A space of spaces

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Bachelor thesis
Leiden University, 17 June 2009

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1 Preface

In elementary functional analysis, one is taught that all normed vector spaces of given finite dimension over the reals or the complex numbers are isomorphic. One therefore could conclude that the theory of finite-dimensional vector spaces is trivial and not worth investigating. In this bachelor thesis, I will try to convince the reader that this is not the case.

For instance, when investigating general Banach spaces one comes upon the field of local theory. This research area studies finite-dimensional normed spaces and the relation between a Banach space and its finite-dimensional subspaces. We will delve into a small part of this theory by introducing the Banach-Mazur distance, a fundamental tool when studying finite-dimensional normed spaces. Methods developed in the study of this distance have helped answer several longstanding questions about Banach spaces, and are used in other areas such as operator theory and harmonic analysis [Tomczak-Jaegermann 1989, p. x,xi].

The Banach-Mazur distance defines a metric on a set of equivalence classes of normed spaces, thereby defining the 'space of spaces' from the title of this thesis. We can then ask ourselves several questions about this space, amongst which whether the space is bounded and compact. The first questions leads to the theorem of John (1948), which gives an upper bound for the diameter of this space. The problem of determining the quality of this upper bound was left open for some time, until it was solved in 1981 by Gluskin. His proof uses an interesting measure-theoretic approach which has since then been used in several other results. This theorem is the main piece of this thesis, and we will work out in more detail his short but very technical proof [Gluskin 1981].

In the paragraph following this introduction, we introduce the Banach-Mazur distance as a way of measuring how different two norms on isomorphic vector spaces are. We explore some of the properties of this distance. Using this distance, we can define a metric on the set of equivalence classes of n-dimensional ($n \in \mathbb{N}$) normed spaces over the reals, and in the third section we go on to explore some of the properties of this metric space. In the fourth paragraph, we introduce the theorem of Gluskin to which we alluded previously, and go on to prove it, using the results from two appendices. We conclude with a paragraph stating some additional results using the same technique as Gluskin's proof.

2 Banach-Mazur distance

In this section we introduce the Banach-Mazur distance. This notion was introduced in [Banach 1932], in collaboration with Mazur, as a quantity expressing how similar two norms on isomorphic vector spaces are. We will use a function of the Banach-Mazur distance to define a metric on a certain set of equivalence classes of normed spaces, thereby creating the space of spaces from the title of this thesis. Then we will go on to investigate some of the properties of this space.

But first let us introduce some notation. For vector spaces E and F, let $L(E, F)$ be the set of linear maps from E to F, and define $L(E) := L(E, E)$. For any normed space $(X, ||\cdot||)$ let $B(X) = \sup\{x \in X : ||x|| \leq 1\}$ be the closed unit ball. If $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ are two normed spaces, for any $T \in L(E, F)$ we can calculate the operator norm $||T|| = \sup\{||Tx||_F : x \in B(E)\}$. If T is invertible, we can do the same for $T^{-1} \in L(F, E)$. If E and F are finite-dimensional T and T^{-1} are automatically bounded, and we have $||T|| \cdot ||T^{-1}|| < \infty$.

We are now ready to introduce the classical definition of the Banach-Mazur distance:

Definition 2.1. Let $(E, ||\cdot||_E)$ and $(F, ||\cdot||_F)$ be normed vector spaces. We define the Banach-Mazur distance $d(E, F)$ between E and F by

$$d(E, F) = \inf\left\{||T|| \cdot ||T^{-1}|| : T \in L(E, F) \text{ invertible with } T^{-1} \text{ bounded}\right\}.$$

If E and F are not isomorphic as vector spaces, we define the Banach-Mazur distance to be ∞.

We immediately give an alternative, equivalent definition, one that gives a little more insight into the use of the Banach-Mazur distance as a measure of the degree to which norms differ.

Proposition 2.2. Let E and F be normed spaces with closed unit balls $B(E)$ respectively $B(F)$, and $c > 1$. Then we have $d(E, F) < c$ if and only if there exist $c_1, c_2 > 0$, with $c_1c_2 < c$, and an invertible $T \in L(E, F)$ such that $\frac{1}{c_1}B(F) \subset TB(E) \subset c_2B(E)$ holds.

Proof:

Suppose we have $d(E, F) < c$. Then there exists an invertible $T \in L(E, F)$ such that $||T|| \cdot ||T^{-1}|| < c$ holds. Take $c_1 = ||T^{-1}|| > 0$ and $c_2 = ||T|| > 0$. We have $TB(E) \subset ||T||B(F) = c_2B(F)$ and $T^{-1}B(F) \subset ||T^{-1}||B(E) = c_1B(E)$. From this we find $\frac{1}{c_1}B(F) \subset TB(E) \subset c_2B(F)$.

Now suppose there exists an invertible $T \in L(E, F)$ and $c_1, c_2 > 0$, $c_1c_2 < c$ such that $\frac{1}{c_1}B(F) \subset TB(E) \subset c_2B(F)$ holds. Then we have $c_2 \geq ||T||$ and $c_1 \geq ||T^{-1}||$, so we find $d(E, F) \leq ||T||\cdot||T^{-1}|| \leq c_1c_2 < c$. □

So we can view the Banach-Mazur distance as a way of measuring how the unit balls of two isomorphic normed spaces fit into each other. Also note that we can assume $c_1 = 1$ holds. For if $T \in L(E, F)$ is invertible such that $\frac{1}{c_1}B(F) \subset TB(E) \subset c_2B(F)$ holds, then $c_1T \in L(E, F)$ is invertible with $B(F) \subset c_1TB(E) \subset c_1c_2B(F)$.

Now we deduce some quantitive properties of this distance.

For instance, since we have $1 = ||T \circ T^{-1}|| \leq ||T|| \cdot ||T^{-1}||$ for all invertible $T \in L(E, F)$, we find

$$d(E, F) \geq 1 \tag{1}$$

for all normed spaces E and F. Also, if E and F are isometric, there exists an invertible $T \in L(E, F)$ such that $||Tx||_F = ||x||_E$ holds for all $x \in E$, or equivalently, an invertible $T \in L(E, F)$ such that $||T|| = ||T^{-1}|| = 1$ holds. This implies we have

$$E \text{ and } F \text{ isometric} \Rightarrow d(E, F) = 1 \tag{2}$$

Clearly we also have

$$d(E, F) = d(F, E) \tag{3}$$

for all normed spaces E and F.
Now for $R : E \to F$ and $S : F \to G$ invertible linear maps between normed spaces, $S \circ R : E \to G$ is linear and invertible, and we have $||S \circ R|| \leq ||R|| \cdot ||S||$. This implies we have

$$d(E, G) \leq d(E, F) \cdot d(F, G). \tag{4}$$

Combining these properties, we find that the Banach-Mazur distance defines a multiplicative pseudometric of sorts; we have (1) instead of $d(E, F) \geq 0$, and the Banach-Mazur distance satisfies the multiplicative triangle inequality (4) instead of the additive one. We will make this statement more precise in a little while.
But first, we can ask ourselves if $d(E, F) = 1$ implies E and F are isometric. If we restrict ourselves to finite-dimensional real normed spaces it does. We note here that, since all the norms on a n-dimensional real normed space are equivalent, any finite-dimensional normed space is complete and thus a Banach space. Therefore, it provides no restriction to regard Banach spaces when examining finite-dimensional real spaces.

Proposition 2.3. Let E and F be n-dimensional real Banach spaces ($n \in \mathbb{N}$) with Banach-Mazur distance $d(E, F) < \infty$. Then there exists an invertible linear map $T \in L(E, F)$ such that $||T|| \cdot ||T^{-1}|| = d(E, F)$ holds.

Proof:
First remark that, since the vector space structure is the same for any finite-dimensional real normed space, we can view any such space as $\mathbb{R}^n$ with a specific norm on it. This implies that we can view any linear map $T : E \to F$ between normed vector spaces of dimension n as an element of $L(\mathbb{R}^n)$, where its operator norm is $||T : E \to F||$, the norm of T viewed as a map from E to F.
Now suppose $E = (\mathbb{R}^n, || \cdot ||_E)$ and $F = (\mathbb{R}^n, || \cdot ||_F)$ are two n-dimensional ($n \in \mathbb{N}$) real spaces, so $d(E, F) < \infty$ holds. Then there exists a sequence $\{T_m\} \subset L(E, F)$ of invertible maps such that

$$\lim_{m \to \infty} ||T_m : E \to F|| \cdot ||T_m^{-1} : F \to E|| = \lim_{m \to \infty} ||T_m|| \cdot ||T_m^{-1}|| = d(E, F)$$

holds. By normalizing these maps we can assume $||T_m : E \to F|| = 1$ holds for all $m \in \mathbb{N}$ (for instance, use $\frac{T_m}{||T_m : E \to F||}$).
Now if we endow $L(\mathbb{R}^n)$ with the operator norm $||T|| = ||T : E \to F||$, which views all maps as maps from E to F, $L(\mathbb{R}^n)$ becomes a finite-dimensional Banach algebra with identity, and thus its closed unit ball is compact (a fact that is equivalent to the finite-dimensionality of the algebra). This implies there exists a subsequence converging to a $T \in L(E, F)$ with respect to this specific operator norm, so we can assume $\{T_m\}$ itself converges with respect to this norm. Since $||T_m|| = 1$ holds for all $m \in \mathbb{N}$, we then find $||T|| = 1$. Now, $L(\mathbb{R}^n)$ is also a finite-dimensional Banach-algebra with the norm $||T||' = ||T : F \to E||$, which views linear maps as maps from F to E. It is a well-known fact that all norms on finite-dimensional are equivalent, so there exists a $C > 0$ such that $||T|| = ||T : E \to F|| \leq C||T : F \to E|| = C||T||'$ holds for all $T \in L(\mathbb{R}^n)$. We will now use this to show T is invertible.
Hereto, we use the following lemma from [Conway 1990, p. 192]:

Lemma 2.4. If A is a Banach algebra with identity $1 \in A$ and we have $x \in A$ such that $||x - 1|| < 1$ holds, then x is invertible.

To use this lemma on T, we note the following holds, where $1 \in L(\mathbb{R}^n)$ is the identity map,

$$||T_m^{-1} \circ T - 1|| = ||T_m^{-1}(T - T_m)|| \leq ||T_m^{-1}|| \cdot ||T - T_m||.$$

Since we have

$$d(E,F) = \lim_{m\to\infty} ||T_m : E \to F|| \cdot ||T_m^{-1} : F \to E|| = \lim_{m\to\infty} ||T_m|| \cdot ||T_m^{-1}||' = \lim_{m\to\infty} ||T_m^{-1}||',$$

for m sufficiently large we have $||T_m^{-1}||' < 2d(E,F)$. Also, since T_m converges to T with respect to the norm $||\cdot||$, for m sufficiently large we have $||T - T_m|| < \frac{1}{2d(E,F)C}$. Now choosing $m \in \mathbb{N}$ large enough, we find

$$||T_m^{-1} \circ T - 1|| \leq ||T_m^{-1}|| \cdot ||T - T_m|| \leq C||T_m^{-1}||' \cdot ||T - T_m|| < 1.$$

This implies, using lemma 2.3, that $T_m^{-1} \circ T$ is invertible in $L(\mathbb{R}^n)$, and thus that T is invertible. Now by theorem 2.2 from the same [Conway 1990, p. 192], we know the map $x \mapsto x^{-1}$ on the invertible elements of $L(\mathbb{R}^n)$ is continuous, so we have $\lim_{m\to\infty} T_m^{-1} = T^{-1}$ with respect to the norm $||\cdot||$ on $L(\mathbb{R}^n)$. However, since the norms $||\cdot||$ and $||\cdot||'$ on $L(\mathbb{R}^n)$ are equivalent and thus define the same topology, this implies T_m^{-1} also converges to T^{-1} with respect to the norm $||\cdot||'$. So we find

$$||T^{-1} : F \to E|| = ||T^{-1}||' = \lim_{m\to\infty} ||T_m^{-1}||' = d(E,F).$$

In conclusion, this means we have found an invertible map $T \in L(\mathbb{R}^n)$ such that $||T|| \cdot ||T^{-1}|| = d(E,F)$ holds. □

Corollary 2.5. Let E and F be finite-dimensional real Banach spaces. Then E and F have Banach-Mazur distance $d(E,F) = 1$ if and only if E and F are isometric.

Now we can expand Proposition 2.2 with these results and the remarks following Proposition 2.2:

Corollary 2.6. Let E and F be finite-dimensional real Banach spaces. Then we have $d(E,F) = c$ if and only if there exists an invertible $T \in L(E,F)$ such that $B(F) \subset TB(E) \subset cB(F)$ holds and c cannot be chosen any smaller.

We have used the finite-dimensionality of E and F in an essential way to produce a convergent subsequence. That the finite-dimensionality is indeed essential can be seen by observing the following real Banach spaces, which have Banach-Mazur distance 1 despite not being isometric. It is an example from [Pelczynski and Bessega 1979] which we will not work out in detail but only state, since it is not essential to our goal of introducing the reader to the Banach-Mazur distance.

Example 2.7. Let $c_0 = \left\{x = (x(1), x(2), \dots) \in \mathbb{R}^{\mathbb{N}} : \lim_{m\to\infty} x(m) = 0\right\}$ be the space of real sequences converging to zero, and consider the following two norms on c_0, for $i = 0, 1$:

$$||x||_i = \sup_{j\in\mathbb{N}} |x(j)| + \left(\sum_{j=1}^{\infty} 2^{-2j} |x(j+i)|^2\right)^{1/2}.$$

Now let E_i be the space c_0 equipped with the norm $||\cdot||_i$, and for $m \in \mathbb{N}$, let $T_m : E_0 \to E_1$ be the bounded operator given by $T_m(x(1), x(2), \dots) = (x(m), x(1), x(2), \dots, x(m-1), x(m+1), \dots)$. Then every T_m is an isomorphism from E_0 onto E_1, and we have $\lim_{m\to\infty} ||T_m|| \cdot ||T_m^{-1}|| = 1$.
However, the norm $||\cdot||_0$ is strictly convex ($||x+y||_0 = ||x||_0 + ||y||_0$ implies we have $y = cx$ for some $c > 0$), while the norm $||\cdot||_1$ is not. Therefore E_0 and E_1 are not isometric.

3 A space of spaces

We introduced the Banach-Mazur distance as a measure of the degree to which norms on isomorphic vector spaces differ. However, the norms on isometric spaces do not differ essentially, and for this reason it makes sense to examine not the set of all n-dimensional normed spaces, but only those normed spaces which are not isometric. Since isometry defines an equivalence relation on the set of all n-dimensional spaces, we can make this more precise.

Definition 3.1. Given $n \in \mathbb{N}$, we define $\mathcal{F}_n$ to be the set of equivalence classes of n-dimensional real Banach spaces with respect to the equivalence relation of isometry between spaces.

Thus in $\mathcal{F}_n$, a point defines in fact a whole equivalence class of spaces. Hence the term space of spaces naturally comes to mind. However, at the moment $\mathcal{F}_n$ is but a set without further structure, and we seek to define a distance of sorts, or more precise, a metric on $\mathcal{F}_n$. To this end, the previous paragraph provides us with the necessary tools:

Definition 3.2. Let $E, F \in \mathcal{F}_n$ be equivalence classes of Banach spaces. Then we define the Banach-Mazur distance $d(E, F) := d(E', F')$ between them to be Banach-Mazur distance between representatives $E' \in E$ and $F' \in F$ of E respectively F.

Now if $E'' \in E$ and $F'' \in F$ also are representatives, we have $d(E'', F'') \leq d(E'', E') \cdot d(E', F') \cdot d(F', F'') = d(E', F')$ and, in the same way, $d(E', F') \leq d(E'', F'')$, since E' and E'' are isometric, as are F' and F''. So Definition 2.7 does not depend on the choice of a particular representative, and the distance is well-defined.
Now we find the properties (1), (3) and (4) hold for the Banach-Mazur distance on $\mathcal{F}_n$ as well, so (2) and corollary 2.5 imply we have $d(E, F) = 1 \Leftrightarrow E = F$ for $E, F \in \mathcal{F}_n$. Now we can make more precise what was meant earlier by a multiplicative metric. For note that if we take the logarithm of the Banach-Mazur distance, a procedure which changes a multiplicative operation into a additive one, we get a function $d' : \mathcal{F}_n \times \mathcal{F}_n \to [0, \infty)$, $d'(E, F) = \log(d(E, F))$, such that the following hold:

$$d'(E, F) = 0 \Leftrightarrow E = F,$$

$$d'(E, F) = d'(F, E),$$

$$d'(E, G) \leq d'(E, F) + d'(F, G).$$

So d' is in fact a well-defined metric on $\mathcal{F}_n$. This means we now finally have found our space of spaces:

Definition 3.3. We define the Minkowski compactum for dimension $n \in \mathbb{N}$ to be the metric space $(\mathcal{F}_n, d')$.

Now the first thing to note is that from now on, we will still mostly speak of the Banach-Mazur distance d when discussing the distance on $\mathcal{F}_n$. On the one hand, this is not a well-defined metric. However, since we can easily produce a well-defined metric d' by taking the logarithm of d, it does not matter essentially whether we discuss d' or d. Therefore, all the statements we make about d can easily be transmuted to corresponding statements about d', and therefore to statements about the metric on $\mathcal{F}_n$. We choose to work with the Banach-Mazur distance since it is easier to work with.
The second thing to note is that the name 'Minkowski compactum' is suggestive, and we will go on to show that this suggestion is correct: that $(\mathcal{F}_n, d')$ is indeed a compact metric space. To this end, we first show that $\mathcal{F}_n$ is a bounded space, i.e. we have $\operatorname{diam}(\mathcal{F}_n) = \{d(E, F) : E, F \in \mathcal{F}_n\} < \infty$.
Hereto, we use the fact that there exists for each n-dimensional real normed space $(E, ||\cdot||)$ an Auerbach system in E: a set of vectors $x_1, \ldots, x_n \in E$ and linear functionals $x_1^*, \ldots, x_n^* \in E^*$ such that $||x_i|| = ||x_i^*|| = 1$ and $x_i^*(x_j) = \delta_{ij}$ for all i, j, where we have $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$[Bollobás 1990, p. 65]. Then $\{x_1, \ldots, x_n\}$ is a basis for E.
Now let $l_1^n = (\mathbb{R}^n, ||\cdot||_1)$ be the Banach space $\mathbb{R}^n$ with the one-norm, i.e. if $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, we have $||\sum_{i=1}^n \lambda_i e_i||_1 = \sum_{i=1}^n |\lambda_i|$.

Proposition 3.4. Let $(E, ||\cdot||)$ be an n-dimensional real Banach space. Then we have $d(X, l_1^n) \leq n$.

Proof:
Let $x_1, \ldots, x_n \in E$, $x_1^*, \ldots, x_n^* \in E^*$ be an Auerbach system. We will show that $J : l_1^n \to E$ given by $J(e_i) = x_i$, is an isomorphism such that $||J|| \leq 1$ and $||J^{-1}|| \leq n$. Since $x_1, \ldots, x_n$ is a basis of E, it is clearly an isomorphism, and for $x = \sum_{i=1}^n \lambda_i e_i$ we have

$$||J(x)|| = ||\sum_{i=1}^n \lambda_i x_i|| \leq \sum_{i=1}^n |\lambda_i| \, ||x_i|| = \sum_{i=1}^n |\lambda_i| = ||x||_1,$$

and thus $||J|| \leq 1$.
Now given $x = \sum_{i=1}^n \lambda_i x_i \in E$, choose for each i an $a_i \in \{-1, 1\}$ such that $a_i \lambda_i = |\lambda_i|$, and define $f = \sum_{i=1}^n a_i x_i^* \in E^*$. Then we have $||f|| = ||\sum_{i=1}^n a_i x_i^*|| \leq \sum_{i=1}^n |a_i| \, ||x_i^*|| = n$ and

$$f(x) = f(\sum_{i=1}^n \lambda_i x_i) = \sum_{i=1}^n \lambda_i f(x_i) = \sum_{i=1}^n \lambda_i a_i = \sum_{i=1}^n |\lambda_i|.$$

From this we find

$$||J^{-1}(x)||_1 = ||\sum_{i=1}^n \lambda_i e_i||_1 = \sum_{i=1}^n |\lambda_i| = f(x) \leq n||x||,$$

and so we have $||J^{-1}|| \leq n$ and $d(l_1, E) \leq n$. □

Corollary 3.5. We have $\operatorname{diam}(\mathcal{F}_n) \leq 2\log(n)$.

Proof:
Choose $E, F \in \mathcal{F}_n$, and let $l_1 \in \mathcal{F}_n$ be the equivalence class of the space l_1^n. Then we find $d(E, F) \leq d(E, l_1) \cdot d(l_1, F) \leq n^2$, and thus, taking the logarithm,

$$d'(E, F) \leq 2\log(n).$$

□

Now that we have established that the space $\mathcal{F}_n$ is bounded, we can show the following.

Theorem 3.6. $\mathcal{F}_n$ is a compact metric space.

Proof:
Let $B_1 := B(l_1^n)$ denote the closed unit ball of l_1^n and let Φ_n denote the set of all norms $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{1}{n}||x||_1 \leq ||x|| \leq ||x||_1.$$

Since B_1 is compact, the set $C(B_1)$ of continuous functions on B_1 is a Banach space with respect to the supremum norm $||\cdot||_\infty$. Now any norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ gives by restriction a continuous function on B_1, since all norms on $\mathbb{R}^n$ are equivalent. This means we can produce, by restricting the elements of Φ_n to B_1, a subset $\Phi_n' = \{f : B_1 \to \mathbb{R} : f(x) = ||x|| \text{ for some norm } ||\cdot|| \in \Phi_n\}$ of $C(B_1)$. We now prove the following lemma:

Lemma 3.7. Φ_n' is compact.

Proof:
Using the Arzela-Ascoli theorem[Conway 1990, p. 175], we need to show that Φ_n' is closed, bounded and equicontinuous in $C(B_1)$.
To show that Φ_n' is closed, let $\{f_m\}_{m=1}^{\infty}$ be a sequence in Φ_n' converging to $f \in C(B_1)$. We will show that f is the restriction of a norm in Φ_n. Hereto, define $||\cdot|| : \mathbb{R}^n \to \mathbb{R}$ by $||x|| = ||x||_1 f(\frac{x}{||x||_1})$, and for any $m \in \mathbb{N}$, let $||\cdot||_m$ be the norm on $\mathbb{R}^n$ whose restriction to B_1 gives f_m.
Now, for any $x \in \mathbb{R}^n$ we have

$$\lim_{m\to\infty} ||x||_m = ||x||_1 \lim_{m\to\infty} \left|\left|\frac{x}{||x||_1}\right|\right|_m = ||x||_1 \lim_{m\to\infty} f_m\left(\frac{x}{||x||_1}\right) = ||x||_1 f\left(\frac{x}{||x||_1}\right) = ||x||.$$

So we find $||x|| = \lim_{m\to\infty} ||x||_m \in [0,\infty)$, since we have $||x||_m \in [0,\infty)$ for each $m \in \mathbb{N}$.
Now if $x \neq 0$ holds, we have $0 < \frac{1}{n}||x||_1 \leq ||x||_m$ and thus also $||x|| = \lim_{m\to\infty} ||x||_m \geq \frac{1}{n}||x||_1 > 0$. Since we also have

$$||\lambda x|| = \lim_{m\to\infty} ||\lambda x||_m = |\lambda| \lim_{m\to\infty} ||x||_m = |\lambda|||x||$$

for any $\lambda \in \mathbb{R}$, we know $||0|| = 0$ and $||\cdot||$ is positive definite.
Concerning the triangle inequality, for $x, y \in \mathbb{R}^n$ we have

$$||x+y|| = \lim_{m\to\infty} ||x+y||_m \leq \lim_{m\to\infty} (||x||_m + ||y||_m) = \lim_{m\to\infty} ||x||_m + \lim_{m\to\infty} ||y||_m = ||x|| + ||y||.$$

So $||\cdot||$ is indeed a norm, and f is clearly its restriction to B_1.
Finally, we have $\frac{1}{n}||x||_1 \leq ||x||_m \leq ||x||_1$ for each $m \in \mathbb{N}$, and thus also

$$\frac{1}{n}||x||_1 \leq ||x|| = \lim_{m\to\infty} ||x||_m \leq ||x||_1,$$

so we find $f \in \Phi_n'$ and Φ_n' is indeed closed.
Now Φ_n' is clearly bounded in $(C(B_1), ||\cdot||_\infty)$, since we have $||f||_\infty = \sup_{x\in B_1} |f(x)| \leq \sup_{x\in B_1} ||x||_1 = 1$ for all $f \in \Phi_n'$.
Now finally, to show Φ_n' is equicontinuous, choose $\epsilon > 0$ and $x_0 \in B_1$. Then we have, for all $f \in \Phi_n'$ and $x \in B_1$,

$$|f(x) - f(x_0)| \leq f(x - x_0) \leq ||x - x_0||_1,$$

so letting $\delta = \epsilon$ we find an open neighbourhood $B_{x_0}(\delta) \subset B_1$ of x_0 such that $|f(x) - f(x_0)| < \epsilon$ holds for all $f \in \Phi_n'$ and $x \in B_{x_0}(\delta)$, so Φ_n' is indeed equicontinuous. □

Proof of Theorem 3.6:
By proposition 3.4 and corollary 2.6 there exists, for every n-dimensional real Banach space $(E, ||\cdot||)$ with closed unit ball $B(E)$, an isomorphism $T \in L(l_1^n, E)$ such that $B(E) \subset TB_1 \subset nB(E)$ holds. By dividing T by n, we may assume we have $\frac{1}{n}B(E) \subset TB_1 \subset B(E)$, which implies that we have

$$\frac{1}{n}||x||_1 \leq ||Tx|| \leq ||x||_1.$$

Since T is an isomorphism, we can define a norm $||\cdot||_F$ on $\mathbb{R}^n$ by $||x||_F = ||Tx||$, and this gives us a Banach space $F = (\mathbb{R}^n, ||\cdot||_F)$ which is isometric to E. Now for F we find

$$\frac{1}{n}||x||_1 \leq ||Tx|| = ||x||_F \leq ||x||_1.$$

This reasoning implies that for every n-dimensional real Banach space E, there exists a norm $||\cdot||_F \in \Phi_n$ such that E is isometric to $F = (\mathbb{R}^n, ||\cdot||_F)$. This means that the natural map $\phi : \Phi_n' \to \mathcal{F}_n$ sending the

restriction to B_1 of a norm $||\cdot||$ on $\mathbb{R}^n$ to the equivalence class of $(\mathbb{R}^n, ||\cdot||)$, is surjective. We show that it is continuous, so we can conclude that $\mathcal{F}_n$ is compact.
Hereto, choose $\epsilon > 0$ and $f_0 \in \Phi'_n$. We show there exists a $\delta > 0$ such that $\sup_{x \in B_1} |f(x) - f_0(x)| < \delta$ for $f \in \Phi'_n$ implies $d'(\phi(f), \phi(f_0)) < \epsilon$.
Choose $\delta > 0$ such that $\log(\frac{1+n\delta}{1-n\delta}) < \epsilon$ and let $f \in \Phi'_n$ be such that $\sup_{x \in B_1} |f(x) - f_0(x)| < \delta$. Now f and f_0 are restriction of norms g respectively g_0 in Φ_n. So for any $x \in \mathbb{R}^n$, $x \neq 0$, we find

$$\frac{|g(x) - g_0(x)|}{||x||_1} = \left| g(\frac{x}{||x||_1}) - g_0(\frac{x}{||x||_1}) \right| = \left| f(\frac{x}{||x||_1}) - f_0(\frac{x}{||x||_1}) \right| < \delta,$$

and thus $|g(x) - g_0(x)| \leq \delta ||x||_1$ for all $x \in \mathbb{R}^n$. Now, since we have $\frac{1}{n}||x||_1 \leq g_0(x)$, this implies $|g(x) - g_0(x)| \leq n\delta g_0(x)$ for all $x \in \mathbb{R}^n$.
Now $(1-n\delta)g_0(x) \leq g(x) \leq (1+n\delta)g_0(x)$ holds for arbitrary $x \in \mathbb{R}^n$. If we let B_g and B_{g_0} be the unit balls of $\mathbb{R}^n$ with the norms g respectively g_0, this implies precisely that we have $(1-n\delta)B_g \subset B_{g_0} \subset (1+n\delta)B_g$. Using proposition 2.2, this implies we have

$$d'(\phi(f), \phi(f_0)) = \log(d(\phi(f), \phi(f_0))) \leq \log(\frac{1+n\delta}{1-n\delta}) < \epsilon.$$

So ϕ is continuous, and we have shown that $\mathcal{F}_n = \phi[\Phi'_n]$ is compact. □

4 Gluskin's theorem

We have estimated the diameter of $\mathcal{F}_n$ by giving an upper bound to the Banach-Mazur distance of any n-dimensional Banach space to l_1^n. One can wonder whether this gives the best bound. Fritz John showed that this is not the case. He improved the estimate by bounding the distance of any space to the n-dimensional real Euclidean space l_2^n.

Theorem 4.1 (John (1948))**.** Let $l_2 \in \mathcal{F}_n$ be the equivalence class of the space l_2^n. Then we have $d(E, l_2) \leq \sqrt{n}$ for any $E \in \mathcal{F}_n$.

Corollary 4.2. $\operatorname{diam}(\mathcal{F}_n) \leq \log(n)$ holds.

We will not prove this result, however a proof can be found in [Bollobás 1990, p. 68], amongst others.
Once again, we can ask ourselves whether this bound is optimal. This question was left open for more than thirty years until Gluskin gave a lower estimate for the diameter of $\mathcal{F}_n$.

Theorem 4.3 (Gluskin(1981))**.** There exists a $c \in \mathbb{R}_{>0}$ such that for every $n \in \mathbb{N}$ there exist n-dimensional real Banach spaces $E_1 = (\mathbb{R}^n, ||\cdot||_{E_1})$ and $E_2 = (\mathbb{R}^n, ||\cdot||_{E_2})$ such that for every $T \in L(\mathbb{R}^n)$ with $|\det(T)| = 1$ we have $||T : E_1 \to E_2|| \geq c\sqrt{n}$ and $||T : E_2 \to E_1|| \geq c\sqrt{n}$.

From this we easily find a lower bound for the diameter of $\mathcal{F}_n$:

Corollary 4.4. There exists a $c' \in \mathbb{R}_{\geq 0}$ such that, for every $n \in \mathbb{N}$, we have $\operatorname{diam}(\mathcal{F}_n) \geq \log(n) - c'$.

Proof:
Let $T \in L(\mathbb{R}^n)$ be invertible and E_1 and E_2 as in Gluskin's Theorem. Then $S := \frac{T}{\sqrt[n]{|\det(T)|}}$ and $S^{-1} = T^{-1} \cdot \sqrt[n]{|\det(T)|}$ are invertible with determinant 1 or -1. We therefore have

$$||T : E_1 \to E_2|| \cdot ||T^{-1} : E_2 \to E_1|| = ||\det(T) \cdot S : E_1 \to E_2|| \cdot ||\frac{1}{\det(T)} S^{-1} : E_2 \to E_1||$$

$$= ||S : E_1 \to E_2|| \cdot ||S^{-1} : E_2 \to E_1|| \geq c^2 n.$$

Taking the infimum over all invertible $T \in L(\mathbb{R}^n)$ and then the logarithm, we find $d(E_1, E_2) \geq \log(n) + 2\log(c)$. Now Corollary 4.2 implies we have $\log(c) \leq 0$ so letting $c' := -2\log(c) \geq 0$, we find the required result. □

The rest of this thesis will expand into the proof of theorem 4.3, a proof which uses an inventive measure-theoretic approach towards showing the existence of the spaces E_1 and E_2.
Hereto, first fix a positive interger $n \in \mathbb{N}$, let $e_1, \ldots, e_n$ denote the standard basis in $\mathbb{R}^n$, S_{n-1} the Euclidean unit sphere and for any $m \in \mathbb{N}$, $\mathcal{A}_m = (S_{n-1})^m$ the set of all sequences of m elements from S_{n-1}. Let λ denote the unique rotation-invariant Borel measure on S_{n-1} such that $\lambda(S_{n-1}) = 1$ holds [Folland 1999, p. 78].
Now we will construct, for each sequence $(f_1, \ldots, f_m) \in \mathcal{A}_m$, a n-dimensional Banach space E having unit ball $B(E) = \operatorname{conv}\{\pm e_i, \pm f_j : i = 1, \ldots, n, j = 1, \ldots, m\}$. To this end, we use the following lemma from [Conway 1990, p. 102]:

Lemma 4.5. Let X be a vector space over $\mathbb{R}$ and V a non-empty convex, balanced set that is absorbing at each of its points. Then there exists a unique seminorm p on X, given by $p(x) = \inf\{t \geq 0 : x \in tV\}$, such that $V = \{x \in X : p(x) < 1\}$ holds.

A set $V \subset X$ is balanced if $\alpha x \in V$ holds for all $x \in V$, $\alpha \in \mathbb{R}$ with $|\alpha| \leq 1$. V is absorbing at $v \in V$ if for every $x \in V$ there exists an $\epsilon > 0$ such that $v + tx \in V$ holds for all $0 \leq t < \epsilon$.
Now we plan to apply this lemma to the interior $\operatorname{int}(B)$ of $B := \operatorname{conv}\{\pm e_i, \pm f_j : i = 1, \ldots, n, j = 1, \ldots, m\}$.
Hereto, first note that any open set is absorbing at each of its points, and that $\operatorname{int}(B)$ is non-empty. Also, the interior of any convex set is itself convex. Now since B contains any convex combination of e_1 and $-e_1$, it contains 0, and so does $\operatorname{int}(B)$. The same holds for αx, where we have $x \in \operatorname{int}(B)$ and $0 \leq \alpha \leq 1$. Since B is a convex combination of $n + m$ elements and minus those elements, we have $-x \in \operatorname{int}(B)$ for $x \in \operatorname{int}(B)$, so $\operatorname{int}(B)$ is balanced. This implies there exists a unique seminorm p on E such that $\operatorname{int}(B) = \{x \in E : p(x) < 1\}$ holds. $B = \overline{\operatorname{int}(B)} = \{x \in E : p(x) \leq 1\}$ is the closed unit ball.
Now to show p is in fact a norm, let $x \in E$ be such that $0 = p(x) = \{t \geq 0 : x \in t \cdot \operatorname{int}(B)\}$. Since $\operatorname{int}(B)$ is bounded with respect to the Euclidean norm $|| \cdot ||_2$, $||y||_2 \leq 1$ holds for $y \in \operatorname{int}(B)$, and we have $||x||_2 \leq \inf\{t \geq 0 : x \in t \cdot \operatorname{int}(B)\} = p(x) = 0$. Thus $x = 0$ holds and p is in fact a norm. Since all finite-dimensional real normed spaces are complete, $(\mathbb{R}^n, p)$ is a Banach space
So now we can construct, for each $A = (f_1, \ldots, f_m) \in \mathcal{A}_m$, a n-dimensional Banach space E_A. From now on, given $A \in \mathcal{A}_m$, E_A will denote such a space and $|| \cdot ||_{E_A}$ its norm.
In this way, elements of $\mathcal{A}_m$ correspond to Banach spaces, and we have a product measure λ^m on $\mathcal{A}_m = (S_{n-1})^m$. So we can attempt to measure the set of Banach spaces which satisfy the requirements of theorem 4.3. To make this more precise, we will show that the $\lambda^m \times \lambda^m$-measure of the subset of $\mathcal{A}_m \times \mathcal{A}_m$ of pairs satisfying theorem 38.4 is greater than 0, and therefore is non-empty. We will use a few technical lemmas to estimate the measure of this set. We prove these lemmas for future use, even though their meaning may not immediately be obvious.
The first lemma gives an upper bound for the λ^m-measure of a certain subset of $\mathcal{A}_m$:

Lemma 4.6. Let $\alpha = \sqrt{\frac{3e^3}{\pi}}$, $\rho > 0$, $m \in \mathbb{N}$, $B = (g_1, \ldots, g_m) \in \mathcal{A}_m$ and $T \in L(\mathbb{R}^n)$ with $|\det(T)| = 1$. Then for all n large enough we have

$$\lambda^m \left\{ A \in \mathcal{A}_m : ||T : E_A \to E_B|| \leq \rho \frac{n^{3/2}}{\alpha(m+n)} \right\} \leq \rho^{mn},$$

where λ^m is the product measure on $(S_{n-1})^m = \mathcal{A}_m$.

Proof:
Let $B_2 = B(l_2^n)$ denote the Euclidian unit ball and define $W_1 := \left\{ x \in B_2 : \frac{x}{||x||_2} \in W \right\}$. If W is Lebesgue measurable, so is W_1, and we have $\lambda(W \cap S_{n-1}) = \frac{\mathrm{vol}\,(W_1)}{\mathrm{vol}\,(B_2)}$, where vol is the usual Lebesgue measure on $\mathbb{R}^n$.
Now set $r = \rho \frac{n^{3/2}}{\alpha(m+n)}$. If we have $A = (f_1, \ldots, f_m) \in \mathcal{A}_m$ with $||T : E_A \to E_B|| \leq r$, then

$$f_j \in \{x \in S_{n-1} : ||Tx||_{E_B} \leq r\}$$

for all $j = 1, \ldots, m$.
It is easy to show that $\{A \in \mathcal{A}_m : ||T : E_A \to E_B|| \leq r\}$ and $\{x \in S_{n-1} : ||Tx||_{E_B} \leq r\}$ are closed subsets of $\mathcal{A}_m$ respectively S_{n-1}. Thus they are λ^m- respectively λ-measurable. From all this we find

$$\lambda^m \{A \in \mathcal{A}_m : ||T : E_A \to E_B|| \leq r\} \leq (\lambda \{x \in S_{n-1} : ||Tx||_{E_B} \leq r\})^m. \tag{5}$$

We now have

$$\lambda \{x \in S_{n-1} : ||Tx||_{E_B} \leq r\} = \lambda \left\{x \in S_{n-1} : x \in rT^{-1}(B(E_B))\right\} = \lambda(rT^{-1}(B(E_B)) \cap S_{n-1}). \tag{6}$$

For any convex, Lebesgue measurable $W \in \mathbb{R}^n$ with $0 \in W$, we have $W_1 \subset W$ and

$$\lambda(W \cap S_{n-1}) = \frac{\mathrm{vol}\,(W_1)}{\mathrm{vol}\,(B_2)} \leq \frac{\mathrm{vol}\,(W)}{\mathrm{vol}\,(B_2)}.$$

Since $B(E_B)$ is closed, convex and contains 0, $rT^{-1}(B(E_B))$ is convex, Lebesgue measurable and contains 0. We find

$$\lambda(rT^{-1}(B(E_B)) \cap S_{n-1}) \leq \frac{\mathrm{vol}(rT^{-1}(B(E_B)))}{\mathrm{vol}(B_2)}. \tag{7}$$

To estimate this latter quantity, we note that $\mathrm{vol}(B_2) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}$. Also, since we have $|\det(T)| = 1$, we find $\mathrm{vol}(rT^{-1}(B(E_B))) = r^n \mathrm{vol}(B(E_B))$. In Appendix A we show that

$$\mathrm{vol}(B(E_B)) \leq (2e^2 \frac{m+n}{n^2})^n. \tag{8}$$

Using Stirling's formula on the Gamma function $\Gamma(z) = \sqrt{\frac{2\pi}{z}} (\frac{z}{e})^z (1 + O(\frac{1}{z}))$, and (7), we find, for n sufficiently large,

$$\lambda(rT^{-1}(B(E_B)) \cap S_{n-1}) \leq r^n \frac{\mathrm{vol}(B(E_B))}{\mathrm{vol}(B_2)} \leq (\alpha r \frac{(m+n)}{n^{3/2}})^n = \rho^n.$$

Combining this with (5) and (6) completes the proof. □

In the next lemma we bound the cardinality of a net in a certain set of operators.
For any $\epsilon > 0$, an ϵ-net N in a metric space (S, d) is a set such that for $x, y \in N$, $x \neq y$, we have $d(x, y) \geq \epsilon$, and for all $x \in S$, there exists an $y \in N$ such that $d(x, y) \leq \epsilon$. Now for $B \in \mathcal{A}_m$, define $M_B := \{T \in L(\mathbb{R}^n) : ||Te_i||_{E_B} \leq \sqrt{n} \text{ for } i = 1, \ldots, n\}$ and let $|| \cdot ||_{\mathrm{op}}$ be the Euclidean operator norm on $L(\mathbb{R}^n)$, i.e. $||T||_{\mathrm{op}} = ||T : l_2^n \to l_2^n||$.

Lemma 4.7. There exists a constant $a > 1$ such that, for $n \in \mathbb{N}$ sufficiently large and all $m \in \mathbb{N}$, if $B \in \mathcal{A}_m$ and $0 < \epsilon \leq 1$ are given, there exists an ϵ-net N_B in $(M_B, || \cdot ||_{\mathrm{op}})$ such that

$$\mathrm{card}\, N_B \leq \left(a \frac{m+n}{n\epsilon} \right)^{n^2}. \tag{9}$$

Proof:
Choose $B \in \mathcal{A}_m$, $\epsilon \in (0,1]$ and let U be unit ball of $(L(\mathbb{R}^n), ||\cdot||_{\mathrm{op}})$. Now indentify $L(\mathbb{R}^n)$ with $\mathbb{R}^{n^2}$ in the usual way: $T \to (T(e_i)) \in (\mathbb{R}^n)^n$. In this way we identify M_B with the subset

$$\overline{M}_B = \left\{(x_1, \ldots, x_n) \in (\mathbb{R}^n)^n : x_i \in \sqrt{n}(B(E_B)) \text{ for all } i = 1, \ldots, n\right\}.$$

Now we construct a sequence of points $y_k \in \overline{M}_B$ in the following manner. Choose $y_1 \in \overline{M}_B$, and given $y_1, \ldots, y_k \in \overline{M}_B$, choose $y_{k+1} \in \overline{M}_B \setminus (\cup_{i=1}^k B_{y_i}(\epsilon))$. Now note that this process stops after a finite number of steps. For since $\overline{M}_B$ is bounded in $(\mathbb{R}^n)^n$, any infinite sequence in $\overline{M}_B$ has a convergent subsequence, and since the elements are ϵ-separated in the norm $||\cdot||_{\mathrm{op}}$, this process has to stop after $K \in \mathbb{N}$ steps. Then the collection $\overline{N}_B := \{y_1, \ldots, y_K\}$ is easily seen to form an ϵ-net, since the elements are all ϵ-separated and the existence of an element $y \in \overline{M}_B$ such that $d(y, y_k) > \epsilon$ for all $k = 1, \ldots, K$ contradicts the maximality of K.
The balls $y + \frac{\epsilon}{2}U$, for $y \in \overline{N}_B$, are disjoint. Now the Cauchy-Schwarz inequality implies, for all $x = \sum_{i=1}^n t_i e_i \in \mathbb{R}^n$,

$$||x||_{E_B} = ||\sum_{i=1}^n t_i e_i||_{E_B} \leq \sum_{i=1}^n |t_i|\,||e_i||_{E_B} = \sum_{i=1}^n |t_i| \cdot 1 \leq \sqrt{\sum_{i=1}^n t_i^2}\sqrt{\sum_{i=1}^n 1} = \sqrt{n}||x||_2.$$

From this we find

$$||T(e_i)||_{E_B} \leq \sqrt{n}||T(e_i)||_2 \leq \sqrt{n}||T : l_2^n \to l_2^n||$$

for $i = 1, \ldots, n$, and this implies that $U \subset \overline{M}_B$. Now we find that each of the balls $y + \frac{\epsilon}{2}U$, for $y \in \overline{N}_B$, are contained in $(1 + \frac{\epsilon}{2})\overline{M}_B$. Now the corresponding set $N_B \in L(\mathbb{R}^n)$ of course forms an ϵ-net in M_B. We now find from the above

$$K\left(\frac{\epsilon}{2}\right)^{n^2} \mathrm{vol}\,(U) = \mathrm{vol}\,\left(\cup_{T \in N_B}(T + \frac{\epsilon}{2}U)\right) \leq \mathrm{vol}\,\left((1 + \frac{\epsilon}{2})\overline{M}_B\right) = \left(1 + \frac{\epsilon}{2}\right)^{n^2} \mathrm{vol}\,(\overline{M}_B).$$

From this we find the inequality $K \leq \left(1 + \frac{2}{\epsilon}\right)^{n^2} \frac{\mathrm{vol}\,(\overline{M}_B)}{\mathrm{vol}\,(U)}$, so it remains to find an upper bound for $\mathrm{vol}\,(\overline{M}_B)$ and a lower estimate for $\mathrm{vol}\,(U)$. The lower estimate $\mathrm{vol}\,(U) \geq (bn)^{n^2/2}\mathrm{vol}\,(U_2)$, where $b > 0$ is a universal constant and U_2 is the Euclidean unit ball in $\mathbb{R}^{n^2}$, is given in Appendix B. For the upper bound, notice that $\overline{M}_B = (\sqrt{n}B(E_B))^n$ holds. We find, from the estimate in Appendix A, for n sufficiently large,

$$\mathrm{vol}\,(\overline{M}_B) = (\mathrm{vol}\,(\sqrt{n}B(E_B)))^n \leq \left(2e^2\frac{m+n}{n^{3/2}}\right)^{n^2}.$$

Combining all this with $\mathrm{vol}\,(U_2) = \frac{\pi^{n^2/2}}{\Gamma(1+n^2/2)}$ and using Stirling's formula for the Gamma function, we find, for n sufficiently large, a universal constant $a > 0$ such that

$$\mathrm{card}\,(N_B) = K \leq \left(1 + \frac{2}{\epsilon}\right)^{n^2}\left(2e^2\frac{m+n}{n^{3/2}}\right)^{n^2}\frac{\Gamma(1+n^2/2)}{(b\pi n)^{n^2/2}} \leq \left(a\frac{m+n}{\epsilon n}\right)^{n^2}$$

holds. Now note we can always choose a such that $a > 1$. □

With these two lemma we are ready to prove Gluskin's theorem.

Proof of theorem 4.3:
Choose n sufficiently large for lemmas 4.6 and 4.7 to hold, set $m = 2n$ and let $\alpha > 1$ and $a > 1$ be

the universal constants as before. Now choose $0 < \rho < \frac{1}{18a\alpha}$ and $6a\rho^2 < \epsilon < \frac{\rho}{3\alpha}$ (the condition on ρ is precisely what makes this possible). Then we have $0 < \epsilon < 1$. Now fix a $B \in \mathcal{A}_{2n}$ and consider the set

$$B' = \left\{ A \in \mathcal{A}_{2n} : ||T : E_A \to E_B|| < \left(\frac{\rho}{3\alpha} - \epsilon\right)\sqrt{n} \text{ for some } T \in L(\mathbb{R}^n) \text{ with } |\det(T)| = 1 \right\}.$$

This set is open in $\mathcal{A}_{2n}$ and thus λ^{2n}-measurable. We want to show that

$$\lambda^{2n}(B') \leq \left(\frac{3a\rho^2}{\epsilon}\right)^{n^2} < \left(\frac{1}{2}\right)^{n^2}. \tag{10}$$

Hereto, let $N_B = \{T_k\} \subset M_B$ be the ϵ-net constructed in lemma 4.7. Choose $A \in B'$ and let $T \in L(\mathbb{R}^n)$ be such that $||T : E_A \to E_B|| < (\frac{\rho}{3\alpha} - \epsilon)\sqrt{n}$. Since we have $\frac{\rho}{3\alpha} - \epsilon < \frac{\rho}{3\alpha} < 1$, we find $T \in M_B$. Thus there exists a $T_k \in N_B$ with $||(T - T_k) : l_2^n \to l_2^n|| < \epsilon$. Now from the inequality $|| \cdot ||_2 \leq || \cdot ||_{E_C} \leq \sqrt{n}|| \cdot ||_2$, for any $C \in \mathcal{A}_{2n}$, we find that $||S : E_A \to E_B|| \leq \sqrt{n}||S : l_2^n \to l_2^n||$ for all $S \in L(\mathbb{R}^n)$. From all this we find

$$||T_k : E_A \to E_B|| \leq ||T : E_A \to E_B|| + ||(T - T_k) : E_A \to E_B|| < \left(\frac{\rho}{3\alpha}\right)\sqrt{n},$$

and thus we have

$$B' \subset \cup_{k=1}^{K} \left\{ A \in \mathcal{A}_{2n} : ||T_k : E_A \to E_B|| \leq \left(\frac{\rho}{3\alpha}\right)\sqrt{n} \right\}.$$

Now we use lemmas 4.6 and 4.7 to conclude that we have

$$\lambda^{2n}(B') \leq \sum_{k=1}^{K} \lambda^{2n} \left\{ A \in \mathcal{A}_{2n} : ||T_k : E_A \to E_B|| \leq \left(\frac{\rho}{3\alpha}\right)\sqrt{n} \right\} \leq \left(\frac{3a}{\epsilon}\right)^{n^2} \rho^{2n^2} = \left(\frac{3a\rho^2}{\epsilon}\right)^{n^2} < \left(\frac{1}{2}\right)^{n^2},$$

so we have shown that (10) holds.
Now consider the subset $G \subset \mathcal{A}_{2n} \times \mathcal{A}_{2n}$ given by

$$G = \left\{ (A, B) \in \mathcal{A}_{2n} \times \mathcal{A}_{2n} : ||T : E_A \to E_B|| < \left(\frac{\rho}{3\alpha} - \epsilon\right)\sqrt{n} \text{ or } \right.$$
$$\left. ||T : E_B \to E_A|| < \left(\frac{\rho}{3\alpha} - \epsilon\right)\sqrt{n} \text{ for some } T \in L(\mathbb{R}^n) \text{ with } |\det(T)| = 1 \right\}.$$

Now Fubini's theorem for $\lambda^{2n} \times \lambda^{2n}$ and inequality (10) imply that

$$(\lambda^{2n} \times \lambda^{2n})(G) < 2\left(\frac{1}{2}\right)^{n^2} \leq 1$$

holds. This implies the complement of G in $\mathcal{A}_{2n} \times \mathcal{A}_{2n}$ is non-empty, and this precisely means that there exists a pair E_A, E_B of spaces satisfying the conditions of theorem 4.3, for $c = \frac{\rho}{3\alpha} - \epsilon > 0$.
Now we have shown theorem 4.3 holds for n sufficiently large, and we can always choose $c > 0$ universally such that this theorem holds for all $n \in \mathbb{N}$. □

5 Additional results

Now that we have shown that Gluskin's theorem holds, the first thing we could ask ourselves is: what do the spaces E_A, for $A \in \mathcal{A}_m$, look like? Fortunately, these spaces are not as exotic as they may appear. In fact, we can show that these spaces are isometric to quotients of l_1^{3n}, the space $\mathbb{R}^{3n}$ with the absolute norm. If we have $A = (f_1, \ldots, f_{2n}) \in \mathcal{A}_{2n}$, the map $Q : l_1^{3n} \to E_A$ given by $Q(e_i) = e_i$ for $i = 1, \ldots, n$

and $Q(e_i) = f_{i-n}$ for $i = n+1, \dots, 3n$, is a quotient map. So in a sense, we can view the spaces E_A and E_B satisfying theorem 4.3 as 'random' quotients of l_1^{3n}.
Also, since we can easily show that the Banach-Mazur distance is invariant under the taking of duals, i.e. $d(X,Y) = d(X^*, Y^*)$ holds for all spaces, and the dual of a quotient of l_1^{3n} is isomorphic to a subspace of l_∞^{3n}, we can find subspaces of l_∞^{3n} satisfying theorem 4.3.
The second thing to remark is that Gluskin's technique has inspired some other results, two of which we shall state here. To this end, we must first remark that we can introduce the Banach-Mazur distance for complex spaces, and most results, including theorem 4.3, transfer unchanged. The two results we discuss are from [Szarek 1986].
The first is a theorem that asserts that, for $n \in \mathbb{N}$, the set of $2n$-dimensional real spaces admitting a complex structure is not dense in $\mathcal{F}_{2n}$.

Theorem 5.1. There exists a constant $c > 0$ such that for all $n \in N$ there exists a $2n$-dimensional real Banach space E which has the following property: whenever F is an n-dimensional complex Banach space and F_R is F treated as a $2n$-dimensional real space, then the (real) Banach-Mazur distance between E and F_R satisfies $d(E, F_R) \geq c\sqrt{n}$.

Now our second result, also due to Szarek, is the following:

Theorem 5.2. There exists a constant $c > 0$ such that for every $n \in \mathbb{N}$, there exist n-dimensional complex Banach spaces E and F which are isometric as $2n$-dimensional real spaces, but with complex Banach-Mazur distance $d(E, F) \geq cn$.

It implies that n-dimensional complex spaces which are isometric as $2n$-dimensional real spaces need not be isometric as complex spaces.

6 Conclusion

We have introduced the Banach-Mazur distance, explored some of its properties and seen how this distance leads to the introduction of the Minkowski compactum $\mathcal{F}_n$, our 'space of spaces'. We have seen that this space is bounded and indeed compact, and have in fact that its diameter is bounded by n. We then wondered how accurate this bound is, and have investigated an answer to this question by Gluskin. He used a meausure-theoretic approach to prove the existence of spaces with Banach-Mazur distance approximately n. We have delved into his proof and expanded the short version he originally gave, expanding it so it can hopefully be understood by a mathematics student at master's level, with some effort.
All in all, we hope to have shared with the reader Gluskin's intriguing technique, and to have raised his or her interest in general functional analysis and measure theory.

A Upper bound for the volume of a unit ball

In this appendix we give, for n sufficiently large, the following upper bound for the volume of the unit ball $B(E_A)$ of E_A, for $A \in \mathcal{A}_m$:

$$\operatorname{vol}(B(E_A)) \leq \left(2e^2 \frac{m+n}{n^2}\right)^n.$$

Hereto, we first show that

$$\operatorname{vol}(B(E_A)) \leq \sum \operatorname{vol}(\sigma(x_1, \dots, x_n)),$$

holds, where the summation runs over all choices $\{x_1, \dots, x_n\} \subset \{\pm e_i, \pm g_j : i = 1, \dots, n, j = 1, \dots, m\}$ of n distinct elements, and where $\sigma(x_1, \dots, x_n) = \operatorname{conv}(0, x_1, \dots, x_n)$.
The first thing to note here is that we have

$$B(E_A) \subset \cup \sigma(x_1, \dots, x_{n+1}),$$

where the summation runs over all choices of $n+1$ distinct elements, by Carathéodory's theorem [Tyrrell Rockafellar 1970, 155]. So it remains to show we have $\sigma(x_1, \dots, x_{n+1}) \subset \cup \sigma(x_{k_1}, \dots, x_{k_n})$, where the union runs over all choices of n distinct elements from $\{x_1, \dots, x_n\}$. To this end, let $T : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be given by $T(e_i) = x_i$ and denote $C := \sigma(e_1, \dots, e_{n+1}) \subset \mathbb{R}^{n+1}$. Then we have $T[C] = \sigma(x_1, \dots, x_{n+1})$, and so we need to show that $T[C] = \cup_{i=1}^{n+1} T(C \cap R_i)$ holds, where $R_i \subset \mathbb{R}^{n+1}$ is the subset of elements whose i-th coordinate is 0.
Hereto, choose $x \in C$ and $v \in \ker(T)$, $v \neq 0$. Then we have $T(x + \lambda v) = T(x)$ for all $\lambda \in \mathbb{R}$, and the line $\cup_{\lambda \in \mathbb{R}}(x + \lambda v) \subset \mathbb{R}^{n+1}$ has non-empty intersection with $\cup_{i=1}^{n+1} R_i$, so there exists an i such that $(\cup_{\lambda \in \mathbb{R}}(x + \lambda v)) \cap R_i \neq \emptyset$ holds. This implies we indeed have $T[C] = \cup_{i=1}^{n+1} T[C \cap R_i]$, and thus also

$$\sigma(x_1, \dots, x_{n+1}) = T[C] = \cup_{i=1}^{n+1} T[C \cap R_i] = \cup_{i=1}^{n+1} \sigma(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1}).$$

From all this we find $B(E_A) \subset \sum \sigma(x_1, \dots, x_{n+1})$.
Each of the simplices $\sigma(x_1, \dots, x_n)$ has volume $\operatorname{vol}(\sigma(x_1, \dots, x_n)) = \operatorname{vol}(\sigma(e_1, \dots, e_n)) \cdot |\det(X)|$, where X is the matrix $\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$. Now Hadamard's inequality implies we have $|\det(X)| \leq \prod_{i=1}^n ||x_i||_2 = 1$, since $x_i \in S_{n-1}$ holds. From all this we find

$$\operatorname{vol}(B(E_A)) \leq \sum \operatorname{vol}(\sigma(x_1, \dots, x_n)) \leq \binom{2(m+n)}{n} \operatorname{vol}(\sigma(e_1, \dots, e_n)).$$

Now, we have $\operatorname{vol}(\sigma(e_1, \dots, e_n)) = \int_{\sigma(e_1, \dots, e_n)} dx = \frac{1}{n!}$, and then Stirling's formula implies, for n sufficiently large, the bound

$$\operatorname{vol}(B(E_A)) \leq \binom{2(m+n)}{n} \frac{1}{n!} \leq \left(2e^2 \frac{m+n}{n^2}\right)^n.$$

B Lower bound for the volume of a unit ball

In this section we give a lower bound for the volume of the unit ball U of the space $(\mathbb{R}^{n^2}, ||\cdot||_{\mathrm{op}})$. Hereto, let U_2 denote the Euclidean unit ball in $\mathbb{R}^{n^2}$. Since both sets are closed, they are Lebesgue measurable. We will go on to prove the following

Lemma B.1. We have

$$\operatorname{vol}(U) \geq \operatorname{vol}(U_2)(bn)^{n^2/2},$$

where $b > 0$ is a universal constant.

For the moment, let λ be the unique normalized rotation invariant measure on $S := S_{n^2-1}$. We first show that

$$\mathrm{vol}(U) = \mathrm{vol}(U_2) \cdot \int_S ||x||_{\mathrm{op}}^{-n^2} d\lambda(x) \tag{11}$$

holds. Since U is Lebesgue measurable, so is its indicator function I. We now integrate in polar coordinates ([Folland 1999, p. 78]) to find a rotation invariant Borel measure σ on S such that

$$\mathrm{vol}\,(U) = \int_{\mathbb{R}^{n^2}} I(x)dx = \int_S \int_0^\infty I(r \cdot x')r^{n^2-1}dr\, d\sigma(x'),$$

where we have $x' = \frac{x}{||x||} \in S$ and $r = ||x|| \in [0, \infty)$. Since this σ is not normalized ($\sigma(S) = \frac{2\pi^{n^2/2}}{\Gamma(n^2/2)}$ [Folland 1999, 79]), we normalize it to find

$$\mathrm{vol}\,(U) = \frac{2\pi^{n^2/2}}{\Gamma(n^2/2)} \cdot \int_S \int_0^\infty I(r \cdot x')r^{n^2-1}dr\, d\lambda(x').$$

Now there exists, for each $x' \in S$, a $R(x') \in [0, \infty)$ such that $I(rx') = 0$ holds for all $r > R(x')$ and $I(rx') = 1$ for all $r \leq R(x')$. We find

$$\mathrm{vol}\,(U) = \frac{2\pi^{n^2/2}}{\Gamma(n^2/2)} \cdot \int_S \int_0^{R(x')} r^{n^2-1}dr\, d\lambda(x') = \frac{2\pi^{n^2/2}}{\Gamma(n^2/2)} \cdot \frac{1}{n^2} \int_S R(x')^{n^2} d\lambda(x').$$

To determine $R(x')$, note that we have $1 = ||R(x') \cdot x'||_{\mathrm{op}} = R(x')||x'||_{\mathrm{op}}$ and thus $R(x') = ||x'||_{\mathrm{op}}^{-1}$. From this we find

$$\frac{2\pi^{n^2/2}}{\Gamma(n^2/2)} \cdot \frac{1}{n^2} \int_S R(x')^{n^2} d\lambda(x') = \frac{2\pi^{n^2/2}}{\Gamma(1+n^2/2)} \cdot \int_S ||x'||_{\mathrm{op}}^{-n^2} d\lambda(x') = \mathrm{vol}\,(U_2) \cdot \int_S ||x'||_{\mathrm{op}}^{-n^2} d\lambda(x'),$$

so we have indeed shown (11).

Now Hölder's inequality and the fact that the L_p-norm is monotonically increasing on S imply that we have

$$1 = \int_S 1 d\lambda(x) \leq (\int_S ||x||_{\mathrm{op}}^2 d\lambda(x))^{1/2} (\int_S ||x||_{\mathrm{op}}^{-2} d\lambda(x))^{1/2} \leq (\int_S ||x||_{\mathrm{op}}^2 d\lambda(x))^{1/2} (\int_S ||x||_{\mathrm{op}}^{-n^2} d\lambda(x))^{1/n^2}$$

for $n > 1$, and thus we find

$$\mathrm{vol}\,(U) = \mathrm{vol}\,(U_2) \cdot \int_S ||x||_{\mathrm{op}}^{-n^2} d\lambda(x) \geq \mathrm{vol}\,(U_2) (\int_S ||x||_{\mathrm{op}}^2 d\lambda(x))^{-n^2/2}.$$

Now we look to give a lower bound for the integral $\int_S ||x||_{\mathrm{op}}^2 d\lambda(x)$. Hereto, we pass to Gaussian variables.

Lemma B.2. Let $||\cdot||$ be a norm on $\mathbb{R}^m$, (Ω, μ) a probability space and $\{g_j\}$, for $j = 1, \ldots, m$, a sequence of independent standard Gaussian random variables on Ω. Then we have

$$\int_{S_{m-1}} ||x||^2 d\lambda(x) = \frac{1}{m} \int_\Omega ||\sum_{j=1}^n g_j(\omega)e_j||^2 d\mu(\omega), \tag{12}$$

where $\{e_j\}$ is an orthonormal basis in $\mathbb{R}^{m-1}$.

Proof:
We first show that, for general continuous $f : S_{m-1} \to \mathbb{R}$ and $\overline{f} : \mathbb{R}^m \to \mathbb{R}$ a 2-homogeneous extension of f to $\mathbb{R}^m$ given by $\overline{f}(x) = f(\frac{x}{||x||_2})||x||_2^2$, we have

$$\int_{S_{m-1}} f(x)d\lambda(x) = \frac{1}{m}\int_{\mathbb{R}^m} \overline{f}(x)d\gamma(x) \tag{13}$$

holds, where γ is the Gaussian measure on $\mathbb{R}^m$.
To see that this is the case, note that $l : C(S_{m-1}) \to \mathbb{R}$ given by $l(g) = \int_{\mathbb{R}^m} \overline{g}(x)d\gamma(x)$, where $\overline{g}$ is the 2-homogeneous extension given before, is a positive linear functional on $C(S_{m-1})$. Also note that this functional is rotation-invariant. By the Riesz representation theorem we find a regular rotation-invariant Borel measure λ' on S_{m-1} such that

$$\int_{\mathbb{R}^m} \overline{g}(x)d\gamma(x) = l(g) = \int_{S_{m-1}} g(x)d\lambda'(x)$$

holds for all $g \in C(S_{m-1})$. Now since a rotation-invariant Borel measure is unique up to multiplication by constants, we find a $K > 0$ such that

$$\int_{S_{m-1}} f(x)d\lambda(x) = K\int_{\mathbb{R}^m} \overline{f}(x)d\gamma(x)$$

holds. Evaluating for $f(x) = ||x||_2$, we indeed find $K = \frac{1}{m}$.
Now because the coordinate functionals $x \mapsto (x, e_j)$ are independent standard Gaussian random variables on $(\mathbb{R}^n, \gamma)$, the map $\psi : \Omega \to \mathbb{R}^m$ given by $\psi(\omega) = (g_1(\omega), \dots, g_m(\omega))$, is $(\mathbb{R}^m, \gamma)$-measurable. Thus we can show that

$$\int_{\mathbb{R}^m} ||x||^2 d\gamma(x) = \int_{\mathbb{R}^m} ||\sum_{j=1}^m (x, e_j)e_j||^2 d\gamma(x) = \int_\Omega ||\sum_{j=1}^m g_j(\omega)e_j||^2 d\mu(\omega). \tag{14}$$

Combining (13), for $f(x) = ||x||^2$, and (14), we indeed find (12). □

Proof of Lemma B.1:
Using Lemma B.2 on the norm $||\cdot||_{\text{op}}$ on $\mathbb{R}^{n^2}$, we find

$$\text{vol}\,(U) \geq \text{vol}\,(U_2) \cdot (\frac{1}{n^2}\int_\Omega ||[g_{ij}(\omega)]||_{\text{op}}^2 d\mu(\omega))^{-n^2/2},$$

where (Ω, μ) is a probability space and $\{g_{ij}\}$, for $i, j = 1, \dots, n$, is a sequence of independent standard Gaussian random variables on Ω.
Now we use an inequality from [Chevet 1977], to find

$$\int_\Omega ||[g_{ij}(\omega)]||_{\text{op}}^2 d\mu(\omega))^{1/2} \leq b'\sqrt{n}, \tag{15}$$

where $b' > 0$ is a universal constant. Another more elementary but still quite involved argument showing (15) can be found as lemma 38.6 in [Tomczak-Jaegermann 1989, p. 288].
From all this we find

$$\text{vol}\,(U) \geq \frac{n^{n^2} \cdot \text{vol}\,(U_2)}{(b'\sqrt{n})^{n^2}} = (bn)^{n^2/2}\text{vol}\,(U_2),$$

where $b > 0$ is a universal constant. □

References

[Banach 1932] S. Banach, *Théorie des operations linéaires*. PWN, Warszawa, 1932.

[Bollobás 1990] B. Bollobás, *Linear Analysis : an Introductory Course. Second Edition*. Cambridge University Press, Cambridge, 1990.

[Chevet 1977] S. Chevet, *Séries de variable aléotoires Gaussiens à valeur dans* $E\check{\otimes}F$. Ecole Polytechnique, Palaiseau, 1977.

[Conway 1990] J. B. Conway, *A Course in Functional Analysis. Second Edition*. Springer-Verlag New York Inc., New York, 1990.

[Folland 1999] G. B. Folland, *Real analysis: modern techniques and their applications. Second edition*. John Wiley & Sons Inc., New York, 1999.

[Gluskin 1981] E. D. Gluskin *The diameter of the Minkowski compactum is approximately equal to* n. Funct Anal Appl (1) **15** (1981), 72-73.

[Pelczynski and Bessega 1979] A. Pelczynski, C. Bessega, *Some aspects of the present theory of Banach spaces, Stefan Banach, Oeuvres*, Vol. II. PWN, Warszawa, 1979.

[Szarek 1986] S. J. Szarek, *On the existence and uniqueness of complex structure and spaces with 'few' operators*. Transactions of the American Mathematical Society **293** (1986), 339-353.

[Tomczak-Jaegermann 1989] N. Tomczak-Jaegermann, *Banach-Mazur distances and finite-dimensional operator ideals*. Longman Scientific & Technical, Burnt Mill, 1989.

[Tyrrell Rockafellar 1970] R. Tyrrell Rockafellar, *Convex Analysis*. Princeton University Press, Princeton, 1970.