Non-linear waves in an atmospheric model
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Preface

Weather forecasts are part of everyday life. People use them to decide on clothing, holiday destinations and as subject of conversations. Nowadays the forecasts are quite accurate up to a week ahead, whereas thirty years ago one could only rely on 2-days forecasts. Of course the results of the forecasts are easy to grasp, but how about the methods to obtain those results, e.g. the underlying theory?

In chapter 2 we will state the governing equations of atmospheric flow. A physical aspect of the system, called the Coriolis effect, will then be described in more detail. After this we start looking for well-behaving solutions; traveling waves. This is done by transforming the system into new coordinates and manipulating the system in such way that we only need to solve one equation. In section 3 we obtain some qualitative results about waves under various conditions using phase plane analysis. Special attention is paid to connections between equilibrium points. Stability of the solutions will be discussed briefly in section 4. In section 5 the model used in the first sections is compared to an other model; the Eady model. The Eady model uses a slightly different approach but will turn out to also be quite similar.

Contents

1 Introduction

In search for more accurate weather forecasts there has been a lot of research on the dynamics in the atmosphere. A well-known theory on the large-scale behavior of the atmosphere found its roots at a work of Richardson [6] (1922). He recognized (amongst others) that the atmosphere could be regarded as a fluid. Nowadays it is indeed clear that the large-scale dynamics of the atmosphere is very much like the dynamics in the ocean. The dynamics in the atmosphere should thus satisfy the basic physical laws of fluid mechanics and thermodynamics. Using these laws, Richardson formed a set of equations that survived upcoming technology, such as computer solutions and improved experimental verifications remarkably well. A few decades later, Hoskins [4] (1975) and others modified some important aspects of the equations.

One adjustment is that they rewrote the system into new coordinates suitable for the atmosphere. In this coordinate system the Earth is assumed to rotate on an axis through the poles. Also, the Earth's surface is assumed to be spherical with perturbations due to varying height of the landscape, e.g. mountains and valleys.

A second adjustment is that the acceleration due to gravitation and centrifugal accelerations are combined and are assumed to act normally to geopotential surfaces (surfaces of constant potential energy). Geopotential surfaces are much like surfaces of constant height, but do not coincide due to the increasing acceleration of gravity when going polewards. The geopotential surfaces in the model are approximated by spherical surfaces. The corresponding equations are rewritten in spherical polar coordinates with the origin at the center of the Earth.

Other modifications included the assumption that the atmosphere consists of a compressible ideal gas and the magnitude of the Rossby number. The Rossby number will be explained shortly. The statements and the justification for the physical assumptions and other simplifications can be found in Gill [2] (1982), Holton [3] (1992) and Pedlosky [5] (1987).

2 Basic equations

As mentioned above, one of the assumptions by Hoskins is the magnitude of the Rossby number. The Rossby number is a measure for the change in momentum in proportion to the Coriolis force. This latter force corresponds to a rotating motion and will be explained in more detail later on. Physically, a large Rossby number corresponds to a system where inertial and centrifugal forces dominate, e.g. a tornado. In a system with small Rossby number the Coriolis force dominates, e.g. a low-pressure system. This latter system is of greater use when modeling large-scale dynamics and more usable for weather forecasts. When the Rossby number is assumed to be small the original set of equations can be modified to so-called semi-geostrophic equations. The semi-geostrophic approximation which is used in this thesis[1] is given by the system

$$\left[\frac{\partial}{\partial t} + (\bar{u}+u)\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right] u_g = -\frac{\partial}{\partial x}\phi + (f_0 + \beta y)v \tag{1}$$

$$\left[\frac{\partial}{\partial t} + (\bar{u}+u)\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right] v_g = -\frac{\partial}{\partial y}\phi - (f_0 + \beta y)u \tag{2}$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}\omega = 0 \tag{3}$$

$$\left[\frac{\partial}{\partial t} + (\bar{u}+u)\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right] \frac{\partial}{\partial z}\phi + S\omega = 0 \tag{4}$$

where u, v and ω are geostrophic coordinates (Y. Xu, Z. Lin and R. Dang [7], 1999). One might think of them as the northward, southward and upward winds[2]. The variables u_g and v_g are the so-called geostrophic wind components. They are northward and eastward components of a theoretical wind that would result from an exact balance between the Coriolis acceleration and the pressure gradient force; the direction and rate of the most rapid pressure change. The constant $\bar{u}$ is a basic zonal flow. It is a translation of u so that $u = 0$ corresponds to a flow along a latitude line. The static stability S is also constant. Static stability is a measure for the ability of a fluid at rest to become turbulent or laminar due to certain forces. The exact nature of S is not of importance in this thesis. Contribution of the Earth's gravity field, the geopotential, is accounted for by the variable ϕ. The constants f_0 and β correspond to the Coriolis parameter. This parameter and the Coriolis effect will now be explained in more detail.

2.1 Coriolis effect

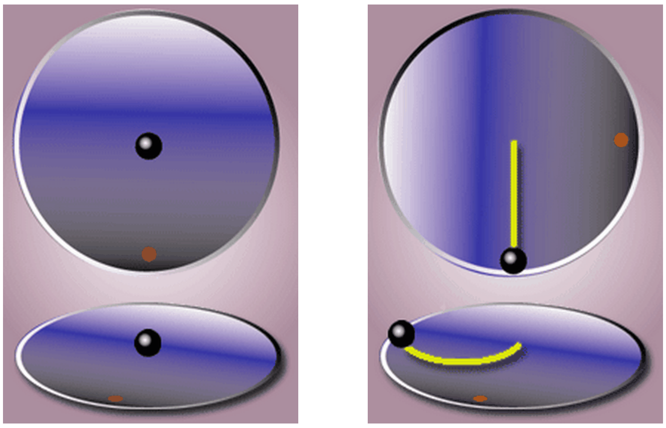
The Coriolis effect is an important factor in the semi-geostrophic equations. It is described in more detail to emphasize the importance to the system. Furthermore the paper by [7] uses a linear approximation of the Coriolis parameter instead of the constant approximation used by Hoskins and Eady [1]. The linear approximation of the Coriolis parameter f is given by

$$f = f_0 + \beta y. \tag{5}$$

where f_0 and β are constants and y is the meridional distance from a fixed latitude. In order to understand the influence of this parameter it is in place to gain more knowledge on the Coriolis effect. The Coriolis effect is an apparent deflection of moving objects when they are viewed from a rotating reference frame. To gain intuition on this effect we will consider a ball on a rotating disc (see figure 1). At the upper image the ball is seen from above whereas in the lower image we view it from the side.

[1]Additional assumptions for this particular approximation include frictionless motion and no heat exchange between different layers in the atmosphere.

[2]Strictly speaking they represent vector velocities.



(a) **(b)**

Figure 1: Illustration of the Corioliseffect. A ball placed on a rotating disc (a) describes the yellow path while rolling to the edge (b).

As can be seen in figure 1b the ball appears to roll from the middle of the disc to the edge in a straight line when we look from above. However, when looking from the side it appears that the ball moved over a non-straight curve. This illustrates the basic idea of the Coriolis effect. Furthermore it illustrates one of its properties when applied to the atmosphere as will be discussed below.

A linear approximation of the Coriolis parameter is given by (5) where

$$f_0 = 2\omega \sin \varphi. \tag{6}$$

and

$$\beta = \frac{2\omega}{R} \cos \varphi. \tag{7}$$

Here ω is the rotation rate of the earth, R is the radius of the earth and φ is the latitude of the particle on which it works. When the Coriolis force works on some particle in the atmosphere it will make the particle bend under radius proportional to $\frac{1}{f}$. This shows that the effect is weak when f is large, e.g. when the particle is close to a pole.

Upon considering the disc in figure 1 to be the earth seen from above, this makes a lot of sense. When the ball stays in the center there is no deflection so that a particle on a pole will not be affected by the Coriolis effect. On the other hand a particle at the equator will be effected strongly as the Coriolis force is small. Also note that from expressions (6) and (7) it follows that $f_0 \geq 0$ and $\beta \geq 0$.

Now that we have obtained some intuition about the Coriolis effect one might wonder how this is implemented in the equations. The earth is a rotating object which should serve as some coordinate system in order to write down equations describing the movement of a particle. For this reason one would like the earth to stand still and instead alter the equations so that they take the rotation into account. In other words, we wish to describe a moving object on a rotating frame. By implementing the Coriolis parameter, this is made possible.

2.2 Traveling waves

Now that we know the basic equations and have more understanding of the underlying physics, we will derive an equation for traveling wave solutions of (1)-(4). Derivation of the equations and the assumptions in this derivation is done analogue to [7].

By differentiating (1) with respect to x, (2) to y and adding them we obtain the vorticity equation

$$\begin{aligned}\left[\frac{\partial}{\partial t}+(\bar{u}+u)\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]&\left(\frac{\partial}{\partial x}v_g-\frac{\partial}{\partial y}u_g\right)\\&+\frac{\partial}{\partial x}u\frac{\partial}{\partial x}v_g+\frac{\partial}{\partial x}v\frac{\partial}{\partial y}v_g-\frac{\partial}{\partial y}u\frac{\partial}{\partial x}u_g-\frac{\partial}{\partial y}v\frac{\partial}{\partial y}u_g\\&\qquad=-(f_0+\beta y)\left(\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v\right)-\beta v\\&\qquad=-f_0\left(\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v\right)-\beta v.\end{aligned}\tag{8}$$

The latter equality is of great importance for the rest of the analysis. However, the exact reason why it holds is unclear. The article by [7] uses this equality without further explanation. One justification for the neglected y-term is an assumption on the area in which the dynamics are described. Recall that y is the meridional distance from a fixed latitude. Considering an area that is relatively close to this fixed latitude, y will be small and $\beta y \approx 0$. For now we will assume (8) is correct under certain assumptions.

Differentiating (1) with respect to y, (2) to x and subtracting them gives the divergence equation

$$\begin{aligned}\left[\frac{\partial}{\partial t}+(\bar{u}+u)\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]&\left(\frac{\partial}{\partial x}u_g+\frac{\partial}{\partial y}v_g\right)\\&+\frac{\partial}{\partial x}u\frac{\partial}{\partial x}u_g+\frac{\partial}{\partial x}v\frac{\partial}{\partial y}u_g+\frac{\partial}{\partial y}u\frac{\partial}{\partial x}v_g+\frac{\partial}{\partial y}v\frac{\partial}{\partial y}v_g\\&\qquad=(f_0+\beta y)\left(\frac{\partial}{\partial x}v-\frac{\partial}{\partial y}u\right)-\beta u-\nabla^2\phi\\&\qquad=f_0\left(\frac{\partial}{\partial x}v-\frac{\partial}{\partial y}u\right)-\beta u-\nabla^2\phi,\end{aligned}\tag{9}$$

where again we obtain the last equality by following [7]. When we look for traveling wave solutions of (1)-(4), we can obtain a simplified system using the relations above. To this end we introduce the traveling wave variables

$$\begin{aligned} &u = U(\rho), \quad v = V(\rho), \quad \omega = \Omega(\rho), \quad \phi = \Phi(\rho) \\ &\rho = kx + my + nz - \nu t \end{aligned} \tag{10}$$

where $k, m, n, \nu \in \mathbb{R}, \nu \geq 0$. The functions U, V, Ω and Φ describe the behavior of waves of u, v, ω and ϕ. We can now obtain an differential equation in Ω only. Furthermore we will find relations between the functions Ω, V and U so that any solution to the differential equation for Ω automatically solves the entire system.

Before doing this the following relations given by Hoskins [4] should be noted:

$$u_g = -\frac{1}{f_0}\frac{\partial}{\partial y}\phi \qquad \text{and} \qquad v_g = \frac{1}{f_0}\frac{\partial}{\partial x}\phi. \tag{11}$$

Substituting the traveling wave assumptions (10) and (11) into equations (3),(4),(8) and (9) gives (see Appendix A)

$$U = -\frac{1}{k}\left(mV + n\Omega\right), \tag{12}$$

and

$$V = b\Omega', \tag{13}$$

where Ω satisfies

$$\Omega'' = c_1\Omega' + c_2\Omega + \frac{\nu_1 - k\bar{u}}{\nu - k\bar{u}}\frac{\Omega^2}{\Omega + \frac{\nu - k\bar{u}}{n}}. \tag{14}$$

The constants are given by

$$\begin{aligned} &c_1 = -\frac{m\beta}{f_0}\frac{K_n^2}{K_2^2K_3^2},\ c_2 = -\frac{n^2\beta^2}{K_2^2K_3^2S} - \frac{v_1 - k\bar{u}}{v - k\bar{u}}, \\ &b = \frac{1}{f_0\beta}\left(\frac{S}{n}K_2^2 + nf_0^2\right),\ v_1 = k\bar{u} - \frac{k\beta}{K_3^2}, \\ &K_2^2 = k^2 + m^2,\ K_3^2 = k^2 + m^2 + \frac{n^2f_0^2}{S},\ K_n^2 = k^2 + m^2 + 2\frac{n^2f_0^2}{S}. \end{aligned}$$

3 Analysis of equilibrium points

We have now reduced system (1)-(4) to just one equation, (14). From now on we only consider this equation. In this section we will analyze the equilibrium points and their corresponding characters. Furthermore we are interested in connections between equilibrium points.

For now assume $c_1 \neq 0$. The degenerated case $c_1 = 0$ will be discussed later. By introducing the following rescaling into equation (14)

$$\tilde{\rho} = \frac{1}{|c_1|}\rho, \qquad \tilde{\Omega}(\tilde{\rho}) = \frac{n}{\nu - k\bar{u}}\Omega(\tilde{\rho})$$

the equation reduces to

$$
\begin{aligned}
&\frac{\partial}{\partial\tilde{\rho}}\tilde{\Omega} = Y \\
&\frac{\partial}{\partial\tilde{\rho}}Y = aY + c\tilde{\Omega} + d\frac{\tilde{\Omega}^2}{\tilde{\Omega}+1}, \\
&a = \operatorname{sgn}(c_1), \quad c = c_1^2 c_2, \quad d = c_1^2\frac{\nu_1 - k\bar{u}}{\nu - k\bar{u}}.
\end{aligned} \tag{15}
$$

Here $sgn(x)$ is the common sign function, in particular, $a^2 = 1$. In this step we also wrote the equation as a system of equations. For notational purposes we drop the tildes in the further analysis.

Note that the system has a singularity at $\tilde{\Omega} = -1$. Moreover, in Appendix A it is shown that $\tilde{\Omega} = -1$ (or $\Omega = -\frac{\nu - k\bar{u}}{n}$) is no solution to (14).

For the system there exist two equilibrium points, namely $(\Omega, Y) = (0,0)$ and $(\Omega, Y) = \left(-\frac{c}{c+d}, 0\right)$. We determine the character of the equilibrium points by analyzing the eigenvalues of the Jacobi matrix J given by

$$
J(\Omega, Y) = \begin{pmatrix} 0 & 1 \\ c + d\Omega\frac{\Omega+2}{(\Omega+1)^2} & a \end{pmatrix}.
$$

For the equilibrium point (0,0) it then follows that the eigenvalues $\lambda_\pm^0$ of $J(0,0)$ are given by

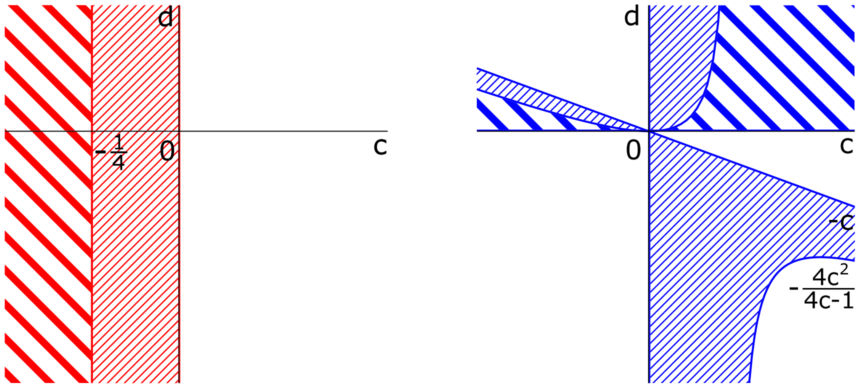
$$
\lambda_\pm^0 = \frac{1}{2}\left[a \pm \sqrt{1+4c}\right]
$$

whereas the eigenvalues $\lambda_\pm^c$ of $J\left(-\frac{c}{c+d}, 0\right)$ are given by

$$
\lambda_\pm^c = \frac{1}{2}\left[a \pm \sqrt{1 - 4c\frac{c+d}{d}}\right].
$$

The character of the equilibrium points is determined by the values of a, c and d. Assuming c and d are independent of each other one can see that both equilibrium points can be a sink, source, (un)stable focus and saddle. The sign of a determines local stability. A diagram with the possible characters is shown in figure 2. The character of $(0,0)$ can be found in figure 2a and depends only on c. The character of $\left(-\frac{c}{c+d}, 0\right)$ can be determined from figure 2b. Phase portraits corresponding to different characters can be found in figure 3. For example consider the focus-sink plot. The equilibrium $(0,0)$ is a focus, so one can see from figure 2a that c lies in the thick striped region. It then follows that $c < -\frac{1}{4}$. In the same way one can see from figure 2b that $-\frac{4c}{4c-1} < d < -c$ corresponds to a sink.

In some cases the equilibrium points are separated by the singularity at $\Omega = -1$. Then, the singularity prevents heteroclinic orbits from occurring. This makes sense when one recalls that $\Omega = -1$ is no solution to equation (14). By continuity a solution Ω will either be smaller than -1 for all ρ or



(a) Equilibrium point (0,0) **(b)** Equilibrium point $\left(0, -\frac{c}{c+d}\right)$

Figure 2: Characters of (0,0) and $\left(0, -\frac{c}{c+d}\right)$ for different values of c and d. For $a = -1$ there are sinks (thin striped) and stable foci (thick striped). For $a = 1$ the equilibrium points have opposite stability. The white areas correspond to saddle nodes.

larger than -1 for all ρ. In particular, a heteroclinic orbit can never occur when the equilibrium states are separated by -1.

The only phase portraits where the singularity does not separate the equilibrium points are those with exactly one saddle node. In those cases a heteroclinic orbit exists and is formed by a branch of the (un)stable manifold of the saddle (depending on the sign of a) which tends to the other equilibrium point forward or backward in time. For example, consider the sink-saddle plot. A heteroclinic orbit then starts at the saddle and tends to the sink. As function of ρ, Ω is then approximately constant for a long time and then rapidly changes to a lower constant value (a kink). In the focus-saddle plot we observe the same behavior though Ω then acts like a damped oscillator around the lower value. Later in this section we consider an other system where a plot of a homoclinic orbit of Ω will be given.

When Ω is known we can find U and V using relations (12) and (13). In the previous examples, the function $V(= b\Omega')$ can be seen to respectively describe a pulse or a pulse followed by oscillations. The function U is a linear combination of Ω and V.

Non-homoclinic solutions either tend to a equilibrium point or tend to $\pm\infty$ (right-hand solutions focus-saddle plot). In the first case, the functions U, V and Ω are eventually constant. Physically, this corresponds to particles moving at constant speed (balanced forces). The latter case corresponds to particles that move down- or upwards with increasing speed. However, this only describes their behavior for a short time. While getting closer to Earth the system does not describe the dynamics correctly. The main reason for this is the decreasing distance to Earth so that the boundaries of the system can

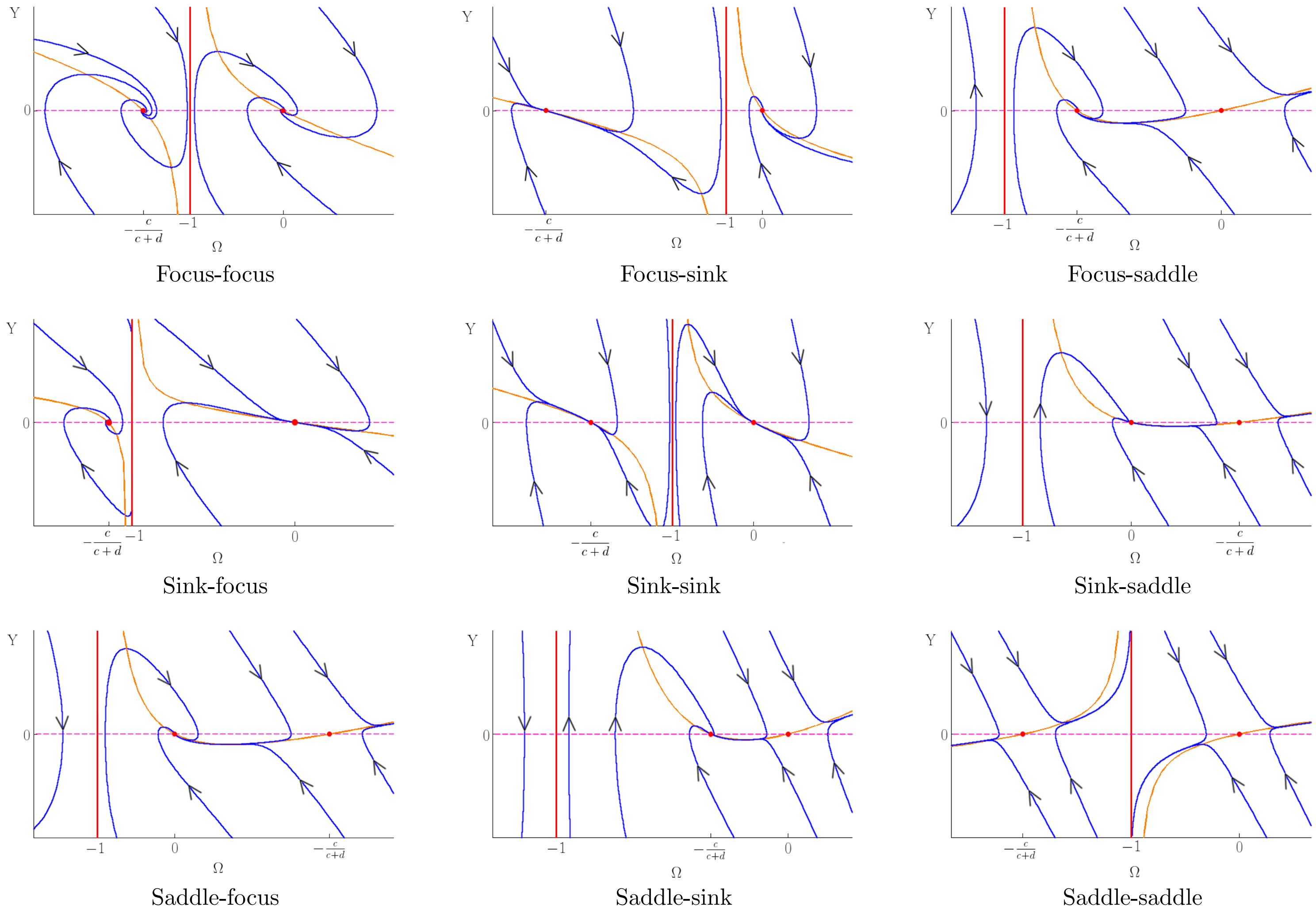


Figure 3: Various phase portraits of (15). The nullclines are given by the orange $\left(\frac{\partial}{\partial\rho}Y = 0\right)$ and dashed pink $\left(\frac{\partial}{\partial\rho}\Omega = 0\right)$ curves. Solutions are in blue. The red line at $x = -1$ shows the location of the discontinuity in the system. $a = -1$ in all plots. The solutions advance in opposite direction if $a = 1$ ($c_1 < 0$). Format of captions: character $(0, 0)$ - character $\left(-\frac{c}{c+d}, 0\right)$. Conditions on c and d can be found in figure 2. For example, the focus-sink plot corresponds to $c < -\frac{1}{4}, -\frac{4c}{4c-1} < d < -c$.

not be neglected.

3.1 Zonal wave

We now analyze the solutions under a specific assumption. As in the paper by Y. Xu, Z. Lin and R. Dang we assume that the waves are independent of the y-coordinate. This is equivalent to setting m equal to zero. This results in $c_1 = 0$ and so the assumption leads to a reduced form of (14). The system is then given by:

$$\Omega'' = c_2\Omega + \frac{\nu_1 - k\bar{u}}{\nu - k\bar{u}}\frac{\Omega^2}{\Omega + \frac{\nu - k\bar{u}}{n}}. \tag{16}$$

When we introduce the rescaling

$$\hat{\rho} = \sqrt{|c_2|}\rho, \qquad \hat{\Omega}(\hat{\rho}) = \frac{n}{\nu - k\bar{u}}\Omega(\hat{\rho})$$

into (16) it yields

$$\frac{\partial^2\hat{\Omega}}{\partial\rho^2} = r\hat{\Omega} + s\frac{\hat{\Omega}^2}{\hat{\Omega}+1} \tag{17}$$

or

$$\begin{aligned} \frac{\partial}{\partial\hat{\rho}}\hat{\Omega} &= Y \\ \frac{\partial}{\partial\hat{\rho}}Y &= r\hat{\Omega} + s\frac{\hat{\Omega}^2}{\hat{\Omega}+1}. \end{aligned}$$

where

$$r = \operatorname{sgn}(c_2), \quad s = |c_2|\frac{\nu_1 - k\bar{u}}{\nu - k\bar{u}}.$$

Again, we will drop the hats in the further analysis. Equilibrium points of this system are given by $(\Omega, Y) = (0,0)$ and $(\Omega, Y) = \left(-\frac{r}{r+s}, 0\right)$. The Jacobi matrix J is given by

$$J(\Omega, Y) = \begin{pmatrix} 0 & 1 \\ r + s\Omega\frac{\Omega+2}{(\Omega+1)^2} & 0 \end{pmatrix}$$

so that the eigenvalues $\lambda^0_\pm$ of the equilibrium point (0,0) are given by

$$\lambda^0_\pm = \pm\sqrt{r}$$

and the eigenvalues $\lambda^s_\pm$ of $J\left(-\frac{r}{r+s}, 0\right)$ are given by

$$\lambda^s_\pm = \pm\sqrt{-\frac{1+rs}{s}}.$$

In contrast to system (15), the equilibrium points can now either be a center or saddle. We are interested in connections in this system. For two centers or

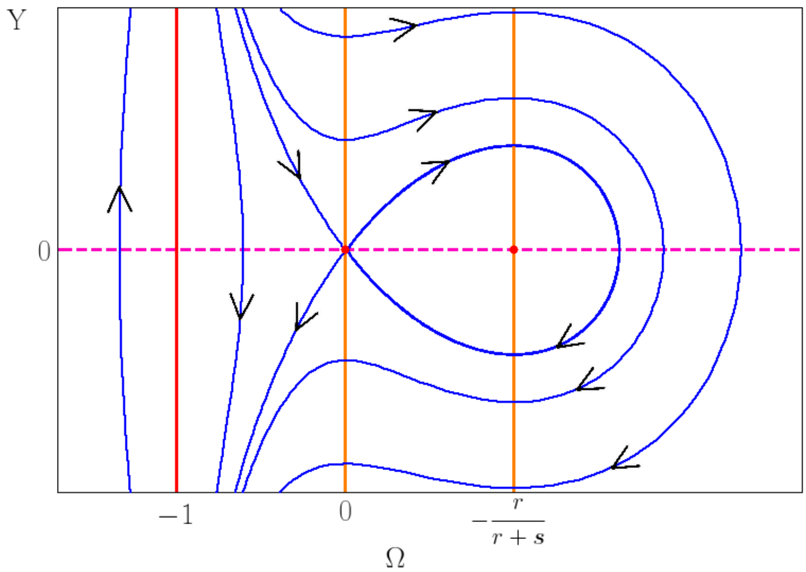


Figure 4: Level curves for (18) where $r = 1$ and $s = -2$. Graphical interpretation is given in the capture of figure 3.

two saddles there can be no connections. It can be shown, using the symmetry at $\Omega' = 0$, that a homoclinic orbit exists when the equilibrium points are a center and a saddle which are not separated by the singularity at $\Omega = -1$. One may check that the saddle and center are always located at the right-hand side of $\Omega = -1$ and so there always is a homoclinic orbit. When the parameters r and s satisfy $r = 1, s \in \mathbb{R}\backslash[-1, 0]$, there is a homoclinic orbit from $(0, 0)$ to itself. When $r = -1$ and $s \in \mathbb{R}\backslash[0, 1]$ the homoclinic orbit tends to $\left(-\frac{r}{r+s}, 0\right)$. A phase portrait of this system containing a homoclinic orbit is plotted in figure 4.

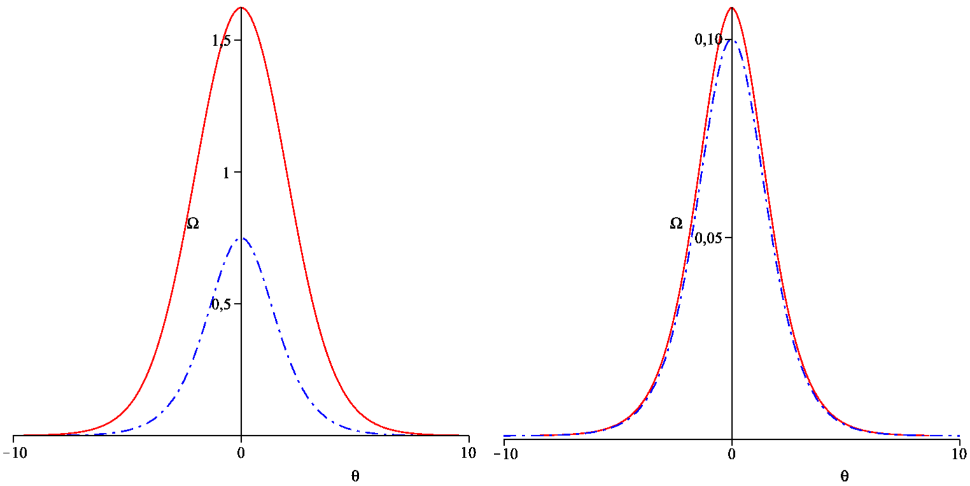
We will now focus on a homoclinic orbit that tends to $(0, 0)$. Multiplying (17) by Ω' and then integrating over ρ gives the first integral, yielding

$$\left(\Omega'\right)^2 = (r + s)\Omega^2 - 2s\Omega + 2s \ln |\Omega + 1| + C \tag{18}$$

where C is an arbitrary constant. It can be seen that $C = 0$ corresponds to the orbits 'through' $(0, 0)$. An implicit expression for the homoclinic orbit can be derived from (18). Solving the equation for Ω' and using the method of separation of variables results into

$$\int_{\Omega_{max}}^{\Omega} \frac{s\bar{\Omega}}{\sqrt{(r + s)\bar{\Omega}^2 - 2s\bar{\Omega} + 2s \ln |\bar{\Omega} + 1| + C}} = -|\rho - \rho_{sym}|$$

Here ρ_{sym} is the symmetry 'time' at which the orbit reaches its maximum value Ω_{max}. The corresponding (ρ, Ω)-plot is given in figure 5. As expected,



(a) $s = -2$ **(b)** $s = -15$

Figure 5: Homoclinic orbit from $(\Omega, Y)(0,0)$ to itself (red) and approximation by a sech-function (blue, dashed). The values of the parameters are $r = 1$ and $C = 0$. The symmetry 'time' ρ_{sym} is set to 0. The maximum value Ω_{max} is approximated numerically and depends on the other parameters. A larger value of $|s|$ results in a lower value of $|\Omega_{max}|$ as can be seen in the two figures.

the solution Ω stays close to the saddle (0,0), rapidly moves away and returns in little time.

Recall that $\rho = kx + nz - \nu t$. We may thus fix x and z and only vary t to observe the behavior of the solution over time. In this way a later time t corresponds to a lower value of ρ. The plot then describes a particle that has constant speed and direction for all time except for a short interval.

It can be seen that the behavior of Ω is very similar to that of a *sech*-function. Indeed the two are much alike as will be shown next. For small Ω we may use the Taylor expansion $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ in equation (17) and obtain

$$\frac{\partial^2 \Omega}{\partial \rho^2} = r\Omega + s\Omega^2 + \mathcal{O}(\Omega^3).$$

Neglecting the higher order terms this differential equation can be solved. The solution is then given by

$$\Omega(\rho) = -\frac{3r}{2s} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{r} \, (\rho - \rho_{sym}) \right). \tag{19}$$

Recall that $r = \pm 1$. For $r = -1$ it can be shown that (19) has infinitely many singularities and thus a solution to this equation does not exist. From figure 5 it can be seen that (19) is indeed a good approximation for small Ω when $r = 1$.

4 Stability

So far we have focused on the qualitative behavior of solutions to (14). Moreover, we found an implicit expression and an explicit approximation for a simplified system. Even more important than finding explicit solutions for the system is to analyze the flows close to a solution. Do solutions starting close to a known solution remain close to that solution for all time? In other words, are solutions stable? This section provides the concept of linear stability and applies it to our model.

An equilibrium solution x^* of a dynamical system $\frac{\partial}{\partial t}x = f(x)$ is called stable when a small perturbation of the initial condition decays over time. That is, any solution starting close to x^* will remain close to or even converge to x^*. Linear stability can be considered as stability in a small, linearized neighborhood of the equilibrium. Under certain smoothness conditions on the system, linear stability implies general stability. We will now describe a method to determine linear stability.

Linear stability is usually proven by considering an initial condition close to the equilibrium solution, $x(t) = x^* + \varepsilon x_d(t)$ ($|\varepsilon| << 1$), and determining the effect for longer time. This is done by substituting x into the corresponding differential system, linearizing this system (neglecting higher order terms) and solving it. One then obtains a leading order differential equation of the form $\frac{dx_d}{dt} = Df(x^*)x_d$ so that the eigenvalues of $Df(x)$ determine what will happen to the $\varepsilon x_d(t)$ perturbation. If all eigenvalues have negative real part then the equilibrium is stable. If at least one eigenvalue has positive real part then the equilibrium is unstable. When all real parts of the eigenvalues are non-positive and at least one eigenvalue equals zero, more advanced theorems such as the Stable Manifold Theorem are needed to conclude results on stability.

The concept of linear stability is the same for wave solutions, although the method has to be extended to partial differential equations and eigenvalues are not so easily found. We would like to know whether solutions starting close to a known orbit will stay close to this orbit. However, to determine stability we have to know how the perturbations evolve *in time*. Equation (14) only depends on the traveling wave variable ρ and thus can not show a solutions time-dependency on a perturbation. The only way to solve this is to reconsider equations (1)-(4) and introduce traveling wave variables that depend on both ρ and t;

$$u = U(\rho, t), \quad v = V(\rho, t), \quad \omega = \Omega(\rho, t), \quad \phi = \Phi(\rho, t)$$
$$\rho = kx + my + nz - \nu t.$$

Repeating the manipulations in Appendix A shows that the earlier relations

$$U(\rho, t) = -\frac{1}{k}\left(mV(\rho, t) + n\Omega(\rho, t)\right)$$

and

$$V(\rho, t) = b\frac{\partial}{\partial \rho}\Omega(\rho, t)$$

still hold. Instead equation (31) becomes

$$\frac{\partial^2\Phi(\rho,t)}{\partial\rho^2}+\frac{\partial}{\partial t}\Phi(\rho,t)=\frac{S\Omega(\rho,t)}{n^2\left(\Omega(\rho,t)+\frac{\nu-k\bar{u}}{n}\right)}.$$

A differential equation in one function is then given by

$$\frac{\partial}{\partial\rho}\left[\frac{nf_0b}{k}(\nu-k\bar{u}+n\Omega)\frac{\partial^2\Omega}{\partial\rho^2}\right]-\frac{nf_0b}{k}\frac{\partial^3\Omega}{\partial\rho^2\,\partial t}$$
$$+(\nu-k\bar{u}+2n\Omega)\frac{n^2\beta}{k^3}\frac{\partial\Omega}{\partial\rho}-\frac{n^2\beta}{k^3}\frac{\partial\Omega}{\partial t}-S\frac{\partial\Omega}{\partial\rho}=0. \quad (20)$$

We will now substitute $\Omega(\rho,t)=\Omega^*(\rho)+\varepsilon\Omega_d(\rho,t)$ into (20). Here $\Omega^*(\rho)$ is a solution of (14) and thus satisfies (32). It can be shown that due to this the $\mathcal{O}(1)$-terms vanish. The $\mathcal{O}(\varepsilon)$-terms are given by

$$\frac{\partial}{\partial\rho}\left[\frac{nf_0b}{k}(\nu-k\bar{u}+n\Omega^*)\frac{\partial^2\Omega_d}{\partial\rho^2}+\frac{n^2f_0b}{k}\Omega_d\frac{d^2\Omega^*}{d\rho^2}\right]$$
$$-\frac{nf_0b}{k}\frac{\partial^3\Omega_d}{\partial\rho^2\,\partial t}+(\nu-k\bar{u}+2n\Omega^*)\frac{n^2\beta}{k^3}\frac{\partial\Omega_d}{\partial\rho}$$
$$+2\frac{n^3\beta}{k^3}\Omega_d\frac{d\Omega^*}{d\rho}-\frac{n^2\beta}{k^3}\frac{\partial\Omega_d}{\partial t}-S\frac{\partial\Omega_d}{\partial\rho}=0.$$

The partial derivatives make it difficult to solve this system. One way to overcome this problem is to substitute the assumption $\Omega_d(\rho,t)=e^{\lambda t}\omega_d(\rho)$ into (20). The perturbation then will decay if $\mathrm{Re}(\lambda)<0$ and grow if $\mathrm{Re}(\lambda)>0$. Performing this substitution gives

$$\frac{d}{d\rho}\left[\frac{nf_0b}{k}(\nu-k\bar{u}+n\Omega^*)\frac{d^2\omega_d}{d\rho^2}+\frac{n^2f_0b}{k}\omega_d\frac{d^2\Omega^*}{d\rho^2}\right]$$
$$-\frac{nf_0b\lambda}{k}\frac{d^2\omega_d}{d\rho^2}+(\nu-k\bar{u}+2n\Omega^*)\frac{n^2\beta}{k^3}\frac{d\omega_d}{d\rho}$$
$$+\frac{2n^3\beta}{k^3}\omega_d\frac{d\Omega^*}{d\rho}-\frac{n^2\beta\lambda}{k^3}\omega_d-S\frac{d\omega_d}{d\rho}=0.$$

When we substitute a known solution Ω^* this expression is still difficult to solve. In particular substitution of a pulse solution leads to expressions that can not be solved without more advanced techniques. Still, it is important to note that this is the correct way to determine stability. The independent t-variable in the wave solutions is required in order to state results on stability.

5 Eady model

In the previous sections we only considered one model. Still, there are lots of models describing the dynamics in the atmosphere. A well-known model

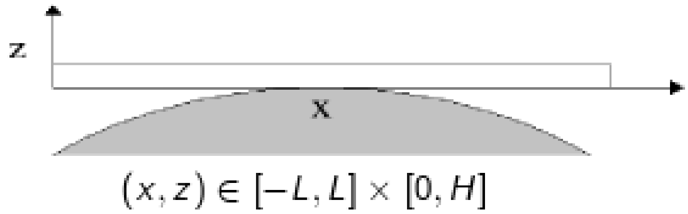


Figure 6: Framework of the two-dimensional Eady model.

concerning the atmosphere is the Eady model [1]. Eady's understanding of the dynamics in the atmosphere and physical insight allowed him to simplify a model similar to the model discussed before. His work is considered as the most pure model to describe baroclinic instability; a phenomenon closely related to weather fronts. It only takes the most relevant forces into consideration and describes the basic mechanism of this instability. The model applies to the mid-latitude areas of the Earth and has a very interesting set-up. Instead of an unbounded three-dimensional plane, the Eady model considers a bounded two-dimensional plane. The dynamics are only described in the x- and z-directions (or rather latitudinal and vertical directions) for fixed distances (see figure 6). It is also for this reason that all functions and variables in the model are independent of y. Other assumptions on the model are much like the ones that give the model of Y. Xu, Z. Lin and R. Dang. Since many assumptions are alike and yet the framework of both models are quite different it is interesting to compare the two. For convenience the model in [7] will be referred to as Xu's model from now on.

The Eady model is given by

$$\begin{aligned}
&\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial z}\right] u = -\frac{\partial}{\partial x}\phi + fv \\
&\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial z}\right] v = Rg\theta_0^{-1}\left(z - \frac{H}{2}\right) - fu \\
&\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial z}\right] \omega = -\frac{\partial}{\partial z}\phi + g\theta\bar{\theta}_0^{-1} \\
&\left[\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + \omega\frac{\partial}{\partial z}\right] \theta - Rv = 0 \\
&\frac{\partial}{\partial x}u + \frac{\partial}{\partial z}\omega = 0
\end{aligned} \tag{21}$$

where u, v and ω are the same as in (1)-(4). The function θ is now related to pressure instead of to gravity. New in this system is the change in potential temperature $\theta(x, z, t)$. An initial reference state is given by $\theta_0^{-1}(x, z)$. R and g are constants. The Coriolis parameter is approximated by the constant f.

As both methods describe systems that are very much alike it is interesting to compare the two. Since the Eady model omits all dependency on y it is best to do the same in Xu's system. Most importantly we set $m = 0$ as we already did in section 3.

We would now like to consider traveling waves in the Eady model. To this end we substitute the traveling wave variables

$$
\begin{aligned}
&u = U(\rho), \quad v = V(\rho), \quad \omega = \Omega(\rho), \quad \phi = \Phi(\rho), \quad \theta = \Theta(\rho) \\
&\rho = kx + nz - \nu t
\end{aligned}
\tag{22}
$$

where $k, n, \nu \in \mathbb{R}, \nu \geq 0$ into the Eady model. The functions U, V, Ω, Φ and Θ describe the behavior of waves of u, v, ω, ϕ and θ as before. It is shown in Appendix B that the relations

$$U = \frac{1}{k}(C - nW)$$

and

$$V = \frac{C - \nu}{R}\Theta' \tag{23}$$

hold. Here C is a constant that is determined by the initial values of U and Ω. Comparing these relations to (12) and (13) (recall $m = 0$) it is clear that the waves are very much alike. The latitudinal velocity U is again proportional to Ω and V equals a constant multiplied with an other function's derivative. It is remarkable, however, that V does not depend on the vertical velocity as in Xu's model, but rather on the change in potential temperature Φ.

One possible explanation for this is the nature of the model. Baroclinic instability is a phenomenon directly related to the temperature differences between layers in the atmosphere. When the temperature is higher, the air will rise. In Xu's model there is no explicit dependence on the temperature. All effects due to temperature are thus implicitly included in the system. Following the preceding reasoning it is natural that most temperature effects are accounted for in the vertical velocity Ω. With the same arguments one could say that the vertical velocity is dominated by the temperature. The two are thus very close related, which is exactly what we see in (13) and (23).

In Appendix B it is also shown that

$$
\begin{aligned}
\frac{Rg}{nf}(C-\nu)K_2^2\frac{d}{d\rho}\left[\left(z-\frac{H}{2}\right)\theta_0^{-1}(x,z)\right] - \frac{1}{nfR}(C-\nu)^3K_2^2\Theta''' \\
- \frac{nf(C-\nu)}{R}\Theta' + kg\bar{\theta}_0^{-1}(x,z)\Theta = 0
\end{aligned}
\tag{24}
$$

holds. This is an equation in Θ, which again appears to be closely related to the function Ω in Xu's model. Furthermore it can be seen that the change in potential temperature Θ is greatly influenced by it's reference state θ_0. This is quite natural and thus it's presence in (24) could be expected.

Note that θ_0 is still a function of x and z instead of the wave variable ρ. Assuming θ_0 is constant we see that (24) becomes

$$\frac{1}{nfR}(C-\nu)^3K_2^2\Theta''' = -\frac{nf(C-\nu)}{R}\Theta' + kg\bar{\theta}_0^{-1}\Theta + \frac{Rg}{n^2f}(C-\nu)K_2^2.$$

The solution to this system is a linear combination of three exponentials and a constant. It is thus not very likely that the initial potential temperature is constant. When substituting a more general θ_0, for instance a linear function in z, (24) is a partial differential equation. One will need more advanced techniques to solve these equations.

As for the comparison with Xu's model we can conclude a few things. First of all, the functions U and V are related to the other functions in the almost exact same way. The biggest difference is that the 'upward wind' from Xu has been replaced by the change in potential temperature. It is reasonable that those two are closely related although this has not been proven in this thesis.

A differential equation for Θ and θ_0 supports this statement although there are still some differences. The non-linear term is not the same as in Xu's model and relatively simple solutions are completely different. The differential equation for Θ can not be easily solved for more complex initial temperatures, so those can not be compared.

6 Conclusion

After a short introduction into the semi-geostrophic equations and its physical background we introduced traveling wave variables in order to find wave solutions of the system. By doing so the system reduced to a single differential equation containing one function. A better understanding of the dynamics was then gained by doing phase plane analysis. From the analysis it followed that there are a lot of possible connections between the steady states. Considering a simplified model allowed us to determine an implicit expression for a homoclinic orbit. The corresponding plot showed the behavior of the homoclinic orbit in time. Now that there was more understanding of wave-solutions of the system, an attempt was made to determine stability. Though this thesis does not provide results on stability of the traveling wave solutions, it did provide the underlying ideas. Finally we compared the model to the two-dimensional Eady model. This model describes different behavior in the atmosphere but shows a lot of similarities as well. Introducing a traveling wave variable into the Eady model resulted in the almost exact relations that were obtained for the model used in the rest of this thesis. This implies a close connection between the two models.

Appendix A

Substituting (11) into (8) gives

$$\left[\frac{\partial}{\partial t}+(\bar{u}+u)\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right]\left(\frac{\partial}{\partial xx}\phi+\frac{\partial}{\partial yy}\phi\right)$$
$$+\frac{\partial}{\partial x}u\frac{\partial}{\partial x}v_g+\frac{\partial}{\partial x}v\frac{\partial}{\partial y}v_g-\frac{\partial}{\partial y}u\frac{\partial}{\partial x}u_g-\frac{\partial}{\partial y}v\frac{\partial}{\partial y}u_g$$
$$=-f_0^2(\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v)-f\beta v$$

which may be rewritten to

$$(k\bar{u}-\nu+kU+mV)(k^2+m^2)\Phi'''+(kU'+mV')(k^2+m^2)\Phi''$$
$$=-f_0^2(kU'+mV')-f\beta V$$

in the new coordinate. This expression then may be simplified to

$$[(k\bar{u}-\nu+kU+mV)K_2^2\Phi'']'+f_0^2(kU'+mV')+f\beta V=0 \tag{25}$$

where $K_2^2=k^2+m^2$. In the same way we may modify (9) to find

$$f_0(kV'-mU')-\beta U-K_2^2\Phi''=0. \tag{26}$$

Applying (10) to equation (4) respectively (3) yields

$$kU'+mV'+n\Omega'=0 \tag{27}$$

respectively

$$(k\bar{u}-\nu+kU+mV)n\Phi''+S\Omega=0 \tag{28}$$

where the former can be integrated with respect to ρ in order to obtain

$$kU+mV+n\Omega=0. \tag{29}$$

The arbitrary integration constant has been set equal to zero. Substituting (27) and (28) into (25) results into

$$-\frac{S}{n}K_2^2\Omega'-nf_0^2\Omega'+f_0\beta V=0$$

or

$$V=b\Omega' \tag{30}$$

where $b=\frac{1}{f_0\beta}\left(\frac{S}{n}K_2^2+nf_0^2\right)$. Substituting (29) into (28) gives

$$-(\nu-k\bar{u}+n\Omega)n\Phi''+S\Omega=0.$$

It can easily be verified that $\Omega=-\frac{\nu-k\bar{u}}{n}$ is no solution to this equation under the assumption $\nu-k\bar{u}\neq 0$. We may thus write Φ'' in terms of Ω:

$$\Phi''=\frac{S\Omega}{n^2\left(\Omega+\frac{\nu-k\bar{u}}{n}\right)}. \tag{31}$$

Using all results above to find an differential equation for Ω. Starting with (26) and substituting (29) we get

$$f_0\left(kV' + \frac{m}{k}\left(n\Omega' + mV'\right)\right) + \frac{\beta}{k}\left(n\Omega + mV\right) - K_2^2\Phi'' = 0$$

which, using (30), can be rewritten as

$$f_0\left(kb\Omega'' + \frac{m}{k}\left(n\Omega' + mb\Omega''\right)\right) + \frac{\beta}{k}\left(n\Omega + mb\Omega'\right) - K_2^2\Phi'' = 0.$$

Substituting (31) gives

$$\frac{f_0}{k}K_2^2 b\Omega'' + \frac{m}{k}\left(f_0 n + \beta b\right)\Omega' + \frac{n\beta}{k}\Omega - K_2^2\frac{S\Omega}{n^2\left(\Omega + \frac{\nu - k\bar{u}}{n}\right)} = 0, \tag{32}$$

which simplifies to (14).

Appendix B

Substituting (22) into the system of equations corresponding to the Eady model yields

$$(kU + nW - \nu)\,U' - fV + k\Phi' = 0 \tag{33}$$
$$(kU + nW - \nu)\,V' + fU - Rg\theta_0^{-1}\varepsilon = 0 \tag{34}$$
$$(kU + nW - \nu)\,\Omega' + n\Phi' - g\bar{\theta}_0^{-1}\Theta = 0 \tag{35}$$
$$(kU + nW - \nu)\,\Theta' - RV = 0 \tag{36}$$
$$kU' + nW' = 0. \tag{37}$$

Integrating (37) over ρ gives

$$kU + nW = C \tag{38}$$

where C is the integration constant. Substituting this into (36) yields

$$V = \frac{C - \nu}{R}\Theta'. \tag{39}$$

Multiplying (33) by n, (35) by k and subtracting gives

$$(kU + nW - \nu)\left(nU' - kW'\right) - nfV + kg\bar{\theta}_0^{-1}\Theta = 0,$$

which, after substituting (37) and (38), becomes

$$\frac{1}{n}(C - \nu)K_2^2 U' - nfV + kg\bar{\theta}_0^{-1}\Theta = 0.$$

Here $K_2^2 = k^2 + n^2$. Substituting (34) into (6) for U then gives

$$\frac{Rg\varepsilon}{nf}(C - \nu)K_2^2\frac{d\theta_0^{-1}(x,z)}{d\rho} - \frac{1}{nf}(C - \nu)^2 K_2^2 V'' - nfV + kg\bar{\theta}_0^{-1}\Theta = 0.$$

When we substitute (39) into this we finally obtain

$$\frac{Rg\varepsilon}{nf}(C - \nu)K_2^2\frac{d\theta_0^{-1}(x,z)}{d\rho} - \frac{1}{nfR}(C - \nu)^3 K_2^2\Theta''' - \frac{nf(C - \nu)}{R}\Theta' + kg\bar{\theta}_0^{-1}\Theta = 0.$$

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