The Banach-Stone Theorem

Meer, E.C.H. van der

E.C.H. van der Meer

The Banach-Stone Theorem

Bachelor's thesis, January 29, 2014

Thesis advisor: Dr. M.F.E. de Jeu

Mathematical Institute, Leiden University

Contents

1 Introduction

In this thesis we will get familiar with the Theorem of Banach and Stone and a nice application. The Theorem is a classic result within the theory of spaces of continuous maps on compact Hausdorff spaces and is named after the mathematicians Stefan Banach and Marshall Stone.

Let $\mathrm{C}(K)$ denote the Banach space consisting of all real- or complex-valued continuous functions on a compact Hausdorff space K, with the supremumnorm $\|f\|_\infty = \sup\{|f(x)| : x \in K\}$. Stefan Banach thought about the case when there is a isometric function between two such spaces. In 1932 he solved his problem for compact metric spaces K by describing that isometric map. This was extended by Marshall Stone in 1937 to general compact Hausdorff spaces K. We will consider this generalized version in this paper.

The contents of this paper are as follows. In the next Section, we will discuss some basic results from different fields in mathematics. We will conclude this Section with a precise formulation of the Banach-Stone Theorem, Theorem 2.14. In Section 3 we consider more advanced results from linear functional analysis. In Section 4 we prove some important theorems from topology which are essential in the proof of the Banach-Stone Theorem. Then we have enough knowledge to prove it in Section 5. We conclude this thesis with Section 6, where we apply the Banach-Stone Theorem to obtain a description of the structure on the group of isometric isomorphisms from $\mathrm{C}(K)$ to $\mathrm{C}(K)$.

Prior to reading this thesis, one should have some basic knowledge of topology and linear functional analysis. A short recap is given, but for details one could consult [6] or [7].

Before we take off, I would like to express my appreciation to my advisor M.F.E. de Jeu for the support and interesting conversations during this project.

2 Basic Results

Before we start introducing some more advanced tools, we will discuss some elementary definitions and facts in this section, which will be concluded by formulating the Theorem of Banach-Stone. We will also state a known result which is going to be important in the proof. The Banach-Stone Theorem and its proof depend on some other special structures, so we start by defining those structures.

2.1. Definition. A topological space $(X, \mathcal{T})$ is called *locally compact* if there exists a compact neighborhood for every point of X.

2.2. Definition. A topological space $(X, \mathcal{T})$ is called *Hausdorff* if, for any $x, y \in X$ with $x \neq y$, there are open sets $U, V \in \mathcal{T}$ with $x \in U$, $y \in V$ and $U \cap V = \varnothing$.

We would like to define a special structure on a locally compact space, but the notion of a general σ-algebra is needed for that.

2.3. Definition. Let Ω be a set. A subset $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is called a *σ-algebra (in Ω)* if it satisfies the following properties

- $\Omega \in \mathcal{A}$,
- $\Omega \setminus A = A^c \in \mathcal{A}$ for any $A \in \mathcal{A}$,
- $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$ for any sequence of sets $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$.

2.4. Definition. Let X be a locally compact space and let $\mathcal{B}(X)$ denote the smallest σ-algebra of subsets of X that contains all open sets of X. Then $\mathcal{B}(X)$ is called the *Borel-σ-algebra of X*, and its elements are called *Borel sets*.

This Borel-σ-algebra of X is essential for our proof of the Banach-Stone Theorem. In measure theory, for example [2], the notion of a measure on a measurable space is introduced. In the following definition we consider a special class of measures on $(X, \mathcal{B}(X))$.

2.5. Definition. A positive measure μ on the Borel-σ-algebra $(X, \mathcal{B}(X))$ is called a *regular Borel measure* if it is

- a *Borel measure*, $\mu(K) < \infty$ for $K \subset X$ compact,
- *inner regular*, $\mu(V) = \sup\{\mu(K) : K \subset V \text{ with } K \text{ compact}\}$ for any $V \in \mathcal{B}(X)$,
- *outer regular*, $\mu(V) = \inf\{\mu(U) : V \subset U \text{ with } U \text{ open in } X\}$ for any $V \in \mathcal{B}(X)$.

If μ is a complex-valued measure, it is regular Borel if $|\mu|$, the *variation of* μ, is a regular Borel measure, see [3, Definitions C.3 and C.10].

2.6. Definition. Let $\mathrm{M}(X)$ denote the set of all complex-valued regular Borel measures on X.

The variation of a measure is also used for defining a norm on this space $\mathrm{M}(X)$, as stated in the next proposition.

2.7. Proposition. $\mathrm{M}(X)$ is a vector space over $\mathbb{C}$ with its natural addition and scalar multiplication and norm $\|\mu\| = |\mu|(X)$ for $\mu \in \mathrm{M}(X)$.

Before we switch from measure theory to linear functional analysis, we recall two propositions about continuous maps between topological spaces.

2.8. Proposition. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces then the following are equivalent for a map $f\colon X \to Y$

1. f is continuous,
2. $f^{-1}(F)$ is closed in X for every closed $F \subseteq Y$,
3. $f^{-1}(U)$ is open in X for every open $U \subseteq Y$.

2.9. Proposition. Let X, Y be topological spaces. Suppose that X is compact, Y is Hausdorff and $f\colon X \to Y$ bijective and continuous. Then f is a homeomorphism.

Proof. The only thing to prove here is that $f^{-1}\colon Y \to X$ is continuous. Let $F \subseteq X$ be a closed set. Note that F is compact and that $(f^{-1})^{-1}(F) = f(F)$ since f is a bijection. Furthermore, note that $f(F)$ is a compact subset of Y since f is a continuous map. Because Y is Hausdorff, $f(F)$ is closed in Y. Now we have that $(f^{-1})^{-1}(F)$ is closed, hence f^{-1} is continuous by Proposition 2.8. □

From now on, we will switch to linear functional analysis. In particular, we will consider the following function spaces.

2.10. Definition. Let X be a topological space. Define the following sets of functions,

$$\begin{aligned}
\mathrm{C}(X) &:= \{f\colon X \to \mathbb{F} : f \text{ is continuous}\},\\
\mathrm{C_b}(X) &:= \{f \in \mathrm{C}(X) : f \text{ is bounded }\},\\
\mathrm{C_0}(X) &:= \{f \in \mathrm{C}(X) : \{x \in X : |f(x)| \geqslant \varepsilon\} \text{ is compact in } X \text{ for all } \varepsilon \in \mathbb{R}_{>0}\}.
\end{aligned}$$

It is easy to verify that both $\mathrm{C_b}(X)$ and $\mathrm{C_0}(X)$ are vector spaces over $\mathbb{F}$ (which will always be either $\mathbb{R}$ or $\mathbb{C}$) with the natural addition and scalar multiplication. Moreover, $\mathrm{C_b}(X)$ and $\mathrm{C_0}(X)$ can be made into normed spaces since the function $\|f\| := \sup_{x\in X}|f(x)|$ defines a norm on $\mathrm{C_b}(X)$.

2.11. Notation. The space $\mathrm{C_b}(X)$ will be called *the dual (space) of* X and denoted by X^*. Also, for $f \in X^*$ and $x \in X$ we will sometimes write $\langle f, x\rangle$ instead of $f(x)$. There are some other interesting properties that are worth mentioning.

2.12. Proposition. Let X be a Hausdorff space. Then both $\mathrm{C_b}(X)$ and $\mathrm{C_0}(X)$ are Banach spaces over $\mathbb{F}$ and $\mathrm{C_0}(X)$ is a closed linear subspace of $\mathrm{C_b}(X)$.

Proof. Let $\varepsilon = 1$, $f \in \mathrm{C_0}(X)$ and define $V := \{x \in X : |f(x)| \geqslant 1\}$. Then, by assumption, we have that V is compact in X. Thus $f(V)$ is compact in $\mathbb{F}$ since f is continuous. Since $\mathbb{F}$ is either equal to $\mathbb{R}$ or homeomorphic to $\mathbb{R}^2$, we conclude that $f(V)$ is always closed and bounded by the Heine-Borel Theorem, [6, Corollary 2.5.12]. Hence f is bounded on V and we have $|f| < 1$ on $X \setminus V$. We conclude that $\mathrm{C_0}(X) \subseteq \mathrm{C_b}(X)$. From now on this is a routine verification, we will not spell it out. For the details, see [3, Example III.1.6]. □

There is another important result which we will need before we are able to state the Banach-Stone Theorem.

2.13. Proposition. For X compact Hausdorff the equality $\mathrm{C}_0(X) = \mathrm{C}_\mathrm{b}(X) = \mathrm{C}(X)$ holds.

Proof. We have already proved that $\mathrm{C}_0(X) \subseteq \mathrm{C}_\mathrm{b}(X) \subseteq \mathrm{C}(X)$. Now let $f \in \mathrm{C}_\mathrm{b}(X)$ and $\varepsilon \in \mathbb{R}_{>0}$. Then $f^{-1}(B_\varepsilon(0)) = \{x \in X : |f(x)| < \varepsilon\}$ is open in X. This is immediate from Proposition 2.8 since $B_\varepsilon(0)$ denotes the open ball around 0 with radius ε and f is continuous. Then its complement is closed and equal to $\{x \in X : |f(x)| \geqslant \varepsilon\}$. Now we have a closed set in a compact space, therefore it is compact. Thus $f \in \mathrm{C}_0(X)$ and we conclude that $\mathrm{C}_\mathrm{b}(X) = \mathrm{C}_0(X)$. For the second equality we note that a continuous $\mathbb{F}$-valued function f on the compact space X has a compact image $f(X) \subseteq \mathbb{F}$. Now we apply the same argument as in the previous proof, with X instead of V. Hence f is bounded on X so $f \in \mathrm{C}_\mathrm{b}(X)$ and $\mathrm{C}_\mathrm{b}(X) = \mathrm{C}(X)$. □

At this moment we have enough knowledge to state and understand the Banach-Stone Theorem. As stated in the Introduction, it will be proved in Section 5.

2.14. Theorem (Banach-Stone). Let X, Y be compact Hausdorff spaces and let T be a surjective isometry $\mathrm{C}(X) \to \mathrm{C}(Y)$. Then there exists a homeomorphism $\tau\colon Y \to X$ and a function $h \in \mathrm{C}(Y)$ such that $|h(y)| = 1$ for all $y \in Y$ and

$$T(f)(y) = h(y)f(\tau(y)), \text{ for all } f \in \mathrm{C}(X) \text{ and } y \in Y.$$

Also the converse is true. Let h and τ as stated, and define the map $T\colon \mathrm{C}(X) \to \mathrm{C}(Y)$ by $T(f)(y) = h(y)f(\tau(y))$ for $f \in \mathrm{C}(X)$ and $y \in Y$. Since τ is a homeomorphism and the assumption $|h(y)| = 1$ for all $y \in Y$ we have that

$$\|T(f)\| = \sup_{y \in Y} |T(f)(y)| = \sup_{y \in Y} |h(y)f(\tau(y))| = \sup_{y \in Y} |h(y)||f(\tau(y))| = \sup_{x \in X} |f(x)| = \|f\|.$$

Thus T is isometric. It is also surjective. For any $g \in \mathrm{C}(Y)$ we define $f := \frac{1}{h \circ \tau^{-1}} \cdot (g \circ \tau^{-1})$. Since h, τ^{-1} and g are continuous maps we have that both $g \circ \tau^{-1}$ and $h \circ \tau^{-1}$ are continuous maps. Thus f is continuous as product of composition of continuous maps, since $h(y) \neq 0$ for all $y \in Y$. Let $y \in Y$, then we have

$$\begin{aligned} T(f)(y) &= h(y)\left(\frac{1}{h \circ \tau^{-1}} \cdot (g \circ \tau^{-1})\right)(\tau(y)) = h(y)\left(\frac{1}{h \circ \tau^{-1}}\right)(\tau(y)) \cdot (g \circ \tau^{-1})(\tau(y)) \\ &= \frac{h(y)}{h(y)} \cdot g(y) = g(y). \end{aligned}$$

Hence $Tf = g$ and the converse of the Banach-Stone Theorem is proved. The proof of the Banach-Stone Theorem is more complicated and involves the Riesz Representation Theorem.

2.15. Theorem (Riesz Representation Theorem). Let X be a locally compact Hausdorff space and $\mu \in \mathrm{M}(X)$. Define

$$\begin{aligned} F_\mu\colon \mathrm{C}_0(X) &\to \mathbb{C} \\ f &\mapsto \int f \,\mathrm{d}\mu. \end{aligned}$$

Then $F_\mu \in \mathrm{C}_0(X)^*$ and the map

$$\begin{array}{rcl} \mathrm{M}(X) & \to & \mathrm{C}_0(X)^* \\ \mu & \mapsto & F_\mu \end{array}$$

is an isometric isomorphism.

The proof of this theorem is an involved construction, which will not be discussed here. For more details, one consults [4, Theorem 7.17 and Corollary 7.18].

3 Linear Functional Analysis

The first part of the proof of the Banach-Stone Theorem is fully covered by general results about normed spaces from Linear Functional Analysis. In this section we will investigate those results and give a rough sketch of the structure of our proof. Before we start, recall that continuous and bounded maps between normed vectorspaces are called *bounded linear operators* or sometimes just *operators* or *(linear) functionals*. We will also use isometric isomorphism instead of surjective isometry, because tey are equivalent. This is clear since every isometric map is always injective. We start with the following theorem, which states the existence of a special map for any bounded linear operator, just like every isomorphism has an inverse.

3.1. Theorem (Adjoint map). Let X, Y be normed spaces and $T\colon X \to Y$ a bounded linear operator. Then there exists a unique operator $T^*\colon Y^* \to X^*$ such that

$$T^*(f)(x) = f(Tx), \text{ for all } f \in Y^*, x \in X.$$

For a proof of this theorem see [7, Theorem 5.50]. We will call T^* the *adjoint of* T. The following lemma is needed in the proof of the next proposition, which states another important property of this dual map.

3.2. Lemma. Let $(X, \|\cdot\|)$ be a normed space with $X \neq \{0\}, \varnothing$ and $f \in X^*$. Then for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $x \in X$ such that $\|x\| = 1$ and $\|f\| - \varepsilon \leqslant |f(x)|$.

Proof. Note that for $f = 0$ we have $\|f\| = 0$, thus any $x \in X$ will do. So assume $f \neq 0$, and $\varepsilon \in \mathbb{R}_{>0}$. Then, by definition of the supremumnorm, there exists $0 \neq y \in X$ such that $\|y\| \leqslant 1$ and $\|f\| - \varepsilon \leqslant |f(y)|$. Since $y \neq 0$, the norm of $x := \frac{y}{\|y\|}$ is equal to 1 and $\frac{1}{\|y\|} \geqslant 1$. Now we have

$$\|f\| - \varepsilon \leqslant \frac{1}{\|y\|}(\|f\| - \varepsilon) \leqslant \frac{f(y)}{\|y\|} = \|f(x)\|. \qquad \square$$

3.3. Proposition. Let X, Y be normed spaces and $T\colon X \to Y$ an isometric isomorphism. Then $T^*\colon Y^* \to X^*$ is again an isometric isomorphism with $(T^*)^{-1} = (T^{-1})^*$.

Proof. Let $S := T^{-1}$ and let S^* and T^* be as in Theorem 3.1. Now, let $f \in X^*, x \in X$. Then, by definition of S,

$$T^*(S^* f)(x) = S^*(f)(Tx) = f((S \circ T)(x)) = f(x)$$

On the other hand, for $g \in Y^*, y \in Y$ we conclude from that definition

$$S^*(T^* g)(y) = T^*(g)(Sy) = g((T \circ S)(y)) = g(y)$$

Hence, we have $S^* \circ T^* = \mathrm{I}_{Y^*}$ and $T^* \circ S^* = \mathrm{I}_{X^*}$. We conclude that $(T^*)^{-1} = S^*$ and $T^* = (S^*)^{-1}$. So T^* is an isomorphism which satisfies $(T^*)^{-1} = (T^{-1})^*$.

In order to prove that T^* is isometric, let $f \in Y^*$ and consider $\|T^*(f)\|$. For $x \in X$ we know that

$$\|T^*(f)(x)\| = \|f(Tx)\| \leqslant \|f\|\|Tx\| \leqslant \|f\|\|T\|\|x\|.$$

Since T is an isometry, we have $\|T\| = 1$. With the result above we have that

$$\begin{aligned}\|T^*(f)\| &= \sup\{\|T^*(f)(x)\| : x \in X \text{ such that } \|x\| \leqslant 1\}\\ &\leqslant \|T\|\|f\| \sup\{\|x\| : x \in X \text{ such that } \|x\| \leqslant 1\}\\ &\leqslant \|T\|\|f\| = \|f\|.\end{aligned}$$

Let $\varepsilon \in \mathbb{R}_{>0}$. Then, by Lemma 3.2, there exists $y \in Y$ such that $\|y\| = 1$ and $\|f\| - \varepsilon \leqslant |f(y)|$. Since T is an isometric isomorphism, $x := T^{-1}y$ is well-defined and we have

$$\|x\| = \|Tx\| = \|TT^{-1}y\| = \|y\| = 1.$$

Now we conclude

$$\begin{aligned}\|f\| - \varepsilon \leqslant |f(y)| = |f(Tx)| &= |T^*(f)(x)|\\ &\leqslant \sup\{|T^*(f)(x)| : x \in X \text{ such that } \|x\| \leqslant 1\}\\ &= \|T^*(f)\|\end{aligned}$$

Then we have

$$\|f\| = \sup_{\varepsilon \in \mathbb{R}_{>0}} \{\|f\| - \varepsilon\} \leqslant \sup_{\varepsilon \in \mathbb{R}_{>0}} \|T^*(f)\| = \|T^*(f)\|.$$

We conclude $\|T^*(f)\| = \|f\|$. Hence T^* is an isometric isomorphism. □

Now that we know some properties of this dual space and the adjoint, consider the following map.

3.4. Definition. Let X be any Hausdorff space and $x \in X$. Define the *evaluation map* by

$$\begin{aligned}\delta_x \colon \mathrm{C_b}(X) &\to \mathbb{F}\\ f &\mapsto f(x).\end{aligned}$$

This map has some convenient properties, as stated in the next proposition.

3.5. Proposition. For δ_x as in Definition 3.4 above, $\delta_x \in \mathrm{C_b}(X)^*$ and $\|\delta_x\| = 1$.

Proof. It is clear from Definition 3.4 that δ_x is linear, since f is linear. On the other hand, by using the operator norm

$$\begin{aligned}\|\delta_x\| &= \sup\{\|\delta_x(f)\| : f \in \mathrm{C_b}(X) \text{ such that } \|f\| \leqslant 1\}\\ &\leqslant \sup\{|f(x)| : f \in \mathrm{C_b}(X) \text{ such that } \|f\| \leqslant 1\}\\ &\leqslant 1.\end{aligned}$$

On the other hand, for $f = 1 \in \mathrm{C_b}(X)$ we have equality. So we conclude $\delta_x \in \mathrm{C_b}(X)^*$ with $\|\delta_x\| = 1$. □

Now it is time to reveal the connection between the previous two sections, especially the one between the regular Borel measures on X and the dual of X. We need the following definitions to make it clear.

3.6. Definition. Let K be a convex set in a vector space X, and $x_1, x_2 \in K$. Let

$$[x_1, x_2] := \{\lambda x_1 + (1 - \lambda)x_2 : \lambda \in [0, 1]\}$$

denote the *line segment* connecting x_1 with x_2. We call it *proper* if $x_1 \neq x_2$. The *open line segment* connecting x_1 with x_2 is $(x_1, x_2) \coloneqq [x_1, x_2] \setminus \{x_1, x_2\}$.

This definition will make more sense with the next definition and theorem.

3.7. Definition. Let X be a vector space and $K \subseteq X$ a convex subset. A point $k \in K$ is called an *extreme point of* K if there exists no proper open line segment that contains k. Let $\operatorname{Ext} K$ be the of extreme points of K.

Recall that, for any normed space X, the closed unit ball of X is denoted by $\operatorname{Ball} X$. Then we have the following theorem.

3.8. Theorem. Let X be a compact Hausdorff space. Then

$$\operatorname{Ext}[\operatorname{Ball} \mathrm{M}(X)] = \{\alpha\delta_x : \alpha \in \mathbb{F} \text{ with } |\alpha| = 1 \text{ and } x \in X\}.$$

The proof of this theorem is in [3, Theorem V.8.4]. Together with the Riesz Representation Theorem, this will be the key to success while proving the Banach-Stone theorem. Before we proceed to this proof we have to consider a special topology on the dual.

4 Nets and weak*-topology on the dual

In metric spaces there is the notion of a (converging) sequence. However, this is not sufficient in general topological spaces. In this section we will discuss a well-known generalization. First of all, we have some definitions.

4.1. Definition. Let $(X, \mathcal{T})$ be a topological space and $x \in X$. A set $N \subseteq X$ is called a *neighborhood of* x if there is a set $U \in \mathcal{T}$ such that $x \in U \subseteq N$. Let $\mathcal{N}_x$ denote the collection of neighborhoods of x.

4.2. Definition. A *directed set* is a partially ordered set $(\mathbb{A}, \leqslant)$ such that if $\alpha_1, \alpha_2 \in I$, then there exists $\alpha_3 \in I$ such that $\alpha_1 \leqslant \alpha_3$ and $\alpha_2 \leqslant \alpha_3$.

4.3. Definition. Let $(X, \mathcal{T})$ be a topological space. A *net* in X is a pair $((\mathbb{A}, \leqslant), x)$ with $\mathbb{A}$ a directed set and x a function $\mathbb{A} \to X$.

We will write x_α instead of $x(\alpha)$ and will use the phrase "let $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ be a net in X". Since $\mathbb{N}$ with its natural ordering is a directed set, any sequence is a net.

4.4. Definition. Let $(X, \mathcal{T})$ be a topological space and $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ a net in X. Then $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ *converges* to $x \in X$ (in symbols, $x = \lim_\alpha x_\alpha$) if for every $N \in \mathcal{N}_x$ there exists an $\alpha_N \in \mathbb{A}$ such that $x_\alpha \in N$ for all $\alpha \in \mathbb{A}$ with $\alpha_N \leqslant \alpha$.

4.5. Proposition. Let $(X, \mathcal{T})$ be a Hausdorff space and $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ a net in X and $x_1, x_2 \in X$ such that the net converges to both x_1 and x_2. Then x_1 and x_2 are equal.

Proof. Let $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ be a net in X such that $\lim_\alpha x_\alpha = x_1$ and $\lim_\alpha x_\alpha = x_2$ with $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Then there are $U, V \in \mathcal{T}$ such that $x_1 \in U, x_2 \in V$ and $U \cap V = \varnothing$ since X is Hausdorff. From the assumption on convergence of the net, we have $\alpha_U, \alpha_V \in \mathbb{A}$ such that $x_\alpha \in U$ for every $\alpha \in \mathbb{A}$ with $\alpha_U \leqslant \alpha$ and $x_\alpha \in V$ for every $\alpha \in \mathbb{A}$ with $\alpha_V \leqslant \alpha$. Since $\mathbb{A}$ is a directed set, there is an index α_0 such that $\alpha_U, \alpha_V \leqslant \alpha_0$. Then we have a contradiction, because for every $\alpha \in \mathbb{A}$ such that $\alpha_0 \leqslant \alpha$ we have that $\alpha \in U \cap V = \varnothing$. We conclude $x_1 = x_2$, so the limit is unique. □

We already have some characterizations for a continuous map, as stated in Proposition 2.8. We are almost ready to to extend it.

4.6. Definition. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces and $x_0 \in X$. A function $f\colon X \to Y$ is called *continuous at* x_0 if $f^{-1}(N) \in \mathcal{N}_{x_0}$ for every $N \in \mathcal{N}_{f(x_0)}$. As usual, we call f *continuous* if it is continuous at every point of X.

4.7. Proposition. Let $(X, \mathcal{T}_X), (Y, \mathcal{T}_Y)$ be topological spaces, $f\colon X \to Y$ and $x_0 \in X$. Then f is continuous at x_0 if and only if $\lim_\alpha f(x_\alpha) = f(x_0)$ for all nets $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ in X such that $\lim_\alpha x_\alpha = x_0$.

Proof. Assume that f is continuous at x_0 and let $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ be a net in X with limit x_0. Also let $V \in \mathcal{T}_Y$ such that $f(x_0) \in V$ and $N \in \mathcal{N}_{f(x_0)}$. According to the definition of continuity at x_0 we have $f^{-1}(N) \in \mathcal{N}_{x_0}$. Then there is an index $\alpha_0 \in \mathbb{A}$ such that $x_\alpha \in f^{-1}(N)$ for all $\alpha \in \mathbb{A}$ with $\alpha_0 \leqslant \alpha$. That is, for such α we have $f(x_\alpha) \in N$. We conclude that $\lim_\alpha f(x_\alpha) = f(x_0)$ since N was arbitrary.

Let $\{x_\alpha\}_{\alpha\in\mathbb{A}}$ be a net in X such that $\lim_\alpha x_\alpha = x_0$. Now assume that $\lim_\alpha f(x_\alpha) = f(x_0)$ and let $\mathcal{U} = \{U \in \mathcal{T}_X : x_0 \in U\}$. Suppose that f is not continuous at x_0. Then there exists an $N \in \mathcal{N}_{f(x_0)}$ such that $f^{-1}(N) \notin \mathcal{N}_{x_0}$. That is, for every $U \in \mathcal{U}$ we have $U \nsubseteq f^{-1}(N)$. Hence for every $U \in \mathcal{U}$ we have that $f(U) \setminus N \neq \varnothing$, that is, there exists $x_U \in U$ such that $f(x_U) \notin N$. Now we have that $\mathcal{U}$ is a directed set: for $U, V \in \mathcal{U}$ define $U \leqslant V$ if $V \subseteq U$. Hence $\{x_U\}_{U\in\mathcal{U}}$ is a net and it converges to x_0 while the net $\{f(x_U)\}_{U\in\mathcal{U}}$ can not converge to $f(x_0)$ by assumption on N. This is a contradiction, and we conclude that f is continuous at x_0. □

Also, we may equip the dual of a normed space with a special topology.

4.8. Definition. Let X be a normed space. The *weak* topology* is the weakest topology on X^* such that the map

$$\begin{aligned} \phi_x \colon X^* &\to \mathbb{F} \\ x^* &\mapsto \langle x^*, x\rangle \end{aligned}$$

is continuous for all $x \in X$.

In case we consider the dual of a normed space X together with this weak* topology, we simply write (X^*, wk^*). Then we have the next lemma.

4.9. Lemma. Let X be a normed space. The weak* topology on X^* is generated by all sets of the form $\{(\phi_x)^{-1}[U] : U \subseteq \mathbb{F} \text{ open and } x \in X\}$.

Proof. Since ϕ_x must be continuous for all $x \in X$ and $U \subseteq \mathbb{F}$ open, those sets are exactly the ones from Proposition 2.8. □

Let us now consider what it means for a net in this topology to converge.

4.10. Lemma. Let X be a normed space and consider (X^*, wk^*). Let $\{x^*_\alpha\}_{\alpha\in\mathbb{A}}$ a net in X^* and $x^* \in X^*$ a function. Then $\{x^*_\alpha\}_{\alpha\in\mathbb{A}}$ is *weak* convergent to* x^* (in symbols $\lim_\alpha x^*_\alpha = x^*$) if and only if

$$\lim_\alpha \langle x^*_\alpha, x\rangle = \langle x^*, x\rangle \text{ for all } x \in X.$$

Proof. Apply [5, Proposition 2.2.4] on X^* and the family $\{x^* \mapsto \langle x^*, x\rangle : x \in X\}$. □

With this equivalence in mind, we define a continuous map between two topological spaces with this weak* topology.

4.11. Definition. Let X, Y be normed spaces and consider $(X^*, \mathrm{wk}^*), (Y^*, \mathrm{wk}^*)$. Let $\{x^*_\alpha\}_{\alpha\in\mathbb{A}}$ a net in X^* converging to $x^* \in X^*$ and $T \colon X^* \to Y^*$ a map. Then T is *weak*-weak* continuous* if $\lim_\alpha Tx^*_\alpha = Tx^*$.

Now we are able to prove the following proposition, which will be very useful.

4.12. Proposition. Let X, Y be topological vector spaces and $T \colon X \to Y$ an isometric isomorphism. Then T^* is a weak*-weak* homeomorphism of $\operatorname{Ball} Y^*$ onto $\operatorname{Ball} X^*$.

Proof. From Lemma 3.3 we have that T^* is an isomorphism such that $(T^*)^{-1} = (T^{-1})^*$. So, it suffices to show that T^* and $(T^*)^{-1}$ are weak*-weak* continuous. Let $\{y^*_\alpha\}_{\alpha\in\mathbb{A}}$ be a net in

Y^* such that $\lim_\alpha y^*_\alpha = y^*$. Let $x \in X$ and consider $\lim_\alpha \langle T^* y^*_\alpha, x\rangle$. Now we use the definition of T^* to find

$$\lim_\alpha \langle T^* y^*_\alpha, x\rangle = \lim_\alpha \langle y^*_\alpha, Tx\rangle = \langle y^*, Tx\rangle = \langle T^* y^*, x\rangle.$$

So we conclude that $\lim_\alpha T^* y^*_\alpha = T^* y^*$. Thus T^* is weak*-weak* continuous.

Now let $\{x^*_\alpha\}_{\alpha \in \mathbb{A}}$ be a net in X^* converging to $x^* \in X^*$. Now let $y \in Y^*$ and use the property of T^* from Lemma 3.3

$$\begin{aligned}\lim_\alpha \langle (T^*)^{-1} x^*_\alpha, y\rangle &= \lim_\alpha \langle (T^{-1})^* x^*_\alpha, y\rangle \\ &= \lim_\alpha \langle x^*_\alpha, T^{-1} y\rangle \\ &= \langle x^*, T^{-1} y\rangle \\ &= \langle (T^*)^{-1} x^*, y\rangle.\end{aligned}$$

So we conclude that $\lim_\alpha (T^*)^{-1} y^*_\alpha = (T^*)^{-1} y^*$. Thus $(T^*)^{-1}$ is weak*-weak* continuous.

By Lemma 3.3 we also have that T^* is an isometry, so the unit ball of Y^* is mapped onto the unit ball of X^*. Hence T^* is a weak*-weak* homeomorphism such that

$$T^*[\mathrm{Ball}\, Y^*] = \mathrm{Ball}\, X^*.$$

□

We have another important property of the evaluation map from Definition 3.4.

4.13. Proposition. Let X be a compact Hausdorff space and consider the map

$$\begin{aligned}\Delta\colon X &\to (\mathrm{C}(X)^*, \mathrm{wk}^*) \\ x &\mapsto \delta_x.\end{aligned}$$

Then Δ is well-defined and a homeomorphism onto its image $(\Delta(X), \mathrm{wk}^*)$.

Proof. First of all, the map is well-defined by Proposition 3.5. First we prove that Δ is continuous. So let $f \in \mathrm{C}(X)$ and $\{x_\alpha\}_{\alpha \in \mathbb{A}}$ a net in X converging to $x \in X$. Because f is continuous, we have that

$$\lim_\alpha \langle \delta_{x_\alpha}, f\rangle = \lim_\alpha \langle f, x_\alpha\rangle = \langle f, x\rangle = \langle \delta_x, f\rangle.$$

So, by Lemma 4.10 the net $\{\delta_{x_\alpha}\}_{\alpha \in \mathbb{A}}$ is weak* convergent to δ_x in the weak* topology on $\mathrm{C}(X)^*$. We conclude that Δ is continuous by Proposition 4.7. Let $x, y \in X$ with $x \neq y$. We will show that $\Delta(x) \neq \Delta(y)$ to conclude that Δ is injective. Since X is Hausdorff, the singletons $\{x\}, \{y\}$ are closed. Then, by Urysohn's Lemma [6, 4.1.2], there exists a continuous function $f\colon X \to \mathbb{F}$ such that $f(X) \subseteq [0,1] \subseteq \mathbb{R} \subseteq \mathbb{C}$ with $f(x) = 0$ and $f(y) = 1$. Thus $\Delta(x) = \delta_x(f) \neq \delta_y(f) = \Delta(y)$. We conclude that Δ is continuous and injective onto its image, hence $\Delta\colon X \to (\Delta(X), \mathrm{wk}^*)$ is a continuous bijection. We also need that $(\Delta(X), \mathrm{wk}^*)$ is Hausdorff. Let $f, g \in X^*$ such that $f \neq g$, so there exist $x \in X$ such that $f(x) \neq g(x)$. Since $\mathbb{F}$ is Hausdorff, there exist $U, V \subseteq \mathbb{F}$ open such that $U \cap V = \varnothing$ and $f(x) \in U$ and $g(x) \in V$. Then $f \in (\phi_x)^{-1}[U]$ and $g \in (\phi_x)^{-1}[V]$, which are open by Lemma 4.9. Also disjoint since U and V are disjoint. We conclude that $(\Delta(X), \mathrm{wk}^*)$ is Hausdorff. So, we have that X is compact, and $(\Delta(X), \mathrm{wk}^*)$ is Hausdorff, hence by Proposition 2.9 Δ is a homeomorphism onto its image $(\Delta(X), \mathrm{wk}^*)$. □

5 Banach-Stone Theorem

Now we have enough knowledge, so it is time to prove the main theorem of this thesis.

5.1. Theorem (Banach-Stone). Let X, Y be compact Hausdorff spaces and let T be a surjective isometry $\mathrm{C}(X) \to \mathrm{C}(Y)$. Then there exists a homeomorphism $\tau\colon Y \to X$ and a function $h \in \mathrm{C}(Y)$ such that $|h(y)| = 1$ for all $y \in Y$ and

$$T(f)(y) = h(y)f(\tau(y)), \text{ for all } f \in \mathrm{C}(X) \text{ and } y \in Y.$$

Proof. First of all, we know that T is an isometric isomorphism and by Proposition 3.3 both T and T^* are isometric isomorphisms, such that $(T^{-1})^* = (T^*)^{-1}$. From Proposition 4.12 we have that T^* is a weak*-weak* homeomorphism such that

$$T^*(\operatorname{Ball}\mathrm{C}(Y)^*) = \operatorname{Ball}\mathrm{C}(X)^*.$$

First note that those closed unit balls are convex. Furthermore, T^* is a linear map, hence it preserves convex combinations. It is therefore clear that

$$T^*(\operatorname{Ext}[\operatorname{Ball}\mathrm{C}(Y)^*]) = \operatorname{Ext}[\operatorname{Ball}\mathrm{C}(X)^*].$$

This is the point where the Riesz Representation Theorem, 2.15, comes in. One finds that

$$T^*(\operatorname{Ext}[\operatorname{Ball}\mathrm{M}(Y)]) = \operatorname{Ext}[\operatorname{Ball}\mathrm{M}(X)].$$

Let $y \in Y$, then by Theorem 3.8 there exists a unique $\tau(y) \in X$ and a unique scalar $h(y) \in \mathbb{F}$ such that $|h(y)| = 1$ and

$$T^*(\delta_y) = h(y)\delta_{\tau(y)}.$$

We will prove now that $h\colon Y \to \mathbb{F}$ is a continuous map and that $\tau\colon Y \to X$ is a homeomorphism. After that we will finish the proof.

Claim. $h\colon Y \to \mathbb{F}$ is continuous.
Proof. Let $y \in Y$ and $\{y_\alpha\}_{\alpha\in\mathbb{A}}$ be a net in Y converging to y. Since Δ is a continuous map, Proposition 4.13, we have that $\lim_\alpha \delta_{y_\alpha} = \delta_y$ in $\mathrm{C}(Y)^*$. Also, since T^* is a weak*-weak* continuous map, we have that

$$\lim_\alpha h(y_\alpha)\delta_{\tau(y_\alpha)} = \lim_\alpha T^*(\delta_{y_\alpha}) = T^*(\delta_y) = h(y)\delta_{\tau(y)}.$$

Now we have, for the constant map $1\colon X \to \mathbb{F}, x \mapsto 1$, which is clearly an element of $\mathrm{C}(X)$,

$$\lim_\alpha h(y_\alpha) = \lim_\alpha \langle h(y_\alpha)\delta_{y_\alpha}, 1\rangle = \langle h(y)\delta_{\tau(y)}, 1\rangle = h(y).$$

We conclude that h is a continuous map by Proposition 4.7, since y was arbitrary.

Claim. $\tau\colon Y \to X$ is a homeomorphism.
Proof. Let $y \in Y$ and $\{y_\alpha\}_{\alpha\in\mathbb{A}}$ be a net in Y converging to y. We will use the results

$\lim_\alpha h(y_\alpha)\delta_{\tau(y_\alpha)} = h(y)\delta_{\tau(y)}$ and $\lim_\alpha h(y_\alpha) = h(y)$ from the previous proposition. Since the scalar multiplication on a topological vector space is continuous, we have that

$$\lim_\alpha \delta_{\tau(y_\alpha)} = \lim_\alpha (h(y_\alpha))^{-1}[h(y_\alpha)\delta_{\tau(y_\alpha)}] = \delta_{\tau(y)}.$$

By Proposition 4.13, we have also that $\lim_\alpha \tau(y_\alpha) = \tau(y)$. Thus τ is continuous by Proposition 4.7 since y was arbitrary.

Now suppose that $y_1, y_2 \in Y$ and $y_1 \neq y_2$. Then, since Δ is injective, we have that $\delta_{y_1} \neq \delta_{y_2}$. So $\overline{h(y_1)}\delta_{y_1} \neq \overline{h(y_2)}\delta_{y_2}$ and since T^* is injective by Proposition 3.3, we conclude

$$\delta_{\tau(y_1)} = \overline{h(y_1)}h(y_1)\delta_{\tau(y_1)} = T^*\left(\overline{h(y_1)}\delta_{y_1}\right) \neq T^*\left(\overline{h(y_2)}\delta_{y_2}\right) = \overline{h(y_2)}h(y_2)\delta_{\tau(y_2)} = \delta_{\tau(y_2)}.$$

By Proposition 4.13, it is immediate that $\tau(y_1) \neq \tau(y_2)$. Thus τ is injective.

Let $x \in X$. From Proposition 3.3 we also have that T^* is surjective, so by the main proof, we conclude that there exists $\mu \in \mathrm{Ext}[\mathrm{Ball}\,\mathrm{M}(Y)]$ such that $T^*\mu = \delta_x$. Moreover, for some $\beta \in \mathbb{F}$ with $|\beta| = 1$ and $y \in Y$ we have that $\mu = \beta\delta_y$. Now we have that

$$\delta_x = T^*(\beta\delta_y) = \beta T^*(\delta_y) = \beta h(y)\delta_{\tau(y)}.$$

That is, $\beta = \overline{h(y)}$ and $x = \tau(y)$. Thus τ is surjective.

Therefore, we have that $\tau\colon Y \to X$ is a continuous bijection and by Proposition 2.9 we conclude that τ is a homeomorphism.

Let us now finish the main proof. So let $f \in \mathrm{C}(X)$ and $y \in Y$. Then one concludes, by using the definition of the adjoint in Definition 3.1, that

$$T(f)(y) = \langle \delta_y, Tf\rangle = \langle T^*\delta_y, f\rangle = \langle h(y)\delta_{\tau(y)}, f\rangle = h(y)\delta_{\tau(y)}(f) = h(y)f(\tau(y)). \qquad \square$$

6 Semi-direct product

In this final section, we will consider a special application of the Banach-Stone Theorem. We will also use that the converse of the Banach-Stone Theorem is true. From now on, suppose that X is a compact Hausdorff space. It turns out that there is a group theoretic connection between the following sets:

$$\begin{aligned}\mathrm{Isom}(C(X)) &:= \{T\colon \mathrm{C}(X) \to \mathrm{C}(X) : T \text{ is an isometric isomorphism}\},\\ \mathrm{Homeo}(X) &:= \{\tau\colon X \to X : \tau \text{ is a homeomorphism}\},\\ \mathrm{Um}(X) &:= \{h \in \mathrm{C}(X) : |h(x)| = 1 \text{ for all } x \in X\}.\end{aligned}$$

In particular, those sets are groups with respect to some simple operations. Let $\circ$ be the composition of maps and $\cdot$ the pointwise multiplication. Functions in the set $\mathrm{Um}(X)$ are sometimes called unimodular functions. Then it is immediate that $(\mathrm{Isom}(C(X)), \circ)$, $(\mathrm{Homeo}(X), \circ)$ and $(\mathrm{Um}(X), \cdot)$ are groups. Recall that, for a group G, the group of automorphisms of G is denoted by

$$\mathrm{Aut}(G) = \{f : G \to G : f \text{ is a bijective homomorphism}\}.$$

We will use this set in the definition of a semi-direct product.

6.1. Proposition. Let G and H be groups and $f\colon H \to \mathrm{Aut}(G)$ a homomorphism. Then the operation $(g_1, h_1)(g_2, h_2) = (g_1 f(h_1)(g_2), h_1 h_2)$ defines a group operation on the product $G \times H$.

The proof of this proposition is in [1, Proposition 8.12]. This group $G \times H$ is called the *semi-direct product of G and H with respect to f* and will be denoted by $G \rtimes_f H$. Let us consider the map

$$\begin{aligned}\sigma\colon \mathrm{Homeo}(X) &\to \mathrm{Aut}(\mathrm{Um}(X))\\ \tau &\mapsto \tau_*,\end{aligned}$$

where the map τ_* is defined as

$$\begin{aligned}\tau_*\colon \mathrm{Um}(X) &\to \mathrm{Um}(X)\\ h &\mapsto h \circ \tau^{-1}.\end{aligned}$$

6.2. Proposition. σ is a homomorphism.

Proof. First of all, we see that τ_* is well-defined, hence σ is a well-defined map. Let $\tau_1, \tau_2 \in \mathrm{Homeo}(X)$ and $h \in \mathrm{Um}(X)$. Then we have that

$$\begin{aligned}\sigma(\tau_1 \circ \tau_2)(h) &= (\tau_1 \circ \tau_2)_*(h)\\ &= h \circ (\tau_1 \circ \tau_2)^{-1}\\ &= (h \circ \tau_2^{-1}) \circ \tau_1^{-1}\\ &= \tau_{1*}(h \circ \tau_2^{-1})\\ &= (\tau_{1*} \circ \tau_{2*})(h)\\ &= (\sigma(\tau_1) \circ \sigma(\tau_2))(h).\end{aligned}$$

Hence $\sigma(\tau_1 \circ \tau_2) = \sigma(\tau_1) \circ \sigma(\tau_2)$ and we have that σ is a homomorphism. $\square$

By applying Proposition 6.1 we have that the semi-direct product $\mathrm{Um}(X) \rtimes_\sigma \mathrm{Homeo}(X)$ is a group with operation $(h_1, \tau_1)(h_2, \tau_2) = (h_1 \cdot \sigma(\tau_1)(h_2), \tau_1 \circ \tau_2)$. We are now able to prove the desired result, where we will need the Banach-Stone Theorem.

6.3. Theorem. The map

$$\begin{array}{ccc} \psi\colon \mathrm{Isom}(C(X)) & \to & \mathrm{Um}(X) \rtimes_\sigma \mathrm{Homeo}(X) \\ \left[\begin{array}{rcl} T\colon \mathrm{C}(X) & \to & \mathrm{C}(X) \\ f & \mapsto & h \cdot (f \circ \tau) \end{array}\right] & \mapsto & (h, \tau^{-1}) \end{array}$$

is a group isomorphism.

Proof. First of all, the map ψ is well-defined and injective, since there exists unique functions $h \in \mathrm{Um}(X)$ and $\tau \in \mathrm{Homeo}(X)$ for every $T \in \mathrm{Isom}(C(X))$ by the Banach-Stone Theorem. It is immediate from the converse of the Banach-Stone Theorem that ψ is surjective, as stated in Section 2. Moreover, ψ is a group homomorphism. Let $T_1, T_2 \in \mathrm{Isom}(C(X))$. Then we assume $\psi(T_1) = (h_1, \tau_1^{-1})$ and $\psi(T_2) = (h_2, \tau_2^{-1})$ for certain maps $\tau_1, \tau_2 \in \mathrm{Homeo}(X)$ and $h_1, h_2 \in \mathrm{Um}(X)$. By definition of the group operation on $\mathrm{Um}(X) \rtimes_\sigma \mathrm{Homeo}(X)$ we have

$$\psi(T_1)\psi(T_2) = (h_1, \tau_1^{-1})(h_2, \tau_2^{-1}) = (h_1 \cdot \sigma(\tau_1^{-1})(h_2), \tau_1^{-1} \circ \tau_2^{-1}) = (h_1 \cdot h_2 \circ \tau_1, (\tau_2 \circ \tau_1)^{-1}).$$

On the other hand, we have for an arbitrary map $f \in \mathrm{C}(X)$ that

$$(T_1 \circ T_2)(f) = T_1(T_2(f)) = T_1(h_2 \cdot f \circ \tau_2) = h_1 \cdot (h_2 \cdot f \circ \tau_2) \circ \tau_1 = h_1 \cdot h_2 \circ \tau_1 \cdot f \circ \tau_2 \circ \tau_1.$$

Then we conclude that

$$\psi(T_1 \circ T_2) = (h_1 \cdot h_2 \circ \tau_1, (\tau_2 \circ \tau_1)^{-1}).$$

Thus ψ is a bijection and a homomorphism, hence it is an isomorphism. $\square$

We conclude that $\mathrm{Isom}(C(X)) \cong \mathrm{Um}(X) \rtimes_\sigma \mathrm{Homeo}(X)$.

References

[1] P. Stevenhagen, *Algebra I*, Universiteit Leiden, 2011.

[2] H. Bauer, *Measure and Integration Theory*, Walter de Gruyter, Berlin · New York, 2001.

[3] J. B. Conway, *Course in Functional Analysis*, Second Edition, Springer Science+Business Media, New York, 2007.

[4] G. B. Folland, *Real Analysis, Modern Techniques and Their Applications*, Second Edition, John Wiley & Sons, Canada, 1999.

[5] R. E. Megginson, *An Introduction to Banach Space Theory*, Springer-Verlag, New York, 1998.

[6] V. Runde, *A Taste of Topology*, Springer Science+Business Media, New York, 2008.

[7] B.P. Rynne, M.A. Youngson, *Linear Functional Analysis*, Second Edition, Springer-Verlag, London, 2008.