



Universiteit
Leiden
The Netherlands

Limiting shapes in the sandpile growth model

Paus, F.H.J. de

Citation

Paus, F. H. J. de. (2014). *Limiting shapes in the sandpile growth model*.

Version: Not Applicable (or Unknown)

License: [License to inclusion and publication of a Bachelor or Master thesis in the Leiden University Student Repository](#)

Downloaded from: <https://hdl.handle.net/1887/3596602>

Note: To cite this publication please use the final published version (if applicable).

F.H.J. de Paus

Limiting shapes in the sandpile growth model

Bachelor thesis, January 22, 2014

Supervisor: prof. dr. E.A. Verbitskiy



**Universiteit
Leiden**

Mathematisch Instituut

Contents

1	Introduction	3
2	Discrete Laplacians	4
2.1	Green's function	5
3	Centrally seeded abelian sandpile growth model	8
3.1	The model	8
4	Analysis of the sandpile equation	11
4.1	Analysis in dimension $d \geq 3$	11
4.2	Analysis critical case in dimension $d \geq 3$	12
4.3	Analysis in dimension $d = 2$	13
4.4	Summary	14
5	A limiting shape	15
5.1	The dissipative case	15
5.2	The critical case	17
5.3	Simulations	18
5.3.1	Behaviour of the odometer u	18
5.3.2	Growth of the support $S_{\sigma(N)}$	19
5.3.3	Height ratios	21
5.3.4	Using the fundamental solution w	23
5.4	Polygonal bound	30
6	Conclusions	31
	Appendices	34
A	Script	35

Chapter 1

Introduction

In the *centrally seeded abelian sandpile growth model*, one adds N grains to the origin of the d -dimensional square lattice \mathbb{Z}^d . If the number of grains at a site $n = (n_1, \dots, n_d)$ is higher than a fixed parameter $k \geq 2d$ the site n is unstable and it ‘topples’: k grains are removed and 1 grain is added to each of the $2d$ neighbours. As long as there are unstable sites the toppling continues. Eventually all sites will become stable. The stable configurations form elaborate patterns and are subject of research its properties, especially in two dimensions.

In this project the asymptotic properties of the stable configurations $\sigma^{(N)}$ as $N \rightarrow \infty$ will be addressed. The most important open problem is the existence of the limiting shape. Namely, whether there exists an $\mathcal{S} \neq \{\emptyset, \mathbb{R}^d\}$, such that

$$\lim_{N \rightarrow \infty} \frac{\{n : \sigma^{(N)}(n) > 0\}}{g(N)} = \mathcal{S} \subset \mathbb{R}^d.$$

Even though lower and upper bounds for the the support of $\sigma^{(N)}$, $S_{u(N)}$, have been established, it remains unclear whether the limiting shape is a disc or a perhaps polygon [8]. The primary aim is to find two functions $f(N)$ and $g(N)$ such that $S_{u(N)}$ lies between two level sets $\{n : |Nw(n)| \leq f(N)\}$ and $\{n : |Nw(n)| \leq g(N)\}$, where w is the fundamental solution of the Discrete Laplace operator (see definitions 1 and 4).

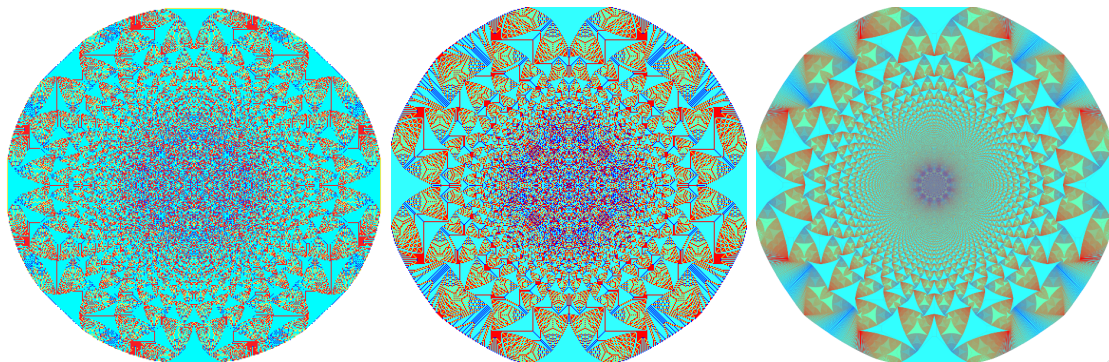


Figure 1.1: From left to right and top to bottom: stable sandpiles ($\sigma^{(N)}$) for $N = 10^6$ simulated by the author and $N = 10^{18}$ and $N = 10^{28}$, simulated by D.B. Wilson, Microsoft Research [10]. All scaled to be the same size. Cyan represents height 4, red height 3, blue height 2 and yellow height 1.

Chapter 2

Discrete Laplacians

Discrete Laplacians are operators on functions on lattices \mathbb{Z}^d and can be viewed as discretisations on \mathbb{R}^d of the Laplacian and appear naturally when working with a graph or (discrete) grid. In what follows we will be working in lattices in \mathbb{Z}^d and therefore base our definitions of the Discrete Laplacian and its properties on this situation.

Definition 1. Consider the square lattice \mathbb{Z}^d , where every vertex or site n has $2d$ neighbours (at graph distance 1). The Discrete Laplace operator Δ acts on real functions $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ as

$$(\Delta f)(n) = 2d \cdot f(n) - \sum_{i=1}^d (f(n + e_i) + f(n - e_i))$$

for all $n \in \mathbb{Z}^d$, where e_i denotes the i -th unit vector.

Figure 2.1: The five-point stencil of site $n = (n_1, n_2) \in \mathbb{Z}^2$.

Definition 2. The Laplace equation, a second order partial difference equation, is

$$\Delta f = 0.$$

Solutions of this equation are called harmonic functions. The set of these functions is the kernel of Δ .

Equivalently, f is harmonic if the value of $f(n)$ is equal to the average of the values of f over the $2d$ nearest neighbours of n . Similarly to \mathbb{R}^d , Liouville's theorem is valid for discrete Laplacians on \mathbb{Z}^d .

Theorem 1 (Liouville's theorem). Every bounded harmonic function is a constant.

Proof. Take an arbitrary bounded harmonic function $f(n)$. Let $M = \max_{m \in \mathbb{Z}^d} f(m)$ and suppose n is such that $f(n) = M$. Since f is harmonic and $f(n) \geq f(m)$ for all $m \in \mathbb{Z}^d$

$$(\Delta f)(n) = 2d \cdot f(n) - \sum_{i=1}^d (f(n + e_i) + f(n - e_i)) = 0 \quad \Rightarrow \quad f(n \pm e_i) \equiv M$$

for $i \in \{1, 2, \dots, d\}$. Therefore f must be constant. \square

Definition 3. For f and e_i defined as above and for any $k \in \mathbb{N}$, $k \geq 2d$, we define Δ_k as

$$\Delta_k f(n) = k \cdot f(n) - \sum_{i=1}^d (f(n + e_i) + f(n - e_i)).$$

Then the functions in the kernel of Δ_k , i.e. those for which $\Delta_k f = 0$, are given by

$$\begin{aligned} k \cdot f(n) - \sum_{i=1}^d (f(n + e_i) + f(n - e_i)) = 0 &\Rightarrow k \cdot f(n) = \sum_{i=1}^d (f(n + e_i) + f(n - e_i)) \\ &\Rightarrow f(n) = \frac{\sum_{i=1}^d (f(n + e_i) + f(n - e_i))}{k}. \end{aligned}$$

We saw that for $k = 2d$ if f is harmonic and bounded it is a constant. For $k > 2d$, one has the following lemma.

Lemma 1. Every bounded function f for which $\Delta_k f(n) = 0$ for all $n \in \mathbb{Z}^d$ with $k > 2d$ is equal to zero.

Proof. Let $k > 2d$, f a non-constant bounded real function and let n be such that $|f(n)| = \max_{m \in \mathbb{Z}^d} |f(m)| > 0$. Then in particular

$$|f(n)| \geq \frac{1}{k} \sum_{i=1}^d (|f(n + e_i)| + |f(n - e_i)|).$$

Then for some $i \in \{1, 2, \dots, d\}$, $|f(n \pm e_i)| > |f(n)|$. This is in contradiction with the assumption that $|f(n)| \geq |f(m)|$ for all $m \in \mathbb{Z}^d$. \square

2.1 Green's function

Definition 4. The function $w : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is called a fundamental solution or the Green function of Δ if for all n

$$\Delta w(n) = \delta_0(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Before we start looking for solutions, let us first introduce some useful notation [3].

We identify the Cartesian product $W_d = \mathbb{R}^{\mathbb{Z}^d}$ with the set of formal real power series in the variables $u_1^{\pm 1}, \dots, u_d^{\pm 1}$ by viewing each $w = (w_n) \in W_d$ as the power series

$$\sum_{n \in \mathbb{Z}^d} w_n u^n \tag{2.1}$$

with $w_n \in \mathbb{R}$ and $u^n = u_1^{n_1} \cdots u_d^{n_d}$ for every $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$.

For every $p \geq 1$ we regard $\ell^p(\mathbb{Z}^d)$ as the set of all $w \in W_d$ with

$$\|w\|_p = \left(\sum_{n \in \mathbb{Z}^d} |w_n|^p \right)^{1/p} < \infty.$$

In particular, $\ell^1(\mathbb{Z}^d)$ is the set of all $w \in W_d$ with $\|w\|_1 = \sum_{n \in \mathbb{Z}^d} |w_n| < \infty$.

Let us now consider the (irreducible) Laurent polynomial for dimension d

$$f^{(d)} = 2d - \sum_{i=1}^d (u_i + u_i^{-1}) \in R_d \quad (2.2)$$

where $R_d = \mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}] \subset \ell^1(\mathbb{Z}^d) \subset W_d$ is the ring of Laurent polynomials with integer coefficients. Every h in any of these spaces will be written as $h = (h_n) = \sum_{n \in \mathbb{Z}^d} h_n u^n$ with $h_n \in \mathbb{R}$ (resp. $h_n \in \mathbb{Z}$ for $h \in R_d$). The map $(m, w) \mapsto u^m \cdot w$ with $(u^m \cdot w)_n = w_{n-m}$ is a \mathbb{Z}^d -action by automorphisms on the additive group W_d which extends linearly to an R_d -action on W_d given by

$$h \cdot w = \sum_{n \in \mathbb{Z}^d} h_n u^n \cdot w \quad (2.3)$$

for every $h \in R_d$ and $w \in W_d$. If w also lies in R_d this definition is consistent with the usual product in R_d .

Equation 2.2 reminds us of the discrete Laplace operator Δ defined in definition 1. Noting that under the embedding of $R_d \hookrightarrow \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ the constant polynomial $1 \in R_d$ corresponds to the element $\delta_0(n) \in \ell^\infty(\mathbb{Z}^d, \mathbb{Z})$ (as in definition 4), we consider

$$f^{(d)} \cdot w = 1. \quad (2.4)$$

The fundamental solutions of this equation (definition 4) can be found by using Fourier analysis.

Definition 5. For every $n = (n_1, \dots, n_d) \in \mathbb{Z}^d$ and $t = (t_1, \dots, t_d) \in \mathbb{T}^d \simeq [0, 1]^d$ we set $\langle n, t \rangle = \sum_{j=1}^d n_j t_j \in \mathbb{T}$. We denote by

$$F^{(d)}(t) = \sum_{n \in \mathbb{Z}^d} f^{(d)}(n) e^{2\pi i \langle n, t \rangle} = 2d - 2 \cdot \sum_{j=1}^d \cos(2\pi t_j), \quad (2.5)$$

the Fourier transform of $f^{(d)}$.

We can find a fundamental solution $w^{(d)}$ of $f^{(d)} w^{(d)} = \delta_0$, as the inverse Fourier transform of $1/F^{(d)}(t)$:

(1) for $d \geq 3$,

$$w^{(d)}(n) = \int_{\mathbb{T}^d} \frac{e^{-2\pi i \langle n, t \rangle}}{F^{(d)}(t)} dt \text{ for every } n \in \mathbb{Z}^d. \quad (2.6)$$

For $d = 2$ the situation is different. One has to modify (2.6), as the corresponding function $1/F^{(2)}(t)$ is not integrable:

(2) for $d = 2$,

$$w^{(2)}(n) = \int_{\mathbb{T}^2} \frac{e^{-2\pi i \langle n, t \rangle} - 1}{F^{(2)}(t)} dt \text{ for every } n \in \mathbb{Z}^2. \quad (2.7)$$

The following theorem gives an insight into the asymptotic behaviour of fundamental solutions: [3]:

Theorem 2. We write $\|\cdot\|$ for the Euclidean norm on \mathbb{Z}^d .

- 1) For every $d \geq 2$, $w^{(d)}$ given by 2.6 and 2.7 satisfies: $f^{(d)} \cdot w = 1$.
- 2) For $d = 2$ and $n \in \mathbb{Z}^2$,

$$w^{(2)}(n) = \begin{cases} 0 & \text{if } n = 0, \\ = -\frac{1}{8\pi} \log \|n\| - \kappa_2 - c_2 \frac{\frac{1}{\|n\|^4} (n_1^4 + n_2^4) - \frac{3}{4}}{\|n\|^2} + \mathcal{O}(\|n\|^{-4}) & \text{if } n \neq 0 \end{cases} \quad (2.8)$$

$$w^{(2)}(n) = \begin{cases} 0 & \text{if } n = 0, \\ = -\frac{1}{2\pi} \log \|n\| - \frac{\log(8)+2\gamma}{4\pi} + \mathcal{O}(\|n\|^{-4}) & \text{if } n \neq 0, \end{cases} \quad (2.9)$$

where γ is the Euler-Mascheroni constant.

- 3) For $d \geq 3$,

$$\|n\|^{d-2} w_n^{(d)} = \kappa_d + c_d \frac{\frac{1}{\|n\|^4} \sum_{i=1}^d n_i^4 - \frac{3}{d+2}}{\|n\|^2} + \mathcal{O}(\|n\|^{-4}) \quad (2.10)$$

where $\kappa_d > 0$, $c_d > 0$.

Having found the fundamental solution w we can now find solution v for equations of the form $f \cdot v = g$, with g being a polynomial, i.e. $g = \sum_{n \in \mathbb{Z}^d} g_n u^n$, with a finite number of non-zero coefficients: if

$$(f v)(n) = g(n)$$

then

$$v(n) = w(n) * g(n) = \sum_{m \in \mathbb{Z}^d} w(n-m) \cdot g(m).$$

Chapter 3

Centrally seeded abelian sandpile growth model

3.1 The model

In this thesis we will be focussing on the *centrally seeded (abelian) sandpile growth model*. The sandpile configuration is a function on the d -dimensional square lattice \mathbb{Z}^d to $\mathbb{Z}_{\geq 0}$, the value indicating the number of grains at - or the *height* of - each site $n = (n_1, \dots, n_d)$.

Definition 6. *The height configuration $\sigma : V \rightarrow \mathbb{Z}$, $V \subset \mathbb{Z}^d$, assigns to each site $n = (n_1, \dots, n_d) \in V$ the number of grains at that site, i.e. the height of site n is $\sigma(n)$.*

A sandpile consists of a (finite) number of grains, $\sum_{n \in V} \sigma(n)$, and it being *centrally seeded* indicates that if further grains are to be added these are only seeded at the origin.

Definition 7. *A site n is called unstable if its height $\sigma(n)$ is greater than a fixed parameter $k \geq 2d$ and stable otherwise.*

From an unstable site k grains are removed and 1 grain is added to each of its $2d$ neighbours. This process is called toppling and can be represented by a toppling matrix.

Definition 8. [5] *A toppling matrix is a symmetric integer valued matrix $\Delta_{n,m}^V$, $n, m \in V \subset \mathbb{Z}^d$ that satisfies the conditions*

1. *For all $n, m \in V$, $n \neq m$, $\Delta_{n,m}^V = \Delta_{m,n}^V \leq 0$,*
2. *For all $n \in V$, $\Delta_{n,n}^V > 0$,*
3. *For all $n \in V$, $\sum_{m \in V} \Delta_{n,m}^V \geq 0$,*

If there exists an unstable site in V , its height configuration is called unstable. This can be expressed in terms of the toppling matrix:

Definition 9. *A configuration σ is called stable if $\sigma(n) \leq \Delta_{n,n}^V = k$ for all $n \in V$. Otherwise it is called unstable toppling starts.*

The Discrete or lattice Laplacian is a toppling matrix with open boundary conditions,

given by

$$\Delta_{n,m} = \begin{cases} 2d & \text{for } n = m, \\ -1 & \text{for } \|n - m\| = 1, \\ 0 & \text{otherwise} \end{cases} \quad (3.1)$$

and the toppling matrix used for this model is its generalization using $k \geq 2d$

$$\Delta_{n,m}^k = \begin{cases} k & \text{for } n = m, \\ -1 & \text{for } \|n - m\| = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

Definition 10. Toppling occurs as long as in V there are unstable sites and can be represented by the mappings $T_n : \mathbb{Z}^V \rightarrow \mathbb{Z}^V$ given by

$$T_n(\sigma) = \begin{cases} \sigma(m) - \Delta_{n,m}^{V,k} & \forall m \in V \text{ if } \sigma(n) > k, \\ \sigma & \text{otherwise,} \end{cases}$$

In other words, $T_n(\sigma) = \sigma'$, where if $\sigma(n) > k$, site n loses k grains, i.e.

$$\sigma'(m) = \sigma(m) - k, \text{ for } m = n$$

and one grain is added to its neighbours, i.e.

$$\sigma'(m) = \sigma(m) + 1, \text{ for } m : \|n - m\| = 1.$$

These topplings have the Abelian Property, i.e. that a stable configuration does not depend on the order of the toppling, i.e. for $n, m \in V$ and σ such that $\sigma(n) > \Delta_{n,n}^V$ and $\sigma(m) > \Delta_{m,m}^V$,

$$T_n \circ T_m(\sigma) = T_m \circ T_n(\sigma).$$

For the proof, see [5].

This matrix representation is useful for understanding the toppling rules and when doing simulations on the model. For further analysis the adapted Discrete Laplace operator Δ_k given in definition 3 is used. Furthermore we will be assuming $V = \mathbb{Z}^d$.

If a start configuration has finite support, i.e. $|\{n : \sigma(n) \neq 0\}| < \infty$, the configuration will always stabilize. The stable end configuration corresponding to start configuration $N\delta_0$ is denoted by $\sigma^{(N)}$, where N is then the number of grains initially in the system. Depending on k the initial number of grains is preserved or grains are lost. Hereto we distinguish two cases:

Definition 11. The critical case, where $k = 2d$ and the initial number of grains is preserved, and the dissipative case, where $k > 2d$, meaning that $k - 2d$ grains are lost from the system every time a site topples.

Definition 12. The number of times a site n topples is denoted by $u(n)$. Function u is also known as the odometer function [4].

The odometer function can be used to describe the complete stabilizing process for start configuration $N\delta_0$, i.e. $\sigma(0) = N$, $N < \infty$, and $\sigma(n) = 0$ for all $n \in \mathbb{Z}^d \setminus 0$ as:

$$N\delta_0 - \Delta_k u^{(N)} = \sigma^{(N)}, \quad (3.3)$$

where

$$\Delta_k u^{(N)}(n) = ku^{(N)}(n) - \sum_{i=1}^d \left(u^{(N)}(n + e_i) + u^{(N)}(n - e_i) \right),$$

describing the change in the height of site n caused by losing grains through toppling itself and gaining grains through toppling neighbours.

Since we know that $\delta_0 = \Delta_k w$, where w is the fundamental solution (cf. Definition 4), we can substitute this in (3.3) and use the properties of the fundamental solution and Δ_k being a linear operator to write

$$\Delta_k Nw - \Delta_k u^{(N)} = \Delta_k v^{(N)}, \quad \text{with } v^{(N)} = w * \sigma^{(N)} \quad (3.4)$$

$$\Rightarrow \Delta_k (Nw - u^{(N)} - v^{(N)}) = 0 \quad (3.5)$$

from which we conclude that $Nw - u^{(N)} - v^{(N)}$ is a harmonic function. We call equation (3.5) the *sandpile equation*.

Chapter 4

Analysis of the sandpile equation

4.1 Analysis in dimension $d \geq 3$

Consider $Nw - u^{(N)} - v^{(N)}$. We know that N is a constant, that w is bounded and that, since the toppling process is finite, $u^{(N)}$ has finite support. Furthermore, since the height function is non-negative and toppling occurs when for a site n , $\sigma(n) > k$ we have $0 \leq \sigma^{(N)}(n) \leq k$ for all $n \in \mathbb{Z}^d$. Therefore $v^{(N)} = w * \sigma^{(N)}$ is also bounded, which makes $Nw - u^{(N)} - v^{(N)}$ a bounded harmonic function and therefore constant.

Let us work the above out a bit more rigorously. For $k \geq 2d$ we have:

a) $\|w\|_\infty = \sup_{n \in \mathbb{Z}^d} |w(n)| < \infty$.

b) $u^{(N)}$ has finite support since the toppling stops and only finitely many sites topple finitely many times. In particular

$$0 \leq u_{(k > 2d)}^{(N)} \leq u_{(k=2d)}^{(N)} \leq N.$$

c) $\|v^{(N)}\|_\infty = \sup_{n \in \mathbb{Z}^d} |v^{(N)}(n)| < \infty$ since

$$\begin{aligned} |v^{(N)}(n)| &= \left| \sum_{m \in \mathbb{Z}^d} \sigma^{(N)}(n-m) \cdot w(m) \right| \\ &\leq \sum_{m \in \mathbb{Z}^d} |\sigma^{(N)}(n-m)| \cdot |w(m)| \\ &\leq \sum_{m \in \mathbb{Z}^d} |\sigma^{(N)}(n-m)| \cdot \sup_{i \in \mathbb{Z}^d} |w(i)| \\ &\leq \|w\|_\infty \sum_{m \in \mathbb{Z}^d} |\sigma^{(N)}(n-m)| \\ &= N \|w\|_\infty, \quad \text{for } k = 2d. \end{aligned}$$

For $k > 2d$, $\sum_{m \in \mathbb{Z}^d} |\sigma^{(N)}(n-m)| < N$.

Combining a, b and c one concludes that $Nw - u^{(N)} - v^{(N)}$ is bounded for $d \geq 3$ and $k = 2d$.

Lemma 2. For $d \geq 3$ and $k \geq 2d$

$$Nw - u^{(N)} - v^{(N)} \equiv \text{constant}.$$

Then using Theorem 1 and Lemma 1 and 2 we get:

Corollary 1. *In dimension $d \geq 3$*

$$Nw - u^{(N)} - v^{(N)} \equiv \begin{cases} \text{constant,} & \text{if critical } (k = 2d), \\ 0, & \text{if dissipative } (k > 2d). \end{cases} \quad (4.1)$$

Later we will show that the constant in equation 4.1 is in fact equal to zero.

4.2 Analysis critical case in dimension $d \geq 3$

In the last section we concluded $Nw - u^{(N)} - v^{(N)}$ to be constant in the critical case, i.e. for $k = 2d$. However, we can say more about its behaviour.

Regarding the fundamental solution w ((2.6),(2.10)) we have

$$|w(n)| \rightarrow 0, \text{ for } |n| \rightarrow \infty \quad (4.2)$$

and since $u^{(N)}$ has finite support, i.e. there exists an $M \in \mathbb{N}$ such that $u^{(N)}(n) = 0$ for all $n \in \mathbb{Z}^d$ with $\|n\|_\infty = \max_{i=1,\dots,d} |n_i| \geq M$, we have

$$u^{(N)}(n) \rightarrow 0, \text{ for } |n| \rightarrow \infty. \quad (4.3)$$

Furthermore, since

$$\begin{aligned} |v^{(N)}(n)| &= \left| \sum_{m \in \mathbb{Z}^d} \sigma^{(N)}(m) \cdot w(n-m) \right| \\ &\leq \sum_{m \in \mathbb{Z}^d} |\sigma^{(N)}(m)| \cdot |w(n-m)| \\ &= \sum_{m \in \mathbb{Z}^d: \sigma^{(N)}(m) > 0} |\sigma^{(N)}(m)| \cdot |w(n-m)| \\ &\leq |2d| \sum_{m \in \mathbb{Z}^d: \sigma^{(N)}(m) > 0} |w(n-m)| \end{aligned}$$

and since the support $S_{\sigma^{(N)}} = \{m \in \mathbb{Z}^d : \sigma^{(N)}(m) > 0\}$ is finite, given that as $|n| \rightarrow \infty$, $|w(n)| \rightarrow 0$, it follows that

$$|v^{(N)}(n)| \rightarrow 0, \text{ for } |n| \rightarrow \infty \quad (4.4)$$

as well and therefore

$$Nw - u^{(N)} - v^{(N)} \rightarrow 0, \text{ for } |n| \rightarrow \infty. \quad (4.5)$$

Then combining (4.1) and (4.5) we for $k = 2d$ get

$$Nw - u^{(N)} - v^{(N)} \equiv \text{constant} \equiv 0$$

and therefore we can state the following:

Lemma 3. *In dimension $d \geq 3$ and for parameter $k \geq 2$:*

$$Nw - u^{(N)} - v^{(N)} \equiv 0.$$

4.3 Analysis in dimension $d = 2$

In dimension $d = 2$, for $k \geq 2d$, $u^{(N)}$ and $\sigma^{(N)}$ also have finite support, however w behaves differently.

In the dissipative case, $k > 2d$, we have

$$\|w\|_1 < \infty, \text{ hence } |w(n)| \rightarrow 0 \text{ for } |n| \rightarrow \infty,$$

from which we can conclude that, analogously to the cases before, in dimension $d = 2$ and for $k > 2d$ one also has

$$Nw - u^{(N)} - v^{(N)} \equiv 0.$$

In the critical case, $k = 2d$, we see from equation (2.9) that $|w(n)|$ behaves logarithmically and is therefore unbounded

$$|w(n)| = \mathcal{O}(\log \|n\|) \rightarrow \infty \text{ for } |n| \rightarrow \infty.$$

Again, since u and σ have finite support, for sufficiently large $|n|$,

$$\begin{aligned} Nw(n) - u^{(N)}(n) - v^{(N)}(n) &= Nw(n) - v^{(N)}(n) = Nw(n) - \sum_{m \in \mathbb{Z}^d} \sigma^{(N)}(m) \cdot w(n-m) \\ &= Nw - \sum_{m \in S_{\sigma^{(N)}}} \sigma^{(N)}(m) \cdot w(n-m) = \mathcal{O}(\log \|n\|). \end{aligned}$$

However, harmonic functions of sublinear growth are also constants. Let us recall the following result, c.f., [1][Theorem 6, p. 199] and [2][Lemma 1, p. 408]:

Lemma 4. *If $f(n)$ is discrete harmonic everywhere and satisfies the inequality*

$$|f(n)| \leq C(1 + \|n\|)^p,$$

for all n , where p is an integer, then $f(n)$ is a polynomial of degree not exceeding p .

This means that in dimension $d = 2$, we have:

$$|Nw - u^{(N)} - v^{(N)}| \leq C \log \|n\| \leq C(1 + \|n\|)^1$$

for $\|n\|$ sufficiently large and where C is a constant. So $Nw - u^{(N)} - v^{(N)}$ is a polynomial of degree not exceeding 1. In fact, there are only 3 harmonic functions with linear growth: 1, n_1 and n_2 . Hence $Nw - u^{(N)} - v^{(N)}$ would be of the form $c_1 + c_2 n_1 + c_3 n_2$. However, since no linear function can be bounded by a logarithmic function, $Nw - u^{(N)} - v^{(N)}$ must be constant.

Now for $k = 2d$, using the facts that

1. $u^{(N)}$ and $\sigma^{(N)}$ have finite support

$$\exists M : u^{(N)}(n), \sigma^{(N)}(n) = 0, \forall n : \|n\| > M$$

2. the amount of grains N is preserved
3. $\sigma^{(N)}(n) \leq 4, \forall n \in \mathbb{Z}^d$

and (also for future reference) defining the bounded (fixed, dependent on N) supports as

$$\begin{aligned} S_{u^{(N)}} &:= \{n \in \mathbb{Z}^d : u^{(N)}(n) > 0\} \\ S_{\sigma^{(N)}} &:= \{n \in \mathbb{Z}^d : \sigma^{(N)}(n) > 0\} \end{aligned}$$

we can rewrite $Nw(n) - u^{(N)}(n) - v^{(N)}(n)$, for $\|n\|$ sufficiently large, as

$$\begin{aligned} \left| Nw(n) - u^{(N)}(n) - v^{(N)}(n) \right| &= \left| Nw(n) - u^{(N)}(n) - \sum_{m \in \mathbb{Z}^d} \sigma^{(N)}(m) \cdot w(n-m) \right| \\ &= \left| Nw(n) - \sum_{m \in S_{\sigma^{(N)}}} \sigma^{(N)}(m) \cdot w(n-m) \right| \\ &= \left| \sum_{m \in S_{\sigma^{(N)}}} \sigma^{(N)}(m) \cdot w(n) - \sum_{m \in S_{\sigma^{(N)}}} \sigma^{(N)}(m) \cdot w(n-m) \right| \\ &= \left| \sum_{m \in S_{\sigma^{(N)}}} \sigma^{(N)}(m) \cdot [w(n) - w(n-m)] \right| \\ &\leq 4 \cdot \sum_{m \in S_{\sigma^{(N)}}} |w(n) - w(n-m)| \\ &= 4 \cdot |S_{\sigma^{(N)}}| \cdot \sup_{m \in S_{\sigma^{(N)}}} |w(n) - w(n-m)| \\ &\leq g(N) \cdot \sup_{m \in S_{\sigma^{(N)}}} |w(n) - w(n-m)| \end{aligned}$$

Now, since the support $S_{\sigma^{(N)}}$ is finite, for sufficiently large $\|n\|$

$$|w(n) - w(n-m)| \simeq C \left| \log \frac{\|n\|}{\|n-m\|} \right| + \mathcal{O}(\|n\|)^{-4}$$

and hence

$$\sup_{m \in S_{\sigma^{(N)}}} |w(n) - w(n-m)| \rightarrow 0, \text{ as } \|n\| \rightarrow \infty.$$

So we can conclude that in dimension $d = 2$, for $k = 2d = 4$, $Nw - u^{(N)} - v^{(N)} \equiv 0$ as well.

4.4 Summary

In this chapter we showed that for sandpile growth models for all $d \geq 2$ and $k \geq 2d$

$$Nw - u^{(N)} - v^{(N)} \equiv 0.$$

Chapter 5

A limiting shape

We are interested in finding limiting shape:

$$\lim_{N \rightarrow \infty} \frac{\{n : \sigma^{(N)}(n) > 0\}}{g(N)} = \mathcal{S} \subset \mathbb{R}^d, \quad \text{with } \mathcal{S} \neq \{\emptyset, \mathbb{R}^d\}.$$

We can use what we know about the behaviour of the harmonic function

$$Nw - u^{(N)} - v^{(N)} \equiv 0 \tag{5.1}$$

to try and find an estimate.

5.1 The dissipative case

In the dissipative case, for $d \geq 2$ the fundamental solution w is non-negative. In a stable end configuration a site n has non-negative integer height $\sigma^{(N)}(n)$ along with a non-negative number of topplings $u^{(N)}(n) \geq 0$. Furthermore, if the site has toppled, it and its neighbours have at least one grain

$$u^{(N)}(n) > 0 \Rightarrow \sigma^{(N)}(n), \sigma^{(N)}(n \pm e_i) > 0 \Rightarrow v^{(N)}(n) > 0.$$

This means that for a given site n , because of (5.1), if $u^{(N)}(n) > 0$, then necessarily $Nw(n) > 1$ since

$$u^{(N)}(n) > 0 \Rightarrow u^{(N)}(n) \geq 1 \Rightarrow Nw(n) = u^{(N)}(n) + v^{(N)}(n) > 1$$

and therefore

$$S_{u^{(N)}} = \{n : u^{(N)}(n) > 0\} \subseteq \{n : Nw(n) > 1\}. \tag{5.2}$$

On the other hand, since w and $\sigma^{(N)}$ are bounded ($\|w\|_\infty < \infty$ and $0 \leq \sigma^{(N)}(n) \leq k$ for

all $n \in \mathbb{Z}^d$) and σ has finite support, for all n one has

$$\begin{aligned}
0 < v^{(N)}(n) &= (w * \sigma)(n) \\
&= \sum_{m \in \mathbb{Z}^d} w(m) \sigma(n - m) \\
&\leq \max \sigma^{(N)} \sum_{n \in \mathbb{Z}^d} |w(m)| \\
&= k \sum_{m \in \mathbb{Z}^d} |w(m)| \\
&= k \|w\|_1 =: C < \infty
\end{aligned}$$

Consequently, if $Nw(n) = u^{(N)}(n) + v^{(N)}(n) > C$ for a site n , then since $v^{(N)}(n) \leq C$, one necessarily has

$$u^{(N)}(n) = Nw(n) - v^{(N)}(n) > C - C = 0.$$

Therefore

$$\{n : Nw(n) > C\} \subseteq \{n : u^{(N)}(n) > 0\} = S_{u^{(N)}}. \quad (5.3)$$

Combining results 5.2 and 5.3 we get the following approximations for $S_{u^{(N)}}$:

$$A_N := \{n : Nw(n) > C\} \subseteq S_{u^{(N)}} \subseteq \{n : Nw(n) > 1\} =: B_N. \quad (5.4)$$

Now, since all sites in $S_{u^{(N)}}$ have at least one direct neighbour, i.e. a site at graph distance 1, which is contained in $S_{\sigma^{(N)}}$, one has

$$S_{u^{(N)}} \subseteq S_{\sigma^{(N)}} \subseteq S_{u^{(N)}} \cup \partial^+ S_{u^{(N)}},$$

where $\partial^+ S_{u^{(N)}} = \partial_1^+ S_{u^{(N)}}$ represents the set of sites $m \in \mathbb{Z}^d \setminus S_{u^{(N)}}$ for which there exists an $n \in S_{u^{(N)}}$ such that $\sum_{i=1}^d |n_i - m_i| = 1$.

Therefore, if the scaling limit of $S_{u^{(N)}}$ exists, then the the scaling limit of $S_{\sigma^{(N)}}$ exists as well, and the limits coincide.

In fact, one can show that the distance between the sets A_N and B_N is bounded by a constant independent of N . In other words

$$A_N \subseteq S_{u^{(N)}} \subseteq B_N \subseteq A_N \cup \partial_p^+ A_N$$

for some p . Therefore, the scaling limits of A_N and $S_{\sigma^{(N)}}$ - if they exist - must coincide as well.

We are going to establish directly the existence of the scaling limits for sets of the form

$$S_{\{|Nw| \leq c\}} = \{n \in \mathbb{Z}^d : |Nw(n)| \leq c\}$$

where $c > 0$. In fact, one can show that for $k > 2d$, there exists an $\mathcal{S} \subset \mathbb{R}^d$, $\mathcal{S} \neq \emptyset, \mathbb{R}^d$, such that for all $c > 0$,

$$\frac{S_{\{|Nw| < c\}}}{\log N} \rightarrow \mathcal{S}$$

where

$$\mathcal{S} = \{x \in \mathbb{R}^d : \alpha(x) \leq 1\},$$

with $\alpha(x)$ defined for $x \in \mathbb{R}^d \setminus \{0\}$ by

$$\alpha(x) = \sum_{j=1}^d x_j \sinh^{-1}(x_j s),$$

where $s > 0$ solves

$$k = 2 \sum_{j=1}^d \sqrt{1 + (x_j s)^2}.$$

The function $\alpha(x)$ was identified by Zerner [7] as the directional limit of w_n :

$$\lim_{\|x\|_\infty \rightarrow \infty} -\frac{\log w(x)}{\alpha(x)} = 1, \quad \left(\|x\|_\infty = \max(|x_1|, \dots, |x_d|) \right), \quad (5.5)$$

5.2 The critical case

Let us first recall some results established in the literature.

Levine and Peres (in [8]) have given the following bounds on the shape of the abelian sandpile in d dimensions.

Theorem 3. *Write $N = \omega_d r^d$, where ω_d is the volume of the unit ball in \mathbb{R}^d . Then for any $\epsilon > 0$ we have*

$$B_{c_1 r - c_2} \subset S_{\sigma(N)} \subset B_{c'_1 r + c'_2}$$

where

$$c_1 = (2d - 1)^{-1/d}, \quad c'_1 = (d - \epsilon)^{-1/d}.$$

The constant c_2 depends only on d , while c'_2 depends only on d and ϵ .

In dimension two, these bounds read as

$$B_{c_1 r - c_2} \subset S_{\sigma(N)} \subset B_{c'_1 r + c'_2}$$

where

$$r = \sqrt{\frac{N}{\pi}}, \quad c_1 = \frac{1}{\sqrt{3}}, \quad c'_1 = \frac{1}{\sqrt{2 - \epsilon}}.$$

and the constants c_2 and c'_2 are independent of N .

Levine and Peres noted that Theorem 3 does not settle the question of the asymptotic shape of $S_{\sigma(N)}$, since the bounds do not converge, as it was the case for the dissipative model. Indeed it is not clear from simulations whether the asymptotic shape in two dimensions is a disc or perhaps a polygon - see figure 1.1 - as conjectured by Fey and Redig [6]. Also the existence of a limit shape has not yet been established.

We would like to understand the relation between $S_{\sigma(N)}$ and level sets of Nw and will use simulations to try and get a better understanding of the shape and size of $S_{\sigma(N)}$.

The aim is to find two functions $f(N)$ and $g(N)$ such that $S_{u(N)}$ is squeezed between two level sets $\{n : |Nw(n)| \leq f(N)\}$ and $\{n : |Nw(n)| \leq g(N)\}$ and explore their properties.

5.3 Simulations

In this section different aspects of the model will be regarded to find out more about the behaviour in the two dimensional critical $k = 2d$ case.

5.3.1 Behaviour of the odometer u

Before analysing the odometer, we first give the following definition.

Definition 13. *The boundary of $S_{\sigma^{(N)}}$ is the set of sites $n \in \mathbb{Z}^d$ which have a direct neighbour (relation \sim) m such that $\sigma^{(N)}(m) = 0$, i.e. outside of $S_{\sigma^{(N)}}$,*

$$\text{boundary of } S_{\sigma^{(N)}} := \{n \in S_{\sigma^{(N)}} : \exists m \in \mathbb{Z}^d \setminus S_{\sigma^{(N)}} : m \sim n\}.$$

The boundary of $S_{u^{(N)}}$ is defined analogously.

The boundary of $S_{\sigma^{(N)}}$ is that of $S_{u^{(N)}} \cup \partial^+ S_{u^{(N)}}$, where $\partial^+ S_{u^{(N)}}$ the *edge*, or outer boundary, of $S_{u^{(N)}}$.

The maximum of $u^{(N)}$, confirmed by simulations, is located at the origin

$$\max(u^{(N)}) = u^{(N)}(0)$$

and the minimum on the boundary of $S_{u^{(N)}}$.

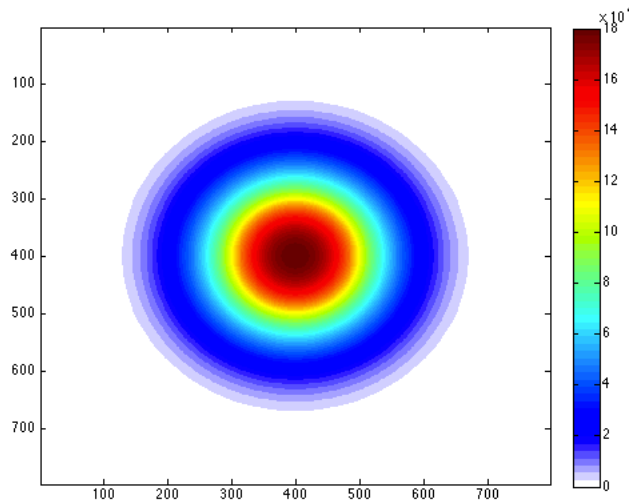


Figure 5.1: The odometer $u^{(N)}$ for $N = 10^6$.

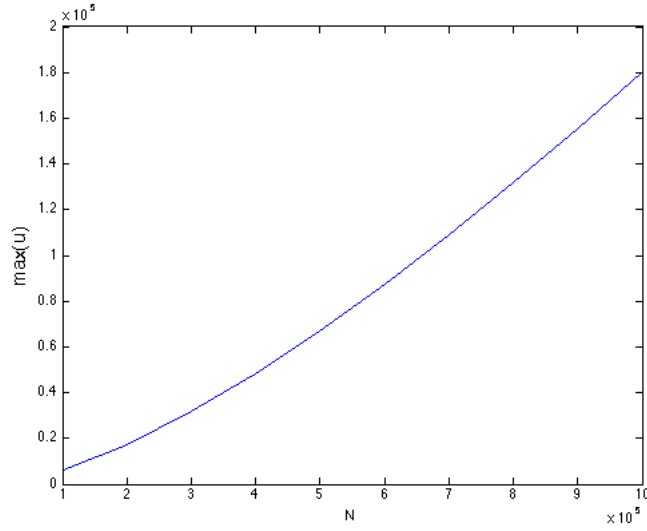


Figure 5.2: Plot of $\max(u^{(N)})$ for $N = [10^5, 10^6]$

5.3.2 Growth of the support $S_{\sigma^{(N)}}$

We would like to establish bounds on $S_{\sigma^{(N)}}$. To get an impression of the growth of $S_{\sigma^{(N)}}$ we the radius.

Definition 14. *The radius $r_{\sigma^{(N)}}$ of $S_{\sigma^{(N)}}$ is defined by*

$$r_{\sigma^{(N)}} := \max\{|n_1| : \sigma^{(N)}((n_1, 0)) \neq 0\},$$

which is equal to $\max\{|n_1| : \sigma^{(N)}((0, n_1)) \neq 0\}$ due to the symmetry of $\sigma^{(N)}$.

We plot the values of $r_{\sigma^{(N)}}$ $N = [10^5, 10^6]$.

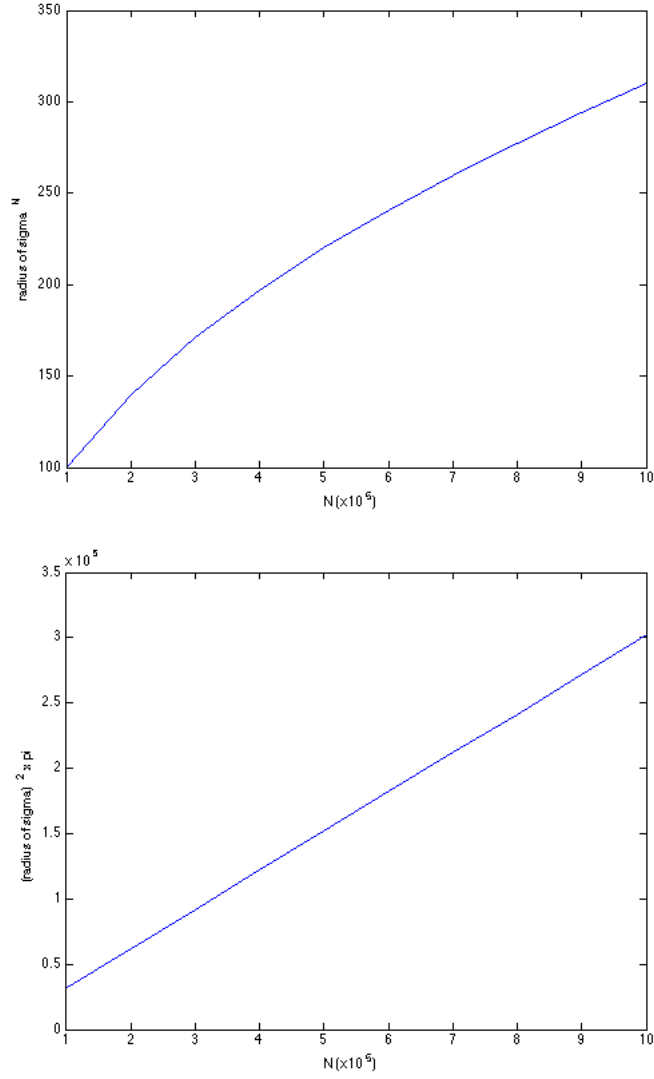


Figure 5.3: Plot of the radius $r_{\sigma^{(N)}}$ of $\sigma^{(N)}$ for $N = [10^5, 10^6]$ and of $\pi \cdot r_{\sigma^{(N)}}^2$ on the same interval, which seems to be just slightly sublinear to the naked eye.

If we would have $\sigma^{(N)}(n) = 1$ for all n in $S_{\sigma^{(N)}}$ and $S_{\sigma^{(N)}}$ were a disc, the radius $r_{\sigma^{(N)}}$, would relate to N as $N \simeq \pi r_{\sigma^{(N)}}^2$. The sites in the support of $\sigma^{(N)}$, however, have heights from 1 to 4 and the radius should therefore be scaled. A fit of the data for the radius $r_{\sigma^{(N)}}$ as a function of N indicates

$$r = c\sqrt{N},$$

$$c = 0.3107 \pm 8.5 \cdot 10^{-4}.$$

Where $(0.3099, 0.3116)$ is the 95% confidence interval. This indicates the radius $r_{\sigma^{(N)}}$ to relate to N as

$$r_{\sigma^{(N)}} = \sqrt{\frac{N}{\rho}} \tag{5.6}$$

where $c = \sqrt{\frac{1}{\rho}} \Rightarrow \rho \approx 10.3590 \approx 3.2974\pi$.

5.3.3 Height ratios

The stable configurations $\sigma^{(N)}$ portray repeating patterns and the ratios of the sites with heights 1 through 4 play a role in determining the total area of the support $S_{\sigma^{(N)}}$. We will analyse the ratios by registering the number of sites with height 1,2,3 and 4 respectively when simulating the stabilization of N grains.

Let $a_i, i = 1, 2, 3, 4$ denote the ratios of the number of sites of height i to the total number of sites in the support of $\sigma^{(N)}$, i.e. the possibility of a site in $\sigma^{(N)}$ having height i ,

$$a_i = \frac{|\{n \in S_{\sigma^{(N)}} : \sigma^{(N)}(n) = i\}|}{A^{(N)}},$$

where $A^{(N)} := |S_{\sigma^{(N)}}|$ is defined as the area of the support of $\sigma^{(N)}$.

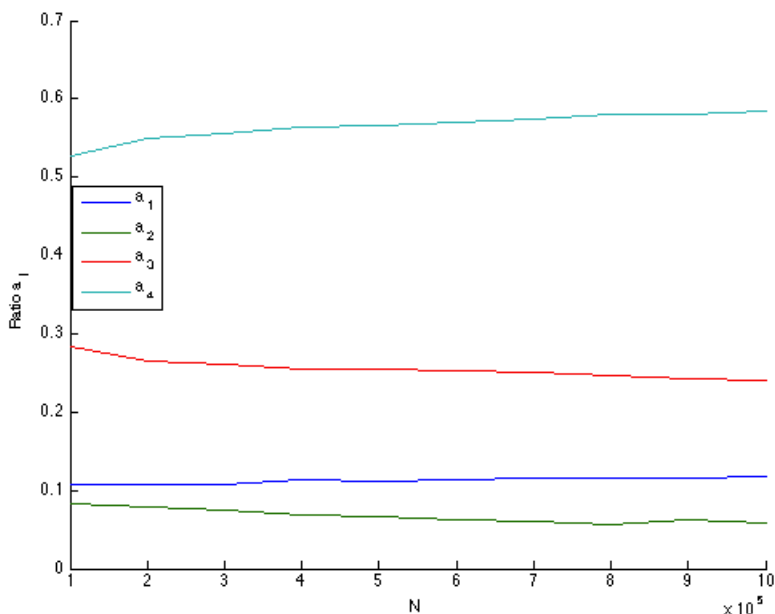


Figure 5.4: The ratios of the heights 1,2,3 and 4 in simulations for $N = [10^5, 10^6]$.

The area $A^{(N)}$ plays an important role in the determination of the limiting shape of $\sigma^{(N)}$ and can be related to N and a_i as follows.

$$\begin{aligned} N &= a_1 A^{(N)} + 2a_2 A^{(N)} + 3a_3 A^{(N)} + 4a_4 A^{(N)} \\ &= A^{(N)} \cdot (a_1 + 2a_2 + 3a_3 + 4a_4) \end{aligned}$$

Since we know N , and the a_i 's tend to a certain ratio, we can predict the surface $A^{(N)}$ by

$$A^{(N)} = \frac{N}{a_1 + 2a_2 + 3a_3 + 4a_4} := \frac{N}{a}. \quad (5.7)$$

Figure 5.5 shows the behaviour of $a = a_1 + 2a_2 + 3a_3 + 4a_4$, based on the data from simulations.

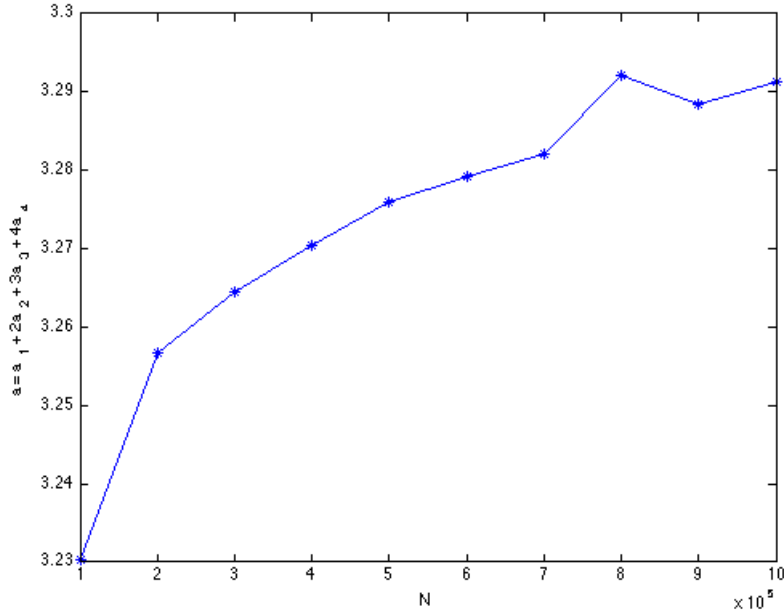


Figure 5.5: Values of $a = a_1 + 2a_2 + 3a_3 + 4a_4$ for $N = [10^5, 10^6]$.

While the data for a is hard to fit, a fit for the values of $A^{(N)}$ gives a slope of 0.3031(0.3025, 0.3036) (figure 5.6), which indicates a to tend to $1/0.3031 \approx 3.2992 \approx \rho/\pi$.

In [9], based on the model where a site n topples when $\sigma(n) \geq 4$ in stead of when $\sigma(n) > 4$, the *stationary density* ζ_s of the sandpile, i.e. the expected number of particles at a fixed site, and the single site height probabilities are given:

$$\begin{aligned}\zeta_s &= 17/8, \\ a_0^\zeta &= \frac{2}{\pi^2} - \frac{4}{\pi^3}, \\ a_1^\zeta &= \frac{1}{4} - \frac{1}{2\pi} - \frac{2}{\pi^2} + \frac{12}{\pi^3}, \\ a_2^\zeta &= \frac{3}{8} + \frac{1}{\pi} - \frac{12}{\pi^3}, \\ a_3^\zeta &= \frac{3}{8} - \frac{1}{2\pi} + \frac{1}{\pi^2} + \frac{4}{\pi^3}.\end{aligned}$$

Assuming $a_i^\zeta = a_{i+1}$, we would expect the a_i to be valued

$$\begin{aligned}a_1 &\approx 0.07363, \\ a_2 &\approx 0.27522, \\ a_3 &\approx 0.30629, \\ a_4 &\approx 0.44617,\end{aligned}$$

though these values do not correspond to the values observed in 5.4. However, the expected number of particles at a fixed site, using these values

$$\hat{a} = 1 \cdot 0.07363 + 2 \cdot 0.27522 + 3 \cdot 0.30629 + 4 \cdot 0.44617 \approx 3.32762, \quad (5.8)$$

does approach our earlier estimated value for a .

One can combine (5.6), (5.7) and (5.8) to relate the radius $r_{\sigma(N)}$ to the area $A(N)$ as

$$A(N) = \pi r_{\sigma(N)}^2,$$

$$r_{\sigma(N)} \approx \sqrt{\frac{N}{3.3\pi}}.$$

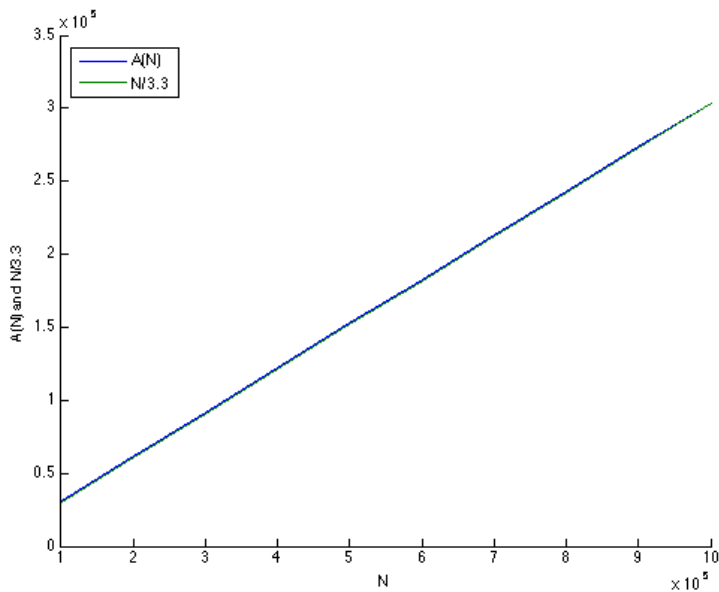


Figure 5.6: $A(N)$ and N/a for $N = [10^5, 10^6]$.

5.3.4 Using the fundamental solution w

To understand the relation between $S_{\sigma(N)}$ and the level sets of Nw and to find functions $f(N)$ and $g(N)$ such that $S_{u(N)}$ is squeezed between two level sets $\{n : |Nw(n)| \leq f(N)\}$ and $\{n : |Nw(n)| \leq g(N)\}$, we first explore the properties of the the fundamental solution w .

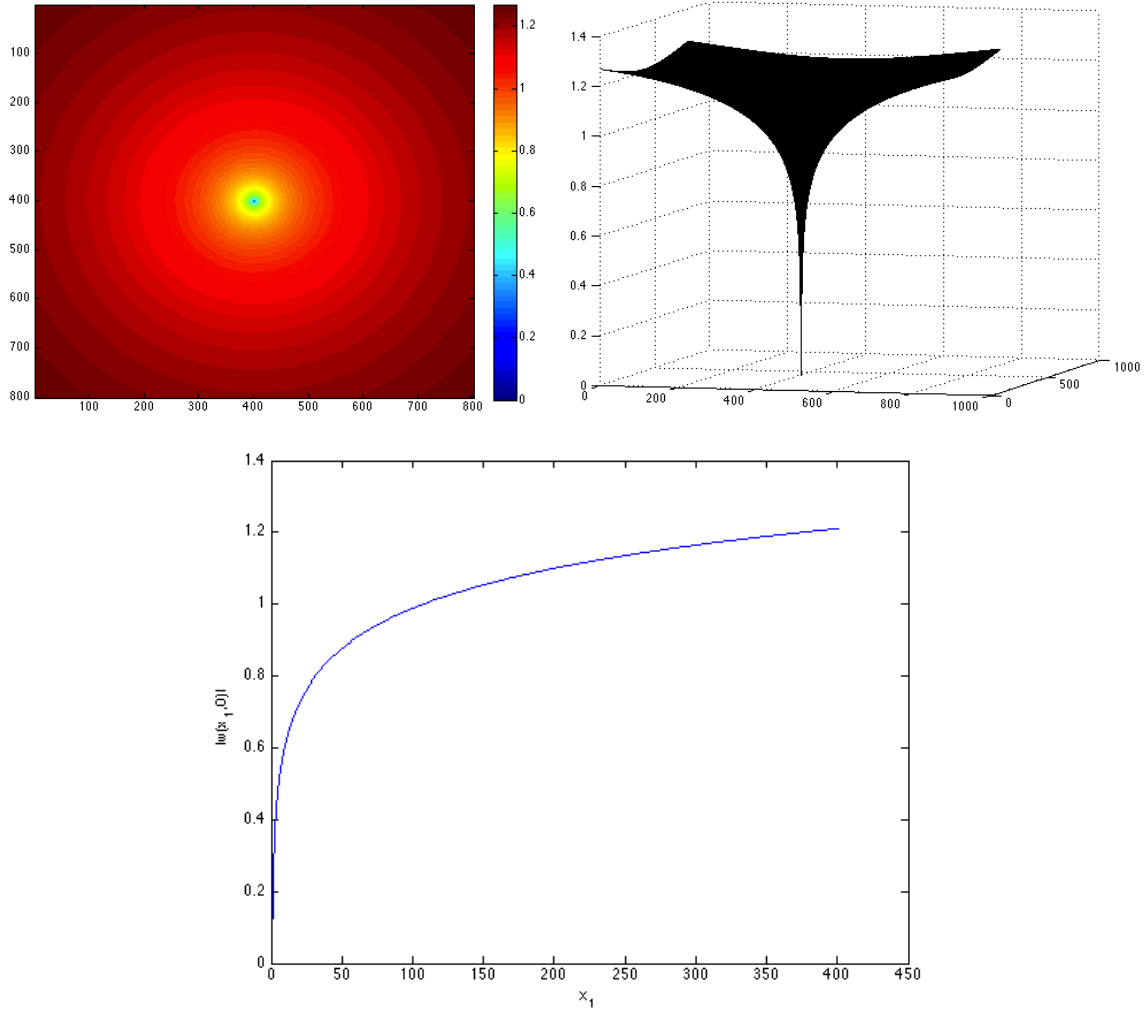


Figure 5.7: Top view (height plot) and surf image of $|w|$ and values on the x_1 -axis $|w(n, 0)|$.

We know that $|w|$ and $\sigma^{(N)}$ are symmetrical in the x_1 , x_2 and $x_1 = \pm x_2$ axes, so we are allowed to limit our analysis to the upper right quarter, which is beneficiary when performing simulations in Matlab.

The level sets of $|Nw|$ are of the form $\{n \in \mathbb{Z}^d : |Nw(n)| \leq C\}$ for a $C \in \mathbb{R}$. We want to find an inner and an outer bound for the support of $\sigma^{(N)}$. Therefore we define two special values, which describe the $|Nw|$ -value on the boundary of the bounding level sets for $S_{\sigma^{(N)}}$:

Definition 15. *The value $C^-(N)$ of $|Nw|$, defined by:*

$$C^-(N) := \max \left\{ C : \{n \in \mathbb{Z}^d : |Nw(n)| \leq C\} \subseteq S_{\sigma^{(N)}} \right\}$$

The value $C^+(N)$ of $|Nw|$, defined by:

$$C^+(N) := \min \left\{ C : \{n \in \mathbb{Z}^d : |Nw(n)| \leq C\} \supseteq S_{\sigma^{(N)}} \right\}$$

A plot of $C^\pm(N)$ for $N = [10^5, 10^6]$ is given in Figure 5.9.

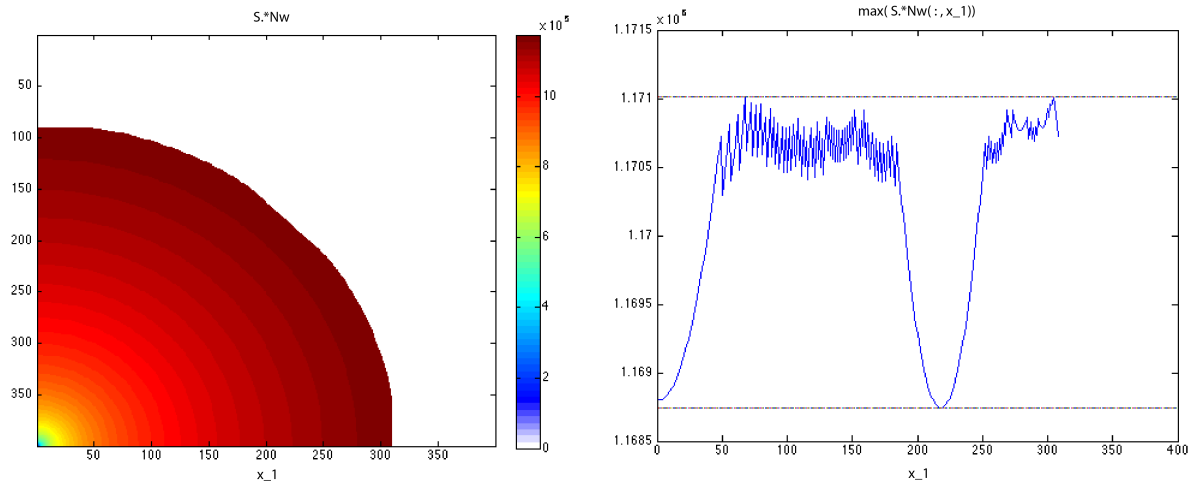


Figure 5.8: Top: 'S.*Nw'. Bottom: The values of Nw ($N = 10^6$) on the boundary of the upper right quarter of $\sigma^{(N)}$ for $N = 10^5$ by taking the maximum of every column of 'S.*Nw'.

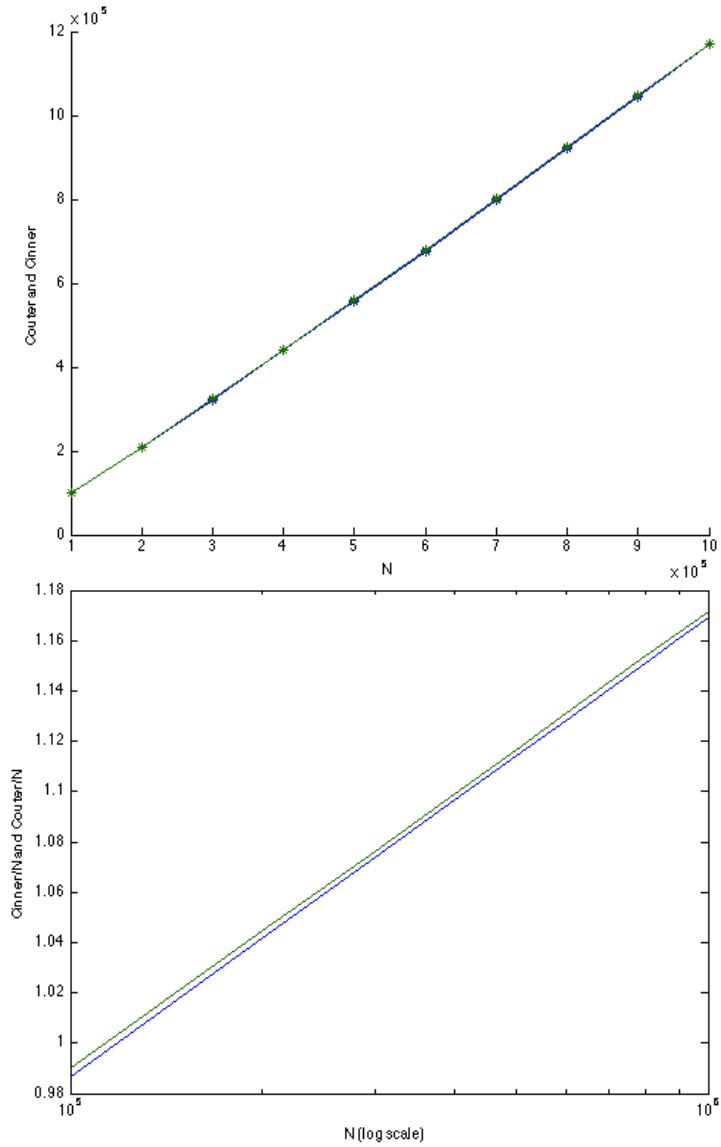


Figure 5.9: Behaviour of C^- and C^+ . Top: C^- and C^+ for $N = [10^5, 10^6]$. Bottom: $\frac{C^-(N)}{N}$ and $\frac{C^+(N)}{N}$ for $N = [10^5, 10^6]$ on a log scale.

Displaying the level sets $\{n : |Nw(n)| \leq C^-\}$ and $\{n : |Nw(n)| \leq C^+\}$ along with \mathfrak{S} , as seen in figure 5.10, we verify that the level sets indeed provide good bounds for $S_{\sigma(N)}$.

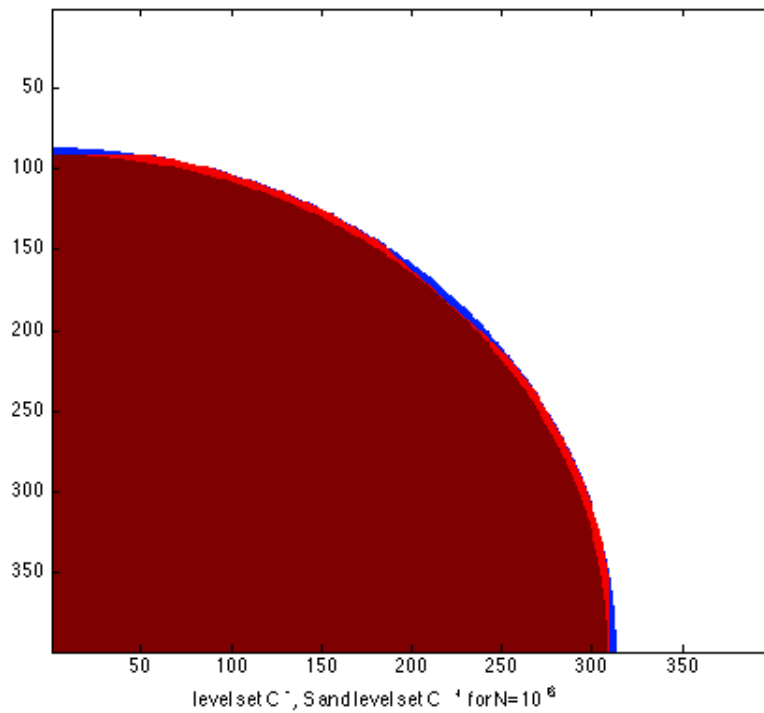
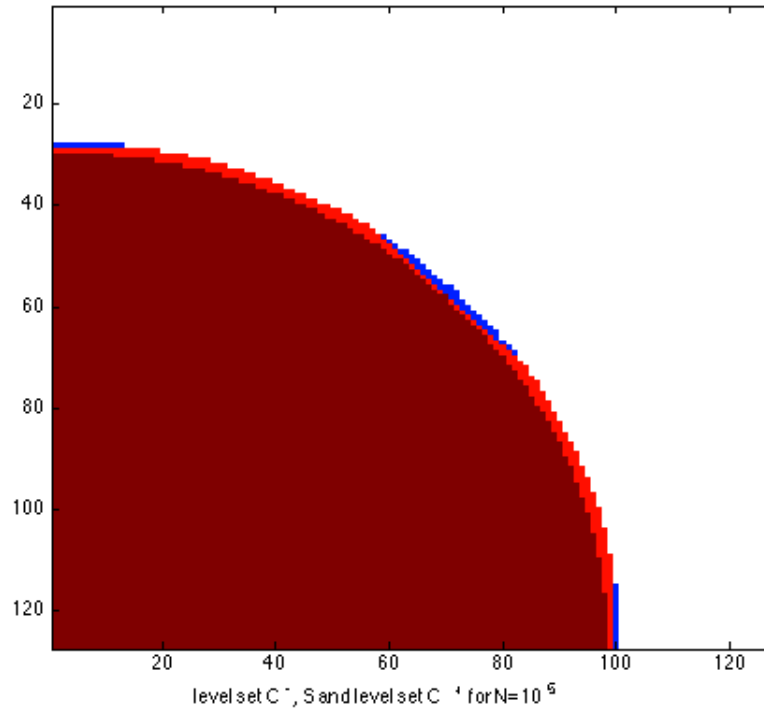


Figure 5.10: The level sets $\{n \in \mathbb{Z}^d : |Nw(n)| \leq C^-\}$ (brown-red) and $\{n \in \mathbb{Z}^d : |Nw(n)| \leq C^+\}$ (blue), bound $S_{\sigma(N)}$ (red) from below and above, for $N = 10^5$ and $N = 10^6$. The radii are r_{C^-} and r_{C^+} respectively, i.e. $|Nw(r_{C^\pm}, 0)| = C^\pm$.

Earlier, in section 5.3.2, we noted the radius of $S_{\sigma(N)}$ to behave like $r_{\sigma(N)} = \sqrt{N/\rho}$ where and the $C^\pm(N)$ are values of $|Nw|$ on the boundary of $S_{\sigma(N)}$, whose growth is equivalent to that of the radius, i.e. $\|n\| \simeq r_{\sigma(N)}$ for the n for which $|Nw(n)| = C^-$, as can be seen in figure 5.10. Therefore, recalling 2.9 for $N \rightarrow \infty$,

$$\frac{C^-(N)}{N} \approx |w(r_{\sigma(N)})| \approx \frac{1}{2\pi} \log \left(\sqrt{\frac{N}{\rho}} \right) + \kappa = \frac{1}{4\pi} (\log(N) - \log(\rho)) + \kappa = \frac{1}{4\pi} \log(N) - \hat{\kappa}, \quad (5.9)$$

where $\frac{1}{4\pi} \approx 0.079577$, $\kappa = \frac{\log(8)+2\gamma}{4\pi} \approx 0.27534$ and $\hat{\kappa} = \frac{\log(\rho)}{4\pi} - \kappa \approx -0.0713$.

A fit of the data for C^\pm verifies the forms

$$\begin{aligned} C^-(N) &= N(\alpha_1 \cdot \log(N) + \beta_1) \\ C^+(N) &= N(\alpha_2 \cdot \log(N) + \beta_2), \end{aligned}$$

with fitted parameters

$$\begin{aligned} \alpha_1 &= 0.07932 & (0.07920, 0.07943), \\ \alpha_2 &= 0.07893 & (0.07886, 0.07900), \\ \beta_1 &= 0.07331 & (0.07177, 0.07484), \\ \beta_2 &= 0.08099 & (0.08006, 0.08192). \end{aligned}$$

These functions $C^\pm(N)$ are in fact the $f(N)$ and $g(N)$ we were looking for such that $S_{\sigma(N)}$ is squeezed between two level sets $F(N) := \{n : |Nw(n)| \leq f(N)\}$ and $G(N) := \{n : |Nw(n)| \leq g(N)\}$, thus

$$f(N) := C^-(N), \quad (5.10)$$

$$g(N) := C^+(N). \quad (5.11)$$

Let the radii r_f and r_g of $F(N)$ and $G(N)$ be defined as

$$r_f = \max \{n : |Nw(n, 0)| \leq f(N)\} \approx \|n\| : |Nw(n)| = f(N), \quad (5.12)$$

$$r_g = \min \{n : |Nw(n, 0)| \geq g(N)\} \approx \|n\| : |Nw(n)| = g(N). \quad (5.13)$$

Combining (2.7) and (5.9) through (5.13) and solving for n , we find the expression for r_f and r_g :

$$\begin{aligned} N \left(\frac{1}{2\pi} \log \|n\| + \kappa \right) &= N \left(\frac{1}{4\pi} \log(N) - \hat{\kappa} \right) \Rightarrow r_f(N) \approx \sqrt{\frac{N}{\rho}} \approx 0.3016\sqrt{N} \\ N \left(\frac{1}{2\pi} \log \|n\| + \kappa \right) &= N (\alpha_2 \log(N) + \beta_2) \Rightarrow r_g(N) \approx \frac{e^{2\pi\beta_2} N^{2\pi\alpha_2}}{2\sqrt{2}} \approx 0.3302\sqrt{N} \end{aligned}$$

A plot of r_f and r_g along with $r_{\sigma(N)}$ is given in figure 5.11.

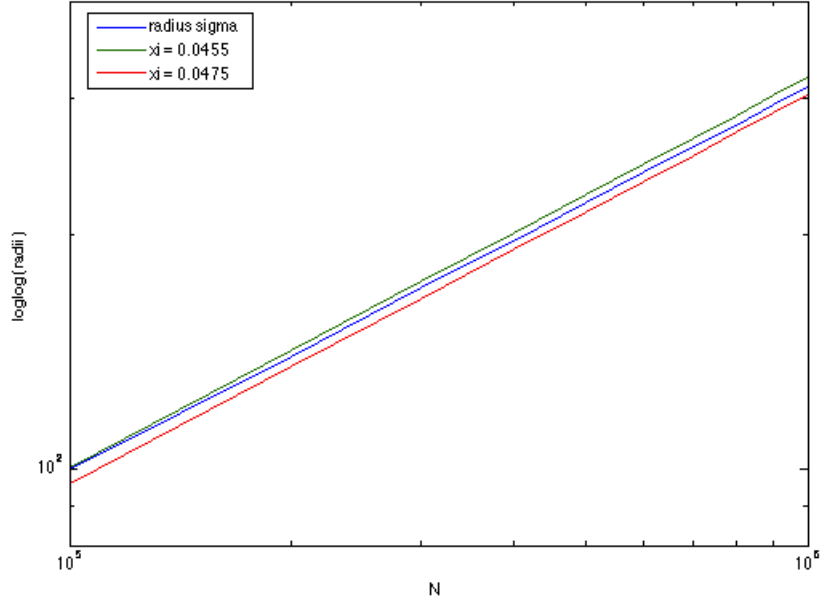


Figure 5.11: The radius $r_{\sigma(N)}$ along with r_f and r_g .

The inner and outer bounds given by Levine and Peres (Theorem 3) indicate radii

$$r_{ib} = \sqrt{\frac{N}{3\pi}} - c_2 \approx 0.3257\sqrt{N} \text{ and}$$

$$r_{ob} = \sqrt{\frac{N}{(2-\epsilon)\pi}} - c'_2 \approx 0.3989\sqrt{N}.$$

Note that these bounds are based on the model where the maximum height is $2d - 1$ in stead of $2d$. We therefore expect the bounds of our model to be smaller. The difference between the radii, of the form $\xi\sqrt{N}$, is smaller too, in particular

$$r_g - r_f \approx 0.0286\sqrt{N},$$

whereas $r_{ob} - r_{ib} \approx 0.0732\sqrt{N}$.

5.4 Polygonal bound

While the examples in figure 1.1 suggest the limiting shape to be dodecagonal (which has also been conjectured by Redig [6]), the plots in figure 5.10 show that the boundary of $S_{\sigma(N)}$ touches the inner bounding level set, with radius $r_f \simeq r_{\sigma(N)}$, at 0 , $\pi/4$ and $\pi/2$ radians and the outer bounding level set, with radius r_g , at $\pi/8$ and $3\pi/8$, which suggests the outer bounding shape to be octagonal (cf. figure 5.12). In particular $S_{\sigma(N)}$ seems to be bound above and below by an octagon of radius (apothem length) $r_{\sigma(N)}$ and its inscribed circle respectively.

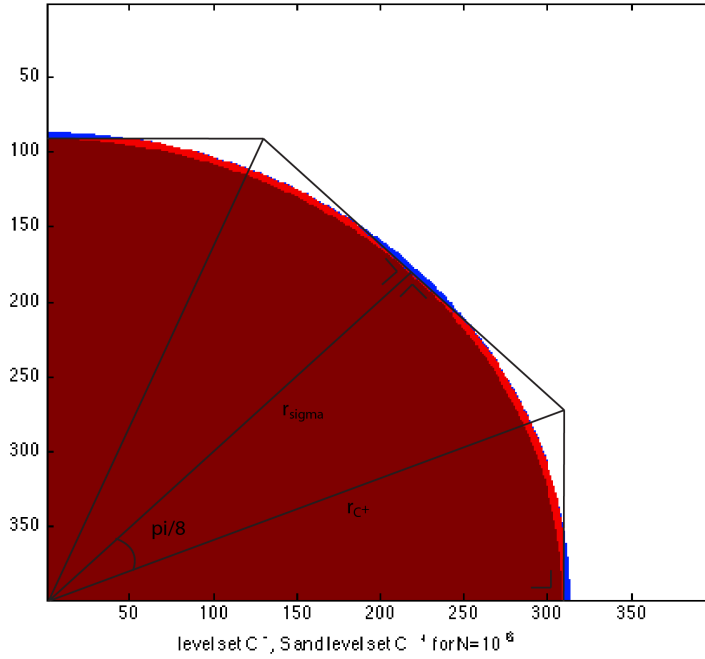


Figure 5.12: Finding a rough upper bound using simple geometry.

In fact, taking a closer look at 5.12 the limiting shape of $S_{\sigma(N)}$ seems to be the intersection between the octagon of radius or apothem length $r_{\sigma(N)}$ and the level set $G(N) = \{n : |Nw(n)| \leq C^+(N)\}$. This would require

$$r_g \leq \frac{r_{\sigma}}{\cos(\frac{\pi}{8})} := r_{C^+},$$

where $r_{\sigma}/\cos(\frac{\pi}{8})$ describes the radius of the circumscribed circle of the concerning octagon. According to our estimates $r_g \approx 0.3302\sqrt{N}$ and $r_{C^+} \approx 0.3362\sqrt{N}$ this is the case.

Chapter 6

Conclusions

In this thesis we performed analysis on the harmonic sandpile equation and showed that for all $d \geq 2$ and $k \geq 2d$

$$Nw - u^{(N)} - v^{(N)} \equiv 0.$$

Furthermore, simulations were performed for the critical, two-dimensional case, with the aim to find a link between the limiting shape and the fundamental solution in the form of level sets. These simulations, performed on the range $N = [10^5, 10^6]$, have shown the following:

The radius of the $S_{\sigma(N)}$ has been estimated at

$$r_{\sigma(N)} \approx \sqrt{\frac{N}{3.3\pi}} \approx 0.3106\sqrt{N}.$$

The two functions $f(N)$ and $g(N)$ such that $S_{u^{(N)}}$ lies between two level sets $\{n : |Nw(n)| \leq f(N)\}$ and $\{n : |Nw(n)| \leq g(N)\}$, have been estimated to be

$$\begin{aligned} f(N) &= C^-(N) \approx N(0.07957 \cdot \log(N) + 0.07130), \\ g(N) &= C^+(N) \approx N(0.07893 \cdot \log(N) + 0.08099), \end{aligned}$$

from which can be deduced that the radii of the level sets can be described by

$$\begin{aligned} r_f(N) &\approx 0.3016\sqrt{N}, \\ r_g(N) &\approx 0.3302\sqrt{N}. \end{aligned}$$

These bounds diverge at a rate of $r_g - r_f \approx 0.0286\sqrt{N}$. We know now that the limiting shape cannot be described in terms of the level sets of the fundamental solution w . We have seen however that for the upper bound, the radius r_g of level set $G(N)$,

$$0.3302\sqrt{N} \approx r_g(N) \leq r_{C^+} = \frac{r_{\sigma}}{\cos(\frac{\pi}{8})} \approx 0.3362\sqrt{N}$$

which leaves the door open for the limiting shape to be an octagon with apothem length $r_{\sigma(N)}$. This is left for another project.

Acknowledgements

Foremost, I would like to thank my advisor for all the patience, understanding, knowledge, guidance and fast replies throughout the writing of this thesis, through thick and thin.

Furthermore I would like to thank pr. dr. M. Zerner for also taking the time to clear up some questions concerning his research on the subject.

Bibliography

- [1] H.A. Heilbronn, On discrete harmonic functions. *Mathematical Proceedings of the Cambridge Philosophical Society*, 45, pp 194-206.
- [2] B.H. Murdoch: Some theorems on preharmonic functions. *J. London Math. Soc.*-1965-Murdoch-407-17
- [3] K. Schmidt, E. Verbitskiy, Abelian Sandpiles and the Harmonic Model. 2009. <http://arxiv.org/abs/0901.3124>
- [4] A. Fey, L. Levine and Y. Peres, Growth rates and explosions in sandpiles. 2009. <http://arxiv.org/abs/0901.3805>
- [5] R. Meester, F. Redig and D. Znamenski: The abelian sandpile model, a mathematical introduction. 2001. <http://arxiv.org/abs/cond-mat/0301481>.
- [6] A. Fey, F. Redig, Limiting shapes for deterministic centrally seeded growth models. *Journal of Statistical Physics* (2008) 130: 579-597. <http://de.arxiv.org/abs/math/0702450>.
- [7] M.P.W. Zerner: Directional decay of the Green's function for a random nonnegative potential on \mathbb{Z}^d . *The Annals of Applied Probability*, 1998, Vol. 8, No. 1, 246-280.
- [8] L. Levine, Y. Peres, Strong Spherical Asymptotics for Rotor-Router Aggregation and the Divisible Sandpile. *Potential Analysis* 30, no. 1 (Jan. 2009), 1-27. <http://arxiv.org/abs/0704.0688>
- [9] A. Fey, L. Levine, D.B. Wilson, The approach to criticality in sandpiles. *Phys. Rev. E* 82, 031121 (2010) (DOI 10.1103/PhysRevE.82.031121). <http://front.math.ucdavis.edu/1001.3401>.
- [10] <http://research.microsoft.com/en-us/um/people/dbwilson/sandpile/>.

Appendices

Appendix A

Script

The following script is the one I use for all my simulations. I only in the really end thought of a way to speed up the process as N grows, but didn't implement that (yet). I had different codes in between - including ones that make movies of the toppling process - but this one turned out to be the most efficient one.

```
1 % Simulating sandpiles
2 % Version 1: 23/3/2013 !! Has been updated a lot since.
3 % Frederique de Paus
4
5 % MAKE SURE TO LOAD 'w.mat' first! (Variable name is Green)
6
7 clc; %clear all;
8
9 % Parameters / initialization
10 N = 1000000;           % End amount of chips
11 n = 100000;           % Start amount of chips
12 batch = 100000;       % Number of chips to add per batch
13 d = 2;                % Dimension
14 h = 0;                % Constant hight at sigma
15 k = 2*d;              % Critical value (height)
16 order = (N/(2*pi))^(1/d); % Order of diameter field
17 M = floor(order);     % To ensure integer
18 r = 2*M+1;            % Length side matrix / field
19 o = M+1;              % Coordinate of origin
20 sigma = h*ones(r);   % Start configuration
21 u = zeros(r);         % Times x topples
22 desiredtime = 100000; % Max time you want to wait for results
23
24 ntimes = max((N-n)/batch+1,1); % 'Step size'
25 nsteps = batch*ones(ntimes,1); % Vector of amount of chips per batch
26 nsteps(1,1) = n;
27
28 umax = zeros(ntimes,1); % Maximum amount of topples
29 swide = zeros(ntimes,1); % Radius of sigma
30
31 Couter = zeros(ntimes,1); % Inner bound of boundary sigma
32 Cinner = zeros(ntimes,1); % Outer bound of boundary sigma
33
34 % Toppling
35 start = cputime;
36 loops = 0;
```

```

37 time = 0;
38 for j = 1:length(nsteps)
39     sigma(o,o) = sigma(o,o)+nsteps(j,1); % Adding chips at origin
40     while ((find(sigma>k)) & (time <= desiredtime))
41         loc = find(sigma>k); % Locations sigma>k
42         for i = 1:length(loc)
43             if floor(loc(i)/r) == loc(i)/r % Transforming to coord.
44                 y = loc(i)/r;
45                 x = r;
46             else
47                 y = floor(loc(i)/r)+1;
48                 x = loc(i) - (y-1)*r;
49             end
50             temp = sigma(x,y);
51             sigma(x,y) = mod(sigma(x,y)-1,k)+1;
52             mult = (temp-sigma(x,y))/k;
53             if x-1 > 0
54                 sigma(x-1,y) = sigma(x-1,y)+mult; end
55             if x+1 < r+1
56                 sigma(x+1,y) = sigma(x+1,y)+mult; end
57             if y-1 > 0
58                 sigma(x,y-1) = sigma(x,y-1)+mult; end
59             if y+1 < r+1
60                 sigma(x,y+1) = sigma(x,y+1)+mult; end
61             u(x,y) = u(x,y)+1;
62         end
63         loops = loops+1;
64         time = cputime-start;
65     end
66
67     % max odometer and radius
68     umax(j,1) = max(u(:)); % For growth u
69     hokje = o; % Start for radius sigma
70     while (sigma(o,hokje)>0 && hokje < o+M) % -
71         hokje = hokje + 1; % -
72     end % -
73     swide(j,1) = hokje-o+1; % end for radius sigma
74
75     % Ratios (Add or remove to save time)
76     aantal = zeros(1,4);
77     aantal(1) = length(find(sigma==1)); % # sites of height 1
78     aantal(2) = length(find(sigma==2)); % # sites of height 2
79     aantal(3) = length(find(sigma==3)); % # sites of height 3
80     aantal(4) = length(find(sigma==4)); % # sites of height 4
81     aandelen(j,6) = sum(aantal); % Area of support
82     aandelen(j,2:5) = aantal./aandelen(j,6);
83
84     % Calculating bound of boundary sigma
85     N = sum(nsteps(1:j,1)); % Amount of grains in system
86     Nw = N.*Green(401-M:401,401:401+M); % Green is 801x801
87     S = sigma(1:o,o:r); S(S>0)=1; % Right upper corner support
88     SedgeNwval = max(S.*Nw); % Max values columns => tresh.
89     SedgeNwval(SedgeNwval==0)=[]; % Remove zero values for min
90     Couter(j,1) = max(SedgeNwval); % Maximal value of Nw on edge
91     Cinner(j,1) = min(SedgeNwval); % Minimal value of Nw on edge
92 end
93
94 loops;
95 time

```