On hypercover-Cech cohomology and dualizing complexes of finite spaces

Baak, K.S.

K. S. Baak

On Hypercover-Čech Cohomology and Dualizing Complexes of Finite Spaces

Master thesis

November 4, 2019

Thesis supervisor:
prof.dr. S.J. Edixhoven

Leiden University
Mathematical Institute

List of Names of Categories

Let $\mathcal{C}$ be a category, $\mathcal{A}$ an abelian category, X a topological space and R a commutative ring. We fix the following notation for some categories that will occur frequently throughout this thesis.

Δ	The simplicial indexing category
Ab	Abelian groups
$Ab(X)$	Abelian sheaves on X
$\mathbf{C}(\mathcal{A})$	(Cochain) Complexes in $\mathcal{A}$
$\mathbf{C}_{\geq 0}(\mathcal{A})$	(Cochain) Complexes in $\mathcal{A}$ that are zero in negative degree
$\mathbf{C}(X, R)$	(Cochain) Complexes of sheaves of R-modules on X
$\mathrm{cSimp}(\mathcal{C})$	Cosimplicial objects in $\mathcal{C}$
$\mathrm{cSh}_R(X)$	Cosheaves of R-modules on X
$\mathbf{D}(\mathcal{A})$	Derived category of $\mathcal{A}$
$\mathbf{D}^+(\mathcal{A})$	Bounded below derived category of $\mathcal{A}$
$\mathbf{D}(X, R)$	Derived category of $\mathrm{Sh}_R(X)$
$\mathbf{D}^+(X, R)$	Bounded below derived category of $\mathrm{Sh}_R(X)$
FTop	Finite topological spaces
$\mathrm{HC}(X)$	Hypercoverings on X with refinement maps as arrows
Mod_R	R-modules
Modfg_R	Finitely generated R-modules
$\mathbf{O}(X)$	Open subsets of X with inclusions as arrows
Pos	Partially ordered sets
$\mathrm{PSh}_R(X)$	Presheaves of R-modules on X
$\mathrm{Simp}(\mathcal{C})$	Simplicial objects in $\mathcal{C}$
$\mathrm{Sh}_R(X)$	Sheaves of R-modules on X
Top	Topological Spaces

Contents

Introduction

In geometry one is often interested in calculating the cohomology groups $H^n(X, \mathcal{F})$ of a pair $(X, \mathcal{F})$ where X is a topological space and $\mathcal{F}$ a sheaf (of abelian groups, R-modules, k-algebras, etc.) on X. The usual cohomology theory of such pairs is called *sheaf cohomology* or *Grothendieck style (sheaf) cohomology*. Sheaf cohomology is defined by taking injective resolutions of sheaves and although one can prove that every sheaf indeed has an injective resolution, finding these resolutions is often hard. Therefore, the definition of sheaf cohomology does not provide a practical way to actually calculate the cohomology groups. Another cohomology theory, *Čech cohomology*, exists and the corresponding Čech cohomology groups $\check{H}^n(X, \mathcal{F})$ are much easier to compute. Luckily, in special cases, the sheaf cohomology and Čech cohomology groups agree. For example, Roger Godement proved that if X is a paracompact Hausdorff space, then sheaf cohomology and Čech cohomology are the same for abelian sheaves. Also in the case of separated k-schemes (for k a field) and quasi-coherent $\mathcal{O}_X$-modules, the Čech cohomology with respect to an open cover of spectra of finitely generated k-algebras agrees with sheaf cohomology. Nevertheless, Čech cohomology is not always powerful enough to calculate sheaf cohomology. Even for fairly simple spaces it does not always work. We consider an example due to Alexander Grothendieck which can be found in his famous Tohoku paper (see example 3.8.3 of [11]).

We define a finite model for the unit disk as follows. Let $\boldsymbol{D}$ be the finite set $\{W, N, E, S, C\}$ and define the partial order $\leq$ on $\boldsymbol{D}$ in the way illustrated in figure 1. That is, we have $x \leq y$ if and only if $x = y$ or there is an arrow $y \to x$ in figure 1. We endow $\boldsymbol{D}$ with the topology defined by the upper sets of $\leq$. The open sets of $\boldsymbol{D}$ are

$$\boldsymbol{D},\ \emptyset,\ \{W, N, S, C\},\ \{E, N, S, C\},\ \{N, C\},\ \{S, C\},\ \{N, S, C\} \text{ and } \{C\}.$$

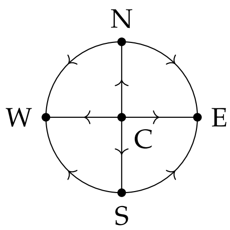

Figure 1: A finite model of the unit disk.

We define the set U_x for $x \in \boldsymbol{D}$ to be the smallest open subset of $\boldsymbol{D}$ containing x and for the ease of notation also write $U = U_C$. Let $j \colon U \hookrightarrow \boldsymbol{D}$ be the inclusion. We extend the sheaf $\mathbb{Z}_U$ given by $U \mapsto \mathbb{Z}$ by zero to get the sheaf $j_!\mathbb{Z}_U$ on $\boldsymbol{D}$. We will now calculate the sheaf cohomology and Čech cohomology of $(\boldsymbol{D}, j_!\mathbb{Z}_U)$. In particular, we will see that sheaf cohomology and Čech cohomology do not agree in degree 2.

We start by calculating the sheaf cohomology. To this end, we first present the following lemma.

Lemma 0.1. *Let $X = \{x, y\}$ with the discrete topology and A an abelian group. We have $H^0(X, A_X) = A \oplus A$ and $H^n(X, A_X) = 0$ for $n > 0$.*

Proof. Recall that for any topological space Y, any open subset $V \subseteq Y$ and any abelian group G, the set of sections $\Gamma(V, G_Y)$ is (isomorphic to) the set of continuous functions $V \to G$, where G is given the discrete topology. As X has two connected components, we see that $H^0(X, A_X) \cong \Gamma(X, A_X) \cong A \oplus A$. Let $U = \{x\}$ and $V = \{y\}$. Then we get a

Mayer-Vietoris sequence

$$\begin{aligned} 0 \longrightarrow A_X(A) &\longrightarrow A_X(U) \oplus A_X(V) \longrightarrow A_X(U \cap V) \\ \longrightarrow H^1(X, A_X) &\longrightarrow H^1(U, A_X|_U) \oplus H^1(V, A_X|_V) \longrightarrow H^1(U \cap V, A_X|_{U \cap V}) \longrightarrow \cdots . \end{aligned}$$

As $U \cap V = \emptyset$ we conclude that we have isomorphisms

$$H^n(X, A_X) \cong H^n(U, A_U) \oplus H^n(U, A_V) \cong 0 \oplus 0 \cong 0$$

for all $n \geq 1$. □

Theorem 0.2. *We have*

$$H^n(\boldsymbol{D}, j_! \mathbb{Z}_U) = \begin{cases} \mathbb{Z} & n = 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. First note that $H^0(\boldsymbol{D}, j_! \mathbb{Z}_U) \cong j_! \mathbb{Z}_U(\boldsymbol{D}) = 0$. Now define $Y = \boldsymbol{D} \setminus U$ and let $i : Y \hookrightarrow \boldsymbol{D}$ be the inclusion. We get a short exact sequence of sheaves on $\boldsymbol{D}$

$$0 \longrightarrow j_! \mathbb{Z}_U \longrightarrow \mathbb{Z}_{\boldsymbol{D}} \longrightarrow i_* \mathbb{Z}_Y \longrightarrow 0. \tag{0.1}$$

As constant sheaves on irreducible spaces are flasque, the long exact sequence of cohomology groups associated to (0.1) gives isomorphisms $H^n(\boldsymbol{D}, i_* \mathbb{Z}_Y) \cong H^{n+1}(\boldsymbol{D}, j_! \mathbb{Z}_U)$ for $n \geq 2$. Also, we have an exact sequence

$$0 \longrightarrow j_! \mathbb{Z}_U(\boldsymbol{D}) \longrightarrow \mathbb{Z}_{\boldsymbol{D}}(\boldsymbol{D}) \longrightarrow i_* \mathbb{Z}_Y(\boldsymbol{D}) \longrightarrow H^1(\boldsymbol{D}, j_! \mathbb{Z}_U) \longrightarrow 0.$$

A short inspection yields $H^1(\boldsymbol{D}, j_1 \mathbb{Z}_U) = 0$. We now consider $n \geq 2$. Since Y is closed, we have an isomorphism $H^n(Y, \mathbb{Z}_Y) \cong H^n(\boldsymbol{D}, i_* \mathbb{Z}_Y)$ for all $n \geq 0$. Now define the subsets $Y_1 = \{W, N, S\} \subseteq Y$ and $Y_2 = \{E, N, S\} \subseteq Y$. We have a Mayer-Vietoris sequence for sheaf cohomology,

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}_Y(Y) &\longrightarrow \mathbb{Z}_Y(Y_1) \oplus \mathbb{Z}_Y(Y_2) \longrightarrow \mathbb{Z}_Y(Y_{12}) \\ \longrightarrow H^1(Y, \mathbb{Z}_Y) &\longrightarrow H^1(Y_1, \mathbb{Z}_Y|_{Y_1}) \oplus H^1(Y_2, \mathbb{Z}_Y|_{Y_2}) \longrightarrow H^1(Y_{12}, \mathbb{Z}_Y|_{Y_{12}}) \longrightarrow \ldots , \end{aligned}$$

where $Y_{12} = Y_1 \cap Y_2$. Recall that for any topological space Z, any open subset $V \subseteq Z$ and any abelian group A, the set of sections $\Gamma(V, A_Z)$ is (isomorphic to) the set of continuous functions $V \to A$, where A is given the discrete topology. Since Y_i is irreducible (and thus connected) we find that $\mathbb{Z}_Y(Y_i) \cong \mathbb{Z}$. Moreover, irreducibility of Y_i gives that $\mathbb{Z}_Y|_{Y_i} = \mathbb{Z}_{Y_i}$ is flasque and thus $H^n(Y_i, \mathbb{Z}_Y|_{Y_i}) = 0$ for all $n > 0$. Furthermore, by lemma 0.1 we have $\mathbb{Z}_Y(Y_1 \cap Y_2) = \mathbb{Z} \oplus \mathbb{Z}$ and $H^n(Y_{12}, \mathbb{Z}_Y|_{Y_{12}}) = 0$ for $n > 0$. Combining all this, we find that

$$H^n(\boldsymbol{D}, j_! \mathbb{Z}_U) \cong H^{n-1}(\boldsymbol{D}, i_* \mathbb{Z}_Y) \cong H^{n-1}(Y, \mathbb{Z}_Y) \cong 0$$

for all $n > 2$ and an exact sequence

$$0 \to \mathbb{Z}_Y(Y) \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \oplus \mathbb{Z} \to H^1(Y, \mathbb{Z}_Y) \to 0.$$

The map $\alpha \colon \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is given by $(n, m) \mapsto (n, n) - (m, m) = (n - m, n - m)$. We conclude

$$H^2(\boldsymbol{D}, j_! \mathbb{Z}_U) \cong H^1(\boldsymbol{D}, i_* \mathbb{Z}_Y) \cong H^1(Y, \mathbb{Z}_Y) \cong \operatorname{coker} \alpha = (\mathbb{Z} \oplus \mathbb{Z}) / \mathbb{Z} \cong \mathbb{Z}.$$

□

We will now calculate the Čech cohomology of $(\boldsymbol{D}, j_!\mathbb{Z}_U)$. Consider the cover $\mathcal{U} = \{U_W, U_E\}$. It is easily seen that $\mathcal{U}$ is a refinement of every possible open cover of $\boldsymbol{D}$. Therefore we have an isomorphism $\check{H}^n(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \check{H}^n(\mathcal{U}, j_!\mathbb{Z}_U)$ for every $n \geq 0$. As the cover $\mathcal{U}$ only has two elements, all information is contained in the map

$$\delta\colon j_!\mathbb{Z}_U(U_1) \times j_!\mathbb{Z}_U(U_2) \longrightarrow j_!\mathbb{Z}_U(U_1 \cap U_2),$$

which is just the map $0 \to 0$. We conclude $\check{H}^2(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \check{H}^2(\mathcal{U}, j_!\mathbb{Z}_U) = 0$.

A solution to this problem does exist. When doing Čech cohomology one can work with *hypercoverings* instead of ordinary open covers. The idea is that instead of taking care of all $(n+1)$-fold intersections of a chosen open cover in degree n of the Čech complex, one can actually take open covers of these intersections. For example,in the case of the finite model of the unit disk, the cover $\{U_W, U_E\}$ computes the Čech cohomology. In degree 1 the (restricted) Čech complex looks like $\mathcal{F}(U_W \cap U_E)$ and in higher degrees it is zero. When working with hypercoverings one can cover $U_W \cap U_E$ by some other opens, for example $\{U_N, U_S\}$ and moreover, one should actually do this to obtain a finest hypercover. As a consequence the complex is no longer zero in degree 2, as for example $U_N \cap U_S \cap U_W$ is considered there. In the first section of this thesis, we will define hypercoverings and the corresponding *hyper(cover)-Čech cohomology*. Although hyper-Čech cohomology always coincides with sheaf cohomology, as proved by Verdier, we will see that the hyper-Čech complex grows very big very fast and is therefore painful to work with when actually calculating cohomology groups.

Although hypercoverings do not provide the solution that might have been hoped for, we will see that sheaf cohomology groups can be calculated using the *Godement resolution*. The second part of this thesis focuses exclusively on finite topological spaces and we will introduce the Godement resolution in section 2.2. The Godement resolution features in a *Grothendieck Duality Theorem* for finite spaces. This Duality Theorem and the *dualizing complexes* it produces, will be the topics of section 2 and 3 respectively. The first duality theorem for sheaf cohomology is the *Serre Duality Theorem*. This classical result states that for a projective scheme X of dimension n over a field k, there exists a coherent sheaf ω_X on X such that for any coherent sheaf $\mathcal{F}$ on X it holds that $H^n(X, \mathcal{F})^\vee \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \omega_X)$. Grothendieck generalized Serre Duality to a statement about proper morphisms of schemes. This *Coherent Duality*, sometimes called *Grothendieck Duality* or *Serre-Grothendeick-Verdier Duality*, states in the language of derived categories that for a proper morphism $f\colon X \to Y$ of schemes the functor $\boldsymbol{R}f_*$ has a right adjoint. The Duality Theorem for finite spaces that will be studied in this thesis states that for any morphism $f\colon X \to Y$ of finite spaces, the functor $\boldsymbol{R}f_*$ has a right adjoint. The theorem was first proved by González, where it had the following form.

Theorem 0.3 (González, [15]). *Let $f\colon X \to S$ be a continuous map of finite topological spaces. Let $\mathcal{I}^\bullet$ be an injective resolution of the sheaf $\mathbb{Z}_S$ on S. There exists a complex of abelian sheaves $\mathcal{D}_f^\bullet$ on X such that for any complex $\mathcal{H}^\bullet$ of abelian sheaves on X we have an isomorphism of complexes*

$$\mathrm{Hom}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet), \mathcal{I}^\bullet) \cong \mathrm{Hom}^\bullet(\mathcal{H}^\bullet, \mathcal{D}_f^\bullet)$$

that is functorial in $\mathcal{H}^\bullet$.

Here $\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$ is the singly graded complex associated to the double complex $G^\bullet(\mathcal{H}^\bullet)$ and G^p is the p-th Godement functor $Ab(X) \to Ab(X)$. We will actually prove a slightly more general version, where the role of $\mathbb{Z}_S$ is replaced by $\mathcal{M}$ for any sheaf of R-modules $\mathcal{M}$ on S (where R is a commutative ring). Instead of $\mathcal{D}_f^\bullet$ we write $f^!\mathcal{M}^\bullet$. In section 2.5 we will show that the theorem can actually be extended to locally finite topological spaces.

If f is the map to the singleton, the complex $f^!\mathcal{M}^\bullet$ is called the *dualizing compex* of X for $\mathcal{M}$. In section 3.1 we will prove that in the context of a commutative triangle of finite spaces

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

the complex $(gf)^!\mathcal{M}^\bullet$ is quasi-isomorphic to $g^!f^!\mathcal{M}^\bullet$ for any sheaf $\mathcal{M}$ on X. In section 3.2 we will give an explicit description of the dualizing complexes of finite spaces. Finally, we use the explicit description of the dualizing complex found in section 3.2 to make a connection between sheaf cohomology and cosheaf homology for finite spaces.

1 Hypercoverings

In this section we introduce hypercoverings and hypercover-Čech cohomology. The *Verdier Hypercovering Theorem*, which we will not prove in this thesis, but which can be found in section 24.10 of [18], states that hypercover-Čech cohomology always coincides with sheaf cohomology. Despite this wanted property, hyper-Čech cohomology is actually not very practical to work with. Even in the case of the finite model of the unit disk $\boldsymbol{D}$ discussed in the introduction, the hyper-Čech complex grows quite big, which makes it hard to calculate the cohomology groups. The main reference for this section is chapter 24 of the Stacks Project [18].

1.1 Simplicial Objects

Starting with an open cover $\mathcal{U}_0$ of a topological space X, a hypercovering of X covers every n-fold intersection of opens in $\mathcal{U}_{n-1}$ and stores all opens used in a collection $\mathcal{U}_n$. Of course, a sequence of collections of opens $(\mathcal{U}_n)_{n\geq 0}$ does not contain enough information of itself. Given an open $U \in \mathcal{U}_n$ for some $n \geq 0$ we would like to know which $(n-1)$-fold intersection of opens in $\mathcal{U}_{n-1}$ it partly covers. We need some form of bookkeeping that allows us to store all necessary information. It turns out that the notion of a *simplicial set* can be used to index a hypercovering. Therefore, before we define the actual hypercoverings, it is important to have a good understanding of (co)simplicial objects in a category.

A simplicial object in a category is a collection of objects $(A_i)_{i\geq 0}$ together with *face maps* $A_i \to A_{i-1}$ and *degeneracy maps* $A_i \to A_{i+1}$ satisfying some properties. We will give a more precise definition.

Definition 1.1. We define the *simplicial indexing category* Δ, whose objects are the sets $[n] = \{0, ..., n\}$ $(n \geq 0)$ and arrows are order preserving maps.

We will now define two classes of important arrows in Δ explicitly.

Definition 1.2. Let $n \geq 1$ and $i \in [n]$. We define the arrow $\delta_i^n \colon [n-1] \to [n]$ as the order preserving map skipping i. Let $n \geq 0$ and $i \in [n]$. We define the arrow $\sigma_i^n \colon [n+1] \to [n]$ as the order preserving map hitting i twice.

For the ease of notation, the superscripts in these maps are often omitted. The following result confirms the importance of these maps.

Proposition 1.3. *Any arrow in Δ is a composition of maps of the form δ_i^n and σ_j^n. More precisely, any arrow $\alpha : [n] \to [m]$ in Δ can be uniquely written as*

$$\alpha = \delta_{i_k} \cdots \delta_{i_1} \sigma_{j_1} \cdots \sigma_{j_\ell}$$

with $0 \leq i_1 < \cdots < i_k \leq m$ and $0 \leq j_1 < \cdots < j_\ell < n$.

Proof. Let $0 \leq i_1 < \cdots < i_k \leq m$ be the elements that are not in the image of α. Then $\alpha = \delta_{i_k} \cdots \delta_{i_1} \beta$ where $\beta : [n] \to [m']$ is surjective and order preserving. Now consider all elements in $[m']$ with more then one element in its preimage. As β is order preserving, this is the same as considering all elements $0 \leq j_1 < \cdots < j_\ell < n$ in $[n]$ that satisfy $\beta(j) = \beta(j+1)$. We have $\beta = \sigma_{j_1} \cdots \sigma_{j_\ell}$ and conclude that indeed

$$\alpha = \delta_{i_k} \cdots \delta_{i_1} \sigma_{j_1} \cdots \sigma_{j_\ell}.$$

This factorization is clearly unique. □

Definition 1.4. A *simplicial object* in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$.

Example 1.5. (i) Let X be an object of a category $\mathcal{C}$. We have the constant simplicial object $[n] \mapsto X$ for all $n \geq 0$ and every morphism in Δ is mapped to the identity id_X.

(ii) One uses a simplicial set when defining singular homology of a topological space. For any $n \geq 0$ we define the *topological n-simplex*

$$\Delta^n = \left\{ \boldsymbol{x} \in \mathbb{R}^{n+1}_{\geq 0} : \sum x_i = 1 \right\} \subseteq \mathbb{R}^{n+1}.$$

For any morphism $\alpha \colon [n] \to [m]$ in Δ, we define $\alpha_* \colon \Delta^n \to \Delta^m$ by $\boldsymbol{x} \mapsto (\sum_{\alpha(i)=j} x_i)_j$. Let X be any topological space. For any $n \geq 0$ we define the set $\mathcal{S}(X)_n$ of continuous functions $\Delta^n \to X$. For any order-preserving map $\alpha \colon [m] \to [n]$ we define the map $\mathcal{S}(X)_n \to \mathcal{S}(X)_m$ by $\sigma \mapsto \sigma \circ \alpha_*$. The set $\mathcal{S}(X)_n$ is called the *singular n-simplex* and $\mathcal{S}(X)$ is an example of a simplicial set.

(iii) Let $f \colon X \to Y$ be a morphism in a category $\mathcal{C}$ and suppose that for any $n \geq 0$ the n-fold fibred product of X along f exists. We define K_n to be the $(n+1)$-fold fibred product and for any order preserving map $\alpha \colon [m] \to [n]$. We define $K(\alpha) = \pi_{\alpha(0)} \times_Y \cdots \times_Y \pi_{\alpha(n)}$, where the π_i are the projections. This makes K a simplicial object in $\mathcal{C}$.

(iv) Contravariant representable functors $\Delta \to \mathrm{Set}$ are simplicial sets. We will spell this out. Let N be a natural number. We define the simplicial set $\Delta[N]$ as follows. For any $n \geq 0$ we set $\Delta[N]_n = \mathrm{Hom}_\Delta([n],[N])$ and for any order preserving map $\alpha \colon [n] \to [m]$ we get an induced map $\alpha^* \colon \Delta[N]_m \to \Delta[N]_n$ given by $f \mapsto f \circ \alpha$.

(v) As a generalization of the previous example, let P be a partially ordered set, then the functor $\mathrm{Hom}_{\mathrm{Pos}}(-,P) \colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is a simplicial set. Here Pos denotes the category with partially ordered sets as objects and order preserving maps as arrows. For $\alpha \colon [n] \to [m]$ order preserving we get $\alpha^* \colon \mathrm{Hom}_{\mathrm{Pos}}([m],P) \to \mathrm{Hom}_{\mathrm{Pos}}([n],P)$ given by $f \mapsto f \circ \alpha$. This simplicial set is called the *nerve* of P.

For any simplicial object A in $\mathcal{C}$, we will write $d_i^n = A(\delta_i^n)$ and $s_i^n = A(\sigma_i^n)$. These maps are called *face maps* and *degeneracy maps*. If it improves readability we will sometimes abuse notation and drop the superscripts of the face and degeneracy maps. If the objects of $\mathcal{C}$ are sets (with possibly some extra structure) we call the elements of A_n *n-simplices*. Elements in the image of a degeneracy map are called *degenerate elements* and elements in the image of a face map are called *faces*. By proposition 1.3 any $A(\alpha)$ for α an arrow in Δ can be written as a composition of face and degeneracy maps. We actually have the following equivalent definition of a simplicial object.

Proposition 1.6. *The notion of a simplicial object A in a category $\mathcal{C}$ is equivalent to a sequence of objects $(A_n)_{n\geq 0}$ of $\mathcal{C}$ together with maps $d_i^n \colon A_n \to A_{n-1}$ and $s_i^n \colon A_n \to A_{n+1}$ for all $i \in [n]$ satisfying the equations*

(i) $d_i \circ d_j = d_{j-1} \circ d_i$ *if* $i < j$,

(ii) $s_i \circ s_j = s_{j+1} \circ s_i$ *if* $i \leq j$,

(iii)

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & i < j \\ \mathrm{id} & i = j, i = j+1 \\ s_j \circ d_{i-1} & i > j+1, \end{cases}$$

whenever they make sense.

Proof. Omitted. □

The definition of the dual notion of a simplicial object is as expected.

Definition 1.7. A *cosimplicial object* in a category $\mathcal{C}$ is a functor $\Delta \to \mathcal{C}$.

Example 1.8. (i) The identity functor $\Delta \to \Delta$ is a cosimplicial object in Δ.

(ii) The functor sending $[n]$ to the topological n-simplex defined in example 1.5.ii is an example of a cosimplicial topological space. This is called the *geometric realization* of Δ.

For a cosimplicial object A in $\mathcal{C}$ we write $\delta_i^n = A(\delta_i^n)$ and $\sigma_i^n = A(\sigma_i^n)$. The context will usually prevent the apparent confusion. Unsurprisingly, these maps are called the *coface maps* and *codegeneracy maps* respectively. In the light of proposition 1.6, we have the following alternative way of thinking about cosimplicial objects.

Proposition 1.9. *The notion of a cosimplicial object A in a category $\mathcal{C}$ is equivalent to a sequence of objects $(A_n)_{n\geq 0}$ of $\mathcal{C}$ together with maps $\delta_i^n\colon A_{n-1} \to A_n$ and $\sigma_i^n\colon A_{n+1} \to A_n$ for all $i \in [n]$ satisfying the equations*

(i) $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$ *if* $i < j$,

(ii) $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$ *if* $i \leq j$,

(iii)

$$\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & i < j \\ \mathrm{id} & i = j, i = j+1 \\ \delta_{i-1} \circ \sigma_j & i > j+1, \end{cases}$$

whenever they make sense.

Proof. Omitted. □

We have a natural notion of morphisms of (co)simplicial objects; they are just natural transformations. For any category $\mathcal{C}$ we get the category $\mathrm{Simp}(\mathcal{C})$ of simplicial objects in $\mathcal{C}$ and the category $\mathrm{cSimp}(\mathcal{C})$ of cosimplicial objects in $\mathcal{C}$.

We conclude this section by introducing some notions that are useful later on. First of all, the category of (co)simplicial objects in $\mathcal{C}$ allows products whenever $\mathcal{C}$ does.

Definition 1.10. Let $\mathcal{C}$ be a category with all binary products. Let A and B be (co)simplicial objects in $\mathcal{C}$. We define the *product* $A \times B$ of A and B to be the (co)simplicial object in $\mathcal{C}$ with $(A \times B)_n = A_n \times B_n$ and as (co)face and (co)degeneracy maps the products of the (co)face and (co)degeneracy maps of A and B.

Of course one should verify that the definition above indeed defines a (co)simplicial object. With the identities of proposition 1.6 and proposition 1.9 this is quite simple. Another important notion is the concept of truncated (co)simplicial objects. For any $n \geq 0$ we let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], \ldots, [n]$.

Definition 1.11. Let $n \geq 0$. An *n-truncated simplicial object* in a category $\mathcal{C}$ is a functor $\Delta_{\leq n}^{\mathrm{op}} \to \mathcal{C}$. An *$n$-truncated cosimplicial object* in $\mathcal{C}$ is a functor $\Delta_{\leq n} \to \mathcal{C}$.

Finally, we show how to build a cochain complex out of a cosimplicial object. Let $\mathcal{A}$ be an abelian category. Given a cosimplicial object A in $\mathcal{A}$, one can associate a cochain complex with A, usually called the *Moore complex*. For any $n \geq 0$ we define the differential maps $\partial^n\colon A_n \to A_{n+1}$ as follows

$$\partial^n = \sum_{i=0}^{n+1} (-1)^i \delta_i^{n+1}.$$

Proposition 1.12. *For all $n \geq 0$ we have $\partial^{n+1}\partial^n = 0$.*

Proof. Using property (i) from proposition 1.9, we find

$$\begin{aligned}
\partial^{n+1}\partial^n &= \sum_{k=0}^{n+2}\sum_{\ell=0}^{n+1}(-1)^{k+\ell}\delta_k\delta_\ell \\
&= \sum_{k=1}^{n+2}\sum_{\ell=0}^{k-1}(-1)^{k+\ell}\delta_k\delta_\ell + \sum_{k=0}^{n+1}\sum_{\ell=k}^{n+1}(-1)^{k+\ell}\delta_k\delta_\ell \\
&= \sum_{k=1}^{n+2}\sum_{\ell=0}^{k-1}(-1)^{k+\ell}\delta_\ell\delta_{k-1} + \sum_{k=0}^{n+1}\sum_{\ell=k}^{n+1}(-1)^{k+\ell}\delta_k\delta_\ell \\
&= \sum_{\ell=0}^{n+1}\sum_{k=\ell+1}^{n+2}(-1)^{k+\ell}\delta_\ell\delta_{k-1} + \sum_{k=0}^{n+1}\sum_{\ell=k}(-1)^{k+\ell}\delta_k\delta_\ell^{n+1} \\
&= -\sum_{\ell=0}^{n+1}\sum_{k=\ell}^{n+1}(-1)^{k+\ell}\delta_\ell\delta_k + \sum_{k=0}^{n+1}\sum_{\ell=k}^{n+1}(-1)^{k+\ell}\delta_k\delta_\ell \\
&= 0.
\end{aligned}$$

□

We conclude that the sequence $(A_n)_{n\geq 0}$ together with the differential maps ∂^n is indeed a cochain complex. The Moore complex of A is denoted by $\mathrm{s}(A)$. Note that a map $f\colon A \to B$ of cosimplicial objects in $\mathcal{A}$ is also naturally a map of cochain complexes $\mathrm{s}(A) \to \mathrm{s}(B)$, and we see that s is actually a functor

$$\mathrm{s}\colon \mathrm{cSimp}(\mathcal{A}) \longrightarrow \mathbf{C}_{\geq 0}(\mathcal{A}).$$

We will usually just write f instead of $\mathrm{s}(f)$.

1.2 Hypercoverings of Topological Spaces

The definition of a hypercovering is a bit technical in nature so we will start by giving a more intuitive explanation. In order to give a hypercovering on a topological space X, one start with an open cover $\mathcal{U}_0$ of X. In ordinary Čech theory one now defines the (unrestricted) Čech complex $\check{C}^\bullet(X,\mathcal{F})$ of an abelian sheaf $\mathcal{F}$ on X by

$$\check{C}^n(\mathcal{U}_0,\mathcal{F}) = \prod_{U_0,\ldots,U_n\in\mathcal{U}_0} \mathcal{F}(U_0\cap\ldots\cap U_n)$$

and the usual differential. The idea behind hypercoverings is to allow an open cover $\mathcal{V}_{(U_0,\ldots,U_n)}$ of the intersection $U_0\cap\ldots\cap U_n$ for any tuple $(U_0,\ldots,U_n)\in\mathcal{U}_0^{n+1}$. Moreover, instead of just intersections of opens in $\mathcal{U}_0$, we consider all intersections of opens occurring in some open cover in the hypercovering so far. We end up with a sequence $\{\mathcal{U}_n\}_{n\geq 0}$ of collections of open subsets of X. For any $U\in\mathcal{U}_n$ we would like to know which intersection of opens in $\mathcal{U}_{n-1}$ it covers. This bookkeeping role is played by the notion of a simplicial set, which was introduced in the previous section. We will now give the formal definition of a hypercovering.

Definition 1.13. A *hypercovering* on a topological space X is a simplicial set I together with a sequences of families $\mathcal{U} = (\mathcal{U}_n = \{U_i\}_{i\in I_n})_{n\geq 0}$ of open subsets of X such that

(H1) For all $n\geq 1$ and all $a\in[n]$ we have $U_i \subseteq U_{d_a^n(i)}$,

(H2) For all $n\geq 0$ and all $a\in[n]$ we have $U_i = U_{s_a^n(i)}$, (H3) The collection $\mathcal{U}_0$ is an open cover of X, that is, $X = \bigcup_{i \in I_0} U_i$,

(H4) For all $i, j \in I_0$ we have

$$U_i \cap U_j = \bigcup_{\substack{k \in I_1 \\ d_0^1(k) = i \\ d_1^1(k) = j}} U_k,$$

(H5) For all $n \geq 1$ and all $i_0, \ldots, i_{n+1} \in I_n$ such that $d_{b-1}^n(i_a) = d_a^n(i_b)$ for all $0 \leq a < b \leq n+1$ we have

$$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup_{\substack{k \in I_{n+1} \\ \forall a \in [n+1]\ d_a^{n+1}(k) = i_a}} U_k.$$

Note that we allow opens U_i to be the empty and that for different $i, j \in I_n$, we can have an equality $U_i = U_j$.

Set Theoretic Remark 1.14. For a given topological space X, let $\mathrm{HC}(X)$ denote the class of all hypercoverings on X. When defining absolute hyper-Čech cohomology groups later on, we take the colimit of a diagram $\mathrm{HC}(X) \to \mathrm{Ab}$, similarly as is done for ordinary Čech cohomology. For this to make sense, we would like $\mathrm{HC}(X)$ to be small, that is, a set. In the case of ordinary Čech cohomology this problem can be solved by only allowing certain indexing sets and showing that any open cover is equivalent to an open cover with an allowed indexing set (for details, see for example page 238 of [7]). In the context of hypercoverings, we can make $\mathrm{HC}(X)$ a set by requiring that all open covers occurring in (H3) till (H5) are open covers in this more restricted sense.

We will now give an example of a hypercovering on a topological space. Note that this example assures that every topological space has hypercoverings.

Example 1.15. Let X be a topological space and suppose that $\mathcal{U} = \{U_a\}_{a \in A}$ is an open cover of X. Endow the index set A with a partial order and let I denote the simplicial set $\mathrm{Hom}_{\mathrm{Pos}}(-, A)$, see example 1.5.vi. For any $n \geq 0$ and any $f \in I_n$ we define $U_f = \bigcap_{i \in [n]} U_{f(i)}$. Let $\mathcal{U}$ denote the collection $(\mathcal{U}_n = \{U_f\}_{f \in I_n})_{n \geq 0}$. We claim that $(I, \mathcal{U})$ is a hypercovering of X. In order to prove this, we simply need to check the conditions (H1)-(H5).

1. Let $n \geq 1$ and $a \in [n]$. We have

$$U_{d_a^n(f)} = U_{f \circ \delta_a^n} = \bigcap_{i \in [n-1]} U_{f(\delta_a^n(i))} = \bigcap_{\substack{i \in [n] \\ i \neq a}} U_{f(i)}$$

and surely $U_f = \bigcap_{i \in [n]} U_{f(i)}$ is contained in this set.

2. Let $n \geq 0$ and $a \in [n]$, we have

$$U_{s_a^n(f)} = U_{f \circ \sigma_a^n} = \bigcap_{i \in [n+1]} U_{f(\sigma_a^n(i))} = \bigcap_{i \in [n]} U_{f(i)} = U_f.$$

3. Note that for any $f \in I_0$ we have $U_f = U_{f(0)}$ and therefore $\mathcal{U}_0 = \mathcal{U}$, which is an open cover of X by assumption.

4. Let $f, g \in I_0$. We have

$$\bigcup_{\substack{h \in I_1 \\ d_0^1(h) = f \\ d_1^1(h) = g}} U_h = \bigcup_{\substack{h \in I_1 \\ h \circ \delta_0^n = f \\ h \circ \delta_1^n = g}} U_h = \bigcup_{\substack{h \in I_1 \\ h(1) = f(0) \\ h(0) = g(0)}} U_h = U_{f(0)} \cap U_{g(0)} = U_f \cap U_g.$$

5. Let $n \geq 1$ and $(f_0, ..., f_{n+1}) \in I_n^{n+2}$ such that $d_{b-1}^n(f_a) = d_a^n(f_b)$ for all $0 \leq a < b \leq n+1$. Let G be the set of all $g \in I_{n+1}$ such that $d_a^{n+1}(g) = f_a$ for all $a \in [n+1]$. For any $a \in [n+1]$ distinct from 0 and any $g \in G$ we have

$$g(a) = g(\delta_{a-1}^{n+1}(a-1)) = d_{a-1}^{n+1}(g)(a-1) = f_{a-1}(a-1).$$

Also $g(0) = g(\delta_1^{n+1}(0)) = d_1^{n+1}(g)(0) = f_1(0)$. We conclude that there exists only one unique g in G and moreover, $U_g = U_{f_1(0)} \cap U_{f_0(0)} \cap U_{f_1(1)} \cap \ldots \cap U_{f_n(n)}$. Let V be the set $\{f_x(x) : x \in [n]\} \cup \{f_1(0)\}$. Let $a \in [n+1]$. If $a > 1$, then we have

$$f_a(0) = f_a(\delta_1^n(0)) = d_1^n(f_a)(0) = d_{a-1}^n(f_1)(0) = f_1(\delta_{a-1}^n(0)) = f_1(0),$$

so we conclude that for all $a \in [n+1]$ we have $f_a(0) \in V$. Now suppose $0 < x < a$. We have

$$\begin{aligned} f_a(x) &= f_a(\delta_{x-1}^n(x-1)) = d_{x-1}^n(f_a)(x-1) = d_{a-1}^n(f_{x-1})(x-1) \\ &= f_{x-1}(\delta_{a-1}^n(x-1)) = f_{x-1}(x-1). \end{aligned}$$

Finally suppose $0 \leq a < x \leq n$, we have

$$f_a(x) = f_a(\delta_{x-1}^n(x-1)) = d_{x-1}^n(f_a)(x-1) = d_a^n(f_x)(x-1) = f_x(\delta_a^n(x-1)) = f_x(x).$$

We conclude that for all $a \in [n+1]$ and all $x \in [n]$ we have $f_a(x) \in V$. Therefore,

$$\bigcap_{a=0}^{n+1} U_{f_a} = \bigcap_{a=0}^{n+1} \bigcap_{x \in [n]} U_{f_a(x)} = \bigcap_{v \in V} U_v = U_g = \bigcup_{h \in G} U_h.$$

We now consider maps of hypercoverings.

Definition 1.16. Let X be a topological space and $K = (I, \mathcal{U})$ and $L = (J, \mathcal{V})$ hypercoverings of X. A *refinement map* $f\colon K \to L$ is a map of simplicial sets $f\colon I \to J$ such that for all $n \geq 0$ and all $i \in I_n$, we have $U_i \subseteq V_{f(i)}$. If such a refinement map exists, we say that K is a *refinement* of L.

From now on $\mathrm{HC}(X)$ denotes the category of hypercoverings on X with refinement maps as arrows. Note that $\mathrm{HC}(X)$ is small by remark 1.14. Just as with ordinary Čech theory, the concept of *finest* (hyper)covers will be useful, for the (hyper-)Čech cohomology groups of a finest (hyper)cover are equal to the absolute (hyper-)Čech cohomology groups. The following proposition makes it easier to recognize finest hypercoverings.

Lemma 1.17. *Let I be a simplicial set. Let $n \geq 0$ and $i \in I_n$. There exists a unique non-degenerate element $x \in I_m$ for some m and unique $0 \leq a_1 \leq \ldots \leq a_k \leq m$ such that*

$$i = s_{a_1} \ldots s_{a_k} x.$$

Proof. See proposition 4.8 of [6]. By using property ii of proposition 1.6 repeatedly, we can make sure that $0 \leq a_1 \leq \ldots \leq a_k \leq m$. □

Proposition 1.18. *Let X be a topological space and let $K = (I, \mathcal{U})$ be a hypercovering of X such that all open covers occurring in (H3), (H4) an (H5) are finest open covers. Then, K is a refinement of every hypercovering on X.*

Proof. Let $L = (J, \mathcal{V})$ be any other hypercovering on X. We will construct a refinement map $f\colon K \to L$ inductively. We first introduce some notation for the sake of readability. Let $n \geq 0$ and $\boldsymbol{i} \in I_n^{n+2}$, we set

$$A_{\boldsymbol{i}}^{n+1} = \{j \in I_{n+1} : d_a(j) = i_a \text{ for all } a \in [n+1]\}.$$

Similarly, for the simplicial set J we define the sets $B_{\boldsymbol{j}}^n$. As $\mathcal{U}_0$ is a finest open cover of X, there exists a map $f_0\colon I_0 \to J_0$ such that $U_i \subseteq V_{fi}$ for all $i \in I_0$. From this map f_0 we will inductively construct a map of simplicial sets $f\colon I \to J$ such that $U_i \subseteq V_{f_n i}$ for all $n \geq 0$ and all $i \in I_n$. We start by constructing f_1. Let $i \in I_1$, we distinguish two cases.

1. Suppose we have $i = s_0 k$ for some $k \in I_0$. We set $f_1 i = s_0 f_0 k$. Note that we have

$$U_i = U_{s_0 k} = U_k \subseteq V_{f_0 k} = V_{s_0 f_0 k} = V_{f_1 i},$$

so f_1 has the desired property on the degenerate elements.

2. Suppose that i is not of this form. Let $\boldsymbol{i} = (d_0 i,\ d_1 i)$. Note that we have

$$\bigcup_{k \in A_{\boldsymbol{i}}^1} U_k = U_{d_0 i} \cap U_{d_1 i} \subseteq V_{f_0 d_0 i} \cap V_{f_0 d_1 i} = \bigcup_{k \in B_{f\boldsymbol{i}}^1} V_k.$$

For any $k \in B_{f\boldsymbol{i}}^1$ we set $W_k = V_k \cap U_{d_0 1} \cap U_{d_1 i}$. Now $\{W_k\}_{k \in B_{f\boldsymbol{i}}^1}$ is an open cover of $U_{d_0 i} \cap U_{d_1 i}$ and by our assumption, we get a map $g\colon A_{\boldsymbol{i}}^1 \to B_{f\boldsymbol{i}}^1$ such that $U_k \subseteq W_{g(k)}$ for all $k \in A_{\boldsymbol{i}}^1$. Note that $i \in A_{\boldsymbol{i}}^1$ and we set $f_1 i = gi$. We have

$$U_i \subseteq W_{g(i)} \subseteq V_{g(i)} = V_{f_1 i},$$

so f_1 has the desired property on all non-degenerate elements.

The map f must be a map of simplicial sets and therefore f_1 must commute with all face and degeneracy maps. Note that commutativity with the degeneracy maps is by definition. We prove that f_1 commutes with the face map. We again distinguish between the degenerate and non-degenerate case.

1. Let $i = s_0 k$ for some $k \in I_0$. We find

$$d_a f_1 i = d_a f_1 s_0 k = d_a s_0 f_0 k = f_0 k = f_0 d_a i \quad \text{for all } a \in [1],$$

so f_1 commutes with all d_a for all degenerate elements.

2. Suppose that i is non-degenerate. We have $d_a f_1 i = f_0 d_a i$, as $f_1 i \in B_{f\boldsymbol{i}}^1$, so f_1 commutes with d_a for all $a \in [1]$.

Suppose that f is constructed up to f_{n-1}. We construct f_n.

1. Let $i \in I_n$ be degenerate and write $i = s_{a_1} \ldots s_{a_k} x$ in the unique form of lemma 1.17. We define $f_n i = s_{a_1} f_{n-1} y$, where $y = s_{a_2} \ldots s_{a_k} x$. It is easily checked that f_n satisfies the desired property. Now suppose $i \in I_{n-1}$ and let $c \in [n]$ and write $s_c i = s_{b_1} \ldots s_{b_{k+1}} x$ in the unique form of lemma 1.17. We have

$$\begin{aligned} s_c f_{n-1} i &= s_c f_n s_{a_1} \ldots s_{a_k} x = s_c s_{a_1} \ldots s_{a_k} f_m x = s_{b_1} \ldots s_{b_{k+1}} f_m x \\ &= f_n s_{b_1} \ldots s_{b_{k+1}} x = f_n s_c s_{a_1} \ldots s_{a_k} x = f_n s_c i. \end{aligned}$$

We conclude that f_n commutes with the degeneracy maps. We show that f_{n-1} commutes with the face maps. Let $b \in [n]$.

(a) Suppose $b < a$. Then

$$d_b f_n i = d_b s_a f_{n-1} k = s_{a-1} d_b f_{n-1} k = s_{a-1} f_{n-2} d_b k$$
$$= f_{n-1} s_{a-1} d_b k = f_{n-1} d_b s_a k = f_{n-1} d_b i.$$

(b) Suppose $b \in \{a, a+1\}$. Then,

$$d_b f_n i = d_b s_a f_{n-1} k = f_{n-1} k = f_{n-1} d_b s_a k = f_{n-1} d_b i.$$

(c) Suppose $b > a+1$. Then,

$$d_b f_n i = d_b s_a f_{n-1} k = s_a d_{b-1} f_{n-1} k = s_a f_{n-2} d_{b-1} k = f_{n-1} s_a d_{b-1} k = f_{n-1} d_b i.$$

2. Suppose that $i \in I_n$ is non-degenerate. Let $\boldsymbol{i} = (d_0 i, \dots, d_n i) \in I_{n-1}^{n+1}$. By the hypercovering property (H5) we have

$$\bigcup_{k \in A_{\boldsymbol{i}}^n} U_k = U_{d_0 i} \cap \dots \cap U_{d_n i} \subseteq V_{f_{n-1} d_0 i} \cap \dots \cap V_{f_{n-1} d_n i} = \bigcup_{k \in B_{f\boldsymbol{i}}^n} V_k.$$

Let $W = U_{d_0 i} \cap \dots \cap U_{d_n i}$. The collection $\{V_k \cap W\}_{k \in B_{f\boldsymbol{i}}^n}$ is an open cover of W and by assumption there exists a map $g \colon A_{\boldsymbol{i}}^n \to B_{f\boldsymbol{i}}^n$ such that $U_k \subseteq V_{gk}$ for all $k \in A_{\boldsymbol{i}}^n$. We define $f_n i = gi$. The fact that $f_n i \in B_{f\boldsymbol{i}}^n$ assures that f_n commutes with the face maps.

□

1.3 Constructing Hypercoverings

The example of a hypercovering given in the previous section is somewhat "primitive" in the sense that all open covers of property (H4) and (H5) are trivial covers. In this section we will give another way to construct hypercoverings. We do this inductively and therefore need the notion of a truncated hypercovering. Recall the definition of trancated simplicial objects 1.11.

Definition 1.19. Let X be a topological space. An *n-truncated hypercovering* on X is an n-truncated simplicial set I together with a collection of families $\mathcal{U} = (\mathcal{U}_k = \{U_i\}_{i \in I_k})_{0 \le k \le n}$ of open subsets of X such that the hypercovering properties (H1) till (H5) are satisfied whenever they make sense.

Let $K = (I, \mathcal{U})$ be an n-truncated hypercovering on a topological space X, we will show how to extend K to an $(n+1)$-truncated hypercovering of X without altering the sets I_k and the collections $\mathcal{U}_k$ for $k \le n$. By induction this provides a method of defining hypercoverings of X from truncated hypercoverings. First we define

$$I_{n+1}^* = \{(i_0, ..., i_{n+1}) \in I_n^{n+2} : d_{b-1}(i_a) = d_a(i_b) \text{ for all } 0 \le a < b \le n+1\}.$$

Note that this definition does not really make sense when $n = 0$, so in this case we just define $I_1^* = I_0^2$. For any $a \in [n+1]$ we define $d_a^{n+1} \colon I_{n+1}^* \to I_n$ by $d_a^{n+1}(\boldsymbol{i}) = i_a$. Moreover, for any $a \in [n]$ we define $s_a^n \colon I_n \to I_{n+1}^*$ by

$$i \longmapsto (s_{a-1} d_0 i,\ s_{a-1} d_1 i,\ ...,\ s_{a-1} d_{a-1} i,\ i,\ i,\ s_a d_{a+1} i,\ s_a d_{a+2} i,\ ..., s_a d_n i).$$

One easily checks that $s_a^n(\boldsymbol{i})$ is indeed an element of I_{n+1}^*. Also the sequence of sets $((I_k)_{0 \le k \le n}, I_{n+1}^*)$ is an $(n+1)$-truncated simplicial set with the face and degeneracy maps just defined. Now let $\boldsymbol{i} \in I_{n+1}^*$ be a degenerate element, then we define $J(\boldsymbol{i}) = \{\boldsymbol{i}\}$ and

$U_{\boldsymbol{i}} = U_i$ where $i \in I_n$ is such that $s_a i = \boldsymbol{i}$ for certain $a \in [n]$. Suppose that $\boldsymbol{i} \in I^*_{n+1}$ is not degenerate and choose an open cover

$$U_{i_0} \cap \ldots \cap U_{i_{n+1}} = \bigcup_{j \in J(\boldsymbol{i})} U_j.$$

We now set

$$I_{n+1} = \coprod_{\boldsymbol{i} \in I^*_{n+1}} J(\boldsymbol{i}).$$

Note that we can replace the codomain of the maps $s^n_a \colon I_n \to I^*_{n+1}$ by I_{n+1}. We also define $d^n_a \colon I_{n+1} \to I_n$ as the map $j \longmapsto d^{n+1}_a(\pi(j))$ where π is the canonical map $I_{n+1} \to I^*_{n+1}$. The sequence $(I_k)_{0 \leq k \leq n+1}$ together with these face and degeneracy maps is now an $(n+1)$-truncated simplicial set. Moreover, by construction we see that the hypercovering properties are satisfied whenever the make sense and we conclude that we have extended K to an $(n+1)$-truncated hypercovering of X.

1.4 Hyper-Čech Cohomology

Let X be a topological space, $K = (I, \mathcal{U})$ a hypercovering of X and $\mathcal{F}$ an abelian sheaf on X. In this subsection, we will define the hyper-Čech cohomology groups $\hat{H}^n(K, \mathcal{F})$ of $\mathcal{F}$ with respect to K and eventually the absolute hyper-Čech cohomology groups $\hat{H}^n(X, \mathcal{F})$. We start by defining abelian groups

$$\mathcal{F}(K)^n = \prod_{i \in I_n} \mathcal{F}(U_i)$$

for all $n \geq 0$. Moreover, for any $n \geq 1$ and any $a \in [n]$ we define the maps

$$\begin{aligned} \delta^n_a \colon \mathcal{F}(K)^{n-1} &\longrightarrow \mathcal{F}(K)^n \\ (\alpha_i)_{i \in I_{n-1}} &\longmapsto (\alpha_{d^n_a(j)}|_{U_j})_{j \in I_n}. \end{aligned}$$

For $n \geq 0$ and $a \in [n]$ we define

$$\begin{aligned} \sigma^n_a \colon \mathcal{F}(K)^{n+1} &\longrightarrow \mathcal{F}(K)^n \\ (\alpha_i)_{i \in I_{n+1}} &\longmapsto (\alpha_{s^n_a(j)})_{j \in I_n}. \end{aligned}$$

Note that these maps are well-defined since $\alpha_{d^n_a(j)} \in \mathcal{F}(U_{d^n_a(j)})$ and $U_j \subseteq U_{d^n_a(j)}$ by (H1). Also $\alpha_{s^n_a(j)} \in \mathcal{F}(U_{s^n_a(j)})$ and $U_{s^n_a(j)} = U_j$ by (H2). One can show that the collection of groups $(\mathcal{F}(K)^n)_{n \geq 0}$ together with these maps satisfy the conditions of proposition 1.9 and therefore forms a cosimplicial abelian group, which we will denote by $\mathcal{F}(K)$. Recalling the notion of a Moore complex of a cosimplicial abelian group, we can now define hyper-Čech cohomology.

Definition 1.20. Let X be a topological space, K a hypercovering of X and $\mathcal{F}$ an abelian sheaf on X, we define the *hyper-Čech cohomology groups* of $\mathcal{F}$ with respect to K as the cohomology groups of the Moore complex $\mathcal{F}(K)$, that is,

$$\hat{H}^n(K, \mathcal{F}) = h^n(\mathrm{s}(\mathcal{F}(K))).$$

As customary after defining a sheaf cohomology theory, we prove the following proposition.

Proposition 1.21. *Let X be a topological space, $K = (I, \mathcal{U})$ a hypercovering on X and $\mathcal{F}$ an abelian sheaf on X. We have*

$$\hat{H}^0(K, \mathcal{F}) \cong \mathcal{F}(X).$$

Proof. Note that we have

$$\hat{H}^0(K, \mathcal{F}) = \ker \partial^0 = \ker \left(\prod_{i \in I_0} \mathcal{F}(U_i) \xrightarrow{\delta_0^1 - \delta_1^1} \prod_{j \in I_1} \mathcal{F}(U_j) \right).$$

Let $i, j \in I_0$ and define $C_{ij} = \{k \in I_1 : d_0^1(k) = i,\ d_1^1(k) = j\}$. Note that by (H3) we have $U_i \cap U_j = \bigcup_{k \in C_{ij}} U_k$. Suppose $s = (s_i)_{i \in I_0} \in \ker \partial^0$. Then we have $s_{d_0^1(k)}|_{U_k} - s_{d_1^1(k)}|_{U_k} = 0$ for all $k \in I_1$. In particular we have $s_i|_{U_k} - s_j|_{U_k} = 0$ for all $k \in C_{ij}$. It follows that $s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j} = 0$. This holds for all $i, j \in I_0$ and we conclude $s \in \ker \varphi$, where φ is the map

$$\begin{aligned} \prod_{i \in i_0} \mathcal{F}(U_i) &\longrightarrow \prod_{i,j \in I_0} \mathcal{F}(U_i \cap U_j) \\ (s_i) &\longmapsto s_i|_{U_i \cap U_j} - s_j|_{U_i \cap U_j}. \end{aligned}$$

So we have $\ker \partial^0 \subseteq \ker \varphi$ and the other inclusion is obvious. By the sheaf property we have $\mathcal{F}(X) \cong \ker \varphi$ and hence

$$\hat{H}^0(K, \mathcal{F}) = \ker \partial^0 = \ker \varphi \cong \mathcal{F}(X).$$

□

The following propositions show that the ordinary Čech cohomology groups occur as hyper-Čech cohomology groups.

Proposition 1.22. *Let X be a topological space and $\mathcal{U}$ an open cover of X. Let K be the hypercovering of X constructed from $\mathcal{U}$ as in example 1.15 by choosing a total order on the index set of $\mathcal{U}$. For any abelian sheaf $\mathcal{F}$ on X and any $n \geq 0$ we have an isomorphism*

$$\hat{H}^n(K, \mathcal{F}) \cong \check{H}^n(\mathcal{U}, \mathcal{F}).$$

Proof. The complex $\mathrm{s}(\mathcal{F}(K))$ is clearly equal to the semi-ordered Čech complex of $\mathcal{U}$. Recall that the semi-ordered Čech complex of $\mathcal{U}$ is given by

$$\check{C}^n_{\text{semi-ord}}(\mathcal{U}, \mathcal{F}) = \prod_{i_0 \leq \ldots \leq i_n} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_n})$$

with the obvious differential. A proof that the semi-ordered Čech complex is homotopy equivalent to the ordered Čech complex can be found in the proof of lemma 20.24.6 of [18]. □

Recall *Leray's Theorem* for ordinary Čech cohomology. If $\mathcal{U}$ is an open cover of a topological space X such that $H^n(U, \mathcal{F}|_U) = 0$ for all $U \in \mathcal{U}$ and all $n > 0$, then $\check{H}^n(\mathcal{U}, \mathcal{F}) \cong H^n(X, \mathcal{F})$ for all $n \geq 0$. We prove a similar statement for hyper-Čech cohomology.

Lemma 1.23. *Let X be a topological space, K a hypercovering on X and $\mathcal{I}$ an injective abelian sheaf on X. Then*

$$\hat{H}^n(K, \mathcal{I}) = 0$$

for all $n > 0$.

Proof. See 24.5.2 of [18]. □

Theorem 1.24. *Let X be a topological space and $\mathcal{F}$ an abelian sheaf on X. Let $K = (I, \mathcal{U})$ be a hypercovering of X such that for any $n \geq 0$ and any $i \in I_n$ we have $H^k(U_i, \mathcal{F}|_{U_i}) = 0$ for all $k > 0$. Then, for all $k \geq 0$ we have a natural isomorphism*

$$\hat{H}^k(K, \mathcal{F}) \cong H^k(X, \mathcal{F})$$

Proof. The case $k = 0$ is proven in Proposition 1.21. Now we consider an exact sequence of sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0 \tag{1.1}$$

with $\mathcal{G}$ injective. Note that existence of such a sequence is ensured as every sheaf embeds into an injective sheaf. Let $n \geq 0$ and $i \in I_n$. Since $H^1(U_i, \mathcal{F}|_{U_i}) = 0$, the long exact sequence of cohomology implies that we have a short exact sequence of abelian groups

$$0 \longrightarrow \mathcal{F}(U_i) \longrightarrow \mathcal{G}(U_i) \longrightarrow \mathcal{H}(U_i) \longrightarrow 0.$$

Taking products we get a short exact sequence of complexes

$$0 \longrightarrow \mathrm{s}(\mathcal{F}(K)) \longrightarrow \mathrm{s}(\mathcal{G}(K)) \longrightarrow \mathrm{s}(\mathcal{H}(K)) \longrightarrow 0.$$

We now get a long exact sequence of hyper-Čech cohomology groups. Lemma 1.23 states that the hyper-Čech cohomology groups of $\mathcal{G}$ vanishes for $k > 0$ and this results in natural isomorphisms $\hat{H}^k(K, \mathcal{F}) \cong \hat{H}^{k-1}(K, \mathcal{H})$ for $k \geq 2$ and an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to \hat{H}^1(K, \mathcal{F}) \to 0. \tag{1.2}$$

Similarly, as $\mathcal{G}$ is injective the cohomology groups $H^k(X, \mathcal{G})$ vanish for $k > 0$ and we get natural isomorphisms $H^k(X, \mathcal{F}) \cong H^{k-1}(X, \mathcal{H})$ for $k \geq 1$ and an exact sequence

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to H^1(X, \mathcal{F}) \to 0. \tag{1.3}$$

The sequences (1.2) and (1.3) imply that there is a natural isomorphism $\hat{H}^1(K, \mathcal{F}) \cong H^1(X, \mathcal{F})$. We now move on by induction. Note that the long exact sequence of cohomology associated to 1.1 together with the assumption of the theorem and the fact that $\mathcal{G}$ is injective imply that $H^k(U_i, \mathcal{H}|_{U_i}) = 0$ for all $k > 0$ and all $i \in I_n$. That is, the sheaf $\mathcal{H}$ also satisfies the assumption of the theorem. Suppose that we have natural isomorphisms $\hat{H}^\ell(K, \mathcal{F}) \cong H^\ell(X, \mathcal{F})$ for all $1 \leq \ell < k$. Then,

$$\hat{H}^k(K, \mathcal{F}) \cong \hat{H}^{k-1}(K, \mathcal{H}) = H^{k-1}(X, \mathcal{H}) \cong H^k(X, \mathcal{F}).$$

□

The following result is an example illustrating the usefulness of Čech cohomology.

Theorem 1.25. *Let k be a field and X a separated k-scheme. Let $\mathcal{U}$ be an open cover of X consisting of spectra of finitely generated k-algebras and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then, for all $n \geq 0$, we have an isomorphism*

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \cong H^n(X, \mathcal{F})$$

Proof. This is a somewhat more general version of Theorem III.4.5 in [12]. In this book, it is exercise III.4.11. □

As a corollary of Theorem 1.24, we find a similar result for hyper Čech cohomology and see that the assumption that X is separated can be dropped. Recall the following theorem, which will be used as an ingredient.

Theorem 1.26. *Let X be a Noetherian affine scheme and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. For all $k > 0$ we have $H^k(X, \mathcal{F}) = 0$.*

Proof. See Theorem 3.7 of [12]. □

Corollary 1.27. *Let k be a field and X a k-scheme. Let $K = (I, \mathcal{U})$ be a hypercovering of X such that for all $n \geq 0$ and all $i \in I_n$ the open set U_i is isomorphic to the spectrum of a finitely generated k-algebra. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then for any $n \geq 0$ we have an isomorphism*

$$\hat{H}^n(K, \mathcal{F}) \cong H^n(X, \mathcal{F}).$$

Proof. Let $n \geq 0$ and $i \in I_n$. By assumption U_i is isomorphic to $\mathrm{Spec}(A_i)$, with A_i a finitely generated k-algebra. So A_i is a Noetherian ring, making U_i an affine Noetherian scheme. As $\mathcal{F}$ is quasi-coherent, so is the restriction sheaf $\mathcal{F}|_{U_i}$ and by theorem 1.26 we find that $H^n(U_i, \mathcal{F}|_{U_i}) = 0$ for all $n > 0$. The result is now an immediate consequence of lemma 1.24. □

1.5 Absolute Hyper-Čech Cohomology

Just as in the case of Čech cohomology we can define absolute cohomology groups by taking an appropiate colimit. Given a refinement map $f\colon K = (I, \mathcal{U}) \to (J, \mathcal{V}) = L$ we get an induced map $\tilde{f}\colon \mathcal{F}(L) \to \mathcal{F}(K)$ given by

$$\tilde{f}^n\colon \prod_{j \in J_n} \mathcal{F}(V_j) \longrightarrow \prod_{i \in I_n} \mathcal{F}(U_i)$$
$$\tilde{f}^n \boldsymbol{s}(i) = \boldsymbol{s}(f(i))|_{U_i}.$$

The fact that $\tilde{f}$ is a map of cosimplicial abelian groups follows from the fact that $f\colon I \to J$ is a map of simplicial sets. The map $\tilde{f}$ in turn induces a map on the Moore complexes $\mathrm{s}(\mathcal{F}(L)) \to \mathrm{s}(\mathcal{F}(K))$ and this induces a map on the hyper-Čech cohomology groups. We conclude for all $n \geq 0$ and all abelian sheaves $\mathcal{F}$ on X we have a diagram

$$\hat{H}^n(-, \mathcal{F})\colon\ \mathrm{HC}(X)^{\mathrm{op}} \xrightarrow{\mathcal{F}(-)} \mathrm{cSimp}(\mathrm{Ab}) \xrightarrow{\mathrm{s}} \mathbf{C}_{\geq 0}(\mathrm{Ab}) \xrightarrow{h^n} \mathrm{Ab}. \qquad (1.4)$$

This diagram actually turns out to be directed. Given two hypercoverings K and L on X we can find a common refinement; the product of the two hypercoverings.

Definition 1.28. Let $K = (I, \mathcal{U})$ and $L = (J, \mathcal{V})$ be two hypercoverings on a topological space X. For any $(i, j) \in (I \times J)_n = I_n \times J_n$ we define $U_{(i,j)} = U_i \cap V_j$. We define $(\mathcal{U} \times \mathcal{V})_n = \{U_{(i,j)} : (i, j) \in I_n \times J_n\}$ and $\mathcal{U} \times \mathcal{V} = ((\mathcal{U} \times \mathcal{V})_n)_{n \geq 1}$. The *product* of the hypercoverings K and L is the hypercovering $K \times L = (I \times J,\ \mathcal{U} \times \mathcal{V})$.

Of course, one should check that $K \times L$ is indeed a well-defined hypercovering. This is straightforward by checking the properties (H1)-(H5). The proof that the diagram 1.4 is directed is completed by a homotopical argument, see lemma 24.9.2 of [18]. We can now take the colimit of 1.4 and define absolute hyper-Čech cohomology groups.

Definition 1.29. Let X be a topological space and $\mathcal{F}$ an abelian sheaf on X. We define the absolute hyper-Čech cohomology groups

$$\hat{H}^n(X,\mathcal{F}) = \operatorname{colim}_{\mathrm{HC}(X)} \hat{H}^n(K,\mathcal{F}),$$

where K runs through all hypercoverings of X.

It turns out that the absolute hyper-Čech cohomology groups are the "right" ones. This result is the *Verdier Hypercovering Theorem.*

Theorem 1.30 (Verdier Hypercovering Theorem)**.** *Let X be a topological space, $\mathcal{F}$ an abelian sheaf on X and $n \geq 0$. Then,*

$$\hat{H}^n(X,\mathcal{F}) \cong H^n(X,\mathcal{F}).$$

Proof. See Theorem 24.10.1 of [18]. □

Corollary 1.31. *Let X be a topological space that allows a finest hypercovering K. Then,*

$$\hat{H}^n(K,\mathcal{F}) \cong H^n(X,\mathcal{F})$$

for all $n \geq 0$.

From this corollary we derive the following example of an explicit computation of sheaf cohomology using hypercoverings.

Proposition 1.32. *Let X be a topological space such that there exists a point $x \in X$ with the property that the only open set of X containing x is X itself. Then, $H^n(X,\mathcal{F}) = 0$ for all abelian sheaves $\mathcal{F}$ on X and all $n \geq 1$.*

Proof. Let K denote the hypercovering $(I^n, \mathcal{U})$, where $I_n = \{*\}$ for all $n \geq 0$ and all face and degeneracy maps are the identity, and where $\mathcal{U}_n = \{X\}$ for all $n \geq 0$. By the assumption of the proposition the open cover $\{X\}$ is the finest open cover of X. By proposition 1.18 it is easily seen that K is a finest hypercovering of X. The complex $\mathrm{s}(\mathcal{F}(X))$ is the complex

$$0 \longrightarrow \mathcal{F}(X) \xrightarrow{0} \mathcal{F}(X) \xrightarrow{\mathrm{id}} \mathcal{F}(X) \xrightarrow{0} \mathcal{F}(X) \xrightarrow{\mathrm{id}} \cdots.$$

We conclude

$$H^n(X,\mathcal{F}) \cong \hat{H}^n(K,\mathcal{F}) = h^n(\mathrm{s}(\mathcal{F}(K))) = 0$$

for all $n \geq 1$. □

1.6 On Computing Hyper-Čech Cohomology

Although the Verdier Hypercovering Theorem states that hypercoverings are the correct tool for calculating sheaf cohomology, actual calculations become rather cumbersome. This is mainly due to the fact that simplicial sets grow quite big when the degree increases. We will illustrate this by calculating the absolute hyper-Čech cohomology of $(\boldsymbol{D}, j_!\mathbb{Z}_U)$, the example from the introduction, up to degree 2. Recall that $\boldsymbol{D}$ consists of the five points W, E, N, S, C and that the open sets are given by

$$\boldsymbol{D},\ \emptyset,\ \{W,N,S,C\},\ \{E,N,S,C\},\ \{N,C\},\ \{S,C\},\ \{N,S,C\} \text{ and } \{C\}.$$

In order to gain some intuition for this space, one could look at figure 1. For any $x \in \boldsymbol{D}$ we denote the smallest open subset of x by U_x. Moreover, we write $U = U_C$ and j for the inclusion $U \to \boldsymbol{D}$. Denote the open cover $\{U_W, U_E\}$ of $\boldsymbol{D}$ by $\mathcal{U}_0$. We now construct a hypercovering K from $\mathcal{U}_0$ in the way explained in section 1.3. In order to do this, it is enough to choose an open cover for every open subset of X. For every $x \in X$ we choose the open cover of U_x just to be the trivial open cover $\{U_x\}$. For $U_W \cap U_E$ we choose the open cover $\{U_N, U_S\}$. All our chosen open covers are finest covers and by lemma 1.18, we conclude that K is a refinement of every possible hypercovering of X. We conclude

$$H^n(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \hat{H}^n(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \hat{H}^n(K, j_!\mathbb{Z}_U)$$

for every $n \geq 0$. Note that we have

$$j_!\mathbb{Z}_U(K)^n \cong \mathbb{Z}^{m(n)},$$

where $m(n) = \#\{i \in I_n : U_i = U\}$. In order to determine the first few values $m(n)$ we take a closer look at the simplicial set I. Following the construction process outlined in section 1.3, we find

$$\begin{aligned}
I_0 &= \{W, E\} \\
I_1 &= \{(W,W),\ (E,E),\ (W,E)_N,\ (W,E)_S,\ (E,W)_N,\ (E,W)_S\} \\
I_2 &= \{[(WW),(WW),(WW)],\ [(EE,EE,EE)]\} \\
&\quad \cup \big\{[(AA),(AB)_x,(AB)_y] : x,y \in \{N,S\}, \{A,B\} = \{W,E\}\big\} \\
&\quad \cup \big\{[(AB)_x,(AA),(BA)_y] : x,y \in \{N,S\}, \{A,B\} = \{W,E\}\big\} \\
&\quad \cup \big\{[(AB)_x,(AB)_y,(BB)] : x,y \in \{N,S\}, \{A,B\} = \{W,E\}\big\}
\end{aligned}$$

where $U_{(W,E)_x} = U_{(E,W)_x} = U_x$ for $x \in \{N,S\}$. This gives

$$m(0) = 0 \quad m(1) = 0 \quad m(2) = 12.$$

It follows that $\hat{H}^0(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \hat{H}^1(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong 0$ and $\hat{H}^2(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \ker \partial^2$ with ∂^2 the second differential of the Moore complex $\mathrm{s}(j_!\mathbb{Z}_U(K))$. Let $\boldsymbol{n} \in \mathbb{Z}^{12}$. We have $\boldsymbol{n} \in \ker \partial^2$ if and only if $\sum_{a=0}^{3} (-1)^i n_{d_a \boldsymbol{i}} = 0$ for all $\boldsymbol{i} \in I_3$. By considering different elements of I_3 we obtain conditions on $\boldsymbol{n}$ for being an element of $\ker \partial^2$. Suppose that $\{A,B\} = \{W,E\}$ and $\{x,y\} = \{N,S\}$, we find the conditions

$$\begin{aligned}
\boldsymbol{i} = [(AA)(AA)(AA)][(AA)(AB)_x(AB)_y][(AA)(AB)_x(AB)_x][(AA)(AB)_y(AB)_x] \\
\Rightarrow n_{[(AA)(AB)_x(AB)_y]} = n_{[(AA)(AB)_y(AB)_x]}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{i} = [(AA)(AB)_x(AB)_x][(AA)(AB)_x(AB)_y][(AB)_x(AB)_x)(BB)][(AB)_x(AB)_y(BB)] \\
\Rightarrow n_{[(AA)(AB)_x(AB)_y]} = n_{[(AB)_x(AB)_y(BB)]}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{i} = [(AB)_x(AA)(BA)_y][(AB)_x(AB)_y(BB)][(AA)(AB_y)(AB_x)][(BA)_y(BB)(AB)_x] \\
\Rightarrow n_{[(AB)_x(AA)(BA)_y]} = n_{[(BA)_y(BB)(AB)_x]}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{i} = [(AB)_x(AA)(BA)_y][(AB)_x(AB)_x(BB)][(AA)(AB)_x(AB)_y][(BA)_y(BB)(AB)_y] \\
\Rightarrow n_{[(AB)_x(AA)(BA)_y]} = -- n_{[(AA)(AB)_x(AB)_y]}.
\end{aligned}$$

Combining all these conditions we find that $\boldsymbol{n}$ must be a constant tuple in $\mathbb{Z}^{12}$. We conclude

$$H^2(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \hat{H}^2(K, j_!\mathbb{Z}_U) \cong \ker \partial^2 \cong \mathbb{Z}.$$

Theorem 0.2 assures that we have successfully found the sheaf cohomology groups of $(\boldsymbol{D}, j_!\mathbb{Z}_U)$ up to degree 2 using hypercoverings. However, from the exposition above it becomes clear that even for fairly small spaces as $\boldsymbol{D}$ the calculation is quite painful.

2 Duality for (Locally) Finite Spaces

From now on we concentrate exclusively on finite topological spaces. In the introduction we saw that Čech cohomology is not always powerful enough to calculate sheaf cohomology, even for finite spaces. In the previous section we saw that this problem can theoretically be solved with the introduction of hypercoverings and the corresponding hyper-Čech cohomology theory. Although calculating Čech cohomology is often doable, section 1.6 showed that calculating hyper-Čech cohomology is another task all together. Fortunately, there does actually exists a complex that can be used to calculate the sheaf cohomology groups of finite spaces: the Godement resolution. This resolution will be introduced in section 2.2.

After our study of the Godement resolution, we will study a Grothendieck Duality Theorem for finite topological spaces, in which the Godement resolution features. This Duality Theorem is due to González and first proved in [15]. We will actually prove a slightly more general form of the theorem, stated for sheaves of R-modules (for R a commutative ring) instead of sheaves of abelian groups. An even more general form, for finite ringed spaces, can be found in [17]. In the last subsection we will show that the stated Duality Theorem can be extended to work for *locally finite* topological spaces.

2.1 Finite Topological Spaces

We start by briefly looking at finite topological spaces in general. It turns out the finite topological (T_0) spaces are equivalent to finite posets. We will make this equivalence explicit.

Definition 2.1. Let X be a topological space. The *specialization order* $\leq_X$ on X is the order defined by $x \leq_X y$ if and only if x is contained in the closure of $\{y\}$. If $x \leq_X y$, then x is called a *specialization* of y and y is called a *generization* of x.

Example 2.2. Let R be a commutative ring and $X = \operatorname{Spec} R$ its spectrum. For any two prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ we have $\mathfrak{q} \subseteq \mathfrak{p}$ if and only if $\mathfrak{p} \leq_X \mathfrak{q}$.

Note that an equivalent definition says that a point x is a specialization of a point y if any open subset containing x also contains y. The subscript X in $\leq_X$ will almost always be omitted. The specialization order of a topological space is a preorder and it is easily seen that it is actually a partial order if and only if the topological space satisfies the T_0 separation axiom. Recall that a topological space is T_0 if any two distinct points are topological distinguishable, i.e. there exists an open that contains exactly one of the two points. For the remainder of this thesis we will assume that all our spaces are T_0.

Assumption 2.3. Any topological space is from now on assumed to be T_0. With the category Top of topological spaces we actually mean the category of T_0 topological spaces.

If one is interested in a space that does not satisfy the T_0 separation axiom, one can move to the *Kolmogorov quotient* of X. The relation $\sim$ on X defined by $x \sim y$ if and only if x and y are topologically indistinguishable is an equivalence relation and the quotient space $KQ(X)$ is easily checked to be a T_0 space. Moreover, moving to the Kolmogorov quotient has no impact on the study of sheaves on X; the notion of a sheaf on X is equivalent to the notion of a sheaf on $KQ(X)$.

Proposition 2.4. *Let X be a topological space and $q\colon X \to KQ(X)$ the quotient map. Let R be a commutative ring. The functors $q_*\colon \mathrm{Sh}_R(X) \to \mathrm{Sh}_R(KQ(X))$ and $q^{-1}\colon \mathrm{Sh}_R(KQ(X)) \to \mathrm{Sh}_R(X)$ form an equivalence of categories.*

Proof. Let $U \subseteq X$ be open and $x \in q^{-1}(qU)$. Then $q(x) \in qU$ so there exists an $y \in U$ such that $q(x) = q(y)$. But then x and y are topological indistinguishable points of X. From $y \in U$ it follows that $x \in U$. We conclude $q^{-1}(qU) \subseteq U$ and thus $q^{-1}(qU) = U$. It follows that the quotient map q is open.

Recall that we write $\boldsymbol{O}(Y)$ for the category of open subsets of a topological space Y with inclusions. We now have an isomorphism of categories

$$\begin{aligned} \boldsymbol{O}(X) &\longrightarrow \boldsymbol{O}(KQ(X)) \\ U &\longmapsto q(U) \\ V \subseteq U &\longmapsto q(V) \subseteq q(U) \end{aligned}$$

$$\begin{aligned} \boldsymbol{O}(KQ(X)) &\longrightarrow \boldsymbol{O}(X) \\ V &\longmapsto q^{-1}(V) \\ W \subseteq V &\longmapsto q^{-1}(W) \subseteq q^{-1}(V). \end{aligned}$$

This isomorphism induces the wanted equivalence. □

Although we require all our spaces to be T_0, the specialization order becomes trivial if we assume that the next separation axiom holds. Indeed, suppose that X is a T_1 space. Recall that a space is T_1 if any two distinct points x and y can be *separated*, that is, there exists an open neighbourhood of x that does not contain y and vice versa. Let $x, y \in X$ such that $x \leq y$. Any open subset containing x also contains y, so by the T_1 axiom we have $x = y$.

We can define the (Krull) dimension of a topological space by means of the specialization order.

Definition 2.5. Let X be a topological space. The *dimension* of a point $x \in X$ is the element of $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined by

$$\dim_X(x) = \sup\{n \in \mathbb{Z}_{\geq 0} : \exists x_0, \ldots, x_n \in X \text{ such that } x_0 < \ldots < x_n = x\}.$$

The dimension of the space X is the element of $\mathbb{Z}_{\geq 0} \cup \{-\infty, \infty\}$ defined by

$$\dim X = \sup\{\dim_X(x) : x \in X\}.$$

Instead of $\dim_X(x)$ we simply write $\dim(x)$ if X is understood.

Remark 2.6. (i) Let X be a finite topological space. For all $x \in X$ we have $\dim(x) \leq |X|$ and thus $\dim X \leq |X|$.

(ii) For a topological space X we have $\dim X = -\infty$ if and only if $X = \emptyset$.

A continuous map of topological spaces preserves the specialization order and therefore we have a functor

$$S\colon \mathrm{Top} \longrightarrow \mathrm{Pos}.$$

We can also go in the other direction.

Definition 2.7. Let P be a poset. The *Alexandrov topology* on P is the collection of all upward closed sets of P.

An order preserving map of posets is continuous with respect to the Alexandrov topologies and we have a functor

$$A\colon \mathrm{Pos} \longrightarrow \mathrm{Top}.$$

The two functors that we found do in general not give an equivalence of categories. For a given topological space, its topology does not necessarily coincide with its Alexandrov topology. However, when restricted to finite topological spaces and finite posets, the functors are inverses, making the categories of finite spaces and finite posets not only equivalent but isomorphic. This observation allows us to use both the language of topological spaces and posets interchangeably when working with finite spaces.

Finite topological spaces have the nice property that arbitrary intersections of open subsets are open.

Definition 2.8. A topological space is called *Alexandrov-discrete* if all intersections of open subsets are open.

Example 2.9. (i) Finite spaces are Alexandrov-discrete

(ii) Discrete spaces are Alexandrov-discrete

(iii) For P any poset, the induced space $A(P)$ is Alexandrov-discrete.

Note that for any point x in an Alexandrov-discrete space X, there exists a minimal open neighbourhood U_x of x. Moreover, we have $x \leq y$ if and only if $U_y \subseteq U_x$.

Notation 2.10. If X is an Alexandrov-discrete space and $x \in X$, then we will write U_x for the smallest open subset of X containing x.

The fact that points in finite spaces have minimal open neighbourhoods, makes the study of sheaves on finite spaces simple. It turns out that sheaves on finite spaces are completely determined by the stalks. Let R be a commutative ring. For any sheaf of R-modules $\mathcal{F}$ on an Alexandrov-discerete space X we have $\mathcal{F}_x \cong \mathcal{F}(U_x)$. Furthermore, if $x \leq y$ then we have a map

$$\mathcal{F}_x \cong \mathcal{F}(U_x) \longrightarrow \mathcal{F}(U_y) \cong \mathcal{F}_y.$$

So a sheaf $\mathcal{F}$ on X induces a functor $\varphi\mathcal{F}\colon S(X) \to \mathrm{Mod}_R$, where the poset $S(X)$ is viewed as a category in the usual way. A morphism of sheaves $\mathcal{F} \to \mathcal{G}$ induces a natural transformation $\varphi\mathcal{F} \to \varphi\mathcal{G}$ and we conclude that we have a functor

$$\varphi\colon \mathrm{Sh}_R(X) \longrightarrow \mathrm{Func}(S(X),\ \mathrm{Mod}_R).$$

Starting with a functor $F\colon S(X) \to \mathrm{Mod}_R$ we define the R-module

$$(\psi F)(U) = \lim_{x \in U} F(x)$$

for any open subset $U \subseteq X$. For any inclusion $V \subseteq U$ of opens in X the universal property of the limit gives a morphism $(\psi F)(U) \to (\psi F)(V)$, making ψF a presheaf on X. We claim that ψF is actually a sheaf. As taking limit is left exact, it is enough to prove that the sheaf property holds for open covers $\{U_i\}_{i \in I}$ of U_x for $x \in X$. For these open covers it is easily seen that the sheaf property holds as there must be at least one $i \in I$ with $U_i = U_x$. A natural transformation $F \to G$ of functors $S(X) \to \mathrm{Mod}_R$ induces a map of sheaves $\psi F \to \psi G$ and we conclude that we have a functor

$$\psi\colon \mathrm{Func}(S(X), \mathrm{Mod}_R) \longrightarrow \mathrm{Sh}_R(X).$$

Theorem 2.11. *Let X be an Alexandrov-discrete topological space and $S(X)$ the associated poset. Let R be a commutative ring. The functors φ and ψ defined above form an equivalence of categories between* $\mathrm{Func}(S(X), \mathrm{Mod}_R)$ *and* $\mathrm{Sh}_R(X)$.

Proof. Follows by construction. □

2.2 The Godement Resolution

In this section we will introduce the *Godement resolution* of a sheaf on a topological space. This resolution can be used to calculate sheaf cohomology groups of finite spaces. Furthermore, it will later occur in the Duality Theorem for finite topological spaces.

Definition 2.12. Let X be a topological space, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. The *Godement sheaf* of $\mathcal{F}$ is the sheaf of R-modules $\mathrm{G}^0(\mathcal{F})$ on X defined by

$$(G^0\mathcal{F})(U) = \prod_{x\in U} \mathcal{F}_x,$$

for open $U \subseteq X$ and with as restriction maps the projections

$$(G^0\mathcal{F})(U) = \prod_{x\in U} \mathcal{F}_x \longrightarrow \prod_{x\in V} \mathcal{F}_x = (G^0\mathcal{F})(V),$$

for $V \subseteq U$.

Of course one should check that $\mathrm{G}^0(\mathcal{F})$ is indeed a sheaf. A map op sheaves $\mathcal{F} \to \mathcal{G}$ induces maps on the stalks and therefore we get a map $\mathrm{G}^0(\mathcal{F}) \to \mathrm{G}^0(\mathcal{G})$. Hence, G^0 is actually a functor $\mathrm{Sh}_R(X) \to \mathrm{Sh}_R(X)$. Note that there exists a canonical injective map of sheaves $\mathcal{F} \to \mathrm{G}^0(\mathcal{F})$ sending a section to the product of all its germs. Call this latter map $d^0_{\mathcal{F}}$. We will almost always omit the subscript $\mathcal{F}$ from notation. Now we get a map $\mathrm{G}^0(\mathcal{F}) \to \operatorname{coker} d^0 \to \mathrm{G}^0(\operatorname{coker} d^0)$, which we call d^1. Going on inductively and defining $G^p\mathcal{F} = \mathrm{G}^0(\operatorname{coker} d^{p-1})$ for $p \geq 1$, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "d^0"] & G^0(\mathcal{F}) \arrow[rr, "d^1"] \arrow[dr] & & G^1\mathcal{F} \arrow[rr, "d^2"] \arrow[dr] & & G^2\mathcal{F} \arrow[r] & \dots \\
& & & \operatorname{coker} d^0 \arrow[ur, hook] & & \operatorname{coker} d^1 \arrow[ur, hook] & &
\end{tikzcd}$$

We see that $\mathcal{F}[0] \to G^\bullet\mathcal{F}$ is actually a resolution and this resolution will be called the *Godement resolution* of $\mathcal{F}$. As the Godement sheaf of any sheaf is easily seen to be flasque, the resolution $\mathcal{F}[0] \to G^\bullet\mathcal{F}$ is flasque and can thus be used for calculating the sheaf cohomology groups of $(X, \mathcal{F})$. This is made even simpler in the case of Alexandrov-discrete spaces with the following Theorem. The complex described in this Theorem is widely known and used (for example in [4], [16] and [17]) but proofs that it equals the Godement resolution are not often given.

Notation 2.13. For any chain $x_0 < \ldots < x_p$ in a space X and any $0 \leq i \leq p$ we write $x_0 < \ldots < \widehat{x_i} < \ldots < x_p$ for the chain $x_0 < \ldots < x_{i-1} < x_{i+1} < \ldots < x_p$.

Theorem 2.14. *Let X be an Alexandrov-discrete space, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. Let $p \geq 0$. For $x \in X$ we have*

$$(G^p\mathcal{F})_x \cong \prod_{x\leq x_0<\ldots<x_p} \mathcal{F}_{x_p} \tag{2.1}$$

and

$$(\operatorname{coker} d^p)_x \cong \prod_{x<x_0<\ldots<x_p} \mathcal{F}_{x_p}.$$

For $x < y$ the maps $(G^p\mathcal{F})_x \to (G^p\mathcal{F})_y$ and $(\operatorname{coker} d^p)_x \to (\operatorname{coker} d^p)_y$ are under these isomorphisms given by

$$\boldsymbol{s}_y(y \leq x_0 < \ldots x_p) = \boldsymbol{s}(x \leq x_0 < \ldots < x_p)$$

and

$$\begin{aligned}\boldsymbol{s}_y(y < x_0 < \dots x_p) = \boldsymbol{s}(x < x_0 < \dots < x_p) \\ - \sum_{i=0}^{p-1} (-1)^i \boldsymbol{s}(x < y < x_0 < \dots < \widehat{x_i} < \dots < x_p) \\ - (-1)^p \boldsymbol{s}(x < y < x_0 < \dots < x_{p-1})_{x_p}\end{aligned} \tag{2.2}$$

respectively. The map

$$q_x^p \colon (G^p\mathcal{F})_x \longrightarrow (\operatorname{coker} d^p)_x$$

is given by

$$\begin{aligned}q_x^p \boldsymbol{s}(x < y_0 < \dots < y_p) = \boldsymbol{s}(x \le y_0 < \dots < y_p) \\ - \sum_{i=0}^{p-1} (-1)^i \boldsymbol{s}(x \le x < y_0 < \dots < \widehat{y_i} < \dots < y_p) \\ - (-1)^p \boldsymbol{s}(x \le x < y_0 < \dots < y_{p-1})_{y_p}.\end{aligned} \tag{2.3}$$

and the differential

$$d_x^{p+1} \colon (G^p\mathcal{F})_x \longrightarrow G^{p+1}(\mathcal{F})_x$$

is given by

$$\begin{aligned}d_x^{p+1} \boldsymbol{s}(x \le x_0 < \dots < x_{p+1}) = \sum_{i=0}^{p} (-1)^i \boldsymbol{s}(x \le x_0 < \dots < \widehat{x_i} < \dots < x_{p+1}) \\ + (-1)^{p+1} \boldsymbol{s}(x \le x_0 < \dots < x_p)_{x_{p+1}}.\end{aligned}$$

Proof. We prove by induction on p. The formula (2.1) holds for $p = 0$ by definition of the Godement sheaf. It is also easily seen that the map between the stalks is as wanted. For any $x \in X$, the map $d_x^0 \colon \mathcal{F}_x \to (G^0\mathcal{F})_x$ is given by

$$(d_x^0 s)(x \le y) = s_y$$

and therefore the map

$$\begin{aligned}q_x^0 \colon \prod_{x \le y} \mathcal{F}_y &\longrightarrow \prod_{x<y} \mathcal{F}_y \\ \boldsymbol{s} &\longmapsto (\boldsymbol{s}(x \le y) - \boldsymbol{s}(x \le x)_y)_{x<y}\end{aligned} \tag{2.4}$$

has $\operatorname{im} d_x^0$ as kernel and induces an isomorphism $(\operatorname{coker} d^0)_x \to \prod_{x<y} \mathcal{F}_y$. Let $x < y$, $\boldsymbol{s} \in \prod_{x<y} \mathcal{F}_y$, and $\boldsymbol{t} \in \prod_{x \le y} \mathcal{F}_y$ such that $q_x^0 \boldsymbol{t} = \boldsymbol{s}$. Then we have

$$\begin{aligned}\boldsymbol{s}_y(y < z) = q_x^0(\boldsymbol{t}_y)(y < z) = \boldsymbol{t}_y(y \le z) - \boldsymbol{t}_y(y \le y)_z \\ = \boldsymbol{t}(x \le z) - \boldsymbol{t}(x \le y)_z = \boldsymbol{s}(x < z) - \boldsymbol{s}(x < y)_z.\end{aligned}$$

So the maps between the stalks of $\operatorname{coker} d^0$ are as wanted. Now suppose that all our wanted formulas hold for $0 \le d < p$. We find

$$\begin{aligned}(G^p\mathcal{F})_x = G^0(\operatorname{coker} d^{p-1})_x = \prod_{x \le y} (\operatorname{coker} d^{p-1})_y \\ \cong \prod_{x \le y} \prod_{y < y_0 < \dots < y_{p-1}} \mathcal{F}_{y_{p-1}} = \prod_{x \le y_0 < y_1 < \dots < y_p} \mathcal{F}_{y_p}.\end{aligned}$$

For the ease of notation we write $\varepsilon_x^{p-1} = d^0_{\operatorname{coker} d^{p-1},x}$. It follows that the differential d^p is given by

$$d_x^p \boldsymbol{s}(x \leq y_0 < \ldots < y_p) = \varepsilon_x^{p-1} q_x^{p-1} \boldsymbol{s}(x \leq y_0 < \ldots < y_p) = (q_x^{p-1}\boldsymbol{s})_{y_0}(y_0 < \ldots < y_p).$$

Using the formulas (2.2) and (2.3) it is now a straightforward but long writing exercise to show that d^p is as claimed. The map (2.4) with the role of $\mathcal{F}$ played by $\operatorname{coker} d^{p-1}$ shows that we have

$$(\operatorname{coker} d^p)_x \cong \prod_{x<y} (\operatorname{coker} d^{p-1})_y = \prod_{x<y} \prod_{y<z_0<\ldots<z_{p-1}} \mathcal{F}_{z_{p-1}} = \prod_{x<y_0<\ldots<y_p} \mathcal{F}_{y_p}$$

and that the map

$$q_x^p \colon (G^p\mathcal{F})_x = \prod_{x \leq y} (\operatorname{coker} d^{p-1})_x \longrightarrow (\operatorname{coker} d^p)_x$$

is given by

$$\begin{aligned} q_x^p \boldsymbol{s}(x < y_0 < \ldots < y_p) &= \boldsymbol{s}(x \leq y_0)(y_0 < \ldots < y_p) - \boldsymbol{s}(x \leq x)_{y_0}(y_0 < \ldots < y_p) \\ &= \boldsymbol{s}(x \leq y_0 < \ldots < y_p) - \boldsymbol{s}(x \leq x < y_1 < \ldots < y_p) \\ &\quad - \sum_{i=1}^{p-1} (-1)^i \boldsymbol{s}(x \leq x < y_0 < \ldots < \widehat{y_i} < \ldots < y_p) \\ &\quad - (-1)^p \boldsymbol{s}(x \leq x < y_0 < \ldots < y_{p-1})_{y_p} \end{aligned}$$

as wanted. □

Corollary 2.15. *Let X be an Alexandrov-discrete space, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. For any open $U \subseteq X$ and any $p \geq 0$ we have*

$$\Gamma(U, G^p\mathcal{F}) = \lim_{x \in U} (G^p\mathcal{F})_x = \lim_{x \in U} \left(\prod_{x \leq x_0 < \ldots < x_p} \mathcal{F}_{x_p} \right) \cong \prod_{U \ni x_0 < \ldots < x_p} \mathcal{F}_{x_p}.$$

Corollary 2.16. *Let X be an Alexandrov-discrete topological space of finite dimension n. Let R be a commutative ring. Then we have $G^p\mathcal{F} = 0$ for all sheaves of R-modules $\mathcal{F}$ on X and all $p > n$. Hence, $d^n_{\mathcal{F}}$ is surjective and we have $G^n\mathcal{F} \cong \operatorname{coker} d^{n-1}_{\mathcal{F}}$. Moreover, as $G^\bullet\mathcal{F}$ is a flasque resolution of $\mathcal{F}$, it follows that $H^p(X, \mathcal{F}) = 0$ for all $p > n$. This proves the well known Theorem of Grothendieck ([11, 3.6.5]) that the cohomological dimension equals at most the Krull dimension in the case of Alexandrov-discrete spaces.*

We turn to the finite model of the unit disk one last time. We will give an explicit calculation of the cohomology groups of $(\boldsymbol{D}, j_!\mathbb{Z}_U)$ using the Godement resolution. Thereafter, we also consider the more general setting of finite models of unit disks in other dimensions. We define the finite model of the 0-disk to be the singleton $\boldsymbol{D}^0 = \{C\}$. Let $n > 0$, we define the finite model of the n-disk to be the topological space

$$\boldsymbol{D}^n = \boldsymbol{D}^{n-1} \cup \{x_0^n, x_1^n\}$$

with $x_i^n \leq y$ for all $y \in \boldsymbol{D}^{n-1}$ and $i = 0, 1$. From this definition it is clear that we have $|\boldsymbol{D}^n| = 2n + 1$ and that $\dim \boldsymbol{D}^n = n$ for all $n \geq 0$. For any $n \geq 0$ the subset $\{C\} \subseteq \boldsymbol{D}^n$ is open and we let $\mathcal{F}^n$ be the sheaf $\mathbb{Z}_{\{C\}}$ extended by zero to the whole of $\boldsymbol{D}^n$. We will prove that $H^p(\boldsymbol{D}^n, \mathcal{F}^n) \cong \mathbb{Z}$ for $p = n$ and 0 otherwise.

We first give an explicit calculation for the case $\boldsymbol{D} = \boldsymbol{D}^2$, which was also considered int he introduction.

(a) $n = 0$ (b) $n = 1$ (c) $n = 2$ (d) $n = 3$

Figure 2: Finite model of the n-disk. A path $y \to \cdots \to x$ means that x is a specialization of y, that is, $x \leq y$.

Example 2.17. As $\dim \boldsymbol{D} = 2$, corollary 2.16 implies that $H^n(\boldsymbol{D}, j_!\mathbb{Z}_U) = 0$ for all $n > 2$. Note that $(j_!\mathbb{Z}_U)_x = \mathbb{Z}$ if $x = C$ and $(j_!\mathbb{Z}_U)_x = 0$ otherwise. The complex $\Gamma(\boldsymbol{D}, G^\bullet(j_!\mathbb{Z}_U))$ looks like

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}^4 \xrightarrow{\beta} \mathbb{Z}^4 \longrightarrow 0 \longrightarrow \cdots,$$

where the first zero lives in degree -1. The $\mathbb{Z}$ in degree 0 is generated by C. The $\mathbb{Z}^4$ in degree 1 is generated by $\{W < C, E < C, N < C, S < C\}$ and the $\mathbb{Z}^4$ in degree 2 is generated by $\{W < N < C, W < S < C, E < N < C, E < S < C\}$. The maps α and β with respect to these bases are

$$\alpha = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \qquad \beta = \begin{pmatrix} -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}.$$

We know that $H^0(\boldsymbol{D}, j_!\mathbb{Z}_U) = j_!\mathbb{Z}_U(\boldsymbol{D}) = 0$. Let $\boldsymbol{n} \in \ker \beta$. Let $x \in \{W, E\}$ and $y \in \{N, S\}$. We have

$$\beta\boldsymbol{n}(x < y < C) = \boldsymbol{n}(x < C) - \boldsymbol{n}(y < C)$$

and thus $\boldsymbol{n}(x < C) = \boldsymbol{n}(y < C)$. It follows that $\ker \beta = \operatorname{im} \alpha \cong \mathbb{Z}$. Hence,

$$H^1(\boldsymbol{D}, j_!\mathbb{Z}_U) \cong \ker \beta / \operatorname{im} \alpha = 0.$$

From $\ker \alpha = 0$ it follows that $\Gamma(\boldsymbol{D}, G^\bullet(j_!\mathbb{Z}_U))$ is quasi isomorphic to

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}^3 \xrightarrow{\gamma} \mathbb{Z}^4 \longrightarrow 0 \longrightarrow \cdots,$$

where the first zero lives in degree 0. From $H^1(\boldsymbol{D}, j_!\mathbb{Z}_U) = 0$ it now follows that $\ker \gamma = 0$ and thus that

$$H^2(\boldsymbol{D}, j|_!\mathbb{Z}_U) \cong \operatorname{coker} \gamma \cong \mathbb{Z}.$$

The more general statement follows from the following theorem.

Theorem 2.18. *Let X be a finite topological space, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. We define $X^+ = X \cup \{x_0, x_1\}$ with $x_0, x_1 \leq y$ for all $y \in X$. Let $j\colon X \to X^+$ denote the inclusion and $j_!\mathcal{F}$ the extension of $\mathcal{F}$ by zero to X^+. We have*

$$H^0(X^+, j_!\mathcal{F}) = 0$$

and for all $n > 0$ there is an isomorphism

$$H^n(X^+, j_!\mathcal{F}) \cong H^{n-1}(X, \mathcal{F}).$$

Proof. Let $p \geq 0$. We define the surjective map

$$\varphi^p : \left(\prod_{(x_0 \leq y_0 < \ldots < y_p) \in X^+} (j_!\mathcal{F})_{y_p} \right) \oplus \left(\prod_{(x_1 \leq y_0 < \ldots < y_p) \in X^+} (j_!\mathcal{F})_{y_p} \right) \longrightarrow \prod_{(z_0 < \ldots < z_p) \in X} \mathcal{F}_{z_p}$$

given by

$$(\varphi^p(\boldsymbol{s}, \boldsymbol{t}))\,(z_0 < \ldots < z_p) = \boldsymbol{s}(x_0 \leq z_0 < \ldots < z_p) - \boldsymbol{t}(x_1 \leq y_0 < \ldots < y_p).$$

Moreover, we define

$$\iota^p : \prod_{(y_0 < \ldots < y_p) \in X^+} (j_!\mathcal{F})_{y_p} \longrightarrow \left(\prod_{(x_0 \leq y_0 < \ldots < y_p) \in X^+} (j_!\mathcal{F})_{y_p} \right) \oplus \left(\prod_{(x_1 \leq y_0 < \ldots < y_p) \in X^+} (j_!\mathcal{F})_{y_p} \right)$$

to be the canonical injective map. It is now easily seen that these maps fit in an exact sequence

$$0 \longrightarrow \Gamma(X^+, G^p(j_!\mathcal{F})) \xrightarrow{\iota^p} (G^p(j_!\mathcal{F}))_{x_0} \oplus (G^p(j_!\mathcal{F}))_{x_1} \xrightarrow{\varphi^p} \Gamma(X, G^p\mathcal{F}) \longrightarrow 0.$$

The collections $(\iota^p)_{p\geq 0}$ and $(\varphi^p)_{p\geq 0}$ define maps of complexes and we conclude that we have an exact sequence

$$0 \longrightarrow \Gamma(X^+, G^\bullet(j_!\mathcal{F})) \xrightarrow{\iota} (G^\bullet(j_!\mathcal{F}))_{x_0} \oplus (G^\bullet(j_!\mathcal{F}))_{x_1} \xrightarrow{\varphi} \Gamma(X, G^\bullet\mathcal{F}) \longrightarrow 0. \tag{2.5}$$

As $G^\bullet(j_!\mathcal{F})$ is a resolution of $j_!\mathcal{F}$, the sequence

$$0 \longrightarrow (j_!\mathcal{F})_x \longrightarrow (G^0(j_!\mathcal{F}))_x \longrightarrow (G^1(j_!\mathcal{F}))_x \longrightarrow (G^2(j_!\mathcal{F}))_x \longrightarrow \ldots$$

is exact for all $x \in X^+$. Hence, $h^n((G^\bullet(j_!\mathcal{F}))_x) = 0$ for all $n \geq 1$ and all $x \in X$. Moreover, as $(j_!\mathcal{F})_x = 0$ for $x \in \{x_0, x_1\}$, we also have $h^0((G^\bullet(j_!\mathcal{F}))_x) = 0$ for these x. Taking cohomology of (2.5) now yields a long exact sequence

$$0 \longrightarrow H^0(X^+, j_!\mathcal{F}) \longrightarrow 0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^1(X^+, j_!\mathcal{F}) \longrightarrow 0 \longrightarrow H^1(X, \mathcal{F}) \longrightarrow \cdots.$$

The wanted statement follows directly from this sequence. □

Corollary 2.19. *Let $n \geq 0$. We have*

$$H^p(\boldsymbol{D}^n, \mathcal{F}^n) \cong \begin{cases} \mathbb{Z} & p = n \\ 0 & \textit{otherwise}. \end{cases}$$

Remark 2.20. The proof of Theorem 2.18 is actually just Mayer-Vietoris with the opens $U = U_{x_0}$ and $V = U_{x_1}$.

As a brief intermezzo, we now exhibit a connection between the Euler characteristic of a finite space and the möbius function of a poset. This connection is already well understood, see for example [1, 3.2] and [2]. Using the Godement resolution we give a cohomological perspective and prove a Theorem that implies the results of [1, 3.2] and [2]. We start by making some definitions. Let R be a principal ideal domain. A sheaf $\mathcal{F}$ of R-modules on a finite space X is called *locally finitely generated* if $\mathcal{F}_x$ is finitely generated for all points $x \in X$.

Definition 2.21. Let X be a finite topological space, R a PID and $\mathcal{F}$ a locally finitely generated sheaf of R-modules on X. The *Euler characteristic* of X with respect to $\mathcal{F}$ is

$$\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i \operatorname{rk}(H^i(X, \mathcal{F})).$$

We can also view the Euler characteristic as an element of the Grothendieck group.

Definition 2.22. Let X be a finite topological space, R a commutative ring and $\mathcal{F}$ a locally finitely generated sheaf of R-modules on X. The *Euler characteristic* of X with respect to $\mathcal{F}$ is the element

$$\boldsymbol{\chi}(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i [H^i(X, \mathcal{F})]$$

in the Grothendieck group $K_0(\text{Modfg}_R)$.

Usually the same notation is used for the two defined Euler characteristics. Note that if R is a PID we have a group homomorphism $\operatorname{rk}\colon K_0(\text{Abfg}) \to \mathbb{Z}$ sending a class $[M]$ to $\operatorname{rk} M$. This morhism links the two Euler characteristics; we have $\boldsymbol{\chi}(X, \mathcal{F}) = \operatorname{rk}(\chi(X, \mathcal{F}))$.

Definition 2.23. Let P be a poset. We define the *möbius function* $\mu\colon P \times P \to \mathbb{R}$ of P recursively by

$$\mu(x,z) = \begin{cases} 0 & x \not\leq z \\ 1 & x = z \\ -\sum_{x \leq y < z} \mu(x,y) & x < z. \end{cases}$$

The following Theorem about the möbius function is important for the connection with the Euler characteristic.

Theorem 2.24 (Hall's Theorem)**.** *Let P be a finite poset and let $x, y \in P$. For any $i \in \mathbb{Z}_{\geq 0}$ let c_i be the number of chains*

$$x = p_0 < \ldots < p_i = y$$

in P. Then,

$$\mu_P(x,y) = \sum_{i \geq 0} (-1)^i c_i.$$

Proof. See lemma 5.2 in [9] or Theorem 3.2 in [2]. □

We now prove the connection between the Euler characteristic and the möbius function as an identity in the Grothendieck group $K_0(\text{Modfg}_R)$.

Theorem 2.25. *Let X be a finite topological space, R a commutative ring and $\mathcal{F}$ a locally finitely generated sheaf of R-modules on X. Let $\widehat{X}$ be the poset $X \coprod \{0\}$ with $0 < x$ for all $x \in X$. We have an equality*

$$\boldsymbol{\chi}(X, \mathcal{F}) = -\sum_{x \in X} \mu_{\widehat{X}}(0, x)[\mathcal{F}_x].$$

in the Grothendieck group $K_0(\text{Modfg}_R)$.

Proof. For all $p \geq 0$ we have an exact sequence

$$0 \longrightarrow \ker d^{p+1} \longrightarrow \Gamma(X, G^p \mathcal{F}) \longrightarrow \operatorname{im} d^{p+1} \longrightarrow 0$$

and for all $p \geq 1$ we have an exact sequence

$$0 \longrightarrow \operatorname{im} d^p \longrightarrow \ker d^{p+1} \longrightarrow H^p(X, \mathcal{F}) \longrightarrow 0.$$

Hence, the following identities holds in the Grothendieck group $K_0(\mathrm{Modfg}_R)$

$$\begin{aligned} [\Gamma(X, G^p\mathcal{F})] &= [\ker d^{p+1}] + [\operatorname{im} d^{p+1}] && \text{for all } p \geq 0, \\ [H^p(X, \mathcal{F})] &= [\ker d^{p+1}] - [\operatorname{im} d^p] && \text{for all } p \geq 1. \end{aligned}$$

We find

$$\begin{aligned} \chi(X, \mathcal{F}) &= \sum_{i \geq 0} (-1)^i [H^i(X, \mathcal{F})] \\ &= [H^0(X, \mathcal{F})] + \sum_{i \geq 1} (-1)^i [H^i(X, \mathcal{F})] \\ &= [\ker d^1] + \sum_{i \geq 1} (-1)^i \left([\ker d^{i+1}] - [\operatorname{im} d^i] \right) \\ &= [\ker d^1] + \sum_{i \geq 1} (-1)^i \left([\Gamma(X, G^i\mathcal{F})] - [\operatorname{im} d^{i+1}] - [\operatorname{im} d^i] \right) \\ &= [\ker d^1] + \sum_{i \geq 1} (-1)^i [\Gamma(X, G^i\mathcal{F})] + [\operatorname{im} d^1] \\ &= [\Gamma(X, G^i\mathcal{F})] - [\operatorname{im} d^1] + \sum_{i \geq 1} (-1)^i [\Gamma(X, G^i\mathcal{F})] + [\operatorname{im} d^1] \\ &= \sum_{i \geq 0} (-1)^i [\Gamma(X, G^i\mathcal{F})]. \end{aligned}$$

For any $i \geq 0$ and any $x \in X$ we define $c_i(x)$ to be the number of chains $0 = x_0 < \ldots < x_i = x$ in $\widehat{X}$ and $d_i(x)$ to be the number of chains $x_0 < \ldots < x_i = x$ in X. Note that we have $c_0(x) = 0$ and $c_i(x) = d_{i-1}(x)$ for all $i \geq 1$. Let $i \geq 0$, we have

$$[\Gamma(X, G^i\mathcal{F})] = \left[\bigoplus_{x_0 < \ldots < x_i} \mathcal{F}_{x_i} \right] = \sum_{x_0 < \ldots < x_i} [\mathcal{F}_{x_i}] = \sum_{x \in X} d_i(x) [\mathcal{F}_x] = \sum_{x \in X} c_{i+1}(x) [\mathcal{F}_x].$$

Hence,

$$\begin{aligned} \chi(X, \mathcal{F}) &= \sum_{i \geq 0} (-1)^i [\Gamma(X, G^i\mathcal{F})] \\ &= \sum_{i \geq 0} \sum_{x \in X} (-1)^i c_{i+1}(x) [\mathcal{F}_x] \\ &= \sum_{x \in X} - \left(\sum_{i \geq 1} (-1)^i c_i(x) \right) [\mathcal{F}_x] \end{aligned}$$

As $c_0 = 0$, it follows that

$$\chi(X, \mathcal{F}) = - \sum_{x \in X} \left(\sum_{i \geq 0} (-1)^i c_i(x) \right) [\mathcal{F}_x].$$

Using Hall's Theorem, we conclude

$$\chi(X, \mathcal{F}) = - \sum_{x \in X} \mu_{\widehat{X}}(0, x) [\mathcal{F}_x].$$

$\square$

Part (i) of the following corollary is Theorem 3.2 in [1] and part (ii) is proposition 3.4 of [2].

Corollary 2.26. *Let X be a finite topological space and let $\widehat{X}$ denote the poset $X \coprod \{0,1\}$ with $0 < x < 1$ for all $x \in X$. Let R be a PID.*

(i) Let $x \in X$ and $\mathrm{Sky}_R(x)$ the skyscraper sheaf taking the value R at x. We have

$$\chi(X, \mathrm{Sky}_R(x)) = -\mu_{\widehat{X}}(0,x).$$

(ii) We have

$$\chi(X, R_X) = \mu_{\widehat{X}}(0,1) + 1.$$

Proof. (i) Obvious.

(ii) For any $i \geq 0$ and any $x \in \widehat{X}$ let $c_i(x)$ be the number of chains $0 = x_0 < \ldots < x_i = x$ in $\widehat{X}$. Note that we have $\sum_{x \in X} c_i(x) = c_{i+1}(1)$ for all $i \geq 1$, $c_1(1) = 1$ and $c_0(x) = 0$ for all $x \in \widehat{X}$ distinct from 0. We have

$$\begin{aligned} \chi(X, R_X) &= -\sum_{x \in X} \mu_{\widehat{X}}(0,x) = -\sum_{x \in X} \sum_{i \geq 1} (-1)^i c_i(x) = -\sum_{i \geq 1} (-1)^i c_{i+1}(1) \\ &= \sum_{i \geq 2} (-1)^i c_i(1) = \mu_{\widehat{X}}(0,1) + c_1(1) = \mu_{\widehat{X}}(0,1) + 1. \end{aligned}$$

□

We now conclude this section by stating some facts about the Godement resolution, that will be used later in our study of the Duality Theorem for finite spaces. By an induction argument one can actually show that the Godement resolution is functorial. So for any topological space X and any commutative ring R, we get functors

$$G^p \colon \mathrm{Sh}_R(X) \longrightarrow \mathrm{Sh}_R(X)$$

for all $p \geq 0$. The functor G^p is called the *p-th Godement functor*.

Proposition 2.27 ([5, 2.2.1])**.** *The p-th Godement functor is exact for all $p \geq 0$.*

Proof. Let X be a topological space and R a commutative ring. We prove by induction on p that the functors G^p and

$$\begin{aligned} \mathrm{Sh}_R(X) &\longrightarrow \mathrm{Sh}_R(X) \\ \mathcal{F} &\longmapsto \operatorname{coker} d^p_{\mathcal{F}} \end{aligned}$$

are exact. As products of exact sequences are exact, it is easily seen that G^0 is indeed an exact functor. Suppose that $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a short exact sequence of sheaves of R-modules on X. Let $x \in X$, we have a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{F}_x \arrow[r] \arrow[d] & \mathcal{G}_x \arrow[r] \arrow[d] & \mathcal{H}_x \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (G^0\mathcal{F})_x \arrow[r] \arrow[d] & (G^0\mathcal{G})_x \arrow[r] \arrow[d] & (G^0\mathcal{H})_x \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (\operatorname{coker} d^0_{\mathcal{F}})_x \arrow[r] \arrow[d] & (\operatorname{coker} d^0_{\mathcal{G}})_x \arrow[r] \arrow[d] & (\operatorname{coker} d^0_{\mathcal{H}})_x \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0, & \end{tikzcd}$$

where the columns and the first two rows are exact. By the Nine Lemma it follows that the bottom row is exact as well. We conclude that $\mathcal{F} \mapsto \operatorname{coker} d^0_{\mathcal{F}}$ is indeed exact. Now it follows that G^1 which sends $\mathcal{F}$ to $G^0(\operatorname{coker} d^0_{\mathcal{F}})$ is exact as the composition of exact functors. Using the Nine Lemma again, we prove that $\mathcal{F} \mapsto \operatorname{coker} d^1_{\mathcal{F}}$ is exact and by induction, we conclude that G^p is indeed exact for all $p \geq 0$. □

2.3 Representable Functors $\mathrm{Sh}_R(X) \to \mathrm{Mod}_R$

An important ingredient in the proof of the Duality Theorem will be the fact that contravariant functors $\mathrm{Sh}_R(X) \to \mathrm{Mod}_R$ sending colimits to limits are representable. That is, the following theorem holds.

Theorem 2.28. *Let X be a topological space and R a commutative ring. A contravariant functor $F\colon \mathrm{Sh}_R(X) \to \mathrm{Mod}_R$ is representable if and only if it sends colimits in $\mathrm{Sh}_R(X)$ to limits in Mod_R.*

The "only if" direction is obvious. This section will be devoted to proving the other direction. The proof mostly follows [8, III.8.19]. Fortunately, the proof will be constructive and provide a way to find the representing object. We will start by introducing the sheaf that will play this role. Recall that for a topological space X, the category of open subsets of X with inclusions is denoted by $\boldsymbol{O}(X)$.

Definition 2.29. Let X be a topological space and R a commutative ring. For any open $U \subseteq X$ we define the presheaf R_U^{pre} by

$$R_U^{pre}(V) = \bigoplus_{\mathrm{Hom}_{\boldsymbol{O}(X)}(V,U)} R$$

with the obvious restriction maps. We define the sheaf R_U as the sheafification of R_U^{pre}.

The sheaves we just defined satisfy the following useful property.

Proposition 2.30. *Let X be a topological space, R a commutative ring, $U \subseteq X$ an open subset and $\mathcal{F}$ a sheaf of R-modules on X. We have*

$$\mathrm{Hom}_{\mathrm{Sh}_R(X)}(R_U, \mathcal{F}) \cong \mathcal{F}(U).$$

Proof. Let $R[\boldsymbol{O}(X)]$ be the category that has the same objects as $\boldsymbol{O}(X)$ and has hom groups

$$\mathrm{Hom}_{R[\boldsymbol{O}(X)]}(V, W) = \bigoplus_{\mathrm{Hom}_{\boldsymbol{O}(X)}(V,W)} R.$$

Viewing R_U^{pre} and $\mathcal{F}$ as presheaves on $R[\boldsymbol{O}(X)]$, the presheaf R_U^{pre} is representable and the Yoneda Lemma gives

$$\mathrm{Hom}_{\mathcal{C}}(R_U^{pre}, \mathcal{F}) \cong \mathcal{F}(U),$$

where $\mathcal{C} = \mathrm{Set}^{R[\boldsymbol{O}(X)]^{\mathrm{op}}}$. We conclude

$$\mathrm{Hom}_{\mathrm{Sh}_R(X)}(R_U, \mathcal{F}) = \mathrm{Hom}_{\mathrm{PSh}_R(X)}(R_U^{pre}, \mathcal{F}) \cong \mathrm{Hom}_{\mathcal{C}}(R_U^{pre}, \mathcal{F}) \cong \mathcal{F}(U).$$

□

Note that in the case of a finite topological space X the sheaf R_U satisfies

$$(R_U)_x = (R_U^{pre})_x = R_U^{pre}(U_x) = \begin{cases} R & x \in U \\ 0 & \text{otherwise.} \end{cases}$$

For open subsets $V \subseteq U \subseteq X$ we have a canonical map of presheaves $\varphi_{V,U}^{pre} \colon R_V^{pre} \to R_U^{pre}$ and this induces a map of sheaves $\varphi_{V,U} \colon R_V \to R_U$. Moreover, this is easily seen to be functorial and we conclude that we have a functor

$$\begin{aligned} R_{(-)} : \boldsymbol{O}(X) &\longrightarrow \mathrm{Sh}_R(X) \\ U &\longmapsto R_U \\ V \subseteq U &\longmapsto \varphi_{V,U}. \end{aligned}$$

Now let $F \colon \mathrm{Sh}_R(X) \to \mathrm{Mod}_R$ be a contravariant functor that sends colimits in $\mathrm{Sh}_R(X)$ to limits in Mod_R. We define the functor $\mathcal{G}_F$ to be the composition

$$\boldsymbol{O}(X) \xrightarrow{R_{(-)}} \mathrm{Sh}_R(X) \xrightarrow{F} \mathrm{Mod}_R.$$

We claim that $\mathcal{G}_F$ is actually a sheaf of R-modules on X. Let $(U_i)_{i\in I}$ be a collection open subsets of X and let $U = \bigcup_{i\in I} U_i$. Consider the sequence of sheaves

$$\bigoplus_{i,j\in I} R_{U_i\cap U_j} \xrightarrow{\alpha} \bigoplus_{k\in I} R_{U_k} \xrightarrow{\beta} R_U \longrightarrow 0$$

where $\alpha = \bigoplus_{i,j\in I} \left(\varphi_{U_i\cap U_j,U_i} - \varphi_{U_i\cap U_j,U_j}\right)$ and $\beta = \bigoplus_{k\in I} \varphi_{U_k,U}$. This sequence is exact on the level of stalks, so it is an exact sequence of sheaves. As F transforms colimits into limits, it is a left exact contravariant functor and we get an exact sequence of abelian groups

$$0 \longrightarrow F(R_U) \xrightarrow{F(\beta)} \prod_{k\in I} F(R_{U_k}) \xrightarrow{F(\alpha)} \prod_{i,j\in I} F(R_{U_i\cap U_j}).$$

This sequence coincides with the sequence

$$0 \longrightarrow \mathcal{G}_F(U) \longrightarrow \prod_{k\in I} \mathcal{G}_F(U_k) \longrightarrow \prod_{i,j\in I} \mathcal{G}_F(U_i \cap U_j),$$

proving that $\mathcal{G}_F$ satisfies the sheaf property.

Before we prove that a contravariant functor $F \colon \mathrm{Sh}_R(X) \to \mathrm{Mod}_R$ sending colimit to limits is representable by $\mathcal{G}_F$, we need the following lemma.

Lemma 2.31. *Let X be a topological space and R a commutative ring. For any sheaf of R-modules $\mathcal{F}$ on X there exists a small category I and a diagram $G \colon I \to \mathrm{Sh}_R(X)$ taking values of the form R_U ($U \subseteq X$ open) such that $\mathcal{F}$ is the colimit of G.*

Proof. Let $\mathcal{I}(\mathcal{F})$ be the category whose objects are pairs (U, a) with $U \subseteq X$ open and $a \in \mathcal{F}(U)$ and arrows $(V, b) \to (U, a)$ are inclusions $V \subseteq U$ such that $a|_V = b$. This category is sometimes denoted by $\int_{\boldsymbol{O}(X)} \mathcal{F}$ and called the *Grothendieck construction* of $\mathcal{F}$. Let $G \colon \mathcal{I}(\mathcal{F}) \to \mathrm{Sh}_R(X)$ be the functor sending (U, a) to the sheaf R_U and $(V, b) \to (U, a)$ to the natural map $\varphi_{V,U} \colon R_V \to R_U$. Let $\mathrm{const}_{\mathcal{F}} \colon \mathcal{I}(\mathcal{F}) \to \mathrm{Sh}_R(X)$ be the constant functor taking the value $\mathcal{F}$. For any open $U \subseteq X$ we have $\mathrm{Hom}_{\mathrm{Sh}_R(X)}(R_U, \mathcal{F}) \cong \mathcal{F}(U)$ and therefore an element $a \in \mathcal{F}(U)$ determines a morphism of sheaves $\tilde{a} \colon R_U \to \mathcal{F}$. We get a natural transformation

$$\boldsymbol{a} \colon G \longrightarrow \mathrm{const}_{\mathcal{F}}$$

with components $\boldsymbol{a}_{(U,a)} = \tilde{a}$. We now get a morphism of sheaves

$$\operatorname{colim}_{(U,a)\in\mathcal{I}(\mathcal{F})} R_U \longrightarrow \operatorname{colim}_{\mathcal{I}(\mathcal{F})} \mathcal{F} = \mathcal{F},$$

which is easily seen to be an isomorphism. $\square$

Theorem 2.32 ([8, III.8.19])**.** *Let X be a topological space and R a commutative ring. let $F\colon \mathrm{Sh}_R(X) \to \mathrm{Mod}_R$ be a contravariant functor sending colimits in $\mathrm{Sh}_R(X)$ to limits in Mod_R. Then, F is representable by the sheaf $\mathcal{G}_F$.*

Proof. We start by a similar argument as in the proof of lemma 2.31. Let $\mathcal{I}$ be the category whose objects are pairs (U,a) with $U \subseteq X$ open and $a \in F(R_U)$ and arrows $f\colon (V,b) \to (U,a)$ are morphisms of sheaves $\overline{f}\colon R_V \to R_U$ such that $F(\overline{f})(b) = a$. Let $H\colon \mathcal{I} \to \mathrm{Sh}_R(X)$ be the functor sending (U,a) to R_U and sending $f\colon (V,b) \to (U,a)$ to $\overline{f}$. Let $\mathrm{const}_{\mathcal{G}_F}$ be the constant functor $\mathcal{I} \to \mathrm{Sh}_R(X)$ with value $\mathcal{G}_F$. For any open $U \subseteq X$ we have

$$\mathrm{Hom}_{\mathrm{Sh}_R(X)}(R_U, \mathcal{G}_F) \cong \mathcal{G}_F(U) = F(R_U),$$

so elements $a \in F(R_U)$ determine morphisms $\tilde{a}\colon R_U \to \mathcal{G}_F$ of sheaves. We get a natural transformation $\boldsymbol{a}\colon H \to \mathrm{const}_{\mathcal{G}_F}$ with components $\boldsymbol{a}_{(U,a)} = \tilde{a}$. This gives a morphism of sheaves

$$\varphi_{\boldsymbol{a}}\colon \operatorname{colim}_{(U,a)\in\mathcal{I}} R_U \longrightarrow \operatorname{colim}_{\mathcal{I}} \mathcal{G}_F = \mathcal{G}_F,$$

which is easily seen to be an isomorphism of sheaves. Using the fact that F transforms colimits into limits, applying F gives an isomorphism of R-modules

$$F(\mathcal{G}_F) \xrightarrow{\sim} F(\operatorname{colim}_{(U,a)\in\mathcal{I}} R_U) = \lim_{(U,a)\in\mathcal{I}} F(R_U).$$

The collection of elements $\{a : (U,a) \in \mathcal{I}\}$ determines a unique element e in the limit $\lim_{(U,a)\in\mathcal{I}} F(R_U) \cong F(\mathcal{G}_F)$. We now define

$$\begin{aligned} \psi_{\mathcal{F}}\colon \mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, \mathcal{G}_F) &\longrightarrow F(\mathcal{F}) \\ f &\longmapsto F(f)(e). \end{aligned}$$

If $\psi_{\mathcal{F}}$ is an isomorphism functorial in $\mathcal{F}$, then F is indeed representable by $\mathcal{G}_F$ and we are done. In the case that $\mathcal{F}$ equals R_U for some $U \subseteq X$, then $\psi_{\mathcal{F}}$ is an isomorphism by construction. As any sheaf of R-modules is the colimit of sheaves of the form R_U by the previous lemma, and as F transform colimits into limits, $\psi_{\mathcal{F}}$ is isomorphism for all $\mathcal{F}$. $\square$

2.4 The Duality Theorem

In this section we will state and prove the Duality Theorem for finite topological spaces as in [15]. An important part of the study of sheaf theory is the study of the so called *six functors*. We recall that five of these six functors are the following. Let X, Y be topological spaces, $f\colon X \to Y$ continuous, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X.

(i) The internal hom functor

$$\underline{\mathcal{H}om}(-,-)\colon \mathrm{Sh}_R(X)^{\mathrm{op}} \times \mathrm{Sh}_R(X) \longrightarrow \mathrm{Sh}_R(X).$$

(ii) The tensor product

$$-\otimes -\colon \mathrm{Sh}_R(X) \times \mathrm{Sh}_R(X) \longrightarrow \mathrm{Sh}_R(X).$$

(iii) The direct image functor

$$f_*\colon \mathrm{Sh}_R(X) \longrightarrow \mathrm{Sh}_R(Y).$$

(iv) The inverse image functor

$$f^{-1}\colon \mathrm{Sh}_R(Y) \longrightarrow \mathrm{Sh}_R(X).$$

(v) The direct image with proper support functor

$$f_!\colon \mathrm{Sh}_R(X) \longrightarrow \mathrm{Sh}_R(Y).$$

We recall the definition of this last functor. Recall that a continuous map between topological spaces is called *proper* if the inverse image of a compact set is compact.

Definition 2.33. Let X be a topological space, R a commutative ring, $\mathcal{F}$ a sheaf of R-modules on X, $U \subseteq X$ an open subset and $s \in \Gamma(U, \mathcal{F})$ a section. The *support* of s is the set

$$\mathrm{supp}(s) = \{x \in U : s_x \neq 0\}.$$

Definition 2.34. Let $f\colon X \to Y$ be a continuous map of topological spaces, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. The *direct image with proper support* of $\mathcal{F}$ is the sheaf $f_!\mathcal{F}$ on Y defined by

$$\Gamma(V, f_!\mathcal{F}) = \{s \in \Gamma(f^{-1}(V), \mathcal{F}) : f|_{\mathrm{supp}(s)}\colon \mathrm{supp}(s) \to Y \text{ is proper}\}.$$

We omit the verifications that $f_!\mathcal{F}$ is indeed a sheaf and that $f_!$ is functorial. We have adjunctions

$$- \otimes \mathcal{F} \dashv \underline{\mathcal{H}om}(\mathcal{F}, -)$$

and

$$f^{-1} \dashv f_*,$$

and might wonder if the functor $f_!$ is also part of an adjunction. Unfortunately, this is in general not the case. However, if we move to the derived categories then the derived functor $\boldsymbol{R}f_!\colon \mathbf{D}(X,R) \to \mathbf{D}(Y,R)$ has a right adjoint under suitable conditions on (X,Y,f,R), see for example [14]. We consider the case where f is a map of finite topological space. Any map f between finite spaces is proper and therefore we have $f_! = f_*$. Proving a Grothendieck Duality Theorem for finite spaces therefore comes down to finding a right adjoint to $\boldsymbol{R}f_*$. We start by recalling some notions of homological algebra.

Definition 2.35. Let $\mathcal{A}$ be an abelian category. A *bicomplex* or *double complex* is a collection of objects $(A^{i,j})_{i,j\in\mathbb{Z}}$ together with a collection of maps $(d^{i,j}_{\mathrm{hor}}\colon A^{i,j} \to A^{i+1,j})_{i,j\in\mathbb{Z}}$ and a collection of maps $(d^{i,j}_{\mathrm{vert}}\colon A^{i,j} \to A^{i,j+1})_{i,j\in\mathbb{Z}}$ such that $A^{\bullet,j}$ is a complex for every $j \in \mathbb{Z}$, $A^{i,\bullet}$ is a complex for every $i \in \mathbb{Z}$ and all squares

$$\begin{tikzcd} A^{i,j} \arrow[r, "d^{i,j}_{\mathrm{hor}}"] \arrow[d, "d^{i,j}_{\mathrm{vert}}"'] & A^{i+1,j} \arrow[d, "d^{i+1,j}_{\mathrm{vert}}"] \\ A^{i,j+1} \arrow[r, "d^{i,j+1}_{\mathrm{hor}}"'] & A^{i+1,j+1} \end{tikzcd} \tag{2.6}$$

commute.

It should be noted that the definition of a bicomplex differs between authors; a lot of authors require the square (2.6) to be anti-commutative. For any $i \in \mathbb{Z}$ the collection $d_{\text{hor}}^{i,\bullet}$ is a morphism of complexes $A^{i,\bullet} \to A^{i+1,\bullet}$ and we have $d_{\text{hor}}^{i+1,\bullet} \circ d_{\text{hor}}^{i,\bullet} = 0$. Hence, a bicomplex in $\mathcal{A}$ is the same as a complex in $\mathbf{C}(\mathcal{A})$. To a bicomplex one can associate a singly graded complex.

Definition 2.36. Let $\mathcal{A}$ be an abelian category and $A^{\bullet,\bullet}$ a bicomplex in $\mathcal{A}$. The *singly graded complex associated to* $A^{\bullet,\bullet}$ or the *(product) total complex* of $A^{\bullet,\bullet}$ is the complex $\mathrm{Tot}(A^{\bullet,\bullet})^{\bullet}$ defined by

$$\mathrm{Tot}(A^{\bullet,\bullet})^n = \prod_{i+j=n} A^{i,j}$$

and differential

$$d^n = \prod_{i+j=n} \left(d_{\text{hor}}^{i,j} + (-1)^i d_{\text{vert}}^{i,j} \right). \tag{2.7}$$

Note that the definition of the total complex above could also be made using the direct sum instead of the direct product. The resulting total complex is called the *direct sum total complex*. We will see that for our purposes, it makes no difference whether we work with the product or the direct sum. The sign in (2.7) is used in the proof that the differential satisfies $dd = 0$. If a bicomplex is defined by requiring the square (2.6) to be anti-commutative, then the sign is not needed.

Proposition 2.37. *Let $\mathcal{A}$ be an abelian category and $A^{\bullet,\bullet}$ a bicomplex in $\mathcal{A}$. The total complex* $\mathrm{Tot}(A^{\bullet,\bullet})^{\bullet}$ *defined in 2.36 is well-defined, that is, $dd = 0$.*

Proof. Let $n \in \mathbb{Z}$. We have

$$\begin{aligned} d^{n+1}d^n = \prod_{i+j=n+1} & d_{\text{hor}}^{i-1,j} d_{\text{hor}}^{i-2,j} + (-1)^{i-1} d_{\text{hor}}^{i-1,j} d_{\text{vert}}^{i-1,j-1} \\ & + (-1)^i d_{\text{vert}}^{i,j-1} d_{\text{hor}}^{i-1,j-1} + (-1)^{2i} d_{\text{vert}}^{i,j-1} d_{\text{vert}}^{i,j-2}. \end{aligned}$$

The argument is concluded by the fact that the rows and columns are complexes and by commutativity of (2.6). ☐

We introduce a certain total complex that will play an important role in the Duality Theorem.

Definition 2.38. Let X be a topological space, R a commutative ring and $\mathcal{H}^{\bullet}$ a complex of sheaves of R-modules on X. We define the complex $\boldsymbol{C}^{\bullet}(\mathcal{H}^{\bullet})$ to be the singly graded complex associated to the double complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & G^0\mathcal{H}^{-1} & \longrightarrow & G^1\mathcal{H}^{-1} & \longrightarrow & G^2\mathcal{H}^{-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & G^0\mathcal{H}^{0} & \longrightarrow & G^1\mathcal{H}^{0} & \longrightarrow & G^2\mathcal{H}^{0} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & G^0\mathcal{H}^{1} & \longrightarrow & G^1\mathcal{H}^{1} & \longrightarrow & G^2\mathcal{H}^{1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Definition 2.39. Let $\mathcal{A}$ be an abelian category. A bicomplex $A^{\bullet,\bullet}$ is called *biregular* if for all $n \in \mathbb{Z}$ we have $A^{i,j} = 0$ for all but finitely many $(i,j) \in \mathbb{Z}^2$ with $i + j = n$.

We see that for biregular complexes the direct sum total complex and direct product total complex coincide. Note that if X is a finite topological space, then $G^i\mathcal{F}$ is only non-zero for $0 \leq i \leq \dim X$. Hence, $\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$ is biregular and we have

$$\boldsymbol{C}^n(\mathcal{H}^\bullet) = \bigoplus_{p=0}^{\dim X} G^p\mathcal{H}^{n-p},$$

for all $n \in \mathbb{Z}$.

We have the following result on biregular complexes.

Proposition 2.40. *Let $\mathcal{A}$ be an abelian category and $A^{\bullet,\bullet}, B^{\bullet,\bullet}$ biregular double complexes in $\mathcal{A}$. If $f\colon A^{\bullet,\bullet} \to B^{\bullet,\bullet}$ is a morphism of double complexes such that $f^{\bullet,j}\colon A^{\bullet,j} \to B^{\bullet,j}$ is a quasi-isomorphism for all $j \in \mathbb{Z}$, then*

$$\mathrm{Tot}(f)\colon \mathrm{Tot}(A^{\bullet,\bullet})^\bullet \longrightarrow \mathrm{Tot}(B^{\bullet,\bullet})^\bullet$$

is a quasi-isomorphism.

Proof. See for example corollary 2.7.27 of [19]. □

Corollary 2.41. *Let X be a finite topological space, R a commutative ring and $\mathcal{H}^\bullet$ a complex of sheaves of R-modules on X. We have a canonical quasi-isomorphism*

$$\mathcal{H}^\bullet \longrightarrow \mathrm{C}^\bullet(\mathcal{H}^\bullet).$$

Remark 2.42. In the language of derived categories: for any complex $\mathcal{H}^\bullet$ of sheaves of R-modules on a topological space X and any map $f\colon X \to Y$ the object $\boldsymbol{R}f_*\mathcal{H}^\bullet$ in $\mathbf{D}(Y,R)$ is defined to be $f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$.

Another important notion of homological algebra, is that of the *hom complex*.

Definition 2.43. Let $\mathcal{A}$ be an abelian category and $A^\bullet$ and $B^\bullet$ complexes in A. The *hom complex* of $A^\bullet$ and $B^\bullet$ is the complex $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ in $\mathcal{A}$ defined by

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_{i+j=n} \mathrm{Hom}_{\mathcal{A}}(A^{-i}, B^j)$$

with differential

$$(d^n \boldsymbol{f})(i,j) = d_B \circ f_{i,j-1} - (-1)^n f_{i-1,j} \circ d_A.$$

Note that the hom complex of $A^\bullet$ and $B^\bullet$ is just the total complex of the bicomplex

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A^{-j+1}, B^{i-1}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(A^{-j+1}, B^{i}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(A^{-j+1}, B^{i+1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A^{-j}, B^{i-1}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(A^{-j}, B^{i}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(A^{-j}, B^{i+1}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(A^{-j-1}, B^{i-1}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(A^{-j-1}, B^{i}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(A^{-j-1}, B^{i+1}) \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with horizontal differential

$$\begin{aligned} d^{i,j}_{\mathrm{hor}} \colon \mathrm{Hom}_{\mathcal{A}}(A^{-j}, B^i) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(A^{-j}, B^{i+1}) \\ f &\longmapsto d^i_B \circ f \end{aligned}$$

and vertical differential

$$\begin{aligned} d^{i,j}_{\mathrm{vert}} \colon \mathrm{Hom}_{\mathcal{A}}(A^{-j}, B^i) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(A^{-j-1}, B^i) \\ f &\longmapsto (-1)^j f \circ d_A^{-j-1}. \end{aligned}$$

We have the following important result on the preservation of quasi-isomorphisms by $\mathrm{Hom}^\bullet(-,-)$.

Theorem 2.44. *Let $\mathcal{A}$ be an abelian category, $P^\bullet$ a bounded above complex of projectives, $I^\bullet$ a bounded below complex of injectives and $A^\bullet \to B^\bullet$ a quasi-isomorphism. The induced maps*

$$\mathrm{Hom}^\bullet(P^\bullet, A^\bullet) \longrightarrow \mathrm{Hom}^\bullet(P^\bullet, B^\bullet) \tag{2.8}$$

and

$$\mathrm{Hom}^\bullet(B^\bullet, I^\bullet) \longrightarrow \mathrm{Hom}^\bullet(A^\bullet, I^\bullet) \tag{2.9}$$

are quasi-isomorphisms.

Proof Sketch. Let $f \colon A^\bullet \to B^\bullet$ be a quasi-isomorphism. The mapping cone $C(f)$ (see for example page 45 of [19]) is acyclic. It follows that $\mathrm{Hom}^\bullet(P^\bullet, C(f))$ and $\mathrm{Hom}^\bullet(C(f), I^\bullet)$ are acyclic. These complexes are the mapping cones of the induced maps (2.8) and (2.9) respectively (see [10, 5.3]). □

We now turn to the proof of the Duality Theorem.

Lemma 2.45. *Let X be a topological space that admits a basis of compact opens and has the property that finite intersections of compact opens are compact. Let R be a commutative ring. Taking filtered colimits of sheaves of R-modules on X commutes with taking sections of compact opens.*

Proof. See lemma 20.20.1 of [18]. □

Corollary 2.46. *Let $f \colon X \to Y$ be a continuous map of finite topological spaces. Then the functor $f_* \colon \mathrm{Sh}_R(X) \to \mathrm{Sh}_R(Y)$ preserves filtered colimits.*

Proof. Let $\mathcal{I} \to \mathrm{Sh}_R(X)$, $i \mapsto \mathcal{F}_i$ be a filtered diagram and $\mathcal{F} = \mathrm{colim}_{\mathcal{I}} \mathcal{F}_i$. As X and Y are finite spaces, all opens are compact and therefore X and Y certainly satisfy the assumptions of lemma 2.45. Let $U \subseteq Y$ open, using lemma 2.45 we have

$$\begin{aligned} \Gamma(U, f_*\mathcal{F}) = \Gamma(f^{-1}(U), \mathrm{colim}_{\mathcal{I}} \mathcal{F}_i) &\cong \mathrm{colim}_{\mathcal{I}} \Gamma(f^{-1}(U), \mathcal{F}_i) \\ &= \mathrm{colim}_{\mathcal{I}} \Gamma(U, f_*\mathcal{F}_i) \cong \Gamma(U, \mathrm{colim}_{\mathcal{I}} f_*\mathcal{F}_i). \end{aligned}$$

These isomorphisms are natural with respect to the restrictions and we conclude

$$f_*\mathcal{F} = \mathrm{colim}_{\mathcal{I}} f_*\mathcal{F}_i.$$

□

Lemma 2.47. *Let $f\colon X \to S$ be a continuous map of finite topological spaces. Let R be a commutative ring and $p \geq 0$. The functor $\mathrm{Sh}_R(X) \to \mathrm{Sh}_R(S)$ given by $\mathcal{F} \mapsto f_*(G^p\mathcal{F})$ is exact and preserves all colimits.*

Proof. We will call the relevant functor A^p. As $G^p\mathcal{F}$ is flasque for any abelian sheaf $\mathcal{F}$ on X, it is f_*-acyclic . The functor G^p is itself exact (proposition 2.27) and it follows that A^p is an exact functor. Hence, it is enough to prove that A^p preserves all filtered colimits. As filtered colimits commute with finite direct products, the functor G^p is easily seen to preserve filtered colimits using Theorem 2.14. Finally, f is a morphism of finite spaces and it follows that f_* preserves filtered colimits by the previous corollary. □

Theorem 2.48 ([4]). *Let $f\colon X \to S$ be a continuous map of finite topological spaces. Let R be a commutative ring and let $\mathcal{M}^\bullet$ be a complex of sheaves of R-modules on S. There exists a complex $f^\nabla(\mathcal{M}^\bullet)^\bullet$ of sheaves of R-modules on X such there is an isomorphism of complexes of R-modules*

$$\mathrm{Hom}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet), \mathcal{M}^\bullet) \cong \mathrm{Hom}^\bullet(\mathcal{H}^\bullet, f^\nabla(\mathcal{M}^\bullet)^\bullet),$$

that is functorial in $\mathcal{H}^\bullet$. Moreover, if $\mathcal{M}^\bullet$ is a complex of injective sheaves, so is $f^\nabla(\mathcal{M}^\bullet)^\bullet$.

Proof. Let $p, q \in \mathbb{Z}$. We define the functor

$$\begin{aligned} F^{p,q}\colon \mathrm{Sh}_R(X) &\longrightarrow \mathrm{Mod}_R \\ \mathcal{F} &\longmapsto \mathrm{Hom}_{\mathrm{Sh}_R(S)}(f_*(G^p\mathcal{F}),\ \mathcal{M}^q). \end{aligned}$$

By lemma 2.47 the functor $\mathcal{F} \mapsto f_*(G^p\mathcal{F})$ preserves colimits. The representable functor $\mathrm{Hom}_{\mathrm{Sh}(S)}(-, \mathcal{M}^q)$ transforms colimits in $\mathrm{Sh}_R(S)$ into limits in Mod_R. We conclude that $F^{p,q}$ is a contravariant functor transforming colimits in $\mathrm{Sh}_R(X)$ into limits in Mod_R. By Theorem 2.32 we conclude that $F^{p,q}$ is representable by the sheaf $\mathcal{D}^{-p,q} = \mathcal{G}_{F^{p,q}}$. Moreover, if $\mathcal{M}^q$ is injective, then $F^{p,q}$ is exact (f_*G^p is exact and taking the Hom into an injective object is exact) and it follows that $\mathcal{D}^{-p,q}$ is actually an injective sheaf. The differential $d^q\colon \mathcal{M}^q \to \mathcal{M}^{q+1}$ induces a differential $\mathcal{D}^{p,q} \to \mathcal{D}^{p,q+1}$, so that for fixed p we have a complex $\mathcal{D}^{p,\bullet}$. Moreover, the natural transformation $f_*G^p \to f_*G^{p+1}$ induces a morphism of sheaves $\mathcal{D}^{p,q} \to \mathcal{D}^{p+1,q}$ and for fixed q we have a complex $\mathcal{D}^{\bullet,q}$. We define the complex $\mathcal{D}_f^\bullet$ to be the singly graded complex associated to $\mathcal{D}^{\bullet,\bullet}$. That is, for any $n \in \mathbb{Z}$ we have

$$\mathcal{D}_f^n = \prod_{p+q=n} \mathcal{D}^{p,q}.$$

Let $\mathcal{H}^\bullet$ be a complex of abelian sheaves on X. and let $n \in \mathbb{Z}$. Using the definition of the hom complex and the complex $\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$ we have

$$\begin{aligned} \mathrm{Hom}^n(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet),\ \mathcal{M}^\bullet) &= \prod_{i\in\mathbb{Z}} \mathrm{Hom}(f_*\boldsymbol{C}^i(\mathcal{H}^\bullet), \mathcal{M}^{n+i}) \\ &= \prod_{i\in\mathbb{Z}} \mathrm{Hom}\left(\bigoplus_{p\in\mathbb{Z}} f_*(G^p\mathcal{H}^{i-p}), \mathcal{M}^{n+i}\right). \end{aligned} \tag{2.10}$$

We can get the direct sum out of the hom and use the representability of $F^{p,n+i}$ to obtain

$$\begin{aligned} \mathrm{Hom}^n(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet),\ \mathcal{M}^\bullet) &= \prod_{i\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} \mathrm{Hom}(f_*(G^p\mathcal{H}^{i-p}), \mathcal{M}^{n+i}) \\ &= \prod_{i\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} F^{p,n+i}(\mathcal{H}^{i-p}) = \prod_{i\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} \mathrm{Hom}(\mathcal{H}^{i-p}, \mathcal{D}^{-p,n+i}). \end{aligned} \tag{2.11}$$

Substituting $q = i - p$ we conclude

$$\begin{aligned}
\mathrm{Hom}^n(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet),\ \mathcal{M}^\bullet) &= \prod_{q\in\mathbb{Z}}\prod_{p\in\mathbb{Z}} \mathrm{Hom}(\mathcal{H}^q, \mathcal{D}^{-p,n+p+q}) \\
&= \prod_{q\in\mathbb{Z}} \mathrm{Hom}\left(\mathcal{H}^q, \prod_{p\in\mathbb{Z}} \mathcal{D}^{-p,n+p+q}\right) \\
&= \prod_{q\in\mathbb{Z}} \mathrm{Hom}(\mathcal{H}^q, \mathcal{D}_f^{n+q}) \\
&= \mathrm{Hom}^n(\mathcal{H}^\bullet, \mathcal{D}_f^\bullet).
\end{aligned}$$

□

Definition 2.49. Let X, S be finite topological space, $f\colon X \to S$ continuous and $g\colon X \to \{*\}$ the map to the singleton. Let R be a commutative ring, M an R-module and $\mathcal{M}^\bullet$ a bounded below complex of sheaves of R-modules on S.

(i) If $\mathcal{I}^\bullet$ is a bounded below complex of injective sheaves on S that is quasi-isomorphic to $\mathcal{M}^\bullet$, then we define the *dualizing complex* of X for $\mathcal{M}^\bullet$ (with respect to f) as the complex $f^!(\mathcal{M}^\bullet)^\bullet = f^\nabla(\mathcal{I}^\bullet)^\bullet$. This dualizing complex is unique up to quasi-isomorphism and, abusing notation somewhat, we simply write $f^!\mathcal{M}^\bullet$. Abusing notation even more, if $\mathcal{M}^\bullet = \mathcal{M}[0]$ for some sheaf of R-modules $\mathcal{M}$ on S we write $f^!\mathcal{M}^\bullet$ for $f^!(\mathcal{M}[0])^\bullet$.

(ii) The complex $g^!M^\bullet_{\{*\}}$ is called the *dualizing complex* of (X, M) and denoted by $\mathcal{D}^\bullet_{M,X}$.

(iii) In a context where R is understood, the complex $g^!R^\bullet_{\{*\}}$ is called the *dualizing complex* of X and denoted by $\mathcal{D}^\bullet_X$.

Remark 2.50. Suppose that in the context of the Duality Theorem $\mathcal{I}^\bullet$ is an injective resolution of some sheaf $\mathcal{M}$ on S. Then $\mathcal{I}^\bullet$ is zero in degree < 0. Moreover, $G^p\mathcal{F}$ is zero for all $p > \dim X$ and all $\mathcal{F}$. We conclude that $f^!\mathcal{M}^\bullet$ is zero in all degrees $< -\dim X$.

Recall that a commutative ring is called *hereditary* if all its ideals are projective as module. For example, all Dedekind domains are hereditary, so in particular all principal ideal domains and all fields.

Corollary 2.51 ([15]). *Let X be a finite topological space, R a commutative hereditary ring and $\mathcal{F}$ a sheaf of R-modules on X. Let M be an R-module. For any $n \geq 0$ we have an isomorphism*

$$h^{-n}(\mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, \mathcal{D}^\bullet_{M,X})) \cong \mathrm{Ext}^1_R(H^{n+1}(X,\mathcal{F}), M) \oplus \mathrm{Hom}_R(H^n(X,\mathcal{F}), M).$$

Proof. Let $P^\bullet$ be a bounded below complex of projective R-modules that is quasi-isomorphic to $\Gamma(X, G^\bullet\mathcal{F})$. By the Universal Coefficient Theorem (see for example theorem 12.43 of [7]) we have a split exact sequence

$$0 \to \mathrm{Ext}^1_R(h^{n+1}(P^\bullet), M) \to h_n(\mathrm{Hom}(P^\bullet, M)) \to \mathrm{Hom}_R(h^n(P^\bullet), M) \to 0.$$

for all $n \geq 0$. Let $I^\bullet$ be an injective resolution of M. Using Theorem 2.44 we have

$$h_n(\mathrm{Hom}(P^\bullet, M)) = h^{-n}(\mathrm{Hom}^\bullet(P^\bullet, M[0])) = h^{-n}(\mathrm{Hom}^\bullet(\Gamma(X, G^\bullet\mathcal{F}), I^\bullet)).$$

From the Duality Theorem it now follows that

$$h_n(\mathrm{Hom}(P^\bullet, M)) = h^{-n}(\mathrm{Hom}^\bullet(\Gamma(X, G^\bullet\mathcal{F}), I^\bullet)) = h^{-n}(\mathrm{Hom}^\bullet(\mathcal{F}[0], \mathcal{D}^\bullet_{M,X})).$$

□

We finish this section with the derived point of view. Let $f\colon X \to Y$ be a map of finite spaces, R a commutative ring and $\mathcal{H}^\bullet, \mathcal{F}^\bullet$ a bounded below complex of sheaves on X. The object $\boldsymbol{R}\mathrm{Hom}(\mathcal{H}^\bullet, \mathcal{F}^\bullet)$ in $\mathbf{D}^+(\mathrm{Mod}_R)$ is by defintion $\mathrm{Hom}^\bullet(\mathcal{H}^\bullet, \mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a bounded below complex of injectives quasi-isomorphic to $\mathcal{F}^\bullet$. As already mentioned in remark 2.42 we have an isomorphism $\boldsymbol{R}f_*\mathcal{H}^\bullet \cong f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$ in $\mathbf{D}^+(X, R)$. The Duality Theorem now states that

$$\boldsymbol{R}\mathrm{Hom}(\boldsymbol{R}f_*(\mathcal{H}^\bullet),\ \mathcal{F}^\bullet) \cong \boldsymbol{R}\mathrm{Hom}(\mathcal{H}^\bullet,\ f^!\mathcal{F}^\bullet).$$

That is, the functor $f^!\colon \mathbf{D}^+(S, R) \to \mathbf{D}^+(X, R)$ is the right adjoint to $\boldsymbol{R}f_*\colon \mathbf{D}^+(X, R) \to \mathbf{D}^+(S, R)$. By introducing *homotopically injective* (also called *K-injective*) sheaves and defining $f^!\mathcal{F}^\bullet$ to be $f^\nabla(\mathcal{I}^\bullet)^\bullet$ for a complex of homotopically injective sheaves that is quasi-isomorphic to $\mathcal{F}^\bullet$, we can actually make everything work and get an adjunction between the non-bounded derived categories.

2.5 Extending the Duality Theorem to Locally Finite Spaces

In this subsection we will show that the Duality Theorem for finite spaces given in the previous subsection can actually be extended to *locally finite* topological spaces. To this end we start with a short general study of locally finite spaces.

Definition 2.52. A topological space X is called *locally finite* if any point $x \in X$ has a finite open neighbourhood.

Example 2.53. (i) Any finite space is locally finite.

(ii) Any discrete space is locally finite.

(iii) Consider $\mathbb{Z}_{\leq 0}$ with the usual partial order $\leq$. The set $\mathbb{Z}_{\leq 0}$ together with the Alexandrov topology induced by $\leq$ is locally finite.

Note that locally finite spaces are not necessarily finite dimensional. Indeed, the space $\mathbb{Z}_{\leq 0}$ in the example above has infinite dimension.

The two functors $S\colon \mathrm{Top} \to \mathrm{Pos}$ and $A\colon \mathrm{Pos} \to \mathrm{Top}$ of section 2.1 are inverses when restricted to locally finite spaces and posets with the property that $\{y : y > x\}$ is finite for all elements x. It follows that locally finite spaces are Alexandrov-discrete and in particular theorem 2.11 implies that sheaves on locally finite spaces are completely determined by the stalks and the maps between the stalks. The following observation will be important later on.

Remark 2.54. A subset of a locally finite space is compact if and only if it is finite.

Any map $f\colon X \to S$ between finite spaces is proper. Therefore, proving a Grothendieck Duality Theorem for finite spaces comes down to finding a right adjoint of $\boldsymbol{R}f_*$. However, not all maps between locally finite spaces are proper. Indeed, if S is the singleton and X is infinite, then remark 2.54 assures that f is not proper and consequently the equality $f_! = f_*$ does not hold. As flasque sheaves are $f_!$-acyclic, the Godement resolution is still a "good" resolution to work with. We can replace the role that $f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$ plays in the Duality Theorem for finite spaces by $f_!\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$.

Remark 2.55. In the language of derived categories: for any complex of sheaves of R-modules on a topological space X and any continuous map $f\colon X \to Y$, the object $\boldsymbol{R}f_!\mathcal{H}^\bullet$ in $\mathbf{D}(Y, R)$ is by definition $f_!\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$.

While studying the proof of the Duality Theorem of the previous subsection, one sees that the finiteness condition on the topological space is only really used twice.

(i) In the proof of the duality theorem, it is used that for any complex $\mathcal{H}^\bullet$ on a finite space X and any $n \in \mathbb{Z}$ we have

$$\boldsymbol{C}^n(\mathcal{H}^\bullet) = \bigoplus_{p\in\mathbb{Z}} G^p\mathcal{H}^{n-p}. \tag{2.12}$$

Recall that by definition we have

$$\boldsymbol{C}^n(\mathcal{H}^\bullet) = \prod_{p\in\mathbb{Z}} G^p\mathcal{H}^{n-p}.$$

The fact that (2.12) holds is used when moving from (2.10) to (2.11).

(ii) An important ingredient for the proof of the Duality Theorem is lemma 2.47, which relies heavily on the finiteness condition.

We first prove that (2.12) also holds when X is locally finite. To this end we introduce the *codimension* of a point in a topological space.

Definition 2.56. Let X be a topological space. The *codimension* of a point $x \in X$ is the element of $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined by

$$\operatorname{codim}_X(x) = \sup\{n \in \mathbb{Z}_{\geq 0} : \exists x_0, \dots, x_n \in X \text{ such that } x = x_0 < \dots < x_n\}.$$

Let X be locally finite and $x \in X$. Let R be a commutative ring and $\mathcal{H}^\bullet$ a complex of sheaves of R-modules on X. As X is locally finite, $\operatorname{codim}_X(x)$ is finite and for any sheaf of R-modules $\mathcal{F}$ on X and any $p > \operatorname{codim}_X(x)$ we have

$$(G^p\mathcal{F})_x = \prod_{x\leq y_0<\dots<y_p} \mathcal{F}_{y_p} = 0.$$

For $n \in \mathbb{Z}$, we now find

$$\begin{aligned}\boldsymbol{C}^n(\mathcal{H}^\bullet)_x &= \left(\prod_{p=0}^{\infty} G^p\mathcal{H}^{n-p}\right)_x \overset{\star}{=} \prod_{p=0}^{\infty}(G^p\mathcal{H}^{n-p})_x = \prod_{p=0}^{\operatorname{codim}_X(x)} G^p(\mathcal{H}^{n-p})_x \\ &= \bigoplus_{p=0}^{\operatorname{codim}_X(x)} (G^p\mathcal{H}^{n-p})_x = \bigoplus_{p=0}^{\infty}(G^p\mathcal{H}^{n-p})_x = \left(\bigoplus_{p=0}^{\infty} G^p\mathcal{H}^{n-p}\right)_x.\end{aligned}$$

These equalities respect restrictions to generizations and we conclude that (2.12) also holds for locally finite spaces. Note that $\star$ is not valid in general, as taking products of sheaves and taking stalks do not generally commute. However, for a family $\{\mathcal{F}_i\}_{i\in I}$ of sheaves on an Alexandrov-discrete spaces we have

$$\left(\prod_{i\in I}\mathcal{F}_i\right)_x = \left(\prod_{i\in I}\mathcal{F}_i\right)(U_x) = \prod_{i\in I}\mathcal{F}_i(U_x) = \prod_{i\in I}(\mathcal{F}_i)_x.$$

We now turn to lemma 2.47, which needs to be proved for locally finite spaces. Recall that instead of working with the direct image f_*, we are working with the proper direct image $f_!$. So we want to prove the following fact.

> For any map $f\colon X \to S$ of locally finite spaces and any $p \geq 0$, the functor $\operatorname{Sh}_R(X) \to \operatorname{Sh}_R(S)$ given by $\mathcal{F} \mapsto f_! G^p\mathcal{F}$ is exact and preserves colimits.

As G^p is an exact functor and $G^p\mathcal{F}$ is $f_!$-acyclic for all sheaves $\mathcal{F}$, exactness of this functor requires no new work. Exact functors preserve finite colimits and it remains to prove that $\mathcal{F} \mapsto f_! G^p\mathcal{F}$ preserves filtered colimits. We first prove that G^p preserves filtered colimits.

Proposition 2.57. *Let X be a locally finite topological space. For any $p \geq 0$ the functor G^p commutes with filtered colimits.*

Proof. Let $\mathcal{I} \to \mathrm{Sh}(X)$ be a diagram with $\mathcal{I}$ a filtered category. Let $\mathcal{F} = \mathrm{colim}_{i\in\mathcal{I}} \mathcal{F}_i$ be its colimit. Let $x \in X$. We have

$$\mathrm{colim}_{i\in\mathcal{I}}(G^p\mathcal{F}_i)_x = \mathrm{colim}_{i\in\mathcal{I}}\Gamma(U_x, G^p\mathcal{F}_i) = \mathrm{colim}_{i\in I} \prod_{x\leq x_0<\ldots<x_p} (\mathcal{F}_i)_{x_p}.$$

As X is locally finite the product in the equation above is finite and since filtered colimits commute with finite limits we get

$$\mathrm{colim}_{i\in\mathcal{I}}(G^p\mathcal{F}_i)_x = \prod_{x\leq x_0<\ldots<x_p} \mathrm{colim}_{i\in\mathcal{I}}(\mathcal{F}_i)_{x_p}.$$

Taking stalks commutes with all colimits and thus

$$\left(\mathrm{colim}_{i\in\mathcal{I}}\, G^p\mathcal{F}_i\right)_x = \mathrm{colim}_{i\in\mathcal{I}}(G^p\mathcal{F}_i)_x = \prod_{x\leq x_0<\ldots<x_p} \mathcal{F}_{x_p} = (G^p\mathcal{F})_x.$$

These isomorphisms respect the maps between the stalks and as sheaves on locally finite spaces are completely determined by the stalks and the maps between these stalks, we conclude

$$\mathrm{colim}_{i\in\mathcal{I}}\, G^p\mathcal{F}_i = G^p\mathcal{F}.$$

□

We will now prove that $f_!$ preserves filtered colimits. To this end we first introduce the module of sections with compact support.

Definition 2.58. Let X be a topological space, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. For any open $U \subseteq X$ we define the sub-R-module

$$\Gamma_c(U,\mathcal{F}) = \{s \in \Gamma(U,\mathcal{F}) : \mathrm{supp}(s) \text{ is compact}\}$$

of $\Gamma(U,\mathcal{F})$. This is called the module of *sections with compact support*.

To see that $\Gamma_c(U,\mathcal{F})$ is indeed a submodule, note that for any $s,t \in \Gamma(U,\mathcal{F})$ and any $r \in R$, we have $\mathrm{supp}(s+t) \subseteq \mathrm{supp}(s) \cup \mathrm{supp}(t)$ and $\mathrm{supp}(rs) \subseteq \mathrm{supp}(s)$ and closed subsets of compact sets are compact. Note that if $f\colon X \to \{*\}$ is the map to the singleton, then

$$\Gamma(\{*\}, f_!\mathcal{F}) = \Gamma_c(X,\mathcal{F}).$$

Taking sections does in general not interact well with filtered colimits (for a situation in which is does act well, see lemma 2.45), we will need the following result on the interaction between taking sections with compact support and filtered colimits.

Theorem 2.59. *Let X be a locally compact topological space, R a commutative ring, $\mathcal{I} \to \mathrm{Sh}_R(X)$, $i \mapsto \mathcal{F}_i$ a filtered diagram and $\mathcal{F}$ its colimit. For any open subset $U \subseteq X$ the canonical map*

$$\mathrm{colim}_{\mathcal{I}}\,\Gamma_c(U,\mathcal{F}_i) \longrightarrow \Gamma_c(U,\mathcal{F})$$

is an isomorphism.

Proof. See proposition 3.10 of [14] or Theorem III.5.1 of [13]. □

We will also need the following proposition.

Proposition 2.60. *Let $f\colon X \to S$ be a continuous map of topological spaces with X locally finite. Let R be a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. For any compact open $V \subseteq S$ we have*

$$\Gamma(V, f_!\mathcal{F}) = \Gamma_c(f^{-1}(V), \mathcal{F}).$$

Proof. Let $s \in \Gamma(V, f_!\mathcal{F}) \subseteq \Gamma(f^{-1}(V), \mathcal{F})$. Then, the map $f|_{\mathrm{supp}(s)}\colon \mathrm{supp}(s) \to S$ is proper. We have $\mathrm{supp}(s) \subseteq f^{-1}(V)$ and thus $\mathrm{supp}(s) = (f|_{\mathrm{supp}(s)})^{-1}(V)$. As V is compact, we conclude that $\mathrm{supp}(s)$ is compact. Hence, $s \in \Gamma_c(f^{-1}(f), \mathcal{F})$.

For the other inclusion, let $s \in \Gamma_c(f^{-1}(V), \mathcal{F})$. Then $\mathrm{supp}(s)$ is compact. Moreover, as X is locally finite and compact sets in locally finite spaces are precisely the finite sets, all subsets of $\mathrm{supp}(s)$ are compact as well. Therefore, the map $f|_{\mathrm{supp}(s)}\colon \mathrm{supp}(s) \to S$ is proper. We conclude $s \in \Gamma(V, f_!\mathcal{F})$. □

We are now able to prove the wanted result.

Proposition 2.61. *Let $f\colon X \to S$ be a continuous map of locally finite spaces. Let R be a commutative ring. The functor $f_!\colon \mathrm{Sh}_R(X) \to \mathrm{Sh}_R(S)$ preserves filtered colimits.*

Proof. Let $\mathcal{I} \to \mathrm{Sh}_R(X)$, $i \mapsto \mathcal{F}_i$ be a filtered diagram and $\mathcal{F}$ its colimit. Let $x \in S$. As U_x is finite and thus compact, we can use proposition 2.60 to find

$$(f_!\mathcal{F})_x = \Gamma(U_x, f_!\mathcal{F}) = \Gamma_c(f^{-1}(U_x), \mathcal{F}).$$

Now Theorem 2.59 and proposition 2.60 gives

$$(f_!\mathcal{F})_x \cong \mathrm{colim}_{\mathcal{I}}\, \Gamma_c(f^{-1}(U_x), \mathcal{F}_i) = \mathrm{colim}_{\mathcal{I}}\, \Gamma(U_x, f_!\mathcal{F}_i).$$

As U_x is compact and any subset of U_x is also compact we have $\Gamma(U_x, \mathcal{H}) = \Gamma_c(U_x, \mathcal{H})$ for all sheaves of R-modules $\mathcal{H}$ on S. Hence,

$$\begin{aligned}(f_!\mathcal{F})_x &= \mathrm{colim}_{\mathcal{I}}\, \Gamma(U_x, f_!\mathcal{F}_i) = \mathrm{colim}_{\mathcal{I}}\, \Gamma_c(U_x, f_!\mathcal{F}_i)\\ &\cong \Gamma_c(U_x, \mathrm{colim}_{\mathcal{I}}\, f_!\mathcal{F}_i) = \Gamma(U_x, \mathrm{colim}_{\mathcal{I}}\, f_!\mathcal{F}_i)\\ &= (\mathrm{colim}_{\mathcal{I}}\, f_!\mathcal{F}_i)_x\,.\end{aligned}$$

The isomorphisms respect the maps between the stalks and we conclude

$$f_!\mathcal{F} \cong \mathrm{colim}_{\mathcal{I}}\, f_!\mathcal{F}_i.$$

□

We conclude that the functor $\mathcal{F} \mapsto f_!G^p\mathcal{F}$ is indeed exact and colimit preserving, as wanted. We can now conclude that we have a Grothendieck Duality Theorem for locally finite spaces.

Theorem 2.62. *Let $f\colon X \to S$ be a continuous map of locally finite topological spaces. Let R be a commutative ring and let $\mathcal{M}^\bullet$ be a complex of sheaves of R-modules on S. There exists a complex $f^\nabla(\mathcal{M}^\bullet)^\bullet$ of sheaves of R-modules on X such that there is an isomorphism of complexes of R-modules*

$$\mathrm{Hom}^\bullet(f_!\boldsymbol{C}^\bullet(\mathcal{H}^\bullet), \mathcal{M}^\bullet) \cong \mathrm{Hom}^\bullet(\mathcal{H}^\bullet, f^\nabla(\mathcal{M}^\bullet)^\bullet),$$

functorial in $\mathcal{H}^\bullet$. If $\mathcal{M}^\bullet$ is a complex of injective sheaves, so is $f^\nabla(\mathcal{M}^\bullet)^\bullet$.

Proof. In this section we have proved that the functor $\mathcal{F} \to f_! G^p \mathcal{F}$ from $\mathrm{Sh}_R(X)$ to $\mathrm{Sh}_R(S)$ preserves colimits for all $p \geq 0$. Consequently, the functor

$$\begin{aligned} \mathrm{Sh}_R(X)^{\mathrm{op}} &\longrightarrow \mathrm{Mod}_R \\ \mathcal{F} &\longmapsto \mathrm{Hom}_{\mathrm{Sh}_R(S)}(f_!(G^p\mathcal{F}), \mathcal{M}^q) \end{aligned}$$

transforms colimits into limits and is therefore representable by a sheaf $\mathcal{D}^{-p,q}$. The natural transformation $f_!\mathcal{G}^p \to f_! G^{p+1}$ and the differential of $\mathcal{M}^\bullet$ make $\mathcal{D}^{\bullet,\bullet}$ a bicomplex and we can show that its total complex is as wanted in precisely the same way as in the proof of Theorem 2.48. In particular, moving from (2.10) to (2.11) is still valid as we have shown that (2.12) is valid for locally finite spaces. □

For any bounded below complex $\mathcal{M}^\bullet$ of sheaves on X, we define $f^!\mathcal{M}^\bullet$ to be $f^\nabla(\mathcal{I}^\bullet)^\bullet$ where $\mathcal{I}^\bullet$ is a injective resolution of $\mathcal{M}^\bullet$ (this is only well defined up to quasi-isomorphism). We get an adjunction $\boldsymbol{R}f_! \dashv f^!$ between the derived categories $\mathbf{D}^+(X,R)$ and $\mathbf{D}^+(S,R)$.

3 Dualizing Complexes

In this section we turn our attention to the dualizing complexes. In the first subsection we give a proof of the fact that $g^!(f^!\mathcal{M}^\bullet)^\bullet$ is quasi-isomorphic to $(gf)^!\mathcal{M}^\bullet$ for any commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "gf"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

of finite spaces and any bounded below complex $\mathcal{M}^\bullet$ of sheaves of R-modules on S. In the second subsection we consider dualizing complexes with respect to maps $f\colon X \to \{*\}$ of finite spaces X to the singleton. We will give a complete description of $f^!M^\bullet_{\{*\}}$ for any R-module M. The last subsection will use this description and corollary 2.51 to obtain a connection between sheaf cohomology and cosheaf homology for finite topological spaces.

3.1 A proof that $(gf)^!\mathcal{M}^\bullet$ is quasi-isomorphic to $f^!g^!\mathcal{M}^\bullet$

Before we prove the desired result, we prove a lemma in which we need the following fact from homological algebra.

Proposition 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $T\colon \mathcal{A} \to \mathcal{B}$ a left exact functor. Let $X^\bullet$ and $Y^\bullet$ be complexes in $\mathcal{A}$ of T-acyclic objects. If $f\colon X^\bullet \to Y^\bullet$ is a quasi-isomorphism in $\boldsymbol{C}(\mathcal{A})$, then $T(f)\colon T(X^\bullet) \to T(Y^\bullet)$ is a quasi-isomorphism in $\boldsymbol{C}(\mathcal{B})$.*

Proof. See for example Theorem 7.5 in [13]. □

Lemma 3.2. *Let R be a commutative ring. Suppose that we have a commutative triangle of finite spaces*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

Let $\mathcal{H}^\bullet$ be a complex of sheaves of R-modules on X. The canonical map of complexes

$$h_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet) \longrightarrow g_*\boldsymbol{C}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet))$$

is a quasi-isomorphism functorial in $\mathcal{H}^\bullet$.

Proof. We have a canonical quasi-isomorphism

$$f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet) \longrightarrow \boldsymbol{C}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet))$$

in $\boldsymbol{C}(Y, R)$ that is functorial in $\mathcal{H}^\bullet$ (see corollary 2.41). For any finite space Z, any complex $\mathcal{F}^\bullet$ of sheaves of R-modules on Z and any $p \in \mathbb{Z}$ the sheaf $\boldsymbol{C}^p(\mathcal{F}^\bullet)$ is the direct sum of flasque sheaves and thus flasque and φ_*-acyclic for any continuous $\varphi\colon Z \to Z'$ with Z' finite. Hence, $\boldsymbol{C}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet))$ is a complex of g_*-acyclic sheaves. Also, $\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$ is a complex of h_*-acyclic sheaves and it follows that $f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)$ is a complex of g_*-acyclic sheaves. The lemma now follows from proposition 3.1 and the fact that $g_*f_* = (gf)_*$. □

Theorem 3.3. *Let R be a commutative ring. Suppose that we have a commutative triangle of finite spaces*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & S. & \end{tikzcd}$$

Let $\mathcal{M}^\bullet$ be a bounded below complex of sheaves of R-modules on S. Then the complexes of sheaves of R-modules $h^!\mathcal{M}^\bullet$ and $f^!g^!\mathcal{M}^\bullet$ on X are quasi-isomorphic.

Proof. Let $\mathcal{I}^\bullet$ be an injective resolution of $\mathcal{M}^\bullet$. From the proof of the Duality Theorem we know that $g^!\mathcal{M}^\bullet$ is quasi-isomorphic to the total complex of the bicomplex $\mathcal{D}^{\bullet,\bullet}$, where $\mathcal{D}^{-p,q}$ is the sheaf representing the functor

$$\begin{aligned} F_g^{p,q} \colon \mathrm{Sh}_R(Y) &\longrightarrow \mathrm{Mod}_R \\ \mathcal{F} &\longmapsto \mathrm{Hom}_{\mathrm{Sh}_R(S)}(g_* G^p(\mathcal{F}),\ \mathcal{I}^q). \end{aligned}$$

We assume that $g^!\mathcal{M}^\bullet$ actually is this complex. Let $\mathcal{H}^\bullet$ be a bounded below complex of sheaves of R-modules on X. Let $n \in \mathbb{Z}$. We have

$$\begin{aligned} \mathrm{Hom}^n(\mathcal{H}^\bullet,\ f^!g^!\mathcal{M}^\bullet) &\cong \mathrm{Hom}^n(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet),\ g^!\mathcal{M}^\bullet) \\ &= \prod_p \mathrm{Hom}_{\mathrm{Sh}_R(Y)}\left(f_*\boldsymbol{C}^p(\mathcal{H}^\bullet),\ \prod_i \mathcal{D}^{-i,p+n+i}\right) \\ &\cong \prod_{p,i} \mathrm{Hom}_{\mathrm{Sh}_R(Y)}(f_*\boldsymbol{C}^p(\mathcal{H}^\bullet),\ \mathcal{D}^{-i,p+n+i}) \\ &= \prod_{p,i} F_g^{i,p+n+i}(f_*\boldsymbol{C}^p(\mathcal{H}^\bullet)) \\ &= \prod_{p,i} \mathrm{Hom}_{\mathrm{Sh}_R(S)}(g_*G^i(f_*\boldsymbol{C}^p(\mathcal{H}^\bullet)),\ \mathcal{I}^{p+n+i}) \end{aligned}$$

Substituting $j = p + i$ we get

$$\begin{aligned} \mathrm{Hom}^n(\mathcal{H}^\bullet,\ f^!g^!\mathcal{M}^\bullet) &\cong \prod_{i,j} \mathrm{Hom}_{\mathrm{Sh}_R(S)}(g_*G^i(f_*\boldsymbol{C}^{j-i}(\mathcal{H}^\bullet)),\ \mathcal{I}^{n+j}) \\ &\cong \prod_j \mathrm{Hom}_{\mathrm{Sh}_R(S)}\left(\bigoplus_i g_*G^i(f_*\boldsymbol{C}^{j-i}(\mathcal{H}^\bullet)),\ \mathcal{I}^{n+j}\right) \\ &= \prod_j \mathrm{Hom}_{\mathrm{Sh}_R(S)}(g_*\boldsymbol{C}^j(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)),\ \mathcal{I}^{n+j}) \\ &= \mathrm{Hom}^n(g_*\boldsymbol{C}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)),\ \mathcal{I}^\bullet). \end{aligned}$$

Hence,

$$\mathrm{Hom}^\bullet(\mathcal{H}^\bullet, f^!g^!\mathcal{M}^\bullet) \cong \mathrm{Hom}^\bullet(g_*\boldsymbol{C}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)), \mathcal{I}^\bullet).$$

Lemma 3.2 and Theorem 3.1 give that we have a quasi isomorphism

$$\mathrm{Hom}^\bullet(g_*\boldsymbol{C}^\bullet(f_*\boldsymbol{C}(\mathcal{H}^\bullet)), \mathcal{I}^\bullet) \longrightarrow \mathrm{Hom}^\bullet(h_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet), \mathcal{I}^\bullet).$$

It follows that we have isomorphisms in $\mathbf{D}^+(\mathrm{Mod}_R)$

$$\boldsymbol{R}\mathrm{Hom}^\bullet(\mathcal{H}^\bullet, f^!g^!\mathcal{M}^\bullet) \cong \boldsymbol{R}\mathrm{Hom}^\bullet(g_*\boldsymbol{C}^\bullet(f_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet)), \mathcal{M}^\bullet) \cong \boldsymbol{R}\mathrm{Hom}^\bullet(h_*\boldsymbol{C}^\bullet(\mathcal{H}^\bullet), \mathcal{M}^\bullet)$$

This isomorphism is functorial in $\mathcal{H}^\bullet$ and it follows that $f^!g^!\mathcal{M}^\bullet$ dualizes the complex $\mathcal{M}^\bullet$ and therefore is quasi-isomorphic to $h^!\mathcal{M}^\bullet$. $\square$

From this theorem it follows that we have a functor

$$\begin{aligned}\mathrm{FTop}^{\mathrm{op}} &\longrightarrow \mathrm{Cat}\\ X &\longmapsto \mathbf{D}^+(X,R)\\ (f\colon X\to S) &\longmapsto (f^!\colon \mathbf{D}^+(S,R)\to \mathbf{D}^+(X,R)),\end{aligned}$$

where Cat is the category of all categories.

3.2 Dualizing Complexes of Spaces

Recall that the *dualizing complex* of a finite topological space X for an R-module M is the complex $f^!M^\bullet_{\{*\}}$ where $f\colon X\to\{*\}$ is the map to the singleton. In this subsection we will give a complete description of the dualizing complex for a module.

Theorem 3.4. *Let X be a finite topological space and $f\colon X\to\{*\}$ the map to the singleton. For any $x\in X$ let i_x denote the inclusion $\{x\}\to X$. Let R be a commutative ring and M an R-module. The complex of sheaves $\mathcal{D}^\bullet_{M,X}$ defined by*

$$\mathcal{D}^{-p}_{M,X} = \bigoplus_{x_0<\ldots<x_p} i_{x_p,*}M$$

with differential $d^p_x\colon \mathcal{D}^p_{M,X,x}\to\mathcal{D}^{p+1}_{M,X,x}$ given by

$$(d_x^{-p}\boldsymbol{s})(x_0<\ldots<x_{p-1}) = \sum_{i=0}^{p-1}(-1)^i\sum_{x_{i-1}<z<x_i}\boldsymbol{s}(x_0<\ldots<x_{i-1}<z<x_i<\ldots<x_{p-1})$$

is a dualizing complex of X for M.

Proof. Let $I^\bullet$ be an injective resolution of M. For any $p,q\in\mathbb{Z}$ define the functor

$$\begin{aligned}F^{p,q}\colon \mathrm{Sh}_R(X) &\longrightarrow \mathrm{Mod}_R\\ \mathcal{F} &\longmapsto \mathrm{Hom}_R(\Gamma(X,G^p(\mathcal{F})),\ I^q).\end{aligned}$$

Let $\mathcal{E}^{-p,q}$ be the sheaf of R-modules representing $F^{p,q}$. The proof of the Duality Theorem states that the singly graded complex $\mathcal{E}^\bullet$ associated to the bicomplex $\mathcal{E}^{\bullet,\bullet}$ dualizes M. We also define the bicomplex $\mathcal{D}^{\bullet,\bullet}$ where for any $q\in\mathbb{Z}$ the complex $\mathcal{D}^{\bullet,q}$ equals $\mathcal{D}^\bullet_{I^q,X}$ and the vertical differentials are induced by the differentials of $I^\bullet$. We will show that the bicomplexes $\mathcal{E}^{\bullet,\bullet}$ and $\mathcal{D}^{\bullet,\bullet}$ are isomorphic. Let $x\in X$ and $p,q\in\mathbb{Z}$ we have

$$\mathcal{E}_x^{-p,q} = \mathcal{G}_{F^{p,q}}(U_x) = F^{p,q}(R_{U_x}) = \mathrm{Hom}_R(\Gamma(X,G^p(R_{U_x})),\ I^q).$$

Applying Theorem 2.14 we get

$$\mathcal{E}_x^{-p,q}\cong \mathrm{Hom}_R\left(\bigoplus_{x_0<\ldots<x_p}(R_{U_x})_{x_p},\ I^q\right)\cong\bigoplus_{x_0<\ldots<x_p}\mathrm{Hom}_R((R_{U_x})_{x_p},\ I^q).$$

The stalk $(R_{U_x})_{x_p}$ equals R if $x_p\in U_x$, that is, if $x\le x_p$, and zero otherwise. Hence,

$$\mathcal{E}_x^{-p,q}\cong\bigoplus_{\substack{x_0<\ldots<x_p\\ x\le x_p}}\mathrm{Hom}_R(R,I^q)\cong\bigoplus_{\substack{x_0<\ldots<x_p\\ x\le x_p}}I^q=\left(\bigoplus_{x_0<\ldots<x_p}i_{x_p,*}I^q\right)_x=\mathcal{D}_x^{-p,q}.$$

Moreover, it is easily seen that the maps between the stalks for points $x \leq y$ agree and we actually have an isomorphism of sheaves $\mathcal{D}^{p,q} \cong \mathcal{E}^{p,q}$ for all $p, q \in \mathbb{Z}$. The differentials agree and we conclude that the bicomplexes $\mathcal{E}^{\bullet,\bullet}$ and $\mathcal{D}^{\bullet,\bullet}$ are isomorphic. It follows that

$$\mathcal{E}^{-p,\bullet} \cong \bigoplus_{x_0 < \ldots < x_p} i_{x_p,*} I^{\bullet}.$$

As $I^{\bullet}$ is quasi-isomorphic to $M[0]$ it follows that $\mathcal{E}^{\bullet}$ is quasi-isomorphic to $\mathcal{D}^{\bullet}_{M,X}$, completing the proof that $\mathcal{D}^{\bullet}_{M,X}$ is a dualizing complex of X for M. □

3.3 Connection to Cosheaf Homology

In this subsection we will use Theorem 3.4 and corollary 2.51 to get a link between the sheaf cohomology groups of $(X, \mathcal{F})$, where X is a topological space and $\mathcal{F}$ an abelian sheaf on X, and the homology groups of $(X, \mathcal{F}^*)$, where $\mathcal{F}^*$ is the dual cosheaf associated to $\mathcal{F}$. We start by introducing cosheaves and the associated homology theory.

Before we turn to the definition of a cosheaf, recall that $\boldsymbol{O}(X)$ denotes the category of open subsets of a topological space X.

Definition 3.5. Let X be a topological space. Let R be a commutative ring. A *precosheaf* of R-modules on X is a functor $\boldsymbol{O}(X) \to \mathrm{Mod}_R$.

If $V \subseteq U$ is an inclusion of opens of a space X and $\mathfrak{F}$ is a precosheaf on X, then the maps $\mathfrak{F}(V) \to \mathfrak{F}(U)$ are called *extensions*. If $x \in \mathfrak{F}(V)$, then its extension to U is denoted by $x|^U$.

Definition 3.6. Let X be a topological space and R a commutative ring. A precosheaf $\mathfrak{F}$ of R-modules on X is called a *cosheaf* if for any open cover $\{U_i\}_{i \in I}$ of an open set $U \subseteq X$, the sequence

$$\bigoplus_{i,j \in I} \mathfrak{F}(U_i \cap U_j) \xrightarrow{\alpha} \bigoplus_{i \in I} \mathfrak{F}(U_i) \xrightarrow{\beta} \mathfrak{F}(U) \longrightarrow 0$$

is exact. Here α is the map given by $(\alpha \boldsymbol{s})(i) = \sum_{j \in I} \boldsymbol{s}(i,j)|^{U_i} - \sum_{j \in I} \boldsymbol{s}(j,i)|^{U_i}$ and β is the map given by $\beta \boldsymbol{s} = \sum_{i \in I} \boldsymbol{s}(i)|^U$.

In the case $R = \mathbb{Z}$ we simply talk about *abelian cosheaves*. If $\mathfrak{F}$ is a cosheaf on a topological space X and $U \subseteq X$ is open, then elements of $\mathfrak{F}(U)$ are called *cosections* and we sometimes use the alternative notation $\Gamma(U, \mathfrak{F})$ for $\mathfrak{F}(U)$. For the sake of simplicity, we immediately go to the context of finite topological spaces. Just as sheaves on finite spaces are determined by the stalks and the maps between the stalks, cosheaves of finite spaces are determined by the costalks and the maps between the costalks.

Definition 3.7. Let X be a topological space, R a commutative ring and $\mathfrak{F}$ a cosheaf of R-modules on X. Let $x \in X$. The *costalk* of $\mathfrak{F}$ at x is the limit

$$\mathfrak{F}_x = \lim_{U \ni x} \mathfrak{F}(U).$$

Theorem 3.8. *Let X be an Alexandrov-discrete space and $S(X)$ the associated poset. Let R be a commutative ring. The category $\mathrm{cSh}_R(X)$ of cosheaves of R-modules on X is equivalent to the category $\mathrm{Func}(S(X)^{\mathrm{op}}, \mathrm{Mod}_R)$ of contravariant functors $S(X) \to \mathrm{Mod}_R$.*

Proof. Dual to Theorem 2.11. Given a cosheaf $\mathfrak{F}$ on X we define the contravariant functor $S(X)^{\mathrm{op}} \to \mathrm{Mod}_R$ by $x \mapsto \mathfrak{F}_x$. Conversely, given a functor $F\colon S(X)^{\mathrm{op}} \to \mathrm{Mod}_R$, we define the precosheaf $\mathfrak{F}$ on X by

$$\mathfrak{F}(U) = \mathrm{colim}_{x \in U} F(x).$$

Using the fact that X is Alexandrov-discrete, one shows that $\mathfrak{F}$ is a cosheaf. □

Notation 3.9. Let X be a finite topological space, R a commutative ring and $\mathfrak{F}$ a cosheaf of R-modules on X. If $x \leq y$, then we write s^x for the image of $s \in \mathfrak{F}_y$ in $\mathfrak{F}_x$.

We see that the global cosection functor $\Gamma(X, -)\colon \mathrm{cSh}_R(X) \to \mathrm{Mod}_R$ is just the functor

$$\mathrm{colim}\colon \mathrm{Func}(S(X)^{\mathrm{op}}, \mathrm{Mod}_R) \to \mathrm{Mod}_R,$$

which is right exact. This gives rise to the definition of cosheaf homology. Note that $\mathrm{cSh}_R(X)$ is abelian and has enough projectives, because $\mathrm{Func}(S(X)^{\mathrm{op}}, \mathrm{Mod}_R)$ has these properties.

Definition 3.10. Let X be a finite topological space, R a commutative ring and $\mathfrak{F}$ a cosheaf of R-modules on X. For any $i \geq 0$ we define the *i-th homology group*

$$H_i(X, \mathfrak{F}) = L^i\Gamma(X, \mathfrak{F}).$$

A common way of calculating sheaf cohomology is using flasque resolutions. Dually, cosheaf homology can be calculated using flasque coresolutions.

Definition 3.11. Let X be a topological space and R a commutative ring. A cosheaf $\mathfrak{F}$ of R-modules on X is called *flasque* or *flabby* if the extension maps $\mathfrak{F}(V) \to \mathfrak{F}(U)$ are injective for all opens $V \subseteq U \subseteq X$.

As an example, we introduce the Godement cosheaves.

Definition 3.12 ([4])**.** Let X be a finite topological space, R a commutative ring and $\mathfrak{F}$ a cosheaf of R-modules on X. Let $p \geq 0$. The *p-th Godement cosheaf* of $\mathfrak{F}$ is the cosheaf on X defined by the costalks

$$(G_p\mathfrak{F})_x = \bigoplus_{x_0 < \ldots < x_p \leq x} \mathfrak{F}_{x_p}.$$

and the maps

$$r_{xy}\colon (G_p\mathfrak{F})_y \longrightarrow (G_p\mathfrak{F})_x$$

given by

$$r_{xy}\boldsymbol{s}(x_0 < \ldots < x_p \leq x) = \boldsymbol{s}(x_0 < \ldots < x_p \leq y)$$

for $x \leq y$

Note that if $x \leq y$, then the map r_{xy} is injective, so $G_p(\mathfrak{F})$ is indeed a flasque cosheaf. Also, for all $x \in X$ we have a canonical surjective map

$$\begin{aligned}(\varepsilon_x\colon G_0\mathfrak{F})_x &\longrightarrow \mathfrak{F}_x \\ (s_y)_{y \leq x} &\longmapsto s_x.\end{aligned}$$

These maps give a surjective map of cosheaves $\varepsilon\colon G_0\mathfrak{F} \to \mathfrak{F}$.

Proposition 3.13. *Let X be a finite topological space, R a commutative ring and $\mathfrak{F}$ a cosheaf of R-modules on X. Then $\mathfrak{F}$ is $\Gamma(X, -)$-acyclic.*

Proof. See lemma 1.5.5 of [4]. □

Corollary 3.14. *Let X be a finite topological space, R a commutative ring, $\mathfrak{F}$ a cosheaf of R-modules on X and $\mathfrak{F}_\bullet$ a flasque coresolution of $\mathfrak{F}$. For any $n \geq 0$ we have*

$$H_n(X, \mathfrak{F}) = h_n(\Gamma(X, \mathfrak{F}_\bullet)).$$

The Godement resolution of a sheaf introduced in section 2.2 is often also called the *canonical flasque resolution*. Similarly, we now introduce a *canonical flasque coresolution* for cosheaves. Let $p \geq 1$ we define

$$(G_p\mathfrak{F})_x \xrightarrow{d_{p,x}} (G_{p-1}\mathfrak{F})_x$$

by

$$\begin{aligned}(d_{p,x}s)(x_0 < \ldots < x_{p-1}) = \sum_{i=0}^{p-1}(-1)^i \sum_{x_i<z<x_{i+1}} s(x_0 < \ldots < x_i < z < x_{i+1} < \ldots < x_{p-1}) \\ + (-1)^p \sum_{x_{p-1}<z} s(x_0 < \ldots < x_{p-1} < z)^{x_{p-1}}.\end{aligned}$$

The collection $(d_{p,x})_{x\in X}$ defines a map of cosheaves $d_p \colon G_p\mathfrak{F} \to G_{p-1}\mathfrak{F}$.

Proposition 3.15. *Let X be a finite topological space, R a commutative ring and $\mathfrak{F}$ a cosheaf of R-modules on X. Then, $(G_\bullet\mathfrak{F}, \varepsilon)$ is a flasque coresolution of $\mathfrak{F}$.*

Proof. Note that this is actually the dual statement of Theorem 2.14. For any $p \geq 0$ define the cosheaf $K_p\mathfrak{F}$ with costalks

$$(K_p\mathfrak{F})_x = \bigoplus_{y_0<\ldots<y_p<x} \mathfrak{F}_{y_p}$$

and maps

$$r_{xy} \colon (K_p\mathfrak{F})_y \longrightarrow (K_p\mathfrak{F})_x$$

for $x < y$ given by

$$\begin{aligned}r_{xy}s(y_0 < \ldots < y_p < x) = s(y_0 < \ldots < y_p < y) \\ - \sum_{i=0}^{p}(-1)^i s(y_0 < \ldots < \widehat{x_i} < \ldots < y_p < x < y)^{y_p}.\end{aligned}$$

We get an injective map of cosheaves $K_p\mathfrak{F} \to G_p\mathfrak{F}$ and it also holds that $G_p\mathfrak{F} = G_0(K_{p-1}\mathfrak{F})$ for all $p \geq 1$. For any $p \geq 1$ the sequence of cosheaves

$$0 \longrightarrow G_p(\mathfrak{F}) \longrightarrow K_{p-1}(\mathfrak{F}) \longrightarrow G_{p-1}(\mathfrak{F}) \longrightarrow 0$$

is exact. Splicing all these exact sequences together, we find the complex $G_\bullet(\mathfrak{F})$. □

We now have build enough theory to work towards the connection between sheaf cohomology and cosheaf homology for finite spaces. Given any sheaf $\mathcal{F}$ on X, we associate a dual cosheaf to $\mathcal{F}$.

Definition 3.16. Let X be a finite topological space, R a commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. The *dual cosheaf* associated to $\mathcal{F}$ is the cosheaf $\mathcal{F}^*$ defined by

$$\mathcal{F}_x^* = \mathrm{Hom}_R(\mathcal{F}_x, R).$$

The dual cosheaf $\mathcal{F}^*$ should not be confused with the dual sheaf $\mathcal{F}^\vee = \underline{\mathcal{H}om}(\mathcal{F}, R_X)$. We now prove the result of this section.

Theorem 3.17. *Let X be a finite topological space, R a hereditary commutative ring and $\mathcal{F}$ a sheaf of R-modules on X. For any $n \geq 0$ we have*

$$H_n(X, \mathcal{F}^*) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H^{n+1}(X, \mathcal{F}), R) \oplus H^n(X, \mathcal{F})^\vee.$$

Proof. Let $f\colon X \to \{*\}$ be the map to the singleton and let $\mathcal{D}_X^\bullet$ denote the dualizing complex $f^! R^\bullet_{\{*\}}$. By corollary 2.51 we have

$$h^{-n}(\mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, \mathcal{D}_X^\bullet)) \cong \mathrm{Ext}^1_{\mathbb{Z}}(H^{n+1}(X, \mathcal{F}), R) \oplus H^n(X, \mathcal{F})^\vee.$$

Let $p \geq 0$, using Theorem 3.4 we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, \mathcal{D}_X^{-p}) &= \mathrm{Hom}_{\mathrm{Sh}_R(X)}\left(\mathcal{F}, \bigoplus_{x_0 < \ldots < x_p} i_{x_p,*}R\right) \\ &\cong \bigoplus_{x_0 < \ldots < x_p} \mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, i_{x_p,*}R).\end{aligned}$$

The adjunction $i_{x_p}^{-1} \dashv i_{x_p,*}$ now gives

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, \mathcal{D}_X^{-p}) &\cong \bigoplus_{x_0 < \ldots < x_p} \mathrm{Hom}_{\mathrm{Sh}_R(\{*\})}(i_{x_p}^{-1}\mathcal{F}, R_{\{*\}}) \\ &\cong \bigoplus_{x_0 < \ldots < x_p} \mathrm{Hom}_R(\mathcal{F}_{x_p}, R) \\ &\cong \bigoplus_{x_0 < \ldots < x_p} \mathcal{F}^*_{x_p} \\ &= \Gamma(X, G_p(\mathcal{F}^*)).\end{aligned}$$

The differential of $\mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, \mathcal{D}_X^\bullet)$ coincides with the differential of $\Gamma(X, G_\bullet(\mathcal{F}^*))$. We conclude

$$h^{-n}(\mathrm{Hom}_{\mathrm{Sh}_R(X)}(\mathcal{F}, \mathcal{D}_X^\bullet)) = h_n(\Gamma(X, G_\bullet(\mathcal{F}^*))) = H_n(X, \mathcal{F}^*).$$

□

Corollary 3.18. *Let X be a finite topological space, R a commutative hereditary ring and $\mathcal{F}$ a sheaf of R-modules on X. We have*

$$H_{\dim X}(X, \mathcal{F}^*) \cong H^{\dim X}(X, \mathcal{F})^\vee.$$

If R is a field (or more generally, a self-injective hereditary commutative ring), then we have

$$H_k(X, \mathcal{F}^*) \cong H^k(X, \mathcal{F})^\vee$$

for all $k \geq 0$.

References

[1] Bacławski, K. (1975). Whitney numbers of geometric lattices. *Advances in Mathematics*, 16:125–138.

[2] Black, A. (2015). The euler characteristic of finite topological spaces. Preprint, on webpage at `http://math.uchicago.edu/~may/REU2015/REUPapers/Black.pdf`.

[3] Bredon, G. E. (1997). *Sheaf Theory*. Springer, New York, second edition.

[4] Carmona Sánchez, V., Maestro Pérez, C., Sancho de Salas, F., and Torres Sancho, J. F. (2018). Homology and cohomology of finite spaces. Preprint, on webpage at `https://arxiv.org/abs/1809.08047`.

[5] El Zein, F. and Tu, L. W. (2014). From sheaf cohomology to the algebraic de rham theorem. In Catani, E., El Zein, F., Griffiths, P. A., and Lê Dũng Tráng, editors, *Hodge Theory*, chapter 2. Princeton University Press.

[6] Friedman, G. (2012). Survey article: An elementary illustrated introduction to simplicial sets. *Rocky Mountain J. Math.*, 42(2):353–423.

[7] Gallier, J. and Quaintance, J. (2019). A gentle introduction to homology, cohomology and sheaf cohomology. Book in progress on webpage at `http://www.cis.upenn.edu/~jean/gbooks/sheaf-coho.html`.

[8] Gelfand, S. I. and Manin, Y. I. (2003). *Methods of Homological Algebra*. Springer, Berlin, second edition.

[9] Godsil, C. D. (2018). An introduction to the moebius function. Preprint, on webpage at `https://arxiv.org/pdf/1803.06664.pdf`.

[10] Goresky, M. (2016). Introduction to perverse sheaves. Lectures Notes, on webpage at `http://www.math.ias.edu/~goresky/math2710/all.pdf`.

[11] Grothendieck, A. (1957). Sur quelques points d'algèbre homologique, i. *Tohoku Math. J. (2)*, 9(2):119–221. English translation (Some aspects of homological algebra) at `http://www.math.mcgill.ca/barr/papers/gk.pdf`.

[12] Hartshorne, R. (1977). *Algebraic Geometry*. Springer-Verlag, New York.

[13] Iversen, B. (1986). *Cohomology of Sheaves*. Springer, Berlin.

[14] Mathew, A. (2011). Verdier duality. Unpublsihed, on webpage at `http://math.uchicago.edu/~amathew/verd.pdf`.

[15] Navarro González, J. A. (1990). Duality and finite spaces. *Order*, 6:401–408.

[16] Sancho de Salas, F. (2014). Ringed finite spaces. Preprint, on webpage at `https://arxiv.org/pdf/1409.4574.pdf`.

[17] Sancho de Salas, F. and Torres Sancho, J. F. (2019). Derived category of finite spaces and grothendieck duality. Preprint, on webpage at `https://arxiv.org/abs/1904.06935`.

[18] Stacks Project Authors (2019). The stacks project. `https://stacks.math.columbia.edu`.

[19] Zheng, W. (2011). Lectures on homological algebra. Lecture Notes, on webpage at `https://server.mcm.ac.cn/~zheng/homalg.pdf`.