Finding closed-form expressions for infinite series

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Notation and Convention

Only symbols not generally agreed upon are mentioned below.

We make the convention that 0 is not a natural number, so $\mathbb{N} = \{1, 2, ...\}$.

When z is a complex number, Rez and Imz will denote the real and imaginary part respectively and $|z|$ will denote its absolute value.

When r is a real and positive number, $\log(r)$ will denote its natural logarithm.

When z is a complex number different from zero, $\log(z)$ will denote the function $\log(r) + i\theta$, where r is the absolute value of z and θ is its argument lying in the interval $(-\pi, \pi]$.

With $\arctan(x)$ we will denote the inverse of the tangent function (over the real numbers) satisfying $-\pi/2 < \arctan(x) < \pi/2$.

If $\{\zeta_n\}$ and $\{z_n\}$ are two sequences such that a number n_0 exists such that $|\zeta_n| < K|z_n|$ whenever $n > n_0$, where K is independent of n, we will write $\zeta_n = O(z_n)$.

If we are given a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers, then we will write $\prod_{n=1}^{\infty} a_n$ to mean $\lim_{N\to\infty} \prod_{n=1}^{N} a_n$, provided this last limit exists. An infinite product $\prod_{n=1}^{\infty} a_n$ is said to converge if the following holds

1. only finitely many factors are equal to zero,
2. if N_0 is so large that $a_n \neq 0$ for $n > N_0$, then $\lim_{N\to\infty} \prod_{n=N_0+1}^{N} a_n$ exists and is nonzero.

An infinite product $\prod_{n=1}^{\infty} (1 + a_n)$ is said to converge absolutely if $\prod_{n=1}^{\infty} (1 + |a_n|)$ converges.

Some other symbols are defined when used.

1 Introduction

The meaning of the term 'closed-form expression' could be defined as follows.

Definition 1.1. *A subfield F of $\mathbb{C}$ is closed under the taking of exponential and logarithm if*

1. $e^x \in F$ *for all* $x \in F$,
2. $\log(x) \in F$ *for all nonzero* $x \in F$.

The field E of numbers in closed-form expression is the intersection of all subfields of $\mathbb{C}$ that are closed under the taking of exponential and logarithm.

It can be shown (see [6]) that this definition fits the intuition of the term 'closed-form expression'. However, we will also be satisfied with some expressions involving the logarithm of the Gamma function and its derivatives (see chapter 5).

Suppose we are given a convergent series $\sum_{n=1}^{\infty} f(n)$ which we want to evaluate in closed-form. Let us, for now, assume that $f(n) \geq 0$, for example $f(n) = 1/n^2$. There are many clever ways to evaluate this particular series (see for example [7]). We will derive a method to rewrite the general series $\sum_{n=1}^{\infty} f(n)$, and this will eventually enable us to evaluate the series in closed-form in some cases.

The series can be interpreted in three dimensions as the sum of the lengths of the line segments l_n between the points $(n,0,0),(n,f(n),0) \in \mathbb{R}^3$. The idea is to project these line segments on a half sphere with radius R and to calculate the sum of the lengths of the corresponding arcs $\sigma_R(l_n)$ on the sphere. We shall denote the length of an arc a on a sphere by $\|a\|$.

Let S be the half sphere $\left\{ (\xi,\eta,\zeta) \in \mathbb{R}^3 \middle| \, \xi^2+\eta^2+(\zeta-R)^2 = R^2 \text{ and } \zeta < \mathrm{R} \right\}$ with radius $R > 0$ and with center $M := (0,0,R)$. One can derive that the stereographic projection of the xy-plane onto S corresponds to the mapping $\sigma_R : \left\{ (x,y,z) \in \mathbb{R}^3 \middle| \, z = 0 \right\} \to S$, where

$$\sigma_R(x,y,0) := \left(\frac{R \cdot x}{\sqrt{R^2+x^2+y^2}}, \frac{R \cdot y}{\sqrt{R^2+x^2+y^2}}, R - \frac{R^2}{\sqrt{R^2+x^2+y^2}} \right).$$

The vector pointing from M to $\sigma_R(n,0,0)$ is parallel to and in the same direction as the vector that starts at $(0,0,0)$ pointing to $(n,0,-R)$. Likewise, the vector starting at M pointing to $\sigma_R(n,f(n),0)$ is parallel to and in the same direction as the vector that starts at $(0,0,0)$ pointing to $(n,f(n),-R)$. It therefore follows that the length of $\sigma_R(l_n)$ is given by

$$\begin{aligned} \|\sigma_R(l_n)\| &= R \arccos\left(\frac{(n,0,-R)\cdot(n,f(n),-R)}{|(n,0,-R)|\,|(n,f(n),-R)|} \right) \\ &= R \arccos\left(\sqrt{\frac{R^2+n^2}{R^2+n^2+f(n)^2}} \right), \end{aligned}$$

where $|\cdot|$ indicates the standard Euclidean norm in three dimensions. This expression for $\|\sigma_R(l_n)\|$ can be rewritten using

$$\arccos(x) = \arctan\left(\sqrt{1-x^2}\Big/x\right), \text{ for } x > 0. \tag{1.1}$$

We then get $\|\sigma_R(l_n)\| = R\arctan\left(f(n)/\sqrt{R^2+n^2}\right)$, where we have made use of our assumption that $f(n) \geq 0$.

Intuitively it is clear that $\lim_{R\to\infty}\|\sigma_R(l_n)\| = f(n)$, because as the radius of the sphere grows larger the sphere itself becomes essentially flat near the origin. So we definitely have that $\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty}\lim_{R\to\infty}\|\sigma_R(l_n)\|$, which raises the question: under what conditions on $f(n)$ do we have that $\sum_{n=1}^{\infty} f(n) = \lim_{R\to\infty}\sum_{n=1}^{\infty}\|\sigma_R(l_n)\|$?

The interest in an answer to this question comes from the observation that if in some particular case we are able to evaluate $\sum_{n=1}^{\infty}\|\sigma_R(l_n)\|$ explicitly (not 'too complicated') as a function of R, then simply taking the limit as R goes to infinity returns us the value of the original series, which would be a closed-form evaluation provided we can calculate this last limit in a closed-form. Things don't seem to have gotten much easier with this since the formulas for $\|\sigma_R(l_n)\|$ seem rather complicated. Therefore we will now adjust the method slightly, resulting in simpler formulas. For this we interpret the series as the sum of the lengths of the line segments b_n between the points $(0,0,0), (f(n),0,0) \in \mathbb{R}^3$ and again we project these line segments on a half sphere of radius R and calculate the sum of the lengths of the corresponding arcs $\sigma_R(b_n)$ on the sphere.

Let S be the same half sphere as before with center $M := (0,0,R)$ and let σ_R be the same mapping of the xy-plane onto S. We now have that the vector pointing from M to $\sigma_R(0,0,0) = (0,0,0)$ is the same as the vector that starts at $(0,0,0)$ pointing to $(0,0,-R)$. The vector starting at M pointing to $\sigma_R(f(n),0,0)$ is parallel to and in the same direction as the vector that starts at $(0,0,0)$ pointing to $(f(n),0,-R)$ (using the definition of the map σ_R). It therefore follows that the length of $\sigma_R(b_n)$ is given by

$$\begin{aligned}\|\sigma_R(b_n)\| &= R\arccos\left(\frac{(0,0,-R)\cdot(f(n),0,-R)}{|(0,0,-R)|\,|(f(n),0,-R)|}\right)\\ &= R\arccos\left(\frac{R}{\sqrt{R^2+f(n)^2}}\right).\end{aligned}$$

Using equation (1.1) and $f(n) \geq 0$, we get $\|\sigma_R(b_n)\| = R\arctan\left(f(n)/R\right)$.

As before it is intuitively clear that $\lim_{R\to\infty}\|\sigma_R(b_n)\| = f(n)$ and again we come to the question whether we can switch the order of this limit and the summation. We will give an answer to this question in the next chapter.

2 Preliminary Lemma's

Dropping the assumption that $f(n) \geq 0$, we have the following.

Proposition 2.1. *Let* $\sum_{n=1}^{\infty} f(n)$ *be an absolutely convergent series and let* $\|\sigma_R(b_n)\| := R \arctan\left(f(n)/R\right).$ *Then*

$$\sum_{n=1}^{\infty} f(n) = \lim_{R\to\infty} \sum_{n=1}^{\infty} \|\sigma_R(b_n)\| \tag{2.1}$$

Proof. First we show that $\lim_{R\to\infty} \|\sigma_R(b_n)\| = f(n)$.

We have for $|x| \leq 1$ the Taylor series $\arctan x = \sum_{m=0}^{\infty} (-1)^m x^{2m+1}/(2m+1)$, and so for R large enough

$$\begin{aligned} |R \arctan\left(\frac{f(n)}{R}\right) - f(n)| &= |R \sum_{m=1}^{\infty} \frac{(-1)^m}{2m+1} \left(\frac{f(n)}{R}\right)^{2m+1} | \\ &= \frac{1}{R^2} | \sum_{m=1}^{\infty} \frac{(-1)^m}{2m+1} \left(\frac{f(n)^{2m+1}}{R^{2m-2}}\right) |, \end{aligned}$$

so $\lim_{R\to\infty} \|\sigma_R(b_n)\| = f(n)$.

Now given $\varepsilon > 0$ we choose, using absolute convergence, $N \in \mathbb{N}$ large enough such that $\sum_{n=N+1}^{\infty} |f(n)| < \varepsilon/3$. Then

$$\begin{aligned} &| \sum_{n=1}^{\infty} R \arctan\left(\frac{f(n)}{R}\right) - \sum_{n=1}^{\infty} f(n)| \\ &\quad = | \sum_{n=1}^{\infty} R \arctan\left(\frac{f(n)}{R}\right) - \sum_{n=1}^{N} f(n) + \sum_{n=1}^{N} f(n) - \sum_{n=1}^{\infty} f(n)| \\ &\quad \leq \sum_{n=1}^{N} |R \arctan\left(\frac{f(n)}{R}\right) - f(n)| + \sum_{n=N+1}^{\infty} \left\{ |f(n)| + |R \arctan\left(\frac{f(n)}{R}\right)| \right\}. \end{aligned}$$

Since for all n, $\lim_{R\to\infty} \|\sigma_R(b_n)\| = f(n)$, we can choose R large enough such that the first sum in this last estimate is smaller than $\varepsilon/3$. The last sum in the estimate is bounded by $2\varepsilon/3$, by virtue of the choice of N and the inequality $|R \arctan\left(f(n)/R\right)| = R \arctan\left(|f(n)|/R\right) \leq |f(n)|$. We see that there exists an $M = M(\varepsilon) \in \mathbb{R}$ such that

$$R > M \Rightarrow | \sum_{n=1}^{\infty} R \arctan\left(\frac{f(n)}{R}\right) - \sum_{n=1}^{\infty} f(n)| < \varepsilon,$$

which is what we set out to prove.

$\square$

Note 2.1. *Since the inverse tangent is an odd function we could also take the limit* $|R| \to \infty$ *in (2.1).*

Before giving an application we state and prove some elementary lemma's and a corollary, see also [1], pages 125-128, and [2], page 38.

Lemma 2.1. *For all* $z \in \mathbb{C}$,

$$e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n .$$

Proof. For $n \in \mathbb{N}$ and $z \in \mathbb{C}$ we have that

$$\begin{aligned} e^z - \left(1 + \frac{z}{n}\right)^n &= \left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right) - \left(\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{z^k}{n^k}\right) \\ &= \left(\sum_{k=0}^{n} c_{nk} \frac{z^k}{k!}\right) + \left(\sum_{k=n+1}^{\infty} \frac{z^k}{k!}\right), \end{aligned}$$

where

$$0 \le c_{nk} = 1 - \frac{n!}{(n-k)!n^k} = 1 - \prod_{j=0}^{k-1} \frac{n-j}{n} \le 1.$$

Now note that for each fixed k and fixed z, $\lim_{n\to\infty} c_{nk} z^k/k! = 0$. Given z and given $\varepsilon > 0$, we choose $m \in \mathbb{N}$ large enough such that $\sum_{k=m+1}^{\infty} |z|^k/k! < \varepsilon/3$. We also choose $N > m$ such that for $n > N$, $\sum_{k=0}^{m} |c_{nk} z^k/k!| < \varepsilon/3$ holds. Using $|c_{nk}| \le 1$, this gives that for $n > N$

$$\left| e^z - \left(1 + \frac{z}{n}\right)^n \right| \le \sum_{k=0}^{m} \left| \frac{c_{nk} z^k}{k!} \right| + \sum_{k=m+1}^{n} \left| \frac{c_{nk} z^k}{k!} \right| + \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,$$

proving the lemma.

□

Before stating and proving the next lemma we derive some basic inequalities, the first three of which are known as Weierstrass's inequalities (see [5], page 104).

If, for $L \in \mathbb{N}$, we are given a sequence $\{a_i\}_{i=0}^{L} \subset [0, 1)$, then with induction one can prove the inequalities

$$\prod_{i=0}^{L} (1 + a_i) \ge 1 + \sum_{i=0}^{L} a_i \tag{2.2}$$

$$\prod_{i=0}^{L}(1-a_i) \geq 1-\sum_{i=0}^{L} a_i. \tag{2.3}$$

Combining (2.3) with $1+a_i \leq (1-a_i)^{-1}$ and assuming $\sum_{i=0}^{L} a_i < 1$, gives

$$\prod_{i=0}^{L}(1+a_i) \leq \prod_{i=0}^{L}(1-a_i)^{-1} \leq \left(1-\sum_{i=0}^{L} a_i\right)^{-1}. \tag{2.4}$$

For some other inequalities we suppose that we are given $0<\varepsilon<1$ and a sequence $\{b_k\}_{k=0}^{L} \subset \mathbb{C}$, with $\sum_{k=0}^{L}|b_k| < \varepsilon/2$. Writing $b_k = \rho_k e^{i\theta_k}$ and $1+b_k = r_k e^{it_k}$, with $\rho_k = |b_k|, r_k = |1+b_k|$ and $\theta_k = \arg(b_k) \in (-\pi,\pi], t_k = \arg(1+b_k) \in (-\pi,\pi]$, we have that $1+b_k = (1+\rho_k\cos\theta_k)+i\rho_k\sin\theta_k$. So using $\rho_k = |b_k| < 1/2$ we have

$$\begin{aligned}
|t_k| = |\arctan\left(\frac{\rho_k\sin\theta_k}{1+\rho_k\cos\theta_k}\right)| &= \arctan\left(\frac{|\rho_k\sin\theta_k|}{1+\rho_k\cos\theta_k}\right) \\
&\leq \arctan\left(\frac{\rho_k}{1-\rho_k}\right) < \arctan(2\rho_k) \leq 2\rho_k.
\end{aligned}$$

From this follows

$$|\sum_{k=0}^{L} t_k\ | < 2\sum_{k=0}^{L}\rho_k < \varepsilon, \tag{2.5}$$

and from (2.4) it follows that

$$\prod_{k=0}^{L} r_k \leq \prod_{k=0}^{L}(1+\rho_k) \leq \left(1-\sum_{k=0}^{L}\rho_k\right)^{-1} < \left(1-\frac{1}{2}\varepsilon\right)^{-1} < 1+\varepsilon. \tag{2.6}$$

Also, using (2.3),

$$\prod_{k=0}^{L} r_k \geq \prod_{k=0}^{L}(1-\rho_k) \geq 1-\sum_{k=0}^{L}\rho_k \geq 1-\varepsilon. \tag{2.7}$$

Writing

$$b := \prod_{k=0}^{L}(1+b_k) = \left(\prod_{k=0}^{L} r_k\right) e^{i\sum_{k=0}^{L} t_k},$$

we have

$$|b-1| \leq\ \left|-1+\cos\left(\sum_{k=0}^{L} t_k\right)\prod_{k=0}^{L} r_k\ \right| +\ \left|\sin\left(\sum_{k=0}^{L} t_k\right)\prod_{k=0}^{L} r_k\ \right|,$$

where by (2.5), (2.6) and (2.7)

$$\left|\sin\left(\sum_{k=0}^{L} t_k\right)\prod_{k=0}^{L} r_k\ \right| \leq \varepsilon(1+\varepsilon) \leq 2\varepsilon \ \text{ and } \ -1+\cos\left(\sum_{k=0}^{L} t_k\right)\prod_{k=0}^{L} r_k \leq \varepsilon,$$

and

$$1-\cos\left(\sum_{k=0}^{L}t_k\right)\prod_{k=0}^{L}r_k \leq 1-\cos(\varepsilon)(1-\varepsilon) \leq 1-(1-\varepsilon)(1-\frac{\varepsilon^2}{2})$$
$$= \frac{\varepsilon^2}{2}-\frac{\varepsilon^3}{2}+\varepsilon \leq 2\varepsilon.$$

We conclude that

$$|b-1| \leq 4\varepsilon. \tag{2.8}$$

The following lemma is the multiplicative analogue of Tannery's theorem, see [5], page 136, for the additive version. In chapter 4, lemma 4.1, we will see and use a continuous analogue.

Lemma 2.2. *For $n \in \mathbb{N}$, let $P_n := \prod_{i=0}^{u(n)}(1+v_i(n))$, where $u(n) \in \mathbb{Z}_{\geq 0}$ is strictly increasing and for all n we have $\{v_i(n)\}_{i=0}^{u(n)} \subset \mathbb{C}$. Suppose there exist sequences $\{w_k\}_{k=0}^{\infty} \subset \mathbb{C}$ and $\{M_k\}_{k=0}^{\infty} \subset \mathbb{R}_{\geq 0}$ satisfying*

1. $\sum_{k=0}^{\infty} M_k < \infty$,
2. $\lim_{n\to\infty} v_k(n) = w_k$, *for all k,*
3. $|v_i(n)| \leq M_i$, *for all n and correspondingly, $i = 0, 1, ..., u(n)$.*

Then

$$\lim_{n\to\infty} P_n = \prod_{k=0}^{\infty}(1+w_k).$$

Proof. Because of assumption 1 we have that $M := \prod_{k=0}^{\infty}(1+M_k)$ converges absolutely (see appendix, lemma 6.1). Assumption 2 and 3 show that for all k, $|w_k| \leq M_k$ and together with assumption 1 and lemma 6.1 we see that $\prod_{k=0}^{\infty}(1+w_k)$ converges absolutely, so it also converges. Now Let $0 < \varepsilon < 1$ be given and let $q = q(\varepsilon) \in \mathbb{N}$ be such that $\sum_{k=q}^{\infty} M_k < \varepsilon/2$. If there is some $l \in \{0, 1, ..., q-1\}$ with $w_l = -1$ then, assuming that $n > q$,

$$|\prod_{i=0}^{q-1}(1+v_i(n)) - \prod_{k=0}^{\infty}(1+w_k)| = |\prod_{i=0}^{q-1}(1+v_i(n))|$$
$$\leq |1+v_l(n)| \prod_{i=0, i\neq l}^{q-1}(1+|v_i(n)|) \leq M\,|1+v_l(n)|,$$

otherwise

$$|\prod_{i=0}^{q-1}(1+v_i(n)) - \prod_{k=0}^{\infty}(1+w_k)|$$

$$\leq |\prod_{k=0}^{q-1}(1+w_k)||\frac{\prod_{i=0}^{q-1}(1+v_i(n))}{\prod_{k=0}^{q-1}(1+w_k)} - \prod_{k=q}^{\infty}(1+w_k)|$$
$$\leq M|\frac{\prod_{i=0}^{q-1}(1+v_i(n))}{\prod_{k=0}^{q-1}(1+w_k)} - 1| + M|1 - \prod_{k=q}^{\infty}(1+w_k)|$$
$$\leq M|\frac{\prod_{i=0}^{q-1}(1+v_i(n))}{\prod_{k=0}^{q-1}(1+w_k)} - 1| + 4\varepsilon M,$$

where we used (2.8) in the last step. Because $\lim_{n\to\infty} v_k(n) = w_k$, we see that in either case there exists an $N_1 = N_1(\varepsilon) \in \mathbb{N}$ bigger than q, such that

$$n > N_1 \Rightarrow |\prod_{i=0}^{q-1}(1+v_i(n)) - \prod_{k=0}^{\infty}(1+w_k)| \leq 5\varepsilon M. \tag{2.9}$$

Since $N_1 > q$, we have that $n > N_1$ implies $u(n) > q$. So that when $n > N_1$,

$$|\prod_{i=0}^{u(n)}(1+v_i(n)) - \prod_{i=0}^{q-1}(1+v_i(n))| \leq |\prod_{i=0}^{q-1}(1+v_i(n))||\prod_{i=q}^{u(n)}(1+v_i(n)) - 1| \leq 4\varepsilon M, \tag{2.10}$$

where we used (2.8) again in the last step. Combining (2.9) and (2.10) gives

$$n > N_1 \Rightarrow |P_n - \prod_{k=0}^{\infty}(1+w_k)| = |\prod_{i=0}^{u(n)}(1+v_i(n)) - \prod_{k=0}^{\infty}(1+w_k)| \leq 9\varepsilon M,$$

proving the lemma.

□

We can use the previous lemma's to prove

Lemma 2.3. *For all* $z \in \mathbb{C}$,

$$\sin z = z \prod_{k=1}^{\infty}\left(1 - \frac{z^2}{\pi^2 k^2}\right). \tag{2.11}$$

Proof. Lemma 2.1 gives that $\sin z = (e^{iz} - e^{-iz})/2i = \lim_{n\to\infty} P_n(z)$, where $P_n(z)$ is the polynomial

$$P_n(z) = \frac{1}{2i}\left(1 + \frac{iz}{n}\right)^n - \frac{1}{2i}\left(1 - \frac{iz}{n}\right)^n.$$

This polynomial vanishes when $z = 0$ and also when $[(1 - iz/n)/(1 + iz/n)]^n = 1$, that is, when $(1 - iz/n)/(1 + iz/n) = w$, or $z = in(w-1)/(w+1)$, with

$w^n = 1, w \neq \pm 1$. We may assume that that n is even, say $n = 2m,\ m \in \mathbb{N}$, so that our polynomial has degree $n-1 = 2m-1$ and the $2m-2$ nonzero roots are given by

$$in\frac{e^{-2ik\pi/2m}-1}{e^{-2ik\pi/2m}+1} = in\frac{e^{-ik\pi/2m}-e^{ik\pi/2m}}{e^{-ik\pi/2m}+e^{ik\pi/2m}} = 2m\tan(k\pi/2m),$$

with $k = \pm 1, \pm 2, ..., \pm(m-1)$. Since at $z = 0$, $P_n(z)/z = 1$ (the coefficient of z in $P_n(z)$), we find the factorization

$$P_n(z) = P_{2m}(z) = z\prod_{k=1}^{m-1}\left(1 - \frac{z^2}{[2m\tan(k\pi/2m)]^2}\right),$$

and thus

$$\sin z = \lim_{m\to\infty} P_{2m}(z) = z\lim_{m\to\infty}\prod_{k=1}^{m-1}\left(1 - \frac{z^2}{[2m\tan(k\pi/2m)]^2}\right).$$

In order to invoke the previous lemma note that $\sum_{k=1}^{\infty}|z|^2/k^2\pi^2 < \infty$ and

$$\frac{|z|^2}{|2m\tan(k\pi/2m)|^2} \leq \frac{|z|^2}{|2m(k\pi/2m)|^2} = \frac{|z|^2}{k^2\pi^2}.$$

Furthermore

$$\lim_{m\to\infty}\frac{z^2}{(2m\tan(k\pi/2m))^2} = \lim_{m\to\infty}\frac{z^2}{k^2\pi^2\left(\frac{\tan(k\pi/2m)}{k\pi/2m}\right)^2} = \frac{z^2}{k^2\pi^2},$$

and lemma 2.2 now concludes the proof.

□

Corollary 2.1. *Let $z, a \in \mathbb{C}$, where a is not an integral multiple of π. Then*

$$\frac{\sin(z+a)}{\sin a} = \frac{z+a}{a}\prod_{k=1}^{\infty}\left\{\left(1 - \frac{z}{k\pi - a}\right)\left(1 + \frac{z}{k\pi + a}\right)\right\}.$$

Proof. By lemma 2.3,

$$\frac{\sin(z+a)}{\sin a} = \frac{(z+a)\prod_{k=1}^{\infty}\left(1 - \frac{(z+a)^2}{k^2\pi^2}\right)}{a\prod_{k=1}^{\infty}\left(1 - \frac{a^2}{k^2\pi^2}\right)} = \frac{z+a}{a}\prod_{k=1}^{\infty}\frac{\left(\frac{k^2\pi^2-(z+a)^2}{k^2\pi^2}\right)}{\left(\frac{k^2\pi^2-a^2}{k^2\pi^2}\right)},$$

and

$$\frac{\left(\frac{k^2\pi^2-(z+a)^2}{k^2\pi^2}\right)}{\left(\frac{k^2\pi^2-a^2}{k^2\pi^2}\right)} = \frac{k^2\pi^2-(z+a)^2}{k^2\pi^2-a^2} = \frac{(k\pi-(z+a))\,(k\pi+(z+a))}{(k\pi-a)(k\pi+a)}$$
$$= \left(1-\frac{z}{k\pi-a}\right)\left(1+\frac{z}{k\pi+a}\right),$$

proving the corollary.

□

3 An Application

First we prove the following proposition (see also [2], page 39), which will also be used in chapter 5.

Proposition 3.1. *Let $x, b \in \mathbb{R}$, where b is not an integral multiple of π. Then*

$$\begin{aligned}&\arctan\left(\tanh(x)\cot(b)\right)\\&\qquad = \arctan\left(\frac{x}{b}\right) + \sum_{k=1}^{\infty}\left\{\arctan\left(\frac{x}{k\pi+b}\right) - \arctan\left(\frac{x}{k\pi-b}\right)\right\}\end{aligned} \tag{3.1}$$

Proof. By corollary 2.1

$$\begin{aligned}&\operatorname{Imlog}\left(\sin(b+ix)/\sin b\right)\\&\quad = \operatorname{Imlog}\left[\left(1+\frac{ix}{b}\right)\prod_{k=1}^{\infty}\left\{\left(1-\frac{ix}{k\pi-b}\right)\left(1+\frac{ix}{k\pi+b}\right)\right\}\right]\\&\quad = \arctan\left(\frac{x}{b}\right) + \sum_{k=1}^{\infty}\left\{\arctan\left(\frac{x}{k\pi+b}\right) - \arctan\left(\frac{x}{k\pi-b}\right)\right\} + l\pi,\end{aligned}$$

with $l \in \mathbb{Z}$ only depending on the first few terms of the infinite product and on the term $1 + (ix/b)$, because as k gets large, the imaginary parts of the factors in the infinite product become arbitrary small in absolute value. On the other hand

$$\begin{aligned}&\operatorname{Imlog}\left(\sin(b+ix)/\sin b\right)\\&\quad = \operatorname{Imlog}\left((\cos ix \sin b + \sin ix \cos b)/\sin b\right)\\&\quad = \operatorname{Imlog}\left(\cos ix + \cot b \sin ix\right)\\&\quad = \operatorname{Imlog}\left(\cosh x + i \cot b \sinh x\right) = \arctan(\cot b \tanh x) + l'\pi,\end{aligned}$$

with $l' \in \mathbb{Z}$. We are done if we show that $l - l' = 0$. When $x = 0$ then (3.1) is clearly true, so in that case $l - l' = 0$. If we now prove that both sides of (3.1) are continuous functions of x, holding b fixed, then $l - l'$ must also be continuous and so equal to its value at $x = 0$, that is zero. To prove continuity it suffices to show that

$$\sum_{k=1}^{\infty}\left\{\arctan\left(\frac{x}{k\pi+b}\right) - \arctan\left(\frac{x}{k\pi-b}\right)\right\} \tag{3.2}$$

is a continuous function of x, holding b fixed. For that we invoke the addition formulas

$$\arctan y + \arctan z = \begin{cases}\arctan\left(\frac{y+z}{1-yz}\right) & \text{if } yz < 1,\\ \arctan\left(\frac{y+z}{1-yz}\right) + \pi sign(y) & \text{if } yz > 1.\end{cases} \tag{3.3}$$

In our case we have that $yz = -x^2/(k^2\pi^2 - b^2)$, and

$$\frac{y+z}{1-yz} = \frac{\frac{x}{k\pi+b} - \frac{x}{k\pi-b}}{1 + \frac{x}{k\pi+b}\frac{x}{k\pi-b}} = \frac{-2bx}{k^2\pi^2 - b^2 + x^2}.$$

So if we choose $K \in \mathbb{N}$, with $K^2\pi^2 - b^2 > 0$, then $yz < 1$ for all $k \geq K$ and so (3.2) equals

$$\sum_{k=1}^{K-1} \left\{ \arctan\left(\frac{x}{k\pi + b}\right) - \arctan\left(\frac{x}{k\pi - b}\right) \right\} + \sum_{k=K}^{\infty} \arctan\left(\frac{-2bx}{k^2\pi^2 - b^2 + x^2}\right).$$

Now it only remains to show the continuity of this last infinite sum. For this we assume $x \in [-N, N]$, $N > 0$, so that for $k \geq K$

$$\arctan\left(\frac{-2bx}{k^2\pi^2 - b^2 + x^2}\right) \leq \frac{|2bx|}{|k^2\pi^2 - b^2 + x^2|} \leq \frac{2N|b|}{k^2\pi^2 - b^2}.$$

Since $\sum_K^\infty 2N|b|/(k^2\pi^2 - b^2) < \infty$, the Weierstrass M-test implies that $\sum_{k=K}^\infty \arctan\left(-2bx/(k^2\pi^2 - b^2 + x^2)\right)$ converges uniformly for $x \in [-N, N]$ and so it is continuous there. Because N was arbitrary we are done.

$\square$

Remark 3.1. *It is clear that the above approach will work for many infinite products to produce series involving the inverse tangent function. In chapter 5 we will see a method that allows us to express many such series in terms of the Gamma function. There exist more methods to derive series like (3.1), see [4] for a nice account of these.*

Example 3.1. *Let us sum the series $\sum_{n=1}^\infty 1/(w^2 - n^2)$ in closed-form, where $w \in \mathbb{R}\backslash\mathbb{Z}$. Proposition 2.1 indicates that we should attempt to sum*

$$\sum_{k=1}^{\infty} \arctan\left(\frac{1}{Rw^2 - Rk^2}\right)$$

and proposition 3.1 and its proof show that we could try to find $b, x \in \mathbb{R}$ such that

$$\frac{-2bx}{k^2\pi^2 - b^2 + x^2} = \frac{1}{Rw^2 - Rk^2} = \frac{1}{\frac{\pi^2}{-2bx}k^2 + \frac{x^2 - b^2}{-2bx}},$$

so we get that $R = \pi^2/2bx$ and $Rw^2 = (b^2 - x^2)/2bx$. Using the quadratic formula one can calculate that a solution is given by

$$x = (1/2)\pi\left(\sqrt{w^2 + (1/R)} - \sqrt{w^2 - (1/R)}\right)$$

and

$$b = (1/2)\pi\left(\sqrt{w^2 + (1/R)} + \sqrt{w^2 - (1/R)}\right).$$

With these values we claim that for $k = 1, 2, \ldots$

$$\arctan\left(\frac{x}{k\pi + b}\right) - \arctan\left(\frac{x}{k\pi - b}\right) = \arctan\left(\frac{-2bx}{k^2\pi^2 - b^2 + x^2}\right).$$

To prove the claim we need to show, by (3.3), that for $k = 1, 2, \ldots,$ $-x^2/(k^2\pi^2 - b^2) < 1$, *that is*

$$\frac{-w^2 + \sqrt{w^4 - (1/R^2)}}{2k^2 - w^2 - \sqrt{w^4 - (1/R^2)}} < 1. \tag{3.4}$$

Because the numerator is always negative and the denominator $2k^2 - w^2 - \sqrt{w^4 - (1/R^2)} > 2k^2 - 2w^2$, *we see that if* $2k^2 - 2w^2 \geq 0$ *then indeed (3.4) holds, so we now assume that for some* k, *the denominator is negative and that* $k^2 < w^2$. *Then for* R *large enough* $(2k^2 - w^2) - \sqrt{w^4 - (1/R^2)} < -w^2 + \sqrt{w^4 - (1/R^2)} < 0$, *so indeed*

$$\frac{-w^2 + \sqrt{w^4 - (1/R^2)}}{2k^2 - w^2 - \sqrt{w^4 - (1/R^2)}} < \frac{-w^2 + \sqrt{w^4 - (1/R^2)}}{-w^2 + \sqrt{w^4 - (1/R^2)}} = 1.$$

Since $w \in \mathbb{R}\backslash\mathbb{Z}$ *we have for sufficiently large* R *that* b *is not an integral multiple of* π, *thus by proposition 2.1 and 3.1*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{w^2 - n^2} &= \lim_{R\to\infty} R \sum_{k=1}^{\infty} \arctan\left(\frac{1}{Rw^2 - Rk^2}\right) \\ &= \lim_{R\to\infty} R\left(\arctan\left(\tanh(x)\cot(b)\right) - \arctan\left(\frac{x}{b}\right)\right).\end{aligned} \tag{3.5}$$

We calculate the limit in (3.5) using the limits $\lim_{R\to\infty} b = \pi|w| \notin \pi\mathbb{Z}$, $\lim_{R\to\infty} x = 0$ *and the standard limits* $\lim_{u\to 0} \arctan(u)/u = \lim_{u\to 0} \tanh(u)/u = 1$. *Usage of l'Hôpital's rule is indicated with **, we get*

$$\begin{aligned}&\lim_{R\to\infty} R\left(\arctan\left(\tanh(x)\cot(b)\right) - \arctan\left(x/b\right)\right) \\ &\quad= \lim_{R\to\infty}\Bigg(R\tanh(x)\cot(b)\frac{\arctan\left(\tanh(x)\cot(b)\right)}{\tanh(x)\cot(b)} \\ &\qquad\qquad - R(x/b)\frac{\arctan\left(x/b\right)}{(x/b)}\Bigg) \\ &\quad= \lim_{R\to\infty}\left(R\tanh(x)\cot(b)\right)\cdot 1 - \lim_{R\to\infty}\left(R(x/b)\right)\cdot 1 \\ &\quad= \cot(\pi|w|)\lim_{R\to\infty}\left(Rx\frac{\tanh(x)}{x}\right) - \lim_{R\to\infty}\left(R(x/b)\right) \\ &\quad= \cot(\pi|w|)\lim_{R\to\infty}(Rx)\cdot 1 - \lim_{R\to\infty}\left(R(x/b)\right)\end{aligned}$$

$$
\begin{aligned}
&= \cot(\pi|w|) \lim_{y\to 0+} \left(\frac{1}{2}\pi \frac{\sqrt{w^2+y}-\sqrt{w^2-y}}{y} \right) \\
&\qquad\qquad - \lim_{y\to 0+} \left(\frac{\sqrt{w^2+y}-\sqrt{w^2-y}}{y\left(\sqrt{w^2+y}+\sqrt{w^2-y}\right)} \right) \\
&\stackrel{**}{=} \cot(\pi|w|) \lim_{y\to 0+} \left(\frac{1}{2}\pi \left(\frac{1}{2\sqrt{w^2+y}} - \frac{-1}{2\sqrt{w^2-y}} \right) \right) \\
&\qquad\qquad - \frac{1}{2|w|} \lim_{y\to 0+} \left(\frac{\sqrt{w^2+y}-\sqrt{w^2-y}}{y} \right) \\
&\stackrel{**}{=} \frac{1}{2}\pi \cot(\pi|w|) \frac{1}{|w|} - \frac{1}{2|w|}\frac{1}{|w|},
\end{aligned}
$$

where every step that replaces a limit of a sum/product by the sum/product of the limits can be justified by reading this string of equalities backwards. The closed-form evaluation thus becomes

$$
\sum_{n=1}^{\infty} \frac{1}{w^2-n^2} = \frac{\pi\cot(\pi w)}{2w} - \frac{1}{2w^2}, \;\; w \in \mathbb{R}\backslash\mathbb{Z}. \tag{3.6}
$$

The significance of (3.6) could be inferred from the following easy corollary.

Corollary 3.1. *For $k \in \mathbb{N}$ let $\zeta(k) := \sum_{n=1}^{\infty} 1/n^k$ and let B_k denote the kth Bernoulli number. Then when k is even*

$$
\zeta(k) = -\frac{1}{2}\frac{(2\pi i)^k B_k}{k!}.
$$

Proof. see [10], page 4.

$\square$

Remark 3.2. *Example 3.1 is by no means the fastest way to prove (3.6): logarithmic differentiation of (2.11) yields this almost at once. However, we are describing methods to find closed-form expressions for infinite series. We started out with a series and found the answer. It is a constructive approach. Also, it shows that proposition 2.1 can be useful. In chapter 5 we will see many more useful applications of proposition 2.1, but not after we establish some properties of the Gamma function.*

4 The Gamma Function

The following is a continuous analogue of lemma 2.2. To avoid confusion we will use the broad setting of the Lebesgue integral.

Lemma 4.1. *For $n \in \mathbb{N}$, let $f_n : (0, \infty) \to \mathbb{C}$ be Lebesgue integrable and put $I_n := \int_0^{p(n)} f_n(t)dt$, where $p(n) \in \mathbb{Z}_{\geq 0}$ is strictly increasing. Suppose that I_n is finite for all n and suppose there exist Lebesgue integrable functions $f : (0, \infty) \to \mathbb{C}$ and $M : (0, \infty) \to \mathbb{R}_{\geq 0}$ satisfying*

1. $\int_0^\infty M(t)dt < \infty$,
2. $\lim_{n\to\infty} f_n = f$ *uniformly on finite interval subsets of* $(0, \infty)$,
3. *for all n we have $|f_n(t)| \leq M(t)$, holding for all $t \in (0, p(n))$.*

Then

$$\lim_{n\to\infty} I_n = \int_0^\infty f(t)dt. \tag{4.1}$$

Proof. Because of conditions 2 and 3, we know that the right hand side of (4.1) is finite. Let $\varepsilon > 0$ be given and let $q = q(\varepsilon) \in \mathbb{N}$ be such that $\int_q^\infty M(t)dt < \varepsilon/3$. Then

$$\begin{aligned} \left|\int_0^q f_n(t)dt - \int_0^\infty f(t)dt\right| &\leq \int_0^q |f_n(t) - f(t)|dt + \int_q^\infty |f(t)|dt \\ &\leq \int_0^q |f_n(t) - f(t)|dt + \varepsilon/3, \end{aligned}$$

Now because of condition 2 we see that there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$n > N \Rightarrow \left|\int_0^q f_n(t)dt - \int_0^\infty f(t)dt\right| \leq 2\varepsilon/3. \tag{4.2}$$

We may assume that N is such that for all $n > N$ we have $p(n) > q$. Then if $n > N$,

$$\left|\int_0^{p(n)} f_n(t)dt - \int_0^q f_n(t)dt\right| \leq \int_q^{p(n)} |f_n(t)|dt \leq \varepsilon/3. \tag{4.3}$$

Combining (4.2) and (4.3) gives

$$n > N \Rightarrow \left|\int_0^{p(n)} f_n(t)dt - \int_0^\infty f(t)dt\right| \leq \varepsilon,$$

proving (4.1).

$\square$

Before we apply lemma 4.1 we first establish a useful inequality (see [5], page 506).

Since the derivative of $1 - e^t(1-t/n)^n$ equals $e^t(1-t/n)^{n-1}(t/n)$, we have for $0 < t < n$ that

$$0 \leq \int_0^t e^v \left(1 - \frac{v}{n}\right)^{n-1} \frac{v}{n} dv = 1 - e^t \left(1 - \frac{t}{n}\right)^n \leq e^t \int_0^t \frac{v}{n} dv = \frac{e^t t^2}{2n},$$

or

$$0 \leq e^{-t} - \left(1 - \frac{t}{n}\right)^n \leq \frac{t^2}{2n}. \tag{4.4}$$

Armed with this last inequality we apply lemma 4.1 to

$$I_n := \int_0^n (1-t/n)^n t^{z-1} dt,$$

where the real part x of the complex number z is assumed positive. So in this case we take $p(n) := n$, $f_n(t) := (1-t/n)^n t^{z-1}$ and $f(t) := e^{-t}t^{z-1}$. We claim that with $M(t) := e^{-t}t^{x-1}$ conditions 1, 2 and 3 of lemma 4.1 are satisfied. Firstly, inequality (4.4) shows that $|f(t) - f_n(t)| \leq t^{x-1}t^2/2n$, so $f_n \to f$ uniformly on finite interval subsets of $(0, \infty)$. Secondly, choosing $C > 0$ such that $t^{x-1} \leq Ce^{t/2}$ for all $t \in (1, \infty)$, we have

$$\int_0^\infty M(t)dt = \int_0^1 e^{-t}t^{x-1}dt + \int_1^\infty e^{-t}t^{x-1}dt \leq \int_0^1 t^{x-1}dt + C\int_1^\infty e^{-t/2}dt < \infty.$$

Finally, for $0 < t < n$, $|f_n(t)| = (1-t/n)^n t^{x-1} \leq e^{-t}t^{x-1} = M(t)$, by inequality (4.4). So we find that

$$\Gamma(z) := \lim_{n\to\infty} I_n = \int_0^\infty e^{-t}t^{z-1}dt, \tag{4.5}$$

which is Euler's integral expression for the Gamma function, valid when the real part of z is positive. Using integration by parts on this expression one can show that the Gamma function satisfies

$$\Gamma(z+1) = z\Gamma(z). \tag{4.6}$$

Starting out again with $I_n = \int_0^n (1-t/n)^n t^{z-1}dt$ and making the substitution

$t = n\tau$, gives that $I_n = n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau$ and repeated integration by parts gives

$$\int\limits_0^1 (1-\tau)^n \tau^{z-1} d\tau = \frac{1}{z}\tau^z(1-\tau)^n\Big|_0^1 + \frac{n}{z}\int\limits_0^1 (1-\tau)^{n-1}\tau^z d\tau$$
$$= \frac{n(n-1)}{z(z+1)}\int\limits_0^1 (1-\tau)^{n-2}\tau^{z+1} d\tau = ...$$
$$= \frac{n!}{z(z+1)...(z+n-1)}\int\limits_0^1 \tau^{z+n-1} d\tau = \frac{n!}{z(z+1)...(z+n)}.$$

So

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)...(z+n)}. \tag{4.7}$$

Letting $\gamma := \lim_{n\to\infty} (\sum_{k=1}^n 1/k) - \log(n)$ be the Euler-Mascheroni constant, we manipulate somewhat further

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} \frac{1}{n^z}\prod_{k=1}^n \left(1+\frac{z}{k}\right) = z \lim_{n\to\infty} e^{-z(\log(n)-\sum_{k=1}^n 1/k)} \prod_{k=1}^n e^{-z/k}\left(1+\frac{z}{k}\right)$$
$$= ze^{\gamma z}\prod_{k=1}^\infty e^{-z/k}\left(1+\frac{z}{k}\right), \tag{4.8}$$

which is Weierstrass' product expression for $\Gamma(z)^{-1}$.

Next we establish some fundamental properties of the Gamma function.

Lemma 4.2. *Weierstrass's product expression extends the Gamma function to a meromorphic function on the entire complex plane, with only simple poles, located at $z = 0, -1, -2, \ldots$. Furthermore we have*

1. $\Gamma(z+1) = z\Gamma(z)$, *for $z \neq 0, -1, -2, ...,$*
2. $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$, *whenever z is not an integer.*

Proof. We let N be a positive integer and take $|z| \leq (1/2)N$. Then if $n > N$,

$$|\log\left(1+\frac{z}{n}\right) - \frac{z}{n}| = |\sum_{k=2}^\infty \frac{(-1)^{k+1}}{k}\left(\frac{z}{n}\right)^k|$$
$$\leq \frac{|z|^2}{n^2}\sum_{k=0}^\infty \left(\frac{|z|}{n}\right)^k \leq \frac{1}{4}\frac{N^2}{n^2}\sum_{k=0}^\infty \left(\frac{1}{2}\right)^k = \frac{1}{2}\frac{N^2}{n^2}.$$

So $\sum_{n=N+1}^{\infty} \{\log(1+z/n) - z/n\}$ converges absolutely and uniformly in the disc $|z| \le (1/2)N$ and because each term of this series is analytic in the disc $|z| < (1/2)N$, it follows that the series itself is analytic in the disc $|z| < (1/2)N$ (see appendix, theorem 6.1). Consequently its exponential $\prod_{k=N+1}^{\infty} e^{-z/k}(1+z/k)$ is analytic in this disc, so $\Gamma(z)^{-1}$ is analytic in every disc $|z| < (1/2)N$, with N a positive integer, which proves the analyticity of $\Gamma(z)^{-1}$ in the entire complex plane.

The convergence of $\sum_{n=N+1}^{\infty} \{\log(1+z/n) - z/n\}$ in the disc $|z| \le (1/2)N$, for every $N \in \mathbb{N}$, reveals that $\Gamma(z)^{-1}$ has only zeros at $z = 0, -1, -2, \dots$, which are all simple, so $\Gamma(z)$ is analytic for $z \in \mathbb{C}\backslash\{0,-1,-2,...\}$, with simple poles at the excluded points.

The analyticity of the Gamma function on $\mathbb{C}\backslash\{0,-1,-2,...\}$ implies the persistence of the functional equation (4.6) here (see appendix, theorem 6.2), proving 1.

Combining the Weierstrass product expression with lemma 2.3 we further deduce

$$\Gamma(z)\Gamma(-z) = -\frac{1}{z^2}\prod_{k=1}^{\infty}\left\{e^{-z/k}\left(1+\frac{z}{k}\right)\right\}^{-1}\prod_{k=1}^{\infty}\left\{e^{z/k}\left(1-\frac{z}{k}\right)\right\}^{-1} = \frac{-\pi}{z\sin(\pi z)},$$

and with the functional equation (4.6) this becomes

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

proving 2.

□

Remark 4.1. *Formula 2 in lemma 4.2 is known as Euler's reflection formula.*

The next theorem is actually another application of Weierstrass's product expansion.

Theorem 4.1. *Let*

$$P := \prod_{n=1}^{\infty}\left\{\frac{(n-a_1)...(n-a_k)}{(n-b_1)...(n-b_k)}\right\}$$

be absolutely convergent, where the a_i and b_i are fixed complex numbers and k is a fixed positive integer, $a_i, b_i \notin \mathbb{N}$. Then

$$P = \prod_{m=1}^{k}\frac{\Gamma(1-b_m)}{\Gamma(1-a_m)}.$$

Proof. Using the geometric series, we have for n large enough that

$$\begin{aligned}\prod_{m=1}^{k}\left\{\frac{n-a_m}{n-b_m}\right\} &= \prod_{m=1}^{k}\left(1-\frac{a_m}{n}\right)\left(1-\frac{b_m}{n}\right)^{-1}\\ &= \prod_{m=1}^{k}\left(1-\frac{a_m}{n}\right)\left(1+\frac{b_m}{n}+A_n^m\right)\\ &= \prod_{m=1}^{k}\left(1-\frac{a_m-b_m}{n}+B_n^m\right) = 1+C_n-\frac{1}{n}\sum_{m=1}^{k}(a_m-b_m),\end{aligned}$$

where all of A_n^m, B_n^m and C_n are $O(n^{-2})$. We know by lemma 6.1 (see appendix) that P is absolutely convergent if and only if $\sum_{n=1}^{\infty}|C_n - n^{-1}\sum_{m=1}^{k}(a_m-b_m)| < \infty$, so $\Delta := \sum_{m=1}^{k}(a_m-b_m) = 0$ is a sufficient condition. It is also necessary since we have

$$\sum_{n=1}^{N}\left\{|C_n| - |\frac{1}{n}\sum_{m=1}^{k}(a_m-b_m)|\right\} \le \sum_{n=1}^{N}|C_n - \frac{1}{n}\sum_{m=1}^{k}(a_m-b_m)|$$

and convergence of the smallest member as $N\to\infty$ is only possible if $\Delta = 0$. Knowing that $\Delta = 0$ we can insert the factor $e^{n^{-1}\Delta} = 1$ into the general factor of the product without altering its value, that is

$$\begin{aligned}P &= \prod_{n=1}^{\infty}\left\{\prod_{m=1}^{k} e^{a_m/n}\left(1-\frac{a_m}{n}\right)e^{-b_m/n}\left(1-\frac{b_m}{n}\right)^{-1}\right\}\\ &= \prod_{m=1}^{k}\left\{\left[\prod_{n=1}^{\infty} e^{a_m/n}\left(1-\frac{a_m}{n}\right)\right]\left[\prod_{n=1}^{\infty} e^{-b_m/n}\left(1-\frac{b_m}{n}\right)^{-1}\right]\right\},\end{aligned}$$

where this last equality holds since according to lemma 4.2 the infinite products on the right all converge and can be expressed in terms of the Gamma function. Doing so and using the functional equation (4.6) together with $\Delta = 0$ we get

$$P = \prod_{m=1}^{k}\left\{\left[-a_m\Gamma(-a_m)e^{-\gamma a_m}\right]^{-1}\left[-b_m\Gamma(-b_m)e^{-\gamma b_m}\right]\right\} = \prod_{m=1}^{k}\frac{\Gamma(1-b_m)}{\Gamma(1-a_m)},$$

which is what we set out to prove. □

Remark 4.2. *Some of the proofs and derivations in this chapter have been taken partially from [11], which however takes Weierstrass's product as the definition of the Gamma function.*

5 More Applications

We will need the following.

Theorem 5.1. *Let*

$$S := \sum_{n=1}^{\infty} f(n)$$

be a real and absolutely convergent series. Then

$$S = \lim_{r\to 0}\left\{\frac{1}{2ir}\log\left[\prod_{n=1}^{\infty}\frac{1+irf(n)}{1-irf(n)}\right]\right\},$$

where r takes only real values.

Proof. We set $\sigma(R) := \sum_{n=1}^{\infty}\arctan(f(n)/R)$, with R real and $|R|$ large enough to ensure $|\sigma(R)| < \pi/2$. With the aid of a picture it is seen that

$$e^{i\arctan(f(n)/R)} = \frac{1+if(n)/R}{\sqrt{1+f(n)^2/R^2}},$$

so that

$$\begin{aligned} e^{2i\sigma(R)} &= \prod_{n=1}^{\infty}\left\{\frac{(1+if(n)/R)^2}{1+f(n)^2/R^2}\right\} \\ &= \prod_{n=1}^{\infty}\left\{\frac{(1+if(n)/R)^2}{(1+if(n)/R)(1-if(n)/R)}\right\} = \prod_{n=1}^{\infty}\left\{\frac{1+if(n)/R}{1-if(n)/R}\right\}. \end{aligned} \tag{5.1}$$

By the assumption on R this never lies on the negative real axis so that we can take the logarithm of both sides. Together with some algebraic manipulation this results in

$$R\sigma(R) = \frac{R}{2i}\log\left\{\prod_{n=1}^{\infty}\left\{\frac{1+if(n)/R}{1-if(n)/R}\right\}\right\}.$$

Letting $|R|$ go to infinity and using proposition 2.1 and note 2.1 we finally get

$$S = \lim_{|R|\to\infty}\frac{R}{2i}\log\left\{\prod_{n=1}^{\infty}\left\{\frac{1+if(n)/R}{1-if(n)/R}\right\}\right\} = \lim_{r\to 0}\frac{1}{2ir}\log\left\{\prod_{n=1}^{\infty}\left\{\frac{1+irf(n)}{1-irf(n)}\right\}\right\},$$

which is the desired result.

□

Note 5.1. *For the upcoming examples, in which we will want to use theorem 4.1 as well, let us show that the infinite product in the statement of theorem 5.1 converges absolutely. We have*

$$|\frac{1+irf(n)}{1-irf(n)}-1| = |\frac{2irf(n)}{1-irf(n)}| = \frac{|f(n)|}{|(1/2ir)-(1/2)f(n)|} \leq |f(n)|,$$

for r in a small enough interval around zero. The absolute convergence of the series for S together with lemma 6.1 in the appendix now furnishes the claimed absolute convergence of the infinite product.

The following examples are applications of some of the preceding material.

Example 5.1. *Let $x,\ b \in \mathbb{R}$, where b is not an integral multiple of π. According to proposition 3.1 and its proof we have for some $m \in \mathbb{Z}$*

$$\sum_{n=1}^{\infty} \arctan\left(\frac{-2bx}{n^2\pi^2-b^2+x^2}\right) = m\pi + \arctan(\tanh(x)\cot(b)) - \arctan\left(\frac{x}{b}\right).$$

With $f(n) := (-2bx)/(n^2\pi^2-b^2+x^2)$ and $R := 1$ in equation (5.1) we get

$$\begin{aligned} e^{2i(\arctan(\tanh(x)\cot(b))-\arctan(x/b))} &= \prod_{n=1}^{\infty}\left\{\frac{n^2\pi^2-b^2+x^2-i2bx}{n^2\pi^2-b^2+x^2+i2bx}\right\} \\ &= \prod_{n=1}^{\infty}\left\{\frac{n^2+(x-ib)^2/\pi^2}{n^2+(x+ib)^2/\pi^2}\right\}. \end{aligned}$$

So with note 5.1 and theorem 4.1 we finally find

$$e^{2i(\arctan(\tanh(x)\cot(b))-\arctan(x/b))} = \frac{\Gamma(1-(-b+ix)/\pi)\Gamma(1-(b-ix)/\pi)}{\Gamma(1-(-b-ix)/\pi)\Gamma(1-(b+ix)/\pi)}, \tag{5.2}$$

which can be confirmed using lemma 4.2.2, although it is cumbersome.

Example 5.2. *Let us, for $k \in \mathbb{N}_{>1}$ and $a \in \mathbb{R}$ not a negative integer, consider the series $S := \sum_{n=1}^{\infty}(n+a)^{-k}$. According to theorem 5.1*

$$S = \lim_{r\to 0}\left\{\frac{1}{2ir}\log\left[\prod_{n=1}^{\infty}\frac{(n+a)^k+ir}{(n+a)^k-ir}\right]\right\}.$$

Since $(-i)^{1/k} = e^{-i\pi/(2k)}e^{2i\pi m/k} = e^{i\pi(-1+4m)/(2k)}$ and $i^{1/k} = e^{i\pi/(2k)}e^{2i\pi m/k} = e^{i\pi(1+4m)/(2k)}$, for $m = 0, 1, \ldots, k-1$, we have the factorization

$$(n+a)^k \pm ir = \prod_{m=0}^{k-1}(n-(r^{1/k}e^{i\pi(\mp 1+4m)/(2k)}-a)).$$

In this last equation and in the remainder of this example, both r and $r^{1/k}$ are taken real and positive. Using note 5.1 and theorem 4.1 we get

$$S = \lim_{r\to 0+}\left\{\frac{1}{2ir}\log\left[\prod_{m=0}^{k-1}\frac{\Gamma(1-(r^{1/k}e^{i\pi(1+4m)/(2k)}-a))}{\Gamma(1-(r^{1/k}e^{i\pi(-1+4m)/(2k)}-a))}\right]\right\}.$$

We calculate the limit

$$\begin{aligned}
S &= \lim_{r\to 0+}\left\{\sum_{m=0}^{k-1}\frac{1}{2ir}\log\left(\frac{\Gamma(1-(r^{1/k}e^{i\pi(1+4m)/(2k)}-a))}{\Gamma(1-(r^{1/k}e^{i\pi(-1+4m)/(2k)}-a))}\right)\right\} \\
&= \sum_{m=0}^{k-1}\lim_{r\to 0+}\frac{1}{2ir^k}\left[\begin{array}{l}\log(\Gamma(1-re^{i\pi(1+4m)/(2k)}+a)) \\ \quad -\log(\Gamma(1-re^{i\pi(-1+4m)/(2k)}+a))\end{array}\right] \\
&= \sum_{m=0}^{k-1}\lim_{r\to 0+}\frac{1}{k!2i}\left[\begin{array}{l}\left(-e^{i\pi(1+4m)/(2k)}\right)^k\psi_{k-1}(1-re^{i\pi(1+4m)/(2k)}+a) \\ \quad -\left(-e^{i\pi(-1+4m)/(2k)}\right)^k\psi_{k-1}(1-re^{i\pi(-1+4m)/(2k)}+a)\end{array}\right],
\end{aligned}$$

where interchanging the limit and the sum will be justified when we show that the limits in the last line exists and where we used l'Hôpital's rule k times in the last step together with the notation

$$\psi_l(z) := \left(\frac{d}{dz}\right)^l\left(\frac{\Gamma'(z)}{\Gamma(z)}\right),\ l\in\mathbb{Z}_{\geq 0}.$$

By lemma 4.1, these functions are continuous for $z\in\mathbb{C}\backslash\{0,-1,-2,...\}$ (even analytic), so

$$\begin{aligned}
S &= \frac{(-1)^k\psi_{k-1}(1+a)}{k!2i}\sum_{m=0}^{k-1}\left[e^{i\pi(1+4m)/2}-e^{i\pi(-1+4m)/2}\right] \\
&= \frac{(-1)^k\psi_{k-1}(1+a)}{k!2i}\sum_{m=0}^{k-1}2i \\
&= \frac{(-1)^k\psi_{k-1}(1+a)}{(k-1)!} = \sum_{n=1}^{\infty}\frac{1}{(n+a)^k} \qquad (5.3)
\end{aligned}$$

Note 5.2. *A nice consequence of (5.3) is the Taylor series*

$$\log\Gamma(1+z) = \sum_{k=1}^{\infty}\frac{\psi_{k-1}(1)}{k!}z^k = -\gamma z + \sum_{k=2}^{\infty}\frac{(-1)^k\zeta(k)}{k}z^k,$$

where we used that $\psi_0(1) = -\gamma$, which can be proved by logarithmic differentiation of Weierstrass's product expansion.

Example 5.3. *In the previous example we have seen that for $k \in \mathbb{N}_{>1}$,*

$$\prod_{n=1}^{\infty} \left\{ \frac{(n+a)^k + ir}{(n+a)^k - ir} \right\} = \prod_{m=0}^{k-1} \left\{ \frac{\Gamma(1 - (r^{1/k} e^{i\pi(1+4m)/(2k)} - a))}{\Gamma(1 - (r^{1/k} e^{i\pi(-1+4m)/(2k)} - a))} \right\}.$$

Taking $r = 1$ and using (5.1) we find

$$e^{2i \sum_{n=1}^{\infty} \arctan\left(1/(n+a)^k\right)} = \prod_{m=0}^{k-1} \left\{ \frac{\Gamma(1 - (e^{i\pi(1+4m)/(2k)} - a))}{\Gamma(1 - (e^{i\pi(-1+4m)/(2k)} - a))} \right\}.$$

For the remainder we assume k to be even and a to be zero, so we have

$$e^{2i \sum_{n=1}^{\infty} \arctan\left(1/n^k\right)} = \prod_{m=0}^{k-1} \left\{ \frac{\Gamma(1 - e^{i\pi(1+4m)/(2k)})}{\Gamma(1 - e^{i\pi(-1+4m)/(2k)})} \right\}.$$

It can be seen that when we replace each element in the set $\{e^{i\pi(1+4m)/(2k)} : m = 0, 1, ..., k-1\}$ by its complex conjugate, we get the set $\{e^{i\pi(-1+4m)/(2k)} : m = 0, 1, ..., k-1\}$ Also, when we replace each element in the set $\{e^{i\pi(1+4m)/(2k)} : m = 0, 1, ..., (2k-4)/4\}$ by its negative, we get the set $\{e^{i\pi(1+4m)/(2k)} : m = 2k/4, 1 + (2k/4), ..., k-1\}$. These observations entail that

$$e^{2i \sum_{n=1}^{\infty} \arctan\left(1/n^k\right)} = \prod_{m=0}^{(2k-4)/4} \left\{ \frac{\Gamma(1 - e^{i\pi(1+4m)/(2k)})\Gamma(1 + e^{i\pi(1+4m)/(2k)})}{\Gamma(1 - e^{-i\pi(1+4m)/(2k)})\Gamma(1 + e^{-i\pi(1+4m)/(2k)})} \right\},$$

so that with $b = -\pi \cos(\pi(1+4m)/(2k))$ and $x = \pi \sin(\pi(1+4m)/(2k))$ in (5.2), this equals

$$\prod_{m=0}^{(2k-4)/4} e^{2i\left\{\arctan\left(\tanh\left[\pi \sin\left(\pi \frac{1+4m}{2k}\right)\right] \cot\left[-\pi \cos\left(\pi \frac{1+4m}{2k}\right)\right]\right) - \arctan\left(\frac{-\sin(\pi(1+4m)/(2k))}{\cos(\pi(1+4m)/(2k))}\right)\right\}}.$$

After taking logarithms we find, for some $\beta_k \in \mathbb{Z}$, the closed-form expression (in the sense of definition 1.1)

$$\beta_k \pi + \sum_{n=1}^{\infty} \arctan\left(\frac{1}{n^k}\right) =$$
$$\sum_{m=0}^{(2k-4)/4} \left\{ \begin{array}{l} -\arctan\left(\tanh\left[\pi \sin\left(\pi \dfrac{1+4m}{2k}\right)\right] \cot\left[\pi \cos\left(\pi \dfrac{1+4m}{2k}\right)\right]\right) \\ +\arctan\left(\tan\left(\pi(1+4m)/(2k)\right)\right) \end{array} \right\},$$

and numerical evidence strongly suggests that β_k is always zero.

Remark 5.1. *This last formula appears to be new. Mathematica 7 gives no closed-form expression, except when k=2, and the author has been unable to find this expression in the literature. The case $k = 2$ can be found in [9]. Notice that the above method, where we use (5.2) to eliminate the Gamma function, heavily exploits the assumption that k is even. The analogies with $\zeta(k)$ are remarkable, which also behaves very different depending on whether k is even or odd. Furthermore, we can evaluate $\zeta(k)$ when k is even using the evaluation (3.6) (corollary 3.1) and we found the above expression using a similar looking evaluation (except for the symbol 'arctan'), appearing in example 5.1.*

Example 5.4. *Let us now turn to the general series $S := \sum_{n=1}^{\infty} n^q (g(n))^{-1}$, where $g(n) := (n-\alpha_1)(n-\alpha_2)...(n-\alpha_k)$, $k \in \mathbb{N}_{>1}$, $q \in \{0, 1, ..., k-2\}$ and $\alpha_1, \alpha_2, ..., \alpha_k \in \mathbb{R}\backslash\mathbb{N}$ are fixed and all different. According to theorem 5.1*

$$S = \lim_{r\to 0} \left\{ \frac{1}{2ir} \log \left[\prod_{n=1}^{\infty} \frac{(n-\alpha_1)(n-\alpha_2)...(n-\alpha_k) + irn^q}{(n-\alpha_1)(n-\alpha_2)...(n-\alpha_k) - irn^q} \right] \right\}.$$

The theory of algebraic functions (see e.g. [3]) tells us that there exist, for $j = 1, 2, \ldots, k$, sequences of complex numbers $\{c_{j,m}\}_{m=1}^{\infty}$ such that near $r = 0$,

$$(n-\alpha_1)(n-\alpha_2)...(n-\alpha_k) + irn^q =$$
$$\left(n - \alpha_1 - \sum_{m=1}^{\infty} c_{1,m} r^m\right) \left(n - \alpha_2 - \sum_{m=1}^{\infty} c_{2,m} r^m\right) ... \left(n - \alpha_k - \sum_{m=1}^{\infty} c_{k,m} r^m\right),$$

where each of the power series converges near $r = 0$. Similarly, for $j = 1, 2, \ldots, k$, there exist sequences of complex numbers $\{d_{j,m}\}_{m=1}^{\infty}$ such that near $r = 0$

$$(n-\alpha_1)(n-\alpha_2)...(n-\alpha_k) - irn^q =$$
$$\left(n - \alpha_1 - \sum_{m=1}^{\infty} d_{1,m} r^m\right) \left(n - \alpha_2 - \sum_{m=1}^{\infty} d_{2,m} r^m\right) ... \left(n - \alpha_k - \sum_{m=1}^{\infty} d_{k,m} r^m\right),$$

where each of the power series converges near $r = 0$. So by note 5.1 and theorem 4.1 we have that near $r = 0$,

$$\prod_{n=1}^{\infty} \frac{(n-\alpha_1)(n-\alpha_2)...(n-\alpha_k) + irn^q}{(n-\alpha_1)(n-\alpha_2)...(n-\alpha_k) - irn^q} = \prod_{j=1}^{k} \left\{ \frac{\Gamma(1 - \alpha_j - \sum_{m=1}^{\infty} d_{j,m} r^m)}{\Gamma(1 - \alpha_j - \sum_{m=1}^{\infty} c_{j,m} r^m)} \right\},$$

and thus we get

$$S = \lim_{r\to 0}\left\{\frac{1}{2ir}\log\left[\prod_{j=1}^{k}\frac{\Gamma(1-\alpha_j-\sum_{m=1}^{\infty}d_{j,m}r^m)}{\Gamma(1-\alpha_j-\sum_{m=1}^{\infty}c_{j,m}r^m)}\right]\right\}$$
$$= \sum_{j=1}^{k}\lim_{r\to 0}\frac{1}{2ir}\log\left[\frac{\Gamma(1-\alpha_j-\sum_{m=1}^{\infty}d_{j,m}r^m)}{\Gamma(1-\alpha_j-\sum_{m=1}^{\infty}c_{j,m}r^m)}\right]$$
$$= \sum_{j=1}^{k}\left\{\frac{(-d_{j,1})\,\Gamma'(1-\alpha_j)-(-c_{j,1})\,\Gamma'(1-\alpha_j)}{2i\Gamma(1-\alpha_j)}\right\},$$

where we used l'Hôpital's rule in the last step. It turns out that all of the coefficients $c_{j,1}, d_{j,1}$ can be determined explicitly in general, as we show now. In the following, the symbol $O(r^2)$ will be used several times to denote a power series in r with only terms of degree two and higher, which converges near $r=0$. This same symbol is used for different power series. With induction one can see that for $p\in\mathbb{N}$,

$$\left(\alpha_j + c_{j,1}r + \sum_{m=2}^{\infty}c_{j,m}r^m\right)^p = (\alpha_j^p + pc_{j,1}\alpha_j^{p-1}r) + O(r^2),$$

Writing $g(n)=\sum_{p=0}^{k}a_pn^p, a_p\in\mathbb{C}$, we have near $r=0$,

$$0 = ir\left(\alpha_j + c_{j,1}r + \sum_{m=2}^{\infty}c_{j,m}r^m\right)^q + \sum_{p=0}^{k}a_p\left(\alpha_j + c_{j,1}r + \sum_{m=2}^{\infty}c_{j,m}r^m\right)^p$$
$$= ir\left((\alpha_j^q + qc_{j,1}\alpha_j^{q-1}r) + O(r^2)\right) + \sum_{p=0}^{k}a_p\left((\alpha_j^p + pc_{j,1}\alpha_j^{p-1}r) + O(r^2)\right)$$
$$= ir\alpha_j^q + g(\alpha_j) + c_{j,1}rg'(\alpha_j) + O(r^2),$$

and by the uniqueness of power series expansions we must have $0 = i\alpha_j^q + c_{j,1}g'(\alpha_j)$ or $c_{j,1} = (-i\alpha_j^q)/g'(\alpha_j)$, which is well defined since we assumed that $g(n)$ has no multiple zeros. Similarly, $d_{j,1} = (i\alpha_j^q)/g'(\alpha_j)$. Together with the notation introduced in the previous example we have found that

$$\sum_{n=1}^{\infty}\frac{n^q}{g(n)} = \sum_{j=1}^{k}\left\{\frac{\left(-(i\alpha_j^q)/g'(\alpha_j)\right)\psi_0(1-\alpha_j) - \left((i\alpha_j^q)/g'(\alpha_j)\right)\psi_0(1-\alpha_j)}{2i}\right\}$$
$$= -\sum_{j=1}^{k}\alpha_j^q\frac{\psi_0(1-\alpha_j)}{g'(\alpha_j)},$$

so that when $h(n)$ is a polynomial of degree at most q with coefficients in $\mathbb{C}$,

$$\sum_{n=1}^{\infty}\left\{\frac{h(n)}{(n-\alpha_1)(n-\alpha_2)...(n-\alpha_k)}\right\} = -\sum_{j=1}^{k}h(\alpha_j)\frac{\psi_0(1-\alpha_j)}{g'(\alpha_j)} \tag{5.4}$$

Note 5.3. *When $g(n)$ has multiple roots we don't have ordinary power series expressions for the zeros like above, but then we have fractional power series (Newton–Puiseux expansions, see [3]), which can also be used to achieve results. Neither time nor place permit further investigation of this. Furthermore, in this case partial fraction decomposition together with (5.3) and (5.4) also does the job of finding similar expressions.*

6 Appendix

Lemma 6.1. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a sequence of complex numbers. The infinite product* $\prod_{n=1}^{\infty}(1+a_n)$ *converges absolutely if and only if* $\sum_{n=1}^{\infty} a_n$ *converges absolutely. If the infinite product converges absolutely, then it converges.*

Proof. See [8], pages 258-260. □

Theorem 6.1. *Let* $f_j : U \to \mathbb{C},\ j \in \mathbb{N}$, *be a sequence of analytic functions on an open set* U *in* $\mathbb{C}$. *Suppose that there is a function* $f : U \to \mathbb{C}$ *such that, for each compact subset* E *of* U, *the sequence* $f_j|_E$ *converges uniformly to* $f|_E$. *Then* f *is analytic on* U.

Proof. See [8], pages 88-89. □

If Z is a subset of the complex numbers then $z \in \mathbb{C}$ is said to be an accumulation point of Z if there is a sequence $\{z_n\}_{n=1}^{\infty} \subseteq Z \backslash \{z\}$ with $\lim_{n\to\infty} z_n = z$.

Theorem 6.2. *Let* $U \subseteq \mathbb{C}$ *be a connected open set and let* f, g *be analytic functions on* U. *If* $\{z \in U : f(z) = g(z)\}$ *has an accumulation point in* U, *then* $f = g$.

Proof. See [8], page 92. □

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