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Glimpses of a solution to the measurement problem in quantum theory

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Contents

1 Introduction

Since the introduction of quantum mechanics, countless many have racked their brains over the interpretation of the theory. An answer to the measurement problem is part of the interpretation and consequently the problem has frequently been discussed. Some however do not see a measurement problem at all and hence the *measurement problem problem* was born: Does the measurement problem exist at all? Belavkin acknowledges a problem and has described a way to solve it in a difficult paper in 2007 (see [1]). His theory is not widely known, probably due to the complexity of his paper. Therefore it seems important to explain Belavkin's framework in the most simple way and applied to the most simple examples to help it become more well-known and understood.

Goal of this thesis is to formulate Belavkin's framework in an easy way. In addition we try to understand several much used models in the quantum theory using his framework. This thesis is not an attempt to reconstruct or simplify the whole theory of Belavkin, only the discrete time and not the continuous time part. At the same time this is not a literature review of other ways to solve the measurement problem.

The thesis has the following structure. The first section gives our view on the measurement problem. That gives directly our take on the measurement problem problem. Belavkin's framework as we will use it in this thesis is constructed in the second section; it is used then in section 3 to give a simple but important application. The main part of the thesis is section 4 which shows that the master equation approach, often used in quantum optics, can be embedded in Belavkin's theory. The constructive way we do this gives us machinery to give a Belavkin-type solution to the problem of modeling a Geiger counter.

2 The measurement problem

In this section we will formulate our version of the measurement problem. It boils down to the question how the wave function collapse can be explained. The treatment of the measurement problem will follow that of S.A. Sabbadini (see [6]). We will illustrate the problem with Schrödinger's cat.

Suppose we have a system S and a system M, the measuring device. M starts always in a certain neutral state $|\Phi_0\rangle$. Let U be the time evolution operator of the system $S \otimes M$ during the interaction where the state of the measuring device becomes $|\Phi_n\rangle$ if S started in $|\phi_n\rangle$. Suppose that $|\phi_n\rangle$ and $|\Phi_n\rangle$ are orthogonal sets. Assuming that this interaction creates no disturbance on the system S, we find

$$U |\phi_n\rangle \otimes |\Phi_0\rangle = |\phi_n\rangle \otimes |\Phi_n\rangle . \tag{1}$$

Therefore, by the linearity of U, if S starts in a superposition of states $|\phi_n\rangle$ we get

$$U \sum_n c_n |\phi_n\rangle \otimes |\Phi_0\rangle = \sum_n c_n |\phi_n\rangle \otimes |\Phi_n\rangle , \tag{2}$$

i.e. the superposition of states corresponding to the possible results. Let ρ be the density matrix corresponding to the right hand side. This represents something completely different from the result of this measurement according to the usual formulation of quantum mechanics, according to which the density matrix of the result is

$$\hat{\rho} = \sum_n |c_n|^2 \left(|\phi_n\rangle \langle\phi_n| \otimes | \Phi_n\rangle \langle\Phi_n|\right), \tag{3}$$

i.e. with probability $|c_n|^2$ the system $S \otimes M$ is in state $|\phi_n\rangle \otimes |\Phi_n\rangle$.

The measurement problem has to do with the compatibility of ρ and $\hat{\rho}$. The first comes from the linearity of U and the second from the measurement axiom of quantum theory. The predictions that ρ and $\hat{\rho}$ make of other (later) measurements are actually the same when the measurements are only on the system S or only on M. On the other hand, if the measurement depends on a correlation between S and M, ρ and $\hat{\rho}$ will generally give different predictions. The measurement problem can now be formulated in two ways:

1. The problem to describe how ρ is transformed into $\hat{\rho}$, which is the collapse of the wave function.
2. The problem to show that ρ and $\hat{\rho}$ are in some sense equivalent or indistinguishable.

Some contemporary physicists see no measurement problem, neither of type 1 nor of type 2. The reasoning is that the current quantum theory 'works'. It predicts correctly the results found in laboratories all over the world. However they have to make a Heisenberg cut, a division between the quantum world with the system to be measured, and the classical world with (at least a part of) the measuring device. They do this, even though physicists know very well that the measuring device is made of particles all subject to quantum laws. Therefore we recognize a measurement problem and our attempt at a solution is of type 2.

2.1 Schrödinger's cat

In this section the measurement problem is illustrated with Schrödinger's cat. Schrödinger's infamous cat is placed in a perfectly isolated box with a device which kills the cat upon the decay of an excited atom. The question is what happens if the atom is in a superposition of states. Let the atom be a two-level system initial in the excited state $|1\rangle$. The initial state of the joint system of the atom and the cat is $|\text{alive}, 1\rangle$. Suppose that after one half-life of the atom the state is the superposition $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, where $|0\rangle$ is the ground state. The cat is killed upon the decay of the atom. So the joint system is after one half-life of the atom in the state $\frac{|\text{dead},0\rangle+|\text{alive},1\rangle}{\sqrt{2}}$. This indicates that the cat is simultaneously alive and dead.

This is really different from the result we get if we take the observation of the decay as a measurement. In that case the cat is dead with probability 1/2 and alive with probability 1/2. The measurement problem here is how these two views on the state of the cat are compatible.

3 Belavkin's framework

In this section we will describe Belavkin's framework, which he presents as an attempt to resolve the measurement problem. We follow the approach of Gill (see [3]). In this framework the measurement and the collapse of the wave function result from a deterministic, unitary evolution. But this evolution has to take place in a mixed classical-quantum system. The classicality corresponds to the restriction of observables to a commuting subset of the bounded operators, which will be a later defined von Neumann algebra. This means that not all quantum superpositions can be detected.

To continue our description of Belavkin's framework we need to define what von Neumann algebras are. Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the set of all bounded operators on $\mathcal{H}$.

Definition 1 *$A \subseteq B(\mathcal{H})$ is called a* *-subalgebra *if it is closed under addition, complex scalar multiplication, composition of operators and the involution * (adjoint).*

Definition 2 *Let $A \subseteq B(\mathcal{H})$. The* commutant *$A'$ is defined by*

$$A' = \{x \in B(\mathcal{H}) : \forall y \in B(\mathcal{H}) \ xy = yx\} . \tag{4}$$

In natural language this means that A' is the collection of all bounded operators on $\mathcal{H}$ which commute with every element of A. Now everything is set for the definition of von Neumann algebras.

Definition 3 *Let $A \subseteq B(\mathcal{H})$ be a *-subalgebra. A is called a* von Neumann algebra *if $A = A'' := (A')'$.*

Next two examples of von Neumann algebras will be given that play an important role later on. We set for both examples $\mathcal{H} = \mathbb{C}^2$. Let M_2 be the set of all 2×2 matrices. It is obvious that M_2 is closed under addition, complex scalar multiplication, composition of operators and the involution *. So M_2 is a *-subalgebra. In order to get M_2', we need to know what bounded operators commute with every element of M_2. The only operators that meet this requirement are all complex multiples of the identity operator $\mathbb{I}_2$. Now we get $M_2'' = \{\mathbb{C}\mathbb{I}_2\}' = M_2$, because everything commutes with the identity operator. So M_2 is our first example of a von Neumann algebra.

Let C_2 be the set of all diagonal 2×2 matrices. Again it is clear that C_2 is a *-subalgebra. The commutant C_2' is C_2 self. Every diagonal matrix commutes with all diagonal matrices and no other matrix does this. This proves that C_2 is a von Neumann algebra, because $C_2'' = C_2' = C_2$.

We continue with the description of Belavkin's framework. $\mathcal{H}$ is again a general Hilbert space. States on $\mathcal{H}$ can be represented in the normal way with a density matrix. Now consider a unitary operator U on $\mathcal{H}$. This U gives the time dynamics of our system. A density matrix ρ representing a state on $\mathcal{H}$ is mapped to $U\rho U^*$ in one time step.

Now we deviate from the usual description of the quantum mechanics. Not all the bounded operators $B(\mathcal{H})$ are observable. We have two von Neumann algebras A and C, with $A' = C$, C commutative and *compatible* with U, which we define in Definition 4. Observe that $C' = A$ holds too. We call C the set of *beables*, these are physical quantities which can be given definite values. Here definite means that all the quantities can be given a value at the same time. C has to be commutative to make this possible. We see C as a classical world within the quantum universe given by $\mathcal{H}$ and time dynamics U. We call A the set of *predictables* and A is the commutant of C. These operators have definite probability relative to C and predict the future of the beables.

Definition 4 *The algebras C and A are called* compatible *with U if*

$$U^*AU \subseteq A \tag{5}$$

and

$$UCU^* \subseteq C. \tag{6}$$

Actually these are not two requirements, but only one, because requirement (5) holds if and only if requirement (6) holds.

Theorem 1 *Suppose A and C are both von Neumann algebras with $A' = C$ commutative. Then $U^*AU \subseteq A \Longleftrightarrow UCU^* \subseteq C$.*

Proof. $U^*AU \subseteq A$ holds iff $(U^*AU)' \supseteq A'$. Because $(U^*AU)' = U^*CU$ we get that $U^*AU \subseteq A$ holds iff $U^*CU \supseteq C$. To prove the theorem we only have to observe that $U^*CU \supseteq C$ holds iff $C \supseteq UCU^*$. ■

In the Heisenberg picture, where all states are time-independent and the operators are time-dependent, an operator B is mapped in one (forward) time step to U^*BU. Equation (5) tells us that all predictables are in the future also predictable. Likewise equation (6) tells us that all beables were in the past also beable. Now Theorem 1 gives that one has to assume only one of these properties to get the other for free.

A state on $\mathcal{H}$ is in Belavkin's framework a mapping from the predictables A to their expectation values. It is still true that a state can be represented by a density matrix, however different density matrices can represent one state. They are then indistinguishable from one another. One can compare this to the following result in the usual description of the quantum world. Different ensembles of quantum states can result in the same density matrix and are thereby indistinguishable.

In this section the discrete time part of Belavkin's framework has been described. A classical world, subject to the same laws of quantum physics as everything else, appears by the choice of beables. Because the beables are compatible with the time dynamics given by U, the past beables are beable and likewise the future predictables are predictable.

4 A simple example of a Belavkin-type solution

In this section we construct a simple example with only a shift on an infinite chain of two-level systems. The goal is to show that stochastic behaviour can emerge from unitary evolution with a suitable choice of the predictables. The Hilbert space $\mathcal{H}$ we consider consists of a countable infinite collection of copies of $\mathbb{C}^2$. The copies are indexed by $n \in \mathbb{Z}$. We take $|0\rangle$ and $|1\rangle$ as two orthonormal vectors of each $\mathbb{C}^2$. Unfortunately the tensor product (as Hilbert spaces) of an infinite number of $\mathbb{C}^2$ is not separable. Therefore we only consider the subset of $\mathcal{H}$ which is the closure of the span of the vectors $|x_n : n \in \mathbb{Z}\rangle$ with only finitely many coordinates x_n equal to $|1\rangle$. Let $\mathcal{H}_S$ denote this subset. We describe $\mathcal{H}_S$ now as the product of two spaces. $\mathcal{H}_S = \mathcal{H}_C \otimes \mathcal{H}_Q$ with $|x_n : n \in \mathbb{Z}\rangle_S = |x_n : n \in \mathbb{Z}_{<0}\rangle_C \otimes |x_n : n \in \mathbb{Z}_{\geq 0}\rangle_Q$.

The unitary evolution U in our example is the left shift defined by $U|x_n : n \in \mathbb{Z}\rangle = |x_{n-1} : n \in \mathbb{Z}\rangle$. The predictables of $\mathcal{H}_S$ are generated by all bounded operators on $\mathcal{H}_Q$, the quantum part of the system, and only classical operators on $\mathcal{H}_C$, i.e. diagonal in the specified basis. Let $\mathbb{C}\mathbb{I}_2^n$ be the algebra of complex multiples of the 2×2 identity matrix on the n^{th} two-level system. Let M_2^n and C_2^n be the algebras of all 2×2 matrices and all diagonal 2×2 matrices respectively on the n^{th} two-level system. Then the predictables on the joint system will be:

$$A = \otimes_{n \in \mathbb{Z}_{<0}} C_2^n \otimes_{n \in \mathbb{Z}_{\geq 0}} M_2^n. \tag{7}$$

The set of beables C of the joint system is the commutant of the algebra A, A'. This can be calculated using the fact (see [5]) that $(D \otimes E)' = D' \otimes E'$. The commutant of M_2^n is $\mathbb{C}\mathbb{I}_2^n$ and that of C_2^n is C_2^n. So we find

$$C = \otimes_{n \in \mathbb{Z}_{<0}} C_2^n \otimes_{n \in \mathbb{Z}_{\geq 0}} \mathbb{I}_2^n. \tag{8}$$

In order to check if this example is compatible with U we have to check if equations (5) and equation (6) hold. By Theorem 1 only one has to be verified. We calculate

$U^*AU = U^*\left(\otimes_{n\in\mathbb{Z}_{<0}} C_2^n \otimes_{n\in\mathbb{Z}_{\geq 0}} M_2^n\right) U = \otimes_{n\in\mathbb{Z}_{<1}} C_2^n \otimes_{n\in\mathbb{Z}_{\geq 1}} M_2^n$. Because $C_2^0 \subseteq M_2^0$ equation (5) holds. The classical observable at position $n=-1$ is shifted to the right into a larger quantum space.

Next we are interested in the states of the system. Let B_k (with $k \in \mathbb{N}$) be the set of positions of the ones in the binary representation of k, where the last position is 0. We define the state $|k,m\rangle$ as the tensor product of $|1\rangle$ at the positions $-B_k - 1$ and B_m and $|0\rangle$ at all other positions. For instance $|3,2\rangle$ has $|1\rangle$ at positions -2,-1 and 1. The general states are of the type $|k, c_0c_1c_2\ldots\rangle = c_0\,|k,0\rangle + c_1\,|k,1\rangle + \ldots$ with $\sum_i |c_i|^2 = 1$. They are of this type by virtue of the special form of the predictables A. Let's see what the left shift does to a state. It is easy to see that $U\,|k,m\rangle = \left|2k + \mathrm{par}(m), \lfloor \frac{m}{2} \rfloor\right\rangle$, where $\mathrm{par}(m)$ is the parity of m. The left shift on $|k, c_0c_1c_2\ldots\rangle$ gives a quantum superposition of states at postion $n=-1$. However the predictables at position $n=-1$ are the diagonal matrices. So at position $n=-1$ the state $\alpha\,|0\rangle + \beta\,|1\rangle$ is indistinguishable from the ensemble: $|0\rangle$ with probability $|\alpha|^2$ and $|1\rangle$ with probability $|\beta|^2$. The left shift on $|k, c_0c_1c_2\ldots\rangle$ results in

$$\begin{cases} |2k, c_0c_2c_4\ldots\rangle & \text{with probability } \displaystyle\sum_{i\ even} |c_i|^2 \\ |2k+1, c_1c_3c_5\ldots\rangle & \text{with probability } \displaystyle\sum_{i\ odd} |c_i|^2 \end{cases}. \tag{9}$$

Actually the normalized states have to be used.

Why did we choose this example of a Belavkin-type solution? Not only because it will be used in Section 5, but also because we believe this to be an essential illustration of Belavkin's framework. Equation (9) shows that stochastic behaviour can result from unitary evolution with a suitable choice of predictables. The future of the classical part is stochastic and the past of the classical part is deterministic. These properties seem typical of Belavkin-type solutions. The Hilbert space $\mathcal{H}_S$ has to be infinite-dimensional to let the quantum part influence the classical part of the system (see again [3]).

5 Master equations

In this section we will show that the master equation approach can be described within Belavkin's framework. Master equations are used in the field of quantum optics to describe non-unitary evolution of a density matrix. This evolution will still be completely positive and trace preserving so the result is again a density matrix. The most used form is the Lindblad form (see [2]), which depicts a quantum system $\mathcal{H}_\rho$ coupled to its environment:

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + \sum_j \left(2L_j\rho L_j^* - L_j^*L_j\rho - \rho L_j^*L_j\right) \tag{10}$$

where ρ is the density matrix of a Hilbert space $\mathcal{H}_\rho$, and H is the Hamiltonian of $\mathcal{H}_\rho$, which is a Hermitian operator representing the unitary part of the evolution, and L_j are the Lindblad operators on $\mathcal{H}_\rho$, resulting from the coupling of the system and environment. One can think of L_j as coming from the coupling of the system to a j^{th} channel.

Note that the master equation is a continuous time description of $\mathcal{H}_\rho$, because it is a differential equation. However we only use discrete time models. Therefore we will approximate the master equation using discrete time Belavkin-type models. To illustrate that master equations can be described within Belavkin's framework, we make the following assumption.

Assumption 1 *$H = 0$ and there is only one Lindblad operator L.*

This is a real restriction of the master equation, but it is easy to produce the general master equation if we can produce master equations fulfilling Assumption 1. H is a Hermitian operator, hence we only have to let H work on the system $\mathcal{H}_\rho$ to account for it. Multiple Lindblad operators can easily be modelled by combining all the models with only one of these operators. Using Assumption 1 the master equation (10) becomes

$$\frac{d\rho}{dt} = 2L\rho L^* - L^*L\rho - \rho L^*L \tag{11}$$

In this section we will use the following notation and definitions, because we will work with two-level systems:

Notation 1 $|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and* $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Definition 5 $\sigma_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *and* $\sigma_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Combining Notation 1 and Definition 5 we find that σ_+ is the raising or creation operator, i.e. $\sigma_+ |0\rangle = |1\rangle$ and $\sigma_+ |1\rangle = 0$. Naturally, we get that σ_- is the lowering or annihilation operator, i.e. $\sigma_- |0\rangle = 0$ and $\sigma_+ |1\rangle = |0\rangle$.

There are several more interesting properties of σ_+ and σ_- we will use in this section:

$$\sigma_+^* = \sigma_- \tag{12}$$

hence, $\langle 0| \sigma_+ = 0$. Additionally the following two equations hold:

$$(\sigma_+)^2 = 0 = (\sigma_-)^2, \tag{13}$$

$$\sigma_+\sigma_- = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{14}$$

This is the projector on the space spanned by $|1\rangle$. In the same way we get that $\sigma_-\sigma_+$ is the projector on the space spanned by $|0\rangle$.

5.1 The model

In this subsection we will state the model for the simple master equation (11). The chain of two-level systems as described in Section 4 is used. Let now all two-level systems start in $|0\rangle$. We make a coupling between the system ρ and the two-level system at the component $n = 0$ of the chain. In one time step a unitary operator U will dictate what ρ and this two-level system do. Then the left shift, which is unitary, is applied to the chain. By the choice of predictables in Section 4 only the diagonal elements of the density matrix of the two-level system at the component $n = -1$ of the chain are detectable. After the left shift ρ is coupled to this component and hence a very particular evolution of ρ has occurred. The last part of the modelling is to take the time step to zero.

The U, which we will show is the right choice, is:

$$U = e^{i\left(\sqrt{2}L\otimes\sigma_+\sqrt{\delta}+\sqrt{2}L^*\otimes\sigma_-\sqrt{\delta}\right)}. \tag{15}$$

δ is here the duration of one time step. U is clearly an operator on the tensor product of ρ and a two-level system.

Lemma 1 *U is a unitary operator.*

Proof. Using equation (12), observe that:

$$\left(\sqrt{2}L \otimes \sigma_+ \sqrt{\delta} + \sqrt{2}L^* \otimes \sigma_- \sqrt{\delta}\right)^* = \left(\sqrt{2}L^* \otimes \sigma_- \sqrt{\delta} + \sqrt{2}L \otimes \sigma_+ \sqrt{\delta}\right) \tag{16}$$

Hence this part is Hermitian. So U is the exponent of i times a Hermitian operator. This gives that U is unitary. ■

We take the Taylor expansion of U. In the end the duration of the time step, δ, goes to zero. Therefore all terms with $\delta^{3/2}$ or δ^2 or higher order of δ are significantly smaller than the other terms. Denote this approximation of U with U':

$$\begin{aligned} U' &= \mathrm{I} + i\left(\sqrt{2}L \otimes \sigma_+ \sqrt{\delta} + \sqrt{2}L^* \otimes \sigma_- \sqrt{\delta}\right) + i^2/2\left(\sqrt{2}L \otimes \sigma_+ \sqrt{\delta} + \sqrt{2}L^* \otimes \sigma_- \sqrt{\delta}\right)^2 \\ &= \mathrm{I} + i\left(\sqrt{2}L \otimes \sigma_+ \sqrt{\delta} + \sqrt{2}L^* \otimes \sigma_- \sqrt{\delta}\right) - LL^* \otimes \sigma_+\sigma_-\delta - L^*L \otimes \sigma_-\sigma_+\delta, \end{aligned} \tag{17}$$

where we used equation (13). Because the left shift is used after applying U, the component $n = 0$ of the chain has *always* a $|0\rangle$ before U is applied. Hence we only have to calculate

$$U(\rho \otimes |0\rangle\langle 0|)U^* \approx U'(\rho \otimes |0\rangle\langle 0|)(U')^*. \tag{18}$$

Using equation (17) in equation (18) gives a large number of terms. Luckily quite a few are zero by using $\sigma_- |0\rangle = 0$ and $\langle 0| \sigma_+ = 0$:

$$\begin{gathered} U'(\rho \otimes |0\rangle\langle 0|)(U')^* = \\ \left(\mathrm{I} + i\sqrt{2}L \otimes \sigma_+ \sqrt{\delta} - L^*L \otimes \sigma_-\sigma_+\delta\right)(\rho \otimes |0\rangle\langle 0|)\left(\mathrm{I} + i\sqrt{2}L^* \otimes \sigma_- \sqrt{\delta} - LL^* \otimes \sigma_-\sigma_+\delta\right). \end{gathered} \tag{19}$$

The calculation can now easily be completed by remembering equations (13) and (14). Important is to keep only the terms with δ^0, $\delta^{1/2}$ or δ^1. The result is written blockwise, dependent on the two-level system at position $n = 0$:

$$\begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & \rho - \delta\left(L^*L\rho + \rho LL^*\right) & -i\sqrt{2}\rho L^* \sqrt{\delta} \\ |1\rangle & i\sqrt{2}L\rho\sqrt{\delta} & 2L\rho L^*\delta \end{array}. \tag{20}$$

Now the left shift is applied to the chain of two-level systems. This transports the component $n = 0$ to $n = -1$. By the choise of the predictables on the components with $n < 0$, only the diagonal elements can be detected. If the two-level system is measured and the outcome is forgotten, the two diagonal elements of equation (20) added give the state of $\rho(a+\delta)$. Here a denotes an arbitrary time point. We get

$$\rho(a+\delta) \approx \rho(a) + \delta\left(2L\rho(a)L^* - L^*L\rho(a) - \rho(a)L^*L\right) \tag{21}$$

hence,

$$\frac{\rho(a+\delta) - \rho(a)}{\delta} \approx 2L\rho(a)L^* - L^*L\rho(a) - \rho(a)L^*L. \tag{22}$$

Now taking the limit of equation (22) as δ goes to zero yields equation (11), the equation we are after.

In this section we have shown a way to embed master equations into Belavkin's framework. Using discrete time Belavkin-type models we approximated the master equation. Again only unitary operations are used, but by the choise of predictables the non-unitary evolution described by master equations is still modelled correctly. The great trick is to use the chain of two-level systems of Section 4.

6 Geiger counter, an application of the master equation

The Geiger counter is an apparatus detecting when an atom in an excited state decays. It gives a macroscopic signal on the detection of a radioactive emission. It is a basic measurement device in quantum mechanics. However none of the basic books on quantum physics give a model for it. We will give an Belavkin-type model for the Geiger counter using the machinery of the master equation developed in section (5). Suppose the atom starts in the initial state $\alpha |0\rangle + \beta |1\rangle$. The requirements on the model are the following:

- With probability $|\alpha|^2$ the atom never decays and no signal is ever emitted by the Geiger counter.
- With probability $|\beta|^2$ the Geiger counter emits a signal on a geometrically distributed random time.
- After the atom decays, the state of the atom is $|0\rangle$.

We will use the same notations as in section (5). It is known that the decay of an atom can be described by the master equation (11) with $L = \sigma_-$. Observe that the system is now also a two-level system, therefore σ_- is an operator on the system. First we calculate the density matrix of the atom in the initial state:

$$\rho = (\alpha |0\rangle + \beta |1\rangle)\left(\bar{\alpha} \langle 0| + \bar{\beta} \langle 1|\right) = \begin{bmatrix} |\alpha|^2 & \alpha\bar{\beta} \\ \bar{\alpha}\beta & |\beta|^2 \end{bmatrix} \tag{23}$$

The next step is to use the Belavkin-type solution of the modelling of master equations. This means we use the now familiar chain of two-level systems. And ρ is changed after one time step into equation (20) with $L = \sigma_-$:

$$\begin{array}{c|cc} & \langle 0| & \langle 1| \\ \hline |0\rangle & \rho - \delta\left(\sigma_-^*\sigma_-\rho + \rho\sigma_-\sigma_-^*\right) & -i\sqrt{2}\rho\sigma_-^*\sqrt{\delta} \\ |1\rangle & i\sqrt{2}\sigma_-\rho\sqrt{\delta} & 2\sigma_-\rho\sigma_-^*\delta \end{array} \tag{24}$$

In this table one sees the state of the atom dependent on the two-level system in the chain.

Now the two-level system is shifted to the place $n = -1$ in the chain and measured. Remember that only the diagonal elements can be measured. This makes it easy to calculate what happens if 0 or a 1 is measured.

6.1 Result of measurement is 1

If a one is measured, the atom should afterwards be in the state $|0\rangle$ (see the requirements). We will verify that this is the case. In equation (24) the not normalized state of the atom after a one is measured is given by $2\sigma_-\rho\sigma_-^*\delta$.

$$2\sigma_-\rho\sigma_-^*\delta = \begin{bmatrix} 2\delta\left|\alpha\right|^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{25}$$

Now we see that the probability of a one being measured is $2\delta\left|\beta\right|^2$. The normalized state of the atom, when this occurs, is $\left|0\right\rangle\left\langle 0\right|$. In natural language this means that after the atom decays, the atom will always be in his ground state and the Geiger teller will afterwards never give a signal.

6.2 Result of measurement is 0

We have seen that the probability of a one being measured is $2\delta\left|\beta\right|^2$. So we find directly that the probability of a zero being measured is $1-2\delta\left|\beta\right|^2$. In equation (24) the not normalized state of the atom after a one is measured is given by $\rho - \delta\left(\sigma_-^*\sigma_-\rho + \rho\sigma_-\sigma_-^*\right)$.

$$\rho - \delta\left(\sigma_-^*\sigma_-\rho + \rho\sigma_-\sigma_-^*\right) = \begin{bmatrix} \left|\alpha\right|^2 & \alpha\bar{\beta}\left(1-\delta\right) \\ \bar{\alpha}\beta\left(1-\delta\right) & \left|\beta\right|^2\left(1-2\delta\right) \end{bmatrix} \tag{26}$$

Remember that this is an approximation which only contains terms with δ^0, $\delta^{1/2}$ or δ^1. So we may add terms with δ^2:

$$\begin{bmatrix} \left|\alpha\right|^2 & \alpha\bar{\beta}\left(1-\delta\right) \\ \bar{\alpha}\beta\left(1-\delta\right) & \left|\beta\right|^2\left(1-2\delta+\delta^2\right) \end{bmatrix} \tag{27}$$

Now this matrix (it is not a density matrix, because the rank is not 1) is equal to $\left|\phi\right\rangle\left\langle\phi\right|$ with $\left|\phi\right\rangle = \alpha\left|0\right\rangle + \beta\left(1-\delta\right)\left|1\right\rangle$. This is the not normalized state of the atom after a 0 is measured. We see that more weight is put on $\left|0\right\rangle$ and less on $\left|1\right\rangle$.

Let us now concentrate on the probability of n measurements all resulting in a zero. We forget about the addition of the δ^2 term and take the original equation (26). Let p_n denote the probability that n times a zero is produced, given that $n-1$ times a zero is produced. Already calculated is that $p_1 = 1-2\delta\left|\beta\right|^2$. After a zero is measured and normalisation $\left|\beta\right|^2$ is replaced by $\frac{R\left|\beta\right|^2}{p_1}$. Therefore $p_2 = 1-2\delta\left(\frac{R\left|\beta\right|^2}{p_1}\right)$, where $R = 1-2\delta$. By iteration it is clear that:

$$p_n = 1 - \frac{2\delta R^{n-1}\left|\beta\right|^2}{\prod_{j=1}^{n-1} p_j} \tag{28}$$

Let a_n denote the probability that n times a zero is produced. Then we get $a_n = \prod_{j=1}^{n} p_j = \prod_{j=1}^{n-1} p_j - 2\delta R^{n-1}\left|\beta\right|^2$. With this observation a recursion formula is constructed, $a_n = a_{n-1} - 2\delta R^{n-1}\left|\beta\right|^2$. This recursion formula yields

$$a_n = 1 - \left|\beta\right|^2 + \left|\beta\right|^2 R^n \tag{29}$$

When taking the limit $n \to \infty$, we see that the probability of the Geiger teller never emitting a signal is $1-\left|\beta\right|^2 = \left|\alpha\right|^2$. The first requirement is fulfilled. The probability to have one signal of the Geiger teller in n steps is $1 - a_n = |\beta|^2 - |\beta|^2 R^n$. Again taking the limit $n \to \infty$, gives us that the probability of ever emitting a signal is $|\beta|^2$. Also it is clear that if it emits a signal the time of this signal is distributed with a geometric distribution, with 2δ as parameter. So all the requirements are fulfilled.

Lastly we will look at the continuous time limit, where δ goes to zero. The probability for one signal in time t is

$$\lim_{\delta \to 0} |\beta|^2 - |\beta|^2 (1 - 2\delta)^{\frac{t}{\delta}} = |\beta|^2 \left(1 - e^{-2t}\right) \tag{30}$$

So in the continuous time limit, the signal is delivered at a exponentially distributed random time, if it is going to come at all.

7 Conclusion

In this thesis we have tried to get Belavkin's theory more known and understood. Belavkin's theory is important, because it is a solution to the measurement problem as stated in section 1. The basic framework consists of a coupled classical-quantum system with a deterministic unitary evolution. Not all bounded operators can be observed, we consider beables and predictables. The beables give definite values to physical quantities and the predictables give definite probabilities to the beables. Important is the requirement that past beables are beable and future predictables also predictable. This is the most simple version of the framework

Next we focussed on examples of Belavkin-type solutions. With the first, an infinite chain of two-level systems, it became clear how this framework can construct measurements as in the usual quantum theory. It shows the way unitary evolution can result in stochastic behaviour. The same chain is used in section 4 to embed the master equation approach to quantum mechanics. The system fulfilling the master equation is coupled to the infinite chain and on this coupled system we find a unitary operator U. This U gives, in the continuous time limit i.e. when the duration of the time step goes to zero, exactly the result of the master equation. We showed this using only Taylor expansion and some linear algebra. Less general is the modelling of a Geiger counter as done in the last section. It uses quite heavily the machinery developed for embedding the master equations. All the requirements one realisticly could make on the model are fulfilled. In total we have shown several examples of Belavkin-type solutions.

More research is needed to understand the power of Belavkin's framework. Belavkin has shown a continuous time framework. One could then look at how to make specific continuous time examples. Furthermore it is important to find which other attempts at a solution to the measurement problem can be embedded in the framework, in the same way we have shown that the master equation approach is embedded. Specifically the works of Hepp (see [4]) seem ripe to be included.

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