle, but creating a Čech cocycle from a G-bundle works only up to cohomology. But the situation is not as bad as it may seem: cocycles in the same cohomology class give isomorphic G-bundles.

Lemma 3.8. *Let $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$ be cohomologous Čech cocycles on some good cover $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ of some locally simply connected space X with coefficients in some group G. Then the G-bundles constructed from $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$ using construction 3.6 are isomorphic.*

Proof. Let $p : Y \to X$ and $p' : Y' \to X$ be the G-bundles we construct from $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$, respectively. We will show that Y and Y' are isomorphic.

For each $\alpha \in A$ we let $h_\alpha : U_\alpha \to G$ be a locally constant function such that $h_\alpha(x) \cdot d_{\alpha\beta}(x) = c_{\alpha\beta}(x) \cdot h_\beta(x)$ holds for each $x \in U_\alpha \cap U_\beta$. Then we define a map $\varphi : Y \to Y'$ by sending the equivalence class of a point (x, g, α) to the class of $(x, g \cdot h_\alpha(x), \alpha)$. To check that this map is well-defined, let (x, g, α) and (x, h, β) elements of the same equivalence class in Y. Then we must have $h = g \cdot c_{\alpha\beta}(x)$. Then we have

$$\begin{aligned}
\varphi\overline{(x, h, \beta)} &= \overline{(x, h \cdot h_\beta(x), \beta)} \\
&= \overline{(x, g \cdot c_{\alpha\beta}(x) h_\beta(x), \beta)} \\
&= \overline{(x, g \cdot h_\alpha(x) d_{\alpha\beta}(x), \beta)} \\
&= \overline{(x, g h_\alpha(x), \alpha)} = \varphi\overline{(x, g, \alpha)},
\end{aligned}$$

so φ is well-defined. It is clear that φ is continuous, since each h_α is locally constant. Also, φ is G-equivariant, as a quick verification shows. Therefore, φ is a morphism of G-bundles, and by proposition 2.8 it is an isomorphism. □

So now we can convert G-bundles over locally simply connected spaces to Čech cohomology classes, and in the other direction we can build G-bundles from cohomology classes. The following theorem, which is the main result of this chapter, gives us the relation between these two constructions.

Theorem 3.9.

1. *Let $Y \to X$ be a G-bundle, and construct a new G-bundle $Y' \to X$ by first applying construction 3.7 and then applying construction 3.6. Then Y' and Y are isomorphic as G-bundles.*
2. *Let $\overline{(c_{\alpha\beta})}$ be a Čech cohomology class, and construct a new class $\overline{(d_{\alpha\beta})}$ by applying construction 3.6 and 3.7. Then $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$ are cohomologous.*

Proof.

1. Let $p : Y \to X$ be a G-bundle, let $\varphi_\alpha : U_\alpha \times G \to p^{-1}(U_\alpha)$ be as in construction 3.7, and let $(c_{\alpha\beta})$ be the resulting Čech cocycle. Then by construction we have

$$\varphi_\alpha(x, g) = \varphi_\beta(x, g \cdot c_{\alpha\beta}(x)) \tag{3.3}$$

for each $g \in G$ and each $x \in U_\alpha \cap U_\beta$.

Let Y' be the G-bundle constructed from $(c_{\alpha\beta})$. Then we define a map $\psi : Y' \to Y$ by

$$\overline{(x, g, \alpha)} \longmapsto \varphi_\alpha(x, g).$$

This is well-defined exactly because of identity 3.3. It is continuous because it is continuous on $p^{-1}(U_\alpha)$ for each $\alpha \in A$, it is G-equivariant because every φ_α is, and it is clearly compatible with the projections, so ψ is a morphism of G-bundles, and therefore an isomorphism.

2. Let $(c_{\alpha\beta})$ be a Čech cocycle, and let $p : Y \to X$ be the G-bundle constructed from it. For each $\alpha \in A$ we define $\varphi_\alpha : U_\alpha \times G \to p^{-1}(U_\alpha)$ by

$$(x, g) \longmapsto \overline{(x, g, \alpha)}$$

This map is clearly continuous, G-equivariant and compatible with the projection maps, so φ_α is a isomorphism of G-bundles over U_α, for each $\alpha \in A$.

For every $\alpha, \beta \in A$ and every $x \in U_\alpha \cap U_\beta$ we have

$$\varphi_\alpha(x, g) = \overline{(x, g, \alpha)} = \overline{(x, g \cdot c_{\alpha\beta}(x), \beta)} = \varphi_\beta(x, g \cdot c_{\alpha\beta}(x)),$$

so $(c_{\alpha\beta})$ is the cocycle that we construct by using the φ_α as trivialisations in construction 3.7. In particular, every other trivialisation leads to a Čech cocycle that is cohomologous to $(c_{\alpha\beta})$, by the last paragraph of construction 3.7. □

Corollary 3.10. *For a group G and a locally simply connected space X with good cover $\mathcal{U}$, constructions 3.6 and 3.7 define bijections between the set of isomorphism classes of G-bundles over X and the set $H^1(\mathcal{U}; G)$ of Čech cohomology classes on $\mathcal{U}$ with coefficients in G.*

Corollary 3.11. *Let G be a group and X a locally simply connected space. Let $\mathcal{U}$ and $\mathcal{U}'$ be good covers. Then there is a canonical bijection between $H^1(\mathcal{U}; G)$ and $H^1(\mathcal{U}'; G)$.*

It has been some work, but we've got what we came for: theorem 3.9 gives a complete classification of G-bundles over spaces that are locally simply connected, in terms of simpler objects. But what if some space X isn't locally simply connected? Well, if $\mathcal{U}$ is any cover of any space X, we can still apply construction 3.6 to any cocycle on $\mathcal{U}$, and that will still give us a G-bundle over X. On the other hand, we can also apply construction 3.7 to any G-bundle over X, provided it is trivial over each open in $\mathcal{U}$. In fact, with some minor adjustments much of the work done in the last few pages can be applied in this case as well to construct a bijection between $H^1(\mathcal{U}; G)$ and isomorphism classes of G-bundles over X *that are trivial over each $U \in \mathcal{U}$.*

But we know from lemma 2.5 that every G-bundle must be trivial over *some* open cover $\mathcal{U}$, and then also over each refinement of $\mathcal{U}$. By taking more and more refined open covers, we can describe G-bundles that are trivial over smaller and smaller opens. We can restrict Čech cocycles on open covers to more refined open covers, and these restriction maps define a so-called direct system, allowing us to take a direct limit. In this limiting process, we finally obtain a correspondence between G-bundles and Čech cohomology. However, this correspondence lacks some of the elegance and calculability of the situation with locally simply connected spaces.

Chapter 4

G-bundles and Manifolds

In the previous chapter we have seen a connection between G-bundles and Čech cocycles. To obtain this connection, we made use of what we called good covers of X: a covering of X by simply connected and locally simply connected opens. In this chapter, we study an application of this theory to manifolds. By definition, a manifold comes with an atlas consisting of charts, and these charts have domains that are already locally simply connected and can be shrunk to be simply connected. In other words, each manifold comes with a natural good cover.

In this chapter by manifolds we will mean smooth second-countable Hausdorff manifolds. Readers not familiar with the ideas and concepts of smooth manifolds can find the necessary background in the first few chapters of the excellent book of Jänich [Jän01].

Definition 4.1. Given a manifold M, a *good atlas* for M will be an atlas for M where every chart domain is simply connected.

Clearly, a good atlas is an example of a good cover in the sense of definition 3.4. The next lemma shows that every manifold has such an atlas.

Lemma 4.2. *Every manifold has a good atlas.*

Proof. We only need to show that we can cover chart domains with simply connected subdomains. Since every chart domain is by definition homeomorphic to an open subset of Euclidean space, it is enough to show that every open subset of Euclidean is the union of simply connected opens. But a subset of Euclidean space is open exactly when it is a union of open balls, and open balls are simply connected. □

Given a G-bundle $p : Y \to X$, we know from Chapter 2 that p is a local homeomorphism. Since being a manifold is mainly a local property, a manifold structure on X can often be lifted to Y. The only catch is that Y will not be second countable if G is too big.

Proposition 4.3. *Let X be a manifold and let $p : Y \to X$ be a G-bundle over X. If G is countable, then the manifold structure of X lifts to Y.*

Proof. First, we prove that Y is Hausdorff and second-countable. Let $x, y \in Y$ with $x \neq y$. If x and y are in the same fiber, then we can find a neighbourhood U of $p(x)$ over which the bundle is trivial. Then x and y are in different components of $p^{-1}(U)$, and these components separate x and y. If $p(x) \neq p(y)$, then $p(x)$ and $p(y)$ can be separated by opens (since X is Hausdorff), and lifting these opens separates x and y. Thus, Y is Hausdorff. Furthermore, the topology of X has a countable base $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$. We can assume that the bundle is trivial over each U_i. Then for each i, $p^{-1}(U_i)$ has countably many connected components, and it is easily checked that the set of all the connected components of all the U_i form a countable base for the topology of Y.

The rest is rather straightforward. Let $\mathcal{A}_X$ be a good atlas for X. If (U, φ) is a chart in $\mathcal{A}_X$, then the bundle is trivial over U, since we assumed that U is simply connected. Therefore $p^{-1}(U)$ is homeomorphic to $U \times G$. Thus every connected component of $p^{-1}(U)$ is homeomorphic to U, which is homeomorphic to some open subset of $\mathbb{R}^n$ via φ. Therefore we let $\mathcal{A}_Y$ consist of all pairs (U', φ') where U' is a connected component of $p^{-1}(U)$ for some chart $(U, \varphi) \in \mathcal{A}_X$, and $\varphi' = \varphi \circ p|_{U'}$. Then clearly this defines a differentiable structure on Y, and p is differentiable and of full rank relative to these charts. ☐

A particular G-bundle we will investigate is the so-called orientation bundle. This is a C_2-bundle that can be constructed over any manifold, and as the name suggests, it has to do with orientations.

Construction 4.4 (Orientation bundle)**.** Let X be a smooth manifold with a good atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$. We define a Čech cocycle on $\mathcal{A}$ with coefficients in $C_2 = \{\pm 1\}$ as follows. Let $\alpha, \beta \in A$ and $x \in U_\alpha \cap U_\beta$. We set $c_{\alpha\beta}(x)$ equal to the sign of the determinant of the Jacobian of the transition map $\varphi_\beta^{-1} \circ \varphi_\alpha$ at $\varphi_a(x)$. That is, if we let $Jf(p)$ denote the Jacobian matrix of a function f at a point p, then we let $c_{\alpha\beta}(x) = +1$ if $\left|J(\varphi_\beta \circ \varphi_\alpha^{-1})(\varphi_\alpha(x))\right|$ is positive, and $c_{\alpha\beta}(x) = -1$ if it is negative. Note that φ_α and φ_β are by assumption diffeomorphisms, so the determinant is never zero. Since the determinant of the Jacobian depends continuously on x and can never by zero, we find (by the intermediate value theorem) that $c_{\alpha\beta}$ is locally constant.

We only need to check the cocycle condition. But this is quickly checked: if $\alpha, \beta, \gamma \in A$ and $x \in U_\alpha \cap U_\beta \cap U_\gamma$ are given, then

$$J(\varphi_\gamma \circ \varphi_\alpha^{-1}) = J(\varphi_\gamma \circ \varphi_\beta^{-1} \circ \varphi_\beta \circ \varphi_\alpha^{-1}) = J(\varphi_\gamma \circ \varphi_\beta^{-1}) \cdot J(\varphi_\beta \circ \varphi_\alpha^{-1}),$$

implying $c_{\alpha\gamma}(x) = c_{\alpha\beta}(x) \cdot c_{\beta\gamma}(x)$.

Definition 4.5. The C_2-bundle constructed from the above cocycle by construction 3.6 will be called the *orientation bundle* over X.

Of course, we would like the orientation bundle to only depend on the differentiable structure for X, and be independent of the actual choice of atlas.

Lemma 4.6. *If two good atlases $\mathcal{A}$ and $\mathcal{B}$ for X are differentiably related, they induce the same orientation bundle.*

Proof. Without loss of generality, we may assume that $\mathcal{B}$ is the maximal good atlas. Then every chart in $\mathcal{A}$ is also a chart in $\mathcal{B}$. In particular, the overlaps between charts in $\mathcal{A}$ are also overlaps between charts in $\mathcal{B}$, and therefore the cocycle we obtain above using $\mathcal{B}$ as an atlas will agree with the cocycle obtained by using $\mathcal{A}$ on all charts in $\mathcal{A}$. Since X is already covered by the charts in $\mathcal{A}$, we see that the orientation bundles we construct using construction 3.6 must agree as well. ☐

Theorem 4.7. *Let X be a connected manifold, and let $p : Y \to X$ be the orientation bundle over X. Then Y is connected if and only if X is* not *orientable.*

Proof. Suppose X is orientable, and that $\mathcal{A}$ is an oriented good atlas for X. Lemma 4.6 tells us that we can freely choose any good atlas from which to construct the orientation bundle, so we might as well choose $\mathcal{A}$. But since $\mathcal{A}$ is oriented, every transition map has a positive Jacobian determinant, and so the cocycle in construction 4.4 is trivial. Thus, the orientation bundle over X is trivial, and a trivial C_2-bundle is not connected.

Now, suppose Y is not connected. We claim that for $y \in Y$, the points y and $-y$ are not connected by a path. To see this, let $x \in Y$ be arbitrary. Then there is a path γ connecting $p(x)$ with $p(y)$, and its lift $\widetilde{\gamma}$ to Y starting at x must terminate at either y or $-y$. If y and $-y$ are connected by a path, then this shows that there is some path connecting x and y. But $x \in Y$ was arbitrary, so this contradicts the fact that Y is not connected. So we see that Y is the disjoint union of two connected components U_1 and U_2, with the property that $x \in U_1 \iff -x \in U_2$, thus that $U_1 = -U_2$. Therefore, Y is the trivial C_2-bundle over X.

Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be the maximal good atlas for X, and let $(c_{\alpha\beta})$ be the Čech cocycle constructed in 4.4. Since the bundle corresponding to this Čech cocycle is trivial by the previous paragraph, the cocycle itself must be cohomologous to the trivial cocycle. Therefore, let $h_\alpha : U_\alpha \to C_2$ be locally constant functions such that $h_\alpha(x) = c_{\alpha\beta}(x) \cdot h_\beta(x)$ for all $x \in U_\alpha \cap U_\beta$. Note that the h_α are in fact constant functions. Removing all charts $(U_\alpha, \varphi_\alpha)$ with $h_\alpha = -1$, we obtain a collection $\mathcal{A}'$ of charts. This is an atlas for X, since for every map $(U_\alpha, \varphi_\alpha)$ that we remove, there is another chart $(U_\alpha, \varphi_\beta)$ in $\mathcal{A}$ with the same chart domain such that the transition map between them has negative determinant. Since $h_\alpha = -1$ and $c_{\alpha\beta}(x) = -1$ for all $x \in U_\alpha$, we find that $h_\beta = +1$. Thus, $\mathcal{A}'$ is an atlas for X that is compatible with the maximal atlas $\mathcal{A}'$. Moreover, one quickly checks that $\mathcal{A}'$ is in fact an oriented atlas, implying that X is orientable. □

In fact, more can be said about the structure of the orientation bundle. For example, the orientation bundle over a manifold is always orientable (given the natural manifold structure from proposition 4.3), independent of the orientability of the original manifold. The proof, which mainly consists of lifting chart domains and and studying their behaviour relative to the overlapping chart domains and their lifts, is easier to visualise than to describe, and will be left as an exercise to the reader. Another fact to note is that choosing a section of the orientation bundle (which exists if and only if the original manifold is orientable, by proposition 2.10) is the same as choosing an orientation of the manifold. In fact, one might even take this as a definition of an orientation.

As a consequence of 4.7, we have the following, somewhat surprising result, which has very little to do with G-bundles or Čech cohomology per se.

Corollary 4.8. *A simply connected manifold is orientable.*

Proof. If X is simply connected, then every C_2-bundle over X is trivial by theorem 3.1. In particular, its orientation bundle is trivial and hence not connected. □

In fact, in corollary 7.7 we will strengthen this result by showing that the fundamental group of any connected non-orientable manifold has a normal subgroup of index 2.

Chapter 5

G-bundles and the Fundamental Group

In chapter 3 we showed that we can think of G-bundles over some locally simply connected space X in terms of Čech cocycles and their cohomology relation. In fact, when one looks at the definition of this relation (equation 3.2), then one might notice the similarity with the definition of natural isomorphisms between functors in category theory. Therefore, we might suspect that Čech cocycles are in fact disguised functors from some category to another. In this chapter, we will show that this is indeed the case. The Čech cocycles turn out to be functors on the so-called fundamental groupoid, which contains informations of all paths through X, modulo path homotopy. As the name suggests, there is some relation between the fundamental groupoid of a space and its fundamental group, and in the case that X is connected, we will show that Čech cocycles define group morphisms from $\pi_1(X)$ to G. These results can also be found in [Ful95], paragraph 14. The use of groupoids in this chapter is inspired by the book of R. Brown on the use of groupoids in topology, see [Bro06].

Definition 5.1. A *groupoid* is a small category (i.e. a category in which the class of objects form a set) in which every morphism is invertible. A *group* is a groupoid with exactly one object.

Note that this definition coincides with the traditional definition of a group: every group can be interpreted as a groupoid with one object, and vice versa. From now on, we will view every group as a one-point groupoid.

Definition 5.2. Let X be a topological space, and $B \subset X$ a non-empty subset of X. The *fundamental groupoid of X with base B* (notation: $\Pi_1(X, B)$) is the category with as its objects the elements of B, and as morphisms the homotopy classes of paths between the elements of B, with natural composition[1].

Since any path can be inverted, the above indeed is a groupoid. In the case that $B = X$, we usually write $\Pi_1(X)$ instead of $\Pi_1(X, X)$. Note that we have $\pi_1(X, x_0) = \Pi_1(X, \{x_0\})$ for every $x_0 \in X$, so we can view the fundamental groupoid as a generalisation of the fundamental group, where we allow for a set of base points instead of just one.

We will fix some notation to avoid repeatedly declaring the same objects over and over again.

Notation 5.3. In the rest of this chapter use the following notation. Let G be a discrete group, let X be a locally simply connected space with a good cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ and let $(c_{\alpha\beta})$ be a Čech cocycle on $\mathcal{U}$ with coefficients in G. Let $p : Y \to X$ be the G-bundle defined by applying construction 3.6 to $(c_{\alpha\beta})$. For each $\alpha \in A$ we choose an $x_\alpha \in U_\alpha$, and define $B = \{x_\alpha\}$.

[1] To clarify, the composition of two classes $[\gamma_1]$ and $[\gamma_2]$ is defined to be $[\gamma_1 \oplus \gamma_2]$, where $\gamma_1 \oplus \gamma_2$ is defined to be the path that *first* traverses γ_1, and *then* traverses γ_2. This seems to be the usual convention in algebraic topology, but it contrasts with the composition rules in category theory, where $f \circ g$ usually means 'first g, then f'. So to fit paths and homotopy classes into the framework of category theory, each class of paths from x to y should actually correspond to a morphism $y \to x$ in the fundamental groupoid. However, this subtle point will not be of importance in the following discussion.

We will show in the rest of this chapter that a Čech cocycle essentially defines a functor from $\Pi_1(X, B)$ to G (remember, every group is a category now, so it makes sense to speak of functors to G).

Construction 5.4. (From Čech cocycle to functor). Let notation be as in 5.3. For each pair $x_\alpha, x_\beta \in B$ and every path $\gamma : I \to X$ from x_α to x_β we define $F_c([\gamma])$ as follows. We let $\widetilde{\gamma}$ be the lift of γ to Y that starts at $\overline{(x_\alpha, e, \alpha)}$, which exists and is unique by lemma 3.2. Then $\widetilde{\gamma}(1)$ is of the form $\overline{(x_\beta, g, \beta)}$ for a unique $g \in G$. Define $F_c([\gamma]) = g$. We will show that this defines a well-defined functor $F_c : \Pi_1(X, B) \to G$.

If $\gamma, \gamma' : I \to X$ are path-homotopic paths from x_α to x_β, then their lifts have the same endpoint by lemma 3.2, so we have $F_c([\gamma]) = F_c([\gamma'])$, which means that F_c is well-defined.

Suppose $\gamma_1 : I \to X$ is a path from x_α to x_β, and $\gamma_2 : I \to X$ is a path from x_β to x_γ. Let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be the lifts of γ_1 and γ_2 starting at $\overline{(x_\alpha, e, \alpha)}$ and $\overline{(x_\beta, e, \beta)}$, respectively. Note that we cannot compose $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ directly if $F_c([\gamma_1]) \neq e$. But if we define $\widetilde{\gamma}'_2$ to be the path $s \mapsto F_c([\gamma_1]) \cdot \widetilde{\gamma}_2(s)$, then we can compose $\widetilde{\gamma}_1$ and $\widetilde{\gamma}'_2$, and we easily see that $\widetilde{\gamma}_1 \oplus \widetilde{\gamma}'_2$ is the lift of $\gamma_1 \oplus \gamma_2$. Since the endpoint of this lift is $F_c([\gamma_1]) \cdot F_c([\gamma_2])$, we get $F_c([\gamma_1 \oplus \gamma_2]) = F_c([\gamma_1]) \cdot F_c([\gamma_2])$.

So F_c is a functor $\Pi_1(X, B) \to G$.

Although the construction of this functor is quite direct, it is a little hard to work with. In the lemma below, we give a more verbose but equivalent definition.

Lemma 5.5. *With notation as in 5.3, let $\gamma : I \to X$ be a path from x_α to x_β. Then there are $n \in \mathbb{N}$, reals $0 = r_0 < \ldots < r_{n+1} = 1$ and indices $\alpha_0, \ldots, \alpha_n \in A$ such that $\alpha_0 = \alpha$, $\alpha_n = \beta$, and $\gamma([r_k, r_{k+1}]) \subset U_{\alpha_k}$ for $k = 0, \ldots, n$. Moreover, if F_c is the functor $\Pi_1(X, B) \to G$ constructed from $(c_{\alpha\beta})$ in 5.4, we have*

$$F_c([\gamma]) = c_{\alpha_0\alpha_1}(\gamma(r_1)) \cdot \ldots \cdot c_{\alpha_{n-1}\alpha_n}(\gamma(r_n)).$$

Proof. Let $\mathcal{V}$ be the set of connected components of all sets $\gamma^{-1}(U_\alpha)$ for $\alpha \in A$. Then $\mathcal{V}$ is a cover of I. Since I is compact, $\mathcal{V}$ has some finite subcover $\mathcal{V}'$. Define $n = \#\mathcal{V}' - 1$. Since the elements of $\mathcal{V}'$ are open intervals, there are numbers a_i and b_i with $i = 0, \ldots, n$ such that $\mathcal{V}' = \{(a_i, b_i)\}$. We can assume without loss of generality that both the a_i and b_i are strictly increasing by rearranging indices and assuming that $\mathcal{V}'$ is a minimal subcover of $\mathcal{V}$. Define $r_0 = 0$, $r_{n+1} = 1$ and $r_i = \frac{1}{2}(a_i + b_{i-1})$ for $i = 1, \ldots, n$. Then it is clear to see that $[r_i, r_{i+1}] \subset (a_i, b_i)$ for each i, so for every i there is some α_i such that $[r_i, r_{i+1}] \subset \gamma^{-1}(U_{\alpha_i})$.

For the second part of the statement, we first consider the case where we have $n \leq 1$. Then there is some $r \in [0, 1]$ such that $\gamma([0, r]) \subset U_\alpha$ and $\gamma([r, 1]) \subset U_\beta$. Now, let $p : Y \to X$ be the G-bundle we get from applying construction 3.6 to $(c_{\alpha\beta})$. We let $\widetilde{\gamma}$ be the lift of γ starting at the point $\overline{(x_\alpha, e, \alpha)}$. Then it is immediate that

$$\widetilde{\gamma}(r) = \overline{(\gamma(r), e, \alpha)} = \overline{(\gamma(r), c_{\alpha\beta}(\gamma(r)), \beta)}.$$

Therefore, we have $\widetilde{\gamma}(1) = \overline{(x_\beta, c_{\alpha\beta}(\gamma(r)), \beta)}$, so $F_c([\gamma]) = c_{\alpha\beta}(\gamma(r))$.

Now, suppose $n > 1$. Because of the first part of the lemma, we can think of γ as the concatenation of n paths, for all of which we can take $n = 1$ and apply the previous case (note that to do so, we actually require that γ passes through each x_{α_k} somewhere between r_{k-1} and r_k, but since each U_{α_k} is simply connected, this is no loss of generality). The statement follows. □

In the last part of the proof, we made use of the fact that any path is the concatenation of 'simpler' paths, namely, those paths for which we can choose $n = 1$ in the lemma. We will call such paths *primitive paths*, and any homotopy class in the fundamental groupoid that contains at least one primitive path will be called a *primitive class*. The previous lemma shows that every path in X from any x_α to any x_β is homotopic to a composition of primitive homotopic paths, or in other words, that the primitive classes generate the fundamental groupoid.

Lemma 5.6. *Every class in $\Pi_1(X, B)$ is the product of primitive classes.* □

Next, we show how to construct a cocycle from a functor.

Construction 5.7. (From functor to cocycle). With notation as in 5.3, let $F : \Pi_1(X, B) \to G$ be a functor. Let $\alpha, \beta \in A$ be such that $U_\alpha \cap U_\beta \neq \emptyset$. Let $x \in U_\alpha \cap U_\beta$ be given, let $\gamma_1 : I \to U_\alpha$ be a path in U_α from x_α to x, and let $\gamma_2 : I \to U_\beta$ be a path in U_β from x to x_β. Define $c_{\alpha\beta}(x) = F([\gamma_1 \oplus \gamma_2])$. Since U_α and U_β are by assumption simply connected, the homotopy class of $\gamma_1 \oplus \gamma_2$ does not depend on the particular paths chosen.

This defines a set of functions $c_{\alpha\beta} : U_\alpha \cap U_\beta \to G$. We claim that it is in fact a cocycle. For this, we need two check two things: firstly, that every map is locally constant; secondly, that the collection satisfies the cocycle condition, whenever applicable.

Let $x, y \in U_\alpha \cap U_\beta$ be in the same path-connected component of $U_\alpha \cap U_\beta$. Let $\gamma_1 : I \to U_\alpha$ be a path from x_α to x, let $\gamma_2 : I \to U_\alpha \cap U_\beta$ be a path in $U_\alpha \cap U_\beta$ from x to y, and let $\gamma_3 : I \to U_\beta$ be a path from y to x_β. Then $\gamma_1 \oplus \gamma_2$ is a path in U_α from x_α to y and $\gamma_2 \oplus \gamma_3$ is a path in U_β from x to x_β, so we have

$$c_{\alpha\beta}(y) = F([(\gamma_1 \oplus \gamma_2) \oplus \gamma_3]) = F([\gamma_1 \oplus (\gamma_2 \oplus \gamma_3)]) = c_{\alpha\beta}(x).$$

So each $c_{\alpha\beta}$ is constant on path-connected components (and therefore on connected components, by local path-connectedness), so each $c_{\alpha\beta}$ is locally constant.

Let $x \in U_\alpha \cap U_\beta \cap U_\gamma$ be given. Let $\gamma_1 : I \to U_\alpha$ be a path from x_α to x, let $\gamma_2 : I \to U_\beta$ be a path form x_β to x, and let $\gamma_3 : I \to U_\gamma$ be a path from x_γ to x. Then we have

$$c_{\alpha\gamma}(x) = F([\gamma_1 \oplus \gamma_3^{-1}]) = F([\gamma_1 \oplus \gamma_2^{-1} \oplus \gamma_2 \oplus \gamma_3^{-1}]) = F([\gamma_1 \oplus \gamma_2^{-1}]) \cdot F([\gamma_2 \oplus \gamma_3^{-1}]) = c_{\alpha\beta}(x) \cdot c_{\beta\gamma}(x).$$

So $(c_{\alpha\beta})_{\alpha,\beta\in A}$ is indeed a Čech cocycle.

Of course, these constructions are only interesting if they are each other's inverse.

Proposition 5.8. *Let the notation be as in 5.3.*

1. *Let $F : \Pi_1(X, B) \to G$ be a functor, $(c_{\alpha\beta})$ be the Čech cocycle obtained by applying construction 5.7 and let $F_c : \Pi_1(X, B) \to G$ be the functor obtained by applying construction 5.4 on $(c_{\alpha\beta})$. Then $F_c = F$.*

2. *Let $(c_{\alpha\beta})$ be a Čech cocycle on $\mathcal{U}$ with coefficients in G, let F_c be the functor obtained by applying 5.4 to $(c_{\alpha\beta})$, and let $(d_{\alpha\beta})$ be the cocycle obtained by applying construction 5.7 to F_c. Then $(c_{\alpha\beta}) = (d_{\alpha\beta})$.*

Proof. 1. Since by lemma 5.6 the primitive classes generate $\Pi_1(X, B)$, a functor is completely determined by its values on the primitive classes. Let γ be a primitive path from x_α to x_β. Then there is some $r \in [0, 1]$ with $\gamma([0, r]) \subset U_\alpha$ and $\gamma([r, 1]) \subset U_\beta$. Now, from lemma 5.5 we see that $F_c([\gamma]) = c_{\alpha\beta}(r)$. But from construction 5.7 we have by definition $c_{\alpha\beta}(r) = F([\gamma])$. Therefore, we have $F_c([\gamma]) = F([\gamma])$, which proves the claim.

2. Let $x \in U_\alpha \cap U_\beta$ be given. Let γ_1 be a path in U_α from x_α to x, and let γ_2 be a path in U_β from x to x_β. By definition, we then have $d_{\alpha\beta}(x) = F_c([\gamma_1 \oplus \gamma_2])$. Now, applying lemma 5.5 to the path $\gamma_1 \oplus \gamma_2$ with $n = 1$ and $r_1 = \frac{1}{2}$, we find $F_c([\gamma_1 \oplus \gamma_2]) = c_{\alpha\beta}(x)$, so $c_{\alpha\beta}(x) = d_{\alpha\beta}(x)$. □

To relate this result to the previous chapters, we should look not only at cocycles, but also at cohomology. We have the following:

Lemma 5.9. *Let $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$ be two Čech cocycles, and let F_c and F_d be the functors $\Pi_1(X, B) \to G$ constructed from these cycles using construction 5.4. Then F_c and F_d are isomorphic as functors if and only if $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$ are cohomologous.*

Proof. If $(c_{\alpha\beta})$ and $(d_{\alpha\beta})$ are cohomologous, there are $h_\alpha \in G$ such that $h_\alpha \cdot d_{\alpha\beta}(x) = c_{\alpha\beta}(x) \cdot h_\beta$. We will prove that we have $h_\alpha \cdot F_d(\gamma) = F_c(\gamma) \cdot h_\beta$ for each homotopy class $\gamma : x_\alpha \to x_\beta$, which proves that the collection (h_α) defines a natural transformation. As usual, we are done if we prove this for the primitive classes. Let γ be a primitive path in X from x_α to x_β, and let $r \in [0, 1]$ be such that $\gamma([0, r]) \subset U_\alpha$ and $\gamma([r, 1]) \subset U_\beta$. Then applying lemma 5.5, we have

$$h_\alpha \cdot F_d([\gamma]) = h_\alpha \cdot d_{\alpha\beta}(\gamma(r)) = c_{\alpha\beta}(\gamma(r)) \cdot h_\beta = F_c([\gamma]) \cdot h_\beta,$$

which shows that the collection (h_α) defines a natural transformation. Since each h_α is invertible, this is an isomorphism of functors.

The proof for the other implication is analogous. □

We can combine the correspondence between functors and Čech cohomology with the relationship between Čech cohomology and G-bundles obtained in chapter 3, and obtain the following. We let $\mathrm{Fun}(\Pi_1(X, B), G)$ be the category of functors from $\Pi_1(X, B)$ to G.

Corollary 5.10. *The constructions 3.6, 3.7, 5.4 and 5.7 gives rise to bijections*

$$\mathrm{Bun}(G, X)/\textit{isomorphism} \quad \longleftrightarrow \quad H^1(\mathcal{U}; G) \quad \longleftrightarrow \quad \mathrm{Fun}(\Pi_1(X, B), G)/\textit{isomorphism}$$

If X is connected, then there is a close connection between functors on the fundamental groupoid and homomorphisms from the fundamental group.

Lemma 5.11. *Let notation be as in 5.3, and assume that X is connected. For each $x_0 \in B$ the inclusion $\pi_1(X, x_0) \hookrightarrow \Pi_1(X, B)$ induces a isomorphism*

$$\mathrm{Fun}(\Pi_1(X, B), G)/\textit{isomorphism} \longleftrightarrow \mathrm{Hom}(\pi_1(X, x_0), G)/\textit{conjugacy}.$$

Proof. For each functor $F : \Pi_1(X, B) \to G$, we define $\Phi(F)$ to be the restriction of F to $\pi_1(X, x_0)$. From the definition of a functor, we get that $\Phi(F)$ is a group homomorphism. Now, suppose $F_1, F_2 : \Pi_1(X, B) \to G$ are two isomorphic functors. Then for each $\alpha \in A$ there is some $g_\alpha \in G$ such that $g_\alpha \cdot F_1([\gamma]) = F_2([\gamma]) \cdot g_\beta$ holds for each γ from x_α to x_β. Then in particular, we have $g_0 \cdot F_1(c) = F_2(c) \cdot g_0$ for each $c \in \pi_1(X, x_0)$. In other words, we have $\Phi(F_1) = g_0^{-1} \cdot \Phi(F_2) \cdot g_0$, so Φ induces a map $\overline{\Phi}$ from $\mathrm{Fun}(\Pi_1(X, B), G)$ modulo isomorphism to $\mathrm{Hom}(\pi_1(X, x_0), G)$ modulo conjugacy.

Now, assume $f : \pi_1(X, x_0) \to G$ is a homomorphism. Choose for each $\alpha \in A$ a path γ_α from x_0 to x_α. For each path γ from x_α to x_β we define $F([\gamma]) = f([\gamma_\alpha] \cdot [\gamma] \cdot [\gamma_\beta^{-1}])$. One easily checks that this gives a well-defined functor $F : \Pi_1(X, B) \to G$ with $\Phi(F) = f$, so Φ is surjective, and therefore $\overline{\Phi}$ is surjective as well. Now, assume we have functors $F_1, F_2 : \Pi_1(X, B) \to G$ and some

$g \in G$ such that $\Phi(F_1) = g^{-1} \cdot \Phi(F_2) \cdot g$. For each $\alpha \in A$ we choose a path γ_α from x_0 to x_α, and define $g_\alpha = F_1([\gamma_\alpha]^{-1}) \cdot g \cdot F_2([\gamma_\alpha])$. We claim that this gives an isomorphism of functors. To see this, let γ be a path from x_α to x_β. Then we have

$$\begin{aligned} F_2([\gamma]) &= F_2([\gamma_\alpha^{-1} \oplus \gamma_\alpha \oplus \gamma \oplus \gamma_\beta^{-1} \oplus \gamma_\beta]) \\ &= F_2([\gamma_\alpha]^{-1}) \cdot F_2([\gamma_\alpha \oplus \gamma \oplus \gamma_\beta^{-1}]) \cdot F_2([\gamma_\beta]) \\ &= F_2([\gamma_\alpha]^{-1}) \cdot g^{-1} \cdot F_1([\gamma_\alpha \oplus \gamma \oplus \gamma_\beta^{-1}]) \cdot g \cdot F_2([\gamma_\beta]) \\ &= g_\alpha^{-1} \cdot F_1([\gamma]) \cdot g_\beta, \end{aligned}$$

which proves the claim. This proves that $\overline{\Phi}$ is also injective, and therefore a bijection. □

This result, together with corollary 5.10, gives a nice classification of G-bundles over connected and locally simply connected spaces.

Corollary 5.12. *Let X be a connected and locally simply connected space, G a group and $x_0 \in X$ a base point. Then there is a canonical bijection between the G-bundles over X up to isomorphism, and the homomorphisms from $\pi_1(X, x_0)$ to G up to inner automorphisms of G.*

Proof. Let notation be as in 5.3, and assume without loss of generality that $x_0 \in B$. Then apply lemmas 5.10 and 5.11. □

Chapter 6

Associated Morphisms and the Universal Cover

In this chapter, we will let G be a group, and X a connected and locally simply connected space with a base point $x_0 \in X$. Also, we will simply write $\pi_1(X)$ for $\pi_1(X, x_0)$.

The previous chapters have set up a chain of bijections between various sets. At the endpoints of this chain is a correspondence between G-bundles over X, up to isomorphism, and homomorphisms $\pi_1(X) \to G$, up to inner automorphisms (corollary 5.12). In contrast to the Čech cocycles and the functors we considered in the previous chapters, these sets do not depend on the choice of a good cover of X, but only on the data at hand, that is, the group G and the connected and locally simply connected space X (admittedly, with a given base point). One purpose of this chapter is to further investigate this connection.

As it stands, we know that there is a correspondence between $\mathrm{Bun}(G, X)$ and $\mathrm{Hom}(\pi_1(X), G)$. However, the precise nature of this bijection is fogged by the details of the various constructions we needed to establish it. The first proposition sheds some light on the interpretation of the correspondence.

Proposition 6.1. *Let $f : \pi_1(X) \to G$ be a morphism of groups, and let $p : Y \to X$ be the G-bundle over X corresponding to f by corollary 5.12. Then there is some $y_0 \in p^{-1}(x_0)$ such that for each loop $\gamma : x_0 \to x_0$ the lift of γ starting at y_0 has terminal point $f([\gamma]) \cdot y_0$.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a good cover of X, and let $B = (x_\alpha)_{\alpha \in A}$ be a set of base points with $x_\alpha \in U_\alpha$. We can assume without loss of generality that $x_0 = x_{\alpha_0}$ for some $\alpha_0 \in A$. Let $F : \Pi_1(X, B) \to G$ be an extension of f, and let $(c_{\alpha\beta})$ be the cocycle corresponding to F by construction 5.7. Let Y' be the G-bundle constructed from $(c_{\alpha\beta})$ in 3.6, and define $y'_0 = \overline{(x_0, e, \alpha_0)}$.

Let γ be a loop in X, and lift it to Y'. Then by construction 5.4 and proposition 5.8, its endpoint is

$$\overline{(x_0, F([\gamma]), \alpha_0)} = F([\gamma]) \cdot y'_0 = f([\gamma]) \cdot y'_0$$

Since Y and Y' are isomorphic, we can take an isomorphism $\varphi : Y' \to Y$. Then $y_0 = \varphi(y'_0)$ will work. □

Proposition 6.1 suggests that we consider G-bundles with some choice of base point in the cover space. Therefore, the following definition more or less suggests itself:

Definition 6.2. Let (X, x_0) be a pointed space and G a group. A *pointed G-bundle over X* is simply a G-bundle $p : Y \to X$, where Y has a distinguished base point y_0 and p maps y_0 to x_0. A *morphism* of pointed G-bundles is a morphism of G-bundles that maps base point to base point.

Many results from chapter 2 about the basic properties of G-bundles have direct analogs for pointed G-bundles. For example, every pointed G-bundle morphism is an isomorphism.

Now, lemma 6.1 not only tells us that we should choose a base point. It also suggest that the maps $f \in \mathrm{Hom}(\pi_1(X), G)$ can best be understood in terms of the endpoints of lifts of elements of $\pi_1(X)$. Therefore, we introduce another definition.

Definition 6.3. Let $p : (Y, y_0) \to (X, x_0)$ be a pointed G-bundle. The induced map $f_p : \pi_1(X) \to G$ such that for each $[\gamma] \in \pi_1(X)$ the lift of γ to y_0 ends in $f([\gamma]) \cdot y_0$ will be called the *associated homomorphism* of $p : (Y, y_0) \to (X, x_0)$.

Of course, calling something a homomorphism doesn't make it one, so we should check that f_p is indeed a homomorphism. Also, we would like for f_p to be the same if we started with an isomorphic pointed G-bundle.

Lemma 6.4. *The associated homomorphism of a pointed G-bundle $p : (Y, y_0) \to (X, x_0)$ is a homomorphism. Moreover, if $p' : Y' \to X$ is an isomorphic pointed G-bundle, then the associated homomorphisms are equal.*

Proof. By lemma 3.2, the endpoint of a lift of a path γ only depends on the path homotopy of γ, so f_p is well-defined. For $[\gamma_1], [\gamma_2] \in \pi_1(X)$, let $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ be the lifts to y_0. Then the endpoint of $\widetilde{\gamma}_1$ is then $f_p([\gamma_1])$, so $\widetilde{\gamma}_1 \oplus (f_p([\gamma_1]) \cdot \widetilde{\gamma}_2)$ is the lift of $\gamma_1 \oplus \gamma_2$. The endpoint of the lift is then $f_p([\gamma_1]) \cdot \widetilde{\gamma}_2(1) = f_p([\gamma_1]) \cdot f_p([\gamma_2]) \cdot y_0$, which proves that f_p is a homomorphism.

Now, let $p' : Y' \to X$ be another pointed G-bundle and $f_{p'}$ its associated homomorphism, with Y and Y' isomorphic. Let $\varphi : Y \to Y'$ be an isomorphism. Let γ be a path in X, let $\widetilde{\gamma}$ be its lift to Y and let $\widetilde{\gamma}'$ be its lift to Y'. Then $\varphi \circ \widetilde{\gamma}$ is also a lift of γ to Y', so by uniqueness we have $\widetilde{\gamma}' = \varphi \circ \widetilde{\gamma}$. In particular, we have $\varphi(\widetilde{\gamma}(1)) = \widetilde{\gamma}'(1)$, so $\varphi(f_p([\gamma])y_0) = f_{p'}([\gamma])y_0'$. Since φ is G-equivariant and $\varphi(y_0) = y_0'$, this gives $f_p([\gamma]) = f_{p'}([\gamma])$. ☐

Lemma 6.1 can now be restated as follows:

Lemma 6.5 (Lemma 6.1 restated). *Let $f \in \mathrm{Hom}(\pi_1(X), G)$, and let $p : Y \to X$ be the G-bundle that corollary 5.12 assigns to it. Then there is a choice of base point in Y such that f equals the associated morphism.* ☐

A natural question to ask is in how far f_p depends on the choice of base point in Y. The next lemma shows that choosing another base point in Y changes the associated morphism by an inner automorphism of G.

Lemma 6.6 (Change of base point). *Let $p : Y \to X$ be a G-bundle. Let $y_0, y_0' \in p^{-1}(x_0)$ with $y_0' = gy_0$, and let f_p and f_p' be the associated homomorphisms obtained from the pointed G-bundles $(Y, y_0) \to X$ and $(Y, y_0') \to X$. Then $f_p' = g \cdot f_p \cdot g^{-1}$. In particular, $f_p = f_p'$ if and only if g commutes with every element in the image of f_p.*

Proof. Let $[\gamma] \in \pi_1(X)$ be fixed. As usual, let $\widetilde{\gamma}$ and $\widetilde{\gamma}'$ be the lifts starting in y_0 and y_0', respectively. Then $\widetilde{\gamma}'(1) = g\widetilde{\gamma}(1)$, so $f_p'([\gamma])y_0' = gf_p'([\gamma])y_0$. Substituting $y_0' = gy_0$ and using freeness gives $f_p([\gamma]) \cdot g = g \cdot f_p([\gamma])$. The statement follows. ☐

Now we are in the position to state a reverse of 6.5:

Lemma 6.7. *Let $p : Y \to X$ be a pointed G-bundle, and let f_p be its associated morphism. Also, let $C \in \mathrm{Hom}(\pi_1(X), G)/\mathrm{Inn}(G)$ be the class of morphisms that corollary 5.12 assigns to this G-bundle. Then $f_p \in C$.*

Proof. Let y_0 be the base point of Y, and let $f \in C$ be given. By lemma 6.5, there is some point $y_0' \in p^{-1}(x)$ such that f is the associated morphism of the bundle $(Y, y_0') \to X$ (notice that this is the same G-bundle over X, but with a possibly different choise of base point in Y). By lemma 6.6, f and f_p differ by an element of $\mathrm{Inn}(G)$, thus $f_p \in C$. ☐

By lemma 6.4 we know that the associated morphisms of isomorphic pointed G-bundles are the same. A converse of this statement is true if the associated morphisms under consideration are surjective.

Lemma 6.8. *Let $p : (Y, y_0) \to X$ and $p' : (Y', y_0') \to X$ be pointed G-bundles with associated morphisms f_p and $f_{p'}$. Suppose f_p is surjective. If $f_p = f_{p'}$, then the bundles are isomorphic.*

Proof. Assume that $f_p = f_{p'}$. With lemma 6.7 and lemma 5.12 we find that Y and Y' are isomorphic as (non-pointed) G-bundles, since f_p and $f_{p'}$ are in the same class in $\mathrm{Hom}(\pi_1(X), G)/\mathrm{Inn}(G)$. Therefore, without loss of generality, we may assume that $Y = Y'$ and $p = p'$, since only the base points of Y and Y' are relevant. Let y_0 and y_0' be the base points corresponding to f_p and f_p', and let $g \in G$ be such that $y_0' = gy_0$. By lemma 6.6, g commutes with every element of G, since f_p is surjective. Now, by example 2.7, the map $y \mapsto g \cdot y$ is an automorphism of G-bundles that maps y_0 to y_0'. In particular, it is an isomorphism of pointed G-bundles. □

The same statement holds if the associated morphisms are not surjective, but the proof does not: multiplication with g is a G-bundle morphism if and only if g is central, but lemma 6.6 can only guarantee that g above commutes with every element in the image of the associated morphism. As a consequence, the proof for the case of a non-surjective associated morphism is a bit more involved. The main observation is that every homomorphism (and in fact any map) becomes surjective when its codomain is restricted to its image.

Proposition 6.9. *If two pointed G-bundles over X have the same associated morphism, then they are isomorphic.*

Proof. Let $p : Y \to X$ and $p' : Y' \to X$ be two pointed G-bundles over X with basepoints y_0 and y_0' respectively, and with the same associated morphism f_p. If f_p is surjective we are done by lemma 6.8, so assume f_p is not surjective. Set $H = \mathrm{im} f_p$, and let $U \subset Y$ be the connected component of Y containing y_0. Then by the definition of the associated morphism, we see that $g \cdot y_0$ is an element of U if and only if g is an element of H. Moreover, if $y \in U$ is arbitrary, then there is some path γ from y to y_0. Then for all $g \in G$ we see that $g \cdot \gamma$ connects gy with gy_0. Therefore, for all $y \in U$ we have $gy \in U$ if and only if $g \in H$. Thus the restriction $p|_U : U \to X$ is an H-bundle. Likewise, if we let U' be the the connected component of Y' containing y_0', then $p'|_{U'} : U' \to X$ is an H-bundle. The associated morphisms of these H-bundles are equal to f_p, but with the codomain restricted to H. Thus, these H-bundles have the same associated morphism, and moreover, this morphism is surjective by definition of H. Thus, we can apply lemma 6.8, and obtain an isomorphism $\varphi_1 : U \to U'$ of H-bundles over X.

We now only need to extend φ_1 to an isomorphism from Y to Y'. To to this, we take an index set I and for each $i \in I$ a $g_i \in G$ such that these g_i form a complete set of representatives of the right cosets $H \setminus G$. In other words, for every $g \in G$ these is an unique $i \in I$ such that $Hg = Hg_i$. Now, we claim that for each $y \in Y$, there is a unique g_i such that $g_i \cdot y \in U$. To see this, let γ be any path from $p(y)$ to x_0, and lift this path to $\widetilde{\gamma}$ with initial point y. Then $\widetilde{\gamma}(1)$ is in the fiber over x_0, and therefore there is a $g \in G$ such that $g \cdot \widetilde{\gamma}(1) = y_0$. Then the image of $g \cdot \widetilde{\gamma}$ lies in g, and in particular we have $gy \in U$. Now, let $i \in I$ be such that $Hg_i = Hg$. Then $g_i \cdot g^{-1} \in H$, and since U is closed under the action of H we find $g_i y = g_i g^{-1} gy \in U$. So there is al least one g_i such that $g_i y \in U$. Now, suppose that $g_j y \in U$ as well. Then we have $g_i g_j^{-1} \in H$, and this gives $Hg_i = Hg_j$, and therefore $i = j$.

So this induces a map $f : Y \to G$ such that for every $y \in Y$ there is some $i \in I$ with $f(y) = g_i$, and such that $f(y) \cdot y \in U$. Notice that f is locally constant, since the G-bundle is locally trivial. We now define $\varphi : Y \to Y$ by setting $\varphi(y) = (f(y))^{-1} \cdot \varphi_1(f(y)y)$. Then φ is continuous since φ_1 and f are. We claim that φ is a G-bundle morphism. Clearly we have $p'(\varphi(y)) = p(y)$, so we only need to check that $\varphi(gy) = g\varphi(y)$. To this end we calculate

$$\varphi(gy) = (f(gy))^{-1} \cdot \varphi_1(f(gy) \cdot gy).$$

It is tempting to take the factors $f(gy)$ and g out of φ_1, but since φ_1 is an H-bundle isomorphism and not a G-bundle isomorphism, we can only take out elements of H. But $f(gy)g \cdot y$ is an element of U, so $f(gy)g \cdot (f(y))^{-1}$ is an element of H. Thus, there is some $h \in H$ such that $h \cdot f(y) = f(gy) \cdot g$. Substituting gives

$$\begin{aligned}\varphi(gy) &= (f(gy))^{-1} \cdot \varphi_1(f(gy) \cdot gy) \\ &= (h \cdot f(y) \cdot g^{-1})^{-1} \cdot \varphi_1(h \cdot f(y) \cdot g^{-1} \cdot gy) \\ &= g \cdot (f(y))^{-1} \cdot h^{-1} \cdot \varphi_1(h \cdot f(y) \cdot y) \\ &= g \cdot (f(y))^{-1} \cdot \varphi_1(f(y) \cdot y) \\ &= g \cdot \varphi(y).\end{aligned}$$

Thus, φ is a G-bundle morphism and thus an G-bundle isomorphism. Moreover, we have $\varphi_1(y_0) = y_0'$, so Y and Y' are isomorphic as pointed G-bundles. □

The previous lemmas and propositions give us a more concrete way of viewing the correspondence between pointed G-bundles and homomorphism $\pi_1(X) \to G$.

Theorem 6.10. *For every homomorphism $f : \pi_1(X) \to G$, there is up to isomorphism exactly one pointed G-bundle on X having f as its associated morphism.*

Proof. By proposition 6.5 there is at least one such G-bundle. By proposition 6.9 every pair of such G-bundles is isomorphic. □

So to give a pointed G-bundle is the same as giving a homomorphism from the fundamental group of X to G. The trivial morphism that sends every element of $\pi_1(X)$ to $e \in G$ corresponds to the trivial G-bundle. But if $G = \pi_1(X)$, then $\mathrm{Hom}(\pi_1(X), G)$ has another distinguished element, namely the identity morphism.

Definition 6.11. Let $G = \pi_1(X)$. The pointed G-bundle with the identity as its associated morphism is called the *universal cover* of X, and denoted $\widetilde{X}$.

Examples 6.12.

1. If X is simply connected, then we have $G = \pi_1(X) = 1$ and $\widetilde{X} = X$. In this case the trivial G-bundle and the universal cover of X coincide. In fact, this is the only case where this happens: the trivial $\pi_1(X)$-cover corresponds to the constant endomorphism on $\pi_1(X)$, while the universal cover corresponds to the identity morphism.
2. Let $X = S^1 \subset \mathbb{C}$ be the unit circle in the complex plane, and $x_0 = 1$. Define a pointed $\mathbb{Z}$-cover $p : (\mathbb{R}, 0) \to X$ by $x \mapsto \exp(2\pi i x)$. We identify $\mathbb{Z}$ with $\pi_1(X)$ in the usual way using the (counterclockwise) winding number around the center of the circle, and we let $\mathbb{Z}$ act on $\mathbb{R}$ by translation. Then this $\mathbb{Z}$-bundle gives us the universal cover of S^1. This is easily seen: if γ is some path from x_0 to x_0 that winds k times around the circle, then the endpoint of a lift that starts at 0 is exactly $k + 0 = k$.

The 'universality' of the universal cover will be further investigated in the next chapter, but for now we note that $\widetilde{X}$ is always simply connected.

Lemma 6.13. *Let $p : Y \to X$ be a pointed morphism, and let f_p be the associated morphism.*

1. *Y is connected if and only if f_p is surjective.*

2. *Every connected component of Y is simply connected if and only if f_p is injective.*
3. *The universal cover of X is simply connected.*

Proof. Let y_0 be the base point of Y.

1. If Y is connected, then for any $g \in G$ there is some path γ from y_0 to gy_0. By definition of the associated morphism, this implies that $f_p([p \circ \gamma]) = g$, so f_p is surjective. On the other hand, if f_p is surjective, then for each $g \in G$ there is some path γ with $f_p([\gamma]) = g$. The lift of this path to y_0 ends in gy_0, so the complete fiber of $p^{-1}(x)$ lies in the same connected component. Now, let $y \in Y$ be arbitrary. Let γ be a path from x_0 to $p(y)$. Then the lift $\widetilde{\gamma}$ of γ to y_0 must end in the same fiber as y. In particular, there is some $g \in G$ with $g \cdot \widetilde{\gamma}(1) = y$. Then $g \cdot \gamma(1)$ connects y and $g \cdot y_0$, so y lies in the same connected component as the fiber $p^{-1}(x)$. Thus, Y is connected.
2. If every component of Y is simply connected, then every loop from y_0 to itself is homotopic to a constant path. Thus, the kernel of f_p is trivial, and f_p is injective. On the other hand, assume f_p is injective. Let $y_0' \in p^{-1}(x)$, and let $g \in G$ such that $y_0' = gy_0$. Let $\widetilde{\gamma}$ be a loop in Y from y_0' to itself, and γ its projection to X. Then $f_p([\gamma]) = e$, so γ is homotopic to a constant loop, and therefore $\widetilde{\gamma}$ is homotopic to a constant loop as well. We see that the component of Y containing y_0' is simply connected. Analogous to the previous paragraph, we can show that any connected component of Y contains some point of $p^{-1}(x_0)$, so therefore every connected component of Y is simply connected.
3. The universal cover has by definition the identity on $\pi_1(X)$ as its associated morphism, which is injective and surjective, so the universal cover of X is simply connected. □

Corollary 6.14. *Let $p : Y \to X$ be a pointed G-bundle, and suppose that Y is simply connected. Then $\pi_1(X)$ is isomorphic to G.*

Proof. The associated morphism $f_p : \pi_1(X) \to G$ is both injective and surjective. □

This corollary can be useful to calculate the fundamental groups of some often occuring spaces. If Y is a locally simply connected space (a manifold, for example) that is simply connected on which G acts evenly, then the canonical map $Y \to G \setminus Y$ is by definition a G-bundle. By the corollary we see that in this case the orbit space has fundamental group isomorphic to G. So for example, $\mathbb{Z}$ acts evenly on $\mathbb{R}$ by translation. Since $\mathbb{R}$ is simply connected and the orbit space is S^1, we find that $\pi_1(S^1) = \mathbb{Z}$. Likewise, the fundamental groups of the real projective spaces $\mathbb{RP}^n = S^n/\{\pm \mathrm{Id}\}$ are isomorphic to C_2, for $n > 1$, since S^n is simply connected for $n > 1$.

Chapter 7

A Galois Connection

As in the previous chapter, we let X be a connected and locally simply connected pointed space with base point x_0, and write $\pi_1(X)$ for $\pi_1(X, x_0)$.

Up to now, we have mainly taken fixed X and G and looked in various ways at the G-bundles over X. In this chapter, we will take a broader view: we will still keep X fixed, but we will allow G to vary. In other words, we will look to general group bundles over X. We will exhibit a correspondence between connected pointed group bundles over X and the normal subgroups of $\pi_1(X)$ that turns out to be a Galois connection: bigger subgroups correspond to 'smaller' group bundles. This chapter is inspired by the exposition in [Sza09], Chapter 2, where a similar connection is established between the Galois covering maps over X and the subgroups of the fundamental group.

To formalize the setting, we will introduce some terminology.

Definition 7.1. A *group bundle* over X is a triple (Y, p, G) such that $p : Y \to X$ is a connected, pointed G-bundle. A *morphism of group bundles* between (Y, p, G) and (Z, q, H) is a pair (f, φ) where $f : G \to H$ is a homomorphism and $\varphi : Y \to Z$ is a continuous map such that $q = p \circ \varphi$, $\varphi(g \cdot y) = f(g) \cdot \varphi(y)$, and φ maps the base point of Y to the base point of Z.

Note that what we call group bundles should more precisely called connected, pointed group bundles. However, in this chapter we will have no need to consider group bundles without a base point or with a disconnected covering space, so will dispense with the adjectives connected and pointed, and just call them group bundles.

In the case of G-bundles, we had a proposition asserting that every morphism is an isomorphism. In the more general setting of group bundles, this is no longer the case. Still, our choice to only allow connected group bundles puts some restraint on the morphisms.

Proposition 7.2. *Let (f, φ) be a group bundle morphism between the group bundles (Y, p, G) and (Z, q, H) over X, and let f_p and f_q be the associated morphisms. Then*

1. *$f \circ f_p = f_q$.*
2. *f and φ are both surjective.*
3. *f is injective if and only if φ is injective.*
4. *If (f', φ') is another group bundle morphism from (Y, p, G) to (Z, q, H), then $f = f'$ and $\varphi = \varphi'$.*

Proof. 1. Let y_0 and z_0 be the base points of Y and Z. Let $[\gamma] \in \pi_1(X)$ be given. Let $\widetilde{\gamma}$ be the lift to Y and $\widetilde{\gamma}'$ be the lift to Z. By uniqueness, we have $\widetilde{\gamma}' = \varphi \circ \widetilde{\gamma}$. Thus,

$$f_q([\gamma]) \cdot z_0 = \widetilde{\gamma}'(1) = \varphi(\widetilde{\gamma}(1)) = \varphi(f_p([\gamma]) \cdot y_0) = f(f_p[\gamma]) \cdot z_0.$$

By freeness, we get $f_q = f \circ f_p$.

2. Since Y and Z are connected, f_p and f_q are surjective. Since $f_q = f \circ f_p$, this implies that f is surjective.

   Now, let $z \in Z$ be given. Let $x = q(z)$, and let $y \in p^{-1}(x)$. Then $\varphi(y)$ and z are both elements of $q^{-1}(x)$, so there is some $h \in H$ such that $z = h \cdot \varphi(y)$. Since f is surjective, there is some $g \in G$ such that $f(g) = h$. In particular, $\varphi(gy) = f(g)\varphi(y) = h\varphi(y) = z$. So φ is surjective.

3. Suppose $f(g) = e$. Then $\varphi(gy) = f(g)\varphi(y) = \varphi(y)$. Therefore, if f is not injective, then φ is not injective eighter. On the other hand, assume $\varphi(y) = \varphi(y')$. Then in particular $p(y) = p(y')$, so there is some $g \in G$ with $y' = g \cdot y$. Thus, $\varphi(y) = \varphi(y') = f(g)\varphi(y)$. In particular, $f(g) = e$. Therefore, if φ is not injective, then f is not injective eighter. □

4. From part 1. we know that $f \circ f_p = f_q = f' \circ f_p$, and since f_p is surjective, this means that $f = f'$. Now let $y \in Y$, and let γ be a path in Y from y_0 to y. Then both $\varphi \circ \gamma$ and $\varphi' \circ \gamma$ are lifts of $p \circ \gamma$ to Z. Moreover, $(\varphi \circ \gamma)(0) = \varphi(y_0) = z_0 = \varphi'(y_0) = (\varphi' \circ \gamma)(0)$, so $\varphi \circ \gamma$ and $\varphi' \circ \gamma$ start at the same point. By unicity of lifts, we find that $(\varphi \circ \gamma)(1) = (\varphi' \circ \gamma)(1)$, thus $\varphi(y) = \varphi'(y)$. Since $y \in Y$ was arbitrary, this implies $\varphi = \varphi'$.

Somewhat surprisingly, group bundle morphisms turn out to be group bundles themselves.

Proposition 7.3. *Let (f, φ) be a morphism from (Y, p, G) to (Z, q, H), and let $N = \ker f$. The restriction of the action of G on Y to N turns the map $\varphi : Y \to Z$ into an N-bundle, with f as its associated morphism.*

Proof. We will apply lemma 2.9. To do so, we only need to check that every point $z \in Z$ has an open neighbourhood U such that $\varphi^{-1}(U) \to U$ gives a trivial N-bundle.

So let $z \in Z$, and let $x = q(z)$. Since group bundles are locally trivial, there is some open neighbourhood $U' \subset X$ such that both bundles are trivial over U'. Let U be the connected component of $q^{-1}(U')$ containing z. We claim that the restriction of φ to $\varphi^{-1}(U) \to U$ defines a trivial N-bundle. For this, choose $y \in \varphi^{-1}(z)$ (note that φ is surjective by proposition 7.2), and let V be the connected component of $\varphi^{-1}(U)$ containing y. By construction, q maps U homeomorphically to U', and p maps V homeomorphically to U'. Since $p = q \circ \varphi$, this means that φ maps V homeomorphically to U. Now, for every $g \in G$ such that $\varphi(gV) = U$ we have $\varphi(gy) = z$, so $z = \varphi(gy) = f(g)\varphi(y) = f(g)z$, so $g \in \ker f$. On the other hand, if $n \in N$, then $\varphi(ny) = \varphi(y) = z$, so $\varphi(nV) = U$. We see that $\varphi^{-1}(U) = \coprod_{n \in N} nV$, so $\varphi^{-1}(U) \to U$ is a trivial N-bundle. Therefore, with lemma 2.9, $\varphi : Y \to Z$ is a N-bundle.

The statement on the associated morphism follows from the fact that $f_p = f \circ f_p$, and the observation that lifting a path from X to Y gives the same result as lifting a path from X to Z, and then lifting the lift from Z to Y. □

Proposition 7.2 states that a morphism from a G-bundle to an H-bundle implies that H is a factor group of G. The other way works as well.

Lemma 7.4. *Let $p : Y \to X$ be a pointed G-bundle with associated morphism f_p, and $N \subset G$ be normal. Then the induced map $q : N \setminus Y \to X$ is a G/N-bundle over X. The natural maps $f : G \to G/N$ and $\varphi : Y \to N \setminus Y$ form a group bundle morphism, and the associated morphism is $f \circ f_p$.*

Proof. The proof is similar to the proof of proposition 7.3, so we will only give a sketch. We choose any point $x \in X$, and any $y \in p^{-1}(x)$. Then there is some open neighbourhood U of x such that $p^{-1}(U)$ is a trivial bundle over U. We let V be the component of $p^{-1}(U)$ containing

y. Then one checks that $q^{-1}(U)$ equals $\coprod_{g \in G} gNU$, where $gNU = \{g \cdot nu, n \in N, u \in U\}$. This shows by lemma 2.9 that $N \setminus Y$ has the structure of a G/N-bundle over X.

That (f, φ) is a morphism between these group bundles follows directly from the definitions. The associated morphism is then $f \circ f_p$ by proposition 7.2.1. □

If we apply the preceding lemma to the universal cover of X, we see that every normal subgroup of $\pi_1(X)$ gives rise to a group bundle over X. On the other hand, given a group bundle, the kernel of the associated morphism is a normal subgroup of $\pi_1(X)$. This correspondence is the core of the following theorem, which has some similarities to the main theorem of Galois theory.

Theorem 7.5. *Let X be a connected and locally simply connected space with base point x_0 and universal cover $\widetilde{X}$. Let $G = \pi_1(X)$ be the fundamental group, $\mathcal{G} = \{N \trianglelefteq G\}$ be the set of normal subgroups of G, and let $\mathrm{Bun}(X)$ be the category of group bundles over X, up to isomorphism. Then*

1. *There is a canonical map $T : \mathcal{G} \to \mathrm{Bun}(X)$ sending N to the G/N-bundle $N \setminus \widetilde{X} \to X$, and a canonical map $S : \mathrm{Bun}(X) \to \mathcal{G}$ sending a group bundle $Y \to X$ to the kernel of the associated morphism. These maps are each others inverse.*
2. *If $K, N \in \mathcal{G}$ are subgroups satisfying $K \subset N$, then there is a unique morphism from $T(K)$ to $T(N)$.*
3. *If (Y, p, H_1) and (Z, q, H_2) are group bundles over X such that there is a morphism (f, φ) from the former to the latter, then $S(Y)$ is a subgroup of $S(Z)$.*

Proof.

1. Let $N \in \mathcal{G}$. From lemma 7.4, we know that $N \setminus \widetilde{X} \to X$ has the structure of a G/N-bundle, with associated morphism the canonical map $G \to G/N$. In particular, the kernel of the associated morphism is N. This shows that $S \circ T$ is the identity on $\mathcal{G}$.

   On the other hand, let $p : Y \to X$ be a H-bundle, with associated morphism f_p. Let $N = \ker f_p$. Then $T(S(Y))$ is the natural G/N-bundle $N \setminus \widetilde{X} \to X$, with associated bundle the map $G \to G/N$. By the isomorphism theorem and the surjectivity of f_p, we get an isomorphism $H \cong G/N$. By using this isomorphism to identify H with G/N, we can view our original H-bundle as an G/N-bundle, and the associated bundle then becomes the canonical map $G \to G/N$. Therefore, Y and $T(S(Y))$ are G/N-bundles with the same surjective associated morphism. By lemma 6.8, Y and $T(S(Y))$ are isomorphic.
2. Let $K, N \in \mathcal{G}$ such that $K \subset N$. Then in particular, K is normal in N. Therefore, we can apply lemma 7.4 to the G/N-bundle $N \setminus \widetilde{X} \to X$. This lemma shows that the natural maps $f : G/K \to G/N$ and $\varphi : K \setminus \widetilde{X} \to N \setminus \widetilde{X}$ define a morphism of group bundles from $T(K)$ to $T(N)$. This morphism is unique by proposition 7.2.4
3. Let (Y, p, H_1) and (Z, q, H_2) be group bundles over X, and (f, φ) a morphism from the former to the latter. Let f_p and f_q be the associated morphisms. Then $f_q = f \circ f_p$, so $S(Y) = \ker f_p \subset \ker f_q = S(Z)$. □

The above shows the universality of the universal cover of X.

Corollary 7.6. *For every group bundle $Y \to X$ there is a unique group bundle morphism from the universal cover of X to Y. Moreover, any other group bundle with this property is isomorphic to the universal cover of X, and this isomorphism is unique.*

Proof. The normal subgroup corresponding to the universal cover is the trivial subgroup $1 \subset \pi_1(X)$. With the correspondence in theorem 7.5, the statement we want to prove translates to the statement that the trivial subgroup is the only normal subgroup of $\pi_1(X)$ that is contained in every normal subgroup of $\pi_1(X)$, which is clearly true. □

We also obtain the promised result on the fundamental group of non-orientable manifolds.

Corollary 7.7. *The fundamental group of any connected non-orientable manifold has a normal subgroup of index 2.*

Proof. Let X be a connected non-orientable manifold and $x_0 \in X$ be fixed. Then the orientation cover over X is connected by theorem 4.7. Therefore, the associated morphism $\pi_1(X, x_0) \to C_2$ is surjective, and the kernel of this morphism is a normal subgroup of index 2. □

Chapter 8

The Seifert-van Kampen Theorem

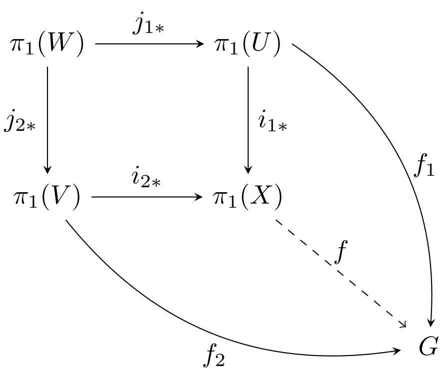
The well-known Seifert-van Kampen theorem relates the fundamental group of a space to the fundamental groups of two connected open subsets and their connected intersection, provided they cover the original space. A proof of this theorem for locally simply connected spaces, which goes back to Grothendieck, can be given using the ideas we have developed so far. The exposition in this chapter is can also be found in [Ful95], paragraph 14.

We will begin by fixing some notation that we will use in the rest of this chapter.

Notation 8.1. Let X be a connected and locally simply connected space, and let $U, V \subset X$ be connected open subsets with connected intersection W such that $X = U \cup V$. Let $x_0 \in W$ be the basepoint of these spaces. Let $i_1 : U \to X$, $i_2 : V \to X$, $j_1 : W \to U$ and $j_2 : W \to V$ be the inclusion maps, and let $i_{1*} : \pi_1(U) \to \pi_1(X)$, $i_{2*} : \pi_1(V) \to \pi_1(X)$, $j_{1*} : \pi_1(W) \to \pi_1(U)$ and $j_{2*} : \pi_1(W) \to \pi_1(V)$ be the maps induced by the inclusions.

$$\begin{array}{ccc} W & \xrightarrow{j_1} & U \\ {\scriptstyle j_2}\downarrow & & \downarrow{\scriptstyle i_1} \\ V & \xrightarrow{i_2} & X \end{array} \qquad\qquad \begin{array}{ccc} \pi_1(W) & \xrightarrow{j_{1*}} & \pi_1(U) \\ {\scriptstyle j_{2*}}\downarrow & & \downarrow{\scriptstyle i_{1*}} \\ \pi_1(V) & \xrightarrow{i_{2*}} & \pi_1(X) \end{array}$$

Theorem 8.2 (Seifert-van Kampen)**.** *Let G be a group and $f_1 : \pi_1(U) \to G$ and $f_2 : \pi_1(V) \to G$ be homomorphisms satisfying $f_1 \circ j_{1*} = f_2 \circ j_{2*}$. Then there is a unique homomorphism $f : \pi_1(X) \to G$ satisfying $f_1 = f \circ i_{1*}$ and $f_2 = f \circ i_{2*}$.*



The Seifert-van Kampen theorem states that the commutative square above is a pushout. For the proof, we will need a few lemma's relating G-bundles to their restriction. The first one establishes how restricting a G-bundle interacts with the associated morphisms.

Lemma 8.3. *With notation as in 8.1, let $p : Y \to X$ a pointed G-bundle, and let $p' : U' \to U$ be the restriction of the bundle to U, where $U' = p^{-1}(U)$. Let $f_p : \pi_1(X) \to G$ and $f'_p : \pi_1(U) \to G$ be the respective associated morphisms. Then $f'_p = f_p \circ i_*$.*

Proof. Let $[\gamma] \in \pi_1(U)$ be arbitrary, and let $\widetilde{\gamma}$ be the lifting to U' starting at y_0. Then $g := f'_p([\gamma])$ is the element of G such that $\widetilde{\gamma}(1) = g \cdot y_0$. But $\widetilde{\gamma}$ is also a lift in Y of γ considered as a path in X, and thus from $\widetilde{\gamma}(1) = g \cdot y_0$ we also obtain $g = f_p(i_*([\gamma]))$. The statement follows. □

The next lemma we need shows that we can patch a G-bundle over U to a G-bundle over V, provided they agree over W.

Lemma 8.4. *Let notation be as in 8.1, and let $p_1 : Y_1 \to U$ and $p_2 : Y_2 \to V$ be pointed G-bundles. If the restrictions of the bundles are isomorphic over W, then they induce a pointed G-bundle over X.*

Proof. Suppose $\varphi : p_1^{-1}(W) \to p_2^{-1}(W)$ is an isomorphism of pointed G-bundles. The idea is to use φ to glue Y_1 and Y_2 together. To make this formal, let $Y' = Y_1 \sqcup Y_2$ be the disjoint union of Y_1 and Y_2, and call $y_1 \in Y_1$ and $y_2 \in Y_2$ equivalent if y_1 is an element of $p_1^{-1}(W)$ and $\varphi(y_1) = y_2$. Then let Y be Y' modulo this equivalence. Now we can set $p : Y \to X$ as $p(y) = p_1(y)$ if $y \in Y_1$ and $p(y) = p_2(y)$ if $y \in Y_2$ (note that this is well-defined because φ is a G-bundle isomorphism), and we let G act on Y in the natural way. To check that $p : Y \to X$ is a G-bundle, we note that Y is locally a trivial G-bundle and apply lemma 2.9. □

The previous lemma gives us a way to glue together G-bundles over U and V, provided they agree on W. The next lemma shows that this patching is unique up to isomorphism.

Lemma 8.5. *Let $p : Y \to X$ and $q : Y' \to X$ be pointed G-bundles such that the restriction of p to U is isomorphic to the restriction of q to U, and the restriction of p to V is isomorphic to the restriction of q to V. Then Y and Y' are isomorphic.*

Proof. Let $\varphi_1 : p^{-1}(U) \to q^{-1}(U)$ and $\varphi_2 : p^{-1}(V) \to q^{-1}(V)$ be the isomorphisms. We want to glue φ_1 and φ_2 together to construct an isomorphism $Y \to Y'$. We claim that $\varphi_1(y) = \varphi_2(y)$ when $p(y) \in W$. To see this, we first note that $\varphi_1(y)$ and $\varphi_2(y)$ are in the same fiber over X, so there is a unique $g \in G$ such that $\varphi_1(y) = g \cdot \varphi_2(y)$. This induces a function $f : p^{-1}(W) \to G$ such that $\varphi_1(y) = f(y) \cdot \varphi_2(y)$ for all $y \in p^{-1}(W)$. Since any G-bundle is locally trivial, we find that f is locally constant. Moreover, for $y \in p^{-1}(W)$ and $g \in G$ we have

$$\varphi_1(gy) = g\varphi_1(y) = g \cdot f(y) \cdot \varphi_2(y) = g \cdot f(y) \cdot g^{-1}\varphi_2(gy),$$

which shows that $f(gy) = g \cdot f(y) \cdot g^{-1}$.

Now, let y_0 and y'_0 be the base points of Y and Y', then we have $\varphi_1(y_0) = y'_0 = \varphi_2(y_0)$. So $f(y_0) = e$, and therefore $f(gy_0) = e$. Thus the value of f is equal to e on the fiber $p^{-1}(x_0)$. Now, let $y \in p^{-1}(W)$ be arbitrary. Since W is connected, there is a path γ from $p(y)$ to x_0. Then the lift of γ to y connects y to some point in $p^{-1}(x_0)$. Since f is locally constant and has value e on the endpoint of the lift of γ, this implies that $f(y) = e$. Since $y \in p^{-1}(W)$ was arbitrary, we find that $\varphi_1(y) = \varphi_2(y)$ for all $y \in p^{-1}(W)$.

Thus, we can define $\varphi : Y \to Y'$ by setting $\varphi(y) = \varphi_1(y)$ if $p(y) \in U$ and $\varphi(y) = \varphi_2(y)$ if $p(y) \in V$. The above shows exactly that this is well-defined. Moreover, φ is a morphism of G-bundles because it locally is. Therefore, Y and Y' are isomorphic. □

With these lemmas, we have enough to prove the theorem.

Proof of theorem 8.2. We will first prove existence. Let $p_1 : Y_1 \to U$ be the pointed G-bundle over U with f_1 as associated morphism, and $p_2 : Y_2 \to V$ the pointed G-bundle over V with f_2 as associated morphism. We claim that the restrictions over W are isomorphic. For this, note that the restriction of p_1 to W has $f_1 \circ j_{1*}$ as associated morphism by lemma 8.3. Similarily, the restriction of p_2 to W has $f_2 \circ j_{2*}$ as associated morphism. But these are the same by assumption, and therefore the restrictions of the bundles to W are isomorphic. Now we apply 8.4 to glue Y_1 and Y_2 together to a pointed G-bundle $p : Y \to X$ over X. Finally, we let f be the associated morphism of this bundle. Applying 8.3 again, we find that $f_1 = f \circ i_{1*}$ and $f_2 = f \circ i_{2*}$.

So it only remains to show that f is unique. To this end, suppose there is another $f' : \pi_1(X) \to G$ such that $f_1 = f' \circ i_{1*}$ and $f_2 = f' \circ i_{2*}$. Then there is some pointed G-bundle $q : Y' \to X$ with f' as associated morphism. The restriction of this bundle to U must by lemma 8.3 have $f' \circ i_{1*} = f_1$ as associated morphism, which is by construction also the associated morphism of the bundle $p_1 : Y_1 \to U$. Therefore, the restriction of q to U is isomorphic to the restriction of p to U. Likewise, the restriction of q to V is isomorphic to the restriction of p to V. Applying lemma 8.5, we find that Y and Y' are isomorphic as G-bundles, and therefore the have the same associated morphism. Thus we have $f = f'$, and this ends the proof. □

The proof above gives us a different way of thinking about the statement of the Seifert-van Kampen theorem. When we have G-bundles over U and V that are compatible over W, then there is exactly one way to patch these bundles to a G-bundle over X. The statement of the theorem is then just a reformulation in terms of associated morphisms.

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