Schubert Calculus

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Contents

Preface

In this Bachelor Thesis, we will explain a calculus named Schubert Calculus. Schubert Calculus is invented by Hermann Cäsar Hannibal Schubert around the end of the nineteenth century. This calculus allowed Schubert and his successors to solve many enumerative problems in geometry, although they didn't have rigorous proofs of the rules in this calculus. This is the reason why Hilbert's 15-th problem concerns with this calculus, and nowadays most of the rules in this calculus are finally formalized (through topology and intersection theory). The main purpose of this Bachelor Thesis is to explain the rules of this Schubert Calculus and solve some enumerative problems.

The first chapter introduces the Grassmann Variety (mainly from [KL]), and the second chapter gives some basic facts about the cohomology ring of this Grassmann Variety (mainly based on [KL], [FU] and [ST]). In the third and the fifth chapter we will develop the calculus in this cohomology ring (mainly from [KL] and [ST]). The fourth chapter shows the power of the Schubert Calculus by solving several enumerative problems (many of which are new).

I have decided not to include complete proofs of the formulae from the second chapter, since the complete proofs I know are very technical (although we will give a sketch). Proofs can be found, for example, in [GH] (although it contains some errors), [FU] (as exercises) and [HP] (but this is hard to read). For more details and proofs of Chapter Five, I suggest to read [FU].
I have also decided not to include (part of) the theory of Schubert Polynomials and Varieties, which is a current area of research, since a detailed introduction can be found in [FU].

Remark. In this thesis, we will work over $\mathbb{C}$.

1 The Grassmann Variety

To solve enumerative problems in geometry, we first need a good description of the d-planes in a $\mathbb{P}^n$, and for this one uses the Grassmann Variety (Grassmannian).

From linear algebra we know that a d-plane S_d in a projective space can be represented by a basis consisting of $d+1$ vectors, say by $p_i = (p_i(0), \ldots, p_i(n)) \in \mathbb{A}^{n+1}$ for $i = 0, \ldots, d$. We can put these vectors in a $(d+1) \times (n+1)$-matrix as follows:

$$M(p_i) = \begin{pmatrix} p_0(0) & p_0(1) & \ldots & p_0(n) \\ p_1(0) & p_1(1) & \ldots & p_1(n) \\ \vdots & \vdots & \ddots & \vdots \\ p_d(0) & p_d(1) & \ldots & p_d(n) \end{pmatrix}$$

Now it is natural to look at the $\binom{n+1}{d+1}$ determinants of the $(d+1) \times (d+1)$-submatrices of this matrix. We claim that this gives us a map from the set of d-planes to a subset of $\mathbb{P}^N$ where $N = \binom{n+1}{d+1} - 1$. First we denote $p(j_0, \ldots, j_d)$ for the determinant of the $(d+1) \times (d+1)$-submatrix of our $M(p_i)$ consisting of the columns $j_0, \ldots, j_d$ where $0 \leq j_0 < j_1 < \ldots < j_d \leq n$. We first need to show that this map is well-defined. Suppose our S_d has another basis q_i for $i = 0, \ldots, d$ then linear algebra gives us an invertible linear map A which converts the first basis into the second one. We now notice that $q(i_0, \ldots, i_d) = \det(A) p(i_0, \ldots, i_d)$, and since we are working in a projective space, our map is well-defined. Also notice that the image of an S_d is never zero, since the p_i defining it are linearly independent (so there is a non-vanishing minor).
The image of this map is even a smooth projective variety in $\mathbb{P}^N$. We call this variety $G_{d,n}$, the *Grassmann Variety* of d-planes in $\mathbb{P}^n$. This important fact is given in the next theorem:

1.1. Theorem. *Let (QR) be the quadratic relations given by*

$$\sum_{i=0}^{d+1} (-1)^i p(j_0, \ldots, j_{d-1}, k_i) p(k_0, k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{d+1}) = 0$$

where $j_0, \ldots, j_{d-1}$ and $k_0, \ldots, k_{d-1}$ are any sequence of integers with $0 \leq j_r, k_s \leq n$. Then the set of d-planes in $\mathbb{P}^n$ is in bijection with the set of points in $\mathbb{P}^N$ satisfying (QR), $G_{d,n}$, under the map just defined.
Moreover, there is a natural bijection from the set of points in $\mathbb{P}^N$ satisfying both (QR) and $p(k_0, \ldots, k_d) \neq 0$ to the affine $(d+1)(n-d)$-dimensional space of $(d+1) \times (n+1)$-matrices $[p_i(j)]$ satisfying the condition that the submatrix $[p_i(k_s)]$ for $s = 0, \ldots, d$ is the identity matrix. The image of a point $(\ldots : p(j_0, \ldots, j_d) : \ldots)$ is given by the $(d+1) \times (n+1)$-matrix with entries given by

$$p_i(j) = \frac{p(k_0, \ldots, k_{i-1} j k_{i+1}, \ldots, k_d)}{p(k_0, \ldots, k_d)}$$

Finally, $G_{d,n}$ is irreducible of dimension $(d+1)(n-d)$ and non-singular.

Proof. The proof is rather technical and can be found for example in [KL], pages 1063–1066. We will sketch why $G_{d,n}$ is a complex manifold of dimension $(d+1)(n-d)$. When fixing a chart, we get all the $(d+1)(n+1)$ matrices of dimension $d+1$ such that some fixed columns give an I_{d+1}. The rest of the matrix consists of $(d+1)(n-d)$ spots which can be filled freely with elements of $\mathbb{C}$. As a consequence, $G_{d,n}$ is even non-singular. $\square$

1.2. Remark. Notice that $G_{0,n} = \mathbb{P}^n$.

1.3. Remark. There is also another natural way to define the Grassmann variety: instead of looking at a basis for a d-plane, we look at the equations defining the d-plane. We can map these equations, using determinants of the minors, to a $\mathbb{P}^{\binom{n+1}{n-d}-1} = \mathbb{P}^{\binom{n+1}{d+1}-1}$. In this Bachelor Thesis, we will only use the notion given earlier.

We now want to define a class of very important subvarieties of $G_{d,n}$, the Schubert Cycles:

1.4. Definition. A *flag* is a chain $A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_d$ of linear subspaces of $\mathbb{P}^n$. We define, using the embedding above, the corresponding *Schubert Cycle* as follows:

$$\Omega(A_0, A_1, \ldots, A_d) := \{L\ d\text{−plane} : \dim(A_i \cap L) \geq i\} \subset G_{d,n}$$

1.5. Theorem. *The set $\Omega(A_0, A_1, \ldots, A_d) \subset G_{d,n}$ is given by linear equations, hence it has the natural structure of a subvariety of $G_{d,n}$. If $\dim(A_i) = a_i$, then $\Omega(A_0, A_1, \ldots, A_d)$ has dimension $\sum_{i=0}^{d}(a_i - i)$.*

Proof. The first part of the proof can be found in [KL], page 1066. For the second part, we choose a basis such that A_i consists of the vectors of the form $(c_0, \ldots, c_{a_i}, 0, \ldots, 0)$ where the $c_i \in \mathbb{C}$. We now choose an affine chart of $\mathbb{P}^N$ defined by $p(a_0, a_1, \ldots, a_d) \neq 0$, and we see that the points of our subvariety can be given by a matrix where the first row looks like $(c_{0,0}, \ldots, c_{0,a_0-1}, 1, 0, \ldots, 0)$, the second row looks like $(c_{1,0}, \ldots, c_{1,a_0-1}, 0, c_{1,a_0+1}, \ldots, c_{a_1-1}, 1, 0, \ldots, 0)$ and so on. We directly see that our dimension (the number of $c_{i,j}$'s which are to be chosen freely) is equal to $a_0 + (a_1 - 1) + \ldots + (a_d - d)$, and this gives us the formula. $\square$

2 Facts from Cohomology

In this section, we will give a short introduction to the cohomology ring of the Grassmann Variety. We will first need some facts from Algebraic Topology (from [FU]):

2.1. Facts.

1. One can assign a cohomology group $H^*(Y, \mathbb{Z}) = \bigoplus_{i=0}^{2\dim(Y)} H^i(Y, \mathbb{Z})$ to a non-singular projective variety Y, and this group becomes a graded ring under the cup product.
2. An irreducible closed subset Z of codimension d in Y determines a class $[Z] \in H^{2d}(Y, \mathbb{Z})$.
3. If Y has dimension n, then $H^{2n}(Y, \mathbb{Z}) \cong \mathbb{Z}$. It is generated by the class corresponding to a point, say $[.]$.
4. If two closed subsets $Z_1, Z_2 \subset Y$ meet transversally in t points, then $[Z_1] \cdot [Z_2] = t[.] \in H^{2n}(Y, \mathbb{Z}) \cong \mathbb{Z}$.
5. If Y has a filtration $\emptyset \subset Y_1 \subset \ldots \subset Y_s = Y$ by algebraic sets, and $Y_i \setminus Y_{i-1}$ is a disjoint union of varieties $U_{i,j}$ each isomorphic to an affine space $\mathbb{C}^{n_{i,j}}$ (we usually refer to this as a cell decomposition), then the classes of the closures of these $U_{i,j}$, $[\bar{U}_{i,j}]$ give an additive basis for $H^*(Y, \mathbb{Z})$.

In our case we have:

2.2. Facts.

1. We can assign a cohomology group to $G_{d,n}$: $H^*(G_{d,n},\mathbb{Z}) = \bigoplus_{i=0}^{2(n-d)(d+1)} H^i(G_{d,n},\mathbb{Z})$.
2. An irreducible closed subset X of $G_{d,n}$ determines a class $[X] \in H^{2((d+1)(n-d)-\dim(X))}$.

2.3. Definition. The element corresponding to $\Omega(A_0, A_1, \ldots, A_d)$ in the cohomology ring only depends on the dimensions of the A_i, say a_i (since we can transform flags into each other by means of linear transformations). We denote the corresponding element by $\Omega(a_0, a_1, \ldots, a_d)$.

This cohomology ring becomes a very important instrument in studying intersection theory, as can be seen from fact 2.1.4. First we want to understand the group structure of this cohomology ring completely:

2.4. Theorem. *(Basis Theorem) Every odd part of the cohomology group is* 0*, and* $H^{2p}(G_{d,n},\mathbb{Z})$ *is a free abelian group with basis the Schubert Cycles* $\Omega(a_0,\ldots,a_d)$ *satisfying* $[(d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)] = p$*. As a consequence, our graded ring becomes commutative.*

Proof. First fix a standard flag $V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{P}^n$, using the standard basis $\{e_0,\ldots,e_n\}$ of $\mathbb{A}^{n+1}$. We will construct a cell decomposition (Fact 2.1.5). We take a trivial filtration, $\emptyset \subset G_{d,n}$. We claim that the cells are given by sets of the form

$$W_{i_0,\ldots,i_d} = \{L\ d\text{−plane} : \dim(V_{i_j} \cap L) = j\} \subset G_{d,n}$$

where $0 \le i_0 < \ldots < i_d \le n$. First notice that the sets are disjoint, and that every d-plane is in a $W_{i_0,\ldots,i_d}$. These cells are complex manifolds of dimension $\sum_{j=0}^{d}(i_j - j)$ (the proof is basically the same as the proof of Theorem 1.5, where we showed how to construct one chart. The construction of the other charts goes analogously). The closure of such a cell is

$$\{L\ d\text{−plane} : \dim(V_{i_j} \cap L) \ge j\} \subset G_{d,n}$$

and these are just the Schubert Cycles. By Fact 2.1.5, we see that the Schubert Cycles with $[(d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)] = p$ form a basis for $H^{2p}(G_{d,n},\mathbb{Z})$ and also that $H^{2i+1}(G_{d,n},\mathbb{Z}) = 0$. A consequence of the fact that the parts of odd degree of the cohomology group are zero, is that the ring becomes commutative: this follows from Theorem 3.14 in [HAT]. □

2.5. Remark. Due to the last theorem, some $H^{2p}(G_{d,n},\mathbb{Z})$ become extremely easy. For example, $H^0(G_{d,n},\mathbb{Z})$ has basis $\{\Omega(n-d, n-d+1,\ldots,n)\}$. The one element belonging to this basis is the class of $G_{d,n}$ itself: every d-plane intersects a $n-d+i$ plane in a space of dimension at least i. Another interesting case is $H^{2(d+1)(n-d)}(G_{d,n},\mathbb{Z})$. This time, our basis is equal to $\{\Omega(0,1,\ldots,d)\}$. There is actually one plane meeting the requirements of the corresponding flag, so it is the class of a point in $G_{d,n}$. So an element x in $H^{2(d+1)(n-d)}(G_{d,n},\mathbb{Z})$ is uniquely represented by $\lambda\Omega(0,1,\ldots,d)$ for a $\lambda \in \mathbb{Z}$, and this λ is called the degree of x. We will usually write just λ instead of $\lambda\Omega(0,1,\ldots,d)$, since this λ is often the solution to an enumerative problem.

2.6. Remark. We define β_i, the i-th *Betti Number* of $G_{d,n}$, as $\beta_i = \text{rank } H^i(G_{d,n},\mathbb{Z})$. We want to calculate this number for even i (since the odd ones are always zero). It follows from the Basis Theorem, that β_{2i} is the number of solutions in integers $0 \le a_0 < a_1 < \ldots < a_d \le n$ to $[(d+1)(n-d) - \sum_{j=0}^{d}(a_j - j)] = i$. We can form the Poincaré series $\sum_i \beta_{2i} q^i$. Using combinatorics, one can show that

$$\sum_i \beta_{2i} q^i = \frac{\prod_{i=0}^{d}(q^{n+1-i} - 1)}{\prod_{i=0}^{d}(q^{d+1-i} - 1)}$$

If q is a prime power, then the number we just got is equal to the number of d-planes of an n-dimensional projective space over $\mathbb{F}_q$. This looks like a coincidence, but the Weil Conjectures show that this is not the case.

2.7. Facts. Let X, Y be subvarieties of $G_{d,n}$ which intersect proper. Then we have that $[X]\cdot[Y] = \sum n(X,Y,W)[W]$, where the sum runs over the irreducible components of $X \cap Y$, and $n(X,Y,W)$ is some intersection multiplictity. If X and Y don't intersect proper, one can 'move' X to another X' with $[X] = [X']$ such that the intersection of X' and Y is proper (by Chow's Moving Lemma, see [HAG], page 427).
In our calculations, we will always end up in the highest nonzero degree, and we find an element of the form $\lambda\Omega(0,1,\ldots,d) = \lambda$. This number tells us how many intersection points varieties which have the same cohomology classes as X and Y in general have. If we end up too high (so we get automatically zero), we have no intersection points in general, and if we end in a class below the highest one, then there are in general an infinite amount of intersection points.

So it is very important to know how to do the multiplication in $H^*(G_{d,n},\mathbb{Z})$, and the next section gives us easy formulae which determine the complete multiplicative structure.

3 Formulae

In this section we will state some formulae which enable us to do calculations in the cohomology ring of the Grassmann Variety. The first formula is Poincaré Duality, which is very useful in many cases:

3.1. Theorem. *(Poincaré Duality) The two bases* $(\ldots,\Omega(a_0,\ldots,a_d),\ldots)$ *of* $H^{2p}(G_{d,n},\mathbb{Z})$ *and* $(\ldots,\Omega(n-a_d,\ldots,n-a_0),\ldots)$ *of* $H^{2((d+1)(n-d)-p)}(G_{d,n},\mathbb{Z})$ *are dual under the cup product.*

Proof. We will give a geometric proof of this duality. Take two Schubert Cycles $\Omega(a_0,\ldots,a_d)$ and $\Omega(b_0,\ldots,b_d)$ such that $\sum_{i=0}^d (a_i - i) = (d+1)(n-d) - \sum_{i=0}^d (b_i - i)$. Rewriting gives us $\sum_{i=0}^d (a_i + b_i - 2i) = (d+1)(n-d)$. Now note that there are d-planes lying in both $\Omega(a_0,\ldots,a_d)$ and $\Omega(b_0,\ldots,b_d)$, in general, if $a_i + b_{d-i} \geq n$ for all i (since $A_i \cap B_{d-i} \neq \emptyset$, we need that $\dim(A_i) + \dim(B_i) \geq n$). But we notice that $\sum_{i=0}^d (n-2i) = (d+1)n - d(d+1) = (d+1)(n-d)$, so the only option in our case is $a_i + b_{d-i} = n$. Now we only have to prove that $\Omega(a_0,\ldots,a_d)$ and $\Omega(n-a_d,\ldots,n-a_0)$ intersect in general in one d-plane. Given two corresponding flags $A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_d$ and $B_0 \subsetneq B_1 \subsetneq \ldots \subsetneq B_d$, we need to find the planes satisfying the corresponding flag conditions. But there is just one plane satisfying these conditions in general: the plane spanned (seen as an affine space) by the points in the intersection of A_i and B_{n-i} (each intersection is just an affine line), and this concludes the proof. $\square$

3.2. Remark. The above theorem just says that $\Omega(a_0,\ldots,a_d)\cdot\Omega(c_0,\ldots,c_d)$ is one if $n - c_i = a_{d-i}$, and otherwise zero (in the case Poincaré Duality applies).

3.3. Corollary. *(Bézout) Let* $Z(f)$ *and* $Z(g)$ *be curves in* $\mathbb{P}^2$*, then they intersect in* $\deg(f)\deg(g)$ *points in general.*

Proof. Using the Basis Theorem, we can represent the class of a curve c by $x = \lambda\Omega(1) \in H^*(G_{0,2},\mathbb{Z})$. To find this λ, we use Poincaré Duality and we see that $\lambda = x \cdot \Omega(1)$. But a line and a curve intersect in general in $\deg(c)$ points, hence $x = \deg(c)\Omega(1)$. So for our original problem, we have to calculate $\deg(f)\deg(g)\Omega(1)^2 = \deg(f)\deg(g)$, where we used Poincaré Duality again. $\square$

We now define the Special Schubert Cycle :

3.4. Definition. Define $\sigma(h) = \Omega(h, n-d+1, \ldots, n)$ for $h = 0, \ldots, n-d$.

3.5. Remark. $\sigma(h)$ is an element of $H^{2p}(G_{d,n}, \mathbb{Z})$, where $p = n-d-h$. Many books have a slightly different notation, and have a notation where one can see the p instead of the h. Furthermore $\sigma(n-d)$ is just $\Omega(n-d, \ldots, n)$, which represents $G_{d,n}$, and $\sigma(n-d-1)$ is the single Schubert Cycle in the class $H^2(G_{d,n}, \mathbb{Z})$.

The next theorem gives a direct formula which expresses a Schubert Cycle in terms of the Special Schubert Cycles:

3.6. Theorem. *(The Determinantal Formula, Giambelli's Formula) The following equality holds in the cohomology ring of the Grassmann Variety:*

$$\Omega(a_0, \ldots, a_d) = \det \begin{pmatrix} \sigma(a_0) & \ldots & \sigma(a_0 - d) \\ \vdots & \ddots & \vdots \\ \sigma(a_d) & \ldots & \sigma(a_d - d) \end{pmatrix} = |\sigma(a_i - j)|$$

where we define $\sigma(h) = 0$ if $h \notin [0, n-d]$.

Now our problem has been reduced to finding a way to calculate the product of Special Schubert Cycles, and for this we use the next theorem (actually, as we will see, the Determinantal Formula is a consequence of Pieri's Formula):

3.7. Theorem. *(Pieri's Formula) For $h = 0, 1, \ldots, n-d$ the next formula holds:*

$$\Omega(a_0, \ldots, a_d)\sigma(h) = \sum \Omega(b_0, \ldots, b_d)$$

where the sum is over the $b_0, \ldots, b_d$ such that $0 \leq b_0 \leq a_0 < b_1 \leq a_1 < \ldots < b_d \leq a_d$ and $\sum_{i=0}^{d} b_i = \sum_{i=0}^{d} a_i - (n-d-h)$.

With these formulae, we are able to solve many enumerative problems. In the next chapter, we will use these formulae in practice. We will first sketch a proof of Pieri's Formula and Giambelli's Formula.

3.1 Sketch of the Proofs

In this section, we will sketch a proof of Pieri's Formula and Giambelli's Formula. We will follow the proof of [GH].

The first step idea is not to fix a d and n directly, and this is done as follows. Instead of looking at finite sequences of the form $a_0, \ldots, a_d$, one takes infinite sequences a_i (and one adopts some conventions when things become zero in a certain $G_{d,n}$). With this notation, one can show that if a formula holds in $G_{d,n+1}$ or $G_{d+1,n+1}$, it will also hold in $G_{d,n}$. Now define $d(a, b; c)$ for sequences a, b and c if $\Omega(a)\Omega(b) = \sum d(a, b; c)\Omega(c)$ holds in all $G_{d,n}$.
To calculate this $d(a, b; c)$, we will use Poincaré Duality, and it comes down to calculating products of the form $\Omega(a)\Omega(b)\Omega(c')$ (which end up in the highest class) in some $G_{d,n}$. Then [GH] give a proof of a first reduction formula, which relates this product to another product in $G_{d-1,n-1}$ (and gives a condition when a coefficient is zero). The reduction mainly focuses on

certain conditions of the coefficients such that a d-plane lying in the intersection of the three Schubert Cycles always contains a certain line: hence we can 'forget' about this line and look at the $(d-1)$-planes in a $\mathbb{P}^{n-1}$. A consequence of this first reduction formula, is a second reduction formula. This formula uses an idea of duality: a d-plane in a $\mathbb{P}^n$ is given by $n-d$ equations, and so one can look at the $(n-d-1)$-planes in a $\mathbb{P}^n$, and one can even translate flag conditions. The proof just translates the intersection conditions to some dual conditions, then one applies the first reduction formula, and translates it back again (in this reduction, one goes from $G_{d,n}$ to a $G_{d,n-1}$).
With this the second reduction formula, one can now prove Pieri's Formula.
The next step is to prove an awkward looking formula (which formally follows from Pieri's Formula using only the box principle). A proof of Giambelli's Formula can then be given by a simple application of induction (the awkward formula is just some alternating sum coming from a determinant).

4 Using the Formulae

In this section we will solve some enumerative problems, using the rules we have found in the previous chapter.

4.1 Lines in $\mathbb{P}^3$

We will start with a very easy example to show how Schubert Calculus works:

Problem A. *How many lines lie on three given* 3*-planes of a* $\mathbb{P}^4$ *in general?*
Solution. The lines lying in a $\mathbb{P}^3$ are represented as $\Omega(2,3)$. And now we just calculate as follows (using the Determinantal Formula):

$$\begin{aligned}
\Omega(2,3)^2 &= \Omega(2,3)\begin{vmatrix} \sigma(2) & \sigma(1) \\ \sigma(3) & \sigma(2) \end{vmatrix} \\
&= \Omega(2,3)\sigma(2)^2 + \Omega(2,3)\sigma(1)\sigma(3) \\
&= \sigma(2)\Omega(1,3) + \Omega(0,3) \\
&= \Omega(1,2) + \Omega(0,3) + \Omega(0,3) \\
&= \Omega(1,2) + 2\Omega(0,3)
\end{aligned}$$

Now we use Poincaré Duality, and we find:

$$\Omega(2,3)(\Omega(1,2) + 2\Omega(0,3)) = 1 + 0 = 1$$

So the answer to this problem is 1. But this is just what we would have expected! The three 3-planes will intersect in a line.

The solution to the following problem is less clear:

Problem B. *How many lines, in general, intersect* 4 *given lines in* $\mathbb{P}^3$*?*
Solution. In this case, we want to look at $G_{1,3}$. The class of lines intersecting a given line is just given by $\Omega(1,3) = \sigma(1)$. So for this problem, we need to calculate $\sigma(1)^4$. But this is no problem

using the above formulae (we only use Pieri's Formula, notice that $n-d-h=3-1-h=2-h$):

$$
\begin{aligned}
\sigma(1)^2 &= \sigma(1)\Omega(1,3) = \Omega(0,3) + \Omega(1,2) \\
\sigma(1)^3 &= \sigma(1)(\Omega(0,3) + \Omega(1,2)) = \Omega(0,2) + \Omega(0,2) = 2\Omega(0,2) \\
\sigma(1)^4 &= \sigma(1)2\Omega(0,2) = 2\Omega(0,1) = 2
\end{aligned}
$$

So the answer is 2. Using Poincaré Duality, there is also another way to calculate this:

$$\sigma(1)^4 = (\sigma(1)^2)^2 = (\Omega(0,3) + \Omega(1,2))^2 = \Omega(0,3)^2 + 2\Omega(0,3)\Omega(1,2) + \Omega(1,2)^2 = 1 + 0 + 1 = 2$$

4.2 Planes in $\mathbb{P}^4$

Problem C. *How many lines, in general, intersect* 6 *given planes in* $\mathbb{P}^4$*?*
Solution. This time, we look at $G_{1,4}$. A line intersecting a plane is given by $\Omega(2,4) = \sigma(2)$. Now notice that $n-d-h = 4-1-h = 3-h$, so our $\sigma(2)$ lowers the sum of the a_i by 1 every time we multiply by it. For this problem, we just need to calculate $\sigma(2)^6$:

$$
\begin{aligned}
\sigma(2)^2 &= \sigma(2)\Omega(2,4) = \Omega(1,4) + \Omega(2,3) \\
\sigma(2)^3 &= \sigma(2)(\Omega(1,4) + \Omega(2,3)) = \Omega(0,4) + \Omega(1,3) + \Omega(1,3) \\
&= \Omega(0,4) + 2\Omega(1,3)
\end{aligned}
$$

Now we can use Poincaré Duality again, and we find:

$$
\begin{aligned}
\sigma(2)^6 &= (\Omega(0,4) + 2\Omega(1,3))^2 \\
&= \Omega(0,4)^2 + 4\Omega(0,4)\Omega(1,3) + 4\Omega(1,3)^2 \\
&= 1 + 0 + 4 = 5
\end{aligned}
$$

So the answer to this problem is 5.

We can also solve a problem where we switch the roles of the line and the plane:

Problem D. *Given* 6 *lines in* $\mathbb{P}^4$*, how many planes intersect all the lines in general?*
Solution. We now take $d = 2, n = 4$, and for this problem we have to calculate $\Omega(1,3,4)^6 = \sigma(1)^6$. We get the following calculation (notice that $n-d-1 = 1$):

$$
\begin{aligned}
\sigma(1)\Omega(1,3,4) &= \Omega(0,3,4) + \Omega(1,2,4) \\
\sigma(1)(\Omega(0,3,4) + \Omega(1,2,4)) &= \Omega(0,2,4) + \Omega(0,2,4) + \Omega(1,2,3) = 2\Omega(0,2,4) + \Omega(1,2,3)
\end{aligned}
$$

Now using Poincaré Duality, we see that the answer to the problem is $2^2 + 1^2 = 5$.

4.3 Another Line Problem

Problem E. *How many lines meet* $2n-2$ *given* $(n-2)$*-planes in* $\mathbb{P}^n$ *in general?*
Solution. For this problem, we put $d = 1$. A line meeting a $(n-2)$-plane is given by $\Omega(n-2,n) = \sigma(n-2)$, so we have to calculate $\sigma(n-2)^{2n-2}$. Notice that $n-d-(n-2) = n-1-(n-2) = 1$, so multiplying by $\sigma(n-2)$ lowers the sums of the coefficients in a Schubert Cycle by 1. We will calculate $\Omega(n-2,n)^{2n-2}$. This can be visualised by drawing a diagram $\mathbb{Z}\times\mathbb{Z}$. Then $\sigma(n-2)^{2n-2}$ is the number of paths from $(n-2,n)$ to $(0,1)$, where a path is a finite sequence $(c_i,d_i) \in \mathbb{Z}\times\mathbb{Z}$, where (by Pieri's Formula) $(c_i,d_i) - (c_{i+1},d_{i+1})$ is equal to $(0,1)$ or $(1,0)$. Such a path shouldn't intersect the diagonal (since Schubert Cycle's of the form $\Omega(a,a)$ are not allowed). In the next section, we will solve a more general problem, and in this case one finds C_{n-1}, where C_{n-1} is the $n-1$-th Catalan number. So the answer for $n = 4$ is 5 (see Section 4.2). Now $n = 5$ gives us 14, and the following numbers are $42, 132, 429$.

4.4 The Degree of a Subvariety of $G_{d,n}$

First of all, the degree of a projective variety is the "general" number of intersection points of the variety with a linear variety of complementary dimension. In the case of $G_{d,n}$, the Special Schubert Cycle $\sigma(n-d-1)$ represents such a linear variety of codimension one, so in terms of the cohomology group, we should just calculate $\sigma(n-d-1)^{(d+1)(n-d)}$. But we might also be interested in the degree of an $\Omega(a_0,\ldots,a_d)$. For this we have to calculate $\Omega(a_0,\ldots,a_d)\cdot\sigma(n-d-1)^{\sum_{i=0}^{d}(a_i-i)}$. We denote the degree of $\Omega(a_0,\ldots,a_d)$ by $d(a_0,\ldots,a_d)$. We first state a more combinatorial way to calculate this number:

4.1. Lemma. *The degree of $\Omega(a_0,\ldots,a_d)$ is the number of paths from $(0,1,\ldots,d)$ to $(a_0,\ldots,a_d)$ where a path is a sequence $\ell_0=(c_{0,0},\ldots,c_{0,d}),\ldots,\ell_s=(c_{s,0},\ldots,c_{s,d})$ where $s=\sum_{i=0}^{d}(a_i-i)$ and the following two properties hold:*
(a) The points on the path do represent a Schubert Cycle (so we have the condition that for a point $(c_0,\ldots,c_d)$ it should hold that $0\le c_0<c_1\ldots<c_d\le n$).
(b) $\ell_s-\ell_{s-1}$ is of the form $(0,\ldots,0,1,0,\ldots,0)$.

Proof. The proof is clear from the definition given above: It only uses the facts that multiplying by $\sigma(n-d-1)$ just lowers the sum of the a_i by 1, and Pieri's Formula. □

Notice that this makes sense if we take $\Omega(0,1,\ldots,d)$ for example: then there is only the constant path, hence the degree is 1.

There is a simple recursive formula for the degree:

4.2. Lemma. *The degree of $\Omega(a_0,\ldots,a_d)$ satisfies:*

$$d(a_0,\ldots,a_d)=\sum_{i=0}^{d}d(a_0,\ldots,a_{i-1},a_i-1,a_{i+1},\ldots,a_d)$$

where $d(c_0,\ldots,c_d)=0$ if $\Omega(c_0,\ldots,c_d)$ is not a Schubert Cycle.

Proof. The proof follows directly from the statements above. □

The degree is finally given in the next theorem:

4.3. Theorem. *Let $0\le a_0\le a_1\ldots a_d\le n$, and let $s=\sum_{i=0}^{d}(a_i-i)$. Then the following formula holds for the degree:*

$$d(a_0,\ldots,a_d)=\frac{s!}{a_0!\ldots a_d!}\prod_{i>j}(a_i-a_j)$$

Proof. The proof of this formula can be found in [HP], pages 364–366. □

4.5 Lines and Quadrics

Problem F. *Let $n=3n'+1$, $n'\ge 1$. How many lines in $\mathbb{P}^n$, in general, will lie on $2n'$ quadric hypersurfaces?*

The answer to this problem is not very easy, and will be calculated in several steps. In [KL] a proof is given for the case $n'=1$, and the answer in that case turns out to be 16. First we need the next theorem:

4.4. Theorem. *The lines on a generic non-singular hypersurface of degree s in $\mathbb{P}^n$ with $s \leq 2n-3$ form an algebraic set of codimension $s+1$ in $G_{1,n}$.*

Proof. The basic idea is the following: It is hard to determine which lines lie on a hypersurface of a certain degree, but it is easy to give all the hypersurfaces containing a given line. We will solve the second problem, and with some theory of algebraic geometry, we will be able to solve the first one as well.
We first notice that $\binom{n+s}{s}$ coefficients determine a hypersurface of degree s in $\mathbb{P}^n$. Now look at the subvariety $X \subset \mathbb{P}^{\binom{n+s}{s}-1} \times G_{1,n}$ which consists of pairs where the second coordinate represents a line which is contained in the hypersurface which is represented by the first coordinate. This variety comes with two natural projections, b_0 onto $\mathbb{P}^{\binom{n+s}{s}-1}$ and b_1 onto $G_{1,n}$ (here we need $s \leq 2n-3$, see [LA], Theorem 0.1). It now follows from Proposition 9.5 of Chapter III of [HAG] that

$$
\begin{aligned}
\dim(G_{1,n}) + \dim(\text{generic fiber of a line under } b_1) &= \dim(X) \\
&= \binom{n+s}{s} - 1 + \dim(\text{generic fiber of a } s\text{−hypersurface under } b_0)
\end{aligned}
$$

We denote dim(generic fiber of a $s-$hypersurface under b_0) by $\dim Y$. Now we have to calculate dim(generic fiber of a line under b_1). We fix a line L, which is given by $n-1$ linear equations $L_0, \ldots, L_{n-2}$ in $\mathbb{P}^n$. All the degree-s hypersurfaces with the property that L lies on it, are given by equations of the form

$$\sum_{i=0}^{n-2} S_i L_i$$

with S_i of degree $s-1$. We have to calculate the dimension of the space of s-hypersurfaces we obtain in this way. After a change of coordinates, we may consider $\sum_{i=0}^{n-2} S_i x_i$, and we need to calculate how many monomials of degree s can be formed by choosing the S_i. This can be done very efficiently: it is just the total number of monomials in $n+1$ variables, minus the ones in two variables (the x_{n-1} and x_n). So this number is just $\binom{n+s}{s} - \binom{1+s}{s} = \binom{n+s}{s} - (s+1)$. Since we work in projective space, we get that the dimension is equal to $\binom{n+s}{s} - (s+1) - 1$.
We finally obtain

$$
\begin{aligned}
\dim(G_{1,n}) - \dim(Y) &= \binom{n+s}{s} - 1 - (\binom{n+s}{s} - (s+1) - 1) \\
&= s+1
\end{aligned}
$$

$\square$

4.5. Remark. Now consider the case $n=3, s=3$ in the above theorem (we can use it, since $s = 2n-3$). We see that the codimension is 4. But the dimension of $G_{1,3}$ is $(1+1)(3-1) = 4$, and we only have a finite number of lines on a typical non-singular cubic in $\mathbb{P}^3$. Actually, one can show that this number is always equal to 27 ([HAG], Chapter V, Theorem 4.9).

4.6. Remark. Actually, we can prove a stronger statement (here we also need [LA]). In the theorem above, if we consider d-planes, we get that the codimension will be $\binom{d+s}{s}$, if $\binom{d+s}{s} \leq (d+1)(n-d)$. If we put $d=1$, we get the condition $1+s \leq 2(n-1)$, or $s \leq 2n-3$.

4.7. Remark. We now want to use apply Fact 2.2.2, but we don't have the irreducibility. In Remark 4.5 for example, we get an algebraic set which is not irreducible, it is the union of 27 points. Theorem 0.1 from [LA] gives us an easy condition for connectedness and smoothness, and this implies irreducibility. One gets irreducibility if strict inequalities hold in Remark 4.6 (or Theorem 4.4), unless $s = 2, n \leq 2d$ or $n = 2d + 1$.

We will continue with the case $s = 2$:

4.8. Theorem. *Let $n \geq 3$. The subvariety X of $G_{1,n}$ which represents the lines in a (non-singular) quadric is represented by $4\Omega(n-3, n-1)$ in $H^6(G_{1,n}, \mathbb{Z})$.*

Proof. From the theorem above, we know that X has codimension 3 (here we need $n \geq 3$), so using the basis theorem, we have to solve $a_0 + a_1 - 1 = (d+1)(n-d) - 3 = 2(n-1) - 3 = 2n-5$. So $a_0 + a_1 = 2n-4$. The only valid solutions are $(a_0, a_1) = (n-3, n-1)$ and $(a_0, a_1) = (n-4, n)$. So we see that the class of X is of the form $[X] = \lambda\Omega(n-3, n-1) + \mu\Omega(n-4, n)$. We will use Poincaré Duality to calculate μ and λ. We get that $\mu = [X] \cdot \Omega(0, 4)$ and $\lambda = [X] \cdot \Omega(1, 3)$.
Now we look at $[X] \cdot \Omega(0, 4)$ geometrically: it consists of all lines in our quadric through a general given point. But if we choose the point outside our quadric, we clearly get no intersection points. Since a point outside a quadric is general enough, we get $\mu = 0$.
We now need to calculate $[X] \cdot \Omega(1, 3)$. This $\Omega(1, 3)$ represents the lines in a $\mathbb{P}^3$ intersecting a given line. Now we look at our quadric in this $\mathbb{P}^3$. This quadric and the line intersect in general in two points. Now in a non-singular quadric in $\mathbb{P}^3$ there are two lines through every point of the quadric ([HAG], Exercise I.2.15), and hence the total number of lines is equal to $2 \cdot 2 = 4$ in general. This shows that $[X] = 4\Omega(n-3, n-1)$ (see Theorem 4.12 for a more general case). □

So to solve the problem, we have to calculate $(4\Omega(n-3, n-1))^{2n'}$. Note that the solution to our problem will be a finite number, since $\Omega(n-3, n-1) \in H^{2\cdot 3}(G_{d,n}, \mathbb{Z})$, so $\Omega(n-3, n-1)^{2n'} \in H^{2\cdot 3\cdot(2n')}(G_{d,n}, \mathbb{Z}) = H^{2\cdot(2n-2)}(G_{d,n}, \mathbb{Z}) = H^{2(d+1)(n-d)}(G_{d,n}, \mathbb{Z})$, which is the top level. The next lemma gives us a way to calculate the powers of $\Omega(n-3, n-1)$:

4.9. Lemma. *Let $0 \leq a_0 < a_1 \leq n$, then the following equality holds:*

$$\Omega(n-3, n-1) \cdot \Omega(a_0, a_1) \quad = \quad \Omega(a_0 - 1, a_1 - 2) + \Omega(a_0 - 2, a_1 - 1)$$

where one should read zero for a $\Omega(b_0, b_1)$ if it does not represent a Schubert Cycle (so when $0 \leq b_0 < b_1 \leq n$ is not fulfilled).

Proof. In Chapter 5, we will give a proof using the Littlewood Richardson rule. A proof can also be given with Pieri's Formula and Giambelli's Formula, but we will leave it to the reader. □

This lemma already gives us a clue what the solution might look like. One would guess that the solution would be of the form $(2 \cdot 4)^{2n'} = 64^{n'}$, but we will be able to give the exact formula.
The idea is now that multiplication by $\Omega(n-3, n-1)$ works as some sort of lowering operator. With the above formula, it is easy to calculate for example the cases $n' = 1, 2$, but we are looking for a more general formula. The next theorem gives the solution to our problem:

4.10. Theorem. *Let $n = 3n'+1$, $n' \geq 1$. Then the number of lines lying on $2n'$ general quadrics in $\mathbb{P}^n$ is equal to $2^{4n'}C_{n'}$ where $C_{n'}$ is equal to the n'-th Catalan number, that is $C_{n'} = \frac{1}{n'+1}\binom{2n'}{n'}$.*

Proof. We want to calculate $\Omega(n-3, n-1)^{2n'}$. Using the previous lemma, we can state this question as follows.
How many paths are there from $\Omega(0,1)$ to $\Omega(n-3, n-1)$, where a path is a sequence of pairs of numbers $(a_1, b_1), \ldots, (a_{2n'-1}, b_{2n'-1})$ such that the following two statements hold:
(1) For all i $(a_i - a_{i-1}, b_i - b_{i-1})$ should either be $(1,2)$ or $(2,1)$.
(2) For all i we have $a_i < b_i$.
We will calculate this number. First notice that every step of the path, there is always a component of the form $(1,1)$, so we can restate the problem as follows:
How many paths are there from $\Omega(0,1)$ to $\Omega(n-3-2n'+1, n-1-2n'+1) = \Omega(n'-1, n'+1)$ where a path is a sequence of pairs of numbers $(a_1, b_1), \ldots, (a_{2n'-1}, b_{2n'-1})$ such that the following two statements hold:
(1) For all i $(a_i - a_{i-1}, b_i - b_{i-1})$ should either be $(0,1)$ or $(1,0)$.
(2) For all i we have $a_i \leq b_i$ (we are now allowed to stay on the diagonal).
Now some clever combinatorics should do the trick. But actually, this is not needed. We already calculated this number, by Lemma 4.1. We just calculate $d(n'-1, n'+1)$, which by Theorem 4.3, is equal to

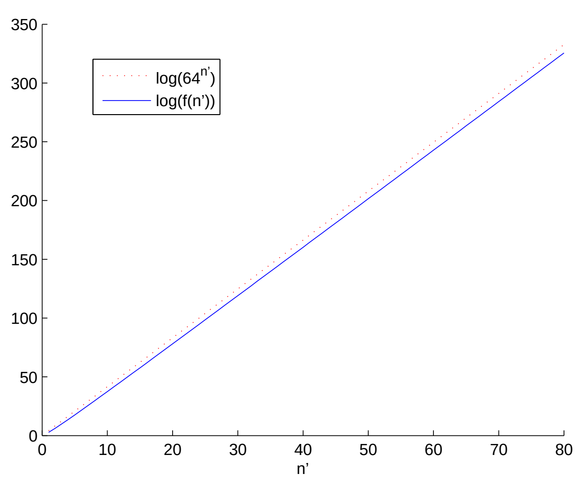
$$\frac{(2n'-1)!}{(n'-1)!(n'+1)!}2 = \frac{2n'}{n'!(n'+1)!} = \frac{1}{n'+1}\binom{2n'}{n'} = C_{n'}$$

.
Putting everything together, we get $(4\Omega(n-3, n-1))^{2n'} = 2^{4n'}C_{n'}$. $\square$

Using our formula, we obtain the following results:

$n' = 1$: $2^4 \cdot 1$
$n' = 2$: $2^8 \cdot 2$
$n' = 3$: $2^{12} \cdot 5$
$n' = 4$: $2^{16} \cdot 14$
$n' = 5$: $2^{20} \cdot 42$
$n' = 6$: $2^{24} \cdot 132$

We have plotted this value as a function of n', say $f(n')$ against $\log(64^{n'})$, and we get the following picture:



The fact that $d(n') \leq 64^{n'}$ (for large n') follows from the observation that $C_{n'} \leq \frac{2^{2n'}}{n'+1} \leq 2^{2n'}$.

4.6 d-Planes and Quadrics

In this section, we will look at the d-planes on a quadric in $\mathbb{P}^n$. First we have the following theorem:

4.11. Theorem. *A non-singular quadric X in $\mathbb{P}^n$ contains no linear subspaces of dimension strictly greater than $(n-1)/2$.*

Proof. Assume that the equation of the non-singular quadric X is given by $\sum_{i,j} q_{ij} X_i X_j$ with $q_{ij} = q_{ji}$. Then for a point $p = (a_0 : \ldots : a_n)$ the tangent space at p is just the hyperplane in $\mathbb{P}^n$ satisfying $\sum_{i,j} q_{ij} a_j X_i = 0$. Since X is non-singular (which means that the rank of the matrix (q_{ij}) is $n+1$), the map sending a point to its tangent space is an injection from X to $\mathbb{P}^{n^*}$.
Now take a d-plane D in X. Then the tangent planes coming from a point in D form a d-dimensional linear subspace of $\mathbb{P}^{n^*}$. Also, the tangent space at any point in D contains D itself. The set of hyperplanes in $\mathbb{P}^n$ containing D is an $(n-d-1)$-dimensional linear subspace of $\mathbb{P}^{n^*}$, and so we obtain $d \leq n-d-1$, or $d \leq \frac{n-1}{2}$. □

We now want to determine the cohomology class $[F]$ of the d-planes on a quadric X. Using Remark 4.6, we see that we have to look at the Schubert Cycles $\Omega(a_0, \ldots, a_n)$ with $\sum_i (a_i - i) = \frac{(d+2)(d+1)}{2}$. Using Poincaré Duality, we have to calculate $[F] \cdot \Omega(b_0, \ldots, b_d)$ with $\sum_i (b_i - i) = \frac{(d+2)(d+1)}{2}$. Now comes the most important observation: suppose that L is a d-plane in F and in $\Omega(b_0, \ldots, b_d)$. Then $L \cap B_i$ is a linear subspace of a $\mathbb{P}^{b_i}$ of dimension at least i. Now $F \cap B_i$ is a smooth quadric in this $\mathbb{P}^{b_i}$, hence by Theorem 4.11, $\frac{b_i - 1}{2} \geq \dim(L \cap B_i) \geq i$.
This gives:

$$\begin{aligned}
\frac{(d+1)(d+2)}{2} &= \sum_{i=0}^{d} (b_i - i) \\
&\geq \sum_{i=0}^{d} (i+1) \\
&= \frac{(d+1)(d+2)}{2}
\end{aligned}$$

Surprisingly, this just gives us $b_i = 2i+1$. So $[X] = \lambda \Omega(n-(2d+1), n-(2d-1), \ldots, n-1) = \lambda \Omega(n-2d-1, n-2d+1, \ldots, n-1)$. If we put $d = 1$, we get $\lambda \Omega(n-3, n-1)$, which is basically Theorem 4.8.
We will now calculate this λ, and do this using Poincaré Duality again, directly from the definition. We have (for a flag V_i) that $\Omega(1, 3, \ldots, 2d+1) = \{L\ d\text{−plane} : \dim(L \cap V_i) \geq i\}$. Then take an L which is in both $\Omega(1, 3, \ldots, 2d+1)$ and X. This L must meet V_1 in one of the points of $V_1 \cap X$, say in p_1 or p_2, and assume it is p_1. Then we look at $T_{p_1}(X)$, which is a $(n-1)$-plane (since X is non-singular), and we take an $(n-2)$-plane in $T_{p_1}(X)$ not meeting p_1, which gives rise to a quadric X_1 in a $\mathbb{P}^{n-2}$. Then we need that V_3 intersects X in a line, but it already meets the $(n-2)$-plane in a line, and this gives two intersection points with X_1, say p_{11}, p_{12}. For any d-plane $L \subset X$ through p_1 we have $L = \mathrm{span}(p_1, L \cap \mathbb{P}^{n-2})$, so any L should contain p_{11} or p_{12}. Assume that this time we take p_{11}.
Now take a d-plane D in X which contains p_1 and p_{11}. Then this d-plane is contained in $T_{p_1}(X) \cap T_{p_{11}}(X)$, which is just a $\mathbb{P}^{n-2}$. Then take a $\mathbb{P}^{n-4}$ in this projective space, which is disjoint from the span of p_1 and p_{11}. This $\mathbb{P}^{n-4}$ intersects X in a non-singular quadric, X_2.

The third condition says that D meets V_5 in a 2-plane, and hence it contains one of the two intersection points of V_5 with X_2, say in p_{21} or p_{22}, and assume it is p_{21}.
The process should be clear by now, and every step we have 2 choices for the points. Since we have $d+1$ steps, in total we find 2^{d+1} d-planes. So we have:

4.12. Theorem. *Let X be a non-singular quadric in $\mathbb{P}^n$ (with $\frac{(d+2)(d+1)}{2} < (d+1)(n-d)$). Then in the cohomology ring of $G_{d,n}$ we have:*

$$[X] = 2^{d+1}\Omega(n-2d-1, n-2d+1, \ldots, n-1)$$

.

We now want to state a meaningful problem, and we propose the following one:

Problem G. *Let $n = 2d+2$. How many d-planes lie on two quadrics in $\mathbb{P}^n$ in general?*
First of all, we obtain a finite number. This follows since

$$2\frac{(d+1)(d+2)}{2} = 2\frac{(d+1)(n-d)}{2} = (d+1)(n-d)$$

The solution in this case is easy, since we only have to square. The solution is $[X]^2 = (2^{d+1}\Omega(n-2d-1, n-2d+1, \ldots, n-1))^2 = 2^{2d+2} = 2^n$. Note that if we put $d = 1$, we get $n = 4$ and we obtain the answer 16 which was mentioned before.

We can also look at the following problem:

Problem H. *Let $n = 2n'+2 \geq 6$. How many 2-planes lie on n' quadrics in $\mathbb{P}^n$ in general?*
Again, this is a meaningful question, since $n'\frac{(d+1)(d+2)}{2} = 6n' = 3(2n') = (d+1)(n-d)$, so we end up in the highest class. To solve this problem, we just need to calculate $(2^3\Omega(n-5, n-3, n-1))^{n'}$. To calculate this, we want to have some easy formula for $\Omega(n-5, n-3, n-1)\Omega(a_0, a_1, a_2)$, but this already is a problem in general. For $n' = 2$, this is not that hard. By Poincaré Duality, we just obtain $2^6 = 64$ (see also the previous problem).

4.7 Lines and Cubics

The last sections were devoted to quadrics, and in this section, we will do the cubic case (although this case is a lot harder). First we have the following theorem:

4.13. Theorem. *The subvariety of lines on a non-singular cubic in $\mathbb{P}^n$ is represented by $27\Omega(n-3, n-2) + 18\Omega(n-4, n-1) \in H^{2\cdot 4}(G_{1,n}, \mathbb{Z})$ for $n \geq 4$.*

Proof. Let X be a cubic in $\mathbb{P}^n$. From Theorem 4.4 we obtain that the lines form an algebraic set of codimension 4, when $n \geq 3$, say F. So we have to look at the Schubert Cycles satisfying $a_0 + a_1 - 1 = 2(n-1) - 4 = 2n - 6$. So our subvariety is represented by a class of the form $\lambda\Omega(n-5, n) + \mu\Omega(n-4, n-1) + \nu\Omega(n-3, n-2)$. We will calculate these constants with Poincaré Duality.
Obviously $\lambda = 0$: We need to calculate $[X] \cdot \Omega(0, 5)$. If $n \geq 5$, and we take a point outside our cubic, we see that the intersection is empty.
For the second coefficient, we need tot calculate $[X] \cdot \Omega(1, 4)$, which turns out to be 18. A line in a $\mathbb{P}^4$ intersects the cubic in 3 points. Through each of these points go six lines which lie on the cubic ([MU], Theorem 1.19), so in total we find $6 \times 3 = 18$ lines.
Finally we calculate $[X]\Omega(2, 3)$: This is just the number of lines of a cubic in $\mathbb{P}^3$, which is 27 ([HAG], Chapter V, Theorem 4.9).

□

We can now state our problem:

Problem I. *Let $n = 2n'+1$ with $n' \geq 2$. How many lines lie on n' non-singular generic cubics?*
The answer to the problem is a finite number, since if $x \in H^{2\cdot 4}(G_{1,n}, \mathbb{Z})$, then we see that $x^{n'} \in H^{2\cdot 4n'}(G_{1,n}, \mathbb{Z}) = H^{2\cdot 2(n-1)}(G_{1,n}, \mathbb{Z}) = H^{2(d+1)(n-d)}(G_{1,n}, \mathbb{Z})$.
We need to calculate a lot for this, but the following lemma is useful:

4.14. Lemma. *Let $0 \leq a < b \leq n$. Then the following formulae hold:*

$$\begin{aligned}
\Omega(n-3, n-2)\Omega(a, b) &= \Omega(a-2, b-2) \\
\Omega(n-4, n-1)\Omega(a, b) &= \Omega(a-3, b-1) + \Omega(a-1, b-3) + (1-\delta_{a+1,b})\Omega(a-2, b-2)
\end{aligned}$$

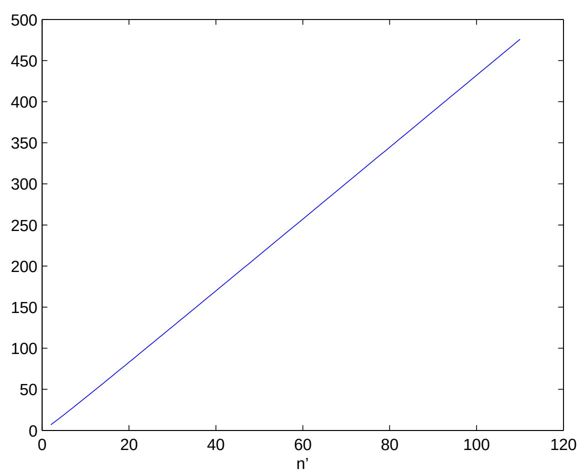
where a $\Omega(a_0, a_1)$ is defined zero if it does not represent a Schubert Cycle.

Proof. One can prove this lemma directly with Pieri's Formula and Giambelli's Formula, but this would give a long and tedious proof. One can give an easy proof using the Littlewood-Richardson rule (basically the same as Example 5.11), and we leave this as an example to the reader. □

The lemma above gives an easy way to calculate solutions for small values of n', but I don't see an easy generalization for general n'. We find the following table (the first one follows directly from Poincaré Duality):

$n' = 2$: $27^2 + 18^2 = 1053$
$n' = 3$: 51759
$n' = 4$: 2893401
$n' = 5$: 174489795

These numbers become big very quickly. Roughly, one would expect the number n' to be about $(2(27+18))^{n'} = 90^{n'}$. We have plotted the real function $\log(f(n'))$ below:



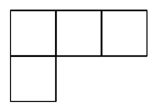
As seen in this picture, we get an exponential behavior, and the picture suggests a behavior of the form $(e^{4.5})^{n'} \approx 90^{n'}$.

5 Symmetric Polynomials

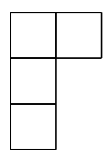
In this chapter, we will construct a ring isomorphism from a quotient of the ring of symmetric polynomials in $(d+1)(n-d)$ variables to the cohomology ring of the Grassmann Variety of d-planes in $\mathbb{P}^n$.
Let R be the ring of symmetric polynomials in $(d+1)(n-d)$ variables.

5.1 Introduction to Partitions

Take a number $m \in \mathbb{Z}_{\geq 0}$. Then λ is a partition of m if λ is an (infinite) nonincreasing sequence of nonnegative integers with sum m. We denote this as $|\lambda| = m$.
We can draw the Young Diagram corresponding to a partition λ, $Y(\lambda)$, which can best be explained in an example. Take $\lambda = (3, 1, 0, 0, \ldots)$ (which we often denote as $(3, 1)$), which is a partition of 4. The Young Diagram we get consist of a 2 rows, the first consists of 3 blocks, the second of just one:



The dual Young Diagram is the diagram reflected in the diagonal (from top to bottom, left to right), so in this case we get:



For two partitions λ, μ we write $\lambda \leq \mu$ if $\lambda_i \leq \mu_i$ for all i. In a diagram, this corresponds to the diagram of λ fitting inside in the diagram of μ. In this case we can remove the diagram of λ from μ, and obtain $Y(\mu/\lambda)$ (see Example 5.5 for an example). We will come back to these notions later.

5.2 Schur Polynomials

We will now define a $\mathbb{Z}$-basis for R:

5.1. Definition. Let λ be a partition of m. We define the corresponding *Schur Polynomial* s_λ as follows:

$$s_\lambda = \sum_\pi M(\pi)$$

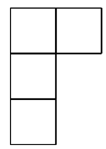
This sum is taken over all ways π of filling the squares of $Y(\lambda)$ with positive integers (up to $(d+1)(n-d)$) such that the integers along any row are non-increasing and along any column are strictly decreasing. Then let $M(\pi) = x_1^{a_1} x_2^{a_2} \ldots$ where a_i entries of π are equal to i.

These polynomials defined above are indeed symmetric:

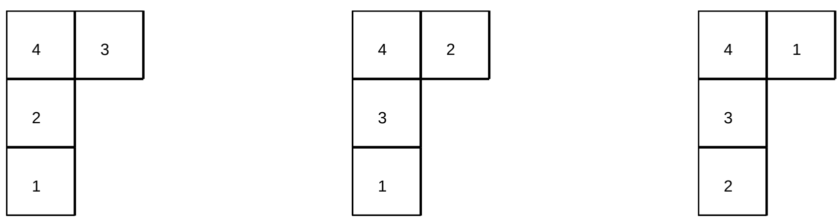
5.2. Theorem. *Let λ be a partition of m, then s_λ is a homogeneous symmetric polynomial of degree m.*

Proof. A proof can be found in [FU]. $\square$

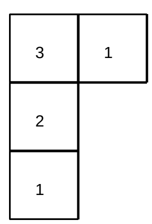
5.3. Example. We will calculate $s_{(2,1,1)}$ when we have 4 variables x_1, x_2, x_3 and x_4. In this case, we have to fill the following Young Diagram:



First we will calculate the coefficient of $x_1x_2x_3x_4$, which amounts to filling the square with a 1, a 2, a 3, and a 4 subject to the rules in the definition. The solutions are:



Hence the coefficient is 3.
For $x_1^2x_2x_3$, we get



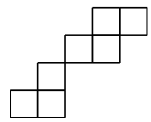
So the coefficient is 1. For $x_1^3x_2$, $x_1^2x_2^2$ and x_1^4 there are no correct fillings. Using the fact that our polynomial is symmetric, we see that $s_{(2,1,1)} = 3\sum_4 x_1x_2x_3x_4 + \sum_4 x_1^2x_2x_3 = 3x_1x_2x_3x_4 + \sum_4 x_1^2x_2x_3$.

The idea now is that the Schur Polynomials behave exactly as the Schubert Cycles (modulo some ideal). We will prove Pieri's Formula for the Schur Polynomials and this will give us an isomorphism. We have the very important rule:

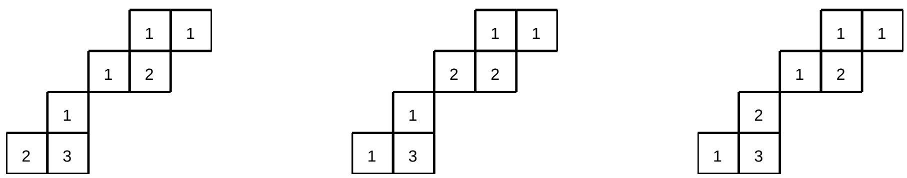
5.4. Theorem. *(Littlewood-Richardson rule) Suppose that $\nu = (\nu_1, \nu_2, \ldots), \mu = (\mu_1, \mu_2, \ldots)$ are partitions. The coefficient of s_λ in $s_\nu s_\mu$ is equal to zero unless $\mu \leq \lambda$ and $|\lambda| = |\nu| + |\mu|$. In that case, then the coefficient is equal to the number of ways of inserting ν_1 1's, ν_2 $2'$s,etc. into the square of $Y(\lambda/\mu)$, subject to the following conditions:*
(1) The numbers are non-decreasing in each row and strictly increasing in each column.
(2) If $a_1, a_2, \ldots$ is the order of the numbers in the diagram, reading from right to left, top to bottom, than for any i, j the number of i's among $a_1, a_2, \ldots, a_j$ is not less then the numbers of $(i+1)$'s among $a_1, a_2, \ldots, a_j$.

Proof. A proof of this theorem can be found in [FU], and is far from trivial. $\square$

5.5. Example. We will correct an example from [ST]. Let us calculate the coefficient of $s_{(5,4,2,2)}$ in $s_{(4,2,1)}s_{(3,2,1)}$, using the above theorem. We then have to fill the following diagram, with 4 ones, 2 two's and 1 three, according to the rules given above:



We can do this in three ways:



Hence the coefficient we were looking for is equal to 3.

5.3 Symmetric Polynomials and Schubert Calculus

In this section, we will connect Symmetric Polynomials with Schubert Calculus. First we need to define an ideal of R:

5.6. Definition. Let λ be the partition with $d+1$ parts equal to $n-d$, so our Young Diagram is just a $(d+1)\times(n-d)$ rectangle. We define $I_{d,n}$ as the $\mathbb{Z}$-module generated by the s_μ such that $\mu \not\leq \lambda$. This latter condition just means that either $\mu_1 > n-d$ or μ has more than $d+1$ parts. We claim that this is an ideal. Suppose $s_\mu \in I_{d,n}$, and suppose s_ρ appears in $s_\mu s_\nu$ for some partition ν. By the Littlewood-Richardson rule, this only happens if $\mu \leq \rho$, and now it also follows that $\rho \not\leq \lambda$, so we indeed have an ideal. Now define $R' = R/I_{d,n}$.

5.7. Remark. R' is a free $\mathbb{Z}$-module, generated by the $[s_\mu]$, for which we write s_μ from now on, such that $\mu \leq \lambda$. Also, for multiplication, use the Littlewood-Richardson rule and just ignore all $s_\mu \in I_{d,n}$.

We can also prove Pieri's Formula:

5.8. Theorem. *(Pieri's Formula) Let $\mu = (n-a_0-d, n-a_1-d+1, \ldots, n-a_d)$. Then $s_\mu s_{(n-d-h)} = \sum s_\lambda$, where the sum runs over the λ such that $\mu \leq \lambda$ and $Y(\lambda/\mu)$ has exactly $n-d-h$ squares and there are no columns with more than one square.*

Proof. We will use the Littlewood-Richardson rule, and this directly gives us that the coefficient is zero unless $\mu \leq \lambda$ and $|\lambda| = |\mu| + (d+1)(n-d) - \sum_{i=0}^{d}(a_i - i)$. Since $|\lambda|$ is a specified number, and we cannot fulfil the non-increasing condition when there is a column with more than one square in $Y(\lambda/\mu)$ (since we only have one number to fill in). If these conditions hold, then there is obviously a unique and correct way of filling such a square (we have no choice), and we get a coefficient 1. This proves our formula. $\square$

As a consequence, we directly have the Determinantal Formula (the proof is basically the same as the proof of Pieri's Formula in the cohomology ring of the Grassmann Variety, since this followed formally from Pieri's Formula):

5.9. Theorem. *(Determinantal Formula)* $s_{(n-a_0-d,n-a_1-d+1,\ldots,n-a_d)} = |s_{(n-d-a_i+j)}|$ *(a determinant) where* $1 \leq i, j \leq d$*.*

Finally we can make the connection with the cohomology ring of the Grassmann Variety:

5.10. Theorem. *The map*

$$\begin{array}{rcl} \varphi : R' & \to & H^*(G_{d,n}, \mathbb{Z}) \\ s_{(n-a_0-d,n-a_1-d+1,\ldots,n-a_d)} & \mapsto & \Omega(a_0, a_1, \ldots, a_d) \end{array}$$

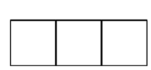
is an isomorphism of rings.

Proof. First note that R' and $H^*(G_{d,n}, \mathbb{Z})$ are both graded rings, where every graded part is a free abelian group. Also, the rank of the degree m-th part of R' is equal to the rank of $H^{2m}(G_{d,n}, \mathbb{Z})$. Now we notice that Pieri's Formula completely determines the multiplicative structure of both rings, so we will just check this formula. Let $\mu = (n - a_0 - d, n - a_1 - d + 1, \ldots, n - a_d)$:

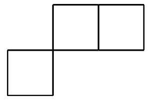
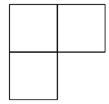
$$\varphi(s_\mu s_{n-d-h}) \quad = \quad \varphi(\sum s_\lambda)$$

We want to translate the conditions for a term appearing in our sum in terms of $\lambda = (n - b_0 - d, n - b_1 - d + 1, \ldots, n - b_d)$. So we have the λ such that $\mu \leq \lambda$ and $\lambda_1 \leq n - d$ and $Y(\lambda/\mu)$ has exactly $n - d - h$ squares and there are no columns with more than one square. Now $\mu \leq \lambda$ just gives us that $b_i \leq a_i$. The $n - d - h$ just gives us a condition on the "dimension", in this case $\sum b_i = \sum a_i - (n - d - h)$. We look at the last condition. In a Young Diagram, this means that $\lambda_{i+1} \leq \mu_i$. Translating this gives $(n - b_{i+1} - d + i + 1) \leq (n - a_i - d + i)$, or just $a_i + 1 \leq b_{i+1}$, so $a_i < b_{i+1}$. On the other hand, if $a_i < b_{i+1}$, then the condition is satisfied, and so we see that Pieri's Formula also holds.
This shows that we have the same rules and hence an isomorphism. $\square$

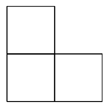
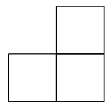
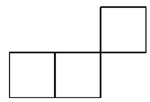
5.11. Example. (Proof of Lemma 4.9) As an example, we will prove Lemma 4.9 using the Littlewood-Richardson rule. We want to calculate $\Omega(n-3, n-1)\Omega(a_0, a_1)$. In R' we have to calculate $s_{(n-(n-3)-1,n-(n-1))} s_{(n-a_0-1,n-a_1)} = s_{(2,1)} s_{(n-a_0-1,n-a_1)}$. Using the Littlewood-Richardson rule, we see that we have to add three squares to $Y(n - a_0 - 1, n - a_1)$. We can only add these squares in the first or second row (since $d + 1 = 2$), and we have a few options to add them. We can add 3 to the first or last row. We then have to fill the following diagram:



According to the first rules, we should fill it as (right to left, top to bottom) $2, 1, 1$, which fails to satisfy the second rule.
Then we can add 2 to the first one, and 1 to the second one. We then have to fill one of the following diagrams:

(Notice that in the first case, the second row can be even more to the left, but columns without any squares in them can be ignored). Both can be filled in a unique way: $1, 1, 2$, and so we get a coefficient 1 in all cases.
The last case is a bit harder, we add one square to the first line one, and two to second one. We then obtain one of the following diagrams:

The second an the third have one filling, which is $1, 2, 1$, and the first does not even represent a partition, so we can say that we get a coefficient one in this case (the isomorphism will just map it to $\Omega(a, a)$ which we define to be zero). Note also that we didn't consider cases where our new diagram has rows whose length is bigger than $n - 1$, but if we translate those back, we get Schubert Cycles with negative entries, which are defined to be zero anyway.
So we can just say $s_{(2,1)} s_{(n-a_0-1,n-a_1)} = s_{(n-a_0+1,n-a_1+1)} + s_{(n-a_0,n-a_1+2)}$. Translating this back, we get $\Omega(n-3, n-1)\Omega(a_0, a_1) = \Omega(a_0 - 2, a_1 - 1) + \Omega(a_0 - 1, a_1 - 2)$.

6 Summary

In this Bachelor Thesis, we have tried to explain the rules of the Schubert Calculus. We have split this up into a chapter on the Grassmann Variety, one on the structure of the cohomology ring of this Grassmann Variety, and two chapters on the formulae in this cohomology ring. The fourth chapter shows how to use the formulae in practice to solve some enumerative problems in geometry. The first few examples are standard (or trivial), but the sections on Quadrics and Cubics are new. The most remarkable formula we have derived is Theorem 4.10, which we state again below:

4.10 Theorem *Let $n = 3n' + 1$, $n' \geq 1$. Then the number of lines lying on $2n'$ non-singular quadrics in $\mathbb{P}^n$ is equal to $2^{4n'} C_{n'}$ where $C_{n'}$ is equal to the n'-th Catalan number, that is $C_{n'} = \frac{1}{n'+1}\binom{2n'}{n'}$.*

The examples in Chapter 4 also show a general method for solving certain enumerative problems in geometry. When a problem is stated, concerning the d-planes on some variety $X \subset \mathbb{P}^n$, one first needs to translate this problem to some calculation in the cohomology ring of $G_{d,n}$. This is almost always done (unless it is trivial) with Poincaré Duality (Theorem 3.1), and this part is in general very hard. Then one "only" has to do some multiplications in this cohomology ring, and this can always be done with Pieri's Formula (Theorem 3.7) and Giambelli's Formula (Theorem 3.6). To give general rules, one can work with the same formulae, but the Littlewood-Richardson rule (Theorem 5.4) and Theorem 5.10 may help, since they can give coefficients directly. Even with the Littlewood-Richardson rule, it is still very hard to give exact formulae, and it becomes even harder when d becomes larger.

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