# Infinitesimally Rigid Construction of the Algebraic Numbers

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**M. Derickx**

# Infinitesimally Rigid Construction of the Algebraic Numbers

**Bachelor thesis, 11 june 2010**

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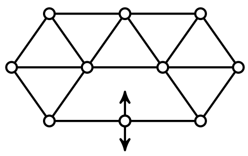
# Contents

# Preface

In my thesis I will generalize a previous result of M. Maehara. In "Distances in a rigid unit-distance graph in the plane"[1] he proved that the distances that occur between vertices in planar rigid unit-distance graphs are precisely the positive real algebraic numbers. A unit-distance graph is a framework of equal length bars which are connected in a flexible way at their endpoints. Such a framework is called rigid if it cannot be deformed without changing the length of the bars. There is also a stronger notion of rigidity which is called infinitesimal rigidity. This stronger notion asks that the framework cannot even be deformed infinitesimally without an infinitesimal change of the lengths of the bars. The picture on this page shows an example of a framework which is rigid but not infinitesimally rigid. The point with the arrows, cannot really move relative to the rest of the construction, although you can do this infinitesimally in the direction indicated by the arrows.



A yet unanswered question was whether Maehara's result also holds for infinitesimally rigid frameworks. It turned out to be true even with this stronger notion of rigidity. I will prove this in my thesis by showing that Maehara's construction is infinitesimally rigid in most cases and give a different construction for the cases where Maehara's construction isn't infinitesimally rigid.

[1] Discrete Applied Mathematics Volume 31, Issue 2, 15 April 1991, Pages 193-200

# 1 Definitions and Conventions

In this thesis we assume that the graphs are always finite and simple. We define a morphism between two graphs to be a map between the vertex sets that sends adjacent points either to adjacent points or to the same point. The graphs form a category with the mentioned morphisms. For graphs $(V, E)$ we view the edge set $E$ as a subset of $\{S \subset V | \#S = 2\}$. Whenever $f : V \mapsto X$ is any function defined on $V$ we sometimes also view $f$ as a function on $E$, sending $\{v_1, v_2\}$ to $\{f(v_1), f(v_2)\}$.

For frameworks we take the following definition:

**Definition 1.1.** A $n$-dimensional framework is an ordered triple $(G, f, \mathbb{R}^n)$ where $G = (V, E)$ is a graph and $f : V \to \mathbb{R}^n$ a map which is called a realization of the graph $G$.

A morphism between two $n$-dimensional frameworks $(G, f, \mathbb{R}^n)$ and $(G', f', \mathbb{R}^n)$ is a tuple $(g, h)$ where $g : G \to G'$ is a graph morphism and $h$ is an isometry of $\mathbb{R}^n$ such that $h \circ f = f' \circ g$.

We also define the category of frameworks to be the category wich has as objects the frameworks and has as morphisms the ones mentioned above.

In the categories of graphs and framewors the fibred sum always exists. People not familiar with categories should see the following propositions as definitions of the fibred sum.

The fibred sum, sometimes called gluing, of two graphs can be more explicitly given. The goal of this thesis is not to handle theory about graphs and frameworks in detail. So we state the following proposition without proof.

**Proposition 1.2.** *Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$ be graphs and $g_1 : G_3 \to G_1$ and $g_2 : G_3 \to G_2$ be two graph morphisms. Then $G_1 \coprod_{G_3} G_2$ given by*

$$G_1 \coprod_{G_3} G_2 = (V_1 \coprod_{V_3} V_2, \{e \in \iota_1(E_1) \cup \iota_2(E_2) | \#e = 2\}) \tag{1}$$

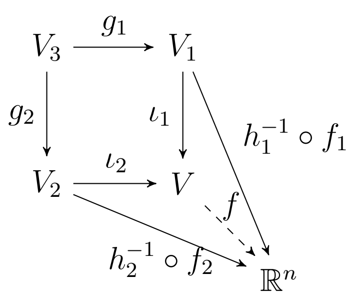
*satisfies the universal property of the fibred sum. Here $V_1 \coprod_{V_3} V_2$ is the usual fibred sum of sets and $\iota_1$ and $\iota_2$ are the two maps which come with this fibered sum. Furthermore $\iota_1$ and $\iota_2$ are also the graph morphisms which come with this fibred sum of graphs.*

In the rest of this thesis $g_1$ and $g_2$ will always be injective. In this case $\iota_1$ and $\iota_2$ will be injective also so we will view $G_1$ and $G_2$ as subgraphs of $G$.

The same can be done for frameworks.

**Proposition 1.3.** *Let the definition of variables be the same as in the previous proposition. Let furthermore* $F_1 = (G_1, f_1, \mathbb{R}^n), F_2 = (G_2, f_2, \mathbb{R}^n)$ *and* $F_3 = (G_3, f_3, \mathbb{R}^n)$ *be frameworks and* $(g_1, h_1) : F_3 \to F_1$ *and* $(g_2, h_2) : F_3 \to F_2$ *be two framework morphisms.*

*Then there is a unique map* $f = h_1^{-1} \circ f_1 \coprod h_2^{-1} \circ f_2 : V \to \mathbb{R}^n$ *which makes the following diagram commute.*




*Also* $F_1 \coprod_{F_3} F_2$ *given by*

$$F_1 \coprod_{F_3} F_2 = (G_1 \coprod_{G_3} G_2, f, \mathbb{R}^n) \tag{2}$$

*satisfies the universal property of the fibred sum. The framework morphisms from* $F_1$ *and* $F_2$ *to* $F_1 \coprod_{F_3} F_2$ *are* $(\iota_1, h_1^{-1})$ *and* $(\iota_2, h_2^{-1})$

We also define two quite similar functions for every framework: the distance and the length function.

**Definition 1.4.** Let $G = (V, E)$ be a graph and $(G, f, \mathbb{R}^n)$ be a framework then the distance function $d_f$ is given by:

$$\begin{aligned} d_f : \quad V \times V &\to \mathbb{R} \\ (v_1, v_2) &\mapsto |f(v_1) - f(v_2)| \end{aligned} \tag{3}$$

where $|\cdot|$ is the standard norm on $\mathbb{R}^n$. It is called the distance function because $d_f(v_1, v_2)$ is the distance between $f(v_1)$ and $f(v_2)$ The length function $l_f$ is given by:

$$\begin{aligned} l_f : \quad E &\to \mathbb{R} \\ \{v_1, v_2\} &\mapsto |f(v_1) - f(v_2)| \end{aligned} \tag{4}$$

Note that for an edge $e$ the value $l_f(e)$ is just the length of the edge.

For any finite set $V$ we will view $\mathrm{Map}(V, \mathbb{R}^n) \cong \mathbb{R}^{n \cdot \#V}$ as an $n \cdot \#V$ dimensional manifold. So we have notions like continuity and differentiability when dealing with functions from or to $\mathrm{Map}(V, \mathbb{R}^n)$.

There is a differentiable function called the edge function which will prove useful later on since it encodes enough information to tell whether a framework is infinitesimally rigid or not.

$$\begin{aligned} h: \ \operatorname{Map}(V, \mathbb{R}^n) &\rightarrow \operatorname{Map}(E, \mathbb{R}) \\ f &\mapsto \big(e \mapsto l_f(e)^2\big) \end{aligned} \tag{5}$$

The fact that $h$ is differentiable follows from $|\cdot|^2$ being differentiable.

There are also a few conventions about tangent spaces of manifolds that I will use. These conventions follow the definition of the geometric tangent space as defined in "Vector Analysis" [2]. Let $M$ be a manifold and $p \in M$ be a point. Then $T_pM$ denotes the tangent space at $M$ in $p$. Elements in $T_pM$ are represented by equivalence classes of differentiable curves passing through $p$. So if we write $[y] \in T_pM$ we implicitly say that $y$ is a differentiable curve in $M$ passing through $p$ and that $[y]$ is its equivalence class in $T_pM$. Now let $N$ be a second manifold, and $f : M \rightarrow N$ a map which is differentiable in $p$ then $df_p : T_pM \rightarrow T_{f(p)}N$ denotes the differential of $f$ at the point $p$, and is given by $df_p([y]) = [f \circ y]$. Now let $U \subseteq V$ be two finite sets then we have a restriction map $\rho : \operatorname{Map}(V, \mathbb{R}^n) \rightarrow \operatorname{Map}(U, \mathbb{R}^n)$. This map is differentiable and hence for every $f \in \operatorname{Map}(V, \mathbb{R}^n)$ it also induces a restriction map $d\rho_f T_f \operatorname{Map}(V, \mathbb{R}^n) \rightarrow T_{\rho(f)} \operatorname{Map}(U, \mathbb{R}^n)$ on the tangent sheaves. Let $y \in T_f \operatorname{Map}(V, \mathbb{R}^n)$ then we use $y|_U := d\rho_f(y)$ as a shorthand notation and if $U = \{u\}$ a set of only one point we also use $y(u) := d\rho_f(y)$ as notation. These restriction maps induce a natural isomorphism:

$$\begin{aligned} \Phi: \ T_f \operatorname{Map}(V, \mathbb{R}^n) &\rightarrow \prod_{p \in V} T_f(p)\mathbb{R}^n \\ [y] &\mapsto (y(p))_{p \in V} \end{aligned} \tag{6}$$

So if we write $V = \{v_1, \ldots, v_n\}$ we can write $y \in T_f \operatorname{Map}(V, \mathbb{R}^n)$ as $y = (y_1, \ldots, y_n) = (y(v_1), \ldots, y(v_n))$. In the context of frameworks $y$ should be thought of as an infinitesimal change of the framework as a whole, while the $y_1$ till $y_n$ are the velocity vectors of $v_1$ till $v_n$.

## 1.1 Infinitesimal Rigidity

Being a textbook on rigidity is not the goal of my thesis. Therefore I will only give a definition of infinitesimal rigidity. For a definition of rigidity and the fact that infinitesimal rigidity implies regular rigidity I refer to the

[2] Vector Analysis by Klaus Jänich ISBN: 0-387-98649-9

book "Counting on Frameworks" [3]. This book gives a detailed explanation of frameworks and rigidity and also a lot of examples.

Let $G = (V, E)$ be a graph and $F = (G, f)$ be an $n$-dimensional framework then the edge function $h : \mathrm{Map}(V, \mathbb{R}^n) \to \mathrm{Map}(E, \mathbb{R})$ is differentiable and hence the differential of $h$ is defined for every $f \in \mathrm{Map}(V, \mathbb{R}^n)$. For $[y] \in T_f \mathrm{Map}(V, \mathbb{R}^n)$ we have $dh_f([y]) = [h \circ y] = 0$ if and only if $[h \circ y](e) = 0$ for all edges $e \in E$. Let $t$ be the parameter on which the curve $y$ depends then $[h \circ y](e) = 0$ if and only if $\frac{dl_y(e)^2}{dt}(0) = 0$. The latter equation can be interpreted as that the square of the length of the edge $e$ doesn't change as a result of the infinitesimal change of the framework. This justifies the following definition.

**Definition 1.5.** An infinitesimal movement of a framework $F = (G, f)$ is a tangent vector $y \in T_f \mathrm{Map}(V, \mathbb{R}^n)$ for which $dh_f(y) = 0$ holds. The vector space of all infinitesimal movements is denoted by by $T^{mov}F$, and we will denote $T_f \mathrm{Map}(V, \mathbb{R}^n)$ by $TF$.

To work with this rather abstract definition in a more explicit case one should just write down a matrix of the linear map $dh_f$ with respect to a basis so you can precisely calculate the infinitesimal movements. Another remark that might give more insight is that for an edge $e = \{v_1, v_2\}$ the expression $\frac{dl_y(e)^2}{dt}(0) = 0$ can be rewritten as

$$\frac{dl_y(e)^2}{dt}(0) = \frac{d|y(v_1) - y(v_2)|^2}{dt}(0) = 2\langle \frac{dy(v_1) - y(v_2)}{dt}(0), f(v_1) - f(v_2)\rangle = 0$$

In other words the difference in velocities of the points $v_1$ and $v_2$ must be perpendicular to $f(v_1) - f(v_2)$ or equivalently, the velocities of $v_1$ and $v_2$ have to be the same in the $f(v_1) - f(v_2)$ direction.

The matrix belonging to $dh_f$ is called the rigidity matrix in the literature, since it encodes enough information to see if a framework is infinitesimally rigid or not as we will see later on.

Some of these infinitesimal movements are not really infinitesimal deformations since they can be obtained by infinitesimally moving the framework as a whole. These infinitesimal movements which come from infinitesimally moving the framework as a whole are called the infinitesimal isometries of the framework. For the formal definition of infinitesimal movements we use that $\mathrm{Isom}(\mathbb{R}^n)$ act in a differentiable way on $\mathrm{Map}(V, \mathbb{R}^n)$. This action can be

[3] Counting on Frameworks: Mathematics to Aid the Design of Rigid Structures, by Jack E. Graver. ISBN: 0-88385-331-0

given more explicit as follows:

$$\begin{aligned} \psi : \quad \operatorname{Isom}(\mathbb{R}^n) \times \operatorname{Map}(V, \mathbb{R}^n) &\to \operatorname{Map}(V, \mathbb{R}^n) \\ (g, f) &\mapsto g \circ f \end{aligned} \tag{7}$$

Write $\varphi := \psi(\_, f)$ then $\varphi$ is a differentiable function from $\operatorname{Isom}(\mathbb{R}^n)$ to $\operatorname{Map}(V, \mathbb{R}^n)$ with $\varphi(\mathrm{Id}) = f$ and hence induces a map $d\varphi_{\mathrm{Id}} : T_{\mathrm{Id}} \operatorname{Isom}(\mathbb{R}^n) \to T_f \operatorname{Map}(V, \mathbb{R}^n)$.

**Definition 1.6.** The elements of $\operatorname{im} d\varphi_{\mathrm{Id}}$ are called infinitesimal isometries and we will use $T^{isom}F := \operatorname{im} d\varphi_{\mathrm{Id}}$ as a shorthand notation.

The elements in $T^{isom}F$ can be thought of as being infinitesimal movements coming from infinitesimal isometries of $\mathbb{R}^n$. As isometries move the points of $\mathbb{R}^n$ around, the infinitesimal isometries give velocities to all points in the $\mathbb{R}^n$. So more formally let $v = [y_t] \in T_{\mathrm{Id}} \operatorname{Isom}(\mathbb{R}^n)$ be an equivalence class of curves then for any $p \in \mathbb{R}^n$ we have that $y_t(p)$ is a curve in $\mathbb{R}^n$ through $p$ and hence gives rise to an equivalence class $[y_t(p)] \in T_p\mathbb{R}^n$. We use $v(p) := [y_t(p)]$ as a shorthand notation for this.

It is clear that $T^{isom}F \subset T^{mov}F$ since the action of $\operatorname{Isom} \mathbb{R}^n$ leaves the lengths of the edges unchanged. So as definition of infinitesimal rigidity we take:

**Definition 1.7.** A framework $F$ is infinitesimally rigid if $\dim T^{isom}F = \dim T^{mov}F$.

## 2 Tools for proving infinitesimal rigidity

Checking (infinitesimal) rigidity of a framework directly from the definition is a rather circuitous business. The goal of this section is to provide us with tools which will come in handy to prove the infinitesimal rigidity of frameworks. These will even allow us to prove the infinitesimal rigidity of large sets of frameworks without explicitly finding the space of infinitesimal movements $T^{mov}F$ or equivalently calculating the rank of the rigidity matrix $dh_f$.

For checking the infinitesimal rigidity of a framework you also have to calculate the dimension of $T^{isom}F$. The dimension of $T^{isom}F$ can be computed in a very general way in all cases needed in the constructions given later on.

**Lemma 2.1.** *Let $(G, f)$ be an $n$-dimensional framework such that the points in $\operatorname{im} f$ are an affine span of $\mathbb{R}^n$ (or equivalently $\operatorname{im} f$ contains at least $n+1$ points in general linear position). Then the dimension of $T^{isom}F$ is given by the formula*

$$\dim T^{isom}F = \binom{n+1}{2}.$$

*Proof.* The isometry group of $\mathbb{R}^n$ has dimension $\binom{n+1}{2}$. So to prove the statement it suffices to show that $d\varphi_{\mathrm{Id}}$ is injective in this case. Suppose that $[y] \in T_{\mathrm{Id}} \operatorname{Isom}(\mathbb{R}^n)$ is an infinitesimal isometry such that $d\varphi_{\mathrm{Id}}([y]) = 0$. Let $v_0, \dots v_n$ be the points such that $f(v_0), \dots, f(v_n)$ are an affine span of $\mathbb{R}^n$. Then in particular $d\varphi_{\mathrm{Id}}([y])(v_i) = [\varphi \circ y](v_i) = [y \circ f](v_i) = 0$. Let $t$ be the parameter on which $y$ depends, then $[y \circ f](v_i) = 0$ implies $\frac{dy \circ f}{dt}(v_i) = 0$, now since only $y$ depends on $t$ and not $f$ we have $\frac{dy}{dt}(f(v_i)) = 0$ and hence $[y](f(vi)) = 0$. Let $p$ be any point in $\mathbb{R}^n$, now since $f(v_0), \dots, f(v_n)$ are an affine span $\mathbb{R}^n$ we can find $a_0, \dots, a_n \in \mathbb{R}$ such that $\sum_{i=0}^n a_i = 1$ and $\sum_{i=0}^n a_i f(v_i) = p$. Now write $y = A + b$ with $A$ in the rotational part and $b$ in the translational part of $\operatorname{Isom}(\mathbb{R}^n)$ then $[y](p) = [A + b](\sum_{i=0}^n a_i f(v_i)) = [A](\sum_{i=0}^n a_i f(v_i)) + [b] = \sum_{i=0}^n a_i [A](f(v_i)) + \sum_{i=0}^n a_i [b] = \sum_{i=0}^n a_i [y](f(v_i)) = 0$. Hence $[y](p) = 0$ for all $p \in \mathbb{R}^n$, this can only happen if the infinitesimal isometry $[y]$ itself was already trivial and hence $d\varphi_{\mathrm{Id}}$ is injective indeed. □

This lemma already makes our work a bit easier, for now we only have to determine the dimension of $T^{mov}F$ if we want to determine if a framework is infinitesimally rigid.

## 2.1 Completely determining subsets

Before I continue with the next tool I will define three different sheaves for every framework. It doesn't matter if you don't know what a sheaf is, because the definitions will be understandable without this knowledge. Recall that in the first section I defined a restriction map $d\rho_f : T_f \operatorname{Map}(V, \mathbb{R}^n) \to T_{\rho(f)} \operatorname{Map}(U, \mathbb{R}^n)$ for all pairs of finite sets $U \subset V$.

**Definition 2.2.** Let $F = (G, f)$ be a framework and $V$ be its set of vertices. We give $V$ the discrete topology.

1. The tangent sheaf of $F$ is denoted by $\mathcal{T}_F$. For any $U \subseteq V$ we define $\mathcal{T}_F(U) := T_f \operatorname{Map}(U, \mathbb{R}^n)$. For $W \subset U$ we define $\rho_{U,W} : \mathcal{T}_F(U) \to \mathcal{T}_F(U)$ to be the restriction map given earlier.

2. The sheaf of movements of $F$ is denoted by $\mathcal{F}_F$. For any $U \subseteq V$ the sheaf we notate $\mathcal{F}_F(U) := \rho_{V,U}(T^{mov} F)$ and $\rho_{U,W} : \mathcal{F}_F(U) \to \mathcal{F}_F(U)$ for the restriction of the map in 1.

3. The sheaf of movements of $F$ is denoted by $\mathcal{I}_F$. For any $U \subseteq V$ the sheaf we notate $\mathcal{I}_F(U) := \rho_{V,U}(T^{isom} F)$ and $\rho_{U,W} : \mathcal{F}_F(U) \to \mathcal{F}_F(U)$ for the restriction of the map in 1.

Completely determining subsets are subsets of the vertex set of a framework with the property that if you prescribe the velocity vectors at these points, then there is at most 1 way to assign velocity vectors to the other points in a way that all these vectors together form an infinitesimal movement. So more formally:

**Definition 2.3.** Let $F = (G, f)$ be a framework and $V$ its vertex set. A set $U \subseteq V$ is said to determine the infinitesimal movement of another set $W \subseteq V$ if and only if $\rho_{W \cup U, U} : \mathcal{F}_F(W \cup U) \to \mathcal{F}_F(U)$ is an isomorphism. $U$ is said to be a determining subset of $F$ if $U$ determines $V$.

This determinacy also satisfies some nice properties, as will be stated and proved below.

**Proposition 2.4.** *Let $F$ be a framework and $V$ its vertex set and let $V_1, V_2, V_3 \subseteq V$ be any three subsets. Then it holds that*

1. *$V_1$ determines $V_1$ (reflexivity).*

2. *if $V_1$ determines $V_2$ then $V_1 \cup V_3$ determines $V_2 \cup V_3$.*

3. *if $V_1$ determines $V_2$ and $V_3 \subset V_2$ then $V_1$ determines $V_3$.*

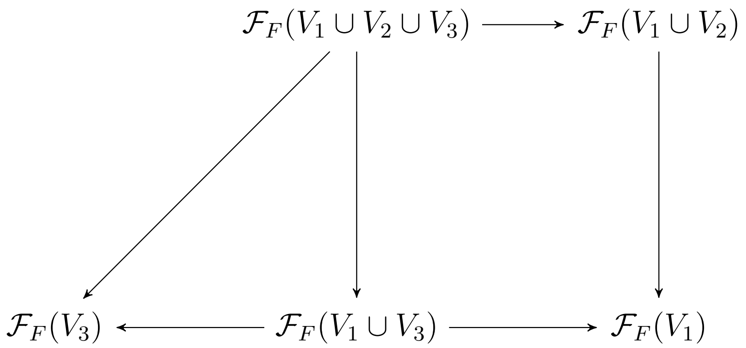
*4. if $V_1$ determines $V_2$ and $V_2$ determines $V_3$ then $V_1$ determines $V_3$ (transitivity).*

*5. if $V_1$ determines $V_2$ and $V_3$ then it determines $V_2 \cup V_3$.*

*Proof.*

1. This follows directly from $\rho_{V_1,V_1} : \mathcal{F}_F(V_1) \to \mathcal{F}_F(V_1)$ being an isomorphism.

2. Consider the following commutative diagram.




   What needs to be shown is that the restriction map from $\mathcal{F}_F(V_1 \cup V_2 \cup V_3)$ to $\mathcal{F}_F(V_1 \cup V_3)$ is an isomorphism. By construction it's already a surjective homomorphism, so we only need to show injectivity. Now take any two elements $a, b \in \mathcal{F}_F(V_1 \cup V_2 \cup V_3)$ such that $a|_{V_1 \cup V_3} = b|_{V_1 \cup V_3}$, then $a|_{V_1} = b|_{V_1}$ and therefore also $a|_{V_1 \cup V_2} = b|_{V_1 \cup V_2}$ since $V_1$ determines $V_2$. We clearly also have $a|_{V_3} = b|_{V_3}$. But since $V_3$ and $V_1 \cup V_2$ form an open cover of $V_1 \cup V_2 \cup V_3$ we have that $a = b$ which shows the injectivity.

3. What needs to be shown is that the restriction map from $\rho_{V_1 \cup V_3, V_1}$ is an isomorphism. By construction it's already a surjective homomorphism, so we only need to show injectivity. Now take any two elements $a, b \in \mathcal{F}_F(V_1 \cup V_3)$ such that $a|_{V_1} = b|_{V_1}$, since the restriction map from $\mathcal{F}_F(V_1 \cup V_2)$ to $\mathcal{F}_F(V_1 \cup V_3)$ is also surjective, we have that there are $a', b' \in \mathcal{F}_F(V_1 \cup V_2)$ such that $a'|_{V_1 \cup V_3} = a$ and $b'|_{V_1 \cup V_3} = b$ but then we have $a'|_{V_1} = b'|_{V_1}$ and since $V_1$ determines $V_2$ also $a' = b'$ but then $a = a'|_{V_1 \cup V_3} = b'|_{V_1 \cup V_3} = b$, showing the injectivity.

4. $V_2$ determines $V_3$ so 2 to proves that $V_1 \cup V_2$ determines $V_2 \cup V_3$. But now we see that $\rho_{V_1 \cup V_2 \cup V_3, V_1} = \rho_{V_1 \cup V_2, V_1} \circ \rho_{V_1 \cup V_2 \cup V_3, V_1 \cup V_2}$ is an isomorphism since it's a composition of two isomorphisms. Thus $V_1$ determines $V_2 \cup V_3$ and therefore also $V_3$

5. Define $V_2' = V_2 - V_3$. According to 3 $V_2'$ is also completely determined by $V_1$. So if we take $v_1 \in \mathcal{F}_F(V_1)$ then there are unique $v_2' \in \mathcal{F}_F(V_1 \cup V_2')$ and $v_3 \in \mathcal{F}_F(V_1 \cup V_3)$ such that $v_2'|_{V_1} = v_1 = v_3|_{V_1}$. Now since $V_1 \cup V_2'$ and $V_1 \cup V_3$ cover $V_1 \cup V_2 \cup V_3$ and $(V_1 \cup V_2') \cap (V_1 \cup V_3) = V_1$ we know that there is a $v \in \mathcal{F}_F(V_1 \cup V_2 \cup V_3)$ such that $v|_{V_1} = v_1$ and that this $v$ is unique. Which shows that the restriction map from $\mathcal{F}_F(V_1 \cup V_2 \cup V_3)$ to $\mathcal{F}_F(V_1)$ is an isomorphism.

□

**Corollary 2.5.** *Part 3 and 5 of the above proposition show that it suffices to check determinacy only for points, since together they immediately imply that $V_1$ determines $V_2$ if and only if $V_1$ determines $\{v_2\}$ for all $v_2 \in V_2$.*

There is another thing to say about complete determinacy. It is preserved under morphisms.

**Proposition 2.6.** *Let $F, G$ be two frameworks, the vertex sets of $F$ and $G$ are $V$ and $W$ respectively. Let $(k, l) : F \to G$ be morphism of frameworks. If $V_1, V_2 \subset V$ are sets such that $V_1$ determines $V_2$ then $k(V_1)$ determines $k(V_2)$.*

*Proof.* Before we really start with the proof, we first need some preparation. The morphism $(k, l)$ induces the following map:

$$\varphi_{(k,l)} : \mathrm{Map}(W, \mathbb{R}^n) \quad \to \quad \mathrm{Map}(V, \mathbb{R}^n) \tag{8}$$
$$f \mapsto l^{-1} \circ f \circ k \tag{9}$$

This map between manifolds also induces a map between their tangent spaces $d\varphi_{(k,l)} : TG \to TF$. This map can of course be extended to a morphism of their tangent sheaves. The map $d\varphi_{(k,l)}$ can even be seen as map between their sheaves of movement because for every edge $\{v_1, v_2\}$ of $F$ either $\{f(v_1), f(v_2)\}$ is also an edge of $G$ or $f(v_1) = f(v_2)$. Which shows that the condition $x \in TG$ is an infinitesimal movement of $G$ is a stronger condition then the condition that $d\varphi_{(k,l)}(x)$ is an infinitesimal movement of $F$. There is also another thing which can be said about $d\varphi(k, l)$ namely that it's injective when viewing it as a map from $\mathcal{T}_G(k(V))$ to $\mathcal{T}_F(V)$. This is a consequence of $k : V \to k(V)$ being surjective. Therefore $\varphi_{(k,l)} : \mathrm{Map}(k(V), \mathbb{R}^n) \to \mathrm{Map}(V, \mathbb{R}^n)$ and also $d\varphi_{(k,l)} : \mathcal{T}_G(k(V)) \to \mathcal{T}_F(V)$ are injective.

What we want to show is that the restriction map from $\mathcal{F}_G(k(V_1) \cup k(V_2))$ to $\mathcal{F}_G(k(V_1))$ is an isomorphism. By construction it's already a surjective homomorphism, so we only need to show injectivity. But this is an easy consequence of the following diagram being commutative:

$$\begin{array}{ccc}
\mathcal{F}_G(k(V_1 \cup V_2)) & \xrightarrow{\ \rho\ } & \mathcal{F}_G(k(V_1)) \\
\Big\downarrow {\scriptstyle d\varphi_{(k,l)}} & & \Big\downarrow {\scriptstyle d\varphi_{(k,l)}} \\
\mathcal{F}_F(V_1 \cup V_2) & \xrightarrow{\ \rho'\ } & \mathcal{F}_F(V_1)
\end{array}$$

where $\rho$ and $\rho'$are the restriction maps. We know that $\rho'$ is an isomorphism and that both the $d\varphi(k,l)$ are injective so $\rho$ also has to be injective. □

The following proposition shows a way to make planar frameworks which are completely determined by two points.

**Proposition 2.7.** *Let $F_1 = (G_1, f_1)$, $F_2 = (G_2, f_2)$ be two infinitesimally rigid frameworks of dimension 2 and let $G_3$ be the graph consisting only consisting of the point $v_3$. Suppose there exist two points $v_1, v_1'$ of $F_1$ and two point $v_2, v_2'$ of $F_2$ such that $f_1(v_1') - f_1(v_1)$ and $f_2(v_2') - f_2(v_2)$ are linear independent and $f_1(v_1) = f_2(v_2)$. Let $F_3 = (G_3, f_3)$ be the framework such that $f_3(v_3) = f_1(v_1) = f_2(v_2)$. Then the maps $(g_1, \mathrm{Id}) : F_3 \to F_1$ and $(g_2, \mathrm{Id}) : F_3 \to F_2$ given by $g_1(v_3) = v_1$ and $g_2(v_3) = v_2$ are framework morphisms and $F = (G, f) := F_1 \amalg_{F_3} F_2$ is completely determined by $v_1'$ and $v_2'$.*

Please recall the remark which allowed us to view $G_1$ and $G_2$ are subgraphs of $G_3$. This is needed to make sense of the statement $v_1'$ and $v_2'$ completely determine $F$, and will also be needed to make sense of the proof below.

*Proof.* Since $F_1$ and $F_2$ are infinitesimally rigid and $f_1(v_1') - f_1(v_1)$ and $f_2(v_2') - f_2(v_2)$ are both nonzero, so we see that $v_1$ and $v_1'$ determine $F_1$ and $v_2$ and $v_2'$ determine $F_2$ (since the infinitesimal isometries of the plane are entirely fixed if you know them at two distinct points). Thus $v_1, v_1'$ and $v_2'$ completely determine as well $F_1$ as $F_2$ and thus entire $F$. What remains to show is that the velocity vector at $v_1$ is completely determined by $v_1'$ and $v_2'$. Since $F_1$ is infinitesimally rigid, we know what the velocity vector at $v_1$ should be in the $f_1(v_1') - f_1(v_1)$ direction, given that we know the vector at $v_1'$. Since $v_1$ and $v_2$ are the same point in $F$, we can make the same remark about the $f_1(v_2') - f_1(v_2)$ direction. So since $f_1(v_1') - f_1(v_1)$ and $f_1(v_2') - f_1(v_2)$ are linearly independent, there is a unique possibility for the velocity vector at $v_1$, which proves the proposition. □

# 3 The Construction of Positive Algebraic Numbers

Until this point in the thesis it hasn't been mentioned what is meant by constructing a number. But now it's time to give the definition.

**Definition 3.1.** A real number $r$ is called (infinitesimally) n-constructible if there exists an (infinitesimally) rigid $n$-dimensional framework $(G, f)$ such that $\text{Im}\ lf = \{1\}$ and $r \in \text{Im}\ df$. When the framework $(G, f)$ is not only rigid but infinitesimally rigid, we call $r$ infinitesimally rigid constructible.

H. Maehara showed what the 2-constructible numbers are, and what I will do in my thesis is showing what the infinitesimally 2-constructible numbers are.

**Example 1.** The question of this thesis is easy to answer in the 1-dimensional case. The graph $G$ of a framework $(G, f)$ has to be connected for the framework $(G, f)$ to be (infinitesimally) rigid. For if it's not connected you could move the connected components around independently of each other. Since the distance between two connected points has to be 1 and all the points of a 1-dimensional framework $(G, f)$ have to lie on the same line, all constructible distances have to be multiples of 1. So all the 1-constructible numbers are a subset of $\mathbb{N}$. The picture below shows that all numbers in $\mathbb{N}$ are infinitesimally rigid 1-constructible. The proof that the framework in this picture is infinitesimally rigid is left as an exercise to the reader.


This example gives us the answer to the 1-dimensional case. The more general $n$-dimensional case is a bit harder, but it's answer is still relatively easy to formulate.

**Theorem 3.2.** *The infinitesimally rigid $n$-constructible numbers are precisely the positive algebraic numbers (the numbers $x \in \mathbb{R}_{\geq 0}$ such that $p(x) = 0$ for some polynomial $p \in \mathbb{Q}[x]$) when $n > 1$.*

I will prove this theorem for $n = 2$. Proving it in higher dimensions as well would make this thesis to long. If you are interested you can try to contact me for a sketch of the proof. The proof is split in several smaller parts, one part is the proof that all constructible numbers are algebraic. The other part will concern the actual infinitesimally rigid construction of the positive algebraic numbers. Since every infinitesimally rigid framework has to be rigid, the theorem 3.2 is then proven.

**Theorem 3.3.** *All* $\{1\}, n$ *constructible numbers are algebraic.*

The proof of this theorem can be found in "Distances in a rigid unit-distance graph in the plane" by H. Maehara[4]. Instead of unit distance frameworks he talks about unit distance graphs and there is only a small difference in definition, but the proof can easily be translated to fit our definition.

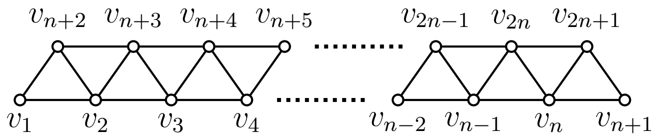
[4]Discrete Applied Mathematics Volume 31, Issue 2, 15 April 1991, Pages 193-200

## 3.1 Constructions in the plane

An arbitrary positive algebraic number can be constructed in an infinitesimally rigid way. The construction that does this is made using several other constructions. We continue with the description of these sub-constructions.

### 3.1.1 Construction of the Natural numbers

Before we are going to construct the positive algebraic numbers we need first to the natural numbers. The natural number $n \in \mathbb{N}$ can be constructed in the plane with the bridge-like construction shown in the picture below.




The graph $G_n$ of the 2-dimensional framework $(G_n, f_n)$ consists of the $2n+1$ points $V_n = \{v_1, \ldots, v_{2n+1}\}$ and the $4n-1$ edges $(v_i, v_{i+1}), (v_{i+n}, v_{i+n+1})$, $(v_i, v_{i+n+1}), (v_{i+n+1}, v_{i+1})$ for $i \in 2, \ldots n$, $(v_1, v_2)$, $(v_1, v_{n+1})$ and $(v_{n+1}, v_2)$. The realization $f_n$ is given by

$$f_n(v_i) = \begin{cases} (i-1, 0) & \text{if} \quad i \leq n+1 \\ (i - n - \frac{3}{2}, \sqrt{\frac{3}{4}}) & \text{else} \end{cases} . \tag{10}$$

**Proposition 3.4.** *$(G_n, f_n)$ is infinitesimally rigid for all $n \in \mathbb{N}$*

*Proof.* The first thing to be shown is that $(G_n, f_n)$ is completely determined by $v_1$ and $v_{n+2}$. This can be done by induction. The induction step is to show that if for $0 \leq i \leq n-2$ the points $v_{2+i}$ and $v_{n+2+i}$ are completely determined by $v_1$ and $v_{n+2}$, then also $v_{2+i+1}$ and $v_{n+2+i+1}$ are completely determined by $v_1$ and $v_{n+2}$. The initial step is to show that $v_{2+0}$ and $v_{n+2+0}$ are completely determined by $v_1$ and $v_{n+2}$. But this is clear since the proof for $v_{n+2+0}$ is determined by itself and for $v_2$ it's proposition 2.7. Now for the induction step. This is straightforward, since 2.7 tells that $v_{n+2+i+1}$ is completely determined by $v_{2+i}$ and $v_{n+2+i}$ and that $v_{n+2+i+1}$ is in it's turn completely determined by $v_{n+2+i+1}$ and $v_{2+i}$. So $v_1$ and $v_{n+2}$ indeed determine $(G_n, f_n)$.

We also have that

$$\dim \mathcal{F}_{(G_n,f_n)} \{v_1, v_{n+2}\} = \dim \mathcal{F}_{(G_n,f_n)}(V_n) \geq \dim \mathcal{I}_{(G_n,f_n)}(V_n) = 3.$$

What also holds is $\dim \mathcal{F}_{(G_n,f_n)}\{v_1, v_{n+2}\} \leq 3$. This is because $\dim \mathcal{T}_{(G_n,f_n)}\{v_1, v_{n+2}\} = 4$ and there is an edge between $v_1$ and $v_{n+2}$ hence

$$\dim \mathcal{F}_{(G_n,f_n)}\{v_1, v_{n+2}\} \leq 4 - 1$$

is at least one dimension smaller. Putting this all together we get $\dim \mathcal{F}_{(G_n,f_n)}(V_n) = \dim \mathcal{I}_{(G_n,f_n)}(V_n)$ so $(G_n, f_n)$ is infinitesimally rigid. □

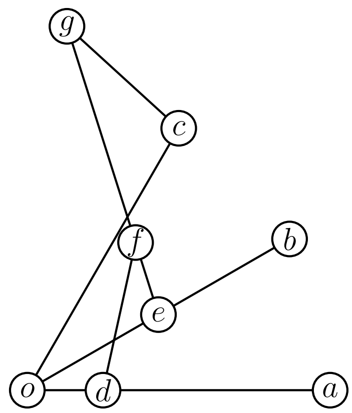
We will now introduce some notation to make it easier to use the frameworks $(G_n, f_n)$ in other constructions. Let $a, b \in \mathbb{R}^2$ such that $|a-b| \in \mathbb{N}$ then we denote by $F(a, b)$ the unique framework which is isometric to the framework $(G_{|a-b|}, f_{|a-b|})$ and has the properties that $f(v_1) = a$, $f(v_{|a-b|+1}) = b$ and such that orientation of $\mathbb{R}^2$ is preserved, by the isometry in the framework morphism.

### 3.1.2 The Reverser

The reverser is not an infinitesimally rigid construction like $F(a,b)$ but it is verry close to being infinitesimally rigid, namely its space of infinitesimal movements is only one dimension larger than its space of infinitesimal isometries. Before defining the framework we introduce yet some more notation.

**Definition 3.5.** Let $a$ and $b$ be two points in $\mathbb{R}^2$ and $x$ and $y$ be two points in $\mathbb{R}_{>0}$ such that $|x-y| < |a-b| < x+y$. We then define $a\angle_x^y b$ to be the intersection point of the circle with center $a$ and radius $x$ and the circle with center $b$ and radius $y$ which lies on the left side of the line from $a$ to $b$.

**Definition 3.6.** Let $a$ and $b$ be two points in $\mathbb{R}^2$ such that $0 < |a-b| < 8$ then we can define the following points: $o := a\angle_4^4 b$, $d := \frac{1}{4}o + \frac{3}{4}a$, $e := \frac{1}{2}o + \frac{1}{2}b$, $f := d\angle_2^1 e$, $g = 4f - 3e$ and $c := g\angle_2^4 o$. We use these points to define the framework $R(a,b)$. The framework $R(a,b)$ is obtained by gluing the frameworks $F(o,a)$, $F(o,b)$, $F(o,c)$, $F(d,f)$, $F(e,g)$, $F(c,g)$ together over the points which map to the same point by construction, thus not over points which might accidentally also map to the same point.




$R(a,b)$

**Proposition 3.7.** *The space of infinitesimal movements of $R(a,b)$ is completely determined by the two vertices mapping to $a$ and $b$ and $\angle aob = \angle boc$.*

*Proof.* It is clear that all the infinitesimal movements of $R(a,b)$ are completely determined by $o, a, b, c, d, e, f$ and $g$. What remains is to show that the infinitesimal movements at $o, c, d, e, f$ and $g$ are determined by $a$ and $b$.

According to 2.7 the infinitesimal movements at $o$ are determined by $a$ and $b$.

According to 2.7 $d$ and $e$ are determined by $o, a$ and $o, b$ respectively.
$f$ on it's turn is determined by $d$ and $e$.
$g$ is determined by $e$ and $f$ and finally
$c$ is determined by $o$ and $g$.

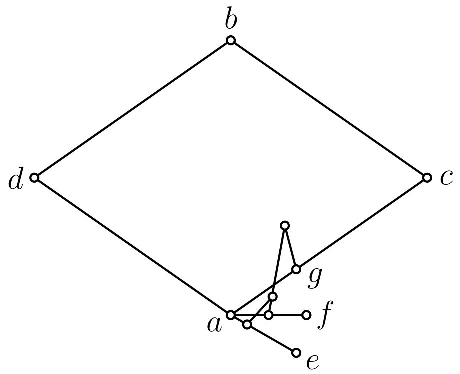
The equality of the two angles is easily proven as follows:
$\triangle fdo \cong \triangle oef$ (side-side-side)
$\triangle geo \cong \triangle ocg$ (side-side-side)
$\triangle oef \sim \triangle geo$ (side-angle-side)
which leads us to:
$\angle aob = \angle doe = \angle dof - \angle eof = \angle eog - \angle cog = \angle eoc = \angle boc$ □

This proposition shows that the space of infinitesimal movements of this object is at most of dimension 4. But since the movement given by $\psi(t) = R(t \cdot a + (1-t)b, b)$ induces a non trivial infinitesimal movement. We see that the space of infinitesimal movements has to be precisely 4.

In the case $|a-c| < 8$, the non trivial movement described previously also induces a non trivial movement on the subframework consisting of points $a$ and $c$. Meaning that, in this case, the dimension of infinitesimal movements of $R(a, b)$ and $a, c$ are the same and the map between the spaces is surjective, so it has to be injective. As a result $a$ and $c$ are also completely determine the framework.

### 3.1.3 The Orthogonalizer

We give yet another useful building block, which as the name suggests contains three points which will be always orthogonal to each other.




**Definition 3.8.** Let $a, b \in \mathbb{R}^2$ and let $n \in \mathbb{N}$ such that $n > |a-b|/2 > 0$. Define the points $c = b\angle_n^n a$, $d = a\angle_n^n b$, $e = \frac{n+4}{n}a - \frac{4}{n}d$, $f = a + 4\frac{c-d}{|c-d|}$ and $g = \frac{n-4}{n}a + \frac{4}{n}c$. The framework $O(a, b, n)$ is obtained by gluing the

frameworks $F(a,c), F(c,b), F(b,d), F(d,e)$ and $R(e,f)$. These frameworks are then glued together over the points $a$ till $g$.

**Proposition 3.9.** *The infinitesimal movements of the framework $O(a,b,n)$ are determined by $a$ and $b$. Also the vectors $b-a$ and $f-a$ are orthogonal.*

*Proof.* The orthogonality follows straight from $f-a=\frac{b-d}{|b-d|}$ and the fact that $b$ and $d$ by construction both lie on the perpendicular line bisector of $a$ and $b$.

It's clear that $a$ till $g$ together completely determine the infinitesimal movements of the entire framework. So it remains to show that the movements at $c$ till $g$ are also completely determined by $a$ and $b$. For $c$ and $d$ it's clear that they are determined by $a$ and $b$. In it's turn $g$ is determined by $a$ and $c$ and $e$ by $a$ and $d$. According to the remark about the reverser, $f$ is determined by $e$ and $g$ since $|g-e|=|\frac{4}{n}(c-a)+\frac{4}{n}(d-a)|=\frac{4}{n}|b-a|<8$. $\square$

The movement $h(t)=O(a,at+b(1-t),n)$ induces a infinitesimal movement $\frac{dh}{dt}$ which is not an infinitesimal isometry. Therefore the infinitesimal movements of $O(a,b,n)$ have to be at least 4-dimensional. As a result of the proposition it's also at most 4-dimensional. So the infinitesimal movements are spanned by this non trivial movement, and the infinitesimal isometries.

### 3.1.4 Construction of the Positive Algebraic numbers

In this section an explicit method for constructing all algebraic distances between 0 and 2 will be given. This construction will only depend on the minimal polynomial of the given algebraic distance. For a given algebraic distance $x\in(0,2)$ there exists a $\theta$ such that $x=2\cos(\theta)$. The main idea is to make an infinitesimally rigid construction which contains this angle $\theta$. For this the following lemma will be useful.

**Lemma 3.10.** *For every polynomial $P\in\mathbb{N}[X]$ of degree $n$ there exist integers $a_0,\dots,a_n$ such that*

$$P(2\cos(\theta))=\sum_{k=0}^{n}a_k\cos(k\theta)$$

*Proof.* This is an easy consequence of the power reduction formulas for the cosine. These two formulas state that:

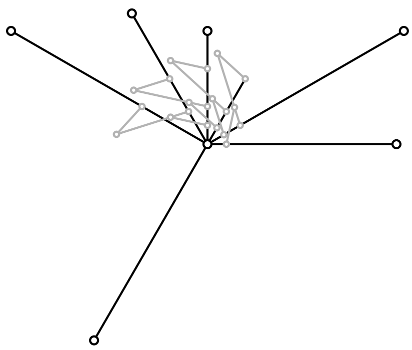
$$\cos^n(\theta)=\frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos((n-2k)\theta)$$

when $n$ is odd and:

$$\cos^n(\theta) = \frac{1}{2^n}\binom{n}{\frac{n}{2}} + \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n}{2}-1}\binom{n}{k}\cos((n-2k)\theta)$$

when $n$ is even. Applying these formulas to every term of the polynomial gives the desired result. □

The first step of the construction is a fan as in the picture below.



$Fan(\frac{\pi}{6}, 5, 6, -6, 3, 4, 6)$

Define $p(\theta) = (\cos(\theta), \sin(\theta)) \in \mathbb{R}^2$. Thus $r \cdot p(\theta)$ is a way to denote the point with radius $r$ and angle $\theta$ polar coordinates.

**Definition 3.11.** The framework $Fan(\theta, a_0, \ldots, a_n)$ consists of the following three sets of frameworks.

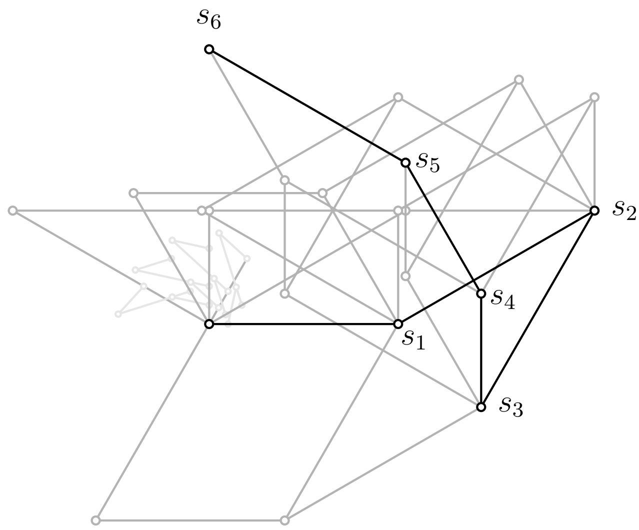
1. For all positive $a_k$ the framework $F_k(0, a_k p(k\theta))$
2. For all negative $a_k$ the framework $F_k(4p(k\theta), a_k p(k\theta))$
3. For all $0 \leq k \leq n-2$ the reverser $R_k(4p(k\theta), 4p((k+1)\theta))$

These frameworks are then glued together over the points which map to the same point in $\mathbb{R}^2$ by construction. So not the points which might accidentally happen to map to the same point.

**Proposition 3.12.** *The space of infinitesimal movements of $Fan(\theta, a_0, \ldots, a_n)$ is completely determined by the two vertices mapping to $4p(0)$ and $4p(\theta)$.*

*Proof.* As soon as the spaces at $4p(k\theta)$ and $4p((k+1)\theta)$ are determined by $4p(0)$ and $4p(\theta)$, the space at $4p((k+2)\theta)$ is also determined by $4p(0)$ and $4p(\theta)$ because this point is part of reverser $R_k$. So by induction, the infinitesimal movements of all the point of each $R_k$ are completely determined by $4p(0)$ and $4p(\theta)$. For $0 \leq k \leq n-2$ the defined $F_k$ have at least two points in common with $R_k$, these two points are 0 and $p(k\theta)$, therefore the space of infinitesimal movements of these $F_k$ are also determined by $4p(0)$ and $4p(k\theta)$. The frameworks $F_{n-1}$ and $F_n$ have, when defined, two points in common with $R_{n-2}$ so the space of infinitesimal movements of these two frameworks are also completely determined by $4p(0)$ and $4p(k\theta)$. □

The next step is to make a construction that "adds" all the vectors $a_k p(k\theta)$ in such a way that the infinitesimal movements of this new construction are still completely determined by $4p(0)$ and $4p(\theta)$. An example of such a construction is shown in the picture below. For more a step by step construction of this picture below see Appendix A.




$Add(\frac{\pi}{6}, 5, 6, -6, 3, 4, 6)$

Given the integers $a_0, \ldots, a_n$ and an angle $\theta$ we define $s_{-1} := 0$ and for $0 \leq k \leq n$ and $s_{-1} = 0$ we define $s_k := s_k(\theta, a_0, \ldots, a_n) := \sum_{i=0}^{k} a_i p(i\theta)$.

**Definition 3.13.** The framework $Add(\theta, a_0, \ldots, a_n)$ consists of the following frameworks.

1. The framework $Fan(\theta, a_0, \ldots, a_n)$ and the following set of frameworks.

2. For every $0 \leq j < k \leq n$ such that $a_j \neq 0$ and$a_k \neq 0$ the two frameworks $F(s_j, s_j + a_k p(k\theta))$ and $F(s_{j-1} + a_k p(k\theta), s_j + a_k p(k\theta))$.

Again, these things are glued together over the points which by construction map to the same point in $\mathbb{R}^2$, thus not gluing over points which might happen to map to the same point by an unlucky choice of $\theta$.

**Proposition 3.14.** *When $\theta$ is an irrational multiple of $\pi$, the infinitesimal movements of the framework $Add(\theta, a_0, \ldots, a_n)$ are completely determined by $4p(0)$ and $4p(\theta)$.*

*Proof.* This is a proof by induction. First note that we already proved that all the points in $Fan(\theta, a_0, \ldots, a_n)$ are already completely determined by $4p(0)$ and $4p(\theta)$. In the case $j = 0$ and $0 < k \leq n$ all the points $s_0 + a_k p(k\theta)$ are completely determined by $4p(0)$ and $4p(\theta)$. This is because these points are either points in $Fan(\theta, a_0, \ldots, a_n)$ (case $a_k = 0$ or $a_0 = 0$) or completely determined by points $Fan(\theta, a_0, \ldots, a_n)$ (the case $a_j \neq 0$ and$a_k \neq 0$). Noting that $k\theta$ cannot be an angle of 0 or $\pi$ degrees and then applying proposition 2.7 to the frameworks $F(s_0, s_0 + a_k p(k\theta))$ and $F(a_k p(k\theta), s_0 + a_k p(k\theta))$ proves the correctness of the claim in the case $a_j \neq 0$ and$a_k \neq 0$.

Now assume that for a $j = i$ and all $i < k \leq n$ the points $s_j + a_k p(k\theta)$ are completely determined. Then for $j = i + 1$ and $i + 1 < k \leq n$, we have that the point $s_j + a_k p(k\theta)$ either coincides with a point of the form $s_i + a_k p(k\theta)$ (case $a_k = 0$ or $a_j = 0$) or these points are connected to points of the form $s_{i+1} + a_k p(k\theta)$ (case $a_k \neq 0$ and $a_j \neq 0$). In the first case the induction step is trivial. In the second case we apply proposition 2.7 again and find that the point in question is completely determined by the points of the form $s_i + a_k p(k\theta)$. □

At this point the construction is almost finished. All what needs to be done is to attach an orthogonalizer to make it infinitesimally rigid.

**Definition 3.15.** Let $x \in (0, 2)$ be algebraic and let $P \in \mathbb{N}[X]$ be a polynomial of minimal degree such that $P(x) = 0$, take $n = \deg P$, let $a_i \in \mathbb{N}$ be as in lemma 5.9 and $\theta \in \mathbb{R}$ such that $x = 2\cos(\theta)$. Then we define the constructing framework of $x$ to be $C(a_0, \ldots, a_n, \theta)$. Where $C(a_0, \ldots, a_n, \theta)$ is the union of $Add(a_0, \ldots, a_n, \theta)$ and $O(0, s_n)$. [5] These two frameworks are glued together over $0, 4p(0)$ and $s_k$.

[5] Remember that $s_k = \sum_{i=1}^{n} a_i p(i\theta)$ with $p(x) = (\cos(x), \sin(x))$

**Proposition 3.16.** *If $C(a_0, \ldots, a_n, \theta)$ is the constructing framework of $x$ such that $\theta$ is an irrational multiple of $\pi$ then it is infinitesimally rigid.*

*Proof.* The dimension of the space of movements $C(a_0, \ldots, a_n, \theta)$ is at most 4 since it's completely determined by the two points which also completely determine $Add(\theta, a_0, \ldots, a_n)$. So for showing the infinitesimal rigidity of $C(a_0, \ldots, a_n, \theta)$ it suffices to show that there is just one movement of the points $4p(0)$ and $4p(\theta)$ which doesn't come from the entire framework. The movement $h(t) = Add(\theta(1-t), a_0, \ldots, a_n)$ induces an infinitesimal movement $g$ of $Add(a_0, \ldots, a_n, \theta(1-\theta))$ and also an infinitesimal movement $g'$ of the two points $4p(0)$ and $4p(\theta)$. Since $4p(0)$ and $4p(\theta)$ determine the infinitesimal movements of $Add(a_0, \ldots, a_n, \theta(1-\theta))$, all the infinitesimal movements off the entire framework $C$ compatible with $g'$ have to restrict to the infinitesimal movement $g$ on $Add$. Now assume there exists such an infinitesimal movement of $C$.

This would also mean that there exists an infinitesimal movement $h$ of $O(0, s_n)$ such that the velocity vector at 0 and $4p(0)$ is zero and at $s_n$ is given by:

$$(\sum_{i=1}^{n} \frac{da_i \cos(i\theta)}{d\theta}, \sum_{i=1}^{n} \frac{da_i \sin(i\theta)}{d\theta})$$

The first coördinate is nonzero which can be proven by contradiction. Assume it's zero then

$$0 = \sum_{i=1}^{n} \frac{da_i \cos(i\theta)}{d\theta} = 2\sin(\theta)P'(2\cos(\theta)) = P'(2\cos(\theta)) = P'(x)$$

The second last step holds since $2\sin(\theta) \neq 0$ implies that $0 = P'(2\cos(\theta))$. Which is in contradiction with the minimality of the degree of $P$. As a result the first coördinate is nonzero indeed . The space of infinitesimal movements of $O(0, s_n)$ is spanned by the infinitesimal isometries and the movement induced by $k(t) = O(0, (1-t)s_n)$ which we will call $k'$. But since $s_n = (\sum_{i=0}^{n} a_i \cos(i\theta), s_{n,2}) = (0, s_{n,2})$ the velocity vector at $s_n$ of $k'$ also has zero as first coordinate. $k(t)$ also leaves the points 0 and $4p(0)$ fixed so the velocity vectors at these points are also zero in the infinitesimal movement $k'$. The infinitesimal movements of $O(0, s_n)$ are spanned by $k'$ and the infinitesimal isometries so $h - \lambda k'$ has to be an infinitesimal isometry for some $\lambda$. The velocity vectors at 0 and $4p(0)$ are zero in both $k'$ and $h$ so the infinitesimal isometry $h - \lambda k'$ has to be the identity. Contradicting the fact that the first coördinate of this movement at $s_n$ is nonzero. As result there are no infinitesimal movement of $C$ such that it is $g'$ at the points 0 and $4p(0)$. Therefore it is infinitesimally rigid. □

**Proposition 3.17.** *If $\theta = \frac{\pi}{k}$ with $k \in \mathbb{N}$ then $Fan(\theta, 1, \ldots, 1)$ with $k$ ones and the two points at $4p(0)$ identified is infinitesimally rigid.*

*Proof.* The infinitesimal movements of the framework are completely determined by $4p(0)$ and $4p(\theta)$ so its dimension can at most be 4. But the infinitesimal movement which keeps $4p(0)$ on its place and rotates $4p(\theta)$ infinitesimally around the origin does not extend to a movement of the entire framework. Since this would mean that one of the two identified points has to have a velocity vector of 0 and the other has to have a velocity vector which is $2k$ times as long as the vector at $4p(\theta)$. □

## 3.2 Proof of the main theorem

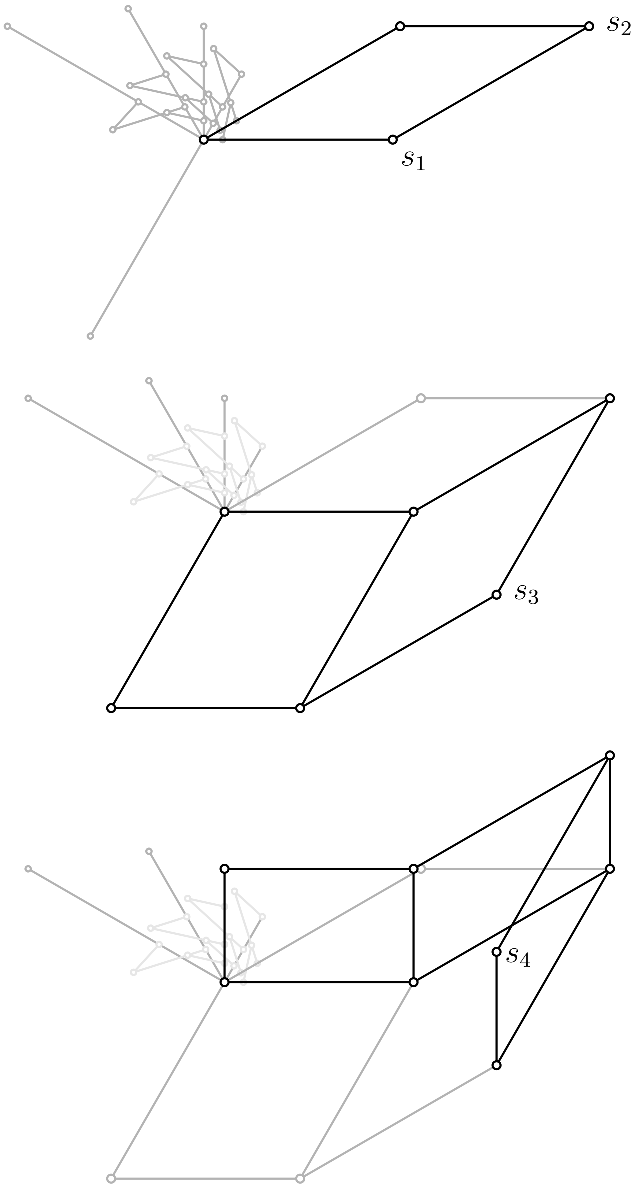
With all the propositions in this thesis and the use of theorem 3.3 the main theorem 3.2 is now easily proven. Theorem 3.3 shows that the constructible numbers need to be algebraic, and since infinitesimal rigidity is a stronger condition then rigidity, infinitesimally rigid constructible numbers also need to be algebraic. For the other way around we will use propositions 3.16 and 3.17. Now let $x \in \mathbb{R} > 0$ be an algebraic distance and $k \in \mathbb{N}$ such that $x < 2k$, and $\theta$ such that $x = k2\cos(\theta)$. Then 3.16 gives us that we have an infinitesimal rigid construction containing $4p(0)$ and $4p(\theta)$ in the case that $\theta$ is an irrational multiple of $\pi$, and 3.17 gives us this in the other case (if $\phi = \frac{a}{b}\pi$ then both points are in $Fan(\frac{1}{b}\pi, 1, \ldots, 1)$). Now to this framework we can glue $R(4p(0), 4p(\theta))$, $F(0, kp(0))$ and $F(4p(2\theta), -kp(2\theta))$ still leaving it infinitesimally rigid. The distance $x$ can now be found between the points $kp(0)$ and $-kp(2\theta)$ since

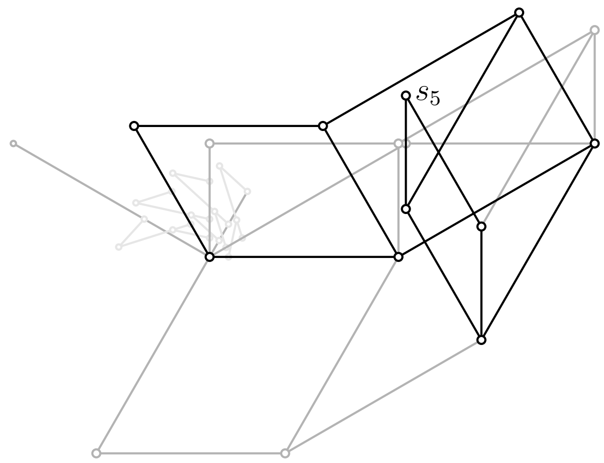
$$|kp(0) + kp(2\theta)| = k|p(-\theta) + p(\theta)| = k(\cos(-\theta) + cos(\theta)) = k2\cos(\theta) = x$$

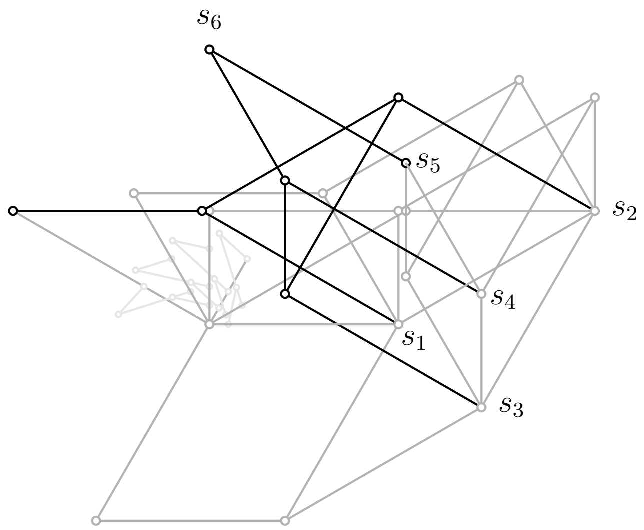
.

# 4 Appendix A

Step by step construction of $Add(\frac{\pi}{6}, 5, 6, -6, 3, 4, 6)$:




$s_5$


$s_6$
$s_5$
$s_2$
$s_4$
$s_1$
$s_3$

# 5 References

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