Entropy for skew tent maps

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Abstract

We consider the family of skew tent maps $T_{\alpha,\beta} : [0,1] \to [0,1]$ defined by

$$T_{\alpha,\beta}(x) = \begin{cases} \alpha x + \frac{\alpha+\beta-\alpha\beta}{\beta} & \text{for } \; x \in [0, 1-\frac{1}{\beta}], \\ \beta - \beta x & \text{for } \; x \in [1-\frac{1}{\beta}, 1] \end{cases}$$

with $\alpha, \beta > 1$ and $\alpha + \beta \geq \alpha\beta$. By A. Lasota and J.A. Yorke [LY73] we know that each skew tent map has a unique acim. We fix the parameter β and show that the measure-theoretic entropy of the skew tent maps, with respect to the unique acim, depends continuously on α on a part of the parameter domain. The stability of the acim under small perturbations plays an important role in showing this result. We also investigate the relation between the measure theoretic entropy and the topological entropy for skew tent maps.

Contents

Chapter 1

Introduction

1.1 Background

In ergodic theory we study dynamical systems and related problems. The systems we study can be used to model physical phenomena whose states change over time. We want to describe the long term behavior of such systems. There are theorems such as the Poincaré Recurrence Theorem that give information about the long term behavior of trajectories. The Poincaré Recurrence Theorem states that in a dynamical system the initial state in finite time almost surely returns to the initial state or a state very close to the initial state. The time elapsed before recurrence to the initial state may depend on the exact initial state. This makes it in general hard to predict the long term behavior of trajectories. It is therefore natural to describe the long term behavior by statistical means. This is done by proving the existence of invariant measures and determining their ergodic properties. In some cases it is easy to find invariant measures, but those measures might be trivial and they do not give relevant information. We look for invariant measures that are absolutely continuous with respect to the Lebesgue measure. These measures are nice to work with from a theoretical point of view, because some properties that are verified for the absolutely continuous invariant measure (acim) also automatically hold for the Lebesgue measure. On the other side these measures are also important from a physical point of view, because computer simulations of orbits of the dynamical system reveal only invariant measures which are absolutely continuous with respect to Lebesgue measure.

In the field of one-dimensional dynamical systems a lot of research has been done on the existence of acims. In showing the existence of acims the Perron-Frobenius operator plays an important role, because the existence of an acim is equivalent to the existence of a fixed point of the Perron-Frobenius operator. For piecewise monotonic maps we get a nice and practical representation for the Perron-Frobenius operator. This representation can be used to show that the Perron-Frobenius operator has fixed points. For non-singular piecewise expanding C^2 maps this was done by A. Lasota and J.A. Yorke [LY73]. They derived an inequality, now known as the Lasota-Yorke inequality, that implies the existence of a fixed point for the Perron-Frobenius operator. Their method has been used by many others to show existence of acims for various classes of non-singular piecewise expanding maps. Once we have established the existence of an acim, the next goal is to classify the number of acims and determine their ergodic properties.

One important property of a dynamical system is the stability of the invariant measures. Suppose we have a dynamical system with a unique acim. We would like to know if the acim of a perturbed dynamical system is in some sense close to the acim of the unperturbed dynamical system. If this is the case, then we call the map acim-stable. In practice this is motivated by the fact that we often observe a perturbation of a dynamical system, due to for example measurement errors, and would like to know if the unperturbed and the perturbed system show the same behavior.

G. Keller [Kel82] showed that piecewise expanding transformations that satisfy a uniform Lasota-Yorke inequality have a stable acim. Many dynamical systems do not satisfy a uniform Lasota-Yorke bound. For those maps we need other conditions to show that the acim is stable or unstable. A mechanism that can cause instability is a periodic critical point in the unperturbed system, for which there exist small neighborhoods (around the orbit of the periodical critical point) that are invariant under the perturbed system. G. Keller [Kel82] used this mechanism to construct a family of W-shaped maps with an unstable acim. G. Keller conjectured that this is the only mechanism that can cause instability of the acim. This conjecture turned out to be not true as shown by P. Eslami and M. Misiurewicz in [EM12]. Other papers that deal with instability of acims for W-shaped maps are for example [LGB$^+$13], [Li13] and [LG13].

In this thesis we look at a family of transformations called skew tent maps. Each skew tent map is a piecewise linear map that depends on two parameters. By A. Lasota and J.A. Yorke [LY73] each skew tent map has an acim and by T.Y. Li and J.A. Yorke [LY78] the acim is unique. We fix one parameter of the skew tent map. By M. Misiurewicz [Mis89] we know that the topological entropy of the skew tent map depends continuously on the non-fixed parameters. We prove that there is a region were the skew tent maps are acim-stable. As a consequence the measure-theoretic entropy of the skew tent map, with respect to the unique acim, depends continuously on the non-fixed parameter on the same region where the skew tent maps are acim-stable. In addition, we investigate the conditions under which the unique acim of the skew tent map is also a measure of maximal entropy.

1.2 Thesis overview

In the next chapter we introduce some basic concepts and results from ergodic theory and dynamical systems. In Chapter 3 we introduce the Perron-Frobenius operator and piecewise expanding maps. We derive some basic properties for the Perron-Frobenius operator and explain how the Perron-Frobenius operator can be used to show the existence of acims for piecewise expanding maps. We also explain what acim-stability is and give some results that can be used to show acim-stability for piecewise expanding maps. In Chapter 4 we look at skew tent maps. We start by showing that the skew tent map has a unique acim and look at properties such as exactness. After that we take one parameter fixed and show that the skew tent map is acim-stable for various values of the non-fixed parameter. As a consequence there exists a region where the measure-theoretic entropy of the skew tent maps depends continuously on the non-fixed parameter. In Chapter 5 we look at the relation between the acim and measures of maximal entropy for skew tent maps.

Chapter 2

Preliminaries

In this section we present the measure-theoretic background on dynamical systems and some general background that is needed throughout this thesis. We assume that the reader is familiar with some basic ideas from measure theory such as σ-algebras and measures. For a more extensive background on dynamical systems and some examples we refer the reader to [BG97] and [LM94].

2.1 Measure preserving transformations

In the study of dynamical systems from a measure-theoretic perspective we are interested in measures for which the probabilities of observable events do not change in time. This idea is formalized in the notion of a measure preserving transformation.

Definition 2.1.1. Let $(X, \mathcal{F}, \mu)$ be a probability space. We call a measurable transformation $T\colon X \to X$ *measure preserving with respect to* μ or say μ *is* T*-invariant* if $\mu(T^{-1}A) = \mu(A)$ for every $A \in \mathcal{F}$. The quadruple $(X, \mathcal{F}, \mu, T)$ is called a *dynamical system*.

Often one would like to have that a dynamical system cannot be split up into smaller dynamical systems. This irreducibility property is called ergodicity.

Definition 2.1.2. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. The transformation T is called *ergodic* with respect to the measure μ if for every $A \in \mathcal{F}$, such that $T^{-1}A = A$ we have $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

The famous Pointwise Ergodic Theorem also known as Birkhoff's Ergodic Theorem tells us that for ergodic transformations the time average equals the space average almost surely. This means that even if we are not able to describe what orbits of transformation do we still have information about what most orbits do on the average.

Theorem 2.1.3 (Pointwise Ergodic Theorem)**.** *Let* $(X, \mathcal{F}, \mu)$ *be a probability space and* $T\colon X \to X$ *a measure preserving transformation. Then, for any* $f \in L^1(X, \mathcal{F}, \mu)$*,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x)) = f^*(x)$$

exists μ*-a.e., is* T*-invariant and* $\int_X f\,d\mu = \int_X f^*\,d\mu$*. Moreover, if* T *is ergodic, then* f^* *is a constant* μ*-a.e. and* $f^* = \int_X f\,d\mu$*.*

Proof. See [Wal82, Theorem 1.14]. □

From the Pointwise Ergodic Theorem we get another characterization of being ergodic.

Corollary 2.1.4. *Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation. Then, T is ergodic if and only if for all each $A, B \in \mathcal{F}$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B).$$

Proof. See [Wal82, Corollarly 1.14.2]. □

From this characterization we see that being ergodic is a weak form of asymptotic independence. An ergodic system is asymptotically independent on average. A dynamical can have stronger notions of being independent, one example is mixing.

Definition 2.1.5. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. Then,

(i) T is called *weakly mixing* if for each $A, B \in \mathcal{F}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0.$$

(ii) T is called *strongly mixing* if for each $A, B \in \mathcal{F}$

$$\lim_{n\to\infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Weakly mixing is also a form of being asymptotically independent on average, while strongly mixing means being asymptotically independent. It can be shown that strongly mixing implies weakly mixing and weakly mixing implies ergodicity. The implications in the other direction are not true in general. There is a notion, introduced by Rokhlin, that is even stronger than mixing called exactness.

Definition 2.1.6. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. The transformation T is called *exact* if $\mathcal{F}_\infty(T) = \cap_{n=0}^{\infty} T^{-n}(\mathcal{F})$ consists of only sets of μ-measure 0 or 1.

Rokhlin studied properties of exact transformations and showed the following characterization of exactness.

Theorem 2.1.7. *Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. The transformation T is exact if and only if for each $A \in \mathcal{F}$ with positive μ-measure and measurable images $TA, T^2A, \ldots$ the following relationship holds:*

$$\lim_{n\to\infty} \mu(T^n A) = 1.$$

Proof. See [Rok61]. □

2.2 Invariant measures

Let $(X, \mathcal{F})$ be a measurable space, $T\colon X \to X$ a measurable transformation and $M(X)$ denote the collection of all measures on $(X, \mathcal{F})$. We are interested in the space of all T-invariant measures, i.e.

$$M(X,T) = \{\mu \in M(X) : \mu \circ T^{-1} = \mu\}.$$

In general it is not clear if there always exists a T-invariant measure. There are cases for which it is easy to prove the existence of invariant measures. One of those cases is when a transformation has periodic orbits. Let $x \in X$ be a point in a periodic orbit of T with order n. Then, the measure defined by

$$\mu = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x}$$

is a T-invariant measure. This follows from observing that for every $A \in \mathcal{F}$ we have

$$\mu(A) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x}(A) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^{i+1} x}(A) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i x}(T^{-1}A) = \mu(T^{-1}A).$$

For continuous transformations on a compact metric space we can also prove the existence of invariant measures. This is due to the following theorem.

Theorem 2.2.1. *Let X be a compact metric space and $T\colon X \to X$ be a continuous transformation. If $\{\mu_n\}_{n=1}^{\infty}$ is a sequence in $M(X)$ and we form the new sequence $\{\tilde{\mu}_n\}_{n=1}^{\infty}$ by $\tilde{\mu}_n = \frac{1}{n}\sum_{i=0}^{n-1} \mu_n \circ T^{-i}$, then any limit point $\tilde{\mu}$ of $\{\tilde{\mu}_n\}$ is a member of $M(X,T)$. Such limit points exist by the compactness of $M(X)$.*

Proof. See [Wal82, Theorem 6.9]. □

If we have an interval map, then the invariant measure we constructed for a periodic orbit is supported on a small set. We are interested in interval maps that have an invariant measure that is supported on a large part of the system. A measure that does this is the Lebesgue measure, but it is not necessary an invariant measure. We look for invariant measures that behave like the Lebesgue measure. This is formalized in the notation of absolutely continuous measures.

Definition 2.2.2. Let μ and ν be two measures on the same measurable space $(X, \mathcal{F})$. We say that μ is *absolutely continuous* with respect to ν if for any $A \in \mathcal{F}$, such that $\nu(A) = 0$, it follows that $\mu(A) = 0$. The absolute continuity of μ with respect to ν is denoted by $\mu \ll \nu$.

We assume absolute continuity with respect to the Lebesgue measure when not explicitly mentioned and abbreviate absolutely continuous invariant measure by acim. It is difficult to find acims under general assumptions or they might not even exist. In Chapter 3 we introduce the piecewise expanding maps and show the existence acims for some maps in this class.

2.3 Entropy

The concept of entropy was introduced in information theory by C.E. Shannon [Sha48]. It measures the amount of information contained in a source. In general, the more certain or deterministic a source is, the less information it will contain. The concept of entropy is now used in different fields of mathematics. We look at the entropy in a measure preserving dynamical system and in a topological dynamical system.

2.3.1 Measure-theoretic entropy

The measure-theoretic entropy expresses the amount of randomness in the system generated by a transformation. For a detailed explanation on measure-theoretic entropy we refer the reader to [Wal82]. We start out by introducing some notation that is used to define measure-theoretic entropy.

Definition 2.3.1. Let $(X, \mathcal{F}, \mu)$ be a measure space and I a finite or countable index set. We call $\alpha = \{\alpha_i : i \in I\}$ *a partition of* X if α is the disjoint union of X up to sets of measure zero.

Let $(X, \mathcal{F}, \mu)$ be a measure space and let $\alpha = \{\alpha_1, \dots, \alpha_n\}$, $\beta = \{\beta_1, \dots, \beta_m\}$ be finite partitions on X. We define a refinement by

$$\alpha \vee \beta := \{\alpha_i \cap \beta_j : 1 \leq i \leq n, 1 \leq j \leq m\}.$$

This refinement is called the join of α and β and is also a partition of X. We can also define the join of a finite partition α and a transformation T by

$$\bigvee_{i=0}^{n-1} T^{-i}\alpha := \alpha \vee T^{-1}\alpha \vee \ldots \vee T^{-(n-1)}\alpha.$$

This set consists of elements that are of the form

$$A_{i_0} \cap T^{-1} A_{i_1} \cap \ldots \cap T^{-(n-1)} A_{i_{n-1}}.$$

For $j, k \geq 0$ we let

$$\sigma\big(\bigvee_{i=j}^{k} T^{-i}\alpha\big) \quad \text{and} \quad \sigma\big(\bigvee_{i=-j}^{-k} T^{-i}\alpha\big)$$

denote the smallest σ-algebras containing the partitions $\bigvee_{i=j}^{k} T^{-i}\alpha$ and $\bigvee_{i=-j}^{-k} T^{-i}\alpha$ respectively. Furthermore, if the transformation T is invertible we let

$$\sigma\big(\bigvee_{i=-\infty}^{\infty} T^{-i}\alpha\big)$$

denote the smallest σ-algebra containing all the elements of all the partitions $\bigvee_{i=j}^{k} T^{-i}\alpha$ and $\bigvee_{i=-j}^{-k} T^{-i}\alpha$ for all $j, k \in \mathbb{N}$. If T is invertible, then we call a partition α a *generator* with respect to T if $\sigma\big(\bigvee_{i=-\infty}^{\infty} T^{-i}\alpha\big) = \mathcal{F}$. If T is non-invertible, then we call a partition α a generator with respect to T if $\sigma\big(\bigvee_{i=0}^{\infty} T^{-i}\alpha\big) = \mathcal{F}$.

We now define the measure-theoretic entropy. This is done in a few steps. The first step is to define the entropy of a finite partition α which is defined as

$$H_\mu(\alpha) := -\sum_{i=1}^{n} \mu(\alpha_i)\log(\mu(\alpha_i)).$$

Next we define the entropy of T with respect to the partition α which is defined as

$$h_\mu(\alpha, T) := \lim_{n\to\infty} \frac{1}{n} H_\mu(\bigvee_{i=0}^{n-1} T^{-i}\alpha).$$

Finally we define the measure-theoretic entropy of a transformation.

Definition 2.3.2. Let $(X, \mathcal{F}, \mu, T)$ be a dynamical system. The *measure-theoretic entropy* of T is given by

$$h_\mu(T) = \sup_\alpha h_\mu(\alpha, T),$$

where the supremum is taken over all partitions with finite entropy.

Calculating the measure-theoretic entropy from the definition is difficult, because we need to take the supremum over all partitions with finite entropy. We get a more practical way of calculating the entropy by using a partition that is a generator.

Theorem 2.3.3 (Kolmogorov-Sinai Theorem)**.** *If the partition α is a generator with respect to the map T and $H_\mu(\alpha) < \infty$, then $h_\mu(T) = h_\mu(\alpha, T)$.*

The next theorem gives another way to calculate the measure-theoretic entropy. It gives an expression for the measure-theoretic entropy in terms of the Jacobian and the invariant measure. This result also holds for higher dimensional maps. We formulate the result for the one-dimensional case, where the Jacobian is simply the derivative of the map.

Theorem 2.3.4 (Rokhlin's formula)**.** *Let $(X, \mathcal{F}, \mu)$ be a probability space and $T\colon X \to X$ a measure preserving map that is locally invertible. If the partition α is a generator with finite entropy and every $\alpha_i \in \alpha$ is an invertibility domain of T, then $h_\mu(T) = \int \log_X |T'| d\mu$.*

Proof. See [VO16, Theorem 9.7.3]. $\square$

We now give an example where we use the Kolomogorov-Sinai Theorem to calculate the measure-theoretic entropy.

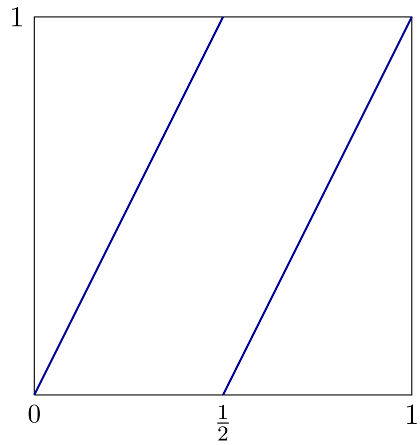

Figure 2.1: The doubling map.

Example 2.3.5. Let $(I, \mathcal{B}(I), \lambda)$ be a probability space and $T\colon [0,1) \to [0,1)$ the *doubling map*, which is defined by

$$T(x) = 2x \bmod 1 = \begin{cases} 2x & \text{if } \ x \in [0, \frac{1}{2}), \\ 2x - 1 & \text{if } \ x \in [\frac{1}{2}, 1]. \end{cases}$$

The graph is shown in Figure 2.1. For any interval $[a, b)$ we have

$$T^{-1}[a,b) = \Big[\frac{a}{2}, \frac{b}{2}\Big) \cup \Big[\frac{a+1}{2}, \frac{b+1}{2}\Big),$$

and

$$\lambda(T^{-1}[a,b)) = b - a = \lambda([a,b)).$$

Using Theorem A.0.1 it can be shown that λ is T-invariant. For the partition $\alpha = \big\{[0, \frac{1}{2}), [\frac{1}{2}, 1)\big\}$ we have

$$\bigvee_{i=0}^{n-1} T^{-i}\alpha = \left\{ \left[\frac{i}{2^n}, \frac{i+1}{2^n}\right) : i = 0, \dots, 2^n - 1 \right\}. \tag{2.1}$$

The intervals in (2.1) are called dyadic intervals and they generate the Borel σ-algebra. The entropy with respect to the partition α and transformation T is given by

$$\begin{aligned} H_\lambda\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) &= -\sum_{i=0}^{2^n-1} \lambda\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right) \log \lambda\left(\left[\frac{i}{2^n}, \frac{i+1}{2^n}\right)\right) \\ &= -\sum_{i=0}^{2^n-1} \left(\frac{1}{2^n}\right) \log\left(\frac{1}{2^n}\right) \\ &= -2^n \left(\frac{1}{2^n}\right) \log\left(\frac{1}{2^n}\right) \\ &= n \log 2. \end{aligned}$$

By the Kolmogorov-Sinai Theorem the entropy of T is given by

$$h_\lambda(T) = h_\lambda(\alpha, T) = \lim_{n\to\infty} \frac{1}{n} H_\lambda\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) = \lim_{n\to\infty} \frac{1}{n} \cdot n \log 2 = \log 2.$$

2.3.2 Topological entropy

The topological entropy is a measure for the complexity of the system. The first definition of topological entropy was given by Adler, Konheim and McAndrew [AKM65] and makes use of open covers for the space X. Their definition requires X to be a compact metric space and the transformation $T\colon X \to X$ to be continuous. Later Bowen [Bow71] and Dinaburg [Din70] independently introduced a different definition using ϵ-separated points. It can also be defined using the dual definition of ϵ-spanning points. Their definition requires (X, d) to be a metric space and $T\colon X \to X$ to be uniformly continuous. In the setting of compact metric spaces and continuous maps, these two definitions agree. The definition by Adler, Konheim and McAndrew can be found in [AKM65]. We follow [Wal82] for the definition introduced by Bowen and Dinaburg. We define the topological entropy using ϵ-spanning points. The definition using ϵ-separated points is analogous and results in the same value for the topological entropy.

Let (X, d) be a metric space and $T\colon X \to X$ a uniformly continuous map. We denote the open ball with centre x and radius r by $B(x, r)$ and the closed ball by $\bar{B}(x, r)$. We define a new metric d_n on X by

$$d_n(x, y) = \max_{0\leq i\leq n-1} d(T^i x, T^i y).$$

The open ball with centre x and radius r in the new metric d_n is $\bigcap_{i=0}^{n-1} T^{-i}B(T^i x, r)$.

Definition 2.3.6. Let n be a natural number, $\epsilon > 0$ and K a compact subset of X. A subset A of X is said to *(n, ϵ)-span* K with respect to T if for every $x \in K$ there is a $y \in A$ such that $d_n(x, y) \leq \epsilon$, i.e.

$$K \subset \bigcup_{y\in A} \bigcap_{i=0}^{n-1} T^{-i}\bar{B}(T^i y, \epsilon).$$

Let $r_n(\epsilon, K, T)$ denote the smallest cardinality of any (n, ϵ)-spanning set for K with respect to T. By compactness of K the open cover $\{\bigcap_{i=0}^{n-1} T^{-i}B(T^i x, r) : x \in X\}$ of K has finite subcover and therefore $r_n(\epsilon, K, T) < \infty$. Next define

$$h(T, K) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{\log r_n(\epsilon, K, T)}{n}.$$

Definition 2.3.7. Let (X, d) be a metric space and $T\colon X \to X$ a uniformly continuous map. Then the *topological entropy* of T is

$$h(T) = \sup_K h(T, K),$$

where the supremum is taken over the collection of all compact subsets of X.

We now give a result that can be used to calculate the topological entropy of piecewise monotone continuous maps. These maps are defined in Section. 3.2

Definition 2.3.8. Let $T : X \to X$ be a piecewise monotone continuous map. The *lap number* of T, which we denote by $c_1(T)$, is the minimal number of intervals on which T is monotone. In other words, $c_1(T) - 1$ is the number of turnings points of T. With $c_n(T)$ we denote the minimal number of intervals on which T^n is monotone.

Theorem 2.3.9 (**Misiurewicz and Szlenk**, [MS80])**.** *Let $T : X \to X$ be a piecewise monotone continuous map, then*

$$h(T) = \lim_{n\to\infty} \frac{1}{n} \log c_n$$

and $\frac{1}{n} \log c_n \geq h(T)$ for any n.

Corollary 2.3.10. *If $T : X \to X$ is a piecewise linear continuous map with slope equal to $\pm s$, then the topological entropy of T is equal to* $\max\{0, \log s\}$.

Proof. See Corollary 7.2 in [dMvS93]. □

Example 2.3.11. Consider the *tent map* $T\colon [0,1] \to [0,1]$, which is defined by

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2), \\ 2 - 2x & \text{if } x \in [1/2, 1]. \end{cases}$$

The graph is shown in Figure 2.2. It follows from Corollary 2.3.10 that the topological entropy of T is given by $h(T) = \log 2$.

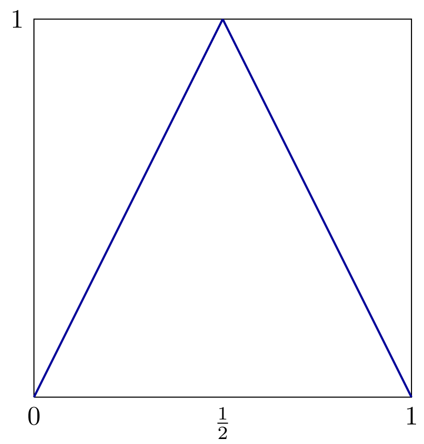

Figure 2.2: The tent map.

2.3.3 Variational principle

The measure-theoretic entropy and topological entropy are related through the Variational Principle for the entropy.

Theorem 2.3.12. *Let $T\colon X \to X$ be a continuous map on a compact metric space X. Then $h(T) = \sup\{h_\mu(T) : \mu \in M(X,T)\}$.*

Proof. See [Wal82, Theorem 8.6]. □

2.4 Isomorphism of dynamical systems

When you have a dynamical system you can study the measure structure and the dynamics of the transformation. It is possible that the measure structure and the dynamics of another dynamical system are the same. Two dynamical systems that are essentially the same are called isomorphic.

Definition 2.4.1. Two dynamical systems $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are *isomorphic* if there exists a map $\psi\colon (X, \mathcal{F}, \mu, T) \to (Y, \mathcal{C}, \nu, S)$ that has the following properties.

(i) ψ is one-to-one and onto almost everywhere. By this we mean that if we remove a suitable μ-null set N_X from X and a suitable ν-null set N_Y from Y, such that $T(X\backslash N_X) \subset X\backslash N_X$ and $S(Y\backslash N_Y) \subset Y\backslash N_Y$, then the map $\psi\colon X\backslash N_X \to Y\backslash N_Y$ is a bijection.

(ii) ψ is bi-measurable, i.e., $\psi^{-1}(C) \in \mathcal{F}$ for all $C \in \mathcal{C}$.

(iii) ψ preserves the measure: $\nu = \mu \circ \psi^{-1}$, i.e., $v(C) = \mu(\psi^{-1}(C))$ for all $C \in \mathcal{C}$.

(iv) ψ preserves the dynamics of T and S, i.e., $\psi \circ T = S \circ \psi$.

The map ψ is called an *isomorphism*.

Theorem 2.4.2. *Let $(X, \mathcal{F}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ be two isomorphic dynamical systems, then $h_\mu(T) = h_\nu(S)$.*

Proof. See [Wal82, Theorem 4.11]. □

The converse of Theorem 2.4.2 is in general not true. If two measure preserving transformations have the same entropy, then they are not necessarily isomorphic.

Chapter 3

Theoretical background

In the first part of this chapter we introduce the Perron-Frobenius operator. It describes the time evolution of densities under a transformation and is therefore an essential tool in the study of absolutely continuous invariant measures. In the second part we introduce piecewise expanding transformations. We study the existence of absolutely continuous invariant measures for piecewise expanding transformations, their ergodic properties and their stability.

3.1 The Perron-Frobenius operator

The motivation for the definition of the Perron-Frobenius operator is as follows. Let $(X, \mathcal{F}, \mu)$ be a probability space and $Y : X \to X$ a random variable with probability density function f. Then, for every $A \in \mathcal{F}$ we have

$$\mathrm{Prob}(\{Y \in A\}) = \int_A f \mathrm{d}\mu.$$

Let $T \colon X \to X$ be a transformation. Then $T \circ Y$ is also a random variable. We wonder if $T(Y)$ also has a probability density function. We have the following

$$\mathrm{Prob}(\{T \circ Y \in A\}) = \mathrm{Prob}(\{Y \in T^{-1}(A)\}) = \int_{T^{-1}A} f \mathrm{d}\mu.$$

This means that if there exists a probability density function f^* such that

$$\int_{T^{-1}A} f \mathrm{d}\mu = \int_A f^* \mathrm{d}\mu, \tag{3.1}$$

then $T \circ Y$ has a probability density function. In general it is not clear if such a probability density function exists. However, if T is non-singular we can say more about the existence of a probability density function such that equation (3.1) holds.

Definition 3.1.1. A measurable transformation $T \colon X \to X$ on a measure space $(X, \mathcal{F}, \mu)$ is called *non-singular* if for all $A \in \mathcal{F}$ we have that $\mu(T^{-1}(A)) = 0$ if and only if $\mu(A) = 0$.

If T is non-singular, then the existence of a probability density function such that equation (3.1) holds is a consequence of the Radon-Nikodym Theorem. The operator that maps the probability density function f to the probability density function f^* such that equation (3.1) holds is called the Perron-Frobenius operator.

Theorem 3.1.2. *Let $(X, \mathcal{F})$ be a measurable space and let μ be a σ-finite measure on $(X, \mathcal{F})$. Let Y be a random variable on X with probability density function f. If the transformation $T\colon X \to X$ is non-singular, then there exists a unique probability density function $P_T f$ such that*

$$\int_{T^{-1}A} f\, d\mu = \int_A P_T f\, d\mu. \tag{3.2}$$

The unique operator $P_T\colon L^1(X, \mathcal{F}, \mu) \to L^1(X, \mathcal{F}, \mu)$ such that (3.2) holds is called the Perron-Frobenius operator corresponding to T.

Proof. Assume that the transformation T is non-singular. For any $A \in \mathcal{F}$ we write

$$\nu(A) = \int_{T^{-1}A} f \ \mathrm{d}\mu.$$

It can be shown that ν is a σ-finite measure. By non-singularity of T it follows that $\nu \ll \mu$. Then, the Radon-Nikodym Theorem [1] gives the existence of a unique $P_T f \in L^1(X, \mathcal{F}, \mu)$ such that

$$\int_A P_T f \mathrm{d}\mu = \int_{T^{-1}A} f \mathrm{d}\mu.$$

□

We now state the most basic properties for the Perron-Frobenius. See e.g. [BG97] and [LM94] for more details.

Proposition 3.1.3 (Linearity). *The Perron-Frobenius operator is a linear operator.*

Proof. Let $f, g \in L^1(X, \mathcal{F}, \mu)$ and take constants $\alpha, \beta \in \mathbb{R}$. For every $A \in \mathcal{F}$ we have

$$\begin{aligned} \int_A P_T(\alpha f + \beta g)\mathrm{d}\mu &= \int_{T^{-1}A} (\alpha f + \beta g)\mathrm{d}\mu \\ &= \alpha \int_{T^{-1}A} f\mathrm{d}\mu + \beta \int_{T^{-1}A} g\mathrm{d}\mu \\ &= \alpha \int_A P_T f\mathrm{d}\mu + \beta \int_A P_T g\mathrm{d}\mu \\ &= \int_A (\alpha P_T f + \beta P_T g)\mathrm{d}\mu. \end{aligned}$$

Since this holds for every measurable set A we have

$$P_T(\alpha f + \beta g) = \alpha P_T f + \beta P_T g \quad \mu\text{-a.e.}$$

□

[1] see Appendix A for the Radon-Nikodym Theorem

Proposition 3.1.4 (Positivity)**.** *For every $f \in L^1(X, \mathcal{F}, \mu)$ with $f \geq 0$ we have $P_T f \geq 0$.*

Proof. For every $A \in \mathcal{B}$ we have

$$\int_A P_T f \mathrm{d}\mu = \int_{T^{-1}A} f \mathrm{d}\mu \geq 0.$$

Since this holds for every measurable set A we have $P_T f \geq 0$. □

Proposition 3.1.5 (Preservation of Integrals)**.** *For the Perron-Frobenius operator we have the preservation of integrals, i.e.*

$$\int_X P_T f \, d\mu = \int_X f \, d\mu.$$

Proof. Using the definition of the Perron-Frobenius operator and the non-singularity of T gives

$$\int_X P_T f \mathrm{d}\mu = \int_{T^{-1}X} f \mathrm{d}\mu = \int_X f \mathrm{d}\mu.$$

□

Proposition 3.1.6 (Contraction Property)**.** *The Perron-Frobenius operator is a contraction, i.e. $||P_T f||_1 \leq ||f||_1$ for any $f \in L^1(X, \mathcal{F}, \mu)$.*

Proof. Let $f \in L^1(X, \mathcal{F}, \mu)$. Define $f^+ = \max\{f, 0\}$ and $f^- = -\min\{0, f\}$. We have $f^+, f^- \in L^1(X, \mathcal{F}, \mu)$ and $f^+, f^- \geq 0$. Note that we can write $f = f^+ - f^-$ and $|f| = f^+ + f^-$. By Proposition 3.1.3 we have

$$P_T f = P_T(f^+ - f^-) = P_T f^+ - P_T f^-.$$

By the triangle inequality and the linearity of $P_T f$ we get

$$|P_T f| = |P_T f^+ - P_T f^-| \leq |P_T f^+| + |P_T f^-| = P_T f^+ + P_T f^- = P_T(f^+ + f^-) = P_T |f|.$$

Combining this with Proposition 3.1.5 gives

$$||P_T f||_1 = \int_X |P_T f| \mathrm{d}\mu \leq \int_X P_T |f| \mathrm{d}\mu = \int_X |f| \mathrm{d}\mu = ||f||_1.$$

□

Proposition 3.1.7 (Composition Property)**.** *For the Perron-Frobenius operator we have $P_{T \circ S} f = P_T \circ P_S f$. In particular, $P_{T^n} f = P_T^n f$.*

Proof. Let $f \in L^1(X, \mathcal{F}, \mu)$ and define the measure ν by

$$\nu(A) = \int_{(T \circ S)^{-1}A} f \mathrm{d}\mu.$$

From the non-singularity of T and S it follows that $\nu \ll \mu$. By the Radon-Nikodym Theorem there exists a function $P_{T \circ S} f$ such that

$$\nu(A) = \int_{(T \circ S)^{-1}A} f \mathrm{d}\mu = \int_A P_{T \circ S} f \mathrm{d}\mu.$$

We have

$$\int_A P_T(P_S f)\mathrm{d}\mu = \int_{T^{-1}A} P_S f \mathrm{d}\mu = \int_{S^{-1}(T^{-1}A)} f \mathrm{d}\mu$$

and

$$\int_A P_{T\circ S} f \mathrm{d}\mu = \int_{(T\circ S)^{-1}A} f \mathrm{d}\mu = \int_{S^{-1}(T^{-1}A)} f \mathrm{d}\mu,$$

from which it follows that $P_{T\circ S} f = P_T P_S f$ μ a.e. By induction, it follows that $P_{T^n} f = P_T^n f$ μ-a.e. □

The following proposition gives an important relation between the fixed points of the Perron-Frobenius operator and the densities of measures that are T-invariant and absolutely continuous with respect to λ.

Proposition 3.1.8. *Let $f \in L^1(X, \mathcal{F}, \mu)$, with $f \geq 0$ and $||f||_1$. Then $P_T f = f$ if and only if $\mu = f\lambda$ is T-invariant.*

Proof. Assume that μ is T-invariant, so for every $A \in \mathcal{B}$ we have

$$\mu(T^{-1}A) = \int_{T^{-1}A} f \mathrm{d}\lambda = \int_A f \mathrm{d}\lambda = \mu(A).$$

By the definition of the Perron-Frobenius operator we have

$$\int_A P_T f \mathrm{d}\lambda = \int_{T^{-1}A} f \mathrm{d}\lambda = \mu(T^{-1}A)$$

Hence, $P_T f = f$ a.e. Now assume that $P_T f = f$ a.e., so

$$\int_A P_T f \mathrm{d}\lambda = \int_A f \mathrm{d}\lambda = \mu(A).$$

Again using the definition of the Perron-Frobenius operator we have

$$\int_A P_T f \mathrm{d}\lambda = \int_{T^{-1}A} f \mathrm{d}\lambda = \mu(T^{-1}A).$$

Hence, μ is T-invariant. □

The previous result allows us to show the existence of absolutely continuous invariant measures if one considers the action of the Perron-Frobenius operator on the right space of functions. We consider the action of the Perron-Frobenius operator on the space of functions of bounded variation.

Definition 3.1.9. Let f be a real (or complex) valued function on $[a, b]$ and $x_0, x_1, \ldots, x_n$ a finite sequence of points such that $a = x_0 < x_1 < \ldots < x_n = b$. The *total variation* of f on $[a, b]$ is defined as

$$\operatorname*{Var}_{[a,b]} f := \sup_{P\in\mathcal{P}} \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|,$$

where $\mathcal{P}$ denotes the collection of all sequences of points $P = \{x_1, \ldots, x_n\}$ with $a = x_0 < x_1 < \ldots < x_n = b$ for all n. The space of *functions with bounded variation* is defined by

$$\mathrm{BV}([a, b]) := \{f \in L^1([a, b], \lambda) : \operatorname*{Var}_{[a,b]} f < \infty\}.$$

In Appendix B we derive some results for functions of bounded variation. By using the Kakutani-Yosida Theorem we can show that Perron-Frobenius operator has fixed points on the space of bounded variation.

Theorem 3.1.10 (Kakutani-Yosida Theorem)**.** *Let X be a Banach space and let $P\colon X \to X$ be a bounded linear operator. Assume that there exists $c > 0$ such that $||P^n|| \leq c$ for each $n \in \mathbb{N}$. Furthermore, if for any $f \in A \subset X$, the sequence $\{f_n\}$, where*

$$f_n = \frac{1}{n}\sum_{i=1}^{n} P^i f,$$

contains a subsequence $\{f_{n_j}\}$ which converges weakly in X, then for any $f \in \bar{A}$,

$$f_n \to f^* \in X$$

(norm convergence) and $P(f^*) = f^*$.

Proof. See [DS64]. □

We work towards showing that the Perron-Frobenius operator on the space of function of bounded variation satisfies all conditions of the Kakutani-Yosida Theorem. We start by finding a norm on the space of functions of bounded variation. A logical choice would be to try the L^1-norm, but the space of functions of bounded variation equipped with the L^1-norm is not closed. By adding the variation to the L^1-norm we get a norm on the space of function of bounded variation.

Proposition 3.1.11. *Let $f \in BV(I)$, then the function $||\cdot||_{BV}\colon BV(I) \to [0,\infty)$ defined by*

$$||f||_{BV} = ||f||_1 + \operatorname*{Var}_I f$$

is a norm and turns $(BV(I), ||\cdot||_{BV})$ into a Banach space.

Proof. See [HK82, Lemma 5(ii)]. □

It follows from the linearity, contraction and composition properties that the Perron-Frobenius is a bounded linear operator on the space of functions of bounded variation and that there exists $c > 0$ such that $||P_T^n|| \leq c$ for each $n \in \mathbb{N}$. The next step is to show that the sequence $\{f_n\}$, where

$$f_n = \frac{1}{n}\sum_{i=1}^{n} P^i f,$$

contains a subsequence $\{f_{n_j}\}$ which converges weakly in the space of functions of bounded variation. This is done by showing that for any $f \in BV(I)$ we have

$$\limsup_{n\to\infty} \sup_{x\in I} P_T^n f(x) < \infty \quad \text{and} \quad \limsup_{n\to\infty} \operatorname*{Var}_I P_T^n f < \infty.$$

By Helly's Selection Theorem [2] the sequence $\{P_T^n f\}$ is relatively compact in the space of functions of bounded variation and by Mazur's Theorem [3] the sequence $\{f_n\}$ is also relatively compact in the space of functions of bounded variation. Since the set of functions with bounded variation is dense in L^1 we have for any $f \in L^1$ that the sequence $\{f_n\}$ is also relatively compact in L^1.

[2] See Appendix A for Helly's Selection Theorem

[3] See Appendix A for Mazur's Theorem

3.2 Piecewise expanding maps

In the next chapter we introduce the family of skew tent maps. They are part of the class of piecewise expanding maps. The piecewise expanding maps fall in the larger category of piecewise monotonic maps. In this section we give a few important results on piecewise expanding maps that we use in the next chapter. We start out by introducing some notation.

Definition 3.2.1. A partition of the interval $[a, b]$ is a finite sequence $x_0, x_1, \ldots, x_n$ of real numbers such that $a = x_0 < x_1 < \ldots < x_n = b$. We write $I_i = (x_{i-1}, x_i)$ and refer to I_i as an open subinterval. We use $P = \{I_i = (x_{i-1}, x_i) : 1 \leq i \leq n\}$ to denote the collection of open subintervals for the partition $x_0, x_1, \ldots, x_n$.

We will only look at piecewise expanding maps on the interval $I = [0, 1]$. For convenience we define everything for the interval I instead of a general interval.

Definition 3.2.2. The map $T\colon I \to I$ is called *piecewise monotonic* if there exists $P = \{I_i = (x_{i-1}, x_i) : 1 \leq i \leq n\}$ and a number $k \geq 1$ such that T satisfies the following conditions for all $1 \leq i \leq n$,

(1) $T_i := T_{|I_i}$ is a C^k function, which can be extended to a C^k function on $[x_{i-1}, x_i]$;

(2) $|T'(x)| > 0$ for all $x \in I_i$.

The map T is called *expanding* if $|T'(x)| > 1$ for all $x \in I$, where $T'(x)$ is defined.

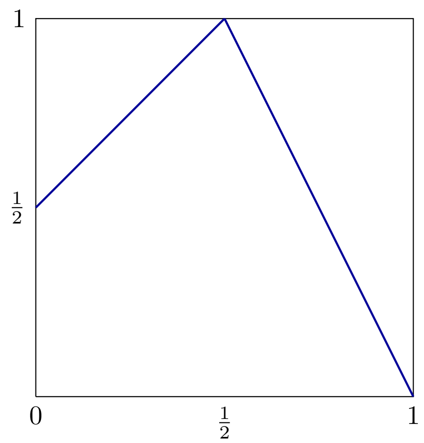

Figure 3.1: A piecewise monotonic map

In Figure 3.1 we have a piecewise monotonic map that consists of two monotonic pieces. The maps that we consider later on fall into the following subclass of piecewise expanding maps.

Definition 3.2.3. The map $T\colon I \to I$ is called *piecewise expanding* $C^{1,1}$ if there exists $P = \{I_i = (x_{i-1}, x_i) : 1 \le i \le n\}$ such that T satisfies the following conditions for all $1 \le i \le n$,

1. $T_i := T_{|I_i}$ is monotonic, C^1, and can be extended to the closed interval $[x_{i-1}, x_i]$ as a C^1 function;
2. $T_i^{'}$ is Lipschitz, i.e., there exists a constant M_i such that $|T_i^{'}(x) - T_i^{'}(y)| \le M_i|x-y|$, for all $x, y \in I_i$;
3. $|T_i^{'}(x)| \ge s_i > 1$ for all $x \in I_i$.

Let $\mathcal{T}(I)$ be the class of piecewise expanding $C^{1,1}$ maps on I. If a family of maps $\{T_\epsilon\}$ satisfies the conditions with uniform constants s_i and M_i (i.e. independent of ϵ), then we shall write $\{T_\epsilon\} \subset \mathcal{T}(I)$ *uniformly.*

The following proposition gives information about the number of ergodic acims for piecewise expanding maps can have. The proof for this proposition and other properties of acims can be found in [BG97, Chapter 8].

Proposition 3.2.4. *Let $T : I \to I$ be a piecewise continuous map defined on a partition $P = \{I_1, \ldots, I_n\}$. Then, the number of distinct ergodic absolutely continuous invariant measures for T is at most $n-1$.*

The following propositions give conditions under which a piecewise expanding map is exact. These results can be found in [HK82, Theorem 3(ii)] and [Kel78, Corollary 2], respectively.

Proposition 3.2.5. *Let $T : I \to I$ be a piecewise monotonic transformation. If (T, μ) is weakly mixing, then (T, μ) is exact.*

Proposition 3.2.6. *Let $T : I \to I$ a piecewise expanding $C^{1,1}$ map defined on a partition $P = \{I_i = (x_{i-1}, x_i) : 1 \le i \le n\}$. For n even the map T is exact if*

$$\inf_{x \in I} |T'(x)| > \sqrt{\frac{n}{2}\Big(\frac{n}{2}+1\Big)}.$$

3.2.1 The Perron-Frobenius operator

Let $T\colon I \to I$ be a piecewise monotonic map and let $P = \{I_i = (x_{i-1}, x_i) : 1 \le i \le n\}$ denote the partition of I into monotonic intervals. For piecewise monotonic transformations we get a nice representation for the Perron-Frobenius operator. Since T is monotonic on each interval I_i we can define an inverse function for each $T(I_i)$. Let $\zeta_i : T(I_i) \to I_i$ be the function

$$\zeta_i = T^{-1}_{|T(I_i)}.$$

For every measurable set $E \subset I$ we can write $E = \bigcup_{i=1}^n T(I_i) \cap E$. This gives the following representation

$$T^{-1}(E) = \bigcup_{i=1}^{n} \zeta_i(T(I_i) \cap E).$$

Combining this with the definition for the Perron-Frobenius operator gives

$$\int_E P_T f \mathrm{d}\mu = \int_{T^{-1}E} f \, \mathrm{d}\mu = \sum_{i=1}^{n} \int_{\zeta_i(T(I_i)\cap E)} f \mathrm{d}\mu.$$

By using a change of variables we obtain

$$\begin{aligned}\int_E P_T f \mathrm{d}\mu &= \sum_{i=1}^{n} \int_{T(I_i)\cap E} f(\zeta_i(x))|\zeta_i^{'}(x)| \, \mathrm{d}\mu(x) \\ &= \sum_{i=1}^{n} \int_E f(\zeta_i(x))|\zeta_i^{'}(x)|\chi_{T([x_{i-1},x_i])}(x) \, \mathrm{d}\mu(x),\end{aligned}$$

where $\chi_{T([x_{i-1},x_i])}$ is the indicator function for the interval $T([x_{i-1}, x_i])$. Rewriting the expression above gives

$$\int_E P_T f \mathrm{d}\mu = \int_E \sum_{i=1}^{n} \frac{f(T_i^{-1}(x))}{|T_i^{'}(T_i^{-1}(x))|} \chi_{T([x_{i-1},x_i])}(x) \, \mathrm{d}\mu.$$

Since E is arbitrary, we conclude that for any $f \in L^1$ we have

$$P_T f(x) = \sum_{i=1}^{n} \frac{f(T_i^{-1}(x))}{|T_i^{'}(T_i^{-1}(x))|} \chi_{T([x_{i-1},x_i])}(x) \quad \mu\text{-a.e.} \tag{3.3}$$

Example 3.2.7. Consider the map $T : I \to I$ defined by

$$T(x) = \begin{cases} x + \frac{1}{2} & \text{if } \ x \in [0, \frac{1}{2}], \\ 2 - 2x & \text{if } \ x \in [\frac{1}{2}, 1]. \end{cases}$$

The graph is shown in Figure 3.1. The map T is a piecewise monotonic map, because T_1 and T_2 are linear functions. Let $I_1 = [0, \frac{1}{2}]$ and $I_2 = [\frac{1}{2}, 1]$, then $T(I_1) = I_2$ and $T(I_2) = I$. Moreover, $|T_1^{'}(x)| = 1$ and $|T_2^{'}(x)| = 2$. The inverse functions are given by $T_1^{-1}(x) = x - \frac{1}{2}$ and $T_2^{-1}(x) = 1 - \frac{1}{2}x$. Substituting these expressions in (3.3) gives the following representation for the Perron-Frobenius operator

$$P_T f(x) = f(x - \tfrac{1}{2})\chi_{[\frac{1}{2},1]} + \tfrac{1}{2} f(1 - \tfrac{1}{2}x).$$

3.3 Piecewise expanding C^2 transformations

The following result by A. Lasota and J.A. Yorke [LY73] gives the existence of absolutely continuous invariant measures for non-singular piecewise C^2 maps.

Theorem 3.3.1. *Let $T\colon I \to I$ be a non-singular piecewise C^2 function such that $\inf_{x\in[0,1]} |T^{'}(x)| > 1$. Then for any $f \in BV(I)$ the sequence*

$$f_n = \frac{1}{n}\sum_{k=0}^{n-1} P_T^k f$$

is convergent in L^1-norm to a function $f^* \in L^1(I, \mathcal{B}, \lambda)$. The limit function has the following properties:

(1) $f \geq 0 \Rightarrow f^* \geq 0$.

(2) $\int_0^1 f^* d\lambda = \int_0^1 f d\lambda$.

(3) $P_T f^* = f^*$ *and consequently the measure* $d\mu^* = f^* d\lambda$ *is invariant under* T.

(4) *The function* f^* *is of bounded variation; moreover, there exists a constant* c *independent of the choice of initial* f *such that the variation of the limiting* f^* *satisfies the inequality*

$$\operatorname*{Var}_I f^* \leq c||f||.$$

In the proof of Theorem 3.3.1 there is an inequality that plays an important, that inequality became known as the Lasota-Yorke inequality. Before we state it we need some notation. Write $s = \inf |T^{'}|$ and choose a number N such that $s^N > 2$. It is easy to see that the function $\phi = T^N$ is a piecewise C^2 function. Denote by $b_0, \ldots, b_q$ the partition corresponding to the intervals of monotonicity of ϕ. Writing ϕ_i for the corresponding C^2 functions we have

$$|\phi_i^{'}(x)| \geq s^N, \quad x \in [b_{i-1}, b_i],\ i = 1, \ldots, q. \tag{3.4}$$

Let $\psi_i = \phi_i^{-1}, \sigma_i(x) = |\psi_i^{'}(x)|$ and $J_i = \phi_i([b_{i-1}, b_i])$, then it follows from (3.4) that

$$|\sigma_i(x)| \leq s^{-N}, \quad x \in J_i,\ i = 1, \ldots, q. \tag{3.5}$$

Computing the Frobenius-Perron operator for ϕ we obtain

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_i(x),$$

where χ_i is the characteristic function of the interval J_i. An upper bound on the variation of $P_\phi f$ is given by the Lasota-Yorke inequality.

Proposition 3.3.2 (Lasota-Yorke Inequality)**.** *Let* $T : I \to I$ *be a non-singular piecewise expanding* C^2 *function. Then for every* $f \in BV(I)$ *there exists* $N \in \mathbb{N}$ *such that*

$$\operatorname*{Var}_I P_{T^N} f \leq 2s^{-N} \operatorname*{Var}_I f + (K + 2h^{-1}) \int_I |f| d\lambda,$$

where $K = \frac{\max_i |\sigma_i^{'}|}{\min_i \sigma_i}$ *and* $h = \min_i (b_{i-1} - b_i)$*.*

Proof. See Appendix C. □

3.4 Piecewise expanding $C^{1,1}$ transformations

In this section we state some results from a recent article by P. Eslami and P. Góra [EG13], in which they prove the Lasota-Yorke inequality for piecewise expanding $C^{1,1}$ transformations with a smaller constant than the previously known $2s^{-N}$ from Proposition 3.3.2.

Let $T \in \mathcal{T}(I)$ and define

$$s := \min_{1 \leq i \leq n} s_i \quad \text{and} \quad M := \max_{1 \leq i \leq n} M_i.$$

These are the values of the branch with the smallest expansion rate and the branch with the largest Lipschitz constant. Let $P = \{I_i = (x_{i-1}, x_i) : 1 \le i \le n\}$ be a partition of I such that T is piecewise expanding $C^{1,1}$. Also let

$$\delta_i^{\pm} = \delta_{\{T(x_i^{\pm}) \notin \{0,1\}\}} = \begin{cases} 0 & \text{if } \ T(x_i^{\pm}) \in \{0,1\}, \\ 1 & \text{if } \ T(x_i^{\pm}) \notin \{0,1\}, \end{cases}$$

where $T(x_i^{\pm})$ means $\lim_{x \to x_i^{\pm}} T(x)$. For example, $\delta_i^{+} = 1$ means that the left endpoint of the $(i+1)$-st branch of T is hanging (it does not touch 0 or 1). Furthermore, let

$$\eta_i := \begin{cases} \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} & \text{if } \ i = 1, \\ \max\left\{\frac{\delta_{q-1}^-}{s_{q-1}}, \frac{\delta_q^-}{s_q}\right\} & \text{if } \ i = n, \\ \max\left\{\frac{\delta_{i-1}^-}{s_{i-1}}, \frac{\delta_i^+}{s_{i+1}}\right\} & \text{if } \ i = 2, \dots, n-1. \end{cases}$$

The following result is a Lasota-Yorke type inequality with a smaller constant than the classical Lasota-Yorke inequality for the class of piecewise expanding $C^{1,1}$ transformations.

Proposition 3.4.1. *Suppose $T \in \mathcal{T}(I)$. Then, for every $f \in BV(I)$,*

$$\operatorname*{Var}_I P_T f \le \max_{1 \le i \le n}\left\{\frac{1}{s_i} + \eta_i\right\} \operatorname*{Var}_I f + \left[\frac{M}{s^2} + \frac{2\max_{1 \le i \le n} \eta_i}{\min_{1 \le i \le n} \lambda(I_i)}\right] \int_I |f| d\lambda. \tag{3.6}$$

Proof. See [EG13, Proposition 3.1]. $\square$

If the Lasota-Yorke inequality in Proposition 3.4.1 holds with coefficient of $\operatorname{Var}_I f$ less than 1, then we have the existence of acims for piecewise expanding $C^{1,1}$ maps.

Theorem 3.4.2. *If a map $T \in \mathcal{T}(I)$ satisfies inequality (3.6) with the coefficient*

$$\max_{1 \le i \le n}\left\{\frac{1}{s_i} + \eta_i\right\} \le \gamma < 1, \tag{3.7}$$

for some $\gamma > 0$, then for any $f \in BV(I)$ and $n \in \mathbb{N}$,

$$||P_T^n f||_{BV} \le \gamma^n ||f||_{BV} + \left(1 + \frac{K + 2h^{-1}}{1 - \gamma}\right) ||f||_1,$$

where $K := M/s^2$ and $h := \min_{1 \le i \le n} \lambda(I_i)$. Furthermore, T admits an acim with a density of bounded variation.

Proof. See [EG13, Theorem 4.1]. $\square$

The following proposition gives a condition such that the Lasota-Yorke inequality in Proposition 3.4.1 holds with coefficient of $\operatorname{Var}_I f$ less than 1.

Proposition 3.4.3. *Suppose $T \in \mathcal{T}(I)$ satisfies the following condition:*

$$\frac{1}{s_i} + \frac{1}{s_{i+1}} \le \gamma < 1, \ \textit{for } i = 1, \dots, q-1.$$

Then (3.7) holds for T or for an extension $(\hat{T}, \hat{I})$ of (T, I) that contains (T, I) as an attractor.

Proof. See [EG13, Theorem 3.2]. $\square$

3.5 The stability of absolutely continuous invariant measures

In this section we explain what it means for a map to be acim-stable and give conditions that ensure acim-stability. In general the setting is as follows. Let T_0 be a piecewise expanding map with unique acim μ_0 and $\{T_\epsilon\}_{\epsilon>0}$ a family of perturbations with acims μ_ϵ. We wonder if the acims μ_ϵ converge to μ_0 when the maps T_ϵ converge to T_0. We formalize this idea and explain what kind of convergence we use for the maps and the measures.

Definition 3.5.1. Given a family of maps $\{T_\epsilon : X \to X\}_{\epsilon\geq 0}$ with corresponding invariant densities $\{f_\epsilon\}_{\epsilon\geq 0}$, we say that T_0 is *acim-stable* if $\lim_{\epsilon\to 0} T_\epsilon = T_0$ implies $\lim_{\epsilon\to 0} f_\epsilon = f_0$. The limits are taken with respect to properly chosen metrics on the spaces of maps and densities, respectively.

We need a notion for the distance between two maps. The Skorokhod metric will be used as a measure of closeness for maps.

Definition 3.5.2. The *Skorokhod distance* $d_S(T_\epsilon, T_0)$ between two maps is the infimum of all positive δ such that there exists a subset $A_\delta \subseteq I$ with $m(A_\delta) > 1-\delta$ and a diffeomorphism $\sigma : I \to I$ such that

$$T_{\epsilon|_{A_\delta}} = T_0 \circ \sigma_{|_{A_\delta}}, |\sigma(x) - x| < \delta, \text{ and } \left| \frac{1}{\sigma'(x)} - 1 \right| < \delta,$$

for all $x \in A_\delta$.

We now consider two situations where the Skorokhod distance goes to zero as ϵ approaches zero. In those situations convergence on the space of continuous and differentiable functions is required.

Definition 3.5.3. Let X be a compact metric space and $k \geq 1$. $C^k(X)$ denotes the space of all k-times continuously differentiable real functions $f : X \to \mathbb{R}$ with the norm

$$||f||_{C^k} = \max_{0\leq i\leq k} \sup_{x\in X} |f^{(i)}(x)|,$$

where $f^{(i)}(x)$ is the i-th derivative of $f(x)$ and $f^{(0)}(x) = f(x)$.

The two situations were the Skorokhod distance goes to zero are mentioned in [EG13] as Example 5.1 and Example 5.2. We formulate them here as two propositions.

Proposition 3.5.4. *Assume that $T_0 \in \mathcal{T}(I)$ satisfies condition (3.7). Assume that $\{T_\epsilon\}_{\epsilon>0}$ is defined on the same partition $\mathcal{P} = \{I_1, I_2, \ldots, I_q\}$ as T_0, and $T_\epsilon \to T_0$ as $\epsilon \to 0$ in $C^1(int(I_i))$ for all $i = 1, 2, \ldots, q$. Then, $d_S(T_\epsilon, T_0) \to 0$ as $\epsilon \to 0$, $\{T_\epsilon\} \subset \mathcal{T}(I)$ uniformly for all $\epsilon \geq 0$.*

Proposition 3.5.5. *Assume that $T_0 \in \mathcal{T}(I)$ satisfies condition (3.7). Assume that T_ϵ is piecewise expanding on the partition $P_\epsilon = \{I_1^{(\epsilon)}, I_2^{(\epsilon)}, \ldots, I_q^{(\epsilon)}\}$, $I_i^{(\epsilon)} = (x_{i-1}^{(\epsilon)}, x_i^{(\epsilon)})$, such that $x_i^{(\epsilon)} \to x_i^0$ as $\epsilon \to 0$ (in particular T_ϵ has the same number of monotonic branches as T_0). Additionally, assume that there exists $\epsilon_1 > 0$ such that for every $0 < \epsilon_0 < \epsilon_1$, $T_\epsilon \to T_0$ in C^1 on the set*

$$\bigcup_{i=1,\ldots,q} \left[\max\left\{x_{i-1}^{(0)}, x_{i-1}^{(\epsilon_0)}\right\}, \min\left\{x_i^{(0)}, x_i^{(\epsilon_0)}\right\} \right].$$

and that $\{T_\epsilon\} \subset \mathcal{T}(I)$ uniformly for all $\epsilon \geq 0$. Then, $d_S(T_\epsilon, T_0) \to 0$ as $\epsilon \to 0$.

In [Kel82] and [KL99] it is shown that we have acim-stability if the family of perturbations $\{T_\epsilon\}_{\epsilon>0}$ satisfies a Lasota-Yorke inequality with uniform constants. The usual conditions that ensure this are $\inf |T_\epsilon^{'}| > 2$ for all $\epsilon > 0$ and that the minimal lengths of the intervals in the partition are uniformly bounded away from zero. If $1 < |T_0^{'}| < 2$, then the usual method is to work with an iterate of T_0 for which the derivative is larger than 2. This method can not be used if the map has a turning fixed or periodic point touching a branch with slope 2 or smaller. The existence of such a point causes the appearance of arbitrary short partition intervals for the iterates of the perturbed maps. The stronger Lasota-Yorke inequality bypasses this problem. If T satisfies the conditions of Theorem 3.4.2 and the Skorokhod distance $d_S(T_0, T_\epsilon) \to 0$ as $\epsilon \to 0$, then the following theorem shows that the map T_0 is acim-stable.

Theorem 3.5.6. *Consider the one-parameter family of maps $\{T_\varepsilon\}_{\varepsilon\geq 0}$ where $\{T_\varepsilon\}_{\varepsilon\geq 0} \subset \mathcal{T}(I)$ uniformly. Suppose there exists $0 < \gamma < 1$ such that*

$$\max_{1\leq i\leq q} \left\{ \frac{1}{s_i} + \eta_i \right\} \leq \gamma < 1, \tag{3.8}$$

Let f_ε be a T_ε-invariant density. If $d_S(T_\epsilon, T_0) \to 0$ as $\epsilon \to 0$, then the following statements hold:

(1) The family $\{f_\epsilon\}_{\epsilon>0}$ is relatively compact in L^1, and any of its limits functions is a T_0-invariant density.

(2) If T_0 is ergodic, then T_ϵ is ergodic for small ϵ and $f_\epsilon \to f_0$ in L^1 as $\epsilon \to 0$ (i.e. T_0 is acim-stable).

(3) If T_0 is weakly mixing, then the eigenvalue gaps of $\{P_{T_\epsilon}\}_\epsilon$, for ϵ small enough, are uniformly bounded, i.e. $0 < \phi < 1 - |\lambda_2^\epsilon|$. As a consequence, there exists a constant $C > 0$ such that for all ϵ small enough and all densities $f \in BV$,

$$||P_{T_\epsilon}^n f - f_\epsilon||_{L^1} \leq C(1-\phi)^n ||f||_{BV}.$$

We have listed the theorem here as mentioned in [EG13, Theorem 4.5], but in fact in the next chapter we will only use statement (2).

Chapter 4

Skew tent maps

The skew tent map is a two-parameter, piecewise linear map. In this section we show that expanding skew tent maps have a unique acim and are exact on a part of the parameters region. We fix one parameter and show that under certain conditions the skew tent map is acim-stable. This result is then used to show that if we fix one parameter and change the other parameter continuously, then there is a part of the parameters region where the measure-theoretic entropy changes continuously.

4.1 Skew tent maps and their properties

A general skew tent map can be defined by taking two straight lines in the plane, one with positive slope α and one with negative slope $-\beta$. The two lines will eventually intersect at some point (x_0, y_0). For $x \leq x_0$ we take the line with positive slope and for $x \geq x_0$ we take the line with negative slope. We can translate the map such that the intersection point is at $(x_0, y_0) = (0, 1)$. This gives a function $F_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ defined by

$$F_{\alpha,\beta}(x) = \begin{cases} 1 + \alpha x & \text{for} \quad x \leq 0, \\ 1 - \beta x & \text{for} \quad x \geq 0. \end{cases}$$

We want the skew map to be an expanding interval map. If we assume that $\alpha, \beta > 1$ and $\alpha + \beta \geq \alpha\beta$, then the map $F_{\alpha,\beta} : [1-\beta, 1] \to [1-\beta, 1]$ is an expanding interval map. It is convenient to study interval maps on the unit interval, which can be done by translating and scaling the map $F_{\alpha,\beta}$. This gives a map $T_{\alpha,\beta} : [0,1] \to [0,1]$ defined by

$$T_{\alpha,\beta}(x) = \begin{cases} \alpha x + \frac{\alpha+\beta-\alpha\beta}{\beta} & \text{for} \quad x \in \left[0, \frac{\beta-1}{\beta}\right], \\ \beta - \beta x & \text{for} \quad x \in \left(\frac{\beta-1}{\beta}, 1\right], \end{cases}$$

with $\alpha, \beta > 1$ and $\alpha + \beta \geq \alpha\beta$. In Figure 4.4 we see some examples of skew tent maps. We now turn to proving the existence of a unique acim and exactness.

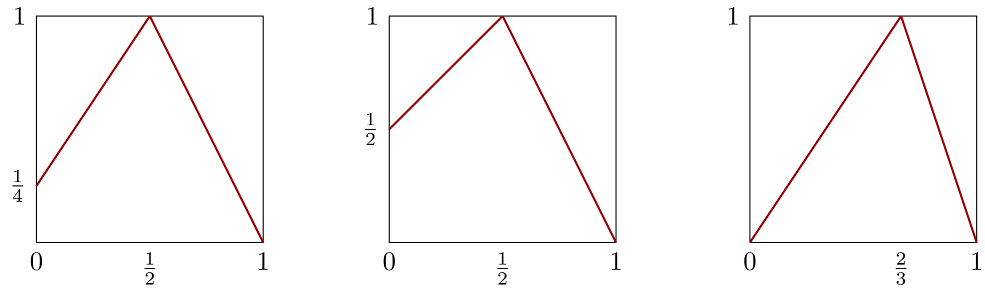

Figure 4.1: Skew tent maps on the unit interval.

Proposition 4.1.1. *The map $T_{\alpha,\beta}$ has a unique acim. Furthermore, $T_{\alpha,\beta}$ is ergodic.*

Proof. Let $b_1 = 0$, $b_2 = (\beta - 1)/\beta$ and $b_3 = 1$. The set $P = \{I_i = (b_{i-1}, b_i) : i = 1, 2\}$ is a partition of I. Let T_i be the restriction of $T_{\alpha,\beta}$ to the interval I_i. The derivative of $T_{\alpha,\beta}$ is given by

$$T'_{\alpha,\beta}(x) = \begin{cases} \alpha & \text{for} \quad x \in [0, \frac{\beta-1}{\beta}], \\ -\beta & \text{for} \quad x \in (1 - \frac{\beta-1}{\beta}, 1]. \end{cases}$$

The skew tent map has the following properties:

1. T_i is linear, so it is monotonic, C^2, and it is clear that it can be extended to the closed interval $[b_{i-1}, b_i]$ as a C^2 function;
2. $|T'_i(x)| \geq \min\{\alpha, \beta\} > 1$ for any i and for all $x \in I_i$.
3. T_i is linear, so $|T'_i(x) - T'_i(y)| = 0$ for all $x, y \in I_i$.

This shows that $T_{\alpha,\beta}$ is a piecewise linear expanding $C^{1,1}/C^2$ map. It is known that linear maps scale the Lebesgue measure, therefore $T_{\alpha,\beta}$ is also non-singular. It follows from Theorem 3.3.1 that $T_{\alpha,\beta}$ has an acim. Moreover, combining this with Theorem 3.2.4 yields that $T_{\alpha,\beta}$ has a unique acim. It follows from the uniqueness of the acim that $T_{\alpha,\beta}$ is ergodic, see [Daj14, Theorem 6.1.6]. □

Recall from Definition 2.1.7 that a map T is exact if and only if for each measurable set A with positive measure such that also all the images $T^n A$ are measurable we have $\lim_{n\to\infty} \mu(T^n A) = 1$. In general it can be difficult to show exactness. However, for the skew tent map there are cases where the map has some nice structure which makes it easier to show that a map is exact or not exact.

Proposition 4.1.2. *If $\alpha, \beta > \sqrt{2}$, then the dynamical system $(T_{\alpha,\beta}, \mu_{\alpha,\beta})$ is exact.*

Proof. The map $T_{\alpha,\beta}$ satisfies the conditions of Proposition 3.2.6. Hence, $(T_{\alpha,\beta}, \mu_{\alpha,\beta})$ is exact. □

Proposition 4.1.3. *If $\alpha\beta - \alpha = 1$, then the dynamical system $(T_{\alpha,\beta}, \mu_{\alpha,\beta})$ is exact.*

Proof. Let $\alpha\beta - \alpha = 1$. We show that the conditions of Theorem 2.1.7 hold. Let $\epsilon > 0$ be arbitrary and take an interval around the turning point, say $J = [\frac{\beta-1}{\beta} - \epsilon, \frac{\beta-1}{\beta} + \epsilon]$. We note that for $\alpha\beta - \alpha = 1$ we have $T_{\alpha,\beta}(0) = \frac{\beta-1}{\beta}$, so that under the map $T_{\alpha,\beta}$ we have

$$\begin{aligned}
&T_{\alpha,\beta}J \supseteq [1-\alpha\epsilon, 1]\\
&T^2_{\alpha,\beta}J \supseteq [0, \alpha\beta\epsilon]\\
&T^3_{\alpha,\beta}J \supseteq \left[\tfrac{\beta-1}{\beta}, \tfrac{\beta-1}{\beta} + \alpha^2\beta\epsilon\right].
\end{aligned}$$

Let $n \geq 0$, then in general we have the following

$$\begin{aligned}
&T^{3n+1}_{\alpha,\beta}J \supseteq [1-\alpha^{n+1}\beta^{2n}\epsilon, 1]\\
&T^{3n+2}_{\alpha,\beta}J \supseteq [0, \alpha^{n+1}\beta^{2n+1}\epsilon]\\
&T^{3n+3}_{\alpha,\beta}J \supseteq \left[\tfrac{\beta-1}{\beta}, \tfrac{\beta-1}{\beta} + \alpha^{n+2}\beta^{2n+1}\epsilon\right].
\end{aligned}$$

For each ϵ we can find $m \geq 0$ such that $m = \min\{n \geq 0 : \alpha^{n+1}\beta^{2n+1}\epsilon \geq \frac{\beta-1}{\beta}\}$, because $\alpha, \beta > 1$. It follows that $T^{3m+3}_{\alpha,\beta}J \supseteq \left[\frac{\beta-1}{\beta}, 1\right]$ and $\mu(T^{3m+4}_{\alpha,\beta}J) = \mu(I) = 1$. □

Proposition 4.1.4. *If $\alpha + \beta \geq \alpha\beta^2$, then the dynamical system $(T_{\alpha,\beta}, \mu_{\alpha,\beta})$ is not exact.*

Proof. Let $\alpha + \beta \geq \alpha\beta^2$. This condition implies that

$$T_{\alpha,\beta}(0) = \frac{\alpha+\beta-\alpha\beta}{\beta} \geq \frac{\beta}{\beta+1},$$

where $\frac{\beta}{\beta+1}$ is the fixed point of the map $T_{\alpha,\beta}$. It also implies that

$$T^2_{\alpha,\beta}(0) = \beta - (\alpha+\beta-\alpha\beta) = \alpha\beta - \alpha > \frac{\beta-1}{\beta}.$$

To see that this condition holds we note that condition above is equivalent to

$$\alpha\beta^2 - (\alpha+1)\beta + 1 > 0,$$

which holds for

$$\beta > \frac{\alpha+1-\sqrt{(\alpha+1)^2-4\alpha}}{2\alpha} = 1 \quad \text{and} \quad \beta < \frac{\alpha+1+\sqrt{(\alpha+1)^2-4\alpha}}{2\alpha} = \frac{1}{\alpha}.$$

We have that $\beta > 1$ is the only valid condition and it always holds. Furthermore,

$$T^3_{\alpha,\beta}(0) = \beta - \beta(\alpha\beta-\alpha) = \beta - \alpha\beta^2 + \alpha\beta > T_{\alpha,\beta}(0).$$

We now have

$$T_{\alpha,\beta}([0, T^2_{\alpha,\beta}(0)]) = [T_{\alpha,\beta}(0), 1] \quad \text{and} \quad T_{\alpha,\beta}([T_{\alpha,\beta}(0), 1]) = [0, T^2_{\alpha,\beta}(0)]$$

This implies that

$$\lim_{n\to\infty} \mu(T^n_{\alpha,\beta}([0, T^2_{\alpha,\beta}(0)])) \neq 1.$$

Hence $T_{\alpha,\beta}$ is not exact. □

4.2 The iterates of the skew tent map

In the following section we use a method that looks at the third iterate of the skew tent map. Therefore we now derive expressions for the second and third iterate. We can distinguish two cases. In the first case $T_{\alpha,\beta}(0) \geq \frac{\beta-1}{\beta} \iff \alpha(\beta-1) \leq 1$ and in the second case $T_{\alpha,\beta}(0) < \frac{\beta-1}{\beta} \iff \alpha(\beta-1) > 1$. In these cases the number of branches for the second and third iterate are different. In the Figure 4.2 we see the graph of $T_{\alpha,\beta}$ for the different cases.

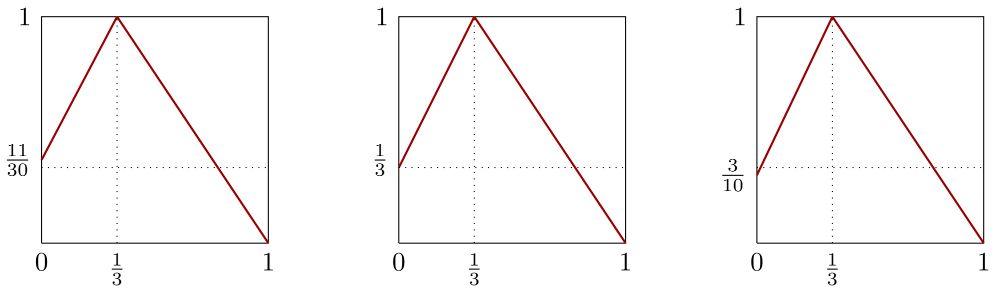

Figure 4.2: Graphs for the cases: $\alpha(\beta-1) < 1$, $\alpha(\beta-1) = 1$ and $\alpha(\beta-1) > 1$.

The map $T_{\alpha,\beta}$ is piecewise linear, this means $T^2_{\alpha,\beta}$ and $T^3_{\alpha,\beta}$ are also piecewise linear maps. The number of branches in $T^n_{\alpha,\beta}$ is equal to the number of branches in $T^{n-1}_{\alpha,\beta}$ plus the number of times the line $y = \frac{\beta-1}{\beta}$ intersect the graph of $T^{n-1}_{\alpha,\beta}$. For $\alpha(\beta-1) \leq 1$ the line $y = \frac{\beta-1}{\beta}$ intersects the graphs as follows:

- The graph of $T_{\alpha,\beta}$ once in the interval $[\frac{\beta-1}{\beta}, 1]$. The second iterate is given by:

$$T^2_{\alpha,\beta}(x) = \begin{cases} -\alpha\beta x + \alpha\beta - \alpha & \text{for } x \in [0, \frac{\beta-1}{\beta}], \\ \beta^2 x - \beta^2 + \beta & \text{for } x \in [\frac{\beta-1}{\beta}, \frac{\beta^2-\beta+1}{\beta^2}], \\ -\alpha\beta x + \frac{\alpha(\beta^2-\beta+1)+\beta}{\beta} & \text{for } x \in [\frac{\beta^2-\beta+1}{\beta^2}, 1], \end{cases}$$

- The graph of $T^2_{\alpha,\beta}$ once in each of the intervals $[0, \frac{\beta-1}{\beta}]$ and $[\frac{\beta-1}{\beta}, \frac{\beta^2-\beta+1}{\beta^2}]$. The third iterate is given by:

$$T^3_{\alpha,\beta}(x) = \begin{cases} \alpha\beta^2 x - \alpha\beta^2 + \alpha\beta + \beta & \text{for } x \in [0, \frac{(\alpha\beta-1)(\beta-1)}{\alpha\beta^2}], \\ -\alpha^2\beta x + \frac{\alpha(\alpha\beta-1)(\beta-1)+\beta}{\beta} & \text{for } x \in [\frac{(\alpha\beta-1)(\beta-1)}{\alpha\beta^2}, \frac{\beta-1}{\beta}], \\ \alpha\beta^2 x - \frac{\alpha(\beta-1)(\beta^2+1)-\beta}{\beta} & \text{for } x \in [\frac{\beta-1}{\beta}, \frac{(\beta-1)(\beta^2+1)}{\beta^3}], \\ -\beta^3 x + \beta^3 - \beta^2 + \beta & \text{for } x \in [\frac{(\beta-1)(\beta^2+1)}{\beta^3}, \frac{\beta^2-\beta+1}{\beta^2}], \\ \alpha\beta^2 x - \alpha\beta^2 + \alpha\beta - \alpha & \text{for } x \in [\frac{\beta^2-\beta+1}{\beta^2}, 1]. \end{cases}$$

In Figure 4.3 the second and third iterate are drawn for the case $\alpha(\beta-1) < 1$.

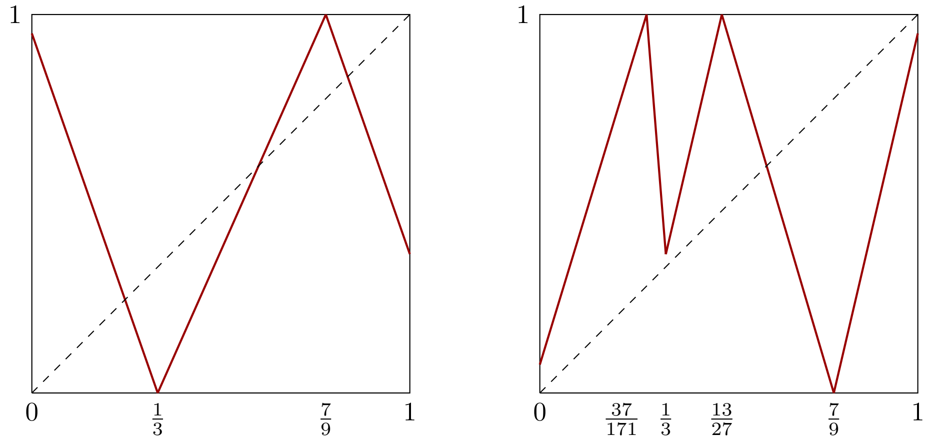

Figure 4.3: The graphs of $T^2_{\alpha,\beta}$ and $T^3_{\alpha,\beta}$ for $\alpha = 1.9$ and $\beta = 1.5$.

For $\alpha(\beta-1) > 1$ the line $y = \frac{\beta-1}{\beta}$ intersects the graphs as follows:

- The graph of $T_{\alpha,\beta}$ once in each of the intervals $[0, \frac{\beta-1}{\beta}]$ and $[\frac{\beta-1}{\beta}, 1]$. The second iterate is given by:

$$T^2_{\alpha,\beta}(x) = \begin{cases} \alpha^2 x + \frac{(\alpha+1)(\alpha+\beta-\alpha\beta)}{\beta} & \text{for } x \in [0, \frac{\alpha\beta-\alpha-1}{\alpha\beta}], \\ -\alpha\beta x + \alpha\beta - \alpha & \text{for } x \in [\frac{\alpha\beta-\alpha-1}{\alpha\beta}, \frac{\beta-1}{\beta}], \\ \beta^2 x - \beta^2 + \beta & \text{for } x \in [\frac{\beta-1}{\beta}, \frac{\beta^2-\beta+1}{\beta^2}], \\ -\alpha\beta x + \frac{\alpha(\beta^2-\beta+1)+\beta}{\beta} & \text{for } x \in [\frac{\beta^2-\beta+1}{\beta^2}, 1]. \end{cases}$$

- The graph of $T^2_{\alpha,\beta}$ once in each of the intervals $[\frac{\alpha\beta-\alpha-1}{\alpha\beta}, \frac{\beta-1}{\beta}]$, $[\frac{\beta-1}{\beta}, \frac{\beta^2-\beta+1}{\beta^2}]$ and $[\frac{\beta^2-\beta+1}{\beta^2}, 1]$. The third iterate is given by:

$$T^3_{\alpha,\beta}(x) = \begin{cases} -\alpha^2\beta x + \alpha^2\beta - \alpha^2 - \alpha & \text{for } x \in [0, \frac{\alpha\beta-\alpha-1}{\alpha\beta}], \\ \alpha\beta^2 x - \alpha\beta^2 + \alpha\beta + \beta & \text{for } x \in [\frac{\alpha\beta-\alpha-1}{\alpha\beta}, \frac{(\alpha\beta-1)(\beta-1)}{\alpha\beta^2}], \\ -\alpha^2\beta x + \frac{\alpha(\alpha\beta-1)(\beta-1)+\beta}{\beta} & \text{for } x \in [\frac{(\alpha\beta-1)(\beta-1)}{\alpha\beta^2}, \frac{\beta-1}{\beta}], \\ \alpha\beta^2 x - \frac{\alpha(\beta-1)(\beta^2+1)-\beta}{\beta} & \text{for } x \in [\frac{\beta-1}{\beta}, \frac{(\beta-1)(\beta^2+1)}{\beta^3}], \\ -\beta^3 x + \beta^3 - \beta^2 + \beta & \text{for } x \in [\frac{(\beta-1)(\beta^2+1)}{\beta^3}, \frac{\beta^2-\beta+1}{\beta^2}], \\ \alpha\beta^2 x - \alpha\beta^2 + \alpha\beta - \alpha & \text{for } x \in [\frac{\beta^2-\beta+1}{\beta^2}, \frac{\alpha\beta^2-\alpha\beta+\alpha+1}{\alpha\beta^2}], \\ -\alpha^2\beta x + \frac{\alpha(\alpha\beta^2-\alpha\beta+\alpha+1)+\beta}{\beta} & \text{for } x \in [\frac{\alpha\beta^2-\alpha\beta+\alpha+1}{\alpha\beta^2}, 1]. \end{cases}$$

In Figure 4.4 the second and third iterate are drawn for the case $\alpha(\beta-1) > 1$.

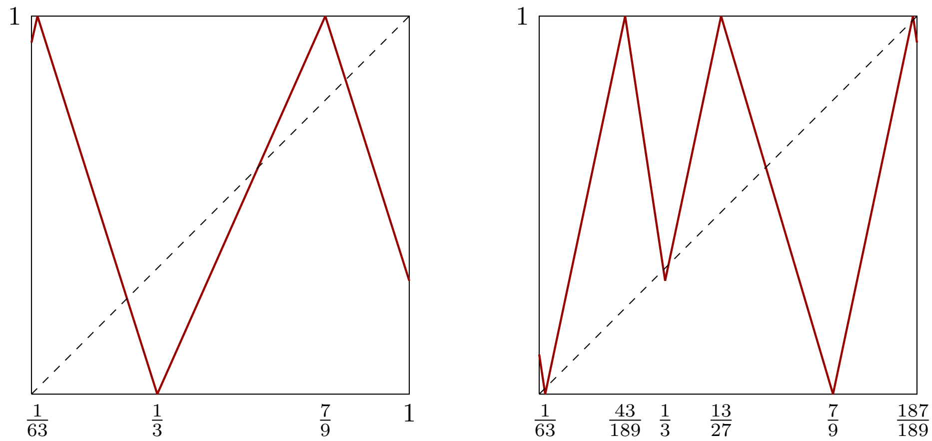

Figure 4.4: The graphs of $T^2_{\alpha,\beta}$ and $T^3_{\alpha,\beta}$ for $\alpha = 2.1$ and $\beta = 1.5$.

4.3 The acim-stability for skew tent maps

In this section we examine the stability of the skew tent maps for fixed β. To indicate that β is fixed we write T_α instead of $T_{\alpha,\beta}$. We show that T_α is acim-stable for $\alpha > 2$. Furthermore, by imposing conditions on the slopes and the unique acims around T_α we are able to show that T_α is acim-stable for some $1 < \alpha < 2$. The proofs in this section are based on the results discussed in Section 3.5.

Theorem 4.3.1. *The map T_α is acim-stable for all $\alpha > 2$.*

Proof. For each $\alpha > 2$ we can find a family of maps $\{T_{\alpha\pm\epsilon}\}_{\epsilon<\epsilon_1}$ around T_α with the property that $\alpha - \epsilon_1 > 2$. The map $T_{\alpha\pm\epsilon}$ has slopes

$$\alpha \pm \epsilon > 2 \quad \text{and} \quad \beta > 1.$$

The condition $\alpha - \epsilon_1 > 2$ guarantees that $\{T_{\alpha\pm\epsilon}\} \subset \mathcal{T}(I)$ uniformly with uniform constants

$$s_1 = \alpha - \epsilon_1 > 2, \quad \beta = s_2 > 1 \quad \text{and} \quad M_1 = M_2 = 0.$$

For the endpoints of the branches we have $\delta_0^+ = 1$, $\delta_1^\pm = \delta_2^- = 0$. It follows that

$$\eta_1 = \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} = \max\left\{\frac{1}{s_1}, \frac{0}{s_2}\right\} = \frac{1}{s_1},$$
$$\eta_2 = \max\left\{\frac{\delta_1^-}{s_1}, \frac{\delta_2^-}{s_2}\right\} = \max\left\{\frac{0}{s_1}, \frac{0}{s_2}\right\} = 0.$$

Since $s_1 > 2$ and $s_2 > 1$ there exists $0 < \gamma < 1$ such that

$$\max_{1\le i\le 2}\left\{\frac{1}{s_i} + \eta_i\right\} = \max\left\{\frac{1}{s_1} + \frac{1}{s_1}, \frac{1}{s_2} + 0\right\} = \max\left\{\frac{2}{s_1}, \frac{1}{s_2}\right\} \le \gamma < 1$$

holds uniformly for the family of maps $\{T_{\alpha\pm\epsilon}\}$. The family of maps $\{T_{\alpha\pm\epsilon}\}$ are all defined on the same partition $P = \{I_1, I_2\}$. This means we can use Proposition 3.5.4 to show that $d_S(T_{\alpha\pm\epsilon}, T_\alpha) \to 0$ as $\epsilon \to 0$. The distance between $T_{\alpha\pm\epsilon}$ and T_α in $\mathcal{C}^1(\mathrm{Int}(I_1))$ is

$$\begin{aligned}
||T_{\alpha\pm\epsilon} - T_\alpha||_{\mathcal{C}^1(\mathrm{Int}(I_1))} &= \max_{0\le k\le 1} \sup_{x\in\mathrm{Int}(I_1)} |T^{(k)}_{\alpha\pm\epsilon}(x) - T^{(k)}_\alpha(x)| \\
&= \max\left\{ \sup_{x\in\mathrm{Int}(I_1)} |T_{\alpha\pm\epsilon}(x) - T_\alpha(x)|, \sup_{x\in\mathrm{Int}(I_1)} |T^{'}_{\alpha\pm\epsilon}(x) - T^{'}_\alpha(x)| \right\} \\
&= \max\left\{ \sup_{x\in\mathrm{Int}(I_1)} |\epsilon\Big(x - \tfrac{\beta-1}{\beta}\Big)|, \sup_{x\in\mathrm{Int}(I_1)} |\epsilon| \right\} \\
&= \epsilon
\end{aligned}$$

and the distance between $T_{\alpha\pm\epsilon}$ and T_α in $\mathcal{C}^1(\mathrm{Int}(I_2))$ is zero, because $T_{\alpha\pm\epsilon} = T_\alpha$ on I_2. It is clear that $T_{\alpha\pm\epsilon} \to T_\alpha$ as $\epsilon \to 0$ in $\mathcal{C}^1(\mathrm{Int}(I_i))$ for all $i = 1, 2$. It is known that each map $T_{\alpha\pm\epsilon}$ is ergodic. We have shown that the family of maps $\{T_{\alpha\pm\epsilon}\}$ satisfies the conditions of Theorem 3.5.6 and therefore T_α is acim-stable. $\square$

For $1 < \alpha < 2$ we can not apply the results of Theorem 3.5.6 to a family of maps $\{T_{\alpha\pm\epsilon}\}$ around T_α, because $\{T_{\alpha\pm\epsilon}\}$ has uniform constant $s_1 < 2$ and therefore

$$\max_{1\le i\le 2}\left\{\frac{1}{s_i} + \eta_i\right\} = \max\left\{\frac{2}{s_1}, \frac{1}{s_2}\right\} > 1.$$

However, if we can show that the conditions of Theorem 3.5.6 hold uniformly for a family of maps $\{T^n_{\alpha\pm\epsilon}\}$ with $n \ge 2$ and where $T^n_{\alpha\pm\epsilon}$ has the same unique absolutely continuous measure as $T_{\alpha\pm\epsilon}$, then it follows that $T_{\alpha\pm\epsilon}$ is acim-stable. In order to use Theorem 3.5.6 we need that maps around T^n_α have the same number of monotonic branches. A condition that ensures that $T^n_{\alpha\pm\epsilon}$ has the same unique absolutely continuous invariant measure as $T_{\alpha\pm\epsilon}$ is exactness.

We show that for $1 < \alpha < 2$ and $\alpha(\beta - 1) \ne 1$ there exists a family of maps $\{T^3_{\alpha\pm\epsilon}\}$ that satisfies the conditions of Theorem 3.5.6 under the assumptions that each $T_{\alpha\pm\epsilon}$ is exact and $(\alpha\pm\epsilon)^2\beta, (\alpha\pm\epsilon)\beta^2 > 2$. The condition $\alpha(\beta - 1) \ne 1$ ensures that maps around T_α have the same number of monotonic branches. For the proof we make a distinction between the cases where T^3_α has five branches and seven branches.

Theorem 4.3.2. *Let $1 < \alpha < 2$ and $\beta > 1$ be given such that $\alpha(\beta - 1) < 1$. Let $\epsilon_1 > 0$ be such that $(\alpha + \epsilon_1)(\beta - 1) < 1$, $(\alpha - \epsilon_1)^2\beta > 2$ and $(\alpha - \epsilon_1)\beta^2 > 2$. If $\{T_{\alpha\pm\epsilon}\}$ is exact, then T_α is acim-stable.*

Proof. Note that $\alpha(\beta - 1) < 1$, so that T^3_α has five branches. For each $0 < \epsilon < \epsilon_1$ the map $T^3_{\alpha\pm\epsilon}$ has slopes

$$(\alpha\pm\epsilon)\beta^2 \ge (\alpha - \epsilon_1)\beta^2 := s_1 = s_3 = s_5, \quad (\alpha\pm\epsilon)^2\beta \ge (\alpha-\epsilon_1)^2\beta =: s_2 \quad \text{and} \quad \beta^3 := s4.$$

The conditions on ϵ_1 now guarantee that $\{T^3_{\alpha\pm\epsilon}\} \subset \mathcal{T}(I)$ uniformly with uniform constants

$$s_1, s_2, s_3, s_5 > 2, \quad s_4 = \beta^3 \quad \text{and} \quad M_1 = M_2 = M_3 = M_4 = M_5 = 0.$$

For the endpoints of the branches we have $\delta_0^+ = \delta_2^\pm = \delta_5^- = 1$ and $\delta_1^\pm = \delta_3^\pm = \delta_4^\pm = 0$. It follows that

$$\begin{aligned}
\eta_1 &= \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} = \max\left\{\frac{1}{s_1}, \frac{0}{s_2}\right\} = \frac{1}{s_1},\\
\eta_2 &= \max\left\{\frac{\delta_1^-}{s_1}, \frac{\delta_2^+}{s_3}\right\} = \max\left\{\frac{0}{s_1}, \frac{1}{s_3}\right\} = \frac{1}{s_3},\\
\eta_3 &= \max\left\{\frac{\delta_2^-}{s_2}, \frac{\delta_3^+}{s_4}\right\} = \max\left\{\frac{1}{s_2}, \frac{0}{s_4}\right\} = \frac{1}{s_2},\\
\eta_4 &= \max\left\{\frac{\delta_3^-}{s_3}, \frac{\delta_4^+}{s_5}\right\} = \max\left\{\frac{0}{s_3}, \frac{0}{s_5}\right\} = 0,\\
\eta_5 &= \max\left\{\frac{\delta_4^-}{s_4}, \frac{\delta_5^-}{s_5}\right\} = \max\left\{\frac{0}{s_4}, \frac{1}{s_5}\right\} = \frac{1}{s_5}.
\end{aligned}$$

Since $s_1, s_2, s_3, s_5 > 2$ and $s_4 > 1$ there exists $0 < \gamma < 1$ such that

$$\max_{1\le i\le 5}\left\{\frac{1}{s_i} + \eta_i\right\} = \max\left\{\frac{2}{s_1}, \frac{1}{s_2} + \frac{1}{s_3}, \frac{1}{s_4}, \frac{2}{s_5}\right\} \le \gamma < 1$$

holds uniformly for the family of maps $\{T^3_{\alpha\pm\epsilon}\}$. Let

$$\begin{aligned}
b_0^{(\alpha\pm\epsilon)} &= 0, & b_3^{(\alpha\pm\epsilon)} &= \frac{(\beta-1)(\beta^2+1)}{\beta^3},\\
b_1^{(\alpha\pm\epsilon)} &= \frac{((\alpha\pm\epsilon)\beta-1)(\beta-1)}{(\alpha\pm\epsilon)\beta^2}, & b_4^{(\alpha\pm\epsilon)} &= \frac{\beta^2-\beta+1}{\beta^2},\\
b_2^{(\alpha\pm\epsilon)} &= \frac{\beta-1}{\beta}, & b_5^{(\alpha\pm\epsilon)} &= 1.
\end{aligned}$$

The map $T^3_{\alpha\pm\epsilon}$ is defined on the partition

$$P_{T^3_{\alpha\pm\epsilon}} = \{I_i^{(\alpha\pm\epsilon)} = (b_{i-1}^{(\alpha\pm\epsilon)}, b_i^{(\alpha\pm\epsilon)}) : 1 \le i \le 5\}.$$

We use Proposition 3.5.5 to show that $d_S(T^3_{\alpha\pm\epsilon}, T^3_\alpha) \to 0$ as $\epsilon \to 0$, because each $T^3_{\alpha\pm\epsilon}$ is defined on a different partition. Let

$$J_i^{(\alpha\pm\epsilon)} := \Big[\max\Big\{b_{i-1}^{(\alpha)}, b_{i-1}^{(\alpha\pm\epsilon)}\Big\}, \min\Big\{b_i^{(\alpha)}, b_i^{(\alpha\pm\epsilon)}\Big\}\Big].$$

The $J_i^{(\alpha\pm\epsilon)}$ are given by

$$\begin{aligned}
J_1^{(\alpha-\epsilon)} &= \big[b_0^{(\alpha)}, b_1^{(\alpha-\epsilon)}\big], & J_3^{(\alpha\pm\epsilon)} &= \big[b_2^{(\alpha)}, b_3^{(\alpha)}\big],\\
J_1^{(\alpha+\epsilon)} &= \big[b_0^{(\alpha)}, b_1^{(\alpha)}\big], & J_4^{(\alpha\pm\epsilon)} &= \big[b_3^{(\alpha)}, b_4^{(\alpha)}\big],\\
J_2^{(\alpha-\epsilon)} &= \big[b_1^{(\alpha)}, b_2^{(\alpha)}\big], & J_5^{(\alpha\pm\epsilon)} &= \big[b_4^{(\alpha)}, b_5^{(\alpha)}\big],\\
J_2^{(\alpha+\epsilon)} &= \big[b_1^{(\alpha+\epsilon)}, b_2^{(\alpha)}\big].
\end{aligned}$$

The distance between $T^3_{\alpha\pm\epsilon}$ and T^3_α in $\mathcal{C}^1(J_i^{(\alpha\pm\epsilon)})$ is given by

$$||T^3_{\alpha\pm\epsilon} - T^3_\alpha||_{\mathcal{C}^1(J_i^{(\alpha\pm\epsilon)})} = \max_{0\le k\le 1} \sup_{x\in J_i^{(\alpha\pm\epsilon)}} |(T^3_{\alpha\pm\epsilon})^{(k)}(x) - (T^3_\alpha)^{(k)}(x)|.$$

From the expressions for $T^3_{\alpha\pm\epsilon}$ in Section 4.2 we can see that for each $\epsilon < \epsilon_1$ and each i the term

$$\max_{0\le k\le 1} \sup_{x\in J_i^{(\alpha\pm\epsilon)}} |(T^3_{\alpha\pm\epsilon})^{(k)}(x) - (T^3_\alpha)^{(k)}(x)|$$

is of the form $c_0\epsilon^2 x + c_1\epsilon x + c_2\epsilon^2 + c_3\epsilon$ with $c_0, c_1, c_2, c_3 \in \mathbb{R}$. Thus $T^3_{\alpha\pm\epsilon} \to T^3_\alpha$ as $\epsilon \to 0$ in $\mathcal{C}^1$ on the set

$$\bigcup_{i=1}^{5} J_i^{(\alpha\pm\epsilon)}.$$

This combined with the fact that $\{T^3_{\alpha\pm\epsilon}\} \subset \mathcal{T}(I)$ uniformly shows that $d_S(T^3_{\alpha\pm\epsilon}, T^3_\alpha) \to 0$ as $\epsilon \to 0$. It is known that each map $T^3_{\alpha\pm\epsilon}$ is ergodic, because $T_{\alpha\pm\epsilon}$ is ergodic. We have shown that the family of maps $\{T^3_{\alpha\pm\epsilon}\}$ satisfies the conditions of Theorem 3.5.6 and therefore T^3_α is acim-stable. Since each $T_{\alpha\pm\epsilon}$ is exact we get acim-stability for T_α. $\square$

Theorem 4.3.3. *Let $1 < \alpha < 2$ and $\beta > 1$ be given such that $\alpha(\beta - 1) > 1$. Let $\epsilon_1 > 0$ be such that $(\alpha - \epsilon_1)(\beta - 1) > 1$, $(\alpha - \epsilon_1)^2\beta > 2$ and $(\alpha - \epsilon_1)\beta^2 > 2$. If $\{T_{\alpha\pm\epsilon}\}$ is exact, then T_α is acim-stable.*

Proof. Note that $\alpha(\beta - 1) > 1$, so that T^3_α has seven branches. For each $0 < \epsilon < \epsilon_1$ the map $T^3_{\alpha\pm\epsilon}$ has slopes

$$(\alpha\pm\epsilon)\beta^2 \ge (\alpha-\epsilon_1)\beta^2 := s_1 = s_3 = s_7, \quad (\alpha\pm\epsilon)\beta^2 \ge (\alpha-\epsilon_1)\beta^2 := s_2 = s_4 = s_6 \quad \text{and} \quad \beta^3 := s_5.$$

The conditions on ϵ_1 now guaranty that $\{T^3_{\alpha\pm\epsilon}\} \subset \mathcal{T}(I)$ uniformly with uniform constants

$$s_1, s_2, s_3, s_4, s_6, s_7 > 2, \quad s_5 = \beta^3 \quad \text{and} \quad M_1 = M_2 = M_3 = M_4 = M_5 = M_6 = M_7 = 0.$$

For the endpoints of the branches we have $\delta_0^+ = \delta_3^\pm = \delta_7^- = 1$ and $\delta_1^\pm = \delta_2^\pm = \delta_4^\pm = \delta_5^\pm = \delta_6^\pm = 0$. It follows that

$$\begin{aligned}
\eta_1 &= \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} = \max\left\{\frac{1}{s_1}, \frac{0}{s_2}\right\} = \frac{1}{s_1},\\
\eta_2 &= \max\left\{\frac{\delta_1^-}{s_1}, \frac{\delta_2^+}{s_3}\right\} = \max\left\{\frac{0}{s_1}, \frac{0}{s_3}\right\} = 0,\\
\eta_3 &= \max\left\{\frac{\delta_2^-}{s_2}, \frac{\delta_3^+}{s_4}\right\} = \max\left\{\frac{0}{s_2}, \frac{1}{s_4}\right\} = \frac{1}{s_4},\\
\eta_4 &= \max\left\{\frac{\delta_3^-}{s_3}, \frac{\delta_4^+}{s_5}\right\} = \max\left\{\frac{1}{s_3}, \frac{0}{s_5}\right\} = \frac{1}{s_3},\\
\eta_5 &= \max\left\{\frac{\delta_4^-}{s_4}, \frac{\delta_5^+}{s_6}\right\} = \max\left\{\frac{0}{s_4}, \frac{0}{s_6}\right\} = 0,\\
\eta_6 &= \max\left\{\frac{\delta_5^-}{s_5}, \frac{\delta_6^+}{s_7}\right\} = \max\left\{\frac{0}{s_5}, \frac{0}{s_7}\right\} = 0,\\
\eta_7 &= \max\left\{\frac{\delta_6^-}{s_6}, \frac{\delta_7^-}{s_7}\right\} = \max\left\{\frac{0}{s_6}, \frac{1}{s_7}\right\} = \frac{1}{s_7}.
\end{aligned}$$

Since $s_1, s_2, s_3, s_4, s_6, s_7 > 2$ and $s_5 > 1$ there exists $0 < \gamma < 1$ such that

$$\max_{1\le i\le 7}\left\{\frac{1}{s_i}+\eta_i\right\} = \max\left\{\frac{2}{s_1}, \frac{1}{s_2}, \frac{1}{s_3}+\frac{1}{s_4}, \frac{1}{s_5}, \frac{1}{s_6}, \frac{2}{s_7}\right\}$$

holds uniformly for the family of maps $\{T^3_{\alpha\pm\epsilon}\}$. Let

$$\begin{aligned}
b_0^{(\alpha\pm\epsilon)} &= 0, & b_4^{(\alpha\pm\epsilon)} &= \frac{(\beta-1)(\beta^2+1)}{\beta^3},\\
b_1^{(\alpha\pm\epsilon)} &= \frac{(\alpha\pm\epsilon)\beta-(\alpha\pm\epsilon)-1}{(\alpha\pm\epsilon)\beta}, & b_5^{(\alpha\pm\epsilon)} &= \frac{\beta^2-\beta+1}{\beta^2},\\
b_2^{(\alpha\pm\epsilon)} &= \frac{((\alpha\pm\epsilon)\beta-1)(\beta-1)}{(\alpha\pm\epsilon)\beta^2}, & b_6^{(\alpha\pm\epsilon)} &= \frac{(\alpha\pm\epsilon)\beta^2-(\alpha\pm\epsilon)\beta+(\alpha\pm\epsilon)+1}{(\alpha\pm\epsilon)\beta^2},\\
b_3^{(\alpha\pm\epsilon)} &= \frac{\beta-1}{\beta}, & b_7^{(\alpha\pm\epsilon)} &= 1.
\end{aligned}$$

The map $T^3_{\alpha\pm\epsilon}$ is defined on the partition

$$P_{T^3_{\alpha\pm\epsilon}} = \{I_i^{(\alpha\pm\epsilon)} = (b_{i-1}^{(\alpha\pm\epsilon)}, b_i^{(\alpha\pm\epsilon)}) : 1\le i\le 7\}.$$

We use Proposition 3.5.5 to show that $d_S(T^3_{\alpha\pm\epsilon}, T^3_\alpha)\to 0$ as $\epsilon\to 0$, because each $T^3_{\alpha\pm\epsilon}$ is defined on a different partition. Let

$$J_i^{(\alpha\pm\epsilon)} := \left[\max\left\{b_{i-1}^{(\alpha)}, b_{i-1}^{(\alpha\pm\epsilon)}\right\}, \min\left\{b_i^{(\alpha)}, b_i^{(\alpha\pm\epsilon)}\right\}\right].$$

The $J_i^{(\alpha\pm\epsilon)}$ are given by

$$\begin{aligned}
J_1^{(\alpha-\epsilon)} &= \left[b_0^{(\alpha)}, b_1^{(\alpha)}\right], & J_4^{(\alpha\pm\epsilon)} &= \left[b_3^{(\alpha)}, b_4^{(\alpha)}\right],\\
J_1^{(\alpha+\epsilon)} &= \left[b_0^{(\alpha)}, b_1^{(\alpha+\epsilon)}\right], & J_5^{(\alpha\pm\epsilon)} &= \left[b_4^{(\alpha)}, b_5^{(\alpha)}\right],\\
J_2^{(\alpha-\epsilon)} &= \left[b_1^{(\alpha-\epsilon)}, b_2^{(\alpha)}\right], & J_6^{(\alpha-\epsilon)} &= \left[b_5^{(\alpha)}, b_6^{(\alpha)}\right],\\
J_2^{(\alpha+\epsilon)} &= \left[b_1^{(\alpha)}, b_2^{(\alpha+\epsilon)}\right] & J_6^{(\alpha+\epsilon)} &= \left[b_5^{(\alpha)}, b_6^{(\alpha+\epsilon)}\right],\\
J_3^{(\alpha-\epsilon)} &= \left[b_2^{(\alpha-\epsilon)}, b_3^{(\alpha)}\right], & J_7^{(\alpha-\epsilon)} &= \left[b_6^{(\alpha-\epsilon)}, b_7^{(\alpha)}\right],\\
J_3^{(\alpha+\epsilon)} &= \left[b_2^{(\alpha)}, b_3^{(\alpha)}\right], & J_7^{(\alpha+\epsilon)} &= \left[b_6^{(\alpha)}, b_7^{(\alpha)}\right].
\end{aligned}$$

The distance between $T^3_{\alpha\pm\epsilon}$ and T^3_α in $\mathcal{C}^1(J_i^{(\alpha\pm\epsilon)})$ is given by

$$||T^3_{\alpha\pm\epsilon} - T^3_\alpha||_{\mathcal{C}^1(J_i^{(\alpha\pm\epsilon)})} = \max_{0\le k\le 1}\sup_{x\in J_i^{(\alpha\pm\epsilon)}} |(T^3_{\alpha\pm\epsilon})^{(k)}(x) - T^3_\alpha)^{(k)}(x)|.$$

From the expressions for $T^3_{\alpha\pm\epsilon}$ in Section 4.2 we can see that for each $\epsilon < \epsilon_1$ and each i the term

$$\max_{0\le k\le 1}\sup_{x\in J_i^{(\alpha\pm\epsilon)}} |(T^3_{\alpha\pm\epsilon})^{(k)}(x) - (T^3_\alpha)^{(k)}(x)|$$

is of the form $c_0\epsilon^2 x + c_1\epsilon x + c_2\epsilon^2 + c_3\epsilon$ with $c_0, c_1, c_2, c_3 \in \mathbb{R}$. Thus $T^3_{\alpha\pm\epsilon} \to T^3_\alpha$ as $\epsilon \to 0$ in $\mathcal{C}^1$ on the set

$$\bigcup_{i=1}^{7} J_i^{(\alpha\pm\epsilon)}.$$

This combined with the fact that $\{T^3_{\alpha\pm\epsilon}\} \subset \mathcal{T}(I)$ uniformly shows that $d_S(T^3_{\alpha\pm\epsilon}, T^3_\alpha) \to 0$ as $\epsilon \to 0$. It is known that each map $T^3_{\alpha\pm\epsilon}$ is ergodic, because $T_{\alpha\pm\epsilon}$ is ergodic. We have shown that the family of maps $\{T^3_{\alpha\pm\epsilon}\}$ satisfies the conditions of Theorem 3.5.6 and therefore T^3_α is acim-stable. Since each $T_{\alpha\pm\epsilon}$ is exact we get acim-stability for T_α. $\square$

For $\alpha(\beta-1) = 1$ the number of monotonic branches for maps around T^3_α are different. If we approach T^3_α by a family of maps $\{T^3_{\alpha-\epsilon}\}$, then the number of monotonic branches of T^3_α and $\{T^3_{\alpha-\epsilon}\}$ are the same. It follows from Theorem 4.3.2 that T_α is acim-stable from the left. The problem lies in the fact that if we approach T^3_α by the family of maps $\{T^3_{\alpha+\epsilon}\}$, then the number of monotonic branches of T^3_α and $\{T^3_{\alpha+\epsilon}\}$ are different. We are therefore not able to apply Theorem 3.5.6. To solve this problem we define an extension $(\hat{T}_{\alpha+\epsilon}, \hat{I})$ of $(T^3_{\alpha+\epsilon}, I)$ that has $(T^3_{\alpha+\epsilon}, I)$ as an attractor and where all extended maps $\hat{T}_{\alpha+\epsilon}$ have the same number of branches. We prove that the extended map $\hat{T}_\alpha$ is acim-stable by showing that the extended family of maps $\{\hat{T}_{\alpha+\epsilon}\}$ satisfies the conditions of Theorem 3.5.6. As a consequence we obtain acim-stability for T_α.

Theorem 4.3.4. *Let $1 < \alpha < 2$ and $\beta > 1$ be given such that $\alpha(\beta-1) = 1$. Let $\epsilon_1 > 0$ be such that $(\alpha+\epsilon_1)^2\beta > 2$ and $(\alpha+\epsilon_1)\beta^2 > 2$. If $\{T_{\alpha\pm\epsilon}\}$ is exact then T_α is acim-stable.*

Proof. It follows from Theorem 4.3.2 that T_α is acim-stable from the left. For the family of maps $\{T^3_{\alpha+\epsilon}\}$ we define an extended family of maps. The construction of the extended maps is as follows. Let $g_1^{(\alpha+\epsilon)}$ and $g_7^{(\alpha+\epsilon)}$ be linear functions which coincide with the first and last branches of $T^3_{\alpha+\epsilon}$, which means

$$g_1^{(\alpha+\epsilon)}(x) = -(\alpha+\epsilon)^2\beta x + c_1^{(\alpha+\epsilon)} \quad \text{and} \quad g_7^{(\alpha+\epsilon)}(x) = -(\alpha+\epsilon)^2\beta x + c_7^{(\alpha+\epsilon)},$$

where

$$c_1^{(\alpha+\epsilon)} = (\alpha+\epsilon)^2(\beta-1) - (\alpha+\epsilon) \quad \text{and} \quad c_7^{(\alpha+\epsilon)} = \frac{(\alpha+\epsilon)^2(\beta^2-\beta+1) + (\alpha+\epsilon) + \beta}{\beta}.$$

For each $\alpha+\epsilon$ we find points $a_0^{(\alpha+\epsilon)}$ and $a_8^{(\alpha+\epsilon)}$ such that

$$g_1^{(\alpha+\epsilon)}(a_0^{(\alpha+\epsilon)}) = a_8^{(\alpha+\epsilon)} \quad \text{and} \quad g_7^{(\alpha+\epsilon)}(a_8^{(\alpha+\epsilon)}) = a_0^{(\alpha+\epsilon)}.$$

This means we need to solve the system of linear equations

$$\begin{aligned} -(\alpha+\epsilon)^2\beta a_0^{(\alpha+\epsilon)} + c_1^{(\alpha+\epsilon)} &= a_8^{(\alpha+\epsilon)}, \\ -(\alpha+\epsilon)^2\beta a_8^{(\alpha+\epsilon)} + c_7^{(\alpha+\epsilon)} &= a_0^{(\alpha+\epsilon)}. \end{aligned}$$

The solution to this system is given by

$$a_0^{(\alpha+\epsilon)} = \frac{c_7^{(\alpha+\epsilon)} - (\alpha+\epsilon)^2\beta c_1^{(\alpha+\epsilon)}}{1-(\alpha+\epsilon)^4\beta^2} \quad \text{and} \quad a_8^{(\alpha+\epsilon)} = \frac{c_1^{(\alpha+\epsilon)} - (\alpha+\epsilon)^2\beta c_7^{(\alpha+\epsilon)}}{1-(\alpha+\epsilon)^4\beta^2}.$$

We extend maps $T^3_{\alpha+\epsilon}$ to $[a_0^{\alpha+\epsilon}, a_8^{\alpha+\epsilon}]$ using the functions $g_1^{\alpha+\epsilon}$ and $g_7^{\alpha+\epsilon}$. Let us call the new maps $\hat{T}_{\alpha+\epsilon}$. The new maps are shown in Figure 4.5. For each $0<\epsilon<\epsilon_1$ the extended map $\hat{T}_{\alpha+\epsilon}$ has slopes

$$(\alpha+\epsilon_1)^2\beta := s_1 = s_3 = s_7, \quad (\alpha+\epsilon_1)\beta^2 := s_2 = s_4 = s_6 \quad \text{and} \quad \beta^3 := s_5.$$

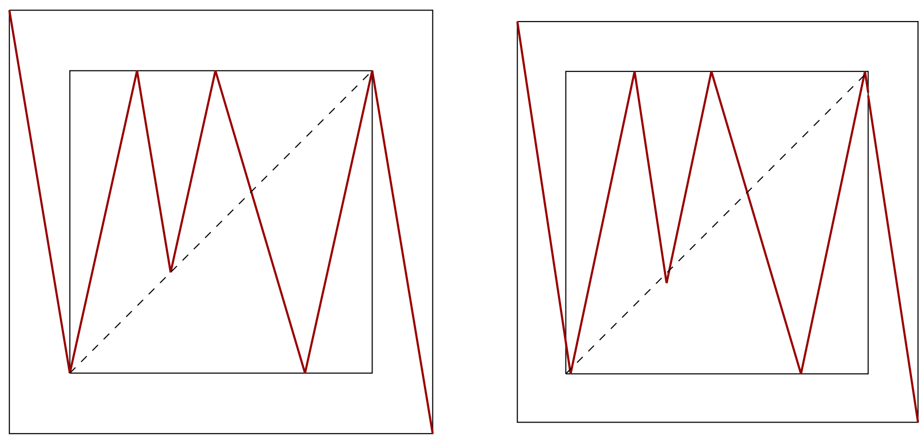

Figure 4.5: The extended graphs of $T^3_{\alpha,\beta}$ and $T^3_{\alpha+\epsilon,\beta}$ for $\alpha=2.0$, $\beta=1.5$ and $\epsilon=0.1$.

The conditions on ϵ_1 now guaranty that $\{\hat{T}_{\alpha+\epsilon}\} \subset \mathcal{T}(I)$ uniformly with uniform constants

$$s_1, s_2, s_3, s_4, s_6, s_7 > 2, \quad s_5 = \beta^3 \quad \text{and} \quad M_1 = M_2 = M_3 = M_4 = M_5 = M_6 = M_7 = 0.$$

For the endpoints of the branches of $\{\hat{T}_{\alpha+\epsilon}\}$ we have $\delta_0^+ = \delta_3^\pm = \delta_7^- = 1$ and $\delta_1^\pm = \delta_2^\pm = \delta_4^\pm = \delta_5^\pm = \delta_6^\pm = 0$. It follows that

$$\begin{aligned}
\eta_1 &= \max\left\{\frac{\delta_0^+}{s_1}, \frac{\delta_1^+}{s_2}\right\} = \max\left\{\frac{1}{s_1}, \frac{0}{s_2}\right\} = \frac{1}{s_1},\\
\eta_2 &= \max\left\{\frac{\delta_1^-}{s_1}, \frac{\delta_2^+}{s_3}\right\} = \max\left\{\frac{0}{s_1}, \frac{0}{s_3}\right\} = 0,\\
\eta_3 &= \max\left\{\frac{\delta_2^-}{s_2}, \frac{\delta_3^+}{s_4}\right\} = \max\left\{\frac{0}{s_2}, \frac{1}{s_4}\right\} = \frac{1}{s_4},\\
\eta_4 &= \max\left\{\frac{\delta_3^-}{s_3}, \frac{\delta_4^+}{s_5}\right\} = \max\left\{\frac{1}{s_3}, \frac{0}{s_5}\right\} = \frac{1}{s_3},\\
\eta_5 &= \max\left\{\frac{\delta_4^-}{s_4}, \frac{\delta_5^+}{s_6}\right\} = \max\left\{\frac{0}{s_4}, \frac{0}{s_6}\right\} = 0,\\
\eta_6 &= \max\left\{\frac{\delta_5^-}{s_5}, \frac{\delta_6^+}{s_7}\right\} = \max\left\{\frac{0}{s_5}, \frac{0}{s_7}\right\} = 0,\\
\eta_7 &= \max\left\{\frac{\delta_6^-}{s_6}, \frac{\delta_7^-}{s_7}\right\} = \max\left\{\frac{0}{s_6}, \frac{1}{s_7}\right\} = \frac{1}{s_7}.
\end{aligned}$$

Since $s_1, s_2, s_3, s_4, s_6, s_7 > 2$ and $s_5 > 1$ there exists $0<\gamma<1$ such that

$$\max_{1\le i\le 7}\left\{\frac{1}{s_i} + \eta_i\right\} = \max\left\{\frac{2}{s_1}, \frac{1}{s_2}, \frac{1}{s_3} + \frac{1}{s_4}, \frac{1}{s_5}, \frac{1}{s_6}, \frac{2}{s_7}\right\}$$

holds for the extended family $\{\hat{T}_{\alpha+\epsilon}\}$. Let

$$\begin{aligned}
b_0^{(\alpha+\epsilon)} &= a_0^{(\alpha+\epsilon)}, & b_4^{(\alpha+\epsilon)} &= \frac{(\beta-1)(\beta^2+1)}{\beta^3},\\
b_1^{(\alpha+\epsilon)} &= \frac{(\alpha+\epsilon)\beta-(\alpha+\epsilon)-1}{(\alpha+\epsilon)\beta}, & b_5^{(\alpha+\epsilon)} &= \frac{\beta^2-\beta+1}{\beta^2},\\
b_2^{(\alpha+\epsilon)} &= \frac{((\alpha+\epsilon)\beta-1)(\beta-1)}{(\alpha+\epsilon)\beta^2}, & b_6^{(\alpha+\epsilon)} &= \frac{(\alpha+\epsilon)\beta^2-(\alpha+\epsilon)\beta+(\alpha+\epsilon)+1}{(\alpha+\epsilon)\beta^2},\\
b_3^{(\alpha+\epsilon)} &= \frac{\beta-1}{\beta}, & b_7^{(\alpha+\epsilon)} &= a_8^{(\alpha+\epsilon)}.
\end{aligned}$$

The map $\hat{T}_{\alpha+\epsilon}$ is defined on the partition

$$P_{\hat{T}_{\alpha+\epsilon}} = \{I_i^{(\alpha+\epsilon)} = (b_{i-1}^{(\alpha+\epsilon)}, b_i^{(\alpha+\epsilon)}) : 1 \le i \le 7\}.$$

We use Proposition 3.5.5 to show that $d_S(\hat{T}_{\alpha+\epsilon}, \hat{T}_\alpha) \to 0$ as $\epsilon \to 0$, because each $\hat{T}_{\alpha+\epsilon}$ is defined on a different partition. Let

$$J_i^{(\alpha+\epsilon)} := \left[\max\left\{b_{i-1}^{(\alpha)}, b_{i-1}^{(\alpha+\epsilon)}\right\}, \min\left\{b_i^{(\alpha)}, b_i^{(\alpha+\epsilon)}\right\}\right].$$

The $J_i^{(\alpha+\epsilon)}$ are given by

$$\begin{aligned}
J_1^{(\alpha+\epsilon)} &= \left[b_0^{(\alpha)}, b_1^{(\alpha+\epsilon)}\right], & J_5^{(\alpha+\epsilon)} &= \left[b_4^{(\alpha)}, b_5^{(\alpha)}\right],\\
J_2^{(\alpha+\epsilon)} &= \left[b_1^{(\alpha)}, b_2^{(\alpha+\epsilon)}\right], & J_6^{(\alpha+\epsilon)} &= \left[b_5^{(\alpha)}, b_6^{(\alpha+\epsilon)}\right],\\
J_3^{(\alpha+\epsilon)} &= \left[b_2^{(\alpha)}, b_3^{(\alpha)}\right], & J_7^{(\alpha+\epsilon)} &= \left[b_6^{(\alpha)}, b_7^{(\alpha)}\right].\\
J_4^{(\alpha+\epsilon)} &= \left[b_3^{(\alpha)}, b_4^{(\alpha)}\right],
\end{aligned}$$

The distance between $T^3_{\alpha+\epsilon}$ and T^3_α in $\mathcal{C}^1(J_i^{(\alpha+\epsilon)})$ is given by

$$||T^3_{\alpha+\epsilon} - T^3_\alpha||_{\mathcal{C}^1(J_i^{(\alpha+\epsilon)})} = \max_{0\le k\le 1} \sup_{x\in J_i^{(\alpha+\epsilon)}} |(T^3_{\alpha+\epsilon})^{(k)}(x) - (T^3_\alpha)^{(k)}(x)|.$$

From the expressions for $T^3_{\alpha+\epsilon}$ in Section 4.2 and the expressions for the functions $g_1^{(\alpha+\epsilon)}$ and $g_7^{(\alpha+\epsilon)}$ we can see that for each $\epsilon < \epsilon_1$ and each i the term

$$\max_{0\le k\le 1} \sup_{x\in J_i^{(\alpha+\epsilon)}} |(T^3_{\alpha+\epsilon})^{(k)}(x) - (T^3_\alpha)^{(k)}(x)|$$

is of the form $c_0\epsilon^2 x + c_1\epsilon x + c_2\epsilon^2 + c_3\epsilon$ with $c_0, c_1, c_2, c_3 \in \mathbb{R}$. We have $\hat{T}_{\alpha+\epsilon} \to \hat{T}_\alpha$ as $\epsilon \to 0$ in $\mathcal{C}^1$ on the set

$$\bigcup_{i=1}^{7} J_i^{(\alpha+\epsilon)}.$$

This combined with the fact that $\{\hat{T}_{\alpha+\epsilon}\} \subset \mathcal{T}(I)$ uniformly shows that $d_S(\hat{T}_{\alpha+\epsilon}, \hat{T}_\alpha) \to 0$ as $\epsilon \to 0$. We already know that each map in $\{\hat{T}_{\alpha+\epsilon}\}$ is ergodic, because $T_{\alpha+\epsilon}$ is ergodic. The family of maps $\{\hat{T}_{\alpha+\epsilon}\}$ satisfies the conditions of Theorem 3.5.6 and therefore $\hat{T}_\alpha$ is acim-stable. Since for all maps $\hat{T}_{\alpha+\epsilon}$ the interval $[0,1]$ is the attractor supporting the unique absolutely continuous invariant measures we obtain acim-stability for T^3_α. By exactness of $T_{\alpha+\epsilon}$ we get acim-stability for T_α. $\square$

Remark 4.3.5. *We had four different cases where we showed acim-stability under certain conditions. Notice that in each case where we show acim-stability the proof gives the existence of a uniform constant* $0 < \gamma < 1$ *such that*

$$\max_{1 \le i \le q} \left\{ \frac{1}{s_i} + \eta_i \right\} \le \gamma < 1$$

holds for all maps $\{T_{\alpha \pm \epsilon}\}_{\epsilon < \epsilon_1}$*. This also means that the maps* $\{T_{\alpha \pm \epsilon}\}_{\epsilon < \epsilon_1}$ *satisfy a Lasota-Yorke type inequality with uniform constant.*

We have made some assumptions to show acim-stability for different values of α. We now indicate the parameter region where our results apply. In Figure 4.6 we have the area where $T_{\alpha,\beta}$ is a piecewise expanding map interval map, i.e. $\alpha, \beta > 1$ and $\alpha + \beta > \alpha\beta$ and have drawn the different assumption areas with different colors.

We showed that T_α is acim-stable for $\alpha > 2$. To show acim-stability for $\alpha \le 2$ we assumed that $\alpha^2\beta > 2$ and $\alpha\beta^2 > 2$. The other assumption we made was that T_α is exact. We know that $T_{\alpha,\beta}$ is exact if $\alpha > \sqrt{2}$ and $\beta > \sqrt{2}$. In that case we also have $\alpha^2\beta > 2$ and $\alpha\beta^2 > 2$. In Figure 4.6 the area where the assumptions $\alpha^2\beta > 2$ and $\alpha\beta^2 > 2$ fail is drawn in purple and the area where we have acim-stability is drawn in green. We know that $T_{\alpha,\beta}$ is not exact for $\alpha + \beta \ge \alpha\beta^2$. In Figure 4.6 the area where $\alpha \le 2$, $\alpha + \beta \ge \alpha\beta^2$, $\alpha^2\beta > 2$ and $\alpha\beta^2 > 2$ is drawn in blue. The green area is where we have acim-stability and in the purple area our results to show acim-stability do not work. In the blue and light-blue area we get acim-stability if T_α^3 has the same acim as T_α. In the light-blue area we do not know if $T_{\alpha,\beta}$ is exact and in the blue area T_α is not exact. If we can show that $T_{\alpha,\beta}$ is exact in the light-blue area, then as a consequence we also get acim-stability.

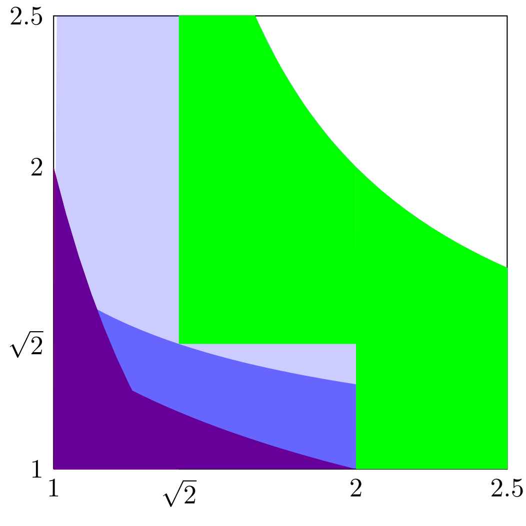

Figure 4.6: The different assumption areas

4.4 Measure-theoretic entropy for skew tent maps

The skew tent map is a piecewise linear interval map. It satisfies the assumptions for Rokhlin's formula and we get the following expression for the measure-theoretic entropy:

$$h_{\mu_{\alpha,\beta}}(T_{\alpha,\beta}) = \int_{[0,1]} \log |T'_{\alpha,\beta}| \, \mathrm{d}\mu_{\alpha,\beta} = \int_0^1 \log |T'_{\alpha,\beta}| f_{\alpha,\beta}(x) \, \mathrm{d}x,$$

where $f_{\alpha,\beta}$ is the invariant density of $\mu_{\alpha,\beta}$. This expression can not be used to calculate the measure-theoretic entropy, because we have no expression for the invariant density. However, we can use this expression to show that for fixed β the measure-theoretic entropy depends continuously on α on the region where T_α is acim-stable. We make use of the following lemma to show this.

Lemma 4.4.1. *Let $\{f_n\}$ be a sequence of functions in $L^1([0,1])$ such that*

1. $||f_n||_\infty \leq K \quad \forall n,$
2. $f_n \xrightarrow{L^1} f$ *for some* $f \in L^1([0,1])$.

Then for any $\psi \in L^1([0,1])$,

$$\int \psi(f_n - f) \to 0.$$

Proof. See [AOT06, Lemma 5.1]. □

Theorem 4.4.2. *Take $\beta > 1$ fixed and let $J = \{\alpha > 1 : \alpha + \beta \geq \alpha\beta\}$. Assume that T_α satisfies the conditions of on of the Theorems 4.3.1, 4.3.2, 4.3.3 or 4.3.4 on a closed interval $K \subseteq J$. Then the map $\alpha \mapsto h_{\mu_\alpha}(T_\alpha)$ is continuous on K.*

Proof. Assume that T_α is acim-stable for all α on a closed interval $K \subseteq J$. Let $\{\alpha_k\}$ be a sequence in K converging to α. We show that $|h_{\mu_\alpha}(T_\alpha) - h_{\mu_{\alpha_k}}(T_{\alpha_k})| \longrightarrow 0$ as $\alpha_k \to \alpha$. We have the following:

$$\begin{aligned}
|h_{\mu_\alpha}(T_\alpha) - h_{\mu_{\alpha_n}}(T_{\alpha_k})| &= \left| \int_0^1 \log |T'_\alpha(x)| f_\alpha(x) \mathrm{d}x - \int_0^1 \log |T'_{\alpha_n}(x)| f_{\alpha_k}(x) \mathrm{d}x \right| \\
&\leq \left| \int_0^1 \log |T'_\alpha(x)| (f_\alpha(x) - f_{\alpha_k}(x)) \mathrm{d}x \right| \\
&\quad + \left| \int_0^1 (\log |T'_\alpha(x)| - \log |T'_{\alpha_k}(x)|) f_{\alpha_k}(x) \mathrm{d}x \right|.
\end{aligned}$$

Since T_α is acim-stable the first term goes to zero as $\alpha_k \to \alpha$. For the second term we note that $T_{\alpha_k} \to T_\alpha$ as $\alpha_k \to \alpha$. If f_{α_k} is uniformly bounded, then it follows that the second term also goes to zero as $\alpha_k \to \alpha$. For $k \geq 1$ we define the Cesàro means

$$f_{k,n} = \frac{1}{n} \sum_{j=0}^{n-1} P^j_{T_{\alpha_k}} 1.$$

Recall that there is a subsequence of $(f_{k,n})_n$ converging to f_{α_k} λ-a.e. By Theorem 3.4.2 there exist $0 < \gamma_k < 1$ and $h > 0$, which is independent of k because each map has the same intervals of monotonicity, such that

$$\begin{aligned}\operatorname*{Var}_I(P^j_{T_{\alpha_k}} 1) + ||P^j_{T_{\alpha_k}} 1||_{L^1} &= ||P^j_{T_{\alpha_k}}||_{\mathrm{BV}} \\ &\leq \gamma_k^j ||1||_{\mathrm{BV}} + \Big(1 + \frac{0 + 2h^{-1}}{1-\gamma_k}\Big)||1||_{L^1} \\ &= 1 + \gamma_k^j + \frac{2h^{-1}}{1-\gamma_k}.\end{aligned}$$

Since $\gamma_k < 1$ it follows that

$$\operatorname*{Var}_I(P^j_{T_{\alpha_k}} 1) \leq 1 + \gamma_k^j + \frac{2h^{-1}}{1-\gamma_k} + ||P^j_{T_{\alpha_k}} 1||_{L^1} \leq 3 + \frac{2h^{-1}}{1-\gamma_k}.$$

We now have

$$\operatorname*{Var}_I(f_{k,n}) \leq \frac{1}{n}\sum_{j=0}^{n-1} \operatorname*{Var}_I(P^j_{T_{\alpha_k}} 1) \leq 3 + \frac{2h^{-1}}{1-\gamma_k}. \tag{4.1}$$

Fix some α. By Remark 4.3.5 we can find in each case $\epsilon_1 > 0$ such that for all $\alpha_k \in [\alpha - \epsilon_1, \alpha + \epsilon_1]$ (4.1) holds with uniform constant γ. Also the supremum of $f_{k,n}$ is bounded with the same uniform constants, so

$$\begin{aligned}\sup |f_{k,n}| &\leq \operatorname*{Var}_I(f_{k,n}) + \int_{[0,1]} f_{k,n} \mathrm{d}\lambda \\ &\leq \operatorname*{Var}_I(f_{k,n}) + \frac{1}{n}\sum_{j=0}^{n-1} \int_{[0,1]} P^j_{T_{\alpha_k}}(1)\mathrm{d}\lambda \\ &\leq 4 + \frac{2h^{-1}}{1-\gamma}.\end{aligned}$$

Since both bounds are independent of α_k and j, we have

$$\operatorname*{Var}_I(f_{\alpha_k}) \leq 4 + \frac{2h^{-1}}{1-\gamma} \quad \text{and} \quad \sup(f_{\alpha_k}) \leq 4 + \frac{2h^{-1}}{1-\gamma}.$$

This shows that $f_{\alpha k}$ is uniformly bounded and it follows that map $\alpha \mapsto h_{\mu_\alpha}(T_\alpha)$ is continuous on K. $\square$

Chapter 5

The absolutely continuous measure and maximal measures

In this section we look at the relation between two different types of measures for the skew tent maps: the unique acim and the measures of maximal entropy. In general it is hard to find a measure of maximal entropy. However, if the skew tent map is Markov and the transition matrix with possible jumps it induces is irreducible, then we can find a unique measure of maximal entropy. This measure is called the Parry measure [Par64]. We investigate when the unique acim is also the Parry measure.

5.1 Markov maps

To simplify the analysis we often look for maps such that the space can be partitioned into a finite number of elements. This makes it possible to study a topologically equivalent system using symbolic dynamics. Such a partition is associated to Markov maps.

Definition 5.1.1. Let $P = \{I_i = (x_{i-1}, x_i) : 1 \leq i \leq n\}$ be a partition of I. A map $T : I \to I$ is called a *Markov map* if $T_{|_{I_i}}$ is a homeomorphism onto some interval $(x_{j(i)}, x_{k(i)})$. The partition $P = \{I_1, \ldots, I_n\}$ is then called a Markov partition. If T is linear on each I_i, then we say that T is a *piecewise linear Markov map*. Let $\mathscr{C}$ denote the family of such maps.

If $T \in \mathscr{C}$, then it induces an $n \times n$ matrix A, where for $1 \leq i, j \leq n$ the entries A_{ij} are defined by

$$A_{ij} = \begin{cases} 1 & \text{if } I_j \subseteq T(I_i), \\ 0 & \text{otherwise.} \end{cases}$$

We will refer to this matrix as the Markov matrix.

Example 5.1.2. Consider the skew tent map $T\colon I \to I$ defined by

$$T(x) = \begin{cases} \frac{1}{\beta-1}x + \frac{\beta-1}{\beta} & \text{for } \ x \in [0, \frac{\beta-1}{\beta}], \\ \beta - \beta x & \text{for } \ x \in [\frac{\beta-1}{\beta}, 1] \end{cases}$$

Let $I_1 = [0, \frac{\beta-1}{\beta}]$ and $I_2 = [\frac{\beta-1}{\beta}, 1]$. The interval I_1 is mapped to I_2 and the interval I_2 is mapped to I. Figure 3.1 shows the example with $\beta = 2$. Since T is a piecewise linear map it is clear that the maps $T_{|I_1}$ and $T_{|I_2}$ are homeomorphisms. This shows that the partition $P = \{I_1, I_2\}$ is a Markov partition. The Markov matrix for this partition is given by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

5.1.1 Subshifts of finite type

Let A be an $n \times n$ matrix with entries in $\{0, 1\}$. Define

$$\Sigma_A^+ := \{(x_0, x_1, \dots) : x_j \in \{1, \dots, n\}, A_{x_j, x_{j+1}} = 1, j \in \mathbb{N}\}.$$

This space is called a one-sided subshift of finite type. It consists of all one-sided infinite sequences of symbols such that the symbol x_j can be followed by the symbol x_{j+1} only if $A_{x_j, x_{j+1}} = 1$. We consider the left shift operator $L \colon \Sigma_A^+ \to \Sigma_A^+$, which is defined by

$$L(x_0, x_1, \dots) = (x_1, x_2, \dots).$$

The subsets of Σ_A^+ for which the first m values are fixed are called cylinder sets of order m and are denoted by

$$[y_1, \dots, y_m] = \{x \in \Sigma_A^+ : x_1 = y_1, \dots, x_m = y_m\}.$$

On Σ_A^+ we consider the σ-algebra generated by the cylinder sets.

Each Markov map induces a Markov matrix A with entries in $\{0, 1\}$. This means that each Markov map can be associated with a subshift of finite type Σ_A^+. On the subshift of finite type we can look for invariant measures, thus leading to a measure-preserving dynamical system.

5.1.2 Markov measures

On the subshifts of finite type we are going to define a large class of L-invariant measures. A measure in this class is called a Markov measure. We assign values to the entries of the Markov matrix that represent probabilities. This gives a stochastic matrix.

Definition 5.1.3. An $n \times n$ matrix P is called *stochastic* if:

1. $P_{i,j} \geq 0 \quad i, j = 1, \dots, n$;
2. $\sum_{j=1}^{n} P_{i,j} = 1, \quad i = 1, \dots, n.$

To define a Markov measure on the subshifts of finite type we use the Perron-Frobenius Theorem.

Theorem 5.1.4 (Perron-Frobenius Theorem)**.** *Let A be a non-negative irreducible $n \times n$ matrix (i.e. $A_{i,j} \geq 0$ for each $1 \leq i, j \leq k$ and there exists k such that $A_{i,j}^k > 0$ for all $1 \leq i, j \leq n$). Then:*

1. *there exists a positive eigenvalue $\lambda > 0$ such that all other eigenvalues $\lambda_i \in \mathbb{C}$ satisfy $|\lambda_i| < \lambda$,*
2. *the eigenvalue λ is simple (i.e. the corresponding eigenspace is one-dimensional),*

3. *there is a unique right-eigenvector* $v = (v_1, \ldots, v_n)^T$ *such that* $v_j > 0$, $\sum_{j=1}^n v_j = 1$ *and* $Av = \lambda v$,

4. *there is a unique left-eigenvector* $u = (u_1, \ldots, u_n)$ *such that* $u_j > 0, \sum_{j=1}^n u_j = 1$ *and* $uA = \lambda u$,

5. *eigenvectors corresponding to eigenvalues other than* λ *are not positive, i.e. at least one coordinate is positive and at least one coordinate is negative.*

We are now going to define a class of L-invariant measures on subshifts of finite type. Let A be the $n \times n$ irreducible matrix with entries in $\{0, 1\}$. This matrix induces a one-sided subshift of finite type Σ_A^+. Let P be a stochastic matrix compatible with A. This means that $P_{i,j} > 0$ if and only if $A_{i,j} = 1$. Since A is irreducible it follows that P is also irreducible. Note that there might be many stochastic matrices that are compatible with A. By the Perron-Frobenius Theorem there exists a unique maximal eigenvalue λ for the matrix P. Since P is stochastic we have that $\lambda = 1$ and it has corresponding right-eigenvector $v = (1, 1, \ldots, 1)^T$. Let $p = (p_1, \ldots, p_n)$ be the corresponding normalized left-eigenvector. We define a probability measure μ_P on the cylinder sets of Σ_A^+ by setting

$$\mu_P[y_l, y_{l+1}, \ldots, y_k] = p_{y_l} P(y_l, y_{l+1}) \ldots P(y_{k-1}, y_k).$$

By the Kolmogorov Extension Theorem this defines a measure on the whole Borel σ-algebra.

We now show that μ_P is L-invariant. It is sufficient to show that μ_P is L-invariant on the cylinder sets, see Theorem A.0.1. For the cylinder sets we have the following:

$$\begin{aligned}
\mu_P(L^{-1}[y_0, \ldots, y_k]) &= \mu_P\Big(\bigcup_{j=1}^n [j, y_0, y_1, \ldots, y_k]\Big) \\
&= \sum_{j=1}^n \mu_P[j, y_0, y_1, \ldots, y_k] \\
&= \sum_{j=1}^n p_j P(j, y_0) P(y_0, y_1) \ldots P(y_{n-1}, y_n) \\
&= \Big[\sum_{j=1}^n p_j P(j, y_0)\Big] P(y_0, y_1) \ldots P(y_{n-1}, y_n) \\
&= p_{y_0} P(y_0, y_1) \ldots P(y_{n-1}, y_n) \\
&= \mu_P[y_0, y_1, \ldots, y_n],
\end{aligned}$$

where we used that p is the left-eigenvector of P, i.e. $pP = p$. This shows that μ_P is L-invariant.

5.2 Parry measure

Given an irreducible matrix A there are a lot of compatible stochastic matrices P each defining a different Markov measure μ_P. The one we are going to look at is called the Parry measure and it is the only invariant measure that maximizes the measure-theoretic entropy.

Let A be an irreducible $n \times n$ matrix with entries in $\{0,1\}$. By the Perron-Frobenius Theorem there exists a unique maximal eigenvalue λ with corresponding left and right eigenvectors $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)^T$, respectively. Let $c = \sum_{i=1}^n u_i v_i$. The Parry measure is defined by

$$P_{i,j} = \frac{A_{ij} v_j}{\lambda v_i}, \quad p_i = \frac{u_i v_i}{c}.$$

Proposition 5.2.1. *The matrix P is a stochastic matrix and p is a normalized left-eigenvector for P.*

Proof. We have $A_{ij} \in \{0,1\}$ and $v_i, v_j, \lambda > 0$ and therefore $P_{i,j} \geq 0$. Since v is the right-eigenvector in the Perron-Frobenius Theorem we have

$$\sum_{j=1}^n A_{ij} v_j = \lambda v_i,$$

from which it immediately follows that

$$\sum_{j=1}^n P_{i,j} = \sum_{j=1}^n \frac{A_{ij} v_j}{\lambda v_i} = 1,$$

for each i. It is clear that p is normalized and since u is a left eigenvector for A it follows that

$$\begin{aligned}
(p_1, \dots, p_n)P &= \Big(\sum_{i=1}^n p_i P_{i,1}, \dots, \sum_{i=1}^n p_i P_{i,n} \Big) \\
&= \Big(\sum_{i=1}^n \frac{u_i v_i}{c} \cdot \frac{A_{i1} v_1}{\lambda v_i}, \dots, \sum_{i=1}^n \frac{u_i v_i}{c} \cdot \frac{A_{i,n} v_n}{\lambda v_i} \Big) \\
&= \Big(\frac{v_1}{\lambda c} \sum_{i=1}^n u_i A_{i1}, \dots, \frac{v_n}{\lambda c} \sum_{i=1}^n u_i A_{in} \Big) \\
&= \Big(\frac{v_1 u_1}{c}, \dots \frac{v_n u_n}{c} \Big) \\
&= (p_1, \dots, p_n).
\end{aligned}$$

This shows that p it is a left-eigenvector for P. □

Theorem 5.2.2. *If A is irreducible, then the Parry measure is the unique measure of maximal entropy for $L : \Sigma_A^+ \to \Sigma_A^+$.*

Proof. See [She13, Theorem 23] □

5.3 The unique acim and the maximal measure

We can use the Parry measure on the subshift of finite type to construct a measure of maximal entropy for a skew Markov tent map with constant slope. The following result by W. Byers and A. Boyarsky [BB85] can be applied to the skew Markov tent maps with constant slopes. It shows that for the skew Markov tent maps with constant slope (and irreducible Markov matrix) the unique acim is the maximal measure.

Theorem 5.3.1. *Let $T \in \mathscr{C}$ be expanding and of constant slope (i.e. the absolute value of the slope is constant). Suppose that the 0-1 matrix A which it induces is irreducible. Then there exists a unique Borel probability measure μ, invariant under T, which maximizes entropy and is equivalent to Lebesgue measure.*

Proof. See [BB85, Theorem 2]. □

We know that skew Markov tent maps with constant slope $\alpha = \beta$ and irreducible Markov matrix have a unique acim that is maximal. We want to know if there are skew Markov tent maps with slopes $\alpha \neq \beta$ and irreducible Markov matrix where the unique acim is a measure of maximal entropy. W. Byers and B. Boyarsky [BB85] give a condition for the unique acim to be maximal.

Let T be a skew Markov tent map with partition $P = \{I_1, \ldots, I_n\}$ and irreducible Markov matrix A. Assume there are integers p and q, $1 \leq p \leq q \leq n$, such that every row of A either consists of a block of 1's, $a_{ij} = 1$ if and only if $j = p, \ldots, q$, or else the row contains a unique nonzero element; $a_{ij} = 1$. Let $J = [x_{p-1}, x_q]$, then we have the following relation for the unique acim and the measure of maximal entropy.

Theorem 5.3.2. *The unique acim for T is maximal if and only if T has constant slope λ on all the intervals $I_i \subset J$, where λ is the largest eigenvalue of A in absolute value.*

Proof. See [BB85, Theorem 3]. □

In the example below we calculate the measure-theoretic entropy for a Markov map to see when it is a measure of maximal entropy.

Example 5.3.3. Let T be the skew tent map from Example 5.1.2. Since the invariant density f is constant on each interval in the Markov partition, see for example [BB85, Lemma 2], it can be written as

$$f(x) = \begin{cases} c_1 & \text{for } \; x \in [0, \frac{\beta-1}{\beta}], \\ c_2 & \text{for } \; x \in [\frac{\beta-1}{\beta}, 1], \end{cases}$$

where $f_1, f_2 > 0$. The measure μ is T-invariant and therefore

$$\mu\Big([0, \tfrac{\beta-1}{\beta}]\Big) = \mu\Big(T^{-1}[0, \tfrac{\beta-1}{\beta}]\Big).$$

Writing these expressions out gives

$$\int_{\left[0, \frac{\beta-1}{\beta}\right]} f \mathrm{d}\mu = \mu\Big([0, \tfrac{\beta-1}{\beta}]\Big) = \mu\Big(T_\beta^{-1}[0, \tfrac{\beta-1}{\beta}]\Big) = \mu\Big([\tfrac{\beta^2-\beta+1}{\beta^2}, 1]\Big) = \int_{\left[\frac{\beta^2-\beta+1}{\beta}, 1\right]} f \mathrm{d}\mu.$$

Calculating the integrals gives

$$\Big(\frac{\beta-1}{\beta}\Big) c_1 = \Big(\frac{\beta-1}{\beta^2}\Big) c_2,$$

which can be simplified to

$$\beta c_1 = c_2.$$

Since μ is a probability measure we have

$$\int_0^1 f \mathrm{d}\mu = \Big(\frac{\beta-1}{\beta}\Big)c_1 + \Big(\frac{1}{\beta}\Big)c_2 = 1.$$

By combining these equations we find

$$c_1 = \frac{\beta}{2\beta-1} \quad \text{and} \quad c_2 = \frac{\beta^2}{2\beta-1}.$$

The measure-theoretic entropy of the acim given by Rokhlin's formula is

$$h_\mu(T) = \int_0^1 \log|T^{'}| \, \mathrm{d}\mu.$$

Calculating the integral gives

$$h_\mu(T) = \frac{1}{2\beta-1}\Big[\beta\log(\beta) - (\beta-1)\log(\beta-1)\Big].$$

The eigenvalues of the Markov matrix can be found by solving the equation

$$\det(A-\lambda I) = -\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0.$$

The solutions for this equation are

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

The topological entropy is given by $h_{\text{top}}(T) = \log\lambda_1$, see [Wal82, Theorem 7.13]. By Misiurewicz [Mis89] we know that the topological entropy of the skew tent maps is continuous. In the example above the measure-theoretic entropy is equal to the topological entropy for $\beta = \lambda_1$. It then follows that $\frac{1}{\beta-1} = \lambda_1$. This is the case were the absolute value of the slope is constant. This shows that the topological entropy and the measure-theoretic entropy of the acim are in general not equal.

Remark 5.3.4. *The results in this section only apply to skew Markov tent maps with irreducible Markov matrix. We do not know anything about the skew tent maps that fall outside this category. There might be results that can be applied to skew tent maps that are not Markov. We are not aware of this at the moment of finishing this thesis.*

Appendix A

Theorems

This appendix contains some theorems that are used in the main text.

Theorem A.0.1. *Let $(X_i, \mathcal{B}_i, \mu_i)$ be a probability space, $i = 1, 2$, and $T\colon X_1 \to X_2$ a transformation. Suppose $\mathcal{S}_2$ is a generating semi-algebra of $\mathcal{B}_2$. Then T is measurable and measure preserving if and only if for each $A \in \mathcal{S}_2$, we have $T^{-1}A \in \mathcal{B}_1$ and $\mu_1(T^{-1}A) = \mu_2(A)$.*

Proof. See [Daj14, Theorem 1.2.2]. □

Theorem A.0.2 (Radon-Nikodym Theorem)**.** *Let $(X, \mathcal{F})$ be a measurable space and let μ and ν be two σ-finite measures on $(X, \mathcal{F})$. If $\nu \ll \mu$, then there exists a unique $f \in L^1(X, \mathcal{F}, \mu)$ such that*

$$\nu(A) = \int_A f\, d\mu \quad \text{for all } A \in \mathcal{F}.$$

Proof. See [DS64]. □

Theorem A.0.3 (Mazur's Theorem)**.** *Let X be a Banach space with $A \subset X$, where the closure of A is compact. Then the closed convex hull of A is compact.*

Proof. See [DS64]. □

Theorem A.0.4 (Helly's First Theorem)**.** *Let an infinite family of functions $F = \{f_n\}$ be defined on an interval $[a, b]$. If all functions of the family and the total variation of all functions of the family are bounded by a singular number, i.e.*

$$|f_n(x)| \leq K, \qquad \operatorname*{Var}_{[a,b]} f_n \leq K, \qquad \forall f_n \in F,$$

then there exists a sequence $\{f_{n_k}\} \in F$ that converges at every point of $[a, b]$ to some function f^* of bounded variation, and $\mathrm{Var}_{[a,b]} f^* \leq K$.

Proof. See [Nat16]. □

Appendix B

Functions of bounded variation

Functions of bounded variation have the following basic properties. These properties and many more can also be found in [Nat16].

Proposition B.0.1. *If f is of bounded variation on $[a, b]$, then f is bounded on $[a, b]$. In fact*

$$|f(x)| \leq f(a) + \operatorname*{Var}_{[a,b]} f$$

for all $x \in [a, b]$.

Proof. For every $x \in [a, b]$ we have

$$\begin{aligned} |f(x)| &= |f(a) + f(x) - f(a)| \\ &\leq |f(a)| + |f(x) - f(a)| \\ &\leq |f(a)| + \operatorname*{Var}_{[a,b]} f \end{aligned}$$

□

Proposition B.0.2. *Let $f\colon [a, b] \to \mathbb{R}$ be of bounded variation and assume $c \in (a, b)$. Then f is of bounded variation on $[a, c]$ and on $[c, b]$ and we have*

$$\operatorname*{Var}_{[a,b]} f = \operatorname*{Var}_{[a,c]} f + \operatorname*{Var}_{[c,b]} f.$$

Proof. Take partition $P_1 = \{x_0, \ldots, x_n\}$ for the interval $[a, c]$ and partition $P_2 = \{y_0, \ldots, y_m\}$ for the interval $[c, b]$. Together these two partitions form a partition for the whole interval $[a, b]$. For the partitions we have the sums

$$V_f(P_1) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|, \qquad V_f(P_2) = \sum_{k=1}^{m} |f(y_k) - f(y_{k-1})|.$$

Let V be the sum corresponding to this method of partition, then by the definition of total variation we have $V = V_f(P_1) + V_f(P_2) \leq \operatorname{Var}_{[a,b]} f$. This holds for any partition, therefore

$$\operatorname*{Var}_{[a,c]} f + \operatorname*{Var}_{[c,b]} f \leq \operatorname*{Var}_{[a,b]} f.$$

For the inequality the other way we take partition $P = \{z_0, \dots, z_p, \dots, z_q\}$, where $c = z_p$. Let $P_1 = \{z_0, \dots, z_p\}$ and $P_2 = \{z_p, \dots, z_q\}$. We can express the sum for the partition P as

$$\begin{aligned} V_f(P) &= \sum_{k=1}^{p} |f(z_k) - f(z_{k-1})| + \sum_{k=p+1}^{q} |f(z_k) - f(z_{k-1})| \\ &= V_f(P_1) + V_f(P_2) \end{aligned}$$

By the definition of total variation we have

$$V_f(P) = V_f(P_1) + V_f(P_2) \leq \operatorname*{Var}_{[a,c]} f + \operatorname*{Var}_{[c,b]} f.$$

The inequality still holds if we take a refinement, therefore

$$\operatorname*{Var}_{[a,b]} f \leq \operatorname*{Var}_{[a,c]} f + \operatorname*{Var}_{[c,b]} f.$$

□

Proposition B.0.3. *Let $f : [a, b] \to \mathbb{R}$ be monotone on $[a, b]$. Then*

$$\operatorname*{Var}_{[a,b]} f = |f(b) - f(a)|.$$

Proof. We give a proof in case the function f is increasing. The proof for f decreasing is analogous. Take an arbitrary partition $P = \{x_0, \dots, x_n\}$ for $[a, b]$. Since f is increasing we have $|f(x_k) - f(x_{k-1})| = f(x_k) - f(x_{k-1})$ and hence

$$V_f(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) = f(b) - f(a).$$

Since $V_f(P)$ does not depend on the partition we conclude that

$$\operatorname*{Var}_{[a,b]} f = f(b) - f(a).$$

□

Proposition B.0.4. *Let $f\colon [a, b] \to \mathbb{R}$ have a continuous derivative f' on $[a, b]$. Then f is of bounded variation and*

$$\operatorname*{Var}_{[a,b]} f = \int_a^b |f^{'}(x)| d\lambda(x).$$

Proof. Since f has a continuous derivative f' it attains a minimum m_1 and maximum m_2. Let $M = \max\{|m_1|, |m_2|\}$. Let $P = \{x_0, \dots, x_n\}$ be an arbitrary partition of $[a, b]$. By the Mean Value Theorem there exists for each interval $[x_{k-1}, x_k]$ a $c_k \in [x_{k-1}, x_k]$ such that

$$f(x_k) - f(x_{k-1}) = f'(c_k)(x_k - x_{k-1}).$$

Combining this with the fact that f' is bounded from above by M gives

$$|f(x_k) - f(x_{k-1})| = |f'(c_k)|(x_k - x_{k-1}) \leq M(x_k - x_{k-1})$$

and hence

$$V_f(P) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \leq M \sum_{k=1}^{n} (x_k - x_{k-1}) = M(b-a).$$

Since the partition is arbitrary it follows that f has bounded variation. For the second part we note that for an arbitrary partition $P = \{x_0, \ldots, x_n\}$ we have

$$\begin{aligned} V_f(P) &= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\ &= \sum_{k=1}^{n} \left| \int_{x_{k-1}}^{x_k} f'(x)\mathrm{d}x \right| \\ &\leq \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} |f'(x)|\mathrm{d}x \\ &= \int_a^b |f'(x)|\mathrm{d}x. \end{aligned}$$

Since the partition is arbitrary it follows that

$$\operatorname*{Var}_{[a,b]} f \leq \int_a^b |f'(x)|\mathrm{d}x.$$

For the inequality the other way we note that the function $|f'|$ is continuous on the closed interval $[a,b]$ interval and therefore Riemann-integrable on $[a,b]$. Let $d(P)$ denote the maximal length of an interval in the partition P. Let $\varepsilon > 0$ be arbitrary. By the Riemann integrability there exists a $\delta > 0$ such that for any partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ with $d(P) < \delta$ and any choice $c_k \in [x_{k-1}, x_k]$ we have

$$\int_a^b |f'(x)|\mathrm{d}x - \varepsilon < \sum_{k=1}^{n} f'(c_k)(x_k - x_{k-1}) < \int_a^b |f'(x)|\mathrm{d}x + \varepsilon.$$

Let $P_1 = \{y_0, \ldots, y_m\}$ be a partition of $[a,b]$ with $d(P') < \delta$. By the Mean Value Theorem

$$V_f(P_1) = \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \sum_{k=1}^{n} f'(\xi_k)(x_k - x_{k-1}).$$

for some $\xi_k \in [x_k - x_{k-1}]$. The right side is equal to the Riemann sum for the partition P_1 and since $d(P_1) < \delta$ we have

$$\int_a^b |f'(x)|\mathrm{d}x - \varepsilon < V_f(P_1) \leq \operatorname*{Var}_{[a,b]} f.$$

Since ε is arbitrary it follows that

$$\int_a^b |f'(x)|\mathrm{d}x \leq \operatorname*{Var}_{[a,b]} f.$$

$\square$

Proposition B.0.5. *Let $f, g\colon [a,b] \to \mathbb{R}$ be of bounded variation, then so are the sum and product and we have the following properties:*

1. $\mathrm{Var}_{[a,b]}(f+g) \leq \mathrm{Var}_{[a,b]} f + \mathrm{Var}_{[a,b]} g$.
2. $\mathrm{Var}_{[a,b]}(f \cdot g) \leq A \cdot \mathrm{Var}_{[a,b]} f + B \cdot \mathrm{Var}_{[a,b]} g$,

where $A = \sup\{g(x) : x \in [a,b]\}$ and $B = \sup\{f(x) : x \in [a,b]\}$.

Proof. 1. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$, then

$$\begin{aligned}
V_{f+g}(P) &= \sum_{k=1}^{n} |(f+g)(x_k) - (f+g)(x_{k-1})| \\
&= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1}) + g(x_k) - g(x_{k-1})| \\
&\leq \sum_{k=1}^{n} \Big(|f(x_k) - f(x_{k-1})| + |g(x_k) - g(x_{k-1})| \Big) \\
&= \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \\
&= V_f(P) + V_g(P).
\end{aligned}$$

Since this holds for any partition P it follows that

$$\operatorname*{Var}_{[a,b]}(f+g) \leq \operatorname*{Var}_{[a,b]} f + \operatorname*{Var}_{[a,b]} g.$$

2. Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$, then

$$\begin{aligned}
V_{f\cdot g}(P) &= \sum_{k=1}^{n} |(f \cdot g)(x_k) - (f \cdot g)(x_{k-1})| \\
&= \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&= \sum_{k=1}^{n} |f(x_k)g(x_k) - f(x_{k-1})g(x_k) + f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})| \\
&= \sum_{k=1}^{n} \Big(|f(x_k)g(x_k) - f(x_{k-1})g(x_k)| + |f(x_{k-1})g(x_k) - f(x_{k-1})g(x_{k-1})| \Big) \\
&\leq \sum_{k=1}^{n} |g(x_k)| \cdot |f(x_k) - f(x_{k-1})| + \sum_{k=1}^{n} |f(x_{k-1})| \cdot |g(x_k) - g(x_{k-1})| \\
&\leq B \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| + A \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \\
&= A \cdot V_f(P) + B \cdot V_g(P).
\end{aligned}$$

Since this holds for any partition P it follows that

$$\operatorname*{Var}_{[a,b]}(f \cdot g) \leq A \cdot \operatorname*{Var}_{[a,b]} f + B \cdot \operatorname*{Var}_{[a,b]} g.$$

$\square$

Appendix C

Lasota-Yorke inequality

Proposition C.0.1 (Lasota-Yorke inequality). *Let $T : I \to I$ be a non-singular piecewise expanding C^2 function. Then for every $f \in BV(I)$ there exists $N \in \mathbb{N}$ such that*

$$\operatorname*{Var}_I P_{T^N} f \leq 2s^{-N} \operatorname*{Var}_I f + (K + 2h^{-1}) \int_I |f| d\lambda,$$

where $K = \frac{\max|\sigma'|}{\min_i \sigma_i}$ and $h = \min_i(b_{i-1} - b_i)$.

Proof. Write $s = \inf |T'|$ and choose a number N such that $s^N > 2$. It is easy to see that the function $\phi = T^N$ is a piecewise C^2 function. Denote by $b_0, \ldots, b_q$ the partition corresponding to the intervals of monotonicity of ϕ. Writing ϕ_i for the corresponding C^2 functions we have

$$|\phi_i'(x)| \geq s^N, \quad x \in [b_{i-1}, b_i],\ i = 1, \ldots, q. \tag{C.1}$$

Let $\psi_i = \phi_i^{-1}, \sigma_i(x) = |\psi_i'(x)|$ and $J_i = \phi_i([b_{i-1}, b_i])$, then it follows from (C.1) that

$$|\sigma_i(x)| \leq s^{-N}, \quad x \in J_i,\ i = 1, \ldots, q. \tag{C.2}$$

Computing the Perron-Frobenius operator for ϕ we obtain

$$P_\phi f(x) = \sum_{i=1}^{q} f(\psi_i(x))\sigma_i(x)\chi_i(x),$$

where χ_i is the characteristic function of the interval J_i. The goal now is to find an upper bound on the variation of $P_\phi f$. Let $0 = y_0 < y_1 < \ldots < y_r = 1$ be an arbitrary partition of I. By the Perron-Frobenius operator we have

$$\begin{aligned}
\sum_{j=1}^{r} |P_\phi f(y_j) - P_\phi f(y_{j-1})| &= \sum_{j=1}^{r} \Big| \sum_{i=1}^{q} f(\psi_i(x_j))\sigma_i(x_j)\chi_{\phi(I_i)}(x_j) \\
&\qquad - \sum_{i=1}^{q} f(\psi_i(x_{j-1}))\sigma_i(x_{j-1})\chi_{\phi(I_i)}(x_{j-1}) \Big| \\
&\leq \sum_{j=1}^{r} \sum_{i=1}^{q} \Big| f(\psi_i(x_j))\sigma_i(x_j)\chi_{\phi(I_i)}(x_j) \\
&\qquad - f(\psi_i(x_{j-1}))\sigma_i(x_{j-1})\chi_{\phi(I_i)}(x_{j-1}) \Big|.
\end{aligned} \tag{C.3}$$

We divide the sum in (C.3) into three parts:

(1) A part for which we have $\chi_{\phi(I_i)}(x_j) = \chi_{\phi(I_i)}(x_{j-1}) = 1$.

(2) A part for which we have $\chi_{\phi(I_i)}(x_j) = 1$ and $\chi_{\phi(I_i)}(x_{j-1}) = 0$.

(3) A part for which we have $\chi_{\phi(I_i)}(x_j) = 0$ and $\chi_{\phi(I_i)}(x_{j-1}) = 1$.

For the first part we have the inequality

$$\sum_{j=1}^{r} \sum_{i=1}^{q} \Big| f(\psi_i(x_j))\sigma_i(x_j) - f(\psi_i(x_{j-1}))\sigma_i(x_{j-1}) \Big| \leq \sum_{i=1}^{q} \operatorname*{Var}_{J_i} f \circ \psi_i \cdot \sigma_i. \tag{C.4}$$

The second and third part occur if two points are on opposite sides of an endpoint of $\chi_{\phi(I_i)}$. For each I_i , the second part happens for at most one pair x_j, x_{j-1} and the third part happens for at most one other pair $x_j^{'}, x_{j-1}^{'}$. Hence for the second and third part we have

$$\begin{aligned}\sum_{j=1}^{r} \sum_{i=1}^{q} \Big| f(\psi_i(x_j))\sigma_i(x_j) - \sum_{i=1}^{q} f(\psi_i(x_{j-1}))\sigma_i(x_{j-1}) \Big| \leq \sum_{i=1}^{q} \Big(&|f(\psi_i(x_j))\sigma_i(x_j)| \\ &+ |f(\psi_i(x_{j-1}^{'}))\sigma_i(x_{j-1}^{'})| \Big).\end{aligned} \tag{C.5}$$

Combining (C.4) and (C.5) we obtain

$$\operatorname*{Var}_{I} P_\phi f \leq \sum_{i=1}^{q} \operatorname*{Var}_{J_i} f \circ \psi_i \cdot \sigma_i + \sum_{i=1}^{q} \Big(|f(\psi_i(x_j))\sigma_i(x_j)| + |f(\psi_i(x_{j-1}))\sigma_i(x_{j-1})| \Big).$$

For the right hand side in (C.4) we get by Proposition B.0.4, the product rule and the triangle inequality

$$\begin{aligned}\operatorname*{Var}_{J_i} f \circ \psi_i \cdot \sigma_i &= \int_{J_i} \Big| [f(\psi_i(x))\sigma_i(x)]' \Big| dx \\ &= \int_{J_i} |(f \circ \psi_i)'(x)\sigma_i(x) + (f \circ \psi_i)(x)\sigma_i^{'}(x)| dx \\ &\leq \int_{J_i} |(f \circ \psi_i)'(x)\sigma_i(x)| dx + \int_{J_i} |(f \circ \psi_i)(x)\sigma_i^{'}(x)| dx.\end{aligned}$$

Let $K = \max \sigma_i^{'} / \min \sigma_i$, then using the chain rule and (C.2) we obtain

$$\operatorname*{Var}_{J_i} f \circ \psi_i \cdot \sigma_i \leq s^{-N} \int_{J_i} |(f' \circ \psi_i)(x)\psi_i^{'}(x)| dx + K \int_{J_i} |(f \circ \psi_i)(x)\sigma_i(x)| dx.$$

Rewriting the equation we obtain

$$\begin{aligned}\operatorname*{Var}_{J_i} f \circ \psi_i \cdot \sigma_i \leq s^{-N} &\int_{\phi_i([b_{i-1}, b_i])} |(f'(\phi_i^{-1}(x))(\phi_i^{-1})'(x)| dx \\ &+ K \int_{\phi_i([b_{i-1}, b_i])} |(f(\phi_i^{-1}(x))|(\phi_i^{-1})'(x)| dx.\end{aligned}$$

Using integration by substitution we obtain

$$\operatorname*{Var}_{J_i} f \circ \psi_i \cdot \sigma_i \leq s^{-N} \int_{b_{i-1}}^{b_i} |f'(x)|dx + K \int_{b_{i-1}}^{b_i} |f(x)|dx.$$

Using Proposition B.0.4 and taking the summation we finally obtain

$$\sum_{i=1}^{q} \operatorname*{Var}_{J_i} f \circ \psi_i \cdot \sigma_i \leq s^{-N} \operatorname*{Var}_{I} f + K \int_I |f(x)|dx. \tag{C.6}$$

For the right hand side in (C.5) we obtain

$$\sum_{i=1}^{q} \Big(|f(\psi_i(x_j))\sigma_i(x_j)| + |f(\psi_i(x_{j-1}))\sigma_i(x_{j-1})| \Big) \leq s^{-N} \sum_{i=1}^{q} (|f(b_{i-1})| + |f(b_i)|).$$

Let $c_i = \text{argmin } \{|f(x)| : x \in [b_{i-1}, b_i]\}$, then

$$\begin{aligned} |f(b_{i-1})| + |f(b_i)| &= |f(b_{i-1}) - f(c_i) + f(c_i)| + |f(b_i) - f(c_i) + f(c_i)| \\ &\leq |f(b_{i-1}) - f(c_i)| + |f(b_i) - f(c_i)| + 2|f(c_i)| \\ &\leq \operatorname*{Var}_{[b_{i-1}, b_i]} f + 2|f(c_i)|. \end{aligned}$$

For $|f(c_i)|$ we have the inequality

$$|f(c_i)| \leq \frac{1}{b_i - b_{i-1}} \int_{b_{i-1}}^{b_i} |f(x)|dx.$$

Let $h = \min_i (b_i - b_{i-1})$, then we obtain

$$\begin{aligned} s^{-N} \sum_{i=1}^{q} (|f(b_{i-1})| + |f(b_i)|) s^{-N} &\leq \sum_{i=1}^{q} \Big(\operatorname*{Var}_{[b_{i-1}, b_i]} f + 2|f(c_i)| \Big) \\ &\leq s^{-N} \sum_{i=1}^{q} \Big(\operatorname*{Var}_{[b_{i-1}, b_i]} f + 2h^{-1} \int_{b_{i-1}}^{b_i} |f(x)|dx \Big) \\ &\leq s^{-N} \operatorname*{Var}_{I} f + 2h^{-1} s^{-N} \int_I |f(x)|dx. \end{aligned} \tag{C.7}$$

Combining (C.4), (C.5), (C.6) and (C.7) we obtain

$$\begin{aligned} \operatorname*{Var}_{I} P_\phi f &\leq s^{-N} \operatorname*{Var}_{I} f + K \int_I |f(x)|dx + s^{-N} \operatorname*{Var}_{I} f + 2h^{-1} s^{-N} \int_I |f(x)|dx \\ &\leq 2s^{-N} \operatorname*{Var}_{I} f + (K + 2h^{-1})||f||. \end{aligned}$$

$\square$

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