Shelah's pcf-theory and the bound on aleph_omega^aleph_o

Bos, F.

Fabian Pepijn Bos

Shelah's pcf-theory and the bound on $\aleph_\omega^{\aleph_0}$

Masterthesis

Supervisors:
Bas Edixhoven
K. P. Hart

Exam date:
June 19, 2019

Mathematisch Instituut, Universiteit Leiden

Contents

Summary

In this masterthesis we explore Shelah's theory of *possible cofinalities* (pcf-theory) to find $\aleph_{\omega_4}$ as an upperbound for $\aleph_\omega^{\aleph_0}$. Some basic knowledge of set theory, among other about ordinal and cardinal numbers, is unavoidable.

In Chapter 1 we introducte the topic of cardinal exponentiation, being the only operation on infinite numbers that is non-trivial. We see Hausdorff's formula, the theorem of Bukovský and Hechler and a theorem on the calculation of cardinal exponentiation. Then we consider Eastons theorem, showing that there is a lot of freedom in cardinal exponentiation of regular cardinals. The theorems of Silver and Galvin and Hajnal however lay limitations on the freedom of singular cardinal exponentiation.

Chapter 2 deals with some background knowledge, and is used to be precise about some widely used definitions. In particular, on a strict linearly and partially ordered set $(X, \prec, \leq)$ there are four notions of a subset Y being cofinal in X:

1. cofinal: for all $x \in X$ exists $y \in Y$ such that $x \leq y$,
2. $\prec$-cofinal: for all $x \in X$ exists $y \in Y$ such that $x \prec y$,
3. true cofinal: cofinal and linearly ordered by $\leq$,
4. $\prec$-true cofinal: $\prec$-cofinal and strict linearly ordered by $\prec$.

We investigate these different notions and find sufficient conditions under which they coincide. When X is an ordinal with standard orderings $<$ and $\leq$, all notions of cofinality coincide.

In the first part of Chapter 3 we start on pcf-theory. The basic definition is

$$\mathrm{pcf}(a) := \{\mathrm{cf}(\prod a/D) : D \text{ is an ultrafilter on } a\},$$

for a set a of ordinals. Here $\prod a/D$ denotes the product of a *reduced* by the ultrafilter D, which means the quotient set of $\prod a$ under the equivalence relation $=_D$, which is given by

$$f =_D g \quad \Leftrightarrow \quad \{\alpha \in a : f(\alpha) = g(\alpha)\} \in D.$$

We usually assume that a is an infinite set of regular cardinals. It turns out to be very useful to look at the ideal

$$J_{<\lambda}(a) := \{b \subseteq a : \text{ If } D \text{ is an ultrafilter on } a \text{ such that } b \in D, \text{ then } \mathrm{cf}(\prod a/D) < \lambda\},$$

the set of subsets of a that 'force' $\mathrm{cf}(\prod a/D)$ below λ, where λ is some cardinal. We show that $\prod a/J_{<\lambda}$ is λ-directed, from which is follows that $\mathrm{cf}(\prod a/D) < \lambda$ if and only if $D \cap J_{<\lambda} \neq \emptyset$. Therefore $|\mathrm{pcf}(a)| \leq 2^{|a|}$, and $\mathrm{pcf}(a)$ has a maximal element. Another important result is that $\mathrm{pcf}(a)$ is an interval of regular cardinals, when a is an interval of regular cardinals such that $|a| < \min(a)$. In the second part of Chapter 3 we see that $J_{<\lambda^+}$ is generated over $J_{<\lambda}$ by a single element, called b_λ. The proof makes use of the existence of universal sequences, that are $<_{J_{<\lambda}}$-increasing sequences $\langle f_\xi : \xi < \lambda \rangle$ that are cofinal in $\prod a/D$ for any ultrafilter D such that $\mathrm{cf}(\prod a/D) = \lambda$.

Model theory will be build up from scratch in Section 4. We will see the basic notions of a language, a structure, formulas, sentences, satisfaction, (elementary) embeddings, definable elements, the Tarski-Vaught test and the theorems of Skolem and Löwenheim. Then we define

$$H(\kappa) := \{x : x \text{ is hereditarily of cardinality less than } \kappa\} = \{x : |\bigcup_{n<\omega} \underbrace{\bigcup ... \bigcup}_{n} x| < \kappa\},$$

which will be frequently used in the remaining chapters.

Chapter 5 forms an intermezzo. We investigate characterizations and the existence of Jónsson algebra's, that are algebra's that yield no strict subagebra's of the same cardinality. There is a nice model-theoretic characterization of the existence of a Jónsson algebra on a cardinal κ, namely κ is Jónsson iff for all elementary substructures M of $H(\kappa^+)$ such that $|M \cap \kappa| = \kappa$, we have $\kappa \subseteq M$. We prove the following theorem using some pcf-theory from Chapter 3: If μ is singular and eventually every $\nu < \mu$ is Jónsson, then μ^+ is Jónsson.

Chapter 6, 7 and 8 delve deep into pcf-theory. First we prove that if $(\min(A))^{|A|} < \sup(A)$ for some interval A, then $\max \mathrm{pcf}(A) = |\prod A|$, using model theory. This already has a non-trivial corollary, namely $\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}$ for any limit ordinal δ. In particular,

$$\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}.$$

In Chapter 7 we prove that

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}.$$

We use a two-player game $G_\xi = G_{\xi,f}$, where $\xi \in \kappa^\omega$ for some fixed κ such that $\kappa^{\aleph_0} = \kappa$ and $f : (\kappa^+)^{<\omega} \to \kappa$. In round n, Player I picks a club $C_n \subseteq \kappa^+$ and Player II responds with an $a_n \in C_n$. Player II wins iff $f(\langle a_0, ..., a_{n-1}\rangle) = \xi_n$ for all $n < \omega$. We prove that for each f there is some ξ such that Player II has a winning strategy in the game G_ξ. Chapter 7 is independent of 8, but contains some interesting concepts of pcf-theory, such as

$$\mathrm{pcf}_\mu(a) := \bigcup \{\mathrm{pcf}(A) : A \subseteq a \text{ and } |A| < \mu\}.$$

In Chapter 8 we will finally prove Shelah's bound $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$ assuming $2^{\aleph_0} < \aleph_\omega$. In general

$$\aleph_\delta^{|\delta|} < \aleph_{|\delta|^{+4}},$$

assuming $2^{|\delta|} < \aleph_\delta$. The proof of the bound is similar to the proof of bound in Chapter 6, but now uses the result that $|\mathrm{pcf}(a)| < |a|^{+4}$ for an interval of regular cardinals a such that $\min(a) > 2^{|a|}$. To prove this, we need some technical lemmas. We prove that we can choose the generating sets b_λ such that $\mu \in b_\lambda$ implies $b_\mu \subseteq b_\lambda$ and $\mathrm{pcf}(b_\lambda) = b_\lambda$. We reason that $\mathrm{pcf}(a) = [\aleph_{\delta+1}, \aleph_{\delta+\rho+1})$ for some $\rho < (2^{|a|})^+$ and define a closure operation $\overline{(.)}$ on $P(\rho + 1)$. The we use the club-guessing sequences from Chapter 2 to show a contradiction of $|\mathrm{pcf}(a)| \geq |a|^{+4}$.

1 Introduction

1.1 Cardinal exponentiation

There are three basic operations on numbers: addition, multiplication and exponentiation. On infinite cardinal numbers two of these operations become trivial, since we have

$$\kappa + \lambda = \kappa \cdot \lambda = \max(\kappa, \lambda)$$

if at least one of κ and λ is an infinite cardinal. However, cardinal exponentiation turns out to be highly non-trivial. For instance, the *Continuum Hypothesis*

$$2^{\aleph_0} = \aleph_1$$

and the Generalized Continuum Hypothesis

$$2^{\aleph_\alpha} = \aleph_{\alpha+1} \quad \text{for all ordinals } \alpha$$

are independent of the axioms of ZFC. Of course there is *Hausdorff's formula* [8, (5.22)]:

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta} \tag{1}$$

for all ordinals α and β, and it's simple generalization

$$\aleph_{\alpha+n}^{\aleph_\beta} = \aleph_{\alpha+n} \cdot \aleph_\alpha^{\aleph_\beta}$$

for all ordinals α and β and all natural numbers n. Bukovský and Hechler independently found the following result:

Theorem 1.1. [1] If κ is a singular infinite cardinal and $\langle 2^\mu : \mu < \kappa \rangle$ becomes constant with value λ, then $2^\kappa = \lambda$.

In general, we have the following theorem restraining the computation of cardinal exponentiation:

Theorem 1.2. [8, Theorem 5.20] Let κ and λ be infinite cardinals. Then

$$\kappa^\lambda = \begin{cases} 2^\lambda, & \text{if } \kappa \leq \lambda, \\ \mu^\lambda, & \text{if } \mu < \kappa \text{ and } \mu^\lambda \geq \kappa, \\ \kappa, & \text{if } \kappa > \lambda, \mu^\lambda < \kappa \text{ for all } \mu < \kappa \text{ and } \mathrm{cf}(\kappa) > \lambda, \\ \kappa^{\mathrm{cf}(\kappa)}, & \text{if } \kappa > \lambda, \mu^\lambda < \kappa \text{ for all } \mu < \kappa \text{ and } \mathrm{cf}(\kappa) \leq \lambda. \end{cases}$$

There appeared to be much freedom in cardinal arithmetic. Cohen's forcing technique [3, 4] was used to counter many potential theorems. In particular, we have the following result by Easton:

Theorem 1.3. [5, Theorem 1] [2, p.207] Assume ZFC is consistent. Suppose $F : \mathrm{Ord} \to \mathrm{Ord}$ satisfies the following:

1. For all $\alpha, \beta \in \mathrm{Ord}$, if $\alpha < \beta$ then $F(\alpha) \leq F(\beta)$.
2. For all $\alpha \in \mathrm{Ord}$, the cofinality of $\aleph_{F(\alpha)}$ is at least $\aleph_{\alpha+1}$.

Then $2^{\aleph_{\alpha+1}} = \aleph_{F(\alpha)}$ for all $\alpha \in \mathrm{Ord}$ is consistent with ZFC.

Note that Easton's theorem only talks about cardinal exponentiation of successor cardinals. It was thought that Easton-like theorems could be generalized to singular cardinals as well, by improving the techniques of forcing and model construction. This came to a halt by the following theorem of Silver:

Theorem 1.4. [9] If κ is singular and of uncountable cofinality, and $2^\lambda = \lambda^+$ for all $\lambda < \kappa$, then $2^\kappa = \kappa^+$.

A more general result was found by Galvin and Hajnal:

Theorem 1.5. [6] If $\aleph_\alpha$ is singular, of uncountable cofinality and such that $2^\lambda < \aleph_\alpha$ for all $\lambda < \aleph_\alpha$, then $2^{\aleph_\alpha} < \aleph_{(2^{|\alpha|})^+}$.

A great contribution to more of such theorems on cardinal exponentiation came by Saharon Shelah (1945), an Israeli mathmeticial. Shelah's study of cofinalities of reduced products of sets of cardinals, Shelah's *pcf-theory*, proved uttermost fruitful. We will see some of these results in this thesis.

1.2 Notational conventions

We have the following notational conventions:

- $P(a)$ denotes the powerset of the set a.
- **ON** denotes the class of all ordinals, Reg denotes the class of all regular cardinals.
- $\mathrm{ot}(E)$ denotes the order-type of E. We use this provided that E is a well-ordered set, so that $\mathrm{ot}(E)$ is an ordinal. If $E \subseteq \alpha$ for some ordinal α, then $\mathrm{ot}(E) \leq \alpha$.
- $a \dot\cup b$ denotes the set $a \cup b$ and simultaneously states that a and b are disjoint. Similarly $\dot\bigcup A$ denotes the set $\bigcup A$ and states that the elements of A are mutually disjoint.
- id_a denotes the identity map on the set a. When a is clear from the context, we just write id.
- The arrows $\twoheadrightarrow$, $\hookrightarrow$ or $\leftrightarrow$ may replace the arrow $\to$ in a function $f : a \to b$, and respectively state that f is surjective, injective or bijective.
- $\subset$ means 'is a strict subset of'. Therefore $a \subset b$ if and only if ($a \subseteq b$ and $a \neq b$).
- $\sup^+$ means 'strict supremum'. For example, if $\alpha_i \in \mathbf{ON}$ for all $i \in I$, then $\sup^+_{i\in I} \alpha_i = \min(\alpha : \alpha_i < \alpha$ for all $i \in I\}$.

2 Background

Before we can start on pcf-theory, we must agree on some basic definitions.

2.1 Orders

Definition 2.1. Let X be a set and let $R \subseteq X \times X$, i.e. R is *(binary) relation on* X. Then R is called

1. *reflexive* iff $(x, x) \in R$ for all $x \in X$,
2. *irreflexive* iff $(x, x) \notin R$ for all $x \in X$,
3. *symmetric* iff $(x, y) \in R$ implies $(y, x) \in R$,
4. *anti-symmetric* iff $(x, y), (y, x) \in R$ implies $x = y$,
5. *transitive* iff $(x, y), (y, z) \in R$ implies $(x, z) \in R$,
6. *total* iff $(x, y) \in R$ or $(y, x) \in R$ for all $x, y \in X$,
7. *trichotomic* iff either $(x, y) \in R$, $(y, x) \in R$ or $x = y$ for all $x, y \in X$,
8. an *equivalence relation* iff R is reflexive, symmetric and transitive,
9. a *quasi order* iff R is reflexive and transitive,
10. a *partial order* iff R is reflexive, anti-symmetric and transitive,
11. a *strict partial order* iff R is irreflexive and transitive (note that this implies non-symmetry and anti-symmetry, since $(x, y), (y, x) \in R$ can never occur),
12. a *linear order* iff R is anti-symmetric, transitive and total (note that totality implies reflexivety),
13. a *strict linear order* iff R is transitive and trichotomic (note that irreflexivity follows from trichotomy),
14. a *well-order* iff R is a strict linear order and for any $Y \subseteq X$ exists $y \in Y$ such that $y \, R \, z$ for all $z \in Y \setminus \{y\}$.

A linearly ordered set is also called a *chain*.

If R is a binary relation on X and $Y \subseteq X$, then the above definitions also apply to Y when we consider $R \cap (Y \times Y)$. For instance we say that R is a quasi order on Y iff the following hold:

1. For all $y \in Y$ we have $(y, y) \in R$.
2. For all $x, y, z \in Y$ we have $(x, y), (y, z) \in R$ implies $(x, z) \in R$.

When R is a relation on X and $x, y \in X$, we also write $x \, R \, y$ instead of $(x, y) \in R$.

When R is an equivalence relation on X, we denote by X/R the set of equivalence classes of X and by x/R the equivalence class of an element $x \in X$.

Quasi orderings and partial orderings are not that different; the only additional property of a partial ordering is that it is anti-symmetric: $(x, y), (y, x) \in R$ implies $x = y$. In fact, suppose X is quasi ordered by R, and define the relation $\sim$ by $x \sim y$ iff $(x, y), (y, x) \in R$. It is easy to see that $\sim$ is an equivalence relation on X. Define $(x/\sim) \leq (y/\sim)$ iff $(x, y) \in R$. This is well-defined, and $\leq$ is a partial order on $X/\sim$. The quotient map $X \to X/\sim$ (given by $x \mapsto x/\sim$) is obviously order preserving: $x \, R \, y$ implies $(x/\sim) \leq (y/\sim)$.

Definition 2.2. Let $\leq$ be a quasi order on X. For $m \in X$, $Y \subseteq X$ and λ a cardinal we say that

1. m is an *upper bound* for Y iff $y \leq m$ for all $y \in Y$,

2. m is a *least upper bound* or *supremum* for Y iff m is an upper bound for Y and any other upper bound m' for Y satisfies $m \leq m'$,
3. m is a *minimal upper bound* for Y iff m is an upper bound for Y and any other upper bound m' for Y satisfies $m' \not\leq m$ or $m \leq m'$,
4. Y is *bounded (in X)* iff there exists $n \in X$ such that n is an upper bound for Y,
5. X is λ-*directed* iff every $Z \subseteq X$ with $|Z| < \lambda$ is bounded in X,

Lower bound, *greatest lower bound* and *lower bound* are defined similarly. When Y is a sequence in X instead of a subset, the same definitions apply to Y and m where we consider the image of Y rather than Y itself.

Suppose $\leq$ is a partial order on X. If m and m' are both least upper bounds for Y, then $m \leq m'$ and $m' \leq m$. Thus we obtain $m = m'$, hence a least upper bound is unique, if it exists. We denote $\sup(Y)$ for the supremum of Y. Similarly we denote $\inf(Y)$ for the greatest lower bound, which is unique if it exists.

2.2 Cofinality

We often encounter a set X on which is strict partial order and a quasi order are defined. The orders may be related, for instance as in the following definition.

Definition 2.3. Let $\prec$ be a strict partial order on X and let $\leq$ be a quasi order on X. Then $(X, \prec, \leq)$ may have the following properties:

(P1) $x \prec y$ implies $x \leq y$.
(P2) ($x \prec y$ and $y \leq z$) implies $x \prec z$; ($x \leq y$ and $y \prec z$) implies $x \prec z$.
(P3) For each x exists y such that $x \prec y$.

For example, when $(X, \leq)$ is a quasi ordered set, then, with $x \prec y$ iff ($x \leq y$ and $y \not\leq x$) by definition, $(X, \prec, \leq)$ satisfies (P1) and (P2).

Note that if $\leq$ is a partial order, then ($x \leq y$ and $y \not\leq x$) if and only if ($x \leq y$ and $x \neq y$).

Definition 2.4. Let $X = (X, \prec, \leq)$ a strict partially and quasi ordered and let $Y \subseteq X$. Then Y is called

1. *cofinal* (in X) iff for all $x \in X$ exists $y \in Y$ such that $x \leq y$,
2. $\prec$-*cofinal* (in X) if for all $x \in X$ exists $y \in Y$ such that $x \prec y$,
3. *true cofinal* (in X) iff it is cofinal and linearly ordered by $\leq$,
4. $\prec$-*true cofinal* (in X) iff it is $\prec$-cofinal and strict linearly ordered by $\prec$.

Note that X is cofinal in X. We furthermore define the

1. *cofinality* of X, denoted $\mathrm{cf}(X)$: The least cardinal κ for which there exists a cofinal set of cardinality κ,
2. $\prec$-*cofinality* of X, denoted $\mathrm{cf}_\prec(X)$: The least cardinal κ for which there exists a $\prec$-cofinal set of cardinality κ; provided there is at least one $\prec$-cofinal set,
3. *true cofinality* of X, denoted $\mathrm{tcf}(X)$: The least cardinal κ for which there exists a true cofinal set of cardinality κ; provided there is at least one true cofinal set,
4. $\prec$-*true cofinality* of X, denoted $\mathrm{tcf}_\prec(X)$: The least cardinal κ for which there exists a $\prec$-true cofinal set of cardinality κ; provided there is at least one $\prec$-true cofinal set.

Sometimes a set X only carries a strict partial order $\prec$. Then we define $x \leq y$ iff ($x \prec y$ or $x = y$), so that $\leq$ is a quasi order on X. When X only has a quasi order $\leq$ take $x \prec y$ iff ($x \leq y$ and $y \not\leq x$) by definition. Then $\prec$ is a strict partial order. Definition 2.4 still applies.

In addition to Definition 2.2 we can now define the notion of an *exact upper bound*:

Definition 2.5. Let $(X, \prec, \leq)$ be a strict partially and quasi ordered set, let $Y \subseteq X$ such that $\prec$ restricted to Y satisfies (P3) and let $m \in X$. Then m is called an *exact upper bound* of Y iff it is a least upper bound of Y and Y is cofinal in $\{x \in X : x \prec m\}$. That is, if $x \prec m$ then there exists a $y \in Y$ such that $x \leq y$.

Now follows a lemma describing some circumstances where the different notions of cofinality coincide.

Lemma 2.6. Let $X = (X, \prec, \leq)$ a strict partially and quasi ordered set. Then the following hold:

1. When (P2) and (P3) hold, any cofinal subset is $\prec$-cofinal. When (P1) holds, any $\prec$-cofinal subset is cofinal.

   Proof. Let $Y \subseteq X$. Suppose Y is cofinal. Given $x \in X$, let $x' \in X$ such that $x \prec x'$ by (P3) and let $x' \leq y$ for $y \in Y$. By (P2), $x \prec y$. So Y is $\prec$-cofinal. Suppose that Y is $\prec$-cofinal. Given $x \in X$, let $x \prec y$ for $y \in Y$. Then $x \leq y$ by (P1). So Y is cofinal. □

2. When (P3) holds, there exists a $\prec$-cofinal set.

   Proof. X itself is cofinal in X. □

3. When $x \prec y$ iff ($x \leq y$ and $y \not\leq x$) and (P3) holds, then the notions of true cofinal and $\prec$-true cofinal coincide.

   Proof. Let $Y \subseteq X$. Suppose Y is true cofinal. Then it is cofinal, hence $\prec$-cofinal by Lemma 2.6.1, since (P2) and (P3) are satisfied. Let $y, y' \in Y$. If $y \leq y'$ and $y' \leq y$, then $y = y'$, since Y is linear. If $y \leq y'$ and $y' \not\leq y$, then $y \prec y'$; if $y' \leq y$ and $y \not\leq y'$, then $y' \prec y$. The last case where $y \not\leq y'$ and $y' \not\leq y$ is not possible, since $\leq$ is a linear order on Y. We have thus show that Y is strict linearly ordered by $\prec$. Hence Y is $\prec$-true cofinal. Now suppose Y is $\prec$-true cofinal. Then it is $\prec$-cofinal, hence cofinal by Lemma 2.6.1, since (P1) is satisfied. Let $y, y' \in Y$. If $y \leq y'$ and $y' \leq y$, then $y \not\prec y'$ and $y' \not\prec y$, so $y = y'$. So $\leq$ is anti-symmetric on Y. Transitivity of $\leq$ on Y is automatic. If $y \prec y'$ then $y \leq y'$, if $y' \prec y$ then $y' \leq y$, if $y = y'$ then $y \leq y'$ (and $y' \leq y$). So $\leq$ is a total order on Y. Thus Y is linearly ordered by $\leq$. Hence Y is true cofinal. □

4. When $\leq$ is linear, then the notions of cofinal and true cofinal coincide.

   Proof. Any subset of a linear set is linear. □

5. When $\prec$ is strict linear, then the notions of $\prec$-cofinal and $\prec$-true cofinal coincide.

   Proof. Any subset of a strict linear set is strict linear. □

We summarize the equivalences in the following diagram:

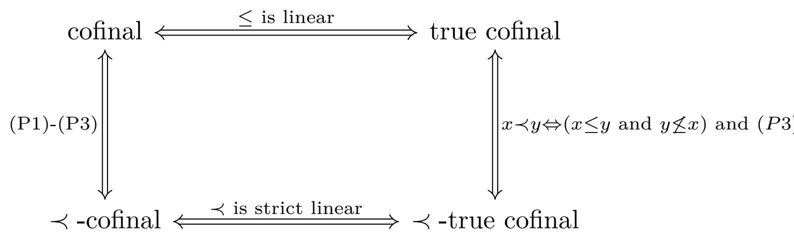

Figure 1: Sufficient conditions for equivalences of notions of *cofinality*.

We now turn our attention to ordinals and ordinal-indexed sequences. For two ordinals α and β, we have $\alpha < \beta$ iff $\alpha \in \beta$, and $\alpha \leq \beta$ iff $\alpha = \beta$ or $\alpha < \beta$. It is well known that any set of ordinals is strict partially ordered by $<$ and quasi ordered (even linearly ordered, even well-ordered) by $\leq$, and that

$$\alpha < \beta \Leftrightarrow \alpha \leq \beta \text{ and } \beta \not\leq \alpha \Leftrightarrow \alpha \leq \beta \text{ and } \alpha \neq \beta.$$

Thus a limit ordinal λ satisfies all the equivalences in Figure 1. We have the very common definition for a limit ordinal λ:

1. λ is called *regular* iff $\mathrm{cf}(\lambda) = \lambda$.
2. λ is called *singular* iff $\mathrm{cf}(\lambda) < \lambda$.

Regular ordinals are automatically cardinals. A similar definition works for successor ordinals, their cofinality and true cofinality is always 1 and they have no $<$-cofinality nor $<$-true cofinality.

Definition 2.7. Let $X = (X, \prec, \leq)$ a strict partially and quasi ordered set. Let S be a set of ordinals. An S-sequence in X, i.e. a map $S \to X$ denoted as $\langle x_\xi : \xi \in S \rangle$, is called

1. *non-decreasing* iff $\xi \leq \xi' \Rightarrow x_\xi \leq x_{\xi'}$,
2. *increasing* iff $\xi < \xi' \Rightarrow x_\xi \prec x_{\xi'}$,
3. *decreasing* iff $\xi < \xi' \Rightarrow x_{\xi'} \prec x_\xi$,
4. *cofinal (in X)* iff it range $\{x_\xi : \xi \in S\}$ is cofinal in X. Similarly, $\prec$-cofinal, true cofinal and $\prec$-true cofinal are defined.

Note that the range of a non-decreasing sequence is automatically totally ordered. Since $\leq$ was already reflexive and transitive, the range is linearly ordered.

Note that the range an increasing sequence is automatically trichotomiccally ordered. Since $\prec$ was already (reflexive and) transitive, the range is strict linearly ordered.

The next three lemmas prove Theorem 2.11, which relates the $\prec$-true cofinality to the existence of an increasing $\prec$-true cofinal sequence of regular length.

Lemma 2.8. Let $X = (X, \prec)$ be a strict partially ordered set satisfying (P3). Suppose $\mathrm{tcf}_\prec(X) = \lambda$. Then there exists an increasing $\prec$-true cofinal λ-sequence in X.

Proof. Let Y be $\prec$-true cofinal in X with $|Y| = \lambda$. Let $i : \lambda \to Y$ be a bijection. Define an increasing sequence $\langle y_\xi : \xi < \lambda \rangle$ by

1. $y_0 := i(0)$.

2. If $\langle y_{\xi'} : \xi' < \xi \rangle$ is already defined and ξ is a successor, let $\xi' + 1 = \xi$. Since Y is $\prec$-true cofinal, let $y_{\xi'} \prec y$ for some $y \in Y$. Let $y_\xi := \max(y, i(\xi))$.
3. If $\langle y_{\xi'} : \xi' < \xi \rangle$ is already defined and ξ is a limit ordinal. Suppose no $y \in Y$ satisfies $y_{\xi'} \prec y$ for all $\xi' < \xi$. Let $x \in X$. Then $x \prec y$ for some $y \in Y$. Then $y_{\xi'} \not\prec y$ for some $\xi' < \xi$. By trichotomy of $\prec$ on Y we have $y_{\xi'} = y$ for $y_{\xi'} \succ y$. Thus $x \prec y_{\xi'}$. Thus $\{y_{\xi'} : \xi' < \xi\}$ would be $\prec$-true cofinal, but this is impossible since $|\xi| < \lambda$. So there exists $y \in Y$ such that $y_{\xi'} \prec y$ for all $\xi' < \xi$. Take $y_\xi := \max(y, i(\xi))$.

Then $\langle y_\xi : \xi < \lambda \rangle$ is an increasing λ-sequence in X and it is $\prec$-true cofinal: For $x \in X$, let $x \prec y$ for some $y \in Y$. Then $\xi := i^{-1}(y) < \lambda$ and $i(\xi) = y$, so $y_\xi = y$ or $y_\xi \succ y$. So $x \prec y_\xi$. □

Lemma 2.9. Let $X = (X, \prec)$ be a strict partially ordered set satisfying (P3). Suppose $\mathrm{tcf}_\prec(X) = \lambda$. Then λ is regular.

Proof. By Lemma 2.8, let $\langle x_\xi : \xi < \lambda \rangle$ be an increasing $\prec$-true cofinal sequence. Let $\langle \xi_\alpha : \alpha < \mathrm{cf}(\lambda) \rangle$ be increasing cofinal in λ.[1] Then $\langle x_{\xi_\alpha} : \alpha < \mathrm{cf}(\lambda) \rangle$ is increasing and $\prec$-true cofinal in X: Obviously increasing and for $x \in X$, let $x \prec x_\xi$ for some $\xi < \lambda$, let $\xi < \xi_\alpha$ for some $\alpha < \mathrm{cf}(\lambda)$, then $x \prec x_\xi \prec x_{\xi_\alpha}$. By minimality of λ it follows that $\mathrm{cf}(\lambda) \geq \lambda$. □

Lemma 2.10. Let $X = (X, \prec)$ be a strict partially ordered set satisfying (P3). Let $\langle x_\xi : \xi < \lambda \rangle$ be increasing and $\prec$-true cofinal in X and suppose λ is regular. Then $\mathrm{tcf}_\prec(X) = \lambda$.

Proof. Certainly $\mu := \mathrm{tcf}_\prec(X) \leq \lambda$. By Lemma 2.8, let $\langle y_\chi : \chi < \mu \rangle$ be an increasing $\prec$-true cofinal sequence. For $\chi < \mu$, recursively define $\xi_\chi < \lambda$ such that $\xi_{\chi'} \prec \xi_\chi$ for all $\chi' < \chi$ and $y_\chi \prec x_{\xi_\chi}$.[2] Then $\langle x_{\xi_\chi} : \chi < \mu \rangle$ is increasing and $\prec$-true cofinal in X: Increasing by construction and for $x \in X$, let $x \prec y_\chi$, then $x \prec x_{\xi_\chi}$. Then $\langle \xi_\chi : \chi < \mu \rangle$ is cofinal in λ: For $\xi < \lambda$, let $x_\xi \prec x_{\xi_\chi}$, then $\xi < \xi_\chi$. Hence $\lambda = \mathrm{cf}(\lambda) \leq \mu$. So $\mu = \lambda$. □

Theorem 2.11. Let $X = (X, \prec)$ be a strict partially ordered set satisfying (P3). Then $\mathrm{tcf}_\prec(X) = \lambda$ if and only if there exists an increasing and $\prec$-true cofinal sequence $\langle x_\xi : \xi < \lambda \rangle$ in X and λ is regular.

Proof. Left to right are Lemma 2.8 and Lemma 2.9. Right to left is Lemma 2.10. □

2.3 Ideals and filters

Definition 2.12. Let X be a set. An *ideal (on X)* is a set I such that

1. $I \subseteq P(X)$, $\emptyset \in I$,
2. $A, B \in I$ implies $A \cup B \in I$,
3. $A \in I$, $B \subseteq A$ implies $B \in I$.

A *filter (on X)* is a set F such that

1. $F \subseteq P(X)$, $X \in F$,
2. $A, B \in F$ implies $A \cap B \in F$,
3. $A \in I$, $A \subseteq B$ implies $B \in F$.

[1] To be precise, here we use Lemma 2.8 with λ for X and $\mathrm{cf}(\lambda)$ for λ and the fact that all notions of cofinality are equivalent for λ.

[2] At stage χ, $\{\xi_{\chi'} : \chi' < \chi\}$ is not cofinal in λ since λ is regular and $\chi < \mu \leq \lambda$.

An ideal can be considered as a notion of smallness; a filter can be considered as a notion of largeness.

If F is an ideal on X and P is a property, then we say that P *holds D-almost everywhere* or P *holds for D-almost all* $x \in X$, iff $\{x \in X : P(x)\} \in D$, so iff the subset of X where P holds is large.

When I is an ideal, then $I^* := \{X \setminus A : A \in I\}$ is a filter, called the *dual* filter of I. Similarly, when F is a filter then $F^* := \{X \setminus A : A \in F\}$ is an ideal called the *dual* ideal of F. Clearly $(I^*)^* = I$ and $(F^*)^* = F$.

Definition 2.13. Let F be a filter on X. Then F is called

1. *proper* iff $F \neq P(X)$, which is equivalent to $\emptyset \notin F$.

Suppose F is proper. Then F is called

2. *maximal* iff there is no proper filter F' on X with $F \subset F'$,
3. *ultra* or an *ultrafilter* iff for all $A \subseteq X$ we have $A \in F$ or $X \setminus A \in F$ (this 'or' is automatically strict),
4. *prime* iff $A \cup B \in F$ implies $A \in F$ or $B \in F$,
5. *concentrated on* B iff $B \in F$,
6. *principal at* B iff $F = \{A \subseteq X : B \subseteq A\}$.
7. *non-principal* iff it is not principal for any B.

Note that each definition for filters applies to ideals as well when we consider their duals.

Lemma 2.14. Let F be a filter on X. Then F is maximal if and only if it is ultra if and only if it is prime.

Proof. Exercise for the reader. □

Definition 2.15. Let X be a set and $G \subseteq P(X)$. Then we say that G *has the finite intersection property* iff $A_1 \cap ... \cap A_n \neq \emptyset$ for all $A_1, ..., A_n \in G$, for all $n \in \mathbb{N}$.

Proposition 2.16. Let $G \subseteq P(X)$ have the finite intersection property. Then G extends to a proper filter F on X.

Proof. Define $F := \{A \subseteq X : \exists n \in \mathbb{N}, \exists A_1, ..., A_n \in G, A \supseteq A_1 \cap ... \cap A_n\}$. Then $G \subseteq F$ and F is a proper filter on X. □

To be ultimately precise, in the definition above one should require $n \neq 0$ or live by the convention that an empty intersection equals X.

Theorem 2.17 (Tarski, Ultrafilter Lemma)**.** Let F be a proper filter on X. Then F extends to an ultrafilter, i.e. there exists an ultrafilter U on X such that $F \subseteq U$.

Proof. Let $P = \{F' : F'$ is a proper filter on X and $F \subseteq F'\}$ and order P by inclusion $\subseteq$. Any $\subseteq$-chain C in P has an upper bound in P, namely $\bigcup C$. Then by Zorn's Lemma, P has a maximal element. This maximal element is a maximal filter, hence an ultrafilter on X extending F. □

In this thesis we will often start with a proper ideal I and extend its dual filter I^*, which is automatically proper, to an ultrafilter U. Any filter extension of I^* is disjoint from I, in particular so is U.

Also we encounter a lot of the following: We have a filter F on Y and $Y \subseteq X$. Then we can extend F to a filter $\hat{F}$ on X, by

$$\begin{aligned}\hat{F} &:= \{A \subseteq X : A \cap Y \in F\} \\ &= \{A \subseteq X : A \supseteq B \text{ for some } B \in F\} \\ &= \{A \cup B : A \in F, B \subseteq (X \setminus Y)\}\end{aligned}$$

Properness and ultraness are preserved under this extension. Note that $F \subseteq \hat{F}$. One could also consider the filter $\{A \cup (X \setminus Y) : A \in F\}$ on X, but this filter is not ultra, except in the trivial case where F is ultra and $Y = X$.

On the other hand, when F is a filter on X and $Y \subseteq X$, then

$$F' := \{A \cap Y : A \in F\}$$

is a filter on Y. Properness is not preserved under this operation, ultraness is. Clearly $F \cap P(Y) \subseteq F'$. We have $F' = F \cap P(Y)$ if and only if $Y \in F$. If $Y \in F$ and F is proper, then F' is proper.

In a similar way, this also works for ideals.

Theorem 2.18. Let I be a proper ideal on X and let $B \subseteq X$ be such that $B \notin I$. Then there exists a proper filter F on X disjoint from I and such that $B \in F$.

Proof. Since $B \notin I$, there is no $A \in I$ such that $B \subseteq A$. So $B \cap (X \setminus A) \neq \emptyset$ for all $A \in I$. So $I^* \cup \{B\}$ has the finite intersection property, hence extends to a proper filter F which is automatically disjoint from I. □

Lemma 2.19. Let I be a filter on a and $B \subseteq a$. Then $I' := \{X \subseteq a : X \subseteq A \cup B \text{ for some } A \in I\}$ is an ideal on a and it is the smallest ideal containing B and I. If $a \setminus B \notin I$ then I' is proper (and I is proper). Let $X, Y \subseteq a$. If $(X \setminus B) \cap Y \notin I$ then $X \cap Y \notin I'$.

Proof. We have

1. Clearly $I' \subseteq P(a)$ and $\emptyset \in I'$.
2. If $X \subseteq A \cup B$ and $X' \subseteq A' \cup B$ for some $A, A' \in I$, then $X \cup X' \subseteq (A \cup B) \cup (A' \cup B) = (A \cup A') \cup B \in I'$ since $A \cup A \in I$.
3. If $X \subseteq A \cup B$ for some $A \in I$ and $X' \subseteq X$ then $X' \subseteq A \cup B$ hence $X' \in I'$.

Clearly I' contains B and I and any other ideal containing B and I must contain I'. If $a \setminus B \notin I$, then $a \setminus B \notin I'$: If $a \setminus B \subseteq A \cup B$ for some $A \in I$, then $a \setminus B \subseteq A$ hence $a \setminus B \in I$, contradiction. If $X \cap Y \in I'$, then $X \cap Y \subseteq A \cup B$ for some $A \in I$. So $(X \setminus B) \cap Y \subseteq A$, so $(X \setminus B) \cap Y \in I$. □

2.4 Club-sets, stationary sets and Fodor's Lemma

The following notions should be familiar to set-theorists.

Definition 2.20. Let C and E be sets and α be an ordinal. We have the following definitions:

1. C is *α-closed* or *closed in α* or simply *closed* when α is understood iff for all $\beta < \alpha$ we have $(\sup(C \cap \beta) = \beta$ implies $\beta \in C)$.
2. C is *α-unbounded* or *unbounded in α* or simply *unbounded* when α is understood iff for all $\beta < \alpha$ there exists some $\gamma \in C$ such that $\beta \leq \gamma < \alpha$.

3. C is *closed unbounded in* α or (an) α*-club(set)* or simply (a) *club(set)* when α is understood when C is α-closed and α-unbounded.
4. C is *bounded* by α iff for all $\beta \in C$ we have $\beta \leq \alpha$.
5. C is *bounded below* α iff there exists some $\beta < \alpha$ such that for all $\gamma \in C$ we have $\gamma \leq \beta$, i.e. iff there exists some $\beta < \alpha$ such that C is bounded by β.
6. E is α*-stationary* or *stationary in* α or simply *stationary* when α is understood iff any α-clubset intersects E.

Note that we do not require an α-club set to be a subset of α. If we desire this, we say C is an α-club subset of α, or simply write $C \subseteq \alpha$. Clearly E is already α-stationary if E intersects only all α-club subsets of α.

Lemma 2.21. Let α be an ordinal. Then there exists a set $C \subseteq \alpha$ which is club in α and which has $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$.

Proof. Let $\langle \alpha_\xi : \xi < \mathrm{cf}(\alpha) \rangle$ be increasing cofinal in α. For $\chi < \mathrm{cf}(\alpha)$ recursively define α'_χ by

1. Base case: $\alpha'_0 := \alpha_0$,
2. Successor case: $\alpha'_{\chi+1} := \min(\{\alpha_\xi : \xi < \mathrm{cf}(\alpha)\} \setminus (\alpha'_\chi + 1))$,
3. Limit case: $\alpha'_\chi := \sup_{\chi' < \chi} \alpha'_{\chi'}$.

Then $\{\alpha'_\chi : \chi < \mathrm{cf}(\alpha)\} \supseteq \{\alpha_\xi : \xi < \mathrm{cf}(\alpha)\}$, so $\langle \alpha'_\chi : \chi < \mathrm{cf}(\alpha) \rangle$ is increasing, cofinal and continuous. Hence $C := \{\alpha'_\chi : \chi < \mathrm{cf}(\alpha)\}$ is club in α and $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$. □

It is well-known that every well-ordered set is isomorphic to a unique ordinal number, and this isomorphism is unique as well. Hence also every well-ordered set carries a notion of club-subsets and stationary subsets, via it's unique isomorphism to a unique ordinal number.

Example 2.22. Let β be a limit ordinal and let $C \subseteq \beta$ be β-club. Then C is well-ordered. So there is a notion of C-clubsets and C-stationary sets, namely: A C-club subset of C is a set $c \subseteq C$ such that

1. for all $\alpha \in C$ exists $\alpha' \in c$ such that $\alpha < \alpha'$,
2. if $\alpha \in C$ and $\sup(c \cap \alpha) = \alpha$, then $\alpha \in c$.

and $e \subseteq C$ is C-stationary iff it intersects every C-club subset. Suppose $c \subseteq C$ is C-club. Then it is clearly β-unbounded, and if $\alpha < \beta$ is such that $\sup(c \cap \beta) = \beta$, then $\sup(C \cap \beta) = \beta$, so $\beta \in C$, thus $\beta \in c$. Thus c is β-club. Now suppose $e \subseteq C$ is C-stationary. Let D be β-club. Then $D \cap C$ is β-club. Thus clearly $D \cap C$ is C-club. So e intersects $D \cap C$, so intersects D. So e is in fact β-stationary. So if $c, e \subseteq C$, then

$$c \text{ is } C\text{-club} \Leftrightarrow c \text{ is } \beta\text{-club},$$
$$e \text{ is } C\text{-stationary} \Leftrightarrow e \text{ is } \beta\text{-stationary}.$$

We have the following lemma on stationary subsets.

Lemma 2.23 (Fodor's Lemma or Pressing Down Lemma)**.** Let κ be a regular uncountable cardinal and let S be κ-stationary. Let f be a regressive function on S, i.e. $f(\alpha) < \alpha$ for all $\alpha \in S \setminus \{0\}$. Then there exists a γ such that $\{\alpha \in S : f(\alpha) = \gamma\}$ is κ-stationary.

A proof relies on *diagonal intersections* of clubsets. For such a proof, see for instance [8, Theorem 8.7]. In Section 4.4 we will prove this lemma for $\kappa = \omega_1$ and $S = \omega_1$. The following lemma is a generalization to cardinals of uncountable cofinility.

Lemma 2.24 (Generalization of Fodor's Lemma)**.** Let κ be a cardinal of uncountable cofinality and let S be κ-stationary. Let f be a regressive function on S, i.e. $f(\alpha) < \alpha$ for all $\alpha \in S \setminus \{0\}$. Then there exists a γ such that $\{\alpha \in S : f(\alpha) \leq \gamma\}$ is κ-stationary.

Proof. Let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be increasing, cofinal and continuous in κ. Then $T := \{i < \mathrm{cf}(\kappa) : \kappa_i \in S\}$ is $\mathrm{cf}(\kappa)$-stationary, namely, if C is $\mathrm{cf}(\kappa)$-club, then $\{\kappa_i : i \in C\}$ is κ-club, hence intersects S, so there is some $i \in C$ such that $\kappa_i \in S$, thus C intersects T. Define

$$\begin{aligned} g : T &\to \mathrm{cf}(\kappa) \\ i &\mapsto \min(j : f(\kappa_i) < \kappa_{j+1}) \end{aligned}$$

Since $f(\kappa_i) < \kappa_i$, we have $g(i) < i$. Thus g is regressive on T, so apply Fodor's Lemma to obtain $j < \mathrm{cf}(\kappa)$ such that $E := \{i \in T : g(i) = j\}$ is $\mathrm{cf}(\kappa)$-stationary. Then $F := \{\kappa_i : i \in E\}$ is κ-stationary, namely, if C is κ-club, then $C \cap \{\kappa_i : i < \mathrm{cf}(\kappa)\}$ is κ-club, so $\{i < \mathrm{cf}(\kappa) : \kappa_i \in C\}$ is $\mathrm{cf}(\kappa)$-club, so intersects E, so F intersects C. Of course, $F \subseteq S$, and if $i \in E$, then $g(\kappa_i) = j$, so $f(\kappa_i) < \kappa_{j+1}$. So $\{\alpha \in S : f(\alpha) \leq \kappa_{j+1}\}$ is κ-stationary. □

2.5 Club-guessing

Lemma 2.25. Let μ and κ be infinite cardinals such that $\mathrm{cf}(\kappa) > \mu$ and μ is regular. Then $S(\kappa, \mu) := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \mu\}$ is κ-stationary.

Proof. Let C be any closed unbounded subset of κ. Then $|C| \geq \mathrm{cf}(\kappa) > \mu$, so let $\langle \alpha_i : i < \mu \rangle$ be the increasing sequence of the first μ elements of C. Then $\alpha := \lim_{i \to \mu} \alpha_i$ satisfies $\alpha < \kappa$, $\mathrm{cf}(\alpha) = \mu$ and $\alpha \in C$. Hence $C \cap E \neq \emptyset$. Since C was arbitrary, E is stationary. □

Definition 2.26. Let μ, κ be regular cardinals such that $\mu < \kappa$. By Lemma 2.25, $S(\kappa, \mu)$ is κ-stationary. Let $T \subseteq S$ be again κ-stationary. The triple (κ, μ, T) may have the *diamond property* $\diamond_{\mathrm{club}}(\kappa, \mu)(T)$, which is defined as:

There exists $\langle S_\alpha : \alpha \in T \rangle$ such that

1. for all $\alpha \in T$ we have $S_\alpha \subseteq \alpha$ and S_α is α-club,
2. for all κ-club C we have $\{\alpha \in T : S_\alpha \subseteq C\}$ is κ-stationary.

We write $\diamond_{\mathrm{club}}(\kappa, \mu)$ for $\diamond_{\mathrm{club}}(\kappa, \mu)(S(\kappa, \mu))$. We call such a sequence $\langle S_\alpha : \alpha \in T \rangle$ a *club-guessing sequence* for T.[3].

In general $\diamond_{\mathrm{club}}(\kappa, \mu)$ is not always true. We will prove that it holds when $\aleph_1 < \mu^+ < \kappa$ via the following lemma.

Theorem 2.27. Let μ, κ be regular, uncountable and such that $\mu^+ < \kappa$. Let $T \subseteq S(\kappa, \mu)$ be κ-stationary. Let $\langle S_\alpha : \alpha \in T \rangle$ be such that $S_\alpha \subseteq \alpha$, S_α is α-club and $|S_\alpha| = \mu$, for all $\alpha \in T$. Then there is a κ-club C such that $\langle (S_\alpha \cap C)' : \alpha \in T \rangle$ is a club-guessing sequence for T, where $(S_\alpha \cap C)' = S_\alpha \cap C$ if this is α-club, and $(S_\alpha \cap C)' = \alpha$ if $S_\alpha \cap C$ is not α-club.

[3]Guessing sequences are in itself an interesting topic in set theory, see for instance [7].

Proof. Suppose not. For all $\beta < \mu^+$ we will define C_β and $\langle S^\beta_\alpha : \alpha \in T\rangle$ such that for all β the following hold:

(i)$_\beta$ C_β is κ-club.

(ii)$_\beta$ For all $\alpha \in T$ we have $S^\beta_\alpha = S_\alpha \cap \bigcap_{\beta'<\beta} C_{\beta'}$.

(iii)$_\beta$ $\{\alpha \in T : S^\beta_\alpha \subseteq C_\beta\}$ is not stationary in κ.

We do this by recursion; suppose it has been done for all $\beta' < \beta$, for some $\beta < \kappa$. Then of course we define $S^\beta_\alpha := S_\alpha \cap \bigcap_{\beta'<\beta} C_{\beta'}$ for all $\alpha \in T$. Now $C := \bigcap_{\beta'<\beta} C_{\beta'}$ is κ-club. So $\hat{C} := \{\alpha \in C : \sup(\alpha \cap C) = \alpha\}$, i.e. the set of limit points of C, is κ-club (here we use that κ is regular and uncountable, or in fact that κ is of uncountable cofinality). Since we assumed the lemma to be false, $\langle (S_\alpha \cap C)' : \alpha \in T\rangle$ is not a club-guessing sequence for T, meaning that there is a κ-club C_β such that $\{\alpha \in T : (S_\alpha \cap C)' \subseteq C_\beta\}$ is not κ-stationary. We will show that it follows that $\{\alpha \in T : S^\beta_\alpha \subseteq C_\beta\}$ is not κ-stationary:

If it were, we prove that $\{\alpha \in T : (S_\alpha \cap C)' \subseteq C_\beta\}$ would be κ-stationary. Let D be κ-club. Then $D \cap \hat{C}$ is κ-club (since $\kappa > \aleph_0$). So $D \cap \hat{C}$ intersects $\{\alpha \in T : S^\beta_\alpha \subseteq C_\beta\}$, say in he point α. Since S_α and $\alpha \cap C$ are α-club, also $S_\alpha \cap (\alpha \cap C) = S_\alpha \cap C$ is α-club (here we use that μ is regular and uncountable, and $\mathrm{cf}(\alpha) = \mu$). Thus $(S_\alpha \cap C)' = S_\alpha \cap C = S^\beta_\alpha \subseteq C_\beta$. Therefore $\alpha \in \{\alpha \in T : (S_\alpha \cap C)' \subseteq C_\beta\}$, hence D intersects $\{\alpha \in T : (S_\alpha \cap C)' \subseteq C_\beta\}$. Since D was arbitrary, we have $\{\alpha \in T : (S_\alpha \cap C)' \subseteq C_\beta\}$ is κ-stationary, a contradiction.

Let $D := \bigcap_{\beta<\mu^+} C_\beta$, then D is κ-club. For each $\alpha \in T$ the sequence $\langle S^\beta_\alpha : \beta < \mu^+\rangle$ is $\subseteq$-decreasing, but $|S^0_\alpha| = |S_\alpha| = \mu$. So we let $\beta_\alpha < \mu^+$ such that $S^{\beta'}_\alpha = S^{\beta_\alpha}_\alpha$ for all $\beta' > \beta_\alpha$. The assignment $\alpha \mapsto \beta_\alpha$ restricts to a map $T \setminus \mu^+ \to \mu^+$, and this a regressive function on the κ-stationary set $T \setminus \mu^+$. By the Pressing Down Lemma (Lemma 2.23), there exists a $\gamma < \mu^+$ such that $E := \{\alpha \in T \setminus \mu^+ : \beta_\alpha = \gamma\}$ is still κ-stationary. Now

$$S^\gamma_\alpha = S^{\gamma+1}_\alpha = S^0_\alpha \cap \bigcap_{\beta'<\gamma+1} C_{\beta'} = S^0_\alpha \cap \left(\bigcap_{\beta'<\gamma} C_{\beta'}\right) \cap C_\gamma = S^\gamma_\alpha \cap C_\gamma, \quad \text{so} \quad S^\gamma_\alpha \subseteq C_\gamma$$

for all $\alpha \in E$. So $\{\alpha \in T : S^\gamma_\alpha \subseteq C_\gamma\}$ is κ-staionary, contradicting (iii)$_\gamma$. □

Corollary 2.28. Let μ, κ be regular, uncountable and such that $\mu^+ < \kappa$. Let $T \subseteq S(\kappa, \mu)$ be κ-stationary. Then $\diamond_{\text{club}}(\kappa, \mu)(T)$ holds.

3 Pcf-theory

3.1 The definition of pcf

3.1.1 Relations and filters

Recall that a filter can be seen as a notion of largeness. So one notion of two functions being 'almost equal' would be to require that they agree on a set that is considered large by some filter. So we will call functions equal *modulo a filter* F if they agree on a set in F. The rigorous definition is as follows:

Definition 3.1. Let X be a set and let F be a filter on X. Let f and g be two functions with domain X. Then f *equals* g *modulo* F, notation $f =_F g$, iff $\{x \in X : f(x) = g(x)\} \in F$.

Note that equality is just *some* relation to compare functions. In fact, we can extend this definition to any kind of relation.

Definition 3.2. Let X be a set and let F be a filter on X. Let R be any binary relation. Let f and g be two functions with domain X. Then we write $f\ R_F\ g$ iff $\{x \in X : f(x)\ R\ g(x)\} \in F$.

An example is when f and g are ordinal-valued functions, and R is taken to be the standard $<$ or $\leq$ on the ordinals.

We often encounter not just two functions, but a set of functions. For example, we often consider $\prod X := \{f : X \to \bigcup X : f(x) \in x \text{ for all } x \in X\}$. Let R be a binary relation on $\bigcup X$, i.e. $R \subseteq \bigcup X \times \bigcup X$. Let F be an ultrafilter on X. Then again define for $f, g \in \prod X$,

$$f\ R_F\ g \Leftrightarrow \{x \in X : f(x)\ R\ g(x)\} \in F.$$

From now on, we use the following notation

$$[f\ R\ g] = \{x \in X : f(x)\ R\ g(x)\}.$$

Some properties of R from Definition 2.1 may be inherited by R_F. We list them here.

1. Reflexivety: Suppose R is reflexive. Then for any $f \in \prod X$ we have $[f\ R\ f] = X \in F$, so $f\ R_F\ f$. So R_F is reflexive.
2. Irreflexivety, assuming F is proper: $[f\ R\ f] = \emptyset \notin F$, so $f \not R_F\ f$.
3. Symmetry: If $f\ R_F\ g$ then $[g\ R\ f] = [f\ R\ g] \in F$, so $g\ R_F\ f$.
5. Transitivity: If $f\ R_F\ g$ and $g\ R_F\ h$ then $[f\ R\ h] \supseteq [f\ R\ g] \cap [g\ R\ h] \in F$, so $f\ R_F\ h$.
6. Totality, assuming F is ultra: If $f \not R_F\ g$, then $[f\ R\ g] \notin F$, so $[g\ R\ f] = X \setminus [f\ R\ g] \in F$, so $g\ R_F\ f$.
8. Equivalence relation: Reflexivity, symmetry and transitivity are inherited.
9. Quasi order: Reflexivity and transitivity are inherited.
11. Strict partial order, assuming F is proper: Irreflexivity and transitivity are inherited.

Other properties of R are not inherited by R_F. Recall that $A \dot\cup B$ simultaneously denotes the union $A \cup B$ of sets A and B and claims that A and B are disjoint. The following properties are not inherited:

4. Anti-symmetry: Suppose R is anti-symmetric. If $f\ R_F\ g$ and $g\ R_F\ f$ then $[f = g] \supseteq [f\ R\ g] \cap [g\ R\ f] \in F$, so $f =_F g$ but not necessarily $f = g$. So R_F is not necessarily anti-symmetric.

7. Trichotomy, even if we assume that F is ultra: $[f\ R\ g] \dot\cup [f = g] \dot\cup [g\ R\ f] = X \in F$, so either $[f\ R\ g] \in F$ or $[f = g] \in F$ or $[g\ R\ f] \in F$, so either $f\ R_F\ g$ or $f =_F g$ (but not necessarily $f = g$) or $g\ R_F\ f$.
10. Partial order: Only anti-symmetry is not inherited.
12. Linear order: Only anti-symmetry is not inherited.
13. Strict linear order, even if we assume that F is ultra: Only trichotomy is not inherited.
14. Well-order: Trichotomy is not inherited.

Note that the relation R_F is $\subseteq$-preserving in the following sense:

$$\text{If } R \subseteq R' \text{ and } F \subseteq F', \text{ then } R_F \subseteq R'_{F'}.$$

When R_F is an equivalence relation, for example when R is, we can look at the set of R_F-equivalence classes in $\prod X$, denoted as $\prod X/R_F$, instead of looking at $\prod X$ itself. We write $\prod X/F := \prod X/R_F$ when R is understood. We write f/F for the equivalence class of an element $f \in \prod X$ in $\prod X/F$.

The following example is of crucial importance in pcf-theory.

Example 3.3. Let a be a non-empty set of ordinals and F a proper filter on a. Consider the relation 'equality' to obtain $\prod a/F = \prod a/{=_F}$. On $\prod a/F$ we define $<$ and $\leq$ by

1. $f/F < g/F$ iff $f <_F g$, where $<$ is standard on ordinals. Then:
   (a) $<$ is well-defined: If $f/F = f'/F$, $g/F = g'/F$ and $f <_F g$, then $[f' < g'] \supseteq [f' = f] \cap [f < g] \cap [g = g'] \in F$, so $f' <_F g'$.
   (b) $<$ is a strict partial order: $<_F$ is a strict partial order since $<$ is a strict partial order.
   (c) $<$ is a strict linear order, assuming F is ultra: Either $f/F < g/F$ or $g/F < f/F$ or $f/F = g/F$.
2. $f/F \leq g/F$ iff $f \leq_F g$, where $\leq$ is standard on ordinals. Then:
   (a) $\leq$ is well-defined: If $f/F = f'/F$ and $g/F = g'/F$ and $f \leq_F g$, then $[f' \leq g'] \supseteq [f' = f] \cap [f \leq g] \cap [g = g'] \in F$, so $[f' \leq g'] \in F$ and $f' \leq_F g'$. Another proof is: Since $\leq$ on ordinals is transitive, so is $\leq_F$. Since $= \subseteq \leq$, we have $=_F \subseteq \leq_F$. If $f/F = f'/F$, $g/F = g'/F$ and $f \leq_F g$, then also $f' \leq_F f$ and $g \leq_F g'$, so $f' \leq_F g'$.
   (b) $\leq$ is a partial order: If $f \leq_F g$ and $g \leq_F f$ then $f =_F g$. Reflexivity and transitivity hold for $\leq_F$, hence for $\leq$ on $\prod a/F$.
   (c) $\leq$ is a linear order, assuming F is ultra: If $f/F \not\leq g/F$, then $[f \leq g] \notin F$, so $[g \leq f] \supseteq [f \not\leq g] \in F$, so $g/F \leq f/F$.

We thus have that $(\prod a/F, <, \leq)$ is a strict partial and quasi ordered set. Does it satisfy (P1)-(P3) from Definition 2.3, and how do $<$ and $\leq$ relate?

We have $[f < g] \dot\cup [f = g] = [f \leq g]$. So $[f < g] \in F$ implies $[f \leq g] \in F$ and $[f = g] \notin F$ (assuming F is proper). So $f/F < g/F$ implies $(f/F \leq g/F$ and $f/F \neq g/F)$. So (P1) always holds. Also it is easy to see that (P2) always holds.

When F is ultra, $([f \leq g] \in F$ and $[f = g] \notin F)$ implies $[f < g] \in F$, i.e. $f/F < g/F$ if and only if $(f/F \leq g/F$ and $f/F \neq g/F)$. Since $\leq$ is anti-symmetric (on $\prod a/F$), we have

$$f/F < g/F \quad \Leftrightarrow \quad f/F \leq g/F \text{ and } f/F \neq g/F \quad \Leftrightarrow \quad f/F \leq g/F \text{ and } g/F \not\leq f/F.$$

If every $\alpha \in a$ is a non-zero limit ordinal, then for any $f/F \in \prod a/F$ we have $f/F < (f+1)/F$, where $(f+1)(\alpha) := f(\alpha) + 1$, so (P3) is also satisfied.

In fact, in the above example it is not necessary to take the domain of the functions f and g to be in $\prod a$. The definition of $<$ and $\leq$ still apply when f and g are just ordinal-valued functions with domain a.

The above example leads to the following theorem:

Theorem 3.4. Suppose $a \neq \emptyset$ consists of non-zero limit ordinals. Let F be a proper filter on a and let F' be any proper filter with $F \subseteq F'$. Suppose $\mathrm{tcf}_<(\prod a/F) = \lambda$. Then $\mathrm{tcf}_<(\prod a/F') = \lambda$. When F' is ultra, we have $\mathrm{cf}(\prod a/F') = \lambda$.

Proof. By Theorem 2.11 let $\langle f_\xi/F : \xi < \lambda\rangle$ be an increasing and $<$-true cofinal sequence in $\prod a/F$ with λ regular. Since $F \subseteq F'$, $f/F < g/F$ implies $f/F' < g/F'$ for any $f, g \in \prod a$. Thus $\langle f_\xi/F' : \xi < \lambda\rangle$ is an increasing and $<$-true cofinal sequence in $\prod a/F'$. Again by Theorem 2.11, $\mathrm{tcf}_<(\prod a/F') = \lambda$. When F' is ultra, $(\prod a/F', <, \leq)$ is strict linearly and linearly ordered, and $f/F' < g/F'$ if and only if $(f/F' \leq g/F'$ and $f/F' \neq g/F')$, thus (P1)-(P3) are satisfied. By Lemma 2.6, all notions of cofinal then coincide. □

3.1.2 Properties of $<_I$-increasing sequences

Note that everything we have done for filters in the previous paragraph, we could have done for ideals, when considering their duals. For example, when I is an ideal on a set X and $f, g \in \prod X$, we have $f =_I g$ iff $[f = g] \in I^*$ iff $[f \neq g] \in I$.

We now discuss some further properties of an $<_I$-increasing sequence in $\prod a/I$.

Definition 3.5. Let X be a set, let I be an ideal on X, let S be a set of ordinals and let $f = \langle f_\xi : \xi \in S\rangle$ be an $<_I$-increasing S-sequence in $\mathbf{ON}^X$. Then f is called *strongly increasing* iff there exists a sequence $\langle Z_\xi : \xi \in S\rangle$ in I such that

$$\xi < \xi', \alpha \in a \setminus (Z_\xi \cup Z_{\xi'}) \quad \Rightarrow \quad f_\xi(\alpha) < f_{\xi'}(\alpha).$$

Clearly strong increase is stronger than (normal) increase. Strong increase requires some uniform increase, each functions f_ξ has only a fixed I-small set Z_ξ where it can behave wild.

Definition 3.6. Let X be a set, let I be an ideal on X, let $\kappa \leq \lambda$ be regular cardinals and let $f = \langle f_\xi : \xi < \lambda\rangle$ be an $<_I$-increasing λ-sequence in $\mathbf{ON}^X$. Then f may satisfy the following star property:

$*_\kappa$: When $S \subseteq \lambda$ is unbounded, there exists $S_0 \subseteq S$ such that $\mathrm{ot}(S_0) = \kappa$ and $\langle f_\xi : \xi \in S_0\rangle$ is strongly increasing.

Definition 3.7. Let A be a set, let I be an ideal on A, let $\kappa \leq \lambda$ be regular cardinals and let $f = \langle f_\xi : \xi < \lambda\rangle$ be an $<_I$-increasing λ-sequence in $\mathbf{ON}^A$. Then f *has the bounding projection property for* κ or *has* bpp_κ iff for all $\langle S_\alpha : \alpha \in A\rangle$ such that $S_\alpha \subseteq \mathbf{ON}$ and $|S_\alpha| < \kappa$ and $\alpha \mapsto \sup S_\alpha$ is an $<_I$-upper bound for f, there exists a $\xi < \lambda$ such that $f^+_\xi = \mathrm{proj}(f_\xi, \langle S_\alpha : \alpha \in A\rangle)$, defined by $\mathrm{proj}(f_\xi, \langle S_\alpha : \alpha \in A\rangle)(\alpha) = \min(S_\alpha \setminus f_\xi(\alpha))$, is an $<_I$-upper bound for f.

Strictly speaking, it is not defined what is means to be an $<_I$-upper bound, since upper bounds were only defined for quasi orders. The definition is of course straightforward: $m \in X$ is an $<$-upper bound for $Y \subseteq X$ iff $y < m$ for all $y \in Y$. In the above situation, an $<_I$-upper bound for f is precisely an $\leq_I$-upper bound for f, since f is $<_I$-increasing. In general: If Y satisfies (P3) and X satisfies (P2), then $<_I$-upper bounds for Y are precisely $\leq_I$-upper bounds for Y.

We will relate $*_\kappa$ and bpp_κ in the following lemmas and Theorem 3.11, using strongly increasing sequences.

Lemma 3.8. Let A be a set, let I be an ideal on A, let $\kappa \leq \lambda$ be regular cardinals and let $f = \langle f_\xi : \xi < \lambda \rangle$ be an $<_I$-increasing λ-sequence in $\mathbf{ON}^A$ satisfying $*_\kappa$. If $|A| < \kappa$, then f has bpp_κ.

Proof. Suppose not and let $S = \langle S_\alpha : \alpha \in A \rangle$ be a counterexample. Modify each f_ξ on an I-set such that $f_\xi(\alpha) < \sup S_\alpha$ for all $\alpha \in A$. Then $\langle f_\xi : \xi < \lambda \rangle$ still satisfies $*_\kappa$.

For any $\xi < \lambda$ we associate a $\bar{\xi} < \lambda$: If $\xi < \lambda$, then $f^+_\xi = \mathrm{proj}(f_\xi, S)$ is not an $<_I$-upper bound for f. Thus $f_{\bar{\xi}} \not<_I f^+_\xi$ for some $\bar{\xi}$. When $\bar{\xi} < \xi'$ then $f_{\bar{\xi}} <_I f_{\xi'}$, so $f_{\xi'} \not\leq_I f^+_\xi$, so $[f^+_\xi < f_{\xi'}] \notin I$.

For $\beta < \lambda$, recursively define $\xi_\beta < \lambda$ such that $\overline{\xi_{\beta'}} < \xi_\beta$ for all $\beta' < \beta$. Then $S := \{\xi_\beta : \beta < \lambda\}$ is unbounded in λ and if $\xi < \xi'$ in S then $[f^+_\xi < f_{\xi'}] \notin I$. By $*_\kappa$, let $T \subseteq S$ be such that $\langle f_\xi : \xi \in T \rangle$ is strongly increasing. Let $\langle Z_\xi : \xi \in T \rangle$ be a sequence in I such that $\xi < \xi', \alpha \in A \setminus (Z_\xi \cup Z_{\xi'}) \Rightarrow f_\xi(\alpha) < f_{\xi'}(\alpha)$.

To any $\xi \in T$ we will associate an $\alpha_\xi \in A$. Let $\xi \in T$, let $\xi^+ := \min(T \setminus (\xi + 1))$, then $[f^+_\xi < f_{\xi^+}] \notin I$ but $Z_\xi, Z_{\xi^+} \in I$, hence we pick some $\alpha_\xi \in [f^+_\xi < f_{\xi^+}] \setminus (Z_\xi \cup Z_{\xi^+})$.

The map $T \to A$ given by $\xi \mapsto \alpha_\xi$ maps a set of size κ into a set of size strictly less than κ. So let $T' \subseteq T$ of size κ and $\alpha \in A$ such that $\alpha_\xi = \alpha$ for all $\xi \in T'$. When $\xi < \xi'$ in T', we have

$$f^+_\xi(\alpha) < f_{\xi^+}(\alpha) \leq f_{\xi'}(\alpha) \leq f^+_{\xi'}(\alpha)$$

where the second inequality holds since $\xi^+ \leq \xi'$ and $\alpha = \alpha_\xi \notin Z_\xi \cup Z_{\xi^+}$ and $\alpha = \alpha_{\xi'} \notin Z_{\xi'} \cup Z_{(\xi')^+}$, so $\alpha \notin Z_{\xi^+} \cup Z_{\xi'}$; and the third inequality holds by definition of $f^+_{\xi'}$. We thus showed that $\langle f^+_\xi(\alpha) : \xi \in T' \rangle$ is an increasing sequence in S_α of length κ, thus $|S_\alpha| \geq \kappa$, contradiction. □

Lemma 3.9. Let A be a set, let I be an ideal on A, let λ be a regular cardinal and let $f = \langle f_\xi : \xi < \lambda \rangle$ be an $<_I$-increasing sequence in $\mathbf{ON}^A$ satisfying $\mathrm{bpp}_{|A|^+}$. If $\lambda > |A|^+$, then f has a $\leq_I$-minimal upper bound.

Proof. Suppose not. Let h_0 be any $<_I$-upper bound for f, for instance $h_0(\alpha) = \sup^+_{\xi<\lambda} f_\xi(\alpha)$. We define $S^\chi = \langle S^\chi_\alpha : \alpha \in A \rangle$ for $\chi < |A|^+$ such that for all χ we have:

1. $|S^\chi_\alpha| \leq |A|$ for all α.
2. For all $\chi' \leq \chi$ we have $S^{\chi'}_\alpha \subseteq S^\chi_\alpha$.
3. $\alpha \mapsto \sup S^\chi_\alpha$ is an $<_I$-upper bound for f.

We do this by recursion.

1. Base case: Take $S^0_\alpha := \{h_0(\alpha)\}$.
2. Successor case: Given $\langle S^\chi_\alpha : \alpha \in A \rangle$ for some χ, employ $\mathrm{bpp}_{|A|^+}$ and let $\xi_\chi < \lambda$ such that $h_\chi := \mathrm{proj}(f_{\xi_\chi}, \langle S^\chi_\alpha : \alpha \in A \rangle)$ is an $<_I$-upper bound for $\langle f_\xi : \xi < \lambda \rangle$. If $\xi_\chi < \xi$, then $f_{\xi_\chi} <_I f_\xi <_I h_\chi$ so

$$h_\chi = \mathrm{proj}(f_{\xi_\chi}, S^\chi) \leq_I \mathrm{proj}(f_\xi, S^\chi) \leq_I \mathrm{proj}(h_\chi, S^\chi) = h_\chi,$$

so in fact we have $=_I$ everywhere. We have h_χ is an $<_I$-upper bound, hence an $\leq_I$-upper bound, but by assumption it is not a minimal upper bound. So there exists an upper bound u_χ such that $u_\chi \leq_I h_\chi$ and $h_\chi \not\leq_I u_\chi$, so $[u_\chi < h_\chi] \notin I$. In fact, we modify u_χ on an I-set such that $u_\chi \leq h_\chi$. Define $S^{\chi+1}_\alpha := S^\chi_\alpha \cup \{u_\chi(\alpha)\}$. We establish some extra property: For $\xi \geq \xi_\chi$ we have

$$[\mathrm{proj}(f_\xi, S^{\chi+1}) < u_\chi] \subseteq [\mathrm{proj}(f_\xi, S^\chi) < u_\chi] \subseteq [\mathrm{proj}(f_\xi, S^\chi) < h_\chi] \subseteq [\mathrm{proj}(f_\xi, S^\chi) \neq h_\chi] \in I$$

since $\mathrm{proj}(f_\xi, S^\chi) =_I h_\chi$. Thus $\mathrm{proj}(f_\xi, S^{\chi+1}) \geq_I u_\chi$. However $\mathrm{proj}(f_\xi, S^{\chi+1}) \leq_I u_\chi$ since u_χ is an $\leq_I$-upper bound for f. Thus $\mathrm{proj}(f_\xi, S^{\chi+1}) =_I u_\chi$.

3. Limit case: Suppose $\chi < |A|^+$ is limit and assume $S^{\chi'}$ has been defined for all $\chi' < \chi$. Take $S^\chi_\alpha := \bigcup_{\chi' < \chi} S^{\chi'}_\alpha$.

Clearly 1.-3. are always satisfied.

From the successor step of the definition above, we have map $|A|^+ \to \lambda$ given by $\chi \mapsto \xi_\chi$, and $\lambda > |A|^+$, thus we can take a $\tilde{\xi} < \lambda$ such that $\xi_\chi < \tilde{\xi}$ for all χ. Let $H_\chi := \mathrm{proj}(f_{\tilde{\xi}}, S^\chi)$ for each χ. Since $\xi_\chi < \tilde{\xi}$, we have $H_\chi =_I h_\chi$. We thus have

$$\begin{aligned} H_\chi =_I h_\chi = \mathrm{proj}(f_\xi, S^\chi) &\text{ for all } \xi \geq \xi_\chi, \text{ in particular for all } \xi \geq \tilde{\xi}. \\ H_{\chi+1} =_I h_{\chi+1} = \mathrm{proj}(f_\xi, S^{\chi+1}) =_I u_\chi &\text{ for all } \xi \geq \xi_{\chi+1}, \text{ in particular for all } \xi \geq \tilde{\xi}. \end{aligned}$$

So $H_{\chi+1} =_I h_{\chi+1} \not\geq_I h_\chi = H_\chi$, thus $[H_{\chi+1} < H_\chi] \notin I$. Let α_χ such that $H_{\chi+1}(\alpha_\chi) < H_\chi(\alpha_\chi)$. Then we have a map $|A|^+ \to A$ given by $\chi \mapsto \alpha_\chi$, and there must be some $\alpha \in A$ such that infinitely many χ's have $\alpha_\chi = \alpha$, and we obtain an infinite decreasing sequence of ordinals, a contradiction. □

Lemma 3.10. Let A be a set, let I be an ideal on A, let λ be a regular cardinal and let $f = \langle f_\xi : \xi < \lambda \rangle$ be an $<_I$-increasing sequence in $\mathbf{ON}^A$ satisfying $\mathrm{bpp}_{\aleph_0}$. Then any $\leq_I$-minimal upper bound is also exact.

Proof. Suppose h is a minimal upper bound and suppose $g <_I h$. We must show that $g \leq_I f_\xi$ for some ξ. Modify g on an I-set to get $g < h$. Let $S_\alpha := \{g(\alpha), h(\alpha)\}$ for all α. Then $|S_\alpha| < \aleph_0$ and f is $<_I$-bounded by the map $\alpha \mapsto \sup S_\alpha$. So let $\xi < \lambda$ such that $f^+_\xi := \mathrm{proj}(f_\xi, \langle S_\alpha : \alpha \in A \rangle)$ is an upper bound for f. Clearly $f^+_\xi \leq h$ but h is minimal, so we must have $h \leq_I f^+_\xi$, thus $f^+_\xi =_I h$. Then by the definition of proj, we must have $g <_I f_\xi$ and hence $g \leq_I f_\xi$. □

Note that in fact we do not need f to satisfy $\mathrm{bpp}_{\aleph_0}$, but only bpp_3.

Theorem 3.11. Let A be a set, let I be an ideal on A, let $\kappa \leq \lambda$ be regular cardinals and let $f = \langle f_\xi : \xi < \lambda \rangle$ be an $<_I$-increasing λ-sequence in $\mathbf{ON}^A$. If $|A|^+ \leq \kappa$ and $|A|^+ < \lambda$, then the following are equivalent:

1. f satisfies $*_\kappa$.
2. f has bpp_κ.
3. f has an exact upper bound g such that $[\mathrm{cf}(g) < \kappa] \in I$.

Here the bracket notation [...] is used is a more general setting; we mean $[\mathrm{cf}(g) < \kappa] = \{\alpha \in a : \mathrm{cf}(g(\alpha)) < \kappa\}$.

Proof. (1. $\Rightarrow$ 2.) This is Lemma 3.8.

(2. $\Rightarrow$ 3.) Lemma 3.9 yields the existence of a minimal upper bound g and g is exact by Lemma 3.10. We will show that without loss of generality we may assume that g is nowhere zero nor a successor ordinal.

For since $f_\xi <_I g$ for any ξ, we have $[f_\xi \geq g] \in I$, hence $[g = 0] \in I$. So we can modify g on an I-set to obtain that g is nowhere 0 and g remains an exact upper bound.

Now suppose that $\{\alpha : g(\alpha) \text{ is a successor ordinal}\} \notin I$. For $\alpha \in a$, define

$$\tilde{g}(\alpha) := \begin{cases} g(\alpha), & \text{if } g(\alpha) \text{ is limit,} \\ \beta, & \text{if } g(\alpha) = \beta + 1. \end{cases}$$

Then $\tilde{g} \leq g$, so $\tilde{g} \leq_I g$ and $[g > \tilde{g}] = \{\alpha : g(\alpha) \text{ is a successor ordinal}\} \notin I$, so $g \not\leq_I \tilde{g}$. Since g is a minimal upper bound, it cannot be that $\tilde{g}$ is an upper bound. So for some ξ we have $f_\xi \not<_I \tilde{g}$, so $[f_\xi \geq \tilde{g}] \notin I$, but $[f_\xi \geq g] \in I$, so $[f_\xi = \tilde{g}] \notin I$. Now $[f_\xi \geq f_{\xi+1}] \in I$ so

$$I \not\ni [f_\xi < f_{\xi+1}] \cap [f_\xi = \tilde{g}] \subseteq [f_{\xi+1} > \tilde{g}] \subseteq [f_{\xi+1} \geq g]$$

(To see $I \not\ni [f_\xi < f_{\xi+1}] \cap [f_\xi = \tilde{g}]$, note that if $X \in I$ and $Y \notin I$, then $X^c \cap Y \notin I$.) But then $f_{\xi+1} \not<_I g$, a contradiction. So $\{\alpha : g(\alpha) \text{ is a successor ordinal}\} \in I$ and again we modify g on an I-set to obtain that g is never a successor ordinal.

We will now show that $P := [\mathrm{cf}(g) < \kappa] \in I$. For $\alpha \in P$ let S_α of cardinality $\mathrm{cf}(g(\alpha))$ be cofinal in $g(\alpha)$, and for $\alpha \notin P$ let $S_\alpha = \{g(\alpha)\}$. Then $|S_\alpha| < \kappa$ and f is $<_I$-bounded by $\alpha \mapsto \sup S_\alpha$. So let ξ such that $f_\xi^+ = \mathrm{proj}(f_\xi, \langle S_\alpha : \alpha \in A\rangle)$ is an $<_I$-upper bound for f. Clearly $f_\xi^+ \leq g$, but g is a minimal upper bound thus $g \leq_I f_\xi^+$, thus $[\mathrm{cf}(g) < \kappa] \subseteq [g > f_\xi^+] \in I$.

(3. $\Rightarrow$ 1.) Modify g on an I-set to have $\mathrm{cf}(g(\alpha)) \geq \kappa$ for all $\alpha \in A$. For each α, let $S_\alpha \subseteq g(\alpha)$ be unbounded of order type $\mathrm{cf}(g(\alpha))$. Let $X \subseteq \lambda$ be unbounded. We need to find an $X_0 \subseteq X$ of order type κ such that $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing. We will define an $<$-increasing sequence $\langle h_\chi : \chi < \kappa\rangle$ in $\prod_{\alpha \in a} S_\alpha$ and a sequence $\langle \xi_\chi : \chi < \kappa\rangle$ in X such that $h_\chi <_I f_{\xi_\chi} <_I h_{\chi+1}$. We do this by recursion:

1. Base case: Let $h_0 \in \prod_{\alpha \in A} S_\alpha$ be arbitrary.
2. Successor case: Suppose $h_\chi \in \prod_{\alpha \in a} S_\alpha$ is defined. Then $h_\chi < g$, so let $\xi_\chi \in X$ such that $h_\chi <_I f_{\xi_\chi}$. Now $f_{\xi_\chi} <_I g$, so $f_{\xi_\chi}^+ = \mathrm{proj}(f_{\xi_\chi}, \langle S_\alpha : \alpha \in A\rangle)$ is well defined modulo I. Let $h_{\chi+1} \in \prod_{\alpha \in A} S_\alpha$ be such that $h_{\chi+1} > \sup(h_\chi, f_{\xi_\chi}^+)$. Then $h_\chi < h_{\chi+1}$ and $h_\chi <_I f_{\xi_\chi} \leq_I f_{\xi_\chi}^+ < h_{\chi+1}$, so certainly $h_\chi <_I f_{\xi_\chi} <_I h_{\chi+1}$.
3. Limit case: Suppose χ is limit and $h_{\chi'}$ has been defined for all $\chi' < \chi$. Then take $h_\chi > \sup_{\chi' < \chi} h_{\chi'}$, this is possible since $\chi < \kappa$ and each S_α has order type $\mathrm{cf}(g(\alpha)) \geq \kappa$.

Then $X_0 := \{\xi_\chi : \chi < \kappa\}$ has order type κ and $\langle f_\xi : \xi \in X_0\rangle$ is strongly increasing by the lemma below, since an $<$-increasing sequence is strongly increasing. ☐

Lemma 3.12. Let A be a set, let I be an ideal on A, let S be a set of ordinals, let $h = \langle h_\xi : \xi \in S\rangle$ and $f = \langle f_\xi : \xi \in S\rangle$ be S-sequences in $\mathbf{ON}^A$ and suppose that h is strongly increasing and $h_\xi <_I f_\xi \leq_I f_{\xi+1}$ for all $\xi \in S$. Then f is also strongly increasing.

Proof. Let $\langle Z_\xi : \xi \in S\rangle$ be a sequence in I such that $\xi < \xi', \alpha \in A \setminus (Z_\xi \cup Z_{\xi'}) \Rightarrow h_\xi(\alpha) < h_{\xi'}(\alpha)$. Let $\langle W_\xi : \xi \in S\rangle$ be a sequence in I such that $\alpha \in A \setminus W_\xi \Rightarrow h_\xi(\alpha) < f_\xi(\alpha) \leq h_{\xi+1}(\alpha)$. Let

$V_\xi := W_\xi \cup Z_\xi \cup Z_{\xi+1}$. Then $\langle V_\xi : \xi \in S\rangle$ is a sequence in I and $\xi < \xi', \alpha \in A \setminus (V_\xi, V_{\xi'}) \Rightarrow f_\xi(\alpha) < h_{\xi+1}(\alpha) \leq h_{\xi'}(\alpha) < f_{\xi'}(\alpha)$, where the first inequality is $\alpha \notin W_\xi$, the second is $\alpha \notin Z_{\xi+1} \cup Z_{\xi'}$ and the third is $\alpha \notin W_{\xi'}$. Thus the sequence $\langle V_\xi : \xi \in S\rangle$ ensures that f is strongly increasing as well. □

3.1.3 Definition and easy properties

It follows from Theorem 3.4 that it is possible to the define the following:

Definition 3.13. Let a be a set of non-zero limit ordinals. Then we define the set of *possible cofinalities* of a as

$$\begin{aligned}\mathrm{pcf}(a) :&= \{\lambda : \text{ there exists a proper filter } F \text{ on } A \text{ such that } \lambda = \mathrm{tcf}_<(\textstyle\prod a/F)\} \\ &= \{\lambda : \text{ there exists a proper ideal } I \text{ on } A \text{ such that } \lambda = \mathrm{tcf}_<(\textstyle\prod a/I)\} \\ &= \{\mathrm{cf}(\textstyle\prod a/U) : U \text{ is an ultrafilter on } a\}.\end{aligned}$$

We can easily prove the following properties of pcf:

Lemma 3.14. Let a be a set of non-zero limit ordinals. The following holds.

1. $\mathrm{pcf}(a)$ is a set of regular cardinals.

   Proof. By Theorem 2.11 a $<$-true cofinality is always a regular cardinal. □

2. $\{\mathrm{cf}(\alpha) : \alpha \in a\} \subseteq \mathrm{pcf}(a)$.

   Proof. Suppose $\alpha \in a$. Let $U := \{A \subseteq a : A \ni \alpha\}$, i.e. U is the principal ultrafilter concentrated on $\{\alpha\}$. Then $\prod a/U \cong \alpha$, hence $\mathrm{cf}(\prod a/U) = \mathrm{cf}(\alpha)$. □

The $\cong$-sign indicates an isomorphism, i.e. two structures only differ in their own name, names for elements, names for orders etc. Clearly cofinality is preserved under isomorphism.

3. If a is finite, then $\{\mathrm{cf}(\alpha) : \alpha \in a\} = \mathrm{pcf}(a)$.

   Proof. The only ultrafilters on a are the filters concentrated on singletons. From the proof of 2. equality follows. □

4. If a consists of regular ordinals[4], then $a \subseteq \mathrm{pcf}(a)$. If a is furthermore finite, then $a = \mathrm{pcf}(a)$.

   Proof. This is a direct corollary of 2. and 3. □

5. If $a \neq \emptyset$, $\min(\mathrm{pcf}(a)) = \min(\mathrm{cf}(\alpha) : \alpha \in a)$. If $a \neq \emptyset$ consists of regular cardinals then $\min(\mathrm{pcf}(a)) = \min(a)$.

   Proof. Suppose $\lambda = \mathrm{tcf}_<(\prod a/F)$. Let $f = \langle f_\xi/F : \xi < \lambda\rangle$ be increasing and cofinal in $\prod a/F$. If $\lambda < \min(\mathrm{cf}(\alpha) : \alpha \in a)$, then the map g defined by $g(\alpha) := \sup^+_{\xi<\lambda} f_\xi(\alpha)$ is an element of $\prod a$ and $f_\xi < g$ for all ξ, which contradicts the assumption that f is cofinal. So $\lambda \geq \min(\mathrm{cf}(\alpha) : \alpha \in a)$. The remaining claims follow easily. □

[4]Regular ordinals are regular cardinals.

6. If $a \subseteq b$, then $\mathrm{pcf}(a) \subseteq \mathrm{pcf}(b)$.

   Proof. Suppose $\lambda \in \mathrm{pcf}(a)$. Let U be an ultrafilter on a such that $\mathrm{cf}(\prod a/U) = \lambda$. Extend U to the ultrafilter $\hat{U}$ on b. We have $\prod a/U \cong \prod b/\hat{U}$, so $\lambda = \mathrm{cf}(\prod a/U) = \mathrm{cf}(\prod b/\hat{U})$, hence $\lambda \in \mathrm{pcf}(b)$. □

7. $\mathrm{pcf}(a \cup b) = \mathrm{pcf}(a) \cup \mathrm{pcf}(b)$.

   Proof. ($\supseteq$) This is 6. ($\subseteq$) Let $\lambda \in \mathrm{pcf}(a \cup b)$. Let U be an ultrafilter on a such that $\mathrm{cf}(\prod(a \cup b)/U) = \lambda$. Then $a \in U$ or $b \in U$, without loss of generality assume $a \in U$. Let U' be the restriction of U to a; then U' is an ultrafilter on a. We have $\prod(a \cup b)/U \cong \prod a/U'$, so $\lambda = \mathrm{cf}(\prod(a \cup b)/U) = \mathrm{cf}(\prod a/U')$, hence $\lambda \in \mathrm{pcf}(a)$. □

8. If $a \neq \emptyset$ consists of regular cardinals, then $\mathrm{pcf}(a \setminus \{\min(a)\}) = \mathrm{pcf}(a) \setminus \{\min(a)\}$.

   Proof. ($\subseteq$) Clearly $\min(a) \notin \mathrm{pcf}(a \setminus \{\min(a)\})$ by 5. and $\mathrm{pcf}(a \setminus \{\min(a)\}) \subseteq \mathrm{pcf}(a)$ by 6. ($\supseteq$) If U is an ultrafilter on $\mathrm{pcf}(a)$, then either $U \ni \{\min(a)\}$, hence $\mathrm{cf}(\prod a/U) = \min(a)$ or $U \ni a \setminus \{\min(a)\}$ so U restricts to an ultrafilter U' on $a \setminus \{\min(a)\}$ and $\mathrm{cf}(\prod a/U) = \mathrm{cf}(\prod(a \setminus \{\min(a)\})/U')$. □

When a is finite, $\mathrm{pcf}(a)$ becomes trivial, so from now on we assume that a is infinite.

Theorem 3.15. Suppose $\min(a) > |\mathrm{pcf}(a)|$. (As we will see in Corollary 3.25, this happens for instance if $\min(a) \geq 2^{|a|}$). So at least every $\alpha \in a$ is uncountable. Then we have $\mathrm{pcf}(\mathrm{pcf}(a)) = \mathrm{pcf}(a)$.

Proof. Let $b = \mathrm{pcf}(a)$. Then b is a set of regular cardinals. Hence ($\supseteq$) is Lemma 3.14.4. To show ($\subseteq$), let $\lambda \in \mathrm{pcf}(\mathrm{pcf}(a))$. Let D be an ultrafilter on $\mathrm{pcf}(a)$ such that $\mathrm{cf}(\prod b/D) = \lambda$ and let $\langle g_\delta/D : \delta < \lambda\rangle$ be increasing cofinal in $\prod b/D$. For $\beta \in b$, let D_β be an ultrafilter on a such that $\mathrm{cf}(\prod a/D_\beta) = \beta$ and let $\langle f^\beta_\delta/D_\beta : \delta < \beta\rangle$ be increasing cofinal in $\prod a/D_\beta$. Define $D^* := \{A \subseteq a : \{\beta \in b : A \in D_\beta\} \in D\}$. We will show that $\mathrm{cf}(\prod a/D^*) = \lambda$. First we show that D^* is an ultrafilter on a:

1. $D^* \subseteq P(a)$; $\{\beta \in b : a \in D_\beta\} = b \in D$ so $a \in D^*$; $\{\beta \in b : \emptyset \in D_\beta\} = \emptyset \notin D$ so $\emptyset \notin D^*$.
2. If $A, A' \in D^*$, then $\{\beta \in b : A \cap A' \in D_\beta\} = \{\beta \in b : A, A' \in D_\beta\} = \{\beta \in b : A \in D_\beta\} \cap \{\beta \in b : A' \in D_\beta\} \in D$ so $A \cap A' \in D^*$.
3. If $A \in D^*$ and $A \subseteq A'$, then $\{\beta \in b : A' \in D_\beta\} \supseteq \{\beta \in b : A \in D_\beta\} \in D$ so $\{\beta \in b : A' \in D_\beta\} \in D$ so $A' \in D^*$.
4. Let $A \subseteq a$. If $A \notin D^*$, then $\{\beta \in b : A \in D_\beta\} \notin D$, so $\{\beta \in b : X \setminus A \in D_\beta\} = \{\beta \in b : A \notin D_\beta\} \in D$, hence $X \setminus A \in D^*$.

For $\delta < \lambda$ and $\alpha \in a$ define

$$h_\delta(\alpha) := \sup_{\beta \in b} f^\beta_{g_\delta(\beta)}(\alpha).$$

Since $g_\delta(\beta) < \beta$ this is well-defined. For all $\alpha \in a$, since $\alpha \geq \min(a) > |\mathrm{pcf}(a)| = |b|$ and α is regular and $f^\beta_{g_\delta(\beta)}(\alpha) < \alpha$ for all $\beta \in b$, we have $h_\delta(\alpha) < \alpha$ for all $\delta < \lambda$, i.e. $h_\delta \in \prod a$ for all $\delta < \lambda$. The followin claim will be proved below.

Claim 3.16. For all $h \in \prod a$, there exists a $\delta^0 < \lambda$ such that for all $\delta < \lambda$ satisfying $\delta^0 \leq \delta$, we have $h \leq_{D^*} h_\delta$.

For $\mu < \lambda$, recursively define $\delta_\mu < \lambda$: $\delta_\mu = \max(\delta^0, \sup^+_{\mu'<\mu} \delta_{\mu'})$, where δ^0 is as in the above claim for $h \in \prod a$ defined by $h(\alpha) := \sup^+_{\mu'<\mu} h_{\delta_{\mu'}}(\alpha)$.
Then $\langle h_{\delta_\mu}/D^* : \mu < \lambda\rangle$ is increasing and cofinal in $\prod a/D^*$. Since λ is regular, we obtain $\mathrm{cf}(\prod a/D^*) = \lambda$ by Theorem 2.11. Thus $\lambda \in \mathrm{pcf}(a)$. □

Proof of Claim 3.16. Fix $h \in \prod a$. For $\beta \in b$, let $\delta_\beta < \beta$ such that $h \leq_{D_\beta} f^\beta_{\delta_\beta}$. Then the map $\delta \mapsto \delta_\beta$ is an element of $\prod b$. So let $\delta^0 < \lambda$ such that $(\delta \mapsto \delta_\beta) \leq_D g_{\delta^0}$. Let $\delta < \lambda$ such that $\delta^0 \leq \delta$. We will show that $h \leq_{D^*} h_\delta$. We have $(\delta \mapsto \delta_\beta) \leq_D g_{\delta^0} \leq_{D^*} g_\delta$ so let $B \in D$ such that $\delta_\beta \leq g_\delta(\beta)$ for all $\beta \in B$. For $\beta \in B$ we have $h \leq_{D_\beta} f^\beta_{\delta_\beta} \leq_{D_\beta} f^\beta_{g_\delta(\beta)}$, so let $A_\beta \in D_\beta$ be such that $h(\alpha) \leq f^\beta_{g_\delta(\beta)}(\alpha)$ for al $\alpha \in A_\beta$. Note that $f^\beta_{g_\delta(\beta)}(\alpha) \leq h_\delta(\alpha)$ for all $\alpha \in A$. So for $\beta \in B$ and $\alpha \in A_\beta$ we have $h(\alpha) \leq h_\delta(\alpha)$. So for $\beta \in B$ we have $A_\beta \subseteq [h \leq h_\delta]$, hence $[h \leq h_\delta] \in D_\beta$. So $\{\beta \in b : [h \leq h_\delta] \in D_\beta\} \supseteq B \in D$, so $[h \leq h_\delta] \in D^*$. So $h \leq_{D^*} h_\delta$. □

3.1.4 The ideal $J_{<\lambda}(a)$

It turns out to be very interesting to look at subsets b of a which large enough to collapse $\prod a/D$ to a small cofinality, when $b \in D$.

Definition 3.17. Let a be an infinite set of non-zero limit ordinals. For any $b \subseteq a$ and any cardinal λ we say that b *forces* $\prod a$ *to have cofinality* $< \lambda$ or b *forces cof* $< \lambda$ iff for any ultrafilter D on a with $b \in D$ we have $\mathrm{cf}(\prod a/D) < \lambda$. Denote

$$J_{<\lambda}(a) := \{b \subseteq a : b \text{ forces cof} < \lambda\} = \{b \subseteq a : \mathrm{pcf}(b) \subseteq \lambda\}.$$

The last equality holds by the following: Assume b forces cof $< \lambda$. Any ultrafilter U on b extends to the ultrafilter $\hat{U}$ on a, and $\mathrm{cf}(\prod a/\hat{U}) = \mathrm{cf}(\prod b/U) < \lambda$. Now assume $\mathrm{pcf}(b) \subseteq \lambda$. If D is an ultrafilter on a with $b \in D$, then we restrict D to an ultrafilter D' on b and $\mathrm{cf}(\prod a/D) = \mathrm{cf}(\prod b/D') < \lambda$.

Proposition 3.18. Let a be an infinite set of non-zero limit ordinals. For any cardinal λ, $J_{<\lambda}(a)$ is an ideal on a.

Proof. We verify the conditions:

1. $J_{<\lambda}(a) \subseteq P(a)$ and no ultrafilter D contains $\emptyset$, so $\emptyset \in J_{<\lambda}(a)$.
2. If $b, c \in J_{<\lambda}(a)$, since any ultrafilter D containing $b \cup c$ contains b or c (or both), we obtain $b \cup c \in J_{<\lambda}(a)$.
3. If $b \in J_{<\lambda}(a)$ and $c \subseteq b$, since any (ultra)filter containing c also contains b, we obtain $c \in J_{<\lambda}(a)$.

So indeed $J_{<\lambda}(a)$ is an ideal. □

Lemma 3.19. Let a be an infinite set of non-zero limit ordinals, let $b \subseteq a$ and let λ be a cardinal. Then $J_{<\lambda}(b) = J_{<\lambda}(a) \cap P(b)$.

Proof. This is obvious from the second equality in definition 3.17. More directly, suppose $c \in J_{<\lambda}(b)$. Then $c \in P(b)$ and $c \subseteq a$. Let D be an ultrafilter on a containing c. Then $D \restriction b$ is an ultrafilter on b and $\prod a/D \cong \prod b/(D \restriction b)$, so $\mathrm{cf}(\prod a/D) = \mathrm{cf}(\prod b/(D \restriction b)) < \lambda$. Hence $c \in J_{<\lambda}(a)$. Now suppose $c \in J_{<\lambda}(a) \cap P(b)$. Let F be an ultrafilter on b containing c. Let F' be the restriction of F to a, then F' is an ultrafilter on a containing c and $\prod a/F' \cong \prod b/F$. So $\mathrm{cf}(\prod b/F) = \mathrm{cf}(\prod a/F') < \lambda$. □

We write J (or $J_{<\lambda}$) for $J_{<\lambda}(a)$ when λ and a (when a) are (is) understood.

Remember that all we have said about $\prod a/F$ for a filter F applies to an ideal as well when we consider its dual filter. So is makes sense to write $\prod a/J_{<\lambda}(a)$.

Theorem 3.20. Let A be an infinite set of regular cardinals such that $\min(A) > |A|$. For any cardinal λ, $\prod a/J_{<\lambda}(a)$ is λ-directed.

Proof. The theorem is equivalent to the statement that if $B \subseteq \prod A$ and $|B| < \lambda$, then there exists an $h \in \prod A$ such that $f \leq_{J_{<\lambda}} h$ for all $f \in B$. We prove this by induction on $|B|$.

1. If $|B| < |A|^+$, then, for $\alpha \in A$, define $h(\alpha) := \sup_{f \in B} f(\alpha)$. Then $h \in \prod A$ and $f \leq h$, so $f \leq_{J_{<\lambda}} h$.
2. If $|A|^+ \leq |B| < \lambda$ and any $E \subseteq \prod A$ with $|E| < |B|$ has an $<_{J_{<\lambda}}$-upper bound, let $B = \{f_i : i < |B|\}$ be an enumeration of B and for $i < |B|$ recursively define f'_i to be an $<_{J_{<\lambda}}$-upper bound for $\{f'_j : j < i\} \cup \{f_i\}$.[5] Then $\langle f'_i : i < |B|\rangle$ is $<_{J_{<\lambda}}$-increasing and any upper bound for $B' = \{f'_i : i < |B|\}$ is an upper bound for B. So we look for a bound for B'. We make a case distinction.
   (a) If $|B|$ is singular, let $C \subseteq |B|$ be cofinal such that $|C| < |B|$, and let h be an upper bound for $\{f'_i : i \in C\}$ by the induction hypothesis. Then h is an upper bound for B' as well.
   (b) If $|B|$ is regular, recursively define g_α for $\alpha < |A|^+$:
      i. $g_0 = f'_0$.
      ii. $g_\alpha = \sup_{\beta<\alpha} g_\beta$ when α is limit. Here we mean a local supremum, i.e. $g_\alpha(x) = \sup_{\beta<\alpha} g_\beta(x)$.
      iii. $g_{\alpha+1}$: If g_α is not an upper bound for B', let $i_\alpha < |B|$ be minimal such that $[g_\alpha < f'_{i_\alpha}] \notin J_{<\lambda}$. Then let D be an ultrafilter on A such that $[g_\alpha < f'_{i_\alpha}] \in D$ but $\mathrm{cf}(\prod A/D) \geq \lambda$. Let $h \in \prod A$ be an $\leq_D$-upper bound for B'. Let $g_{\alpha+1} = \max(g_\alpha, h)$, again locally.

      If the recursion continues up to $|A|^+$, then $\langle g_\alpha : \alpha < |A|^+\rangle$ is $\leq$-increasing. For $i < |B|$ and $\alpha < |A|^+$ define

$$b_i^\alpha := [g_\alpha < f'_i]$$

      Then $b_i^\alpha \supseteq b_i^{\alpha'}$ if $\alpha < \alpha'$. Let $\alpha < |A|^+$ be arbitrary.
      i. If $i \geq i_\alpha$, then $b_i^\alpha = [g_\alpha < f'_i] \supseteq_{J_{<\lambda}} [g_\alpha < f'_{i_\alpha}] \in D$ and $D \cap J_{<\lambda} = \emptyset$, so $b_i^\alpha \in D$.
      ii. For any i, $b_i^{\alpha+1} = [g_{\alpha+1} < f'_i] \subseteq [h < f'_i] \notin D$.

      So $b_i^\alpha \neq b_i^{\alpha+1}$ for $i \geq i_\alpha$. Since $|A|^+ \leq |B|$ and $\alpha \to i_\alpha$ is a map $|A| \to |B|$, there is an $\bar{\imath} < |B|$ such that $i_\alpha \leq \bar{\imath}$ for all α. Then $\langle b_{\bar{\imath}}^\alpha : \alpha < |A|^+\rangle$ is a $\subset$-decreasing $|A|^+$-sequence of subsets of A. This is of course impossible. So some g_α was already an upper bound for B'.

□

Corollary 3.21. Let a be an infinite set of regular cardinals. If $\mathrm{cf}(\prod a/D) < \lambda$ then there exists some $b \in D$ such that b forces $\mathrm{cof} < \lambda$.

Proof. Suppose not, then $D \cap J_{<\lambda}(a) = \emptyset$. Let $\mu := \mathrm{cf}(\prod a/D)$ and let $\{g_\xi/D : \xi < \mu\}$ be cofinal in $\prod a/D$. Then by Theorem 3.20, $\{g_\xi/J : \xi < \mu\}$ is bounded in $\prod a/J$; let g/J be an upper bound.

[5] An $\leq_{J_{<\lambda}}$-upper bound exists by the induction hypothesis, then simply add 1 everywhere so that it becomes an $<_{J_{<\lambda}}$-upper bound.

Then $[g_\xi \not\leq g] \in J$, so $[g_\xi \not\leq g] \notin D$, so $[g_\xi \leq g] \in D$, so g/D is an upper bound for $\{g_\xi/D : \xi < \mu\}$ in $\prod a/D$, thus $\{g_\xi/D : \xi < \mu\}$ is not cofinal in $\prod a/D$, a contradiction. $\square$

So we have

$$\mathrm{cf}(\prod a/D) < \lambda \quad \Leftrightarrow \quad D \cap J_{<\lambda}(a) \neq \emptyset. \tag{2}$$

Clearly $\mu \leq \lambda \Rightarrow J_{<\mu}(a) \subseteq J_{<\lambda}(a)$.

Proposition 3.22. Let a be an infinite set of non-zero limit ordinals. For limit cardinals λ we have $J_{<\lambda}(a) = \bigcup_{\mu<\lambda} J_{<\mu}(a)$.

Proof. Inclusion from right to left is evident. Suppose $b \in J_{<\lambda}(a) \setminus \bigcup_{\mu<\lambda} J_{<\mu}(a)$. Note that $\bigcup_{\mu<\lambda} J_{<\mu}(a)$ is a proper ideal. Recall Theorem 2.18, and let D be an ultrafilter such that $b \in D$ and $D \cap \bigcup_{\mu<\lambda} J_{<\mu}(a) = \emptyset$. Then $\mathrm{cf}(\prod a/D) < \lambda$ and by Corollary 3.21 $\mathrm{cf}(\prod a/D) \geq \mu$ for all $\mu < \lambda$, which is a contradiction. $\square$

Lemma 3.23. Let a be an infinite set of non-zero limit ordinals. Then there exists a proper ideal I with $\mathrm{tcf}_<(\prod a/I) = \lambda$ if and only if $J_{<\lambda}(a) \neq J_{<\lambda^+}(a)$.

Proof. ($\Leftarrow$) Let $b \in J_{<\lambda^+}(a) \setminus J_{<\lambda}(a)$. Let D be an ultrafilter with $b \in D$ such that $\mathrm{cf}(\prod a/D) \geq \lambda$. Since $\mathrm{cf}(\prod a/D) < \lambda^+$, we obtain $\mathrm{cf}(\prod a/D) = \lambda$. For the dual ideal I of D we thus have $\mathrm{cf}(\prod a/I) = \mathrm{tcf}_<(\prod a/I) = \lambda$. ($\Rightarrow$) Since I is proper, let D be an ultrafilter extending the dual filter I^* of I. We have $\prod a/I \cong \prod a/I^*$ and by Theorem 3.4, $\lambda = \mathrm{tcf}_<(\prod a/I) = \mathrm{tcf}_<(\prod a/I^*) = \mathrm{cf}(\prod a/D)$. By Corollary 3.21, since $\mathrm{cf}(\prod a/D) < \lambda^+$, there exists some $b \in D$ which forces cof $< \lambda^+$. Hence $b \in J_{<\lambda^+}(a) \setminus J_{<\lambda}(a)$. $\square$

3.1.5 First results using $J_{<\lambda}(a)$

We have two direct corollaries of Lemma 3.23 on $\mathrm{pcf}(a)$:

Corollary 3.24. We have

$$\begin{aligned}\mathrm{pcf}(a) &= \{\mathrm{cf}(\prod a/D) : D \text{ is an ultrafilter on } a\} \\ &= \{\lambda : \lambda = \mathrm{tcf}_<(\prod a/F) \text{ for some proper filter } F \text{ on } a\} \\ &= \{\lambda : \lambda = \mathrm{tcf}_<(\prod a/I) \text{ for some proper ideal } I \text{ on } a\} = \{\lambda : J_{<\lambda}(a) \subset J_{<\lambda^+}(a)\}.\end{aligned}$$

Corollary 3.25. We have $|\mathrm{pcf}(a)| \leq 2^{|a|}$.

Proof. For any $\lambda \in \mathrm{pcf}(a)$ choose a $b_\lambda \in J_{<\lambda^+}(a) \setminus J_{<\lambda}(a)$. If $\lambda < \lambda'$, then $J_{<\lambda}(a) \subset J_{<\lambda^+}(a) \subseteq J_{<\lambda'}(a) \subset J_{<\lambda'^+}(a)$, so $b_\lambda \neq b_{\lambda'}$. So $b \mapsto b_\lambda$ is an injective function $\mathrm{pcf}(a) \to P(a)$. $\square$

Since $\lambda < \lambda'$ implies $J_{<\lambda} \subseteq J_{<\lambda'}$, and $J_{<\lambda} \subseteq P(a)$, there must be a λ such that $J_{<\lambda} = J_{<\lambda'}$ for all $\lambda' > \lambda$. The following lemma shows that that for this λ, $J_{<\lambda}$ is maximal, namely $P(a)$.

Lemma 3.26. Let a be an infinite set of non-zero limit ordinals. There exists λ such that $J_{<\lambda}(a) = P(a)$.

Proof. If not, then $\bigcup_\lambda J_{<\lambda}(a)$, where we take union over all cardinals λ, is a proper ideal. So let D be an ultrafilter extending its dual filter. Let $\lambda := \mathrm{cf}(\prod a/D)$. By Corollary 3.21 some $b \in D$ forces cof $< \lambda^+$, and $b \in J_{<\lambda^+}(a)$. But then $b \in D \cap \bigcup_\lambda J_{<\lambda}(a)$, a contradiction. □

Theorem 3.27. Let a be an infinite set of non-zero limit ordinals. Then $\mathrm{pcf}(a)$ has a maximal element.

Proof. Let λ be minimal such that $J_{<\lambda}(a) = P(a)$; this exists by Lemma 3.26. Then λ can not be a limit cardinal, for this would violate $J_{<\lambda}(a) = \bigcup_{\mu<\lambda} J_{<\mu}(a)$, i.e. Proposition 3.22, since then we must have $a \in J_{<\mu}(a)$ for some $\mu < \lambda$. So $\lambda = \kappa^+$. Since $J_{<\kappa}(a) \neq J_{<\kappa^+}(a)$, we have $\kappa \in \mathrm{pcf}(a)$ by Lemma 3.23, and again by this lemma we have that κ is maximal. In particular, also κ is regular. □

Example. Let $a := \{\aleph_{2n} : 0 < n < \omega\}$. Then a is an infinite set of regular cardinals and $|a|^+ = \aleph_0^+ = \aleph_1 < \aleph_2 = \min(a)$. Let $0 < k < \omega$; we will show that $\aleph_{2k+1} \notin \mathrm{pcf}(a)$. Let D be any ultrafilter on a. If D contains a finite set, then $D = \{b \subseteq a : b \ni \aleph_{2n}\}$ for some $0 < n < \omega$ and $\mathrm{cf}(\prod a/D) = \aleph_{2n} \neq \aleph_{2k+1}$. If D contains no finite set, then it contains all cofinite sets. Suppose $\langle f_\xi/D : \xi < \aleph_{2k+1}\rangle$ is increasing in $\prod a/D$. Then each for n with $2n > 2k+1$, since $\aleph_{2k+1} < \aleph_{2n}$ and $\aleph_{2n}$ is regular and $f_\xi(n) < \aleph_{2n}$ for all ξ, we have $\sup_{\xi<\aleph_{2k+1}} f_\xi(n) < \aleph_{2n}$. For $0 < n < \omega$ define

$$f(n) := \begin{cases} 0, & \text{if } 2n \leq 2k+1 \\ \sup_{\xi<\omega_{2k+1}} f_\xi(n), & \text{if } 2n > 2k+1 \end{cases}$$

then $f \in \prod a$. For any $\xi < \aleph_{2n}$, since $[f_\xi \leq f]$ is cofinite, we have $[f_\xi \leq f] \in D$. Hence $f_\xi/D \leq f/D$ for all $\xi < \aleph_{2k+1}$. Thus $\langle f_\xi/D : \xi < \aleph_{2k+1}\rangle$ cannot be cofinal in $\prod a/D$. So there exists no increasing and cofinal sequence in $\prod a/D$ of length $\aleph_{2k+1}$, so $\aleph_{2k+1} \notin \mathrm{pcf}(a)$. We have shown that $\mathrm{pcf}(a) \cap \aleph_\omega = a$, and $\mathrm{pcf}(a)$ need not be an interval. Furthermore, Theorem 3.27 yields the existence of $\max \mathrm{pcf}(a)$. We have $\max \mathrm{pcf}(a) \geq \sup(a) = \aleph_\omega$, and even $> \aleph_\omega$ since $\aleph_\omega$ is singular.

3.1.6 When a is in interval

In this paragraph we will see a result which shows that $\mathrm{pcf}(a)$ is an interval of regular cardinals under some additional assumptions on a.

First we introduce the idea of the limit of an ultrafilter on a set of ordinals. Let a be a non-empty set of non-zero ordinals and D an ultrafilter on a. Then $(0, \sup a] \cap a \in D$, but $(0, 0] = \emptyset \notin D$. So let μ be minimal such that $(0, \mu] \cap a \in D$. For any $\nu < \mu$ we have $((0,\nu] \cap a) \dot\cup ((\nu, \mu] \cap a) \in D$ but $((0, \nu] \cap a) \notin D$, so $(\nu, \mu] \cap a \in D$. We write $\mu = \lim_D a$ and call μ the *D-limit of a*. If μ is such that $(\nu, \mu] \cap a \in D$ for every $\nu < \mu$, then $\lim_D a = \mu$.

Theorem 3.28. Let D be an ultrafilter on a set of regular cardinals a such that $|a| < \min(a)$, let $\lambda = \mathrm{cf}(\prod a/D)$ and let $\mu = \lim_D a$. Suppose λ' is regular and $\mu < \lambda' < \lambda$. Then there is a set a' of regular cardinals such that $|a'| \leq |a|$ and an ultrafilter D' on a' such that $\lim_{D'} a' = \mu$ and $\mathrm{cf}(\prod a'/D') = \lambda'$.

A proof of this theorem will follow from Lemmas 3.35 and 3.36.

By '$a = (\nu, \mu)$ is an interval of regular cardinals' we mean that $a = \{\kappa : \nu < \kappa < \mu \text{ and } \kappa \text{ is regular}\}$.

Corollary 3.29. Let $a = (\nu, \mu)$ be an interval of regular cardinals such that $|a| < \min(a)$, $\lambda \in \mathrm{pcf}(a)$ and λ' regular such that $\mu < \lambda' < \lambda$. Then $\lambda' \in \mathrm{pcf}(a)$.

Proof. Let D be an ultrafilter on a such that $\lambda = \mathrm{cf}(\prod a/D)$. Then $\tilde{\mu} := \lim_D(a) \leq \sup(a) \leq \mu$. So $\tilde{\mu} < \lambda' < \lambda$. Then Theorem 3.28 yields a' and D' such that $|a'| \leq |a|$, $\lim_{D'} a' = \tilde{\mu}$ and $\mathrm{cf}(\prod a'/D') = \lambda'$. Define $D'' := \{A \subseteq a : A \cap a' \in D'\}$. Then D'' is an ultrafilter on a:

1. $D'' \subseteq P(a)$, $\emptyset \cap a' = \emptyset \notin D'$ so $\emptyset \notin D''$, $a \cap a' = (\nu, \mu) \cap a' \supseteq (\nu, \tilde{\mu}) \cap a' \in D'$ so $a \in D''$.[6]
2. If $A, B \in D''$, then $(A \cap B) \cap a' = (A \cap a') \cap (B \cap b') \in D'$ so $A \cap B \in D'$.
3. If $A \in D''$ and $A \subseteq B$, then $B \cap a' \supseteq A \cap a' \in D'$ so $B \in D''$.
4. If $A \notin D''$, then $(a \setminus A) \cap a' = (a \cap a') \setminus (A \cap a') \in D'$ since $a \cap a' \in D'$ but $A \cap a' \notin D'$, so $a \setminus A \in D''$.

We have $\prod a/D'' \cong \prod a'/D'$, since $a \cap a' \in D'$ and $a \cap a' \in D''$. Hence $\lambda' = \mathrm{cf}(\prod a'/D') = \mathrm{cf}(\prod a/D'') \in \mathrm{pcf}(a)$. □

In the above situation, when $a = (\nu, \mu)$, it is not immediately clear that $\mu \in \mathrm{pcf}(a)$, so that $\mathrm{pcf}(a)$ is an interval of regular cardinals. We have the following cases:

1. μ is singular. Then of course $\mu \notin \mathrm{pcf}(a)$, but $\mathrm{pcf}(a)$ is an interval of regular cardinals.
2. μ is a successor cardinal, say $\mu = \rho^+$. Consider $a' := (\nu, \rho)$. Let $\lambda = \max \mathrm{pcf}(a)$. If $\lambda = \mu$, then $\mu \in \mathrm{pcf}(a)$ and indeed $\mathrm{pcf}(a)$ is an interval. If $\lambda = \rho$, then it is also obvious that $\mathrm{pcf}(a)$ is an interval. If $\lambda > \mu$, then $\rho < \mu < \lambda$ and $\lambda \in \mathrm{pcf}(a')$ as well (an ultrafilter U on a such that $\mathrm{cf}(\prod a/U) = \lambda$ can be restricted to a', since $\{\rho\} \notin U$). So Corollary 3.29 yields that $\mu \in \mathrm{pcf}(a') \subseteq \mathrm{pcf}(a)$. Thus $\mathrm{pcf}(a)$ is an interval.
3. μ is a regular limit cardinal. However, let $\mu = \aleph_\delta$, then $|a| = \delta$ and $\min(a) < \aleph_\delta = \delta$. So the assumption that $\min(a) > |a|$ rules out this possibility.

We conclude that if $a = (\nu, \mu)$ is an interval of regular cardinals such that $|a| < \min(a)$, then $\mathrm{pcf}(a)$ is an interval of regular cardinals. So if $\nu < \aleph_{|\mathrm{pcf}(a)|^+}$, then $\mathrm{pcf}(a) \subseteq (\nu, \aleph_{|\mathrm{pcf}(a)|^+})$, and in fact there is some $\delta < |\mathrm{pcf}(a)|^+$ such that $\mathrm{pcf}(a) = (\nu, \aleph_\delta]$.

Corollary 3.30. Let a be an interval of regular cardinals such that $\min(a) > 2^{|a|}$. Suppose that the least cardinal in a is a successor cardinal; let $\min(a) = \aleph_{\delta+1}$. Then $\mathrm{pcf}(a)$ contains no limit cardinals.

Proof. Since $|\mathrm{pcf}(a)| \leq 2^{|a|}$ by Corollary 3.25 and $\mathrm{pcf}(a)$ is an interval of regular cardinals by Corollary 3.29, we have $\mathrm{pcf}(a) \subseteq \{\aleph_{\delta+\alpha} : 1 \leq \alpha < \left(2^{|a|}\right)^+\}$. Let $1 \leq \alpha < \left(2^{|a|}\right)^+$ such that $\aleph_{\delta+\alpha}$ is a limit cardinal. Thus $\delta + \alpha$ is a limit ordinal. Then

$$\mathrm{cf}(\aleph_{\delta+\alpha}) = \mathrm{cf}(\delta + \alpha) = \mathrm{cf}(\alpha) \leq |\alpha| \leq 2^{|a|} < \min(a) = \aleph_{\delta+1} \leq \aleph_{\delta+\alpha}.$$

Thus $\aleph_{\delta+\alpha}$ is a singular cardinal, and therefore does not belong to $\mathrm{pcf}(a)$. □

In the next definition we use the *class* **ON** of all ordinals. We use this class to express that certain elements are ordinals, certain sequences are ordinal-valued, to express that a set consists of ordinals, etc. All 'things' we actually work with are still sets and not proper classes.

[6] To have $(\nu, \tilde{\mu}) \cap a' \in D'$, one needs $\nu < \tilde{\mu}$. But this follows from $(0, \tilde{\mu}] \cap (\nu, \mu) \in D$.

Definition 3.31. Let κ and λ be cardinals such that $\lambda > \kappa^+$ and let D be an ultrafilter on κ. For $\alpha < \lambda$, let $f_\alpha : \kappa \to \mathbf{ON}$. Suppose that $\alpha < \alpha'$ implies $f_\alpha <_D f_{\alpha'}$, i.e. $\langle f_\alpha/D : \alpha < \lambda\rangle$ is increasing in $\mathbf{ON}^\kappa/D$. Let $h/D \in \mathbf{ON}^\kappa/D$, $\mathcal{A} \subseteq \mathbf{ON}^\kappa/D$. Then we say that

1. h/D *cuts* $\langle f_\alpha/D : \alpha < \lambda\rangle$ iff there exist $\alpha, \alpha' < \lambda$ such that $f_\alpha/D < h/D < f_{\alpha'}/D$,
2. $\mathcal{A}$ *cofinally cuts* $\langle f_\alpha/D : \alpha < \lambda\rangle$ iff for all $\alpha < \lambda$ there exists $h/D \in \mathcal{A}$ such that $f_\alpha/D < h/D$ and h/D cuts $\langle f_\gamma/D : \gamma < \lambda\rangle$.

Lemma 3.32. Let κ and λ be cardinals such that $\lambda > \kappa^+$, let D be an ultrafilter on κ. Let $\langle f_\alpha/D : \alpha < \lambda\rangle$ be increasing in $\mathbf{ON}^\kappa/D$. Then

1. $\langle f_\alpha/D : \alpha < \lambda\rangle$ has a least upper bound in $\mathbf{ON}^\kappa/D$, or
2. there exists sets $S_\delta \subseteq \mathbf{ON}$ for $\delta < \kappa$ such that $|S_\delta| \leq \kappa$ and $\prod_{\delta<\kappa} S_\delta/D$ cofinally cuts $\langle f_\alpha/D : \alpha < \lambda\rangle$.

Proof. Assume both options do not occur. Then, for $\beta < \kappa^+$, we will recursively define $h_\beta \in \mathbf{ON}^\kappa$ such that each h_β/D is an upper bound for $\langle f_\alpha/D : \alpha < \lambda\rangle$ and $\langle h_\beta : \beta < \kappa^+\rangle$ is $<_D$-decreasing. Then we will arrive at a contradiction.

1. Base case: Take h_0 such that $f_\alpha(\delta) < h_0(\delta)$ for all $\alpha < \lambda$ and all $\delta < \kappa$. For instance, let $h_0(\delta) := \sup^+_{\alpha<\lambda} f_\alpha(\delta)$.
2. Successor case: Given h_β, since h_β/D is an upper bound, but not a least upper bound for $\langle f_\alpha/D : \alpha < \lambda\rangle$, let $h_{\beta+1}/D$ be an upper bound for $\langle f_\alpha/D : \alpha < \lambda\rangle$ with $h_\beta/D \not\leq h_{\beta+1}/D$. Then $h_{\beta+1}/D < h_\beta/D$, since D is an ultrafilter.
3. Limit case: Given h_γ for $\gamma < \beta$, define
   (a) for $\delta < \kappa$, $S_\delta := \{h_\gamma(\delta) : \gamma < \beta\}$
   (b) for $\alpha < \lambda$ and $\delta < \kappa$, $g_\alpha(\delta) := \min(S_\delta \setminus f_\alpha(\delta))$.
   Note that by definition of h_0, $S_\delta \setminus f_\alpha(\delta)$ is never empty. Note that $g_\alpha \in \mathbf{ON}^\kappa$ for all $\alpha < \lambda$. Note that:
   (a) For $\alpha < \lambda$ we have $f_\alpha \leq g_\alpha$ and thus $f_\alpha \leq_D g_\alpha$.
   (b) For $\alpha < \alpha' < \lambda$ we have $f_\alpha <_D f_{\alpha'}$, so $g_\alpha \leq_D g_{\alpha'}$.
   (c) For $\alpha < \lambda$ and $\gamma < \beta$ we have $[g_\alpha \leq h_\gamma] \supseteq [f_\alpha < h_\gamma] \in D$, so $g_\alpha \leq_D h_\gamma$. Since $h_{\gamma'}/D < h_\gamma/D$ for $\gamma < \gamma' < \beta$, we even have $g_\alpha <_D h_\gamma$ for all $\alpha < \lambda$ and $\gamma < \beta$.

   Suppose $\langle g_\alpha/D : \alpha < \lambda\rangle$ is not eventually constant. Then $|S_\delta| \leq \kappa$ and $\prod_{\delta<\kappa} S_\delta/D$ cofinally cuts $\langle f_\alpha/D : \alpha < \lambda\rangle$, which is possibility 2, which we assumed not to occur: Let $\alpha < \lambda$ be given. Then $g_\alpha \in \prod_{\delta<\kappa} S_\delta$ and $f_\alpha <_D g_\alpha$. Let α' such that $\alpha < \alpha' < \lambda$ and such that $g_\alpha \neq_D g_{\alpha'}$. Then $g_\alpha <_D g_{\alpha'}$. Thus $[g_\alpha < g_{\alpha'}] \in D$, hence, by definition of the g_α's, $[g_\alpha < f_{\alpha'}] \in D$. Hence $g_\alpha <_D f_{\alpha'}$. So $f_\alpha/D < g_\alpha/D < f_{\alpha'}/D$.
   So $\langle g_\alpha/D : \alpha < \lambda\rangle$ is eventually constant, and we call this constant value h_β/D. For all $\alpha < \lambda$ we have $f_\alpha <_D h_\beta$. Let α be such that $g_\alpha/D = h_\beta/D$. For $\gamma < \beta$ we have $f_\alpha <_D h_\gamma$. So $h_\gamma(\delta) \in (S_\delta \setminus f_\alpha(\delta))$ for D-a.e. δ. So $g_\alpha(\delta) \leq h_\gamma(\delta)$ for D-a.e. δ. Thus $h_\beta =_D g_\alpha \leq_D h_\gamma$. Since $\gamma < \gamma'$ implies $h_\gamma >_D h_{\gamma'}$, we even have $h_\beta <_D h_\gamma$ for all $\gamma < \beta$.

Now similarly define

1. for $\delta < \kappa$, $\bar{S}_\delta := \{h_\beta(\delta) : \beta < \kappa^+\}$,
2. for $\alpha < \lambda$ and $\delta < \kappa$, $\bar{g}_\alpha(\delta) := \min \bar{S}_\delta \setminus f_\alpha(\delta)$.

Note again that by definition of h_0, $\bar{S}_\delta \setminus f_\alpha(\delta)$ is never empty. Note again that $\bar{g}_\alpha <_D h_\beta$ for all $\beta < \kappa^+$. Given $\alpha < \lambda$ and $\delta < \kappa$, let $\beta(\delta) < \kappa^+$ be such that $\bar{g}_\alpha(\delta) = h_{\beta(\delta)}(\delta)$. Let $\beta(\alpha) := \sup_{\delta<\kappa} \beta(\delta) + \omega$. This is an ordinal $< \kappa^+$ since κ^+ is regular, and by adding ω we ensure that it is a limit ordinal. Then $\alpha \mapsto \beta(\alpha)$ is a map $\lambda \to \kappa^+$. Since we assumed $\lambda > \kappa^+$, there exist $\beta < \kappa^+$ such that unboundedly many $\alpha < \lambda$ satisfy $\beta(\alpha) = \beta$, and this β is a limit ordinal. Note that for these α's, for all $\delta < \kappa$ there exists $\beta' < \beta$ such that $\bar{g}_\alpha(\delta) = h_{\beta'}(\delta)$. Thus for these α's, by definition of h_β, we have $g_\alpha = \bar{g}_\alpha$. But $\langle g_\alpha/D : \alpha < \lambda\rangle$ was eventually constant, so there exists an α such that $h_\beta =_D g_\alpha = \overline{g}_\alpha$. But then we have the following contradiction: $h_\beta/D = g_\alpha/D = \bar{g}_\alpha/D < h_\beta/D$. □

Definition 3.33. Given an ordinal λ, for $\alpha < \lambda$, let C_α be club in α with $\mathrm{ot}(C_\alpha) = \mathrm{cf}(\alpha)$. For $\alpha < \lambda$, define $\mathcal{C}_\alpha := \{C_\beta \cap \alpha : \beta < \lambda\}$. Then $\mathcal{C}_\alpha \subseteq P(\alpha)$, $|\mathcal{C}_\alpha| \leq \lambda$, every $\mathcal{C}_\alpha$ contains a C which is club in α and $\mathrm{ot}(C) = \mathrm{cf}(\alpha)$ (namely $C = C_\alpha = C_\alpha \cap \alpha$), and for any $E \in \mathcal{C}_\alpha$ and $\beta < \alpha$ we have $E \cap \beta \in \mathcal{C}_\beta$ ($E = C_\gamma \cap \alpha$ for some γ, so $E \cap \beta = C_\gamma \cap \alpha \cap \beta = C_\gamma \cap \beta \in C_\beta$). We call $\langle \mathcal{C}_\alpha : \alpha < \lambda\rangle$ a *silly square λ-sequence*.

We now start with the proof of Theorem 3.28. Let D be an ultrafilter on a set of regular cardinals a such that $|a| < \min(a)$, $\lambda = \mathrm{cf}(\prod a/D)$, $\mu = \lim_D a$ and λ' regular such that $\mu < \lambda' < \lambda$. Since $\lim_D a = \mu < \lambda = \mathrm{cf}(\prod a/D)$, we must have that D is non-principal, and μ is a limit cardinal.

We will construct an increasing sequence $\langle f_\alpha/D : \alpha < \lambda'\rangle$ in $\prod a/D$ which has a least upper bound g/D, and $\{\mathrm{cf}(g(\alpha)) : \alpha \in a\}$ is cofinal in μ, and $\mathrm{cf}(\prod_{\alpha\in a} \mathrm{cf}(g(\alpha))/D) = \lambda'$.

Let $\langle \mathcal{C}_\beta : \beta < \lambda'\rangle$ be a silly square λ'-sequence. For $\alpha < \lambda'$ recursively define f_α as follows:

1. $f_0/D \in \prod a/D$ arbitrary.
2. Given f_γ/D for $\gamma < \beta$, the set $\{f_\gamma/D : \gamma < \beta\}$ is not cofinal in $\prod a/D$ since $\mathrm{cf}(\prod a/D) = \lambda > \lambda' > \beta$. So let $h_\beta/D \in \prod a/D$ be such that $h_\beta \not<_D f_\gamma$, hence $f_\gamma \leq_D h_\beta$, for all $\gamma < \beta$. Then $f_\gamma/D \leq h_\beta/D$ for all $\gamma < \beta$. By adding 1 to h_β, we can arrange without loss of generality $f_\gamma/D < h_\beta/D$ for all $\gamma < \beta$. For $\alpha \in a$ and $E \in \mathcal{C}_\beta$ define

$$g^\beta_E(\alpha) := \begin{cases} h_\beta(\alpha), & \text{if } \alpha \leq \mathrm{ot}(E), \\ \max(h_\beta(\alpha), \sup_{\gamma\in E} f_\gamma(\alpha)), & \text{if } \alpha > \mathrm{ot}(E). \end{cases}$$

   If $\alpha > \mathrm{ot}(E)$, then $\alpha > \mathrm{ot}(E) \geq |E|$, and $f_\gamma(\alpha) \in \alpha$ for all γ, and α is regular, hence $\sup_{\gamma\in E} f_\gamma(\alpha) < \alpha$. So $g^\beta_E \in \prod a$. Again $\{g^\beta_E/D : E \in \mathcal{C}_\beta\}$ is not cofinal in $\prod a/D$, since $|\mathcal{C}_\beta| \leq \lambda' < \lambda$. So let $f_\beta/D \in \prod a/D$ such that $g^\beta_E/D < f_\beta/D$ for all $E \in \mathcal{C}_\beta$.

Lemma 3.34. There are no subsets $S_\alpha \subseteq \alpha$ for $\alpha \in a$ and $\mu' < \mu$ such that $|S_\alpha| \leq \mu'$ and $\prod_{\alpha\in a} S_\alpha/D$ cofinally cuts $\langle f_\gamma/D : \gamma < \lambda'\rangle$.

Proof. For $\alpha \in a$, let $S_\alpha \subseteq \alpha$ such that $\prod_{\alpha\in a} S_\alpha/D$ cofinally cuts $\langle f_\gamma/D : \gamma < \lambda'\rangle$, and let μ' be a cardinal such that $|a| < \mu' < \mu$ (note that $|a| < \min(a) < \mu$). We will show that for at least one α we have $|S_\alpha| > \mu$.

For $i < \lambda'$ recursively define $\beta_i < \lambda'$ by

1. β_0 arbitrary,
2. β_{i+1} such that $\beta_{i+1} \geq i + 1$ and $f_{\beta_i} \leq_D k \leq_D f_{\beta_{i+1}}$ for some $k \in \prod_{\alpha\in a} S_\alpha$,
3. $\beta_i = \sup_{j<i} \beta_j$ if i is limit.

Then $B := \{\beta_i : i < \lambda'\}$ is λ'-club and if $i < j$ then $f_{\beta_i} <_D k <_D f_{\beta_j}$ for some $k \in \prod_{\alpha\in a} S_\alpha$. Let $\beta := \beta_{(\mu')^+}$ (note that $(\mu')^+ < \mu$ since μ is a limit cardinal). Then $\mathrm{cf}(\beta) = (\mu')^+$. Let $E \in \mathcal{C}_\beta$ such that $\mathrm{ot}(E) = \mathrm{cf}(\beta)$. Then $E \cap B$ is β-club and we enumerate $E \cap B$ as $\langle \gamma_i : i < \mathrm{cf}(\beta)\rangle$ increasingly. Fix $i < \mathrm{cf}(\beta)$. Let $k_i \in \prod_{\alpha\in a} S_\alpha$ be such that $f_{\gamma_i} \leq_D k_i \leq_D f_{\gamma_{i+1}}$. Now

1. $g^{\gamma_i}_{E\cap\gamma_i} <_D f_{\gamma_i}$ since f_{γ_i} is an $<_D$-upper bound for $\{g^{\gamma_i}_F : F \in \mathcal{C}_{\gamma_i}\}$,
2. $g^{\gamma_i}_{E\cap\gamma_i}(\alpha) \geq f_{\gamma_j}(\alpha)$ for all α such that $\alpha > \mathrm{ot}(E \cap \gamma_i)$ and $j < i$.

Since $\mathrm{ot}(E) = \mathrm{cf}(\beta) < \mu$ and $\lim_D a = \mu$, let $\alpha_i \in a$ be such that $\alpha_i > \mathrm{ot}(E)$ and $f_{\gamma_i}(\alpha_i) \leq k_i(\alpha_i) \leq f_{\gamma_{i+1}}(\alpha_i)$ and $f_{\gamma_i}(\alpha_i) > g^{\gamma_i}_{E\cap\gamma_i}(\alpha_i)$.

Do this for all $i < \mathrm{cf}(\beta)$, to obtain a map $\mathrm{cf}(\beta) \to a$ given by $i \mapsto \alpha_i$. Note that $|a| < \mathrm{cf}(\beta)$, so let $I \subseteq \mathrm{cf}(\beta)$ of size $\mathrm{cf}(\beta)$ consist of limit ordinals and let $\alpha \in a$ be such that $\alpha_i = \alpha$ for all $i \in I$. Then for all $i, j \in I$ such that $i < j$ we have

$$k_i(\alpha) \leq f_{\gamma_{i+1}}(\alpha) \leq g^{\gamma_j}_{E\cap\gamma_j}(\alpha) < f_{\gamma_j}(\alpha) \leq k_j(\alpha).$$

So $\{k_i(\alpha) : i \in I\} \subseteq S_\alpha$ has size $\mathrm{cf}(\beta) = (\mu')^+ > \mu'$. So not for all $\alpha \in a$ we have $|S_\alpha| \leq \mu'$. □

Clearly $\langle f_\alpha : \alpha < \lambda'\rangle$ is $<_D$-increasing: $f_\alpha <_D h_\beta \leq g^\beta_E <_D f_\beta$ for $\alpha < \beta$ (and any $E \in \mathcal{C}_\beta$). Now D is an ultrafilter on a; it transposes to an ultrafilter $\tilde{D}$ on $|a|$ via a bijection $i : a \leftrightarrow |a|$. Any $f \in \mathbf{ON}^a$ transposes to an $\tilde{f} \in \mathbf{ON}^{|a|}$ by $\tilde{f}(x) := f(i^{-1}(x))$. Then $\langle \tilde{f}_\alpha : \alpha < \lambda'\rangle$ is $<_{\tilde{D}}$-increasing in $\mathbf{ON}^{|\alpha|}$. Also $\lambda' > \mu > \min a > |a|$, so $\lambda' > |a|^+$. Hence Lemma 3.32 yields that $\langle \tilde{f}_\alpha/\tilde{D} : \alpha < \lambda'\rangle$ has a least upper bound in $\mathbf{ON}^{|a|}/\tilde{D}$ or there exist sets S_δ for $\delta < |a|$ such that $|S_\delta| \leq |a|$ and $\prod_{\delta<|a|} S_\delta/\tilde{D}$ cofinally cuts $\langle \tilde{f}_\alpha/\tilde{D} : \alpha < \lambda'\rangle$.

Suppose the second. Consider $T_\alpha := S_{i(\alpha)} \cap \alpha \subseteq \alpha$ and $\prod_{\alpha\in a} T_\alpha$. We will show that $\prod_{\alpha\in a} T_\alpha$ cofinally cuts $\langle f_\gamma/D : \gamma < \lambda'\rangle$, which contradicts Lemma 3.34. Let $\alpha < \lambda'$. Let $h \in \prod_{\delta<|a|} S_\delta$ such that $\tilde{f}_\alpha <_{\tilde{D}} h$ and $h/\tilde{D}$ cuts $\langle \tilde{f}_\gamma/\tilde{D} : \gamma < \lambda'\rangle$. Then $h <_{\tilde{D}} f_{\alpha'}$ for some $\alpha' < \lambda'$. So $[h < f_{\alpha'}] \in \tilde{D}$. So without loss of generality we assume that $h < \tilde{f}_{\alpha'}$ everywhere. Let $h'(x) := h(i(x)) < \tilde{f}_{\alpha'}(i(x)) = f_{\alpha'}(x) < x$, so $h \in \prod_{\alpha\in a} T_\alpha$. So $\prod_{\alpha\in a} T_\alpha$ cofinally cuts $\langle f_\gamma/D : \gamma < \lambda'\rangle$.

Thus $\langle f_\gamma/D : \gamma < \lambda'\rangle$ has a least upper bound g/D in $\mathbf{ON}^{|a|}/D$. Since $\mathrm{cf}(\prod a/D) = \lambda > \lambda'$, we may assume $g(\alpha) < \alpha$ and $g/D \in \prod a/D$. Also, $\{\alpha \in a : g(\alpha) \text{ is limit}\} \in D$: If not, then $g'(\alpha) + 1 = g(\alpha)$ for some $g' \in \prod a/D$, for all $\alpha \in A$, for some $A \in D$. Then $g' <_D g$ and g' is also an upper bound for $\langle f_\gamma/D : \gamma < \lambda'\rangle$, since this sequence in increasing. So we assume without loss of generality that $g(\alpha)$ is a limit ordinal for all $\alpha \in a$.

For $\alpha \in a$, let $S_\alpha \subseteq g(\alpha)$ be club in $g(\alpha)$ and of order type $\mathrm{cf}(g(\alpha))$. Enumerate $S_\alpha = \langle S_\alpha(i) : i < \mathrm{cf}(g(\alpha))\rangle$. Since $g(\alpha)$ is always a limit ordinal, it is easy to show that $\prod_{\alpha\in a} S_\alpha/D$ cofinally cuts $\langle f_\alpha/D : \alpha < \lambda'\rangle$.

Define $a' := \{\mathrm{cf}(g(\alpha)) : \alpha \in a\}$ and $D' = \{\{\mathrm{cf}(g(\alpha)) : \alpha \in A\} : A \in D\}$. Then D is an ultrafilter on a'.

Lemma 3.35. We have $\lim_{D'} a' = \mu$.

Proof. Let $\mu' := \lim_{D'} a'$. Since $\mathrm{cf}(g(\alpha)) \leq g(\alpha) < \alpha$ for all $\alpha \in a$, we have $\mu' \leq \mu$. We have $\mathrm{cf}(g(\alpha)) \leq \mu'$ for all $\alpha \in d$ for some $d \in D$. Define $T_\alpha := S_\alpha$ if $\alpha \in d$ and $T_\alpha := \{0\}$ if $\alpha \in a \setminus d$. Then $\prod_{\alpha\in a} T_\alpha/D \cong \prod_{\alpha\in a} S_\alpha/D$, which cofinally cuts $\langle f_\alpha/D : \alpha < \lambda'\rangle$. But $|T_\alpha| \leq \mu'$, so Lemma 3.34 yields $\mu' \geq \mu$. □

Lemma 3.36. We have $\mathrm{cf}(\prod a'/D') = \lambda'$.

Proof. Let $\beta < \lambda'$. We have $f_\beta <_D g$. So for D-almost all $\alpha \in a$ we have $f_\beta(\alpha) < g(\alpha)$. For these α, let $i < \mathrm{cf}(g(\alpha))$ be minimal such that $f_\beta(\alpha) < S_\alpha(i)$ and define $\bar{f}_\beta(\alpha) := S_\alpha(i)$. Then $\bar{f}_\beta/D \in \prod_{\alpha\in a} S_\alpha/D$ is defined. Observe the following:

1. $\langle \bar{f}_\beta/D : \beta < \lambda' \rangle$ is cofinal in $\prod S_\alpha/D$: Given $f \in \prod S_\alpha$, we have $f <_D g$. Since g is a least upper bound, f is not an upper bound for $\langle f_\gamma : \gamma < \lambda' \rangle$. So $f_\gamma \not<_D f$ for some γ, hence

$$f \leq_D f_\gamma <_D f_{\gamma+1} \leq_D \bar{f}_{\gamma+1}.$$

2. Any $S \subseteq \prod S_\alpha/D$ with $|S| < \lambda'$ is not cofinal in $\prod S_\alpha/D$: Given $f \in S$, let $\beta_f < \lambda'$ such that $f \leq_D \bar{f}_{\beta_f}$. Given $\beta < \lambda'$ we have $\bar{f}_\beta <_D g$, so let $\xi(\beta) < \lambda'$ such that $\bar{f}_\beta <_D f_{\xi(\beta)}$ (this exists by the same reasoning as in 1.). Let $\beta_1 := \sup_{f \in S} \xi(\beta_f) < \lambda'$. Then for all f,

$$f \leq_D \bar{f}_{\beta_f} <_D f_{\xi(\beta_f)} \leq_D f_{\beta_1} \leq_D \bar{f}_{\beta_1}.$$

So S is bounded by $\bar{f}_{\beta_1} \in \prod S_\alpha/D$.

We conclude that $\mathrm{cf}(\prod S_\alpha/D) = \lambda'$. For $\beta < \lambda'$ and $\alpha' \in a'$, define

$$\bar{f}'_\beta(\alpha') := \sup(i < \alpha' : S_\alpha(i) = \bar{f}_\beta(\alpha) \text{ for some } \alpha \in a \text{ with } \mathrm{cf}(g(\alpha)) = \alpha').$$

This is a supremum of $\leq |a|$-many elements. Since $\mu' < \mu$, since $\lim_{D'} a' = \mu$, D'-almost all $\alpha' \in a'$ satisfy $\alpha' \geq \mu'$. Note that $|a| < \min a \leq \mu$. Thus D'-almost all $\alpha' \in a'$ satisfy $\alpha' > |a|$. Thus $\bar{f}'_\beta(\alpha') < \alpha'$ for D'-almost every $\alpha' \in a'$. So $\bar{f}'_\beta/D' \in \prod a'/D'$.

To an $f \in \prod a'$ we associate an $\hat{f} \in \prod S_\alpha$ by defining $\hat{f}(\alpha) := S_\alpha(f(\mathrm{cf}(g(\alpha))))$. Given $f \in \prod a'$, let $\beta < \lambda'$ such that $\hat{f} <_D \bar{f}_\beta$. Then $f <_{D'} \bar{f}'_\beta$:

If $\alpha \in a$ is such that $\hat{f}(\alpha) < \bar{f}_\beta(\alpha)$, then $\hat{f}(\alpha) = S_\alpha(F(\mathrm{cf}(g(\alpha))))$, so $f(\mathrm{cf}(g(\alpha)))$ is the i such that $S_\alpha(i) = \hat{f}(\alpha)$. On the other hand, $\bar{f}_\beta(\alpha) = S_\alpha(j)$ for some j and thus $S_\alpha(j) > S_\alpha(i)$ thus $j > i$. And we have $\bar{f}'_\beta(\mathrm{cf}(g(\alpha))) = \sup(k : S_\alpha(k) = \bar{f}_\beta(\mathrm{cf}(g(\gamma))) \text{ and } \mathrm{cf}(g(\gamma)) = \mathrm{cf}(g(\alpha))) \geq j$. So $f(\mathrm{cf}(g(\alpha))) < \bar{f}'_\beta(\alpha)$. So

$$\{\alpha' \in a' : f(\alpha') < \bar{f}'_\beta(\alpha')\} \supseteq \{\mathrm{cf}(g(\alpha)) : \alpha \in [\hat{f} < \bar{f}_\beta]\} \in D.$$

So $\{\bar{f}'_\beta : \beta < \lambda'\}$ is cofinal in $\prod a'/D'$. Any $S' \subseteq \prod a'/D'$ with $|S'| < \lambda'$ is not cofinal in $\prod a'/D'$: $S := \{\bar{f} : f \in S'\}$ satisfies $S \subseteq \prod a/D$ and $|S| < \lambda'$, so S is not cofinal in $\prod a/D$. So S is bounded by some $\bar{f}_\beta$, thus S' is bounded by some $\bar{f}'_\beta$. So indeed $\mathrm{cf}(\prod a'/D') = \lambda'$. □

Noting $|a'| \leq |a|$ and combining with the two lemmas above, we have proved Theorem 3.28.

3.2 Generating $J_{<\lambda^+}$ over $J_{<\lambda}$

3.2.1 Universal sequences

In paragraph 3.2.2 we will need the notion of and some results on *universal sequences*, which we will describe here.

Definition 3.37. Let a be an infinite set of regular cardinals, let $\lambda \in \mathrm{pcf}(a)$ and let $f = \langle f_\xi : \xi < \lambda \rangle$ be an $<_{J_{<\lambda}}$-increasing sequence in $\prod a$. Then f is called λ-*universal* iff f is cofinal in $\prod a/D$ for all ultrafilters D on a such that $\mathrm{cf}(\prod a/D) = \lambda$.

Theorem 3.38. Let a be an infinite set of regular cardinals such that $|a|^+ < \min(a)$, let $\lambda \in \mathrm{pcf}(a)$. Then there exists a λ-universal sequence.

Proof. If $\lambda = \min(a)$, define $f = \langle f_\xi : \xi < \lambda \rangle$ by $f_\xi(a) = \xi$. Then f is $<$-increasing. If $\mathrm{cf}(\prod a/D) = \lambda = \min(a)$, then $D \ni \{\min(a)\}$ and f is cofinal in $\prod a/D$.

So assume $|a|^+ < \min(a) < \lambda$. Assume that no sequence is λ-universal. We will construct sequences of functions $f^\alpha = \langle f^\alpha_\xi : \xi < \lambda \rangle$ in $\prod a$ for $\alpha < |a|^+$ and ultrafilters D_α on a for $\alpha < |a|^+$ such that:

1. $\mathrm{cf}(\prod a/D_\alpha) = \lambda$ for each $\alpha < |a|^+$,
2. f^α is $<_{J_{<\lambda}}$-increasing and $<_{D_\alpha}$-bounded by $f^{\alpha+1}_0$ for each $\alpha < |a|^+$,
3. $f^{\alpha+1}$ is cofinal in $\prod a/D_\alpha$ for each $\alpha < |a|^+$,
4. $\langle f^\alpha_\xi : \alpha < |a|^+ \rangle$ is $\leq$-increasing for each $\xi < \lambda$.

This is visualized in the following figure:

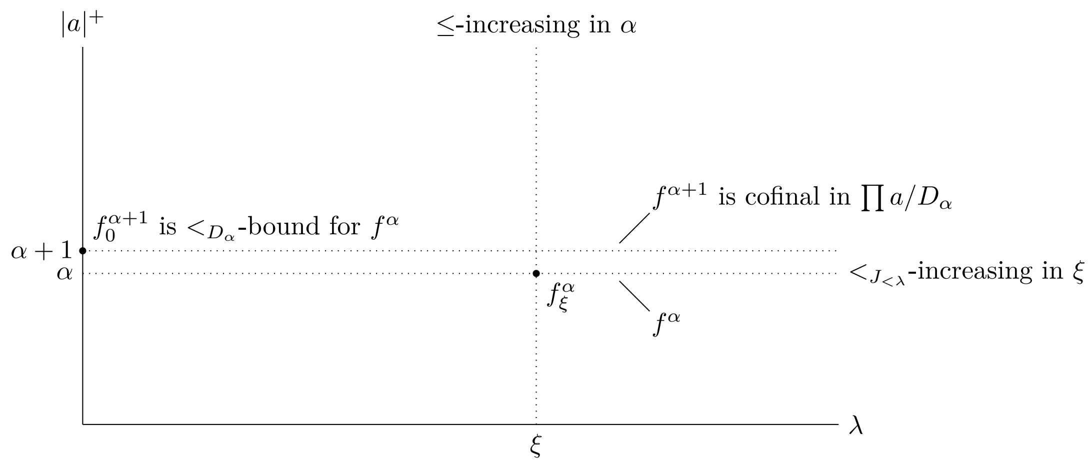

Figure 2: Visualization of $(f^\alpha_\xi)_{\alpha<|a|^+,\xi<\lambda}$.

The construction is of course by recursion, and we always assume that the functions we have constructed satisfy 1.-4.:

i. Base case: Define $f^0_0(x) = 0$ for all $x \in a$. Given $\langle f^0_\xi : \xi < \xi_0 \rangle$ for some $\xi_0 < \lambda$, let $f^0_{\xi_0}$ be an $<_{J_{<\lambda}}$-upper bound of $\langle f^0_\xi : \xi < \xi_0 \rangle$, which exists since $\prod a/J_{<\lambda}$ is λ-directed by Theorem 3.20.

ii. Successor case: Suppose we have defined f^α for some $\alpha < |a|^+$. Let D_α be an ultrafilter such that $\mathrm{cf}(\prod a/D_\alpha) = \lambda$ and that f^α is not cofinal in $\prod a/D_\alpha$. Let $f^{\alpha+1}_0$ be an $<_{D_\alpha}$-upper bound for f^α. Let $\langle \bar{f}^{\alpha+1}_\xi : \xi < \lambda \rangle$ be cofinal in $\prod a/D_\alpha$. Suppose we have defined $f^{\alpha+1}_\xi$ for all $\xi < \xi_0$ for some $0 < \xi_0 < \lambda$. Let $f^{\alpha+1}_{\xi_0}$ be an $<_{J_{<\lambda}}$-upper bound of $\{f^{\alpha+1}_\xi : \xi < \xi_0\} \cup \{\bar{f}^{\alpha+1}_\xi : \xi < \xi_0\} \cup \{f^\alpha_{\xi_0}\}$ and modify $f^{\alpha+1}_{\xi_0}$ on a set in $J_{<\lambda}$ such that $f^{\alpha+1}_{\xi_0} \geq f^\alpha_{\xi_0}$.

iii. Limit case: Suppose we have defined f^α for all $\alpha < \alpha_0$ for some $\alpha_0 < |a|^+$. Suppose we have defined $f^{\alpha_0}_\xi$ for all $\xi < \xi_0$ for some $\xi_0 < \lambda$. Then let $f^{\alpha_0}_\xi$ be an $<_{J_{<\lambda}}$-upper bound for $\{f^{\alpha_0}_\xi : \xi < \xi_0\} \cup \{\sup(f^\alpha_{\xi_0} : \alpha < \alpha_0)\}$ and modify $f^{\alpha_0}_{\xi_0}$ on an $J_{<\lambda}$-set so that $f^{\alpha_0}_{\xi_0} \geq \sup(f^\alpha_{\xi_0} : \alpha < \alpha_0)$.

Then define $h := \sup(f^\alpha_0 : \alpha < |a|^+) \in \prod a$. For $\alpha < |a|^+$ let $\xi_\alpha < \lambda$ such that $h <_{D_\alpha} f^{\alpha+1}_{\xi_\alpha}$, which exists by 3. Then $\bar{\xi} := \sup^+_{\alpha<|a|^+} \xi_\alpha < \lambda$. For eacht α, since $\xi_\alpha < \bar{\xi}$ we have $f^{\alpha+1}_{\xi_\alpha} <_{J_{<\lambda}} f^{\alpha+1}_{\bar{\xi}}$ by 2, thus $f^{\alpha+1}_{\xi_\alpha} \leq_{D_\alpha} f^{\alpha+1}_{\bar{\xi}}$, thus $h <_{D_\alpha} f^{\alpha+1}_{\bar{\xi}}$ as well. For each $\alpha < |a|^+$ define $A^\alpha := [h \leq f^\alpha_{\bar{\xi}}]$. Then

$\langle A^\alpha : \alpha < |a|^+\rangle$ is $\subseteq$-increasing by 4. We have

1. $f^\alpha_{\bar\xi} <_{D_\alpha} f^{\alpha+1}_0 \leq h$ by 2., so $h \not\leq_{D_\alpha} f^\alpha_{\bar\xi}$, so $A^\alpha \notin D_\alpha$,
2. $h <_{D_\alpha} f^{\alpha+1}_{\bar\xi}$ so $A^{\alpha+1} \in D_\alpha$.

So $\langle A^\alpha : \alpha < |a|^+\rangle$ is even $\subset$-increasing. This is a contradiction since $A^\alpha \subseteq a$ for all $\alpha < |a|^+$. $\square$

Lemma 3.39. Let a be an infinite set of regular cardinals such that $|a| < \min(a)$, let $\lambda \in \mathrm{pcf}(a)$ and let μ be minimal such that $a \cap \mu \notin J_{<\lambda}(a)$ (indeed, $a \notin J_{<\lambda}(a)$ and $\emptyset \in J_{<\lambda}(a)$ so there exists such a μ). Then there exists a λ-universal sequence which satisfies $*_\kappa$ with respect to $J_{<\lambda}$, for all regular $\kappa < \mu$. We can take $\kappa = |a|^+$.

Proof. Let D be such that $\mathrm{cf}(\prod a/D) = \lambda$. Then $\{\alpha \in a : \alpha > \lambda\} \notin D$, so $a \cap (\lambda+1) \in D$, so $a \cap (\lambda+1) \notin J_{<\lambda}$, so $\mu \leq \lambda+1$. $\mu = \lambda$ is impossible: λ is regular and $|a| < \min(a) \leq \lambda$, so $a \cap \lambda$ is bounded by some $\mu' < \lambda$; this contradicts the minimality of μ.

If $\mu = \lambda+1$, then $\lambda \in a$, $a \cap \lambda \in J_{<\lambda}$ and even $J_{<\lambda} = P(a \cap \lambda)$, since for any $A \subseteq a$ with $A \not\subseteq a \cap \lambda$, there is some $\lambda' > \lambda$ such that $\lambda' \in A$, and thus the ultrafilter D' concentrated on λ' satisfies $A \in D'$ and $\mathrm{cf}(\prod a/D') = \lambda'$, thus $A \notin J_{<\lambda}$. Take $f = \langle f_i : i < \lambda\rangle$ to be any λ-universal sequence. Then $*_\lambda$ holds, since f is strongly increasing by the simple nature of $J_<\lambda$. Thus $*_\kappa$ holds for any $\kappa < \mu = \lambda+1$.

So we are left with the case $\mu < \lambda$ and $a \cap \mu$ is unbounded in μ. Thus μ is a limit cardinal. Let $\langle g_i : i < \lambda\rangle$ be any λ-universal sequence. By the theorem below, for $I = J_{<\lambda}$, there exists an $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_i : i < \lambda\rangle$ such that $g_i <_{J_{<\lambda}} f_{i+1}$, so f is λ-universal, and such that f satisfies $*_\kappa$ if κ is a regular cardinal such that $\kappa^{++} < \lambda$ and $\{\alpha \in a : \alpha \leq \kappa^{++}\} \in J_{<\lambda}$. If $\kappa < \mu$, then $\kappa^{++} < \mu < \lambda$ and $\{\alpha \in a : \alpha \leq \kappa^{++}\} \in J_{<\lambda}$. So $*_\kappa$ holds for all $\kappa < \mu$. We have $a \cap |a|^+ \subseteq \min(a)$, so $a \cap |a|^+ = \emptyset$, thus $|a|^+ < \mu$. So we can take $\kappa = |a|^+$. $\square$

Theorem 3.40. Let a be an infinite set of regular cardinals, let λ be a regular cardinal, let I be a proper ideal on a and suppose that $\prod a/I$ is λ-directed. Let $\langle g_i : i < \lambda\rangle$ be a sequence in $\prod a$. Then there exists an $<_I$-increasing sequence $\langle f_i : i < \lambda\rangle$ in $\prod a$ such that $g_i <_I f_{i+1}$ for all i and that satisfies $*_\kappa$ if κ is a regular cardinal such that $\kappa^{++} < \lambda$ and $\{\alpha \in a : \alpha \leq \kappa^{++}\} \in I$.

Proof. We do this by recursion:

1. f_0 is arbitrary.
2. f_{i+1} is such that $g_i, f_i <_I f_{i+1}$.
3. If i is limit, then we have the following cases:
   (a) $\mathrm{cf}(i) = \kappa^{++}$ for some regular κ and $\{\alpha \in a : \alpha \leq \kappa^{++}\} \in I$, then let $E_i \subseteq i$ be i-club and such that $\mathrm{ot}(E_i) = \kappa^{++}$. Define $f_i(\alpha) = \sup(f_j(\alpha) : j \in E_i)$ for $\alpha > \kappa^{++}$, then $f_i(\alpha) < \alpha$ for $\alpha > \kappa^{++}$. Thus $f_i \in \prod A/I$ is well-defined.
   (b) We are not in the above situation. Then take f_i to be an $\leq_I$-upper bound for $\{f_j : j < i\}$.

Then $\langle f_i : i < \lambda\rangle$ is clearly $<_I$-increasing, $g_i <_I f_{i+1}$ for all i and it satisfies $*_\kappa$ for all regular cardinals κ such that $\kappa^{++} < \lambda$ and $\{\alpha \in a : \alpha \leq \kappa^{++}\} \in I$, by the following lemma. $\square$

Lemma 3.41. Let a be an infinite set of regular cardinals, let I be an ideal on a, let κ and λ be regular such that $\kappa^{++} < \lambda$, let $f = \langle f_i : i < \lambda\rangle$ be an $<_I$-increasing sequence in $\mathbf{ON}^a$ such that the following holds: If $\mathrm{cf}(i) = \kappa^{++}$, then there exists $E_i \subseteq i$ which is i-club and $\sup(f_j : j \in E_i) <_I f_{i'}$ for some $i' \geq i$. Then $*_\kappa$ holds for f.

Proof. Suppose $\kappa > \aleph_0$ and let $S := S(\kappa^{++}, \kappa)$ (recall Lemma 2.25). By Theorem 2.27, there exists a club-guessing sequence $\langle C_\alpha : \alpha \in S\rangle$ for S. Let U be any unbounded set of λ. We have to find an $U_0 \subseteq U$ of order type κ such that $\langle f_\xi : \xi \in U_0\rangle$ is strongly increasing. For $i < \kappa^{++}$ define $\xi_i < \lambda$ by

1. ξ_0 arbitrary,
2. $\xi_i = \sup_{j<i} \xi_j$ if i is limit,
3. Given ξ_j for all $j \leq i$, define $h_\alpha := \sup(f_{\xi_j} : j \leq i, j \in C_\alpha\}$ for all $\alpha \in S$. If $h_\alpha <_I f_\eta$ for some $\eta < \lambda$, let η_α be minimal with this property. If $h_\alpha \not<_I f_\eta$ for all $\eta < \lambda$, then let $\eta_\alpha := \xi_i + 1$. Let $\xi_{i+1} > \sup(\eta_\alpha : \alpha \in S)$ and such that $\xi_{i+1} \in U$.

Then $\{\xi_i : i < \kappa^{++}\}$ is club in $\xi := \sup_{i<\kappa^{++}} \xi_i$ and $\mathrm{cf}(\xi) = \kappa^{++}$. By assumption there exists $E_\xi \subseteq \xi$ which is ξ-club and $\sup(f_\eta : \eta \in E_\xi) <_I f_{\xi'}$ for some $\xi' \geq \xi$. Then $\{\xi_i : i < \kappa^{++}\} \cap E_\xi$ is ξ-club, and thus $C := \{i < \kappa^{++} : \xi_i \in E_\xi\}$ is κ^{++}-club. So there is an $\alpha \in S$ such that $C_\alpha \subseteq C$, and thus $\sup(f_\eta : \eta \in C_\alpha) <_I f_{\xi'}$. Let $N_\alpha := \{i \in C_\alpha : \sup(C_\alpha \cap i) < i\} = \{i \in C_\alpha : (\exists j \in C_\alpha : (j, i) \cap C_\alpha = \emptyset)\}$. For $i \in N_\alpha$, let $j_i \in N_\alpha$ be such that $(j_i, i) \cap C_\alpha = \emptyset$. In defining ξ_{j_i+1}, we had $h_\alpha := \sup(f_{\xi_j} : j \leq j_i, j \in C_\alpha) \leq \sup(f_{\xi_j} : j \in C_\alpha\} <_I f_{\xi'}$. So there is some η such that $h_\alpha < f_\eta$. Thus $\sup(f_{\xi_j} : j \leq j_i, j \in C_\alpha) <_I f_{\xi_{j_i+1}} \leq_I f_{\xi_i}$, since f is $<_I$-increasing. Let $Z_i = [\sup(f_{\xi_j} : j \leq j_i, j \in C_\alpha) \not< f_{\xi_i}]$. Then $\langle Z_i : i \in N_\alpha\rangle$ is a sequence in I, and if $i, j \in N_\alpha$ and $i < j$ and $\alpha \in Z_i \cap Z_j$, then $f_{\xi_i}(\alpha) < f_{\xi_j}(\alpha)$. Thus $\langle f_{\xi_i} : i \in N_\alpha\rangle$ is strongly increasing. Since $f_{\xi_i} <_I f_{\xi_{i+1}} \leq_I f_{\xi_{i^+}}$, where $i^+ = \min(C_\alpha \setminus (i + 1))$, Lemma 3.12 yields that $\{f_{\xi_{i+1}} : i \in N_\alpha\}$ is strongly increasing as well, and this is a subsequence of $\langle f_\xi : \xi \in U\rangle$ of order type κ. □

3.2.2 Existence of generators

Definition 3.42. Let a be a set, let I, J be ideals on a and let $b \subseteq a$. We say that b *generates* J *over* I when one of the following equivalent requirements holds:

1. the ideal generated by $I \cup \{b\}$ is exactly J, i.e. $J = \bigcap\{K : K$ is an ideal on a and $I \cup \{b\} \subseteq K\}$.
2. $J = \{X \subseteq a : (\exists Y \in I : X \subseteq Y \cup b\}$.
3. $J = \{X \subseteq a : X \setminus b \in I\}$.

The ideal generated by $I \cup \{b\}$ is sometimes denoted as $I + b$, and thus a fourth equivalent requirement is

4. $I + b = J$.

In this paragraph we will show that there always exist generators for $J_{<\lambda^+}$ over $J_{<\lambda}$. We have a simple test for checking whether b is a generator:

Lemma 3.43. Let a be a set of regular cardinals, let $b \subseteq a$ and let $\lambda \in \mathrm{pcf}(a)$. Then

$$b \text{ generates } J_{<\lambda^+} \text{ over } J_{<\lambda} \quad \Leftrightarrow \quad b \in J_{<\lambda^+} \text{ and } [\lambda = \mathrm{cf}(\prod a/D) \Rightarrow b \in D].$$

Proof. ($\Rightarrow$) Clearly $b \in J_{<\lambda^+}$. Suppose $\lambda = \mathrm{cf}(\prod a/D)$. Then $D \cap J_{<\lambda^+} \neq \emptyset$, so let $X \in D \cap J_{<\lambda^+}$. Then $X \setminus b \in J_{<\lambda}$. But $D \cap J_{<\lambda} = \emptyset$, so $X \setminus b \notin D$, so $b \supseteq X \cap b \in D$.

($\Leftarrow$) Since $J_{<\lambda} \subseteq J_{<\lambda^+}$ and $b \in J_{<\lambda^+}$, we have $J_{<\lambda} + b \subseteq J_{<\lambda^+}$ automatically. Now let $X \in J_{<\lambda^+}$ be arbitrary, we will show that $X \in J_{<\lambda} + b$ by showing $X \setminus b \in J_{<\lambda}$. Let D be such that $X \setminus b \in D$. Then $X \in D$, so $\mathrm{cf}(\prod a/D) < \lambda^+$. We have $b \notin D$, so $\mathrm{cf}(\prod a/D) \neq \lambda$ (by the assumption $[\lambda = \mathrm{cf}(\prod a/D) \Rightarrow b \in D]$). Hence $\mathrm{cf}(\prod a/D) < \lambda$. Since D was arbitrary, we have $X \setminus b \in J_{<\lambda}$. Since X was arbitrary, we have that $J_{<\lambda} + b \supseteq J_{<\lambda^+}$. Thus b generates $J_{<\lambda^+}$ over $J_{<\lambda}$. □

Theorem 3.44. Let a be a set of regular cardinals such that $|a| < \min(a)$ and let $\lambda \in \mathrm{pcf}(a)$. Then there exists a b which generates $J_{<\lambda^+}$ over $J_{<\lambda}$.

Proof. If $\lambda = \min(a)$, then $J_{<\lambda^+} = \{\{\min(a)\}, \emptyset\}$ and $J_{<\lambda} = \{\emptyset\}$, so $b_\lambda = \{\min(a)\}$ works. So suppose $\lambda > \min(a) > |a|$. By Lemma 3.39, let $\langle f_\xi : \xi < \lambda \rangle$ be a λ-universal sequence satisfying $*_{|a|^+}$ with respect to $J_{<\lambda}$. By Theorem 3.11, $\langle f_\xi : \xi < \lambda \rangle$ has an $<_{J_{<\lambda}}$-exact upper bound h. Since $f_\xi \in \prod a$ for all ξ, the identity function $\mathrm{id} : a \to a, \alpha \mapsto \alpha$ is an upper bound for $\langle f_\xi : \xi < \lambda \rangle$. So $h \leq_{J_{<\lambda}} \mathrm{id}$, so we may assume $h(\alpha) \leq \alpha$ for all $\alpha \in a$. We define $b := \{\alpha \in a : h(\alpha) = \alpha\}$ and prove that b satisfies the conditions of Lemma 3.43.

Suppose D is an ultrafilter on a such that $b \in D$. If $D \cap J_{<\lambda} \neq \emptyset$ then $\mathrm{cf}(\prod a/D) < \lambda$. If $D \cap J_{<\lambda} = \emptyset$ then $\langle f_\xi/D : \xi < \lambda \rangle$ is $<_D$-increasing and h/D is still its exact upper bound. So $\langle f_\xi/D : \xi < \lambda \rangle$ is $<_D$-increasing and cofinal in $\prod\{h(\alpha) : \alpha \in a\}/D$. But $h =_D \mathrm{id}$, so in fact $\langle f_\xi/D : \xi < \lambda \rangle$ is $<_D$-increasing and cofinal in $\prod a/D$ as well, and $\mathrm{cf}(\prod a/D) = \lambda$. Since D was arbitrary, we have shown $b \in J_{<\lambda^+}$.

Now suppose $\mathrm{cf}(\prod a/D) = \lambda$ and assume $b \notin D$. Then $\{\alpha \in a : h(\alpha) < \alpha\} \in D$ so $h/D \in \prod a/D$. We have $D \cap J_{<\lambda} = \emptyset$, so $f_\xi <_D h$ for all ξ, so $\langle f_\xi/D : \xi < \lambda \rangle$ is not cofinal in $\prod a/D$. This contradicts the universality of $\langle f_\xi : \xi < \lambda \rangle$. So we have shown $[\lambda = \mathrm{cf}(\prod a/D) \Rightarrow b \in D]$. □

3.2.3 Properties of generators

Now that we know the existence of generators, we can look without hesitation for properties and extra assumptions on the generators.

In fact, the first two results are provable and interesting without knowing that generators always exist. However, we present them here among the other properties on generators.

In this paragraph, a always denotes an infinite set of regular cardinals such that $|a| < \min(a)$ and $\langle b_\lambda : \lambda \in \mathrm{pcf}(a) \rangle$ is a sequence of generators.

Lemma 3.45. If $J_{<\lambda^+}(a) = J_{<\lambda}(a) + b$ and $c \subseteq a$, then $J_{<\lambda^+}(c) = J_{<\lambda}(c) + (b \cap c)$. So 'generators restrict'.

Proof. By Lemma 3.43, we need to show $b \cap c \in J_{<\lambda^+}(c)$ and $[\lambda = \mathrm{cf}(\prod c/D) \Rightarrow b \cap c \in D]$. If $b \cap c \in D$, extend D to an ultrafilter $\hat{D}$ on a, then $b \in \hat{D}$ so $\mathrm{cf}(\prod c/D) = \mathrm{cf}(\prod a/\hat{D}) < \lambda^+$. So indeed $b \cap c \in J_{<\lambda^+}(c)$. If $\mathrm{cf}(\prod c/D) = \lambda$, then again extend D to an ultrafilter $\hat{D}$ on a. Then $\lambda = \mathrm{cf}(\prod c/D) = \mathrm{cf}(\prod a/\hat{D})$, so $b \in \hat{D}$. Hence $b \cap c \in D$. So $[\lambda = \mathrm{cf}(\prod c/D) \Rightarrow b \cap c \in D]$. □

Lemma 3.46. If $J_{<\lambda^+} = J_{<\lambda} + b = J_{<\lambda} + c$, then $b =_{J_{<\lambda}} c$.

Proof. We need to show that $b \Delta c = b \setminus c \cup c \setminus b \in J_{<\lambda}$. Let D such that $b \Delta c \in D$. Then $b \cup c \in D$. Note that $b, c \in J_{<\lambda^+}$, so $b \cup c \in J_{<\lambda^+}$. Hence $\mathrm{cf}(\prod a/D) < \lambda^+$. If $\mathrm{cf}(\prod a/D) = \lambda$, then $b, c \in D$ by Lemma 3.43. Hence $b \cap c \in D$, but since $b \Delta c \in D$ we get $\emptyset \in D$, a contradiction. Hence $\mathrm{cf}(\prod a/D) < \lambda$, and $b \Delta c \in J_{<\lambda}$. □

We now give a nice characterization of $\mathrm{cf}(\prod a/D)$ using generators. Remember that b_λ generates $J_{<\lambda^+}$ over $J_{<\lambda}$.

Lemma 3.47. We have

$$\mathrm{cf}(\prod a/D) = \min(\{\lambda : b_\lambda \in D\}).$$

Proof. Let $\mathrm{cf}(\prod a/D) = \lambda$. Then $D \cap J_{<\lambda^+} \neq \emptyset$, so let $b \in D \cap J_{<\lambda^+}$. Since b_λ generates $J_{<\lambda^+}$ over $J_{<\lambda}$, we have $b \setminus b_\lambda \in J_{<\lambda}$. Since $D \cap J_{<\lambda} = \emptyset$, we must have $b \setminus b_\lambda \notin D$, so $b_\lambda \supseteq b \cap b_\lambda \in D$. If $\mu < \lambda$, then $b_\mu \in D$ implies $\mathrm{cf}(\prod a/D) < \mu^+ \leq \lambda$, which is not true. So $b_\mu \notin D$. □

Lemma 3.48. When $c \subseteq a$, there are $\lambda_1, ..., \lambda_n \in \mathrm{pcf}(c)$ such that $c \subseteq b_{\lambda_1} \cup ... \cup b_{\lambda_n}$.

Proof. Define $I := \{b \subseteq c :$ there exist $\lambda_1, ..., \lambda_n \in \mathrm{pcf}(c)$ such that $b \subseteq b_{\lambda_1} \cup ... \cup b_{\lambda_n}\}$. Then I is an ideal on c. If $c \in I$, then we are done. If $c \notin I$, then extend the dual filter of I to an ultrafilter disjoint from I, and extend this to an ultrafilter D on a. Then $c \in D$ and $\lambda := \mathrm{cf}(\prod a/D)$. Also $b_\lambda \in D$ by Lemma 3.47. Hence $c \cap b_\lambda \in D$. Therefore we have $\lambda \in \mathrm{pcf}(c \cap b_\lambda)$. Since $c \cap b_\lambda \subseteq b_\lambda$, we have $b_\lambda \in I$. But this contradicts $D \cap I = \emptyset$. □

We prove a simple extension lemma for ideals, which will be used in the proof of the next theorem, but which will be useful later on as well.

Lemma 3.49. Let X be a set, let I be an ideal on X, let κ be an infinite cardinal and let $\langle c_\alpha : \alpha < \kappa \rangle$ be an $\subseteq_I$-decreasing sequence such that $c_\alpha \notin I$ for all α. Then there is an ultrafilter D on X such that $D \supseteq \{c_\alpha : \alpha < \kappa\}$ and $D \cap I = \emptyset$.

Proof. We need to show that $\{X \setminus A : A \in I\} \cup \{c_\alpha : \alpha < \kappa\}$ has the finite intersection property. Then it extends to a filter and to an ultrafilter with the desired properties. Clearly $(X \setminus A) \cap c_\alpha \neq \emptyset$ for any $A \in I$ and $\alpha < \kappa$. Since I is an ideal, the intersection of finitely many elements of $\{X \setminus A : A \in I\}$ is again in $\{X \setminus A : A \in I\}$. So it remains to check that $(X \setminus A) \cap \bigcap_{i=1}^n c_{\alpha_i} \neq \emptyset$ for any $A \in I$ and $\alpha_i < \kappa$. Let $\alpha = \max \alpha_i$. Then

$$\bigcap_{i=1}^n c_{\alpha_i} = c_\alpha \setminus [(c_\alpha \setminus c_{\alpha_1}) \cup ... \cup (c_\alpha \setminus c_{\alpha_n})] \notin I,$$

since $c_\alpha \setminus c_{\alpha_i} \in I$ for all i and $c_\alpha \notin I$. Thus $(X \setminus A) \cap \bigcap_{i=1}^n c_{\alpha_i} \neq \emptyset$. □

Theorem 3.50. Let a be an infinite set of regular cardinals such that $|a| < \min(a)$, let $\lambda \in \mathrm{pcf}(a)$. Then $\mathrm{tcf}(\prod b_\lambda / J_{<\lambda}(b_\lambda)) = \lambda$.

Note. We use $J_{<\lambda}(b_\lambda)$ be ensure that we have an ideal on b_λ. However, $J_{<\lambda}(b_\lambda) = J_{<\lambda}(a) \cap P(b_\lambda)$ and there is no real difference between $\prod b_\lambda / J_{<\lambda}(b_\lambda)$ and $\prod b_\lambda / J_{<\lambda}(a)$.

Proof. Let $b := b_\lambda$. Let $\langle f_\xi : \xi < \lambda \rangle$ be λ-universal. Then $\langle f_\xi : \xi < \lambda \rangle$ is $<_{J_{<\lambda}(a)}$-increasing. Recall that $J_{<\lambda}(b_\lambda) = J_{<\lambda}(a) \cap P(b_\lambda)$, so that we have

$$\begin{aligned} f_\xi <_{J_{<\lambda}(a)} f_\chi \quad \Rightarrow \quad [f_\xi \not< f_\chi] \in J_{<\lambda}(a) \quad &\Rightarrow \quad [f_\xi \restriction b \not< f_\chi \restriction b] = [f_\xi \not< f_\chi] \cap b \in J_{<\lambda}(b) \\ &\Rightarrow \quad f_\xi \restriction b <_{J_{<\lambda}(b)} f_\chi \restriction b. \end{aligned}$$

Thus the sequence $\langle f_\xi \restriction b : \xi < \lambda \rangle$ is $<_{J_{<\lambda}(b)}$-increasing. We will show that it is also cofinal in $\prod b / J_{<\lambda}(b)$.

Let $h \in \prod b$. Suppose $h \not<_{J_{<\lambda}(b)} f_\xi \restriction b$ for all ξ. Then $[h \not< f_\xi \restriction b] \notin J_{<\lambda}(b)$ for all ξ. Furthermore $\langle [h \not< f_\xi \restriction b] : \xi < \lambda \rangle$ is $\subseteq_{J_{<\lambda}}$-decreasing: If $\xi < \chi$, then

$$\begin{aligned}[h \not< f_\chi \restriction b] \setminus [h \not< f_\xi \restriction b] &= \{\alpha \in b : f_\xi(\alpha) > h(\alpha) \geq f_\chi(\alpha)\} \\ &\subseteq \{\alpha \in a : f_\chi(\alpha) < f_\xi(\alpha)\} \cap b \\ &\subseteq [f_\xi \not< f_\chi] \cap b \in J_{<\lambda}(b).\end{aligned}$$

So Lemma 3.49 applies and we obtain an ultrafilter on b disjoint from $J_{<\lambda}(b)$ and containing $[h \not< f_\xi \restriction b]$ for each ξ. Extend it an ultrafilter D on a. Then $D \cap J_{<\lambda}(a) = \emptyset$ and $b \in D$, so $\mathrm{cf}(\prod a/D) = \lambda$. Since $D \ni [h \not< f_\xi \restriction b]$, any extension $\hat{h}$ of h to a function on a satisfies $h \geq_D f_\xi$, for all ξ. So $\langle f_\xi/D : \xi < \lambda \rangle$ is not cofinal in $\prod a/D$, which contradicts the universality of the sequence $\langle f_\xi : \xi < \lambda \rangle$. □

3.2.4 Another proof

We now present a more direct approach to show the existence of a generator for $J_{<\lambda^+}$ over $J_{<\lambda}$. It is more basic and shorter than the one above, but the final step of the proof relies on the assumption that $2^{|A|} < \min A$.

Let A be a set of regular cardinals such that $|A|^+ < \min A$.

Lemma 3.51. Let I be an ideal on A, let λ be a regular cardinal and let $\langle f_i : i < \lambda \rangle$ be a sequence in $\prod A$ such that it is $<_I$-increasing and $\leq_I$-unbounded in $\prod A/I$. Then there exists a sequence $\langle b_\gamma : \gamma < \lambda \rangle$ in $P(A)$ and a function $g \in \prod A$ such that

1. $b_0 \notin I$,
2. $b_\gamma \subseteq_{J_{<\lambda}} b_{\gamma'}$ for $\gamma < \gamma'$,
3. $\langle f_i \restriction b_\gamma : i < \lambda \rangle$ is increasing and cofinal in $\prod b_\gamma/I$, for all γ,
4. $\langle f_i : i < \lambda \rangle$ is bounded by g in $\prod A/(I + \{b_\gamma : \gamma < \lambda\})$. $I + \{b_\gamma : \gamma < \lambda\}$ is the ideal generated by $I \cup \{b_\gamma : \gamma < \lambda\}$.

We postpone the proof of this lemma to the end of this section.

Corollary 3.52. Let I be an ideal on A such that $\prod A/I$ is λ-directed, let D be an ultrafilter on A such that $I \cap D = \emptyset$ and assume $\lambda = \mathrm{cf}(\prod A/D)$. Then there exists a $b \in D$ such that $\mathrm{tcf}(\prod b/(I \restriction b)) = \lambda$.

Proof. Let $\langle f_i : i < \lambda \rangle$ be increasing and cofinal in $\prod A/D$. Recursively define $f'_i \in \prod A$ for $i < \lambda$ by

1. $f'_0 = f_0$,
2. f'_i is an upper bound in $\prod A/I$ for $\{f'_j : j < i\} \cup \{f_i\}$, which exists by the λ-directedness of $\prod A/I$.

Then $\langle f'_i : i < \lambda \rangle$ is $<_I$-increasing and $<_D$-increasing. Since $f_i \leq_I f'_i$, thus $f_i \leq_D f'_i$, the sequence is still cofinal in $\prod A/D$, hence can not have an upper bound. Thus the sequence is unbounded in $\prod A/I$ as well, since any bound in $\prod A/I$ would also be a bound in $\prod A/D$. So let $\langle b_\gamma : \gamma < \lambda \rangle$ and g be as in Lemma 3.51. If $D \cap (I + \{b_\gamma : \gamma < \lambda\}) = \emptyset$, then g would be an upper bound for $\langle f_i : i < \lambda \rangle$ in $\prod A/D$. So there exists a $d \in D$, $x \in I$ and $\gamma_1, ..., \gamma_n < \lambda$ such that $d \subseteq x \cup b_{\gamma_1} \cup ... \cup b_{\gamma_n}$. Since $\langle b_\gamma : \gamma < \lambda \rangle$ is $\subseteq_{J_{<\lambda}}$-increasing, this reduces to $d \subseteq x' \cup b_\gamma$ for some $x' \in I$ and some $\gamma < \lambda$. Since

$D \cap I = \emptyset$, it follows that $b_\gamma \in D$. Since $\langle f_i \restriction b_\gamma : i < \lambda \rangle$ is increasing and cofinal in $\prod b_\gamma / I$, we have $\mathrm{tcf}(\prod b_\gamma / I) = \lambda$. □

Corollary 3.53. Let I be a proper ideal on A such that if D is an ultrafilter on A and $D \cap I = \emptyset$, then $\mathrm{cf}(\prod A/D) = \lambda$. Then $\mathrm{tcf}(\prod A/I) = \lambda$.

Proof. We have $J_{<\lambda} \subseteq I$: If $b \in J_{<\lambda} \setminus I$, let D be an ultrafilter on A such that $b \in D$ and $D \cap I = \emptyset$, then $\mathrm{cf}(\prod A/D) < \lambda$, which contradicts the assumption that $\mathrm{cf}(\prod A/D) = \lambda$. It follows that $\prod A/I$ is λ-directed, since $\prod A/J_{<\lambda}$ is λ-directed. Let $I' := \{B \subseteq A : B \in I$ or $[B \notin I$ and $\mathrm{tcf}(\prod B/(I \restriction B)) = \lambda]\}$. Then I' is an ideal:

1. $I' \subseteq P(A)$, $\emptyset \in I'$,
2. If $X, Y \in I'$, then
   (a) If $X, Y \in I$, then $X \cup Y \in I$, so $X \cup Y \in I'$,
   (b) $X, Y \notin I$ and $\mathrm{tcf}(\prod X/(I \restriction X)) = \mathrm{tcf}(\prod Y/(I \restriction Y)) = \lambda$, so $\mathrm{tcf}(\prod (Y \cup X)/(I \restriction (Y \cup X))) = \lambda$ and $X \cup Y \notin I$, hence $X \cup Y \in I'$,
   (c) without loss of generality $X \in I$, $Y \notin I$ and $\mathrm{tcf}(\prod Y/(I \restriction Y)) = \lambda$, so $\mathrm{tcf}(\prod (X \cup Y)/(I \restriction (X \cup Y))) = \lambda$ and $X \cup Y \notin I$, so $X \cup Y \in I'$,
3. If $X \in I'$ and $Y \subseteq X$, then
   (a) $X \in I$, so $Y \in I$, so $Y \in I'$,
   (b) $\mathrm{tcf}(\prod X/(I \restriction X) = \lambda$, so
      i. $Y \in I$, so $Y \in I'$,
      ii. $Y \notin I$, so $\mathrm{tcf}(\prod Y/(I \restriction Y)) = \mathrm{tcf}(\prod X/(I \restriction X)) = \lambda$, so $Y \in I'$.

Only cases 2.(b), 2.(c) and 3.(b)ii. are non-trivial, and require a bit of thought about how we could manipulate the true cofinal sequences.

If $I' \neq P(A)$, then any ultrafilter D on A such that $D \cap I' = \emptyset$ satisfies $D \cap I = \emptyset$ so $\mathrm{cf}(\prod A/D) = \lambda$. Therefore there exists a $b \in D$ such that $\mathrm{tcf}(\prod b/I) = \lambda$ by Corollary 3.52. So $D \cap I' \neq \emptyset$, a contradiction. So $I' = P(A)$ and thus $\mathrm{tcf}(\prod A/I) = \lambda$. □

Corollary 3.54. Suppose $b \in J_{<\lambda^+}(a) \setminus J_{<\lambda}(a)$. Then $\mathrm{tcf}(\prod b/J_{<\lambda}(b)) = \lambda$.

Proof. Consider $I := J_{<\lambda} + (A \setminus b)$. For any ultrafiler D on A such that $D \cap I = \emptyset$, we have $D \cap J_{<\lambda} = \emptyset$, thus $\mathrm{cf}(\prod A/D) \not< \lambda$. Furhermore, $D \not\ni (A \setminus b)$, hence $D \ni b$, so $\mathrm{cf}(\prod A/D) < \lambda^+$. So $\mathrm{cf}(\prod A/D) = \lambda$. By Corollary 3.53 $\mathrm{tcf}(\prod A/I) = \lambda$. But $\prod b/J_{<\lambda}(b) \cong \prod A/I$, so we are done. □

Theorem 3.55. If $2^{|A|} < \min A$, then $J_{<\lambda^+}$ is generated over $J_{<\lambda}$ by a single set, for any λ.

Proof. The proof completely relies on the assumption and Lemma 3.56 below. We have $J_{<\lambda^+} \subseteq J_{<\lambda^+}$ and $|J_{<\lambda^+}| \leq |P(A)| = 2^{|A|} < \min A \leq \lambda$, so by lemma 3.56 there exists a $b \in J_{<\lambda^+}$ such that $c \subseteq_{J_{<\lambda}} b$ for all $c \in J_{<\lambda^+}$. Thus $J_{<\lambda} + b = J_{<\lambda^+}$. □

Lemma 3.56. Let $\mu < \lambda$ be cardinals and let $\{b_\alpha : \alpha < \mu\} \subseteq J_{<\lambda^+}$. Then there exists a $b \in J_{<\lambda^+}$ such that $b_\alpha \subseteq_{J_{<\lambda}} b$ for all α.

Proof. Without loss of generality $\{b_\alpha : \alpha < \mu\} \subseteq J_{<\lambda^+} \setminus J_{<\lambda}$. By Corollary 3.54 $\mathrm{tcf}(\prod b_\alpha / J_{<\lambda}) = \lambda$ for all α. Let $\langle f_i^\alpha : i < \lambda \rangle$ in $\prod A$ such that $\langle f_i^\alpha \restriction b_\alpha : i < \lambda \rangle$ is increasing and cofinal in $\prod b_\alpha / J_{<\lambda}$. Let $\langle f_i^* : i < \lambda \rangle$ such that $\{f_i^\alpha : \alpha < \mu\} \cup \{f_j^* : j < i\}$ is bounded by f_i^* in $\prod A/J_{<\lambda}$ (use the λ-directedness of $\prod A/J_{<\lambda}$). Since $\langle f_i^\alpha \restriction b_\alpha : i < \lambda \rangle$ is not bounded in $\prod b_\alpha / J_{<\lambda}$, it follows that

$\langle f_i^\alpha : i < \lambda \rangle$ is not bounded in $\prod A/J_{<\lambda}$. Thus $\langle f_i^* : i < \lambda \rangle$ is not bounded in $\prod A/J_\lambda$. Thus let $\langle c_\gamma : \gamma < \lambda \rangle$ and g be as in Lemma 3.51.

Assume for now the following claim: For all $\alpha < \mu$ exists $\gamma_\alpha < \lambda$ such that $b_\alpha \subseteq_{J_{<\lambda}} c_{\gamma_\alpha}$. Since $\mu < \lambda$, let $\gamma < \lambda$ such that $\gamma_\alpha \leq \gamma$ for all α. Then $b_\alpha \subseteq_{J_{<\lambda}} c_\gamma$ for all α.

We have $c_\gamma \in J_{<\lambda^+}$: Let D be an ultrafilter on A such that $c_\gamma \in D$. If $D \cap J_{<\lambda} \neq \emptyset$ then $\mathrm{cf}(\prod A/D) < \lambda$; if $D \cap J_{<\lambda} = \emptyset$, then $\mathrm{tcf}(\prod c_\gamma/J_{<\lambda}) = \mathrm{cf}(\prod c_\gamma/D)$. But by definition of c_γ, $\langle f_i^* \restriction c_\gamma : i < \lambda \rangle$ is increasing and cofinal in $\prod c_\gamma/J_{<\lambda}$. So $\lambda = \mathrm{tcf}(\prod c_\gamma/J_{<\lambda}) = \mathrm{cf}(\prod c_\gamma/D)$. Thus in both cases $\mathrm{cf}(\prod A/D) \leq \lambda$ hence $c_\gamma \in J_{<\lambda^+}$.

It remains to show that for all $\alpha < \mu$ exists $\gamma_\alpha < \lambda$ such that $b_\alpha \subseteq_{J_{<\lambda}} c_{\gamma_\alpha}$. Suppose not. Let α be such that $b_\alpha \setminus c_\gamma \notin J_{<\lambda}$ for all γ. By Lemma 3.49, let D be an ultrafilter such that $D \ni (b_\alpha \setminus c_\gamma)$ for all γ and $D \cap J_{<\lambda} = \emptyset$. Then $\mathrm{cf}(\prod A/D) \not< \lambda$. Since $\langle f_i^\alpha \restriction b_\alpha : i < \lambda \rangle$ is cofinal in $\prod b_\alpha/J_{<\lambda}$, so is $\langle f_i^* \restriction b_\alpha : i < \lambda \rangle$. Since $b_\alpha \in D$ and $D \cap J_{<\lambda} = \emptyset$, $\langle f_i^* : i < \lambda \rangle$ is cofinal in $\prod A/D$. But this contradicts that $\langle f_i^* : i < \lambda \rangle$ is bounded by g in $\prod A/(J_{<\lambda} + \{c_\gamma : \gamma < \lambda\})$. □

Proof of Lemma 3.51. First note that we must have $\lambda \geq \min A$ otherwise $\alpha \mapsto \sup_{i<\lambda} f_i(\alpha)$ would be an upper bound in $\prod A$ for $\langle f_i : i < \lambda \rangle$. Assume the theorem is false. Recursively define $g_\alpha \in \prod A$ for $\alpha < |A|^+$ as follows:

1. Let $g_0 \in \prod A$ be arbitrary.
2. Let $g_\alpha(x) := \sup_{\beta<\alpha} g_\beta(x)$ when α is limit, for all $x \in A$.
3. Given g_α, define $b_i^\alpha := [g_\alpha < f_i]$ for all $i < \lambda$. For some minimal i_α, $b_{i_\alpha}^\alpha \notin I$ since $\langle f_i : i < \lambda \rangle$ is unbounded in $\prod A/I$. For $i < \lambda$, let

$$b_{i,\alpha} := \begin{cases} b_i^\alpha, & \text{if } i \geq i_\alpha, \\ b_{i_\alpha}^\alpha, & \text{if } i \leq i_\alpha. \end{cases}$$

   Then $\langle b_{i,\alpha} : i < \lambda \rangle$ is a sequence in $P(A)$ such that $b_{0,\alpha} \notin I$, $b_{i,\alpha} \subseteq_I b_{j,\alpha}$ when $i < j$ since $\langle f_i : i < \lambda \rangle$ is increasing in $\prod A/I$ and $\langle f_i : i < \lambda \rangle$ is bounded by g_α in $\prod A/(I + \{b_{i,\alpha} : i < \lambda\})$. Since we assumed the lemma is false, there exists a $\gamma_\alpha \geq i_\alpha$ such that $\langle f_i \restriction b_{\gamma_\alpha,\alpha} : i < \lambda \rangle$ is not cofinal in $\prod b_{\gamma_\alpha,\alpha}/I$. So let $h_\alpha \in \prod A$ be such that $h_\alpha \restriction b_{\gamma_\alpha,\alpha} \not\leq_I f_i \restriction b_{\gamma_\alpha,\alpha}$ for all i. So $[h_\alpha > f_i] \cap b_{\gamma_\alpha,\alpha} \notin I$. Let $g_{\alpha+1} := \max(g_\alpha, h_\alpha)$.

Now we have the map $|A|^+ \to \lambda$ given by $\alpha \mapsto \gamma_\alpha$ and $\lambda \geq \min A > |A|^+$, so let $\gamma < \lambda$ be such that $\gamma_\alpha \leq \gamma$ for all α and consider $\langle b_\gamma^\alpha : \alpha < |A|^+ \rangle$. We have

$$b_\gamma^\alpha = [g_\alpha < f_\gamma] \supseteq [g_{\alpha+1} < f_\gamma] = b_\gamma^{\alpha+1}$$

for all α. We have $[h_\alpha > f_i] \cap b_{\gamma,\alpha} \notin I$ for all i, in particular for $i = \gamma$; let $x \in b_{\gamma,\alpha} = b_\gamma^\alpha = [g_\alpha < f_\gamma]$ be such that $h_\alpha(x) > f_\gamma(x)$. Then $x \in b_\gamma^\alpha$ but $x \notin b_\gamma^{\alpha+1}$. So $b_\gamma^\alpha \supset b_\gamma^{\alpha+1}$. So $\langle b_\gamma^\alpha : \alpha < |A|^+ \rangle$ is a strictly decreasing sequence of subsets of A of length $|A|^+$, which is impossible. □

4 Model theory

Model theory is a field of study in mathematics and logic. The first model theoretic work was done in the first half of the twentieth century, by Skolem, Löwenheim, Gödel and Tarski. Model Theory formalizes the idea that by assuming some basic rules (the *axioms*) we can look for a universe (a *model*) where the axioms are satisfied. To formulate the axioms we need a *language*, and it is standard to assume that every language contains a symbol $=$ for *is equal to*. The language of set theory also contains a symbol $\in$ for *is element of*. An example of an axiom of set theory is that if two sets contain exactly the same elements, then they are equal. To check whether a model satisfies an axiom we need an *interpretation* of the language in the model. By the theorems of Skolem and Löwenheim it turns out that the cardinality of a model is not determined by the axioms: As soon as there is an infinite model satisfying the axioms, there are models of every infinite cardinality larger than or equal to the cardinality of the language.

4.1 Basic definitions

We now rigorously build up the definitions of model theory.

Definition 4.1. A *language* consists of a set of *constants*, of a set of *function symbol* and of a set of *relation symbols*. Each function symbol and each relation symbol is equipped with an *arity*, which is a natural number. The *cardinality* of a language is the sum of the cardinalities of the set of constants, set of function symbols and set of relation symbols. The language of set theory has no constant, no function symbols and one binary relation symbol $\in$.

We denote a language as $L = (C_L, F_L, R_L)$ and its cardinality as $|L| = |C_L| + |F_L| + |R_L|$, and the *language of set theory* is $L_{\text{Set}} = (\emptyset, \emptyset, \{\in\})$.

From a language we can form *terms* and *formulas*. We can perform substitutions in both terms and formulas, and terms and formulas may be *closed*.

To form terms and formulas we need auxiliary symbols:

1. *Variables*: Once and for all we fix a countable set V of variables.
2. *Logical symbols*: $=$, $\vee$, $\neg$ and $\exists$.
3. *Symbols for notation*: (,) and ,.

Terms and formulas will be certain finite sequences in $\mathcal{S} := C_L \cup F_L \cup R_L \cup V \cup \{=, \vee, \neg, \exists, (,), ,\}$. Let $\mathcal{S}^*$ denote the set of finite sequences of $\mathcal{S}$. An element of $\mathcal{S}^*$ *is* a function $f : n \to \mathcal{S}$ for some natural number n, but form now on we will denote such a sequence simply by writing its image elements from left to right: $f(0)f(1)f(2)...f(n-1)$.

Definition 4.2. The set of *L-terms* is the smallest subset of $\mathcal{S}^*$ that satisfies the following:

1. If c is a constant, then c is a term.
2. If x is a variable, then x is a term.
3. If $t_1, ..., t_n$ are terms and f is a function symbol of arity n, then $f(t_1, ..., t_n)$ is a term.

Note that in Definition 4.2.1. we first mean c as a constant, and then c as the sequence $f : 1 \to \mathcal{S}$ given by $f(0) = c$. Something similar holds in 4.2.2. Furthermore, any intersection of a collection of sets satisfying 1.-3. again satisfies 1.-3., so there indeed exists a smallest subset.

Definition 4.3. The set of *L-formulas* is the smallest subset of $\mathcal{S}^*$ that satisfies the following:

1. If t_1 and t_2 are terms, then $t_1 = t_2$ is a formula.
2. If $t_1, ..., t_n$ are terms and R is a relation symbol of arity n, then $R(t_1, ..., t_n)$ is a formula.
3. If ϕ is a formula, then $\neg(\phi)$ is a formula.
4. If ϕ_1 and ϕ_2 are formulas, then $(\phi_1) \vee (\phi_2)$ is a formula.
5. If ϕ is a formula and x is a variable, then $\exists x(\phi)$ is a formula.

Now that we have defined terms and formulas, we define substitution.

Definition 4.4. Let t and s be terms and let x be a variable. We recursively define the *substitution of x by s in t*, and denote this as $t[s/x]$, as follows:

1. If $t = c$, then $t[s/x] = t$.
2. If $t = x$, then $t[s/x] = s$. If $t = y$ and $y \neq x$, then $t[s/x] = t$.
3. If $t = f(t_1, ..., t_n)$ then $t[s/x] = f(t_1[s/x], ..., t_n[s/x])$.

Informally then a substitution of x by s in a term t is just replacing every x in t by s.

Definition 4.5. Let ϕ be a formula, let t be a term and let x be a variable. Define recursively the *substitution of x by t in ϕ*, denoted as $\phi[t/x]$, as follows:

1. If $\phi = \{t_1 = t_2\}$, then $\phi[t/x] = \{t_1[t/x] = t_2[t/x]\}$. (The { and } are only used to express the beginning and end of the formula.)
2. If $\phi = R(t_1, ..., t_n)$, then $\phi[t/x] = R(t_1[t/x], ..., t_n[t/x])$.
3. If $\phi = \neg(\psi)$, then $\phi[t/x] = \neg(\psi[t/x])$.
4. If $\phi = \phi_1 \vee \phi_2$, then $\phi[t/x] = \phi_1[t/x] \vee \phi_2[t/x]$.
5. If $\phi = \exists x(\psi)$, then $\phi[t/x] = \exists x(\psi)$.
   If $\phi = \exists y(\psi)$ and $y \neq x$, then $\phi[t/x] = \exists y(\psi[t/x])$.

Definition 4.6. A term t is closed iff $t[y/x] = t$ for all variables y and x. A formula ϕ is closed iff $\phi[y/x] = \phi$ for all variables y and x. Closed formulas are also called *sentences*.

Let x and y be distinct variables and let ϕ be a formula. Then $\phi[y/x]$ equals ϕ except that some, but maybe not all, x's are replaces by y's. Any occurrence of x in ϕ that is still x in $\phi[y/x]$ is called *bound*; if it changes to y then the occurrence is called *free*. A variable is called *free in ϕ* if there is a free occurrence of this variable in ϕ. When ϕ is a formula, we denote by $\phi(x_1, ..., x_n)$ the same formula, but indicate that its free variables are among $x_1, ..., x_n$. When $c_1, .., c_n$ are constants and $\phi(x_1, ..., x_n)$ is a formula, then $\phi(c_1, ..., c_n) := \phi[c_1/x_1]...[c_n/x_n]$, that is for every i we replace all free occurrences of x_i by the constant c_i. Clearly a closed formula is precisely a formula which has no free variables and clearly $\phi(c_1, ..., c_n)$ is a closed formula.

Definition 4.7. Let L be a language. A *structure* for L is a set M together with *interpretations* for the constants, functions symbols and relation symbols of L. An interpretation of a constant is an element of M, an interpretation of a function symbol of arity n is a function $M^n \to M$ and an interpretation of a relation symbol of arity n is a subset of M^n. The *cardinality* of a structure is just the cardinality of the set M.

For a constant c, function symbol f and relation symbol R we now denote their respective interpretations in M as c^M, f^M, R^M. Later on, we no longer indicate the difference between a symbol and its interpretation. When M is a structure for L, we denote by L_M the language which is L together with a new constant symbol for each element of M. By interpreting such a constant symbol in L_M as itself, we get that M is also a structure for the language L_M.

When there are only 'few' language symbols, we may write $(M, c, d, ..., f, g, ..., R, S, ...)$ instead of just M to stipulate that M is a structure in this language, and not an other language. For instance we will encounter the language of set theory with one additional function symbol h. A structure for this language is denoted as $(M, h, \in)$, whereas a structure for just the language of set theory is simply denoted by M.

We extend the interpretation of constants, function symbols and relation symbols to all closed terms, by recursion:

1. If $t = c$, then t is closed and $t^M := c^M$.
2. If $t = x$, then t is not closed.
3. If $t = f(t_1, ..., t_n)$ and t is closed, then t_i is closed for all i, and their interpretation has already been defined by the induction hypothesis, so we may define $t^M := f^M(t_1^M, ..., t_n^M)$.

Now we define what it means for a structure to satisfy a sentence. This is done in two stages, first for L_M-sentences.

Definition 4.8. Let M be a structure for L. Now consider M as an L_M-structure. We recursively define when an L_M-sentence ϕ *is satisfied in* M, and write $M \vDash \phi$:

1. $M \vDash t_1 = t_2$ iff $t_1^M = t_2^M$.
2. $M \vDash R(t_1, ..., t_n)$ iff $(t_1^M, ..., t_n^M) \in R^M$.
3. $M \vDash \phi \vee \psi$ iff $M \vDash \phi$ or $M \vDash \psi$.
4. $M \vDash \neg(\phi)$ iff not $M \vDash \phi$, this is also denoted as $M \nvDash \phi$.
5. $M \vDash \exists x(\phi)$ iff there is some $m \in M$ such that $M \vDash \phi[m/x]$.

Note that if $\exists x(\phi)$ is closed, then $\phi[m/x]$ is closed as well. Note that $m \in M$ is just an element of M, whereas the m in $\phi[m/x]$ is in fact the sequence consisting of the constant m. When ϕ is satisfied in M, we also say that M *models* ϕ.

Definition 4.9. Let M be a structure for L. We recursively define when an L-sentence *is satisfied in* M, and write $M \vDash \phi$:

1. $M \vDash t_1 = t_2$ iff $t_1^M = t_2^M$.
2. $M \vDash R(t_1, ..., t_n)$ iff $(t_1^M, ..., t_n^M) \in R^M$.
3. $M \vDash \phi \vee \psi$ iff $M \vDash \phi$ or $M \vDash \psi$.
4. $M \vDash \neg(\phi)$ iff not $M \vDash \phi$, this is also denoted as $M \nvDash \phi$.
5. $M \vDash \exists x(\phi)$ iff there is some $m \in M$ such that $M \vDash \phi[m/x]$.

Now in item 5., $\phi[m/x]$ may not be an L-sentence any more. So for $M \vDash \phi[m/x]$ to make sense we must use Definition 4.8, and indeed here we mean M as an L_M-structure and $\phi[m/x]$ as an L_M-sentence. Again when ϕ is satisfied in M we say M *models* ϕ and write $M \vDash \phi$.

Since any L-sentence ϕ is also an L_M-sentence, the expression $M \vDash \phi$ is defined in two ways, namely M as an L_M-structure and ϕ as an L_M-sentence, or just M as an L-structure and ϕ as an L-sentence. Fortunately the definitions are equivalent; we can never have $M \vDash \phi$ in one interpretation of this notation and $M \nvDash \phi$ is the other.

We have the following abbreviations for formulas:

1. If t_1 and t_2 are terms, $t_1 \neq t_2 := \neg(t_1 = t_2)$.
2. If ϕ is a formula and x is a variable, then $\forall x(\phi) := \neg(\exists x(\neg(\phi)))$.

3. If ϕ and ψ are formulas, then
   (a) $(\phi) \wedge (\psi) := \neg((\neg(\phi)) \vee ((\neg(\psi)))$.
   (b) $(\phi) \to (\psi) := (\neg(\phi)) \vee (\psi)$.
   (c) $(\phi) \leftrightarrow (\psi) := ((\phi) \to (\psi)) \wedge ((\phi) \to (\psi))$.
4. If ϕ is a formula and x is a variable then $\exists! x(\phi) := \exists x((\phi) \wedge (\forall y((\phi[y/x]) \to (y = x)))$.

Specifically in the language of set theory we have the abbreviations: When t_1 and t_2 are terms, then

1. $t_1 \in t_2 := \in(t_1, t_2)$,
2. $t_1 \subseteq t_2 := \forall x((x \in t_1) \to (x \in t_2))$,
3. $t_1 \subset t_2 := (t_1 \subseteq t_2) \wedge (t_1 \neq t_2)$.

A *theory* is a set of sentences. A structure M *satisfies* a theory T or M *is a model of* T, denoted as $M \vDash T$, iff $M \vDash \phi$ for all $\phi \in T$. Given a structure M, we write $\mathrm{Th}(M)$ for the set of a sentences satisfied in M. Obviously $M \vDash \mathrm{Th}(M)$ for any L-structure M.

Definition 4.10. Let L be a language and let M be a structure for L. Let $X \subseteq M$. Then X is called *definable* iff there is a formula $\phi(x)$ such that

$$M \vDash \phi(m) \Leftrightarrow m \in X,$$

and we say ϕ *defines* X. An element $m \in M$ is definable iff $\{m\}$ is definable, and if ϕ defines $\{m\}$ then we also say ϕ defines m.

4.2 Elementary embeddings

As often when one defines mathematical objects, one can ask how two of these objects relate to each other. It turs out that in model theory, not *isomorphism* (identical up to names of elements), but *elementary equivalence* (satisfying the same sentences) is the best notion of sameness. We have the following definitions:

Definition 4.11. Let L be a language and let M and N be structures for L. Let $A : M \to N$ be a map. Then A is called a *morphism* iff

1. $A(c^M) = c^N$ for all constants c,
2. $A(f^M(m_1, ..., m_n)) = f^N(A(m_1), ..., A(m_n))$, for all $(m_1, ..., m_n) \in M^n$ and all function symbols f,
3. if $(m_1, ..., m_n) \in R^M$ then $(A(m_1), ..., A(m_n)) \in R^N$, for all $(m_1, ..., m_n) \in M^n$ and all relation symbols R.

Definition 4.12. Let L be a language, let M and N be structures for L and let $A : M \to N$ be a morphism. Then A is called an *embedding* iff

1. A is injective,
2. if $(A(m_1), ..., A(m_n)) \in R^N$ then $(m_1, ..., m_n) \in R^M$, for all $(m_1, ..., m_n) \in M^n$ and all relation symbols R.

When $M \subseteq N$ and the inclusion $i : M \to N$ is an embedding, we say that M is a *substructure* of N and N is an *extension* of M. Note that by a renaming of elements, any embedding can be regarded as an inclusion.

Definition 4.13. Let L be a language, let M and N be structures for L and let $A : M \to N$ be an embedding. Then A is called *elementary* iff for all L-formulas $\phi(x_1, ..., x_n)$ and all $m_1, .., m_n \in M$,

$$M \vDash \phi(m_1, ..., m_n) \Leftrightarrow N \vDash \phi(A(m_1), ..., A(m_n)).$$

When an inclusion $i : M \to N$ is elementary, we denote this as $M \prec N$ and say that M is an *elementary substructure* of N.

When a morphism $A : M \to N$ is bijective and its inverse A^{-1} is also a morphism, then A is called an *isomorphism* and M and N are called *isomorphic*. An isomorphism is always an elementary embedding, but not every elementary embedding is an isomorphism. As mentioned, isomorphisms are relatively unimportant compared to elementary embeddings.

Note that if $M \prec N$, then M contains all definable elements of N: For if $\phi(x)$ defines some $n \in N$, then $N \vDash \exists x(\phi(x) \wedge (\forall y(\phi[y/x] \to y = x)))$, hence $M \vDash \exists x(\phi(x) \wedge (\forall y(\phi[y/x] \to y = x)))$, hence $M \vDash \phi(m) \wedge (\forall y(\phi[y/x] \to y = m))$ for some $m \in M$, hence $N \vDash \phi(m) \wedge (\forall y(\phi[y/x] \to y = m))$, but $N \vDash \phi(n)$, so $m = n$ thus $n \in M$.

4.3 The theorems of Skolem and Löwenheim

The theorems of Skolem an Löwenheim express that every infinite structure M has elementary extensions of any cardinality larger than the cardinality of M and M has elementary substructures of any cardinality larger than or equal to the cardinality of the language. We state the theorems without proof. See for instance [10] for a thorough proof.

Theorem 4.14 (Upwards Skolem-Löwenheim)**.** Let L be a language and let M be an infinite structure for L. Let $\kappa > \max(|M|, |L|)$. Then there exists a structure N for L such that $M \prec N$ and $|N| = \kappa$.

Theorem 4.15 (Downwards Skolem-Löwenheim)**.** Let L be a language and let M be an infinite structure for L. Let $X \subseteq M$. Then there exists a structure N for L such that $X \subseteq N$, $N \prec M$ and $|N| \leq \max(|X|, |L|, \aleph_0)$.

So if M is a structure for L and κ is a cardinal such that $\max(\aleph_0, |L|) \leq \kappa \leq |M|$, let $X \subseteq M$ be of size κ. Then there is an structure N for L such than $X \subseteq N$, $N \prec M$ and $|N| \leq \max(|X|, |L|, \aleph_0) = \kappa$. So $|N| = \kappa$ and M has a substructure of size κ.

In fact a proof of the upwards theorem uses the downward theorem. A proof of the downwards theorem uses the Tarski-Vaught test for determining whether an embedding is elementary: When an existential sentence is true in N, it is already witnessed by an element in M.

Lemma 4.16 (Tarski-Vaught test)**.** Let L be a language, let M and N be models in L and let $A : M \to N$ be an embedding. Then A is elementary if and only if the following holds: For every L_M formula $\phi(x)$ such that $N \vDash \exists x(\phi)$, we have $N \vDash \phi(m)$ for some $m \in M$.

Note that in this definition we interpret the embedding $A : M \to N$ as an inclusion $i : M \to N$, so that the L_M-formulas $\exists x(\phi)$ and $\phi(m)$ are also L_N-formulas.

Using the Tarski-Vaught test, we can also prove the following lemma.

Lemma 4.17. Let L be a language, let M be a structure for L, let i be an ordinal let $\langle N_j : j < i\rangle$ be a $\subseteq$-chain of elementary substructures of M. Then $N := \bigcup_{j<i} N_j$ with natural interpretations is an L-structure and $N \prec M$ as well.

This lemma may also be seen al a consequence of the *elementary system lemma*, which we will not show here. This lemma will be used implicitly in for instance Section 6 and in the proof of Theorem 8.21.

4.4 $H(\kappa)$

Recall that a set x is *transitive* iff $y \in x$ implies $y \subseteq x$. Given x, recursively define $x_0 = x$, $x_{n+1} = \bigcup x_n$. Then $\mathrm{trcl}(x) := \bigcup_{n<\omega} x_n$, the *transitive closure* of x is the smallest transitive set containing x.

Definition 4.18. Let κ be a cardinal. We define $H(\kappa) := \{x : |\mathrm{trcl}(x)| < \kappa\}$ and consider $H(\kappa)$ as a model in the language of set theory.

If $|\mathrm{trcl}(x)| < \kappa$ then x is *hereditarily of cardinality less than κ*. $H(\kappa)$ is the set of sets hereditaliy of cardinality less then κ.

Example 4.19.

$$\begin{aligned} V_\omega = H(\aleph_0) &\models \mathrm{ZFC} - \mathrm{Inf}, \\ H(\theta) &\models \mathrm{ZFC} - \mathrm{P} \text{ for } \theta > \aleph_0, \\ V_\kappa = H(\kappa) &\models \mathrm{ZFC} \text{ for inaccessible } \kappa. \end{aligned}$$

In the language of set theory, we can express many properties of sets by a formula. For example we can express that

1. x is a subset of y,
2. x is transitive,
3. x is well-ordered by $\in$,
4. x is an ordinal,
5. x is a successor ordinal,
6. x is a limit ordinal,
7. f is a function from a to b,
8. f is a injection (surjection, bijection) from a to b,
9. x is a cardinal number,
10. x is a successor cardinal number,
11. x is a limit cardinal number.

Furthermore, all natural numbers are definable.

As an example of the strength and elegance of $H(\kappa)$ and model theory, we will prove the Pressing Down Lemma by using Downwards Skolem-Löwenheim.

Theorem 4.20. Let $f : \omega_1 \to \omega_1$ such that $f(\alpha) < \alpha$ for all $0 < \alpha < \omega_1$. Then there exists a stationary $S \subseteq \omega_1$ such that $f \restriction S$ is constant.

Proof. Note that $|\mathrm{trcl}(f)| < \aleph_2$ so $f \in H(\aleph_2)$. By Downwards Skolem-Löwenheim there exists an $M \prec H(\aleph_2)$ such that $f \in M$ and $|M| = \aleph_0$. Since ω and ω_1 are definable in $H(\aleph_2)$, they belong to M as well. Furthermore every $n < \omega$ is definable in $H(\aleph_2)$, so we have $\omega \subseteq M$. If $\alpha \in M \cap \omega_1$, the $H(\aleph_2)$ contains a surjection $\omega \twoheadrightarrow \alpha$, hence so does M. Since $\omega \subseteq M$, we obtain $\alpha \subseteq H(\aleph_2)$.

Thus $M \cap \omega_1$ is an initial segment of ω_1. Since $|M| = \aleph_0$, we have $\delta := M \cap \omega_1 < \omega_1$. Let $\epsilon := f(\delta)$. Then $\epsilon < \delta$ hence $\epsilon \in M$. Then $S := \{\alpha : f(\alpha) = \epsilon\} \in M$.[7] We will show that S is stationary.

Let $\text{club}(x, y)$ be the following formula:

$$\Big(\forall \alpha \in y\, \exists \beta \in x\, (\alpha \in \beta)\Big) \wedge \Bigg(\forall \alpha \in y\, \Big((\forall \beta \in \alpha\, \exists \gamma \in x\, (\beta \in \gamma \wedge \gamma \in \alpha)) \to \alpha \in x\Big)\Bigg)$$

The enlargement of the parentheses is just to ease the readability of the formula. Now $\text{club}(x, \omega_1)$ expresses that x is club in ω_1. Suppose $C \in M$ is such that $M \vDash \text{club}(C, \omega_1)$. Then $H(\aleph_2) \vDash \text{club}(C, \omega_1)$ hence C is club in ω_1. Now for all $\alpha \in (\omega_1 \cap M)$ exists $\beta \in C \cap M$ such that $\alpha \in \beta$. But $\omega_1 \cap M = \delta$ and $C \cap M = C \cap \delta$. So δ is a limit point of C. Since C is closed, we obtain $\delta \in C$. Therefore $H(\aleph_2) \vDash C \cap S \neq \emptyset$, so $M \vDash C \cap S \neq \emptyset$ as well. We have shown that for $C \in M$, if $M \vDash \text{club}(C, \omega_1)$ then $M \vDash C \cap S \neq \emptyset$. Therefore $M \vDash \forall C(\text{club}(C, \omega_1) \to C \cap S \neq \emptyset)$, hence $H(\aleph_2) \vDash \forall C(\text{club}(C, \omega_1) \to C \cap S \neq \emptyset)$. This exactly says that S is stationary. □

[7] This is a standard result: We have $H(\aleph_2) \vDash \exists! y(\alpha \in y \leftrightarrow (\alpha, \epsilon) \in f)$, namely $y = S$. Since $f, \epsilon \in M$, we must have $M \vDash \exists! y(\alpha \in y \leftrightarrow (\alpha, \epsilon) \in f)$, so there is a $T \in M$ such that $M \vDash (\alpha \in T \leftrightarrow (\alpha, \epsilon) \in f)$. Then $H(\theta) \vDash (\alpha \in T \leftrightarrow (\alpha, \epsilon) \in f)$, and thus $T = S$ and $S \in M$.

5 Jónsson algebras

In this section we use model theory and $H(\kappa)$ to investigate the existence of Jónsson algebras. Jónsson algebras are algebras which do not permit strict subalgebras of the same cardinality. In the proof of Theorem 5.10 we will use Theorem 3.28, a result on pcf-theory.

5.1 Definition and characterization

Definition 5.1. Let A be a set. An *algebra* on A is a sequence $\langle f_n : n < \omega \rangle$, where each f_n is a function $A^{a(n)} \to A$ for some $a(n) < \omega$. An algebra $(f_n)_{n<\omega}$ on A is called *Jónsson* iff there is no $B \subset A$ with $|B| = |A|$ such that $(f_n \restriction B)_{n<\omega}$ is an algebra on B, or equivalently iff there is no proper *subalgebra* of the same cardinality. A cardinal κ is called Jónsson iff there is a Jónsson algebra $(f_n)_{n<\omega}$ on some set A with $|A| = \kappa$, or equivalently iff there is a Jónsson algebra $\langle f_n : n < \omega \rangle$ on κ.

It is more common to say that the tuple $(A, (f_n)_{n<\omega})$, or even $(A, f_0, f_1, ...,)$, is a (Jónsson) algebra. We slightly deviate so that we can use the fact that '$(f_n)_{n<\omega}$ is a (Jónsson) algebra on κ' is expressable in the language of set theory, when κ is definable.

When we say that $(A, f_0, f_1, ..., f_k)$ is a (Jónsson) algebra we mean that, when choosing f_l for each $l \in (k, \omega)$ arbitrary, $(A, (f_n)_{n<\omega})$ is a (Jónsson) algebra.

Note that there is a Jónsson algebra on $\aleph_0$: Define $f_0(n) = n + 1$ and $f_1(n + 1) = n$ and $f_1(0) = 0$. Then $(\aleph_0, f_0, f_1)$ has no proper subalgebra of size $\aleph_0$.

We now turn to a model theoretic characterization of Jónsson algebras.

Theorem 5.2. Let κ be a cardinal. Then the following are equivalent:

1. κ is Jónsson.
2. For all regular $\theta \geq \kappa^+$ and all $M \prec H(\theta)$ we have: ($\kappa \in M$ and $|M \cap \kappa| = \kappa$) implies $\kappa \subseteq M$.
3. For all $M \prec H(\kappa^+)$ we have: $|M \cap \kappa| = \kappa$ implies $\kappa \subseteq M$.
4. For some regular $\theta \geq \kappa^+$ and all $M \prec H(\theta)$ we have: ($\kappa \in M$ and $|M \cap \kappa| = \kappa$) implies $\kappa \subseteq M$.

Recall that $H(\kappa)$ is the set of sets hereditarily of cardinality less than κ.

Proof. (1. $\Rightarrow$ 2.) Let $\theta \geq \kappa^+$ be regular, let $M \prec H(\theta)$, assume $\kappa \in M$ and $|M \cap \kappa| = \kappa$. Let $(\kappa, (f_n)_{n<\omega})$ be a Jónsson algebra. Then $(f_n)_{n<\omega} \in H(\theta)$. Being a Jónsson algebra on κ is expressable in the language of set theory, so there is a Jónsson algebra $(g_n)_{n<\omega}$ on κ in M as well. Then $(\kappa \cap M, (g_n : (\kappa \cap M)^{a(n)} \to (\kappa \cap M))_{n<\omega})$ is a subalgebra of $(\kappa, (g_n : \kappa^{a(n)} \to \kappa)_{n<\omega})$. But $(\kappa, (g_n)_{n<\omega})$ is a Jónsson algebra and $|\kappa \cap M| = \kappa$, so we have $\kappa \cap M = \kappa$, thus $\kappa \subseteq M$.

(2. $\Rightarrow$ 3.) Take $\theta = \kappa^+$. Then κ is definable in $H(\kappa^+)$, so we have $\kappa \in M$.

(3. $\Rightarrow$ 4.) Automatic.

(4. $\Rightarrow$ 1.) Let $\theta \geq \kappa^+$ be regular such that for all $M \prec H(\theta)$ we have ($\kappa \in M$ and $|M \cap \kappa| = \kappa$) implies $\kappa \subseteq M$. Using Downwards Skolem-Löwenheim, let $M \prec H(\theta)$ with $\kappa \in M$, $\kappa \subseteq M$ and with $|M| = \kappa$. Add a function symbol h to the language of set theory, and let h^M be such that $h^M \restriction \kappa$ is a bijection $\kappa \leftrightarrow M$. In fact we require that $h^M(0) = 0$, $h \restriction (\kappa \setminus \{0\})$ is a bijection $\kappa \setminus \{0\} \leftrightarrow M \setminus \{0\}$ and $h^M(a) = 0$ for all $a \in M \setminus \kappa$. Then $(M, h, \in)$ has no proper substructure of size $|M| = \kappa$:

Let $(N, h, \in) \prec (M, h, \in)$ with $N \subseteq M$ and $|N| = \kappa$ (see Figure 3). By elementarity, $h^N = h^M \upharpoonright N$ and also $\kappa \in N$, since κ is definable in M (it is the least non-zero ordinal α such that $h^M(\alpha) = 0$). We have $M \vDash \forall y\, \exists x(h(x) = y)$, so $N \vDash \forall y\, \exists x(h(x) = y)$. So $\kappa = |(h^N)^{-1}[N \setminus \{0\}]| = |(N \cap \kappa) \setminus \{0\}|$. Since $N \prec H(\theta)$, $\kappa \in N$ and $|N \cap \kappa| = \kappa$, we have $\kappa \subseteq N$ and thus $M = h^M[\kappa] = h^M[\kappa \cap N] = h^N[\kappa \cap N] \subseteq N$, so in fact $N = M$.

We will now define a Jónsson algebra on M. For any formula $\phi(x, x_1, ..., x_n)$ in the language $\{\emptyset, \{h\}, \{\in\}\}$ define its *Skolem-function*, the function $f_\phi : M^n \to M$ by sending $(m_1, ..., m_n) \in M^n$ to a witness of $\phi(x, m_1, ..., m_n)$ if $M \vDash \exists x \phi(x, m_1, ..., m_n)$ and to $0 \in M$ if $M \nvDash \exists x \phi(x, m_1, ..., m_n)$. Then $(M, (f_\phi)_{\phi \text{ a formula}})$ is a Jónsson algebra: If $N \subseteq M$ is a subalgebra of M such that $|N| = |M|$, then $(N, h, \in)$ is a substructure of $(M, h, \in)$, since h itself is also a Skolem-function (of the formula $x = h(y)$), thus indeed $h^M(n) \in N$ for $n \in N$. The embedding $(N, h, \in)$ in $(M, h, \in)$ passes the Tarski-Vaught test: If $(M, h, \in) \vDash \exists x \phi(x, n_1, ..., n_n)$ for $n_1, ..., n_n \in N$, then $(M, h, \in) \vDash \phi(f_\phi(n_1, ..., n_n), n_1, ..., n_n)$ and $f_\phi(n_1, ..., n_n) \in N$. So the embedding is elementary. But $(M, h, \in)$ has no proper substructures of size κ. So $N = M$, and $(M, (f_\phi)_{\phi \text{ a formula}})$ has no proper subalgebras of size $|M| = \kappa$. So κ is Jónsson. $\square$

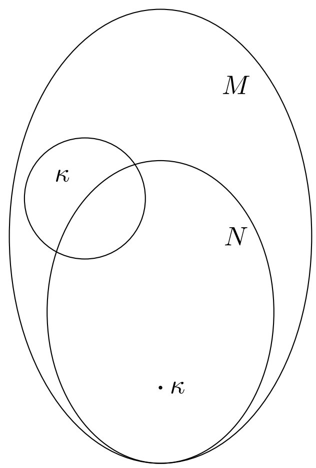

Figure 3: $(M, \in, h)$ with substructure $(N, \in, h)$, subset κ and element κ. If $|N| = \kappa$, then $N = M$.

5.2 Existence results

We use this characterization very often to investigate the existence of Jónsson algebras. For example, the existence of a Jónsson algebra on κ implies the existence of a Jónsson algebra on κ^+:

Theorem 5.3. If κ is Jónsson, then so is κ^+.

Proof. Let $M \prec H(\kappa^{++})$ and suppose $|M \cap \kappa^+| = \kappa^+$. Then there is some minimal $\alpha \in M \cap \kappa^+$ such that $|M \cap \alpha| = \kappa$, and for all $\alpha' \in M \cap \kappa^+$ with $\alpha \leq \alpha'$ we also have $|M \cap \alpha'| = \kappa$. For all such α', $H(\kappa^{++})$ contains a bijection $\kappa \leftrightarrow \alpha'$. Note that κ is definable in $H(\kappa^{++})$, thus $\kappa \in M$ as well. So M contains a bijection $\kappa \leftrightarrow \alpha'$. The bijection $\kappa \leftrightarrow \alpha$, together with $|M \cap \alpha| = \kappa$, ensures $|M \cap \kappa| = \kappa$. Since also $\kappa \in M$ and there is a Jónsson algebra on κ, we obtain $\kappa \subseteq M$. But then the bijection $\kappa \leftrightarrow \alpha'$ ensures $\alpha' \subseteq M$. Since there are cofinally many such $\alpha' < \kappa^+$, we obtain $\kappa^+ \subseteq M$. Hence, κ^+ has a Jónsson algebra. □

Since there is a Jónsson algebra on $\aleph_0$, by this theorem we thus have that $\aleph_n$ has a Jónsson algebra for each $n < \omega$.

Another sufficient condition for the existence of a Jónsson algebra is the existence of a particularly shaped stationary set.

Definition 5.4. Let κ be a cardinal, let S be stationary in κ. Then S is called

1. *reflecting at* α, for some $\alpha < \kappa$, iff $S \cap \alpha$ is stationary in α,
2. *reflecting* iff it is reflecting at α for some $\alpha < \kappa$,
3. *non-reflecting* iff it is not reflecting, i.e. when for all $\alpha < \kappa$, $S \cap \alpha$ is not stationy in α, i.e. when for all $\alpha < \kappa$ there exists $C_\alpha \subseteq \alpha$ closed unbounded in α such that $C_\alpha \cap S = C_\alpha \cap (S \cap \alpha) = \emptyset$.

Theorem 5.5. Let κ be a regular cardinal and suppose there is a non-reflecting stationary subset of κ. Then there is a Jónsson algebra on κ.

Proof. As we have shown there is a Jónsson algebra on $\aleph_0$, we assume κ is uncountable. Let $M \prec H(\kappa^+)$ and suppose $|M \cap \kappa| = \kappa$. Since $H(\kappa^+)$ contains a non-reflecting stationary subset of κ, so does M; let $E \in M$ be such. Let $C := \{\alpha < \kappa : \sup(M \cap \alpha) = \alpha\}$. Then C is closed unbounded in κ:

1. Let $\beta < \kappa$ and suppose $\sup(C \cap \beta) = \beta$. We have to show that $\beta \in C$, i.e. $\sup(M \cap \sup(C \cap \beta)) = \sup(C \cap \beta)$. Let $\alpha < \sup(C \cap \beta)$. Let $\gamma \in C \cap \beta$ such that $\alpha \leq \gamma < \sup(C \cap \beta)$. Since $\gamma \in C$ we have $\sup(M \cap \gamma) = \gamma$. So let $\delta \in M$ such that $\alpha \leq \delta < \gamma$. For an arbitrary $\alpha < \sup(C \cap \beta)$ we have found a $\delta \in M$ with $\alpha < \delta < \sup(C \cap \beta)$. Thus $\sup(M \cap \sup(C \cap \beta)) \geq \sup(C \cap \beta)$. Also $\sup(M \cap \sup(C \cap \beta)) \leq \sup(C \cap \beta)$ is evident.
2. Let $\beta < \kappa$. Then $|M \cap \beta| \leq |\beta| < \kappa$, so $|M \cap (\kappa \setminus \beta)| = \kappa$. For $n < \omega$, recursively define $\alpha_n := \min((M \cap (\kappa \setminus \beta) \setminus \{a_0, ..., a_{n-1}\})$ and $\alpha := \lim_{n \to \omega} \alpha_n < \kappa$; here we use that κ is regular and uncountable. Then $\sup(M \cap \alpha) = \alpha$, so $\alpha \in C$ by definition, and $\beta < \alpha$.

Now, $C \cap E \subseteq M$.

> If not, let $\alpha \in (C \cap E) \setminus M$. Since $|M \cap \kappa| = \kappa$ and $|M \cap \alpha| \leq |\alpha| < \kappa$, let $\gamma \in M$ be minimal with $\alpha < \gamma$; then $\gamma < \kappa$. Since $H(\kappa^+)$ contains a closed unbounded subset of γ disjoint from E, so does M; let $C_\gamma \subseteq \gamma$ be such. Note that $H(\kappa^+) \vDash \forall \beta \in \gamma \, \exists \beta' \in C_\gamma((\beta = \beta' \vee \beta \in \beta') \wedge \beta' \in \gamma)$. So this holds in M as well, meaning that $[\beta, \gamma) \cap C_\gamma \cap M$ is non-empty for any $\beta \in M \cap \gamma$. Since γ is minimal above α, such elements are $< \alpha$. So $C_\gamma \cap \alpha$ is unbounded in α. Since C_γ is closed, this means $\alpha \in C_\gamma$. But then $\alpha \in C_\gamma \cap E = \emptyset$, a contradiction.

The theorem of Solovay says that E is the disjoint union of κ many stationairy subsets. So $H(\kappa^+)$, and hence M, contains a sequence $(E_\alpha)_{\alpha < \kappa}$ with $E = \dot{\bigcup}_{\alpha < \kappa} E_\alpha$ and E_α stationairy, and automatically non-reflecting, for all $\alpha < \kappa$. Let $\alpha < \kappa$ be arbitrary. Certainly $C \cap E_\alpha \subseteq M$ and $C \cap E_\alpha \neq \emptyset$, so suppose $\gamma \in M \cap E_\alpha$. Since $M \vDash \forall \delta \in E \, \exists! \beta (\delta \in E_\beta)$, we must have that $\alpha \in M$ as well. So $\kappa \subseteq M$. Hence there is a Jónsson algebra on κ. □

Non-reflecting stationairy subsets exist at least on successors of regular cardinals, as we will show in Theorem 5.6. Recall from Lemma 2.25 that for regular cardinals κ and λ such that $\kappa \geq \lambda$, the set $S(\kappa, \lambda) := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \lambda\}$ is κ-stationary.

Theorem 5.6. If κ is regular, κ^+ has a non-reflecting stationairy subset.

Proof. Define $S := \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$. Then $S \subseteq \kappa^+$ is stationary. To prove that S is non-reflecting, let $\alpha < \kappa^+$. Let $\langle \alpha_i : i < \mathrm{cf}(\alpha)\rangle$ be increasing cofinal in α and such that $\alpha_i = \sup_{j<i} \alpha_j$ whenever i is a limit ordinal, and such that α_{i+1} is a successor ordinal for all i. [To obtain such a sequence, let $\langle \beta_i : i < \mathrm{cf}(\alpha)\rangle$ be increasing cofinal in α and let $\alpha_i = \sup_{j<i} \beta_j$ when i is a limit and let $\alpha_{i+1} = \beta_{i+1} + 1$ for all i.]. Then $C_\alpha := \{\alpha_i : i < \mathrm{cf}(\alpha)\}$ is closed unbounded in α. For any $i < \mathrm{cf}(\alpha)$, we have $\mathrm{cf}(\alpha_{i+1}) = 1$ and $\mathrm{cf}(\alpha_i) = \mathrm{cf}(\sup_{j<i} \alpha_j) \leq \mathrm{cf}(i) \leq i < \mathrm{cf}(\alpha) = \kappa$ when i is a limit. So $\alpha_i \notin S$ for all i, so $C_\alpha \cap S = \emptyset$. Since α was arbitrary, we have shown that S is non-reflecting. $\square$

Combining the previous two theorems, we obtain the following corollary:

Corollary 5.7. If κ is regular, then κ^+ is Jónsson.

Theorem 5.3 and Corollary 5.7 ensure that the successors of Jónsson or regular cardinals are Jónsson. The following theorem has a more specific assumption.

Theorem 5.8 (Erdős, Hajnal, Rado)**.** If $2^\kappa = \kappa^+$ then there is a Jónsson algebra on κ^+.

Proof. We have $|[\kappa^+]^\kappa| = (\kappa^+)^\kappa = (2^\kappa)^\kappa = \kappa^+$. Enumerate $[\kappa^+]^\kappa$ as $\langle S_\beta : \kappa \leq \beta < \kappa^+\rangle$ and such that $S_\beta \subseteq \beta$. (Exercise for the reader that this is possible). Fix $\alpha \in (\kappa, \kappa^+)$. Enumerate $[\kappa, \alpha) \times \alpha$ as $\langle (\beta_i, \nu_i) : i < \kappa\rangle$. For $i < \kappa$, recursively choose $\delta_i \in S_{\beta_i} \setminus \{\delta_j : j < i\}$. Let $f(\delta_i, \alpha) = \nu_i$. Note that $\delta_i \in S_{\beta_i} \subseteq \beta_i < \alpha$. Do this for all $\alpha \in (\kappa, \kappa^+)$. Then extend this to a function $f : (\kappa^+)^2 \to \kappa^+$. Then f satisfies

$$\forall \alpha \in (\kappa, \kappa^+)\, \forall \beta \in [\kappa, \alpha)\, \forall \nu < \alpha\, \exists \delta < \alpha (f(\delta, \alpha) = \nu \text{ and } \delta \in S_\beta).$$

Then f is a Jónsson algebra on κ^+: Suppose $A \subseteq \kappa^+$ is such that $|A| = \kappa^+$. We will show $f[A \times A] = \kappa^+$.

Let $\nu < \kappa^+$ be arbitrary. Let $\beta \in [\kappa, \kappa^+)$ be such that $S_\beta \subseteq A$. Let $\alpha \in (\kappa, \kappa^+) \cap A$ such that $\beta, \nu < \alpha$, this is possible since $|A| = \kappa^+$. Then there is some $\delta < \alpha$ with $\delta \in S_\beta$ and $f(\delta, \alpha) = \nu$. So $f[A \times A] \supseteq f[S_\beta \times A] \ni v$.

Thus if A is a subalgebra of (κ^+, f) with $|A| = \kappa^+$, then $A = \kappa^+$. So indeed (κ^+, f) is a Jónsson algebra. $\square$

Under some assumptions, we can also prove that the successor of a singular cardinal is Jónsson. These assumptions are satisfied for the successor of the first infinite singular cardinal.

Theorem 5.9. There is a Jónsson algebra on $\aleph_{\omega+1}$.

Proof. Let $\mu = \aleph_\omega$ and let $\theta = (2^\mu)^+$. Then θ is regular and $\mu^+ = \aleph_\omega^+ \leq 2^{\aleph_\omega} = 2^\mu < (2^\mu)^+ = \theta$ so $\mu^+ \in H(\theta)$. So let $M \prec H(\theta)$ and assume $\mu^+ \in M$ and $|M \cap \mu^+| = \mu^+$. We need to show that $\mu^+ \subseteq M$. Let $a := \{\aleph_n : n < \omega\}$. Note that $a \in H(\theta)$ and $\aleph_n \in H(\theta)$ for each $n < \omega$. Since $\aleph_0$ is definable and successor cardinals of definable cardinals are definable, each $\aleph_n$ is definable in $H(\theta)$, so we have $a \subseteq M$. Also a is definable: Every infinite cardinal below $(\aleph_\omega)^+$ distinct from $\aleph_\omega$ is in a and a consists entirely of such cardinals. Note that we use that $(\aleph_\omega)^+ \in M$ by assumption and thus also $\aleph_\omega \in M$. Every ultrafilter on a is an element of $H(\theta)$. There is an ultrafilter D on a such that $\mathrm{cf}(\prod(a/D)) = \aleph_\omega^+ = \mu^+$, since $\mathrm{pcf}(a)$ has a maximal element by Theorem 3.27 and by

Corollary 3.29 pcf(a) is in interval of regular cardinals. So there is also such an ultrafilter D in M and a sequence $\langle f_\beta : \beta < \mu^+\rangle \in M \cap \prod a$ which is increasing and cofinal in $\prod a/D$. We will show that for cofinally many $\alpha \in a$ we have $|M \cap \alpha| = \alpha$.

For suppose for all large α we have $\sup(M \cap \alpha) < \alpha$. Then for these large α let $g(\alpha) = \sup(M \cap \alpha)$ and extend g to an element in $\prod a$. Then $g/D \in \prod a/D$, so $g <_D f_\beta$ for some $\beta < \mu^+$, and since $|M \cap \mu^+| = \mu^+$ we may assume $\beta \in M$. Then there is a large $\alpha \in a$ such that $g(\alpha) < f_\beta(\alpha)$. But $f_\beta(\alpha) \in M \cap \alpha$, contradicting $g(\alpha) = \sup(M \cap \alpha)$.

So for cofinally many $\alpha \in a$ we have $|M \cap \alpha| = \alpha$. Since for all $\alpha \in a$ there is a Jónsson algebra on α, we have $\alpha \subseteq M$ for cofinally many $\alpha \in a$. Hence $\mu \subseteq M$. For all $\xi \in M \cap [\mu, \mu^+)$ there is a bijection $\mu \leftrightarrow \xi$ in $H(\theta)$, hence in M. Since $\mu \subseteq M$ we obtain $\xi \subseteq M$. Since $|M \cap \mu^+| = \mu^+$, there are cofinally many such ξ, hence $\mu^+ \subseteq M$. Hence there is a Jónsson algebra on $\aleph_{\omega+1}$. □

As promised we can generalize this result.

Theorem 5.10. Let μ be a singular cardinal and suppose there exists a $\kappa < \mu$ such that every regular $\nu \in (\kappa, \mu)$ is Jónsson. Then μ^+ is Jónsson.

Proof. Define

$$\lambda := \min(\{\kappa : \kappa \geq \mathrm{cf}(\mu) \text{ and if } \nu \in (\kappa, \mu) \text{ is regular then } v \text{ is Jónsson}\}.$$

Let $\theta := (2^\mu)^+$. For ϕ a formula in the language of set theory, let f_ϕ be a Skolem-function for ϕ in $H(\theta)$. Let $F = \{F_i : i < \omega\}$ be the set of all compositions of the Skolem-formulas f_ϕ, that is F is the smallest set such that $\{f_\phi : \phi \text{ a formula}\} \subseteq F$ and if $f \in F$ is of arity n, and $f_1, ..., f_n \in F$, then $f(f_1, ..., f_n) \in F$ as well. Here $f(f_1, ..., f_n)$ denotes the function

$$(\alpha_{1,1}, ..., \alpha_{1,m_1}, ..., \alpha_{n,1}, ..., \alpha_{n,m_n}) \mapsto f(f_1(\alpha_{1,1}, ..., \alpha_{1,m_1}), ..., f_n(\alpha_{n,1}, ..., \alpha_{n,m_n})).$$

Let $M \prec (H(\theta), \in, F_0, F_1, F_2, ...)$, where the F_i's are function symbols and their interpretation is defined recursively, starting with the standard interpretation of the Skolem-functions. Since $\mu^+ \leq 2^\mu < (2^\mu)^+ = \theta$, we have $\mu \in H(\theta)$. Assume $\mu^+ \in M$ and $|M \cap \mu^+| = \mu^+$. We need to show that $\mu^+ \subseteq M$. Since $\mu^+ \in M$, we have $\mu, \mathrm{cf}(\mu) \in M$ and some increasing μ-cofinal seqeunce $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle \in M$ consisting of regular cardinals. Consider $a = \{\mu_i : i < \mathrm{cf}(\mu)\} \in M$. Let $D \in M$ be an ultrafilter on a containing all the tails of $\langle \mu_i : i < \mathrm{cf}(\mu)\rangle$. Then $\lim_D a = \mu$. Furthermore we have $\mathrm{cf}(\prod a/D) \geq \mu_i$ for all $i < \mathrm{cf}(\mu)$, hence $\mathrm{cf}(\prod a/D) \geq \mu$, hence $\mathrm{cf}(\prod a/D) > \mu$ since μ is singular. So either $\mu^+ = \mathrm{cf}(\prod a/D)$ or $\mu < \mu^+ < \mathrm{cf}(\prod a/D)$. Using Theorem 3.28, let $a' \subseteq \sup(a) = \mu$ be a set of regular cardinals such that $|a'| \leq |a| = \mathrm{cf}(\mu)$, let D' be an ultrafilter on a' such that $\lim_{D'} a' = \mu$ and $\mathrm{cf}(\prod a'/D') = \mu^+$. By elementarity, choose $a', D' \in M$. Note that since $\lim_{D'} a' = \mu$, we have $|a'| = \mathrm{cf}(\mu)$. We arrange that $|a'|, \lambda < \min(a')$:

We already have ensured that $|a'| \leq \mathrm{cf}(\mu) \leq \lambda$, so we only need $\lambda < \min(a')$. Since $\lim_{D'} a' = \mu$ we have $a'' = \{\alpha \in a' : \lambda < \alpha \leq \mu\} \in D'$. Since $\lambda \in M$, $a'' \in M$. Let D'' be the restriction of D' to a'', then $D'' \in M$. Then $\mathrm{cf}(\prod a''/D'') = \mu^+$ and $\lim_{D''} a'' = \mu$ and $|a''|, \lambda < \min(a'')$. So without loss of generality we could have assumed a' and D' to satisfy $|a'|, \lambda < \min(a')$.

Let $\langle f_\beta/D : \beta < \mu^+\rangle \in M$ be increasing and cofinal in $\prod a'/D'$. Define $A = \{\alpha \in M \cap a' : \sup(M \cap \alpha) = \alpha\}$.

Claim 5.11. A is cofinal in μ.

The proof of this claim follows below. So $\sup(A) = \mu$. For any $\alpha \in A$ with $\alpha > \lambda$ we have $\alpha \in M$ and $|M \cap \alpha| = \alpha$, and α is regular and $\lambda < \alpha < \mu$ so α is Jónsson, so $\alpha \subseteq M$. Hence $\mu \subseteq M$. Again there are bijections $\mu \leftrightarrow \xi$ in $H(\theta)$ for all $\xi \in [\mu, \mu^+)$, thus in M for all $\xi \in [\mu, \mu^+) \cap M$. Combined with the assumption $|M \cap \mu^+| = \mu^+$, we obtain $\mu^+ \subseteq M$. Thus μ^+ is Jónsson. $\square$

Proof of Claim 5.11. Suppose not, let $\sup^+(A) =: \mu' < \mu$, where $\sup^+(A)$ is the strict supremum of A. Let M' be the closure of $M \cup a'$ under the F_i's. Since the F_i's are closed under composition, we have $M' = \bigcup_{i<\omega} F_i[M \cup a']$. (Of course, F_i may be of arity $n \neq 1$, but then $F_i[M \cup a']$ is supposed to mean $\{F_i(\bar{a}) : \bar{a} \in (M \cup a')^n.\}$ If $\alpha \in a' \cap M \cap [\mu', \mu)$, then $\sup(M \cap \alpha) < \alpha$. For any $\alpha \in a'$ and any finite $S \subseteq M$, since $|a'| < \min(a') \leq \alpha$ and α is regular, we have that $F_i[S \cup a'] \cap \alpha$ is bounded by some $\alpha_{S,i} < \alpha$ and by elementarity we may assume $\alpha_{S,i} \in M$. Also each $x \in M'$ is in $F_i[S \cup a']$ for some finite $S \subseteq M$ and some $i < \omega$. Hence $\sup(M' \cap \alpha) \leq \sup(\alpha_{S,i} : S \subseteq M \text{ finite}, i < \omega) \leq \sup(M \cap \alpha)$, for any $\alpha \in a'$. Thus if $\alpha \in a' \cap M \cap [\mu', \mu)$, then $\sup(M' \cap \alpha) < \alpha$.

Claim 5.12. There is some $\mu'' \in [\mu', \mu)$ such that any $\beta \in a' \cap [\mu'', \mu)$ satisfies $\sup(M' \cap \beta) < \beta$.

Proof. Suppose not, then for any $\mu'' \in [\mu', \mu)$ there is a $\beta \in a' \cap [\mu'', \mu)$ with $\sup(M' \cap \beta) = \beta$. Since $\lambda < \min(a')$, $\beta \in a'$ is regular and $a' \subseteq \mu$, we have $\lambda < \beta < \mu$ and thus there is a Jónsson algebra on β. Since $M' \prec H(\theta)$, $|M' \cap \beta| = \beta$ and $\beta \in M'$, we obtain $\beta \subseteq M'$. So $\mu \subseteq M'$. Now $M \cap a' \subseteq \mu'$:

If there exists $\alpha \in a' \cap M \cap [\mu', \mu)$, then $\alpha \subseteq \mu \subseteq M'$, then $\sup(M' \cap \alpha) = \sup \alpha = \alpha$ and on the other hand $\sup(M' \cap \alpha) < \alpha$ as shown above. Contradiction, so $M \cap a' \subseteq \mu'$.

Recall that $a' \in M$. Since $H(\theta) \vDash \exists x(x \in a')$, we have $M \vDash \exists x(x \in a')$, so $M \cap a' \neq \emptyset$. So $\mu' \neq 0$ and thus $A \neq \emptyset$. Let $\alpha \in A$, i.e. $\alpha \in M \cap a'$ and $\sup(M \cap \alpha) = \alpha$. Notice that $\lambda < \alpha < \mu$ and α is regular, so α is Jónsson. Also $|M \cap \alpha| = \alpha$ and $\alpha \in M$. Therefore $\alpha \subseteq M$. We have $\mathrm{cf}(\mu) = |a'| < \min(a') \leq \alpha \subseteq M$. It follows that $\mathrm{cf}(\mu) \subseteq M$. Since we have a cofinal sequence $\langle \mu_i : i < \mathrm{cf}(\mu) \rangle \in M$, we obtain that $M \cap \mu$ is cofinal in μ. Since $H(\theta) \vDash \forall \beta \in \mu \, \exists \alpha \in a'(\beta \in \alpha \wedge \alpha \in \mu)$, by elementarity we obtain that $M \cap a'$ is cofinal in μ. But since we assumed that Claim 5.12 does not hold, $A = \{\alpha \in M \cap a' : \sup(M \cap \alpha) = \alpha\}$ is cofinal in $M \cap \alpha$. So $\sup(A) = \mu$, contradicting that $\sup^+(A) = \mu' < \mu$. $\square$

We conclude that for all large $\alpha \in a'$ we have $g(\alpha) := \sup(M' \cap \alpha) < \alpha$. Extend g so that $g \in \prod a'$. Then $g/D' \in \prod a'/D'$ and for some $\beta < \mu^+$ we have $g/D' < f_\beta/D'$ and since $|M \cap \mu^+| = \mu^+$ we can choose $\beta \in M \subseteq M'$. In particular, there is some large $\alpha \in a' \subseteq M'$ for which $g(\alpha) < f_\beta(\alpha)$. But $f_\beta(\alpha) \in M' \cap \alpha$ and $g(\alpha) = \sup(M' \cap \alpha)$, contradiction. $\square$

6 Pcf-theory applied to cardinal arithmetic

Recall Theorem 3.27, which says that $\mathrm{pcf}(a)$ has a maximal element. In this section we will find conditions under which $|\prod a| = \max \mathrm{pcf}(a)$. Note that $|\prod a|$ may be an interesting cardinal exponentiation, for instance $|\prod a| = \prod_{1<n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}$ for $a = \{\aleph_n : 1 < n < \omega\}$. Also using that $|\mathrm{pcf}(a)| \leq 2^{|a|}$, we find non-trivial results on cardinal exponentiation.

6.1 Calculation of $\max(\mathbf{pcf}(a))$

In this subsection we will prove the following theorem.

Theorem 6.1. Let $A = [\min(A), \sup(A))$ be an interval of regular cardinals. Suppose that $(\min(A))^{|A|} \leq \sup(A)$. Then $|\prod A| = \max \mathrm{pcf}(A)$.

To prove this theorem, we use a weakened version of it.

Theorem 6.2. Let $A = [\min(A), \sup(A))$ be an interval of regular cardinals such that $2^{|A|} < \min(A)$ and $(\min A)^{|A|} \leq \sup A$. Then $|\prod A| = \max \mathrm{pcf}(A)$.

Proof of Theorem 6.1 assuming Theorem 6.2. Let

$$A_0 := [\min A, (\min A)^{|A|}] \quad ; \quad A_1 := ((\min A)^{|A|}, \sup A).$$

Then $\prod A_0 \leq ((\min A)^{|A|})^{|A_0|} = (\min A)^{|A|} < \min A_1$, so $|\prod A| = |\prod A_0| \cdot |\prod A_1| = |\prod A_1|$. Also $2^{|A_1|} \leq 2^{|A|} \leq (\min A)^{|A|} < \min(A_1)$ and by Hausdorff's formula (1) we have

$$\begin{aligned}(\min(A_1))^{|A_1|} = (((\min A)^{|A|})^+)^{|A_1|} &= (\min(A)^{|A|})^+ \cdot (\min(A)^{|A|})^{|A_1|} \\ &= \min(A_1) \cdot \min(A)^{|A|} = \min(A_1) \leq \sup(A_1).\end{aligned}$$

So Theorem 6.2 applies and yields $|\prod A_1| = \max \mathrm{pcf}(A_1)$. So $\max \mathrm{pcf}(A) \leq |\prod A| = |\prod A_1| = \max \mathrm{pcf}(A_1) \leq \max \mathrm{pcf}(A)$. □

So we focus on proving Theorem 6.2. For this we use model theory and $H(\kappa)$, introduced in section 4.

Let A be an interval of regular cardinals such that $2^{|A|} < \min(A)$ and $(\min A)^{|A|} \leq \sup A$. Let θ be a large enough and regular cardinal. Recall that $H(\kappa)$ is the set of sets hereditarily of cardinality less than κ. We consider the structure $H(\theta)^* = (H(\theta), \in, <^*)$ where $<^*$ is a binary relation symbol and its interpretation in $H(\theta)$ is a well-order on $H(\theta)$.

Definition 6.3. Let $N \prec H(\theta)^*$. Then N is called *nice* iff

1. $|N| = \min(A)$,
2. $A \in N$,
3. $\min(A) \subseteq N$,
4. There is a sequence $\langle N_i : i < |A|^+ \rangle$ such that
   (a) $N_i \prec H(\theta)^*$ for all i (the sequence is elementary),
   (b) $N_i \subseteq N_j$ whenever $i < j$ (the sequence is a $\subseteq$-chain),
   (c) $N_i = \bigcup_{j<i} N_j$ whenever i is a limit (the sequence is continuous),

(d) $N = \bigcup_{i<|A|^+} N_i$,
(e) $\langle N_j : j < i \rangle \in N$ for all i.

For any $N \prec H(\theta)^*$, we define its *characteristic function* by

$$\chi_N : A \to \mathrm{Ord}, \\ \alpha \mapsto \chi_N(\alpha) := \sup(N \cap \alpha)$$

By $\mathcal{N}$ we denote the set of all nice elementary substructures of $H(\theta)^*$. On $\mathcal{N}$, we define the equivalence relation $\sim$ given by

$$N \sim M \quad \Leftrightarrow \quad \chi_N = \chi_M.$$

We now investigate some properties of nice elementary substructures N. Since $A \in N$, we have $\min A, \mathrm{pcf}(A) \in N$. Since $\min A > 2^{|A|} \geq |\mathrm{pcf}(A)|$, we have that $H(\theta)^*$, and thus N, contains a surjection $\min A \twoheadrightarrow \mathrm{pcf}(A)$. Since $\min(A) \subseteq N$ by assumption, we have $\mathrm{pcf}(A) \subseteq N$. Since $A \subseteq \mathrm{pcf}(A)$, we also have $A \subseteq N$. Since $|A|^+ \leq 2^{|A|} < \min(A)$, we have $|A|^+ \subseteq N$ as well. Hence also $N_i \in N_{i+1}$ for all i.

Since the map $\lambda \mapsto J_{<\lambda}(A)$ is definable in $H(\theta)^*$, it is in each nice N. Let $\langle b_\lambda : \lambda \leq \max \mathrm{pcf}(A) \rangle \in H(\theta)^*$ be the $<^*$-least element of $H(\theta)^*$ such that b_λ generates $J_{<\lambda^+}(A)$ over $J_{<\lambda}(A)$ for all $\lambda < \max \mathrm{pcf}(A)$ and $b_{\max \mathrm{pcf}(A)} = a$. Then $\langle b_\lambda : \lambda \leq \max \mathrm{pcf}(A) \rangle \in N$ as well. It follows that for $\lambda \in \mathrm{pcf}(A)$, since $\mathrm{pcf}(A) \subseteq N$, we have $J_{<\lambda}(A), b_\lambda \in N$.

The following three lemmas form the proof of Theorem 6.2.

Lemma 6.4. Every $x \in H(\theta)^*$ is contained in some nice N. Hence every $f \in \prod A$ is contained in some nice N.

Lemma 6.5. Given some nice N, there are at most $(\max \mathrm{pcf}(A))$-many $f \in \prod A$ for which there exists a nice M such that $M \sim N$ and $f \in M$.

Lemma 6.6. There are at most $(\max \mathrm{pcf}(A))$-many equivalence classes in $\mathcal{N}$.

Proof of Theorem 6.2. By the above lemmas, we have

$$\begin{aligned} |\prod A| &\leq |\mathcal{N}/{\sim}| \cdot \sup(|\{f : (\exists M : M \sim N \text{ and } f \in M\}| : N \in \mathcal{N}) \\ &\leq \max(\mathrm{pcf}(A)) \cdot \max(\mathrm{pcf}(A)) = \max(\mathrm{pcf}(A)). \end{aligned}$$

The other direction is trivial. □

Proof of Lemma 6.4. Let $x \in H(\theta)^*$. By Downwards Skolem-Löwenheim, let N_0 be such that $\{x, A\} \cup \min(A) \subseteq N_0 \prec H(\theta)^*$ with $|N_0| = \min(A)$. Given N_i, use Downwards Skolem-Löwenheim to find N_{i+1} such that $N_i \cup \{\langle N_j : j < i \rangle\} \subseteq N_{i+1} \prec H(\theta)^*$ with $|N_{i+1}| = \min(A)$. Let $N_i = \bigcup_{j<i} N_j$ when i is a limit. Since $|A|^+ < \min(A)$ and $\min(A)$ is regular, we find a continuous elementary $\subseteq$-chain $\langle N_i : i < |A|^+ \rangle$ and $N := \bigcup_{i<|A|^+}$ is a nice elementary substructure of $H(\theta)^*$ which contains x. □

Proof of Lemma 6.5. Let some nice N be given. Suppose $f \in \prod A$, M nice and $f \in M$ and $M \sim N$. Since $f \in M$, $f : A \to \sup A$ and $A \subseteq M$, it follows that $f \in (M \cap \sup A)^A$. By the lemma below, $M \cap \sup A = N \cap \sup A$, hence $f \in (N \cap \sup A)^A$. Since

$$|(N \cap \sup A)^A| = |N \cap \sup A|^{|A|} \leq |N|^{|A|} \leq \min(A)^{|A|} \leq \sup A \leq \max \mathrm{pcf}(A),$$

there are at most $\max \mathrm{pcf}(A)$ such f's. □

Lemma 6.7. Let M and N be nice and such that $M \sim N$. Then $M \cap \sup A = N \cap \sup A$.

Proof. We will prove the lemma by proving that $M \cap \lambda = N \cap \lambda$ for all cardinals λ such that $\min A \leq \lambda \leq \sup A$. We prove this by induction.

1. Base case: $M \cap \min A = \min A = N \cap \min A$.
2. Successor case: Suppose $M \cap \lambda = N \cap \lambda$. We need to prove $M \cap \lambda^+ = N \cap \lambda^+$. We have $\lambda^+ \in A \subseteq M, N$. Since $\langle \sup(N_i \cap \lambda^+) : i < |A|^+ \rangle$ is increasing and cofinal in $\sup(N \cap \lambda^+)$, we have $\mathrm{cf}(\sup(N \cap \lambda^+)) = |A|^+$ and $E_N := \{\sup(N_i \cap \lambda^+) : i < |A|^+\} \subseteq N$. Similarly define E_M. Then $E := E_M \cap E_N \subseteq N \cap M$ is cofinal in $\sup(N \cap \lambda^+) = \chi_N(\lambda^+) = \chi_M(\lambda^+) = \sup(M \cap \lambda^+)$. So $N \cap M$ is cofinal in $N \cap \lambda^+$ and $M \cap \lambda^+$. Given $\alpha \in N \cap M$, $\alpha < \lambda^+$, there exists $f : \lambda \twoheadrightarrow \alpha$ and the $<^*$-smallest such surjection is in both M and N. Since $M \cap \lambda = N \cap \lambda$, we find $N \cap \alpha = M \cap \alpha$. Since α was arbitrary and $N \cap M$ is cofinal, $M \cap \lambda^+ = N \cap \lambda^+$ follows.
3. Limit case: $M \cap \lambda = \bigcup_{\mu < \lambda} M \cap \mu = \bigcup_{\mu < \lambda} N \cap \mu = M \cap \lambda$.

We conclude that $M \cap \lambda = N \cap \lambda$ for all cardinals λ such that $\min A \leq \lambda \leq \sup A$. □

Proof of Lemma 6.6. Let N be nice. For each $\lambda \in \mathrm{pcf}(A)$, let $\langle f_i^\lambda : i < \lambda \rangle$ be the $<^*$-least sequence such that

(A) $\langle f_i^\lambda \restriction b_\lambda : i < \lambda \rangle$ is increasing and cofinal in $\prod b_\lambda / J_{<\lambda}$,
(B) When $\mathrm{cf}(i) = |A|^+$, then

$$f_i^\lambda(\alpha) = \min(\sup(f_j^\lambda(\alpha) : j \in C) : C \text{ is } i\text{-club and } |C| = \mathrm{cf}(i))$$

for all $\alpha \in A$.

Such a sequence exists by Lemma 6.8. Note that this sequence is definable in $H(\theta)^*$ since $b_\lambda, J_{<\lambda} \in N$. We define a sequence $\langle (\lambda_m, \rho_m, A_m) : m \leq n \rangle$ such that

1. $\lambda_m \in \mathrm{pcf}(A)$,
2. $\rho_m < \lambda_m$,
3. $A_m \subseteq A$,
4. $A_m \in J_{<\lambda_m}$,
5. $\langle \lambda_m : m \leq n \rangle$ is decreasing,

as follows, by recursion:

1. $\lambda_0 := \max \mathrm{pcf}(A)$, $\rho_0 := \sup(N \cap \lambda_0)$ and $A_0 := [f_{\rho_0}^{\lambda_0} < \chi_N]$. Clearly $\lambda_0 \in \mathrm{pcf}(A)$, $\rho_0 < \lambda_0$, $A_0 \subseteq A$ and $A_0 = A_0 \cap b_{\lambda_0} \in J_{<\lambda_0}$ by Lemma 6.9.
2. Given (λ_m, ρ_m, A_m) and provided that $A_m \neq \emptyset$ and $A_m \neq \{\min A\}$, define λ_{m+1} such that $A_m \in J_{<\lambda_{m+1}^+} \setminus J_{<\lambda_{m+1}}$. Then $\lambda_{m+1} \in \mathrm{pcf}(A)$, $\lambda_{m+1} \neq \min A$ and since $A_m \in J_{<\lambda_m}$ but $A_m \notin J_{\lambda_{m+1}}$, we have $\lambda_{m+1} < \lambda_m$. Define $\rho_{m+1} := \sup(N \cap \lambda_{m+1})$ and $A_{m+1} = [f_{\rho_{m+1}}^{\lambda_{m+1}} < \chi_N] \cap A_m$. Clearly $\rho_{m+1} < \lambda_{m+1}$ and $A_{m+1} \subseteq A$. By Lemma 6.9, $[f_{\rho_{m+1}}^{\lambda_{m+1}} < \chi_N] \cap b_{\lambda_{m+1}} \in J_{<\lambda_{m+1}}$. Also $A_m \subseteq_{J_{<\lambda_{m+1}}} b_{\lambda_{m+1}}$. Therefore $A_{m+1} \in J_{<\lambda_{m+1}}$.

Since $\langle \lambda_m \rangle$ is stricty decreasing, it must stop at some finite n, and there $A_n = \emptyset$ or $A_n = \{\min A\}$. By Lemma 6.9, $f^{\lambda_m}_{\rho_m} \leq \chi_N$ for all $m \leq n$. By definition of the A_m's, we have $\chi_N \restriction (A \setminus A_0) = f^{\lambda_0}_{\rho_0} \restriction (A \setminus A_0)$, $\chi_N \restriction (A_0 \setminus A_1) = f^{\lambda_1}_{\rho_1} \restriction (A_0 \setminus A_1), ..., \chi_N \restriction (A_{n-1} \setminus A_n) = f^{\lambda_n}_{\rho_n} \restriction (A_{n-1} \setminus A_n)$. Therefore $\chi_N \restriction (A \setminus A_n) = \max(f^{\lambda_0}_{\rho_0}, f^{\lambda_1}_{\rho_1}, ..., f^{\lambda_n}_{\rho_n}) \restriction (A \setminus A_n)$. Since $A_n = \emptyset$ or $A_n = \{\min A\}$, and $\chi_N(\min A) = \min A$, we have that χ_N is completely determined by $\{f^{\lambda_0}_{\rho_0}, f^{\lambda_1}_{\rho_1}, ..., f^{\lambda_n}_{\rho_n}\}$.
Therefore

$$\begin{aligned} |\mathcal{N}/{\sim}| = |\{\chi_N : N \in \mathcal{N}\}| &\leq |\{\text{finite subsets of } \{(\rho, \lambda) : \rho < \lambda \in \text{pcf}(A)\}\}| \\ &= |\{(\rho, \lambda) : \rho < \lambda \in \text{pcf}(A)\}\}| \\ &\leq \sum_{\lambda \in \text{pcf}(A)} \max \text{pcf}(A) \leq 2^{|A|} \cdot \max \text{pcf}(A) \leq \max \text{pcf}(A). \end{aligned}$$

So indeed there are at most $(\max \text{pcf}(A))$-many equivalence classes of nice elementary substructures of $H(\theta)^*$. □

Lemma 6.8. Let $\lambda \in \text{pcf}(A)$. Then there exists a sequence $\langle f_i : i < \lambda \rangle$ which satisfies (A) and (B).

Proof. By Theorem 3.50, $\text{tcf}(\prod b_\lambda / J_{<\lambda}) = \lambda$. So let $\langle g_i : i < \lambda \rangle$ in $\prod A$ such that $\langle g_i \restriction b_\lambda : i < \lambda \rangle$ is increasing and cofinal in $\prod b_\lambda / J_{<\lambda}$. For $i < \lambda$, recursively define $f_i \in \prod A$ as follows:

1. $f_0 = g_0$.
2. Given f_i, let $f_{i+1} = f_i + 1$. That is, $f_{i+1}(\alpha) = f_i(\alpha) + 1$ for all $\alpha \in A$.
3. Suppose i is limit.
   (a) If $\text{cf}(i) \neq |A|^+$, consider $\{f_j \restriction b_\lambda : j < i\} \cup \{g_i \restriction b_\lambda\}$. This set has a strict upperbound in $\prod b_\lambda / J_{<\lambda}$, extend this to a function f_i defined on A.
   (b) If $\text{cf}(i) = |A|^+$, let

$$f_i(\alpha) = \min(\sup(f_j(\alpha) : j \in C) : C \text{ club in } i \text{ and } |C| = \text{cf}(i)).$$

Since $|C| = \text{cf}(i) = |A|^+ < \min A$ for all C and $f_j(\alpha) < \alpha$ for all α, we obtain $f_i \in \prod A$.

We now prove that $\langle f_i : i < \lambda \rangle$ satisfies (A) and (B). (B) is trivial by definition of $\langle f_i : i < \lambda \rangle$. We have

1. Only if $\text{cf}(i) = |A|^+$ it is not immediately clear that $f_j <_{J_{<\lambda}} f_i$ for all $j < i$. So let $\text{cf}(i) = |A|^+$. For each $\alpha \in A$, let C_α be i-club such that $|C| = |A|^+$ and $f_i(\alpha) = \sup(f^\lambda_j : j \in C_\alpha)$. Then $C := \bigcap_{\alpha \in A} C_\alpha$ is an intersection of fewer than $|i|$-many i-clubsets, hence C is an i-clubset and $f_i(\alpha) = \sup(f_j(\alpha) : j \in C)$. So if $j \in C$, then $f_j \leq f_i$. Since C is cofinal in i and $\langle f_j \restriction b_\lambda : j < i\}$ is $<_{J_{<\lambda}}$-increasing (by the induction hypothesis), we obtain $f_j \restriction b_\lambda <_{J_{<\lambda}} f_i \restriction b_\lambda$ for all $j < i$.
2. There are cofinaly many i's in λ such that $\text{cf}(i) \neq |A|^+$.

So (A) is also satisfied. □

Lemma 6.9. Let $\lambda \in \text{pcf}(A) \setminus \min(A)$ and let $\rho := \sup(N \cap \lambda)$. Let $\langle f^\lambda_i : i < \lambda \rangle$ be the $<^*$-least sequence satisfying (A) and (B). Then $f^\lambda_\rho \leq \chi_N$ and $[f^\lambda_\rho < \chi_N] \cap b_\lambda \in J_{<\lambda}$.

Proof. Since $\langle \sup(N_i \cap \lambda) : i < |A|^+ \rangle$ is increasing and cofinal in $\sup(N \cap \lambda) = \rho$, we have $\text{cf}(\rho) = |A|^+$. Notice that $\{\sup(N_i \cap \lambda) : i < |A|^+\}$ is ρ-club. For each $\alpha \in A$, let C_α be ρ-club such that $|C| = |A|^+$ and $f^\lambda_\rho(\alpha) = \sup(f^\lambda_i : i \in C_\alpha)$. Then $C := \bigcap_{\alpha \in A} C_\alpha$ is an intersection of fewer than $|\rho|$-many ρ-clubsets, hence C is a ρ-clubset and $f_\rho^\lambda(\alpha) = \sup(f_i^\lambda(\alpha) : i \in C)$ for all α. Now $C' := C \cap \{\sup(N_i \cap \lambda) : i < |A|^+\}$ is again ρ-club, lies entirely in N and $f_\rho^\lambda(\alpha) = \sup(f_i^\lambda(\alpha) : i \in C')$ for all α. Since $|A|^+ \leq \min A \subseteq N$ and $\langle f_i^\lambda : i < \lambda\rangle \in N$, we have $f_i^\lambda \in N$ for all $i \in N \cap \lambda$ and $f_i^\lambda(\alpha) \in N$ for all α and $i \in N \cap \lambda$. So $f_\rho^\lambda(\alpha) = \sup(f_i^\lambda(\alpha) : i \in C') \leq \sup(N \cap \alpha) = \chi_N(\alpha)$ for all α, which shows the first assertion. Now consider $c := [f_\rho^\lambda < \chi_N] \cap b_\lambda$. We need to show that $c \in J_{<\lambda}$. For $\alpha \in c$, let $\gamma_\alpha \in N \cap \alpha$ be such that $f_\rho^\lambda(\alpha) < \gamma_\alpha$. Then $|\{\gamma_\alpha : \alpha \in c\}| \leq |A| < |A|^+$, so there is an $i < |A|^+$ such that $\gamma_\alpha \in N_i$ for all α. Then $f_\rho^\lambda \restriction c < \chi_{N_i} \restriction c$ and $\chi_{N_i} \restriction b_\lambda <_{J_{<\lambda}} f_j^\lambda \restriction b_\lambda$ for some $j \in N$, $j < \lambda$. Thus $\chi_{N_i} \restriction b_\lambda <_{J_{<\lambda}} f_\rho^\lambda \restriction b_\lambda$. Hence we must have $c \in J_{<\lambda}$. □

Note. In the above proof it would not work to simply say $\chi_N \restriction b_\lambda < f_i^\lambda \restriction b_\lambda$ for some $i < \lambda$, since it would not be clear whether $i \in N$. Also it is not clear whether χ_N is an element of N. (χ_{N_i} is definable since $N_i \in N$, but this does not work for N, since clearly $N \notin N$.)

6.2 Corollaries

We now prove three corollaries of Theorem 6.1.

Corollary 6.10. $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$.

Proof. Suppose $2^{\aleph_0} \geq \aleph_\omega$. Then $\aleph_\omega^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0} < (2^{\aleph_0})^+ \leq \aleph_{(2^{\aleph_0})^+}$ as wanted. So suppose $2^{\aleph_0} < \aleph_\omega$. Let $a = [\aleph_2, \aleph_\omega)$. By Hausdorff's formula (1) we have

$$\aleph_2^{\aleph_0} = \aleph_2 \cdot \aleph_1^{\aleph_0} = \aleph_2 \cdot \aleph_1 \cdot \aleph_0^{\aleph_0} = \aleph_2 \cdot \aleph_1 \cdot 2^{\aleph_0} = \max(\aleph_2, 2^{\aleph_0}) < \aleph_\omega.$$

Thus $\min(a)^{|a|} = \aleph_2^{\aleph_0} < \aleph_\omega = \sup(a)$. Also $|a|^+ = \aleph_0^+ = \aleph_1 < \aleph_2 = \min(a)$. Thus Theorem 6.1 applies and yields $\max(\text{pcf}(a)) = |\prod a| = \prod_{1<n<\omega} \aleph_n = \aleph_\omega^{\aleph_0}$. Recall that $\text{pcf}(a)$ is an interval of regular cardinals, that $a \subseteq \text{pcf}(a)$ and that $|\text{pcf}(a)| \leq 2^{|a|}$. Therefore we have

$$\aleph_\omega^{\aleph_0} = \max(\text{pcf}(a)) < \aleph_{\omega+|\text{pcf}(a)|^+} \leq \aleph_{\omega+(2^{\aleph_0})^+} = \aleph_{(2^{\aleph_0})^+}.$$

□

This result generalizes to arbitrary limit ordinals δ instead of just ω:

Corollary 6.11. Let δ be a limit ordinal. Then $\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}$.

Proof. The proof is a generalization of the proof of Corollary 6.10. Suppose $2^{|\delta|} \geq \aleph_\delta$. Then $\aleph_\delta^{|\delta|} \leq (2^{|\delta|})^{|\delta|} = 2^\delta < (2^{|\delta|})^+ \leq \aleph_{(2^{|\delta|})^+}$ as wanted. So suppose $2^{|\delta|} < \aleph_\delta$. We prove the theorem under the assumption that δ is the disjoint union of $|\delta|$ many δ-cofinal subsets. This assumption is made without loss of generality, since for every ordinal δ there exists an ordinal $\delta' := \delta + |\delta| > \delta$ which has the same cardinality as δ and is the disjoint union of $|\delta|$ many δ-cofinal subsets. Let $a = [(2^{|\delta|})^+, \aleph_\delta)$. By Hausdorff's formula (1), we have

$$((2^{|\delta|})^+)^{|\delta|} = (2^{|\delta|})^+ \cdot (2^{|\delta|})^{|\delta|} = (2^{|\delta|})^+ \cdot 2^{|\delta|} = (2^{|\delta|})^+ < \aleph_\delta.$$

Hence $\min(a)^{|a|} = ((2^{|\delta|})^+)^{|\delta|} < \aleph_\delta = \sup(a)$. Also $|a|^+ \leq |\delta|^+ \leq 2^{|\delta|} < (2^{|\delta|})^+ = \min(a)$. Thus Theorem 6.1 applies and yields $\max(\text{pcf}(a)) = |\prod a| = \aleph_\delta^{|\delta|}$ (in the last equality we use that δ is the disjoint union of $|\delta|$-many δ-cofinal subsets). Recall that $\mathrm{pcf}(a)$ is an interval of regular cardinals, that $a \subseteq \mathrm{pcf}(a)$ and that $|\mathrm{pcf}(a)| \le 2^{|a|}$. Therefore we have

$$\aleph_\delta^{|\delta|} = \max(\mathrm{pcf}(a)) < \aleph_{\delta+|\mathrm{pcf}(a)|^+} \le \aleph_{\delta+(2^{|a|})^+} \le \aleph_{(2^{|\delta|})^+}.$$

□

Corollary 6.12. Let $a = [\min(a), \sup(a))$ be an interval of regular cardinals such that $\min(a)^{|a|} < \sup(a)$. Then $|\prod(a)|$ is a regular cardinal and in particular, if $2^{\aleph_0} < \aleph_\omega$, then $\aleph_\omega^{\aleph_0}$ is regular.

Proof. By Theorem 6.1, $|\prod a| = \max(\mathrm{pcf}(a)) \in \mathrm{pcf}(a)$, so $|\prod a|$ is a regular cardinal. As in the proof of Corollary 6.10, $\aleph_\omega^{\aleph_0} = \max(\mathrm{pcf}(a))$ where $a = [\aleph_2, \aleph_\omega)$, hence $\aleph_\omega^{\aleph_0}$ is regular. □

It is consistent that $\aleph_\omega$ is a strong limit cardinal and $2^{\aleph_\omega} = \aleph_{\omega+\omega+2}$ [2, Remark 5.3]. In this case, let $a = [\aleph_{\omega+1}, \aleph_{\omega+\omega})$. It has been shown that $\max(\mathrm{pcf}(a)) \le \aleph_{\omega+\omega+1}$. Furthermore, $|\prod a| = \prod_{0<n<\omega} \aleph_{\omega+n} \ge \aleph_\omega^{\aleph_0} = \aleph_\omega^{\mathrm{cf}(\aleph_\omega)} = 2^{\aleph_\omega} = \aleph_{\omega+\omega+2}$, where the penultimate equality is (5.23) in Jech. So $\max(\mathrm{pcf}(a)) \ne |\prod a|$. Indeed, $\min(a)^{|a|} = \aleph_{\omega+1}^{\aleph_0} \ge \aleph_\omega^{\aleph_0} = 2^{\aleph_\omega} = \aleph_{\omega+\omega+2} > \aleph_{\omega+\omega} = \sup(a)$, so $\min(a)^{|a|} \not< \sup(a)$. Note that $2^{|a|} = 2^{\aleph_0} < \aleph_{\omega+1} = \min(a)$. So the assumption $\min(a)^{|a|} < \sup(a)$ of Theorem 6.1 cannot be replaced by $2^{|a|} < \min(a)$.

7 Pcf-theory applied to cardinal arithmetic 2

7.1 Games

Let κ be a cardinal such that $\kappa^{\aleph_0} = \kappa$. Let $f : (\kappa^+)^{<\omega} \to \kappa$ be a function. For each sequence $\xi = \langle \xi_n : n < \omega \rangle \in \kappa^\omega$, we define the following two-player game G_ξ of full information and with either Player I or Player II as winner: The game consists of ω rounds; for all $n < \omega$, in round n Player I picks a κ^+-club subset C_n of κ^+, and Player II responds by picking an element $\alpha_n \in C_n$. Player II wins precisely when $f(\langle \alpha_0, ..., \alpha_{n-1} \rangle) = \xi_n$ for all $n < \omega$. By $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle)$ we denote the game G_ξ where for all $i < n$, in round i, I picks C_i and II picks a_i.

Let $\mathcal{C}$ be the set of all κ^+-club subsets of κ^+.

Now a winning strategy for Player I for the game G_ξ would be a function $\sigma : (\kappa^+)^{<\omega} \to \mathcal{C}$ such that any play $C_0, \alpha_0, C_1, \alpha_1, ...$ of the game G_ξ where $C_n = \sigma(\langle a_0, ..., a_{n-1} \rangle)$ for all $n < \omega$ is winning for Player I. A winning strategy for Player II would be a function $\sigma : \mathcal{C}^{<\omega} \to \kappa^+$ such that $\sigma(\langle C_0, ..., C_n \rangle) \in C_n$ for all $\langle C_0, ..., C_n \rangle \in \mathcal{C}^{<\omega}$ and any play $C_0, \alpha_0, C_1, \alpha_1, ...$ of the game G_ξ where $\alpha_n = \sigma(\langle C_0, ..., C_n \rangle)$ for all $n < \omega$ is winning for Player II. A winning strategy for Player I for the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle)$ would be a function $\sigma : (\kappa^+)^{<\omega} \to \mathcal{C}$ such that any play $C_0, \alpha_0, ..., C_{n-1}, a_{n-1}, C_n, \alpha_n, ...$ of the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle)$ where $C_m = \sigma(\langle a_n, ..., a_{m-1} \rangle)$ for all $m < \omega$ such that $n \leq m$ is winning for Player I.

Lemma 7.1. Given $\xi \in \kappa^\omega$, $n < \omega$ and $\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle$ such that $a_i \in C_i$ for all $i < n$, if there exists a C_n such that for all $a_n \in C_n$ there is a winning strategy for Player I in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1}, C_n, a_n \rangle)$, then there is a winning strategy for Player I in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle)$.

Proof. Let C_n be such. For $a_n \in C_n$, let σ_{a_n} be a winning strategy for Player I in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1}, C_n, a_n \rangle)$. Then define

$$\begin{aligned} \sigma : (\kappa^+)^{<\omega} &\to \mathcal{C} \\ \emptyset &\mapsto C_n, \\ \langle a_n, a_{n+1}, ..., a_{m-1} \rangle &\mapsto \sigma_{a_n}(a_{n+1}, ..., a_{m-1}) \qquad (n+1 \leq m < \omega) \end{aligned}$$

That is, Player I picks C_n in round n and from then on follows strategy σ_{a_n} if Player II responds with a_n in round n. Of course σ is a winning strategy for Player I in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle)$. □

Corollary 7.2. Given $\xi \in \kappa^\omega$, $n < \omega$ and $\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle$ such that $a_i \in C_i$ for all $i < n$, if Player I has no winning strategy in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle)$, then for all $C_n \in C$ exists $a_n \in C_n$ such that Player I has no winning strategy in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1}, C_n, a_n \rangle)$.

Lemma 7.3. If Player I has no winning strategy in G_ξ, then Player II has.

Proof. Recursively define a function $\sigma : \mathcal{C}^{<\omega} \to \kappa^+$ such that if Player I has no winning strategy in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1} \rangle)$ and $a_i = \sigma(\langle C_0, ..., C_i \rangle)$ for all $i < n$, and Player I picks C_n in round n of that game, then $\sigma(\langle C_0, ..., C_{n-1}, C_n \rangle) = a_n$ for some a_n such that Player I has no winning strategy in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1}, C_n, a_n \rangle)$. Then σ is a winning strategy for Player II:

Let $C_0, a_0, C_1, a_1, ...$ be a play of the game G_ξ such that $a_n = \sigma(\langle C_0, ..., C_n\rangle)$ for all $n < \omega$. We prove by induction on $n < \omega$ that for all $n < \omega$, Player I has no winning strategy in $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1}\rangle)$ and thus $f(\langle a_0, ..., a_{n-1}\rangle) = \xi_n$.

1. Player I is assumed to have no winning strategy for G_ξ, thus $f(\emptyset) = \xi_0$.
2. If $f(\langle a_0, ..., a_{n-1}\rangle) = \xi_n$ for some $n < \omega$ and Player I has no winning strategy in the game $G_\xi(\langle C_0, a_0, ..., C_{n-1}, a_{n-1}\rangle)$, then Player I has no winning strategy in the game $G_\xi(\langle a_0, ..., a_{n-1}, C_n, a_n\rangle)$ since $a_n = \sigma(\langle C_0, ..., C_{n-1}\rangle)$. In particular, $f(\langle a_0, ..., a_{n-1}, a_n\rangle) = \xi_{n+1}$, otherwise Player I would win.

□

So the game G_ξ is determined: There is always a player with a winning strategy. Of course Lemma 7.3 has the following logical consequence.

Corollary 7.4. If Player II has no winning strategy for the game G_ξ, then Player I has.

Lemma 7.5. There exists $\xi \in \kappa^\omega$ such that Player II has a winning strategy for the game G_ξ.

Proof. Suppose that for all $\xi \in \kappa^\omega$, Player II has no winning strategy for the game G_ξ. For each $\xi \in \kappa^\omega$, let σ_ξ be a winning strategy for Player I for the game G_ξ. Using Skolem-Löwenheim, let $\langle M_n : n < \omega\rangle$ be a $\subseteq$-chain of elementary substructures of $H(\theta)$ such that

1. $\{\sigma_\xi : \xi \in \kappa^\omega\} \subseteq M_0$,
2. $M_n \in M_{n+1}$ for all $n < \omega$,
3. $|M_n| = \kappa$ for all $n < \omega$,
4. there is a set M of size κ such that $M \subseteq M_0$ and $M \in M_0$. (For instance, let $\{\sigma_\xi : \xi \in \kappa^\omega\} \in M_0$.)

Since $M_n, \kappa^+ \in M_{n+1}$, we have $M_n \cap \kappa^+ \in M_{n+1}$. Note that $M_n \cap \kappa^+ \in \kappa^+$: We assumed that M_0, hence M_n, contains a set M of size κ such that $M \subseteq M_0 \subseteq M_n$. If $\beta \in M_n \cap \kappa^+$, then M_n contains a surjection $M \twoheadrightarrow \beta$. Since $M \subseteq M_n$, we obtain $\beta \subseteq M_n$. Thus $M_n \cap \kappa^+$ is an initial segment of κ^+ and it is proper since $|M_n| = \kappa < \kappa^+$. So $\alpha_n := M_n \cap \kappa^+ < \kappa^+$.

Now define $\xi = \langle \xi_n : n < \omega\rangle \in \kappa^\omega$ by $\xi_n = f(\langle \alpha_0, ..., \alpha_{n-1}\rangle)$ for all $n < \omega$. We will show that $C_0, \alpha_0, C_1, \alpha_1, ...$, where $C_n = \sigma_\xi(\langle \alpha_0, ..., \alpha_{n-1}\rangle)$ for all $n < \omega$, is a play of the game G_ξ. But although Player I follows strategy σ_ξ, this game is won by Player II, which is a contradiction.

It remains to show that $C_0, \alpha_0, C_1, \alpha_1, ...$, where $C_n = \sigma_\xi(\langle \alpha_0, ..., \alpha_{n-1}\rangle)$ for all $n < \omega$, is a play of the game G_ξ. Clearly C_n is a κ^+-club subset of κ^+. Since $\sigma_\xi, \alpha_0, ..., \alpha_{n-1} \in M_n$, we have $C_n = \sigma_\xi(\langle \alpha_0, ..., \alpha_{n-1}\rangle) \in M_n$. Since $H(\theta) \models \forall \alpha \in \kappa^+ \, \exists \beta \in \kappa^+ \cap C_n \, (\alpha \in \beta)$, this holds in M_n as well, yielding the following: $\forall \alpha \in \kappa^+ \cap M_n \, \exists \beta \in \kappa^+ \cap C_n \cap M_n \, (\alpha \in \beta)$. But since $\alpha_n = \kappa^+ \cap M_n$, this says exactly that α_n is a limit point of C_n, and since C_n is closed, we get $\alpha_n \in C_n$. Thus $C_0, \alpha_0, C_1, \alpha_1, ...$ is a play of the game. □

Lemma 7.6. There exists $\xi \in \kappa^\omega$ and a set $T \subseteq (\kappa^+)^{<\omega}$ such that when $t \in T \cap (\kappa^+)^n$, then $f(t) = \xi_n$ (thus 'T is ξ-homogeneous') and furthermore $\{\alpha \in \kappa^+ : t^\frown\{\alpha\} \in T\}$ is κ^+-stationary for all $t \in T$.

Note. By $t^\frown\{\alpha\}$ we mean a concatenation of sequences: $t^\frown\{\alpha\}_i = t_i$ for $0 \leq i < n$ and $t^\frown\{\alpha\}_n = \alpha$. Also we require that T is not empty, because otherwise the existence of T would be trivial.

Proof. By Lemma 7.5, let $\xi \in \kappa^\omega$ and let σ be a winning strategy for Player II in the game G_ξ. Let T be set of finite sequences induced by σ:

$$T := \bigcup_{n<\omega} \{t \in (\kappa^+)^n : (\exists \langle C_0, ..., C_{n-1}\rangle \in \mathcal{C}^n : \sigma(\langle C_0, ..., C_m\rangle) = t_m \text{ for all } m < n\}.$$

Since σ is a winning strategy for Player II, clearly T is ξ-homogeneous. Let $t \in T$ and let $C \subseteq \kappa^+$ be κ^+-club. Let $\langle C_0, ..., C_{n-1}\rangle \in \mathcal{C}^n$ be such that $\sigma(\langle C_0, ..., C_m\rangle) = t_m$ for all $m < n$. Let $\alpha = \sigma(\langle C_0, ..., C_{n-1}, C\rangle)$. Then $\alpha \in C$ and $t^\frown\{\alpha\} \in T$. Hence $C \cap \{\alpha \in \kappa^+ : t^\frown\{\alpha\} \in T\}$ is non-empty, which proves that $\{\alpha \in \kappa^+ : t^\frown\{\alpha\} \in T\}$ is κ^+-stationary. □

7.2 A covering lemma

Lemma 7.7. Let λ be a regular cardinal and let W be a set such that $|W| < \aleph_\lambda$. Then there exists a set $B(W, \lambda) \subseteq [W]^\lambda$ such that $|B(W, \lambda)| \leq |W|$ and for all $A \in [W]^\lambda$ exists $B \in B(W, \lambda)$ such that $|A \cap B| = \lambda$.

Thus $B(W, \lambda)$ is some sort of covering set for $[W]^\lambda$ of limited size.

Proof. We prove this by induction on $|W|$.

1. If $|W| < \lambda$, then $[W]^\lambda = \emptyset$ and $B(W, \lambda) := \emptyset$ works.
2. If $|W| = \lambda$, then $B(W, \lambda) := \{W\}$ works.
3. Suppose $\lambda < |W|$ and the claim has been shown for all W' such that $|W'| < |W|$.
   (a) If $|W|$ is regular, let $W = \{w_i : i < |W|\}$ be an enumeration of W. Then $B(\{w_j : j < i\}, \lambda)$ exists for all $i < |W|$, and $B(W, \lambda) := \bigcup_{i<|W|} B(\{w_j : j < i\}, \lambda)$ works.
   (b) If $|W|$ is singular, let $W = \bigcup_{i<\mathrm{cf}|W|} W_i$ where $|W_i| < |W|$ for all i. Then $B(W_i, \lambda)$ exists for all i, and $B(W, \lambda) := \bigcup_{i<\mathrm{cf}|W|} B(W_i, \lambda)$ works:
      i. $|B(W_i, \lambda)| \leq |W_i|$, so $|B(W, \lambda)| \leq |W|$.
      ii. Let $\alpha < \lambda$ be such that $|W| = \aleph_\alpha$. Then $\mathrm{cf}|W| = \mathrm{cf}(\aleph_\alpha) = \mathrm{cf}(\alpha) \leq \alpha < \lambda$. Therefore, since λ is regular, any $A \in [W]^\lambda$ must satisfy $|A \cap W_i| = \lambda$ for some i. Hence $|A \cap W_i \cap B| = \lambda$ for some $B \in B(W_i, \lambda)$, thus $|A \cap B| = \lambda$ for some $B \in B(W, \lambda)$.

□

7.3 Main theorem

The following theorem is a generalization of Corollary 6.11.

Theorem 7.8. Let δ be a limit ordinal. Then

$$\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}.$$

The remainder of this section is devoted to the proof of this theorem.

Let $\mu := \mathrm{cf}(\delta)$. If $2^\mu \geq \aleph_\delta$, then the theorem is easily proven:

$$\aleph_\delta^{\mathrm{cf}(\delta)} \leq (2^\mu)^\mu = 2^\mu \leq |\delta|^\mu \leq \aleph_{|\delta|^\mu} < \aleph_{(|\delta|^{\mathrm{cf}(\delta)})^+}.$$

So assume $2^\mu < \aleph_\delta$. Define $a := (2^\mu, \aleph_\delta)$. Note that $|a| \leq |\delta|$. Define $\mathcal{J} := [a]^{\leq\mu} \setminus \{\emptyset\}$ and $\mathrm{pcf}_\mu(a) := \bigcup_{A\in\mathcal{J}} \mathrm{pcf}(A)$.

Claim 7.9. $\langle J_{<\lambda} \cap [a]^{\leq\mu} : \lambda \in \mathrm{pcf}_\mu(a)\rangle$ is $\subset$-increasing.

Proof. Of course the sequence is $\subseteq$-increasing. If $\lambda \in \mathrm{pcf}_\mu(a)$, then $\lambda \in \mathrm{pcf}(A)$ for some $A \in [a]^{\leq\mu}$, so $J_{<\lambda} \cap P(A) = J_{<\lambda}(A) \subset J_{<\lambda^+}(A) = J_{<\lambda^+} \cap P(A)$, so $J_{<\lambda} \cap [a]^{\leq\mu} \subset J_{<\lambda^+} \cap [a]^{\leq\mu}$. □

Hence $|\mathrm{pcf}_\mu(a)| \leq |[a]^{\leq\mu}| = |a|^\mu \leq |\delta|^\mu$.

Claim 7.10. $\mathrm{pcf}_\mu(a)$ is an interval of regular cardinals.

Proof. We have $a \subseteq \mathrm{pcf}_\mu(a)$ and $\aleph_\delta$ is singular: $\mathrm{cf}(\aleph_\delta) = \mu < 2^\mu < \aleph_\delta$. So we need to show that if $\lambda \in \mathrm{pcf}_\mu(a)$ and λ' is regular such that $\aleph_\delta < \lambda' < \lambda$, then $\lambda' \in \mathrm{pcf}_\mu(a)$. We have $\mathrm{cf}(\prod A/D) = \lambda$ for some $A \in [a]^{\leq\mu}$ and we must have $\lim_D A \leq \sup A = \aleph_\delta$. So $\lim_D A < \lambda' < \lambda$ and Theorem 3.28 yields a set of regular cardinals A' and an ultrafilter D' on A' such that $|A'| \leq |A|$, $\lim_{D'} A' = \lim_D A$ and $\mathrm{cf}(\prod A'/D') = \lambda'$. Since $\aleph_\delta$ is singular and $\lim_{D'} A' = \lim_D A \leq \aleph_\delta$, we may assume that $A' \subseteq \aleph_\delta$, and even $A' \subseteq a$, because a is an interval. So $\lambda' \in \mathrm{pcf}(A') \subseteq \mathrm{pcf}_\mu(A)$. □

If follows directly from Definition 3.17 that for any cardinal κ we have $\mathcal{J} \subseteq J_{<\kappa} \Leftrightarrow \mathrm{pcf}_\mu(a) \subseteq \kappa$. Let κ be minimal such that it is regular and $\mathrm{pcf}_\mu(a) \subseteq \kappa$. Then $\kappa < \aleph_{(|\delta|^\mu)^+}$ by Claim 7.10. We also have $\aleph_\delta < \kappa$: Since $\mathrm{cf}(\aleph_\delta) = \mathrm{cf}(\delta) = \mu$, there exists a set $A \subseteq a$ which is cofinal in $\aleph_\delta$ and of cardinality μ. Thus $A \in \mathcal{J}$ and $\mathrm{pcf}(A)$ has a maximum $\geq \aleph_\delta$. Thus $\kappa > \aleph_\delta$. Since $\aleph_\delta$ is singular, in fact there exists a cardinal $> \aleph_\delta$ in $\mathrm{pcf}_\mu(a)$.

Claim 7.11. $\aleph_\delta^\mu \leq \kappa \cdot |\delta|^\mu$.

To prove this claim we need to do some hard work. The proof is therefore postponed to the next subsection. Now since $\kappa < \aleph_{(|\delta|^\mu)^+}$ and $|\delta|^\mu < \aleph_{(|\delta|^\mu)^+}$, it follows that $\aleph_\delta^{\mathrm{cf}(\delta)} < \aleph_{(|\delta|^\mu)^+}$. So we have proved Theorem 7.8 assuming Claim 7.11.

7.4 Proof of Claim 7.11

Suppose Claim 7.11 does not hold, so assume $\aleph_\delta^\mu > \kappa$ and $\aleph_\delta^\mu > |\delta|^\mu$. We will ultimately show a contradiction. We have $|\delta|^\mu < \aleph_\delta < \kappa$. So $\lambda := (|\delta|^\mu)^{++} < \aleph_\delta < \kappa$. We have $\lambda^\mu = ((|\delta|^\mu)^{++})^\mu = (|\delta|^\mu)^{++} \cdot (|\delta|^\mu)^\mu = (|\delta|^\mu)^{++} = \lambda$ by Hausdorff's formula (1).

Claim 7.12. For $A \in \mathcal{J}$ and ν a cardinal, $A \in J_{<\nu}$ if and only if there is a subset $F_A \subseteq \prod A$ which is $<$-cofinal in $\prod A$ and of size $< \nu$.

Proof. ($\Leftarrow$) Suppose such F_A exists and D is an ultrafilter on A. Then $\{f/D : f \in F_A\}$ is clearly cofinal in $\prod A/D$. ($\Rightarrow$) We prove this by induction on ν. If $A \in J_{<\nu}$, then $\min(A) < \nu$.

1. If $\nu = (\min(A))^+$, then $A = \{\min(A)\}$ and $|\prod A| = |\min A| < \nu$ so $F_A = \prod A$ works.
2. Suppose for every $B \in J_{<\nu}$ exists such F_B. Let $A \in J_{<\nu^+}$. We may assume $A \notin J_{<\nu}$, so $A \in J_{<\nu^+} \setminus J_{<\nu}$ and thus $\mathrm{tcf}(\prod A/J_{<\nu}) = \nu$ by Corollary 3.54, so let $\langle f_i : i < \nu\rangle$ be $<_{J_{<\nu}}$-increasing and cofinal. Let

$$F_A := \{f_i \restriction (A \setminus B) \cup g : i < \nu, \emptyset \neq B \in P(A) \cap J_{<\nu}, g \in F_B\}.$$

Then $|F_A| \le |\nu| \cdot |P(A) \cap J_{<\nu}| \cdot \sup_{B \in P(A) \cap J_{<\nu}} |F_B| = \nu < \nu^+$, since $|P(A) \cap J_{<\nu}| \le 2^{|A|} \le 2^\mu < \min(A) < \nu^+$. Also F_A is cofinal in $\prod A$: Let $f \in \prod A$, then $f <_{J_{<\nu}} f_i$ for some i, so $f \restriction (A \setminus B) < f_i \restriction (A \setminus B)$ for some $B \in J_{<\nu} \cap P(A)$ and $f \restriction B < g$ for some $g \in F_B$. So $f < f_i \restriction (A \setminus B) \cup g \in F_A$.

3. If ν is a limit cardinal, then $J_{<\nu} = \bigcup_{\nu' < \nu} J_{<\nu'}$ and the claim holds by the induction hypothesis.

□

Since $\mathcal{J} \subseteq J_{<\kappa}$, for each $A \in \mathcal{J}$, let $F_A \subseteq \prod A$ be cofinal such that $|F_A| < \kappa$, which exist by Claim 7.12. For each $A \in \mathcal{J}$, $|F_A| < \kappa < \aleph_{(|\delta|^\mu)^+} < \aleph_\lambda$ and λ is regular. So Lemma 7.7 applies and we obtain sets $B(F_A, \lambda)$.

For all ordinals α and β, let $\mathcal{I}(\alpha^\beta)$ be the set of increasing sequences in α^β, and similarly $\mathcal{I}(\alpha^{<\beta})$ the set of increasing sequences in $\alpha^{<\beta}$. For a increasing sequence η, let $\mathrm{oran}(\eta)$ be the least ordinal α such that $\mathrm{ran}(\eta) \subseteq \alpha$. Note that $\mathrm{dom}(\eta)$ is a limit ordinal precisely if $\mathrm{oran}(\eta)$ is a limit ordinal. Let θ be a large enough regular cardinal and let $H(\theta)^* = (H(\theta), \in, <^*)$ where $<^*$ is a well-order on $H(\theta)$.

Let $\langle S_i : i < \kappa \rangle \in ([\aleph_\delta]^\mu)^\kappa$ be a κ-sequence of distinct elements in $[\aleph_\delta]^\mu$. We will ultimately show that for almost all i, $S_i \subseteq S$ for some S such that $|P(S)| < \kappa$ (in case $\mu > \aleph_0$), or $S_i \subseteq N$ for some N such that $|[N]^\mu| < \kappa$ (in case $\mu = \aleph_0$). This is of course a contradiction. Define $\mathcal{J}_{>\lambda} := \{A \in \mathcal{J} : A \cap \lambda = \emptyset\}$. Note that $\mathcal{J} \not\ni \emptyset$, so no $A \in \mathcal{J}_{>\lambda}$ is empty. For $i < \kappa$, recursively define, for $\alpha < \lambda$:

1. M^i_η for all $\eta \in \mathcal{I}(\lambda^{<\mu})$ which satisfy $\mathrm{oran}(\eta) = \alpha$,
2. f^i_α,
3. $f^i_{\alpha,A}$ for all $A \in \mathcal{J}_{>\lambda}$,

such that the following holds:

1. For all $\theta \in \mathcal{I}(\lambda^\mu)$, the sequence $\langle M^i_\eta : \eta \subset \theta \rangle$ is a continuous $\subseteq$-chain of elementary substructures. To be precise:
   (a) $M^i_\eta \prec H(\theta)^*$ for all $\eta \in \mathcal{I}(\lambda^{<\mu})$.
   (b) $M^i_\eta = \bigcup_{\zeta \subset \eta} M^i_\zeta = \bigcup_{\beta < \mathrm{dom}(\eta)} M^i_{\eta \restriction \beta}$ for all $\eta \in \mathcal{I}(\lambda^{<\mu})$ such that $\mathrm{dom}(\eta)$ is a limit ordinal. (Of course, by $\zeta \subset \eta$ we mean $\zeta \subset \eta$ and $\zeta \in \lambda^{<\mu}$.)
   (c) If $\eta, \eta^\frown\langle\beta\rangle \in \mathcal{I}(\lambda^{<\mu})$, then $M^i_\eta \subseteq M^i_{\eta^\frown\langle\beta\rangle}$.
2. $|M^i_\eta| = \mu$ for all $\eta \in \mathcal{I}(\lambda^{<\mu})$.
3. $\{a, \lambda\} \cup S_i \cup \mu \subseteq M^i_\emptyset$ (note that $|S_i| \le \mu$).
4. $\mathrm{ran}(\eta) \subseteq M^i_\eta$ for all $\eta \in \mathcal{I}(\lambda^{<\mu})$ (note that $|\mathrm{ran}(\eta)| < \mu$).
5. $\langle M^i_\zeta : \zeta \subset \eta \rangle = \langle M^i_{\eta \restriction \beta} : \beta < \mathrm{dom}(\eta) \rangle, \langle f^i_\beta : \beta < \alpha \rangle, \langle f^i_{\beta,A} : \beta < \alpha, A \in \mathcal{J}_{>\lambda} \rangle \in M^i_\eta$ for all $\eta \in \mathcal{I}(\lambda^{<\mu})$ such that $\mathrm{oran}(\eta) = \alpha$ and α is a successor.
6. $f^i_\alpha \in \prod(a \setminus \lambda)$ and for all $\rho \in a \setminus \lambda$ we have $f^i_\alpha(\rho) = \sup(\bigcup\{M^i_\eta \cap \rho : \eta \in \mathcal{I}(\lambda^{<\mu})$ and $\mathrm{oran}(\eta) \le \alpha\})$. Since $|M^i_\eta| \le \mu$ and $\rho > \lambda > |\delta|^\mu \ge 2^\mu > \mu$ and $\rho > \lambda = \lambda^\mu = |\lambda^{<\mu}|$, we indeed have $f^i_\alpha(\rho) < \rho$.
7. For each $A \in \mathcal{J}_{>\lambda}$ we have $f^i_{\alpha,A} \in F_A$ such that $f^i_\alpha \restriction A < f^i_{\alpha,A}$ and $f^i_{\beta,A} < f^i_{\alpha,A}$ for all $\beta < \alpha$. Indeed, for $A \in \mathcal{J}_{>\lambda}$ and $\rho \in A$, $\sup(\{f^i_\alpha(\rho)\} \cup \{f^i_{\beta,A}(\rho) : \beta < \alpha\} < \rho$, since $\alpha < \lambda < \rho$. So F_A contains a function $> \sup(\{f^i_\alpha \restriction A\} \cup \{f^i_{\beta,A} : \beta < \alpha\})$.

For $i < \kappa$ and $\theta \in \mathcal{I}(\lambda^\mu)$, define $M^i_\theta = \bigcup_{\eta \subset \theta} M^i_\eta = \bigcup_{\beta < \mu} M^i_{\theta \restriction \beta}$. This is again an elementary substructure of $H(\theta)^*$. Note that we use the symbol θ for an element in $\mathcal{I}(\lambda^\mu)$ and for a cardinal.

It will be clear from the context which one we mean.

For $A \in \mathcal{J}_{>\lambda}$, $\langle f^i_{\alpha,A} : \alpha < \lambda \rangle$ is $<$-increasing. Hence $\{f^i_{\alpha,A} : \alpha < \lambda\}$ has size λ. So let $t^i_A \in B(F_A, \lambda)$ be such that $t^i_A \cap \{f^i_{\alpha,A} : \alpha < \lambda\}$ has size λ. Enumerate each t^i_A as $\{g^i_{\alpha,A} : \alpha < \lambda\}$ such that if $t^i_A = t^j_A$, then t^i_A and t^j_A have equal enumeration. Define

$$\begin{aligned} C^i_A := &\{\beta < \lambda : [\forall \alpha < \beta \, \exists \xi, \zeta \in (\alpha, \beta) : (f^i_{\xi,A} = g^i_{\zeta,A})]\} \\ &\cap \{\beta < \lambda : [\forall \alpha < \beta : (\exists \xi < \lambda (g^i_{\alpha,A} < f^i_{\xi,A})) \Rightarrow (\exists \xi < \beta (g^i_{\alpha,A} < f^i_{\xi,A}))]\}. \end{aligned}$$

Claim 7.13. For all $i < \kappa$ and $A \in \mathcal{J}_{>\lambda}$, C^i_A is λ-club.

Proof. Let $i < \kappa$ and $A \in \mathcal{J}_{>\lambda}$. We will show that both $X := \{\beta < \lambda : [\forall \alpha < \beta \, \exists \xi, \zeta \in (\alpha, \beta) : (f^i_{\xi,A} = g^i_{\zeta,A})]\}$ and $Y := \{\beta < \lambda : [\forall \alpha < \beta : (\exists \xi < \lambda (g^i_{\alpha,A} < f^i_{\xi,A})) \Rightarrow (\exists \xi < \beta (g^i_{\alpha,A} < f^i_{\xi,A}))]\}$ are λ-club.

X is λ-unbounded: Let $\gamma_0 < \lambda$ be arbitrary. We define $\gamma_n < \lambda$ for $n < \omega$: Given γ_n, let $\xi_n, \zeta_n \in (\gamma_n, \lambda)$ be such that $f^i_{\xi_n,A} = g^i_{\zeta_n,A}$. This is possible since there are $t^i_A \cap \{f^i_{\alpha,A} : \alpha < \lambda\}$ has size λ. Given $\xi_n, \zeta_n < \lambda$, let $\gamma_{n+1} := \max(\xi_n, \gamma_n)$. Let $\beta := \sup_{n<\omega} \gamma_n$. Since λ is uncountable regular, we have $\beta < \lambda$, and clearly $\gamma < \beta$. Also it is easy to see that $\beta \in X$.

X is λ-closed: Let $\beta < \lambda$ be arbitrary and suppose $\sup(X \cap \beta) = \beta$. Let $\alpha < \beta$ be arbitrary. Then $\alpha < \beta'$ for some $\beta' \in X \cap \beta$. So there exist $\xi, \zeta \in (\alpha, \beta')$ such that $f^i_{\xi,A} = g^i_{\zeta,A}$. But then $\xi, \zeta \in (\alpha, \beta)$. So $\beta \in X$.

Y is λ-unbounded: Let $\gamma_0 < \lambda$. Define the map $\lambda \to \lambda$ by $\alpha \mapsto \xi_\alpha$, where ξ_α is the minimal ξ such that $g^i_{\alpha,A} < f^i_{\xi,A}$. Of course such ξ does not necessarily exists, in which case we take $\xi_\alpha = 0$. We define $\gamma_n < \lambda$ for $n < \omega$: Given γ_n, let $M_n := \{\xi_\alpha : \alpha < \gamma_n\}$. Given M_n, let $\gamma_{n+1} := \sup M_n$. Since λ is regular and $|M_n| < \lambda$, we have $\gamma_{n+1} < \lambda$. Let $\beta := \sup_{n<\omega} \gamma_n$. Since λ is uncountable and regular, we have $\beta < \lambda$. Clearly $\gamma_0 < \beta$. It is easy to verify that $\beta \in X$.

Y is λ-closed: Let $\beta < \lambda$ be arbitrary and suppose $\sup(Y \cap \beta) = \beta$. Let $\alpha < \beta$ be arbitrary. Then $\alpha < \beta'$ for some $\beta' \in Y \cap \beta$. So if $g^i_{\alpha,A} < f^i_{\xi,A}$ for some $\xi < \lambda$, then it is possible to choose $\xi < \beta'$. But then $\xi < \beta$. Thus $\beta \in Y$. Hence $C^i_A = X \cap Y$ is λ-club. □

For $i < \kappa$, define $C^i := \bigcap_{A \in \mathcal{J}_{>\lambda}} C^i_A$. Since $\lambda > |\delta|^\mu = |\mathcal{J}| \geq |\mathcal{J}_{>\lambda}|$, C^i is λ-club as well.

We now have to consider two cases: We have $\mu > \aleph_0$ or $\mu = \aleph_0$. These will be considered in the following two paragraphs.

7.4.1 The case $\mu > \aleph_0$

Assume $\mu > \aleph_0$. Since $\mu < \lambda$, for each $i < \kappa$, let $\beta_i \in C^i$ be such that $\mathrm{cf}(\beta_i) = \mu$. Consider the following maps, for any $A \in \mathcal{J}_{>\lambda}$, $\beta < \lambda$ and $\theta \in \mathcal{I}(\lambda^\mu)$:

$$\begin{array}{ccc} \kappa & \to & \lambda \\ i & \mapsto & \beta_i \end{array} ; \quad \begin{array}{ccc} \kappa & \to & \mathcal{J}_{>\lambda} \\ i & \mapsto & (M^i_\theta \cap a) \setminus \lambda \end{array} ; \quad \begin{array}{ccc} \kappa & \to & B(F_A, \lambda) \\ i & \mapsto & t^i_A \end{array} ; \quad \begin{array}{ccc} \kappa & \to & F_A \\ i & \mapsto & f^i_{\beta,A} \end{array} ; \quad \begin{array}{ccc} \kappa & \to & \mathcal{J} \\ i & \mapsto & M^i_\theta \cap \lambda \end{array}$$

Note that each domain has cardinality less than κ: $\lambda < \kappa$, $|\mathcal{J}_{>\lambda}| \leq |\mathcal{J}| \leq |\delta|^\mu < \lambda$, $|B(F_A, \lambda)| \leq |P_A| < \kappa$. So we let $I \subseteq \kappa$, $\beta < \lambda$, $A \in J_{>\lambda}$, $\theta \in \mathcal{I}(\lambda^\mu)$ such that $\mathrm{oran}(\theta) = \beta$, $t \in B(F_A, \lambda)$ and $f \in F_A$ such that $|I| = \kappa$, and $\beta_i = \beta$, $(M^i_\theta \cap a) \setminus \lambda = A$, $t^i_A = t$ and $f^i_{\beta,A} = f$ for all $i \in I$, and $M^i_\theta \cap \lambda$ is independent of $i \in I$. Define $t_\alpha := \{g^i_{\beta,A} : \beta < \alpha\}$; this is independent of $i \in I$ since $t^i_A = t$ for all i, thus each t^i_A has the same enumeration.

Claim 7.14. $f^i_\beta \restriction A$ is independent of $i \in I$.

Proof. Let $\rho \in A$ be arbitrary. We have the following equalities, which will be justified below:

$$\begin{aligned} f^i_\beta(\rho) &\overset{1.}{=} \sup(\bigcup\{M^i_\eta \cap \rho : \eta \in \mathcal{I}(\lambda^{<\mu}) \text{ and } \mathrm{oran}(\eta) \leq \beta\}) \\ &\overset{2.}{=} \sup(\sup(\bigcup\{M^i_\eta \cap \rho : \eta \in \mathcal{I}(\lambda^{<\mu}) \text{ and } \mathrm{oran}(\eta) \leq \alpha\}) : \alpha < \beta) \\ &\overset{3.}{=} \sup(f^i_\alpha(\rho) : \alpha < \beta) \\ &\overset{4.}{=} \sup(f^i_{\alpha,A}(\rho) : \alpha < \beta) \\ &\overset{5.}{=} \sup(g(\rho) : g \in t_\beta, g < f^i_{\alpha,A} \text{ for some } \alpha < \beta) \\ &\overset{6.}{=} \sup(g(\rho) : g \in t_\beta, g < f). \end{aligned}$$

1. By definition.
2. Since $\mathrm{cf}(\beta) = \mu$, any $\eta \in \mathcal{I}(\lambda^{<\mu})$ does not have $\mathrm{oran}(\eta) = \beta$.
3. By definition.
4. $(\leq)$ $f^i_\alpha \restriction A < f^i_{\alpha,A}$ for all $\alpha < \beta$. $(\geq)$ Consider $f^i_{\alpha,A}(\rho)$ for some $\alpha < \beta$. Recall that $\mathrm{oran}(\theta) = \beta$, and that $\rho \in A \subseteq M^i_\theta = \bigcup_{\eta \subset \theta} M^i_\eta$. Thus let $\xi < \mu$ be such that $\alpha < \theta(\xi)$ and $\rho \in M^i_{\theta\restriction\xi+1}$. Then $\langle f^i_{\alpha',A'} : \alpha' < \mathrm{oran}(\theta \restriction \xi+1) = \theta(\xi)+1, A' \in \mathcal{J}_{>\mathcal{J}}\rangle \in M^i_{\theta\restriction\xi+1}$ by definition. Therefore
$$f^i_{\alpha,A}(\rho) < \sup(f^i_{\alpha',A'}(\rho) : \alpha' < \theta(\xi)+1, A' \in \mathcal{J}_{>\mathcal{J}}) < \sup(M^i_{\theta\restriction\xi+1} \cap \rho) \leq f^i_{\theta(\xi)+1}(\rho).$$
5. $(\geq)$ $g < f^i_{\alpha,A}$ for some $\alpha < \beta$. $(\leq)$ Let $\alpha < \beta$. We have $\beta \in C^i$ thus $\beta \in C^i_A$, thus let $\xi, \zeta \in (\alpha, \beta)$ be such that $f^i_{\xi,A} = g^i_{\zeta,A}$. Then $g := g^i_{\zeta,A}$ satisfies $g < f^i_{\alpha',A}$ for some $\alpha' < \beta$ (for instance $\alpha' = \xi+1$), and $f^i_{\alpha,A}(\rho) < f^i_{\xi,A}(\rho) = g(\rho)$.
6. $(\geq)$ If $g < f^i_{\alpha,A}$, then $g < f^i_{\beta,A} = f$. $(\leq)$ As $g \in t_\beta$, we have $g = g^i_{\alpha,A}$ for some $\alpha < \beta$. Since $\beta \in C^i_A$ and $g = g^i_{\alpha,A} < f = f^i_{\beta,A}$ let $\xi < \beta$ such that $g = g^i_{\alpha,A} < f^i_{\xi,A}$.

The last expression is independent of $i \in I$. □

Claim 7.15. For $\rho \in A$ we have $f^i_\beta(\rho) = \sup(M^i_\theta \cap \rho)$.

Proof. Let $\rho \in A$ be arbitrary. We have

$$\begin{aligned} f^i_\beta(\rho) &\overset{1.}{=} \sup(\bigcup\{M^i_\eta \cap \rho : \eta \in \mathcal{I}(\lambda^{<\mu}) \text{ and } \mathrm{oran}(\eta) \leq \beta\}) \\ &\overset{2.}{=} \sup(\bigcup\{M^i_{\theta\restriction\xi} \cap \rho : \xi < \mu\}) \\ &\overset{3.}{=} \sup(M^i_\theta \cap \rho), \end{aligned}$$

since

1. By definition of $f^i_\beta(\rho)$.
2. $(\geq)$ Clear. $(\leq)$ Let $\eta \in \mathcal{I}(\lambda^{<\mu})$ such that $\mathrm{oran}(\eta) \leq \beta$. Since $\mathrm{cf}(\beta) = \mu$, we have $\mathrm{oran}(\eta) < \beta$. So let $\xi < \mu$ be such that $\rho \in M^i_{\theta\restriction\xi+1}$ and $\mathrm{oran}(\eta) < \theta(\xi)$. Since also $\langle f^i_\alpha : \alpha < \mathrm{oran}(\theta \restriction \xi+1) = \theta(\xi)+1\rangle, \theta(\xi) \in M^i_{\theta\restriction\xi+1}$, we have $\sup(M^i_\eta \cap \rho) \leq f^i_{\theta(\xi)}(\rho) \in M^i_{\theta\restriction\xi+1} \cap \rho$ by definition of $f^i_{\theta(\xi)}(\rho)$,

3. By definition of M^i_θ.

$\square$

Claim 7.16. For $\rho \leq \aleph_\delta$, $M^i_\theta \cap \rho$ is independent of $i \in I$.

Proof. We prove this by induction on ρ. We already have $M^i_\theta \cap \lambda$ is independent of $i \in I$. Thus for $\rho \leq \lambda$ there is nothing to be done. If ρ is limit and for all $\rho' < \rho$, $M^i_\theta \cap \rho'$ is independent of $i \in I$, then $M^i_\theta \cap \rho = \bigcup_{\rho' < \rho} M^i_\theta \cap \rho$ is independent of $i \in I$. Suppose $\rho > \lambda$ and $M^i_\theta \cap \rho$ is independent of ρ. If for all $i_0, i \in I$, $M^{i_0}_\theta \cap \rho^+ = M^i_\theta \cap \rho^+$, then we are done. So let $i_0 \in I$ and $\alpha \in M^{i_0}_\theta \cap \rho^+ \setminus \rho$. Then $|\alpha| = \rho$, so $\rho, \rho^+ \in M^{i_0}_\theta$. For all $i \in I$, $(M^i_\theta \cap a) \setminus \lambda = A$, so any $i \in I$ has $\rho^+ \in M^i_\theta$, hence $\rho \in M^i_\theta$. For any $i \in I$, $\langle \sup(M^i_{\theta\restriction\alpha} \cap \rho^+) : \alpha < \mu\rangle$ is continuous and cofinal in $\sup(M^i_\theta \cap \rho^+)$. So for all $i \in I$, $\{\sup(M^i_{\theta\restriction\alpha} \cap \rho^+) : \alpha < \mu\} \subseteq M^i_\theta \cap \rho^+$ is club in $\sup(M^i_\theta \cap \rho^+) = f^i_\beta(\rho^+)$ (Claim 7.15), and thus $\mathrm{cf}(f^i_\beta(\rho^+)) = \mu$. Note that $\sup(M^i_\theta \cap \rho^+) = f^i_\beta(\rho)$ is independent of $i \in I$ by Claim 7.14. So $K := \{\sup(M^{i_0}_{\theta\restriction\alpha} \cap \rho^+) : \alpha < \mu\} \cap \{\sup(M^i_{\theta\restriction\alpha} \cap \rho^+) : \alpha < \mu\} \subseteq M^{i_0}_\theta \cap M^i_\theta \cap \rho^+$ is cofinal in $M^{i_0}_\theta \cap \rho^+$ and in $M^i_\theta \cap \rho^+$. For $\alpha \in K$, let h be the $<^*$-least bijection $\rho \leftrightarrow \alpha$ in $H(\theta)^*$. Then $M^i_\theta \cap \alpha = h[M^i_\theta \cap \rho] = h[M^{i_0}_\theta \cap \rho] = M^{i_0}_\eta \cap \alpha$ for all i. Since K is cofinal in both $M^{i_0}_\theta \cap \rho^+$ and $M^i_\theta \cap \rho^+$, we obtain $M^{i_0}_\theta \cap \rho^+ = M^i_\theta \cap \rho^+$. $\square$

So define $S := M^i_\theta \cap \aleph_\delta$. For each $i \in I$, $S_i \subseteq M^i_\emptyset \subseteq M^i_\theta$ and $S_i \subseteq \aleph_\delta$, so $S_i \subseteq S$. So $\{S_i : i \in I\} \subseteq P(S)$. But this contradicts $|P(S)| \leq 2^\mu < \aleph_\delta < \kappa$. So Claim 7.11 is now shown under the assumption that $\mu > \aleph_0$.

7.4.2 Case $\mu = \aleph_0$

Assume $\mu = \aleph_0$. Since $(|\delta|^\mu)^+ < \lambda$, for each $i < \kappa$, let $\beta_i \in C^i$ be such that $\mathrm{cf}(\beta_i) = (|\delta|^\mu)^+$. Consider the map $\kappa \to \lambda$ given by $i \mapsto \beta_i$; let $I \subseteq \kappa$ and $\beta < \lambda$ be such that $|I| = \kappa$ and $\beta_i = \beta$ for all $i \in I$. Let $C \subseteq \beta$ be β-club and such that $|C| = (|\delta|^\mu)^+$. Note that $(|\delta|^\mu)^\mu = |\delta|^\mu$ and $|[a]^{\leq\mu}| = |a|^\mu \leq |\delta|^\mu$. For $i \in I$, let $f^i : C^{<\omega} \to [a]^{\leq\mu}$ be given by $\eta \mapsto (M^i_\eta \cap a) \setminus \lambda$ if η is increasing and $\eta \mapsto \emptyset$ if η is not increasing. Then Lemma 7.6 applies and yields a nonempty $T^i \subseteq C^{<\omega}$ and a sequence $\langle A^i_n : n < \omega\rangle$ in $[a]^{\leq\mu}$ such that $f^i(\eta) = A^i_n$ for all $\eta \in T^i \cap \beta^n$ and $\{\alpha \in C : \eta^\frown\{\alpha\} \in T^i\}$ is C-stationary for all $\eta \in T^i$. Since there must be increasing sequences of every length in T^i, we in fact have that $A^i_n = f(\eta) = (M^i_\eta \cap a) \setminus \lambda$ for such sequences. Since $(M^i_\eta \cap a) \setminus \lambda \neq \emptyset$, we in fact have that every sequence in T^i is increasing. Consider the map $I \to ([a]^{\leq\mu})^{\aleph_0}$ given by $i \mapsto \langle A^i_n : n < \omega\rangle$. Since $|I| = \kappa$ and $|([a]^{\leq\mu})^{\aleph_0}| = (|a|^\mu)^{\aleph_0} = |a|^\mu \leq \delta^\mu < \lambda < \kappa$, thin out[8] I an let $\langle A_n : n < \omega\rangle$ such that $\langle A^i_n : n < \omega\rangle = \langle A_n : n < \omega\rangle$ for all $i \in I$.
Let $\bar{A} := \bigcup_{n<\omega} A_n \in [a]^{\leq\mu} \setminus \{\emptyset\}$. Consider the maps

$$\begin{array}{ccc} I & \to & B(F_{\overline{A}}, \lambda) \\ i & \mapsto & t^i_{\overline{A}} \end{array} \quad ; \quad \begin{array}{ccc} I & \to & F_{\overline{A}} \\ i & \mapsto & f^i_{\beta,\overline{A}} \end{array}.$$

Since $|B(F_{\overline{A}})| \leq |F_{\overline{A}}| < \kappa = |I|$, thin out I again and let $t \in B(F_{\overline{A}})$ and $f \in F_{\overline{A}}$ be such that $f^i_{\beta,\overline{A}} = f$ and $t^i_{\overline{A}} = t$ for all $i \in I$. Recall that we enumerated $t = t^i_{\overline{A}} = \{g^i_{\beta,\overline{A}} : \beta < \lambda\}$; let $t_\alpha := \{g^i_{\beta,\overline{A}} : \beta < \alpha\}$.

[8] This means that we take a $J \subseteq I$ with the desired property and such that $|J| = |I| = \kappa$, and we rename $I = J$.

Claim 7.17. $f^i_\beta \restriction \overline{A}$ is independent of i.

Proof. Given $i \in I$, let $\theta \in \beta^\omega$ be such that $\theta \restriction n \in T^i \cap \beta^n$ for all $n < \omega$. This is possible since $\{\alpha : \eta^\frown\{\alpha\} \in T^i\}$ is stationary for all $\eta \in T^i$. The proof is now the same as the proof of Claim 7.14. ☐

Let $f = f^i_\beta \restriction \overline{A}$. For all $\rho \in \bar{A}$, we have $f(\rho) = f^i_\beta(\rho) = \sup_{\alpha<\beta} f^i_\alpha(\rho)$.

Claim 7.18. For any i, for $\rho \in \bar{A}$, $\langle f^i_\alpha(\rho) : \alpha < \beta\rangle$ is eventually increasing. Hence $\mathrm{cf}(f^i_\beta(\rho)) = \mathrm{cf}(\beta)$.

Proof. Fix some i. Let $n < \omega$ be such that $\rho \in A_n$. There is an $\eta \in T^i \cap \beta^n$, and $A_n = (M^i_\eta \cap a) \setminus \lambda$. We will show that $\langle f^i_\alpha(\rho) : \eta(n-1) < \alpha < \beta\rangle$ is increasing.
Let $\eta(n-1) < \alpha_1 < \alpha_2 < \beta$. We have $\langle M^i_{\eta^\frown\{\alpha_1\}\restriction k} : k \le n\rangle \in M^i_{\eta^\frown\{\alpha_1\}}$, $n \in M^i_{\eta^\frown\{\alpha_1\}}$ (since $n < \omega$ is definable), so $M^i_\eta \in M^i_{\eta^\frown\{\alpha_1\}}$, and $a, \lambda \in M^i_{\eta^\frown\{\alpha_1\}}$, so $A_n = (M^i_\eta \cap a) \setminus \lambda \in M^i_{\eta^\frown\{\alpha_1\}}$. Also $\mathrm{ran}(\eta^\frown\{\alpha_1\}) \subseteq M^i_{\eta^\frown\{\alpha_1\}}$, so $\alpha_1 \in M^i_{\eta^\frown\{\alpha_1\}}$. And $\langle f^i_{\alpha,A} : \alpha < \alpha_1 + 1, A \in \mathcal{J}_{>\lambda}\rangle \in M^i_{\eta^\frown\{\alpha_1\}}$. So $f^i_{\alpha_1,A_n} \in M^i_{\eta^\frown\{\alpha_1\}}$. Since $\rho \in A_n = (M^i_\eta \cap a) \setminus \lambda \subseteq M^i_{\eta^\frown\{\alpha_1\}}$, we obtain $f^i_{\alpha_1,A_n}(\rho) \in M^i_{\eta^\frown\{\alpha_1\}}$. So we have

$$f^i_{\alpha_1}(\rho) < f^i_{\alpha_1,A_n}(\rho) \le \sup(M^i_{\eta^\frown\{\alpha_1\}} \cap \rho) \le f^i_{\alpha_1+1}(\rho) \le f^i_{\alpha_2}(\rho).$$

☐

For $\rho \in \bar{A}$, let $C_\rho \subseteq f(\rho)$ be $f(\rho)$-club and such that $|C_\rho| = \mathrm{cf}(\beta)$. By Downwards Skolem-Löwenheim, let N be such that $|N| = \lambda$ and

$$\overline{A} \cup \lambda \cup \bigcup_{\rho\in\bar{A}} C_\rho \subseteq N \prec H(\theta)^*.$$

Since $|\overline{A}| \le \mu = \aleph_0$ and $|C_\rho| = \mathrm{cf}(\beta) \le \beta < \lambda$, this is possible.
Fix $i \in I$. We will show that $S_i \subseteq N$.

Claim 7.19. For all $\eta \in T^i \cap \beta^n$ and $\rho \in A_n$ exists $\alpha < \beta$ such that $\rho \setminus \sup(M^i_\eta \cap \rho) \cap N \cap M^i_{\eta^\frown\{\alpha\}}$ is non-empty.

Proof. Since $\{f^i_\alpha(\rho) : \alpha < \beta\}$ and C_ρ are club in $f^i_\beta(\rho)$, and $\mathrm{cf}(f^i_\beta(\rho)) = \mathrm{cf}(\beta) = (|\delta|^\mu)^+$, we have $\{f^i_\alpha(\rho) : \alpha < \beta\} \cap C_\rho$ is club in $f^i_\beta(\rho)$, and $\{\alpha < \beta : f^i_\alpha(\rho) \in C_\rho\}$ is β-club. Also $\sup(M^i_\eta \cap \rho) < \beta$, since $|M^i_\eta| = \mu$. So there exists an $\alpha < \beta$ such that $f^i_\alpha(\rho) \in C_\rho$, $\sup(M^i_\eta \cap \rho) < f^i_\alpha(\rho)$ and $\eta^\frown\{\alpha\} \in T^i$. Then clearly $f^i_\alpha(\rho) \in \rho \setminus \sup(M^i_\eta \cap \rho) \cap N$. Since $\langle f^i_\gamma : \gamma < \mathrm{oran}(\eta^\frown\{\alpha\}) = \alpha + 1\rangle \in M^i_{\eta^\frown\{\alpha\}}$, $\rho \in A_n = (M^i_\eta \cap a) \setminus \lambda \subseteq M^i_{\eta^\frown\{\alpha\}}$ and $\alpha \in \mathrm{ran}(\eta^\frown\{\alpha\}) \subseteq M^i_{\eta^\frown\{\alpha\}}$, we have $f^i_\alpha(\rho) \in M^i_{\eta^\frown\{\alpha\}}$. ☐

Observe that if $\rho \setminus \sup(M^i_\eta \cap \rho) \cap N \cap M^i_{\eta^\frown\{\alpha\}}$ is non-empty and $\zeta \subseteq \eta$ and $\eta^\frown\{\alpha\} \subseteq \zeta'$, then $\rho \setminus \sup(M^i_\zeta \cap \rho) \cap N \cap M^i_{\zeta'}$ is non-empty as well.
Let $\zeta \in T^i \cap \beta^n$ and $\eta \in T^i \cap \beta^m$ such that $\zeta \subseteq \eta$. Then $A_n = (M^i_\zeta \cap a) \setminus \lambda \subseteq (M^i_\eta \cap a) \setminus \lambda = A_m$. So $\langle A_n : n < \omega\rangle$ is $\subseteq$-increasing.

From these two observations and Claim 7.19 it follows that there is a sequence $\langle \eta_n : n < \omega \rangle$ such that if $n \leq m$ then $\eta_n \subseteq \eta_m$, for all n we have $\eta_n \in T^i \cap \beta^n$ and if $\rho \in A_n$, then there exists $m > n$ such that $\rho \setminus \sup(M^i_{\eta_n} \cap \rho)) \cap M^i_{\eta_m} \cap N$ is non-empty.

Let $\theta = \bigcup_{n<\omega} \eta_n \in \beta^\omega$, then $M^i_\theta = \bigcup_{n<\omega} M^i_{\eta_n}$ and $\overline{A} = (M^i_\theta \cap a) \setminus \lambda$.

Claim 7.20. For all $\rho \in \overline{A}$ we have that $M^i_\theta \cap N \cap \rho$ is cofinal in $M^i_\theta \cap \rho$.

Proof. Let $\rho \in A_k$. Let $\sigma \in M^i_\theta \cap \rho$, then $\sigma \in M^i_{\eta_n} \cap \rho$ for some n. Then $\rho \in A_{\max(k,n)}$ and $\sigma \in M^i_{\eta_{\max(k,n)}}$. For some $m > \max(k,n)$ we have $\rho \setminus \sup(M^i_{\eta_{\max(k,n)}} \cap \rho) \cap M^i_{\eta_m} \cap N$ is non-empty. So there exists a $\tau > \sigma$, $\tau \in M^i_\theta \cap N \cap \rho$. $\square$

Claim 7.21. We have $M^i_\theta \cap \aleph_\delta \subseteq N \cap \aleph_\delta$.

Proof. We prove this by induction on all cardinals $\rho \leq \aleph_\delta$. For $\rho \leq \lambda$ this is trivial since $\lambda \subseteq N$, so $M^i_\theta \cap \lambda \subseteq \lambda = N \cap \lambda$. If ρ is a limit cardinal, this follows easily from the induction hypothesis. So suppose $\aleph_\delta > \rho > \lambda$ and $M^i_\theta \cap \rho \subseteq N \cap \rho$. We will show $M^i_\theta \cap \rho^+ \subseteq N \cap \rho^+$. Suppose $\alpha \in M^i_\theta \cap (\rho^+ \setminus \rho)$. Then $|\alpha| = \rho \in M^i_\theta$, so $\rho^+ \in (M^i_\theta \cap a) \setminus \lambda = \overline{A}$. By Claim 7.20, let $\beta \in M^i_\theta \cap N \cap \rho^+$ be such that $\alpha < \beta$. Then $|\beta| = \rho \in M^i_\theta \cap N$. So the $<^*$-least bijection $h : \rho \leftrightarrow \beta$ is in M^i_θ and in N by elementarity. So $M^i_\eta \cap \beta = h[M^i_\theta \cap \rho] \subseteq h[N \cap \rho] = N \cap \beta$. So $M^i_\eta \cap \alpha \subseteq N \cap \beta \subseteq N \cap \rho^+$. Since α was arbitrary, we have $M^i_\eta \cap \rho^+ \subseteq N \cap \rho^+$. $\square$

Since $S_i \subseteq M^i_\emptyset \subseteq M^i_\theta$ and $S_i \subseteq \aleph_\delta$, we have $S_i \subseteq M^i_\theta \cap \aleph_\delta \subseteq N \cap \aleph_\delta$, so $S_i \subseteq N$.

Now $|[N]^\mu| = \lambda^\mu = \lambda < \kappa = |\{S_i : i \in I\}|$. This contradicts $S_i \subseteq N$ for all $i \in I$.

So Claim 7.11 is now shown under the assumption that $\mu = \aleph_0$.

8 Shelah's bound on $\aleph_\omega^{\aleph_0}$

In this section we will prove Shelah's renowned and surprising bound $\aleph_\omega^{\aleph_0} < \aleph_{\omega_4}$, assuming $2^{\aleph_0} < \aleph_\omega$.

8.1 First essential lemma

In this subsection we let λ be a singular cardinal of uncountable cofinality, i.e. $\lambda > \mathrm{cf}(\lambda) > \omega$.

Lemma 8.1. There exists a λ-cofinal closed set $C \subseteq [\mathrm{cf}(\lambda), \lambda)$ of order type $\mathrm{cf}(\lambda)$ consisting of singular cardinals.

Proof. Let $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ be increasing and cofinal in λ. For $i < \mathrm{cf}(\lambda)$ define $\mu_i < \lambda$ by

1. $\mu_0 = \mathrm{cf}(\lambda)$,
2. $\mu_{i+1} = \lim_{n<\omega} \mu^{+n}$ where $\mu := \max(\mu_i, \lambda_i)$,
3. $\mu_i = \lim_{j<i} \mu_j$ if i is limit.

Then $\langle \mu_i : i < \mathrm{cf}(\lambda) \rangle$ is increasing, cofinal in λ, and λ_i is singular for all i. □

For any set of cardinals A and any natural number n, let $A^{+n} := \{\lambda^{+n} : \lambda \in A\}$ and let $A^{-n} := \{\lambda : \lambda^{+n} \in A\}$.

Let C be a λ-cofinal closed subset of $[\mathrm{cf}(\lambda), \lambda)$ consisting of singular cardinals as in Lemma 8.1. Define $c := \bigcup_{1 \le k < \omega} C^{+k}$. Then c is an infinite set of regular cardinals, so pcf-theory comes into play. For all cardinals μ, we have a generator $b_\mu(c)$ of $J_{<\mu^+}(c)$ over $J_{<\mu}(c)$. If $\mu \notin \mathrm{pcf}(c)$, then $b_\mu(c) = \emptyset$. In particular, we have generators $b_{\lambda^{+k}}(c)$ for all $1 \le k < \omega$.

The remainder of this subsection will be the proof of the following theorem.

Theorem 8.2. For any $1 \le n < \omega$ we have that $\{\rho \in C : \rho^{+n} \in \bigcup_{1 \le k \le n} b_{\lambda^{+k}}(c)\}$ has a subset which is λ-club.

Assume the theorem is false and let n be minimal such that $\{\rho \in C : \rho^{+n} \in B\}$ contains no λ-club subset, where $B := \bigcup_{1 \le k \le n} b_{\lambda^{+k}}(c)$. Define $a = C^{+n}$ and

$$\begin{aligned} I :&= \{A \subseteq a : \{\rho(\in C) : \rho^{+n} \in A\} \text{ is not stationary in } \lambda\} \\ &= \{A \subseteq a : A^{-n}(\cap C) \text{ is not stationary in } \lambda\}. \end{aligned}$$

'$\in C$' and '$\cap C$' are in brackets because they are automatically satisfied.

Then I is easily shown to be a proper ideal on a. By assumption, $\{\rho \in C : \rho^{+n} \in B\} = \{\rho \in C : \rho^{+n} \in (B \cap a)\}$ contains no λ-club subset. So any λ-club subset D of λ satisfies $D^{+n} \cap a \setminus B \neq \emptyset$, so $a \setminus B \notin I$. On the other hand, subsets of a which are bounded below λ are in I. If $A \in J_{<\lambda}(a)$, then $\mathrm{pcf}(A) \subseteq \lambda$. Since $\mathrm{pcf}(A)$ has a maximum, we have in fact that $\mathrm{pcf}(A)$, and thus A itself, is bounded below λ. Thus $J_{<\lambda}(a) \subseteq I$.

Note that B generates $J_{<\lambda^{+n+1}}(c)$ over $J_{<\lambda^+}(c) = J_{<\lambda}(c)$ and that $B \cap a$ generates $J_{<\lambda^{+n+1}}(a)$ over $J_{<\lambda^+}(a) = J_{<\lambda}(a)$. (For instance, use the third characterization in Definition 3.42).

Let $I^* = I + (B \cap a)$, then I^* is still proper (since $a \setminus B \notin I$, so $a \notin I^*$). Since $I \supseteq J_{<\lambda}(a)$, we have $I^* \supseteq J_{<\lambda^{+n+1}}(a)$. By Theorem 3.20 we therefore have that $\prod a/I^*$ is λ^{+n+1}-directed.

As in Definition 3.33 let $\langle C_\beta : \beta < \lambda^{+n} \rangle$ be a silly square sequence, i.e.

1. $C_\beta \subseteq P(\beta)$,
2. $|C_\beta| \leq \lambda^{+n}$,
3. C_β contains a closed β-unbounded set of order type $\mathrm{cf}(\beta)$,
4. $E \in C_\beta$, $\gamma < \beta$ implies $E \cap \gamma \in C_\gamma$.

We will define an $<_{I^*}$-increasing sequence $\langle f_\alpha : \alpha < \lambda^{+n}\rangle$ in $\prod a$. Let $f_0 \in \prod a$ be arbitrary. Given $\langle f_\gamma : \gamma < \beta\rangle$ for some $\beta < \lambda^{+n}$, let $h \in \prod a$ be such that $f_\gamma <_{I^*} h$ for all $\gamma < \beta$, this is possible since $\prod a/I^*$ is λ^{+n+1}-directed. For $E \in C_\beta$ and $\alpha \in a$, let

$$g^\beta_E(\alpha) := \begin{cases} h(\alpha), & \text{if } \alpha \leq \mathrm{ot}(E), \\ \max(h(\alpha), \sup_{\gamma \in E} f_\gamma(\alpha)), & \text{if } \alpha > \mathrm{ot}(E). \end{cases}$$

Then $g^\beta_E(\alpha) < \alpha$, hence $g^\beta_E \in \prod a$. Since $|C_\beta| \leq \lambda^{+n}$ and $\prod a/I^*$ is λ^{+n+1}-directed, let $f_\beta \in \prod a$ be an $<_{I^*}$-upper bound for $\{g^\beta_E : E \in C_\beta\}$.

Lemma 8.3. Let D be any ultrafilter such that $D \cap I^* = \emptyset$. Then there are no $\mu < \lambda$ and $S_\alpha \subseteq \alpha$ for each $\alpha \in a$ such that $|S_\alpha| \leq \mu$ and $\prod_{\alpha\in a} S_\alpha/D$ cofinally cuts $\langle f_\alpha/D : \alpha < \lambda^{+n}\rangle$.

Proof. Suppose not, let $S_\alpha \subseteq \alpha$ be such that $\prod_{\alpha\in a} S_\alpha/D$ cofinally cuts $\langle f_\alpha/D : \alpha < \lambda^{+n}\rangle$ and such that $|S_\alpha| \leq \mu$ for some $\mu < \lambda$. Assume that $\mu \leq |a|$, this is without loss of generality since $|a| = \mathrm{cf}(\lambda) < \lambda$. Given any $\beta < \lambda^{+n}$, there is a $k \in \prod_{\alpha\in a} S_\alpha$ and $\beta' < \lambda^{+n}$ such that $f_\beta <_D k <_D f_{\beta'}$. So there exists a λ^{+n}-club set $B \subseteq \lambda^{+n}$ such that for all $\beta, \beta' \in B$ such that $\beta < \beta'$, there exists $k \in \prod_{\alpha\in a} S_\alpha$ such that $f_\beta <_D k <_D f_{\beta'}$. Note that $|B| \geq \mathrm{cf}(\lambda^{+n}) = \lambda^{+n}$. Let β be the supremum of the smallest μ^+ elements of B, then $\mathrm{cf}(\beta) = \mu^+$ and $\mu^+ < \lambda$ since λ is singular. Let $E \in C_\beta$ be a closed β-unbounded set of order type $\mathrm{cf}(\beta)$. Then $B \cap \beta$ and E are β-club, so $B \cap \beta \cap E = E \cap B$ is β-club and $\mathrm{ot}(E \cap B) = \mathrm{cf}(\beta)$, so let $E \cap B = \{\beta_i : i < \mathrm{cf}(\beta)\}$ be an increasing enumeration. For each i, choose $k_i \in \prod_{\alpha\in a} S_\alpha$ such that $f_{\beta_i} <_D k_i <_D f_{\beta i+1}$. We already have $f_{\beta_i} >_D g^{\beta_i}_{E\cap\beta_i}$ for all i, and $g^{\beta_i}_{E\cap\beta_i}(\alpha) \geq f_{\beta_j}(\alpha)$ for all $\alpha > \mathrm{ot}(E \cap \beta_i)$, in particular for $\alpha > \mathrm{ot}(E)$, and all $j < i$. Since $D \cap I^* = \emptyset$, D contains no bounded sets, hence for each $i < \mathrm{cf}(\beta)$, there is an $\alpha_i > \mathrm{ot}(E)$ such that $f_{\beta_i}(\alpha_i) < k_i(\alpha_i) < f_{\beta_{i+1}}(\alpha_i)$ and $f_{\beta_i}(\alpha_i) > g^{\beta_i}_{E\cap\beta_i}(\alpha_i)$. Let $\ell(\mathrm{cf}(\beta))$ be the set of limit ordinals in $\mathrm{cf}(\beta)$. Then consider the map $\ell(\mathrm{cf}(\beta)) \to a$ given by $i \mapsto \alpha_i$. Since $|\ell(\mathrm{cf}(\beta))| = |\mathrm{cf}(\beta)| = \mu^+ > \mu$, there must be a set $I \subseteq \ell(\mathrm{cf}(\beta))$ such that $|I| = \mathrm{cf}(\beta)$ and there is an $\alpha \in a$ such that $\alpha_i = \alpha$ for all $i \in I$. Then we have

$$k_i(\alpha) \leq f_{\beta_{i+1}}(\alpha) \leq g^{\beta_j}_{E\cap\beta_j}(\alpha) < f_{\beta_j}(\alpha) \leq k_j(\alpha)$$

for $i, j \in I$ such that $i < j$. Thus $\langle k_i(\alpha) : i \in I\rangle$ is increasing, which contradicts $|S_\alpha| \leq \mu < \mu^+ = \mathrm{cf}(\beta)$. □

Let D be an ultrafilter on a such that $D \cap I^* = \emptyset$. Then $\langle f_\alpha : \alpha < \lambda^{+n}\rangle$ is $<_D$-increasing. Now D is an ultrafilter on a; it transposes to an ultrafilter $\tilde{D}$ on $|a|$ via a bijection $i : a \leftrightarrow |a|$. Any $f \in \mathbf{ON}^a$ transposes to an $\tilde{f} \in \mathbf{ON}^{|a|}$ by $\tilde{f}(x) := f(i^{-1}(x))$. Then $\langle \tilde{f}_\alpha : \alpha < \lambda^{+n}\rangle$ is $<_{\tilde{D}}$-increasing in $\mathbf{ON}^{|\alpha|}$. Also $\lambda^{+n} \geq \lambda^+ > \mathrm{cf}(\lambda)^+ = |a|^+$, and λ^{+n} and $|a|$ are regular. Hence Lemma 3.32 yields that $\langle \tilde{f}_\alpha/\tilde{D} : \alpha < \lambda{+}n\rangle$ has a least upper bound in $\mathbf{ON}^{|a|}/\tilde{D}$ or there exist sets S_δ for $\delta < |a|$ such that $|S_\delta| \leq |a|$ and $\prod_{\delta<|a|} S_\delta/\tilde{D}$ cofinally cuts $\langle \tilde{f}_\alpha/\tilde{D} : \alpha < \lambda^{+n}\rangle$.

Suppose the second. Consider $T_\alpha := S_{i(\alpha)} \cap \alpha \subseteq \alpha$ and $\prod_{\alpha\in a} T_\alpha$. We will show that $\prod_{\alpha\in a} T_\alpha$ cofinally cuts $\langle f_\gamma/D : \gamma < \lambda^{+n}\rangle$, which contradicts Lemma 3.34. Let $\alpha < \lambda^{+n}$. Let $h \in \prod_{\delta<|a|} S_\delta$

such that $\tilde{f}_\alpha <_{\tilde{D}} h$ and $h/\tilde{D}$ cuts $\langle \tilde{f}_\gamma/\tilde{D} : \gamma < \lambda^{+n}\rangle$. Then $h <_{\tilde{D}} f_{\alpha'}$ for some $\alpha' < \lambda^{+n}$. So $[h < f_{\alpha'}] \in \tilde{D}$. So without loss of generality we assume that $h < \tilde{f}_{\alpha'}$ everywhere. Let $h'(x) := h(i(x)) < \tilde{f}_{\alpha'}(i(x)) = f_{\alpha'}(x) < x$, so $h \in \prod_{\alpha\in a} T_\alpha$. So $\prod_{\alpha \in a} T_\alpha$ cofinally cuts $\langle f_\gamma/D : \gamma < \lambda^{+n}\rangle$.

Thus $\langle \tilde{f}_\gamma/\tilde{D} : \gamma < \lambda^{+n}\rangle$ has a least upper bound $\mathbf{ON}^{|a|}/\tilde{D}$, and thus also $\langle f_\gamma/D : \gamma < \lambda^{+n}\rangle$ has a least upper bound g/D in $\mathbf{ON}^a/D$. Now $\prod a/I^*$ is λ^{+n+1}-directed, hence there is an upper bound $h/I^* \in \prod a/I^*$. Thus $g \leq_D h$ and thus $g \leq h$ everywhere and $g \in \prod a$ without loss of generality, i.e. $g(\alpha) < \alpha$ for all $\alpha \in a$. If $\{\alpha \in a : \mathrm{cf}(g(\alpha)) \leq |a|\} \in D$, then this would violate Lemma 8.3:

For these α, we let S_α be cofinal in $g(\alpha)$ of size $\mathrm{cf}(g(\alpha))$ and for all other α let $S_\alpha = \{0\}$. Then $|S_\alpha| \leq |a|$. Then for any $\alpha < \lambda^{+n}$ there exists an $h \in \prod_{\alpha\in a} S_\alpha$ such that $f_\alpha <_D h$. But $h <_D g$ so there exists $\beta < \lambda^{+n}$ such that $h <_D f_\beta$. Thus $\prod_{\alpha \in a} S_\alpha$ cofinally cuts $\langle f_\alpha/D : \alpha < \lambda^{+n}\rangle$, a contradiction.

We thus must have $\{\alpha \in a : \mathrm{cf}(g(\alpha)) > |a|\} \in D$ and thus $\mathrm{cf}(g(\alpha)) > |a|$ for all α without loss of generality.

Consider $\{\mathrm{cf}(g(\alpha)) : \alpha \in a\} = \{\mathrm{cf}(g(\rho^{+n})) : \rho \in C\}$. Define $S_0 := \{\rho \in C : \mathrm{cf}(g(\rho^{+n})) < \rho\}$ and for $1 \leq k < n$, define $S_k := \{\rho \in C : \mathrm{cf}(g(\rho^{+n})) = \rho^{+k}\}$. Then $C = S_0 \,\dot\cup\, \dot\bigcup_{k=1}^{n-1} S_k$ and $a = S_0^{+n} \,\dot\cup\, \dot\bigcup_{k=1}^{n-1} S_k^{+n}$. Exactly one S_k^{+n} $(0 \leq k < n)$ must be in D.

Lemma 8.4. We have $S_0^{+n} \in D$.

Proof. If not, let $1 \leq k < n$ such that $S_k^{+n} \in D$. Since $k < n$, we have $\{\rho \in C : \rho^{+k} \in \bigcup_{m=1}^k b_{\lambda^{+m}}(c)\}$ has a λ-club subset; let $K \subseteq C$ be λ-club such that $K^{+k} \subseteq \bigcup_{m=1}^k b_{\lambda^{+m}}(c)$. Let $a' := C^{+k}$ and

$$D' := \{A \subseteq a' : \{\rho^{+n} : \rho \in C, \rho^{+k} \in A\} \in D\} = \{A^{-(n-k)} : A \in D\},$$

i.e. we shift D so that it becomes an ultrafilter on a'. For any $f \in \prod a$, we define $f'(\rho^{+k}) = f(\rho^{+n})$ for $\rho \in C$. Then $f' : a' \to \bigcup a$, but $f' \notin \prod a'$ is possible. For $f \in \prod a'$ we define $\hat{f}(\rho^{+n}) := f(\rho^{+k})$ for $\rho \in C$, then always $\hat{f} \in \prod a$. Clearly $(.)'$ and $\hat{(.)}$ are each others inverses. Also $f <_D g$ if and only if $f' <_{D'} g'$. We have the sequence $\langle f_\alpha/D : \alpha < \lambda^{+n}\rangle$ in $\prod a/D$, which is increasing and has least upper bound g/D. Another fact is that we assumed that $\{\rho^{+n} : \rho \in C, g(\rho^{+n}) = \rho^{+k}\} \in D$, so $\{\rho^{+k} : \rho \in C, g'(\rho^{+k}) = \rho^{+k}\} \in D'$. Since $f_\alpha <_D g$ for all α, we have $f'_\alpha <_{D'} g' =_D \mathrm{id}_{a'}$ for all α. Thus $f'_\alpha \in \prod a'$. Hence $\langle f'_\alpha/D : \alpha < \lambda^+\rangle$ is an increasing sequence in $\prod a'/D'$. Now suppose there is some h such that $f'_\alpha <_{D'} h$ for all α. Then $f_\alpha <_D \hat{h}$ for all α, so $g \leq_D \hat{h}$, so $h \notin \prod a'$. Therefore $\langle f'_\alpha/D : \alpha < \lambda^+\rangle$ is increasing and cofinal (since D' is an ultrafilter) in $\prod a'/D'$ thus $\mathrm{cf}(\prod a'/D') = \lambda^{+n}$. Consider $A \in D$. Then $A \notin I$, thus A^{-n} is stationary in λ, hence intersects K. Thus K^{+n} intersects A. Thus K^{+n} intersects each element of D. By the finite intersection property and the fact that D is maximal, we must have $K^{+n} \in D$ as well. Hence $K^{+k} \in D'$. Hence $\bigcup_{m=1}^k b_{\lambda^{+m}}(c) \cap a' \in D'$, by which $b_{\lambda^{+m}}(a') = b_{\lambda^{+m}}(c) \cap a' \in D'$ for some $1 \leq m \leq k$, hence $\mathrm{cf}(\prod a'/D') < \lambda^{+m+1} \leq \lambda^{+n}$. This contradicts $\mathrm{cf}(\prod a'/D') = \lambda^{+n}$. □

Definition 8.5. Let $b \subseteq a$. Then we call $\langle f_\beta : \beta < \lambda^{+n}\rangle$ *cofinal in* $\prod b/I^*$ *below* g iff for any $k \in \prod b$ such that $k <_{I^*} g \restriction b$, we have $k \leq_{I^*} f_\beta \restriction b$ for some $\beta < \lambda^{+n}$.

Lemma 8.6. We have $\langle f_\beta : \beta < \lambda^{+n}\rangle$ cofinal in $\prod b/I^*$ below g for some $b \in D$.

Proof. Suppose not. We recursively define a sequence $\langle h_\beta : \beta < |a|^+\rangle$ in $\prod a$ and a sequence $\langle i_\beta : \beta < |a|^+\rangle$ in λ^+ such that

1. $\langle h_\beta : \beta < |a|^+\rangle$ is $\leq$-increasing,
2. $h_\beta < g$ for all β,
3. For all β and $\alpha \in (i_\beta, \lambda^{+n})$ we have $[f_\alpha > h_{\beta+1}] \subset [f_\alpha > h_\beta]$.

We do this as follows:

1. Base case: $h_0(\gamma) = 0$ for all γ.
2. Successor case: Suppose $\langle h_{\beta'} : \beta' \leq \beta\rangle$ and $\langle i_{\beta'} : \beta' < \beta\rangle$ have been defined. If $[f_\alpha > h_\beta] \notin D$ for all α, then $f_\alpha \leq_D h_\beta$, hence $f_\alpha <_D h_\beta$ for all α. But $h_\beta < g$, contradicting the minimality of g. So let $i_\beta < \lambda^{+n}$ such that $b^\beta_{i_\beta} := [f_{i_\beta} > h_\beta] \in D$. By assumption $\langle f_\gamma : \gamma < \lambda^{+n}\rangle$ is not cofinal in $\prod b^\beta_{i_\beta}/I^*$ below g. Let $h \in \prod b^\beta_{i_\beta}$ such that $h <_{I^*} g \restriction b^\beta_{i_\beta}$ but $h \not\leq_{I^*} f_\gamma \restriction b^\beta_{i_\beta}$ for all $\gamma < \lambda^{+n}$ and assume without loss of generality that $h < g \restriction b^\beta_{i_\beta}$ everywhere. Now extend h to $\prod a$ by $h(\gamma) = 0$ for $\gamma \in a \setminus b^\beta_{i_\beta}$. Then let $h_{\beta+1} = \max(h_\beta, h)$. Then $h_{\beta+1} \in \prod a$, $h_{\beta+1} \geq h_\beta$ and $h_{\beta+1} < g$. Let $\alpha \in (i_\beta, \lambda^{+n})$ be arbitrary. We have $h \not\leq_{I^*} f_\alpha \restriction b^\beta_{i_\beta}$ so $[h > f_\alpha] \cap b^\beta_{i_\beta} \notin I^*$. But $f_\alpha >_{I^*} f_{i_\beta}$. So there is some $\gamma \in [f_\alpha > h_\beta]$ such that $h(\gamma) > f_\alpha(\gamma)$. Since $h_{\beta+1} \geq h$ we obtain $[f_\alpha > h_{\beta+1}] \subset [f_\alpha > h_\beta]$.
3. Limit case: Suppose $\langle h_{\beta'} : \beta' < \beta\rangle$ and $\langle i_{\beta'} : \beta' < \beta\rangle$ have been defined for some limit ordinal $\beta < |a|^+$. Let $h_\beta(\gamma) := \sup(\{h_{\beta'}(\gamma) : \beta' < \beta\})$. Since $\beta < |a|^+ \leq \mathrm{cf}(g(\gamma))$, we have $h_\beta(\gamma) < g(\gamma) < \gamma$. So $h_\beta \in \prod a$ and $h_\beta < g$.

Let $i := \sup_{\beta<|a|^+} i_\beta$. Since $i_\beta < \lambda^{+n}$ for all β, $|a|^+ = \mathrm{cf}(\lambda)^+ < \lambda < \lambda^{+n}$ and λ^{+n} is regular, we have $i < \lambda^{+n}$. Then for any $\alpha \in (i, \lambda^{+n})$ we have that $\langle [f_\alpha > h_\beta]\rangle_{\beta<|a|^+}$ is a $\subset$-decreasing sequence of subsets of a, which is a contradiction. ☐

So let $b \in D$ such that $\langle f_\beta : \beta < \lambda^{+n}\rangle$ is cofinal in $\prod b/I^*$ below g. For $\alpha < \lambda^{+n}$ let

$$d_\alpha := \{\gamma \in b : f_\alpha(\gamma) < g(\gamma)\} = \{\gamma \in a : f_\alpha(\gamma) < g(\gamma)\} \cap b \in D.$$

Since $S_0^{+n} \in D$ we also have $d'_\alpha := d_\alpha \cap S_0^{+n} \in D$, thus $d'_\alpha \notin I^*$ so $d'_\alpha \setminus B \notin I$. Hence $S^\alpha := (d'_\alpha \setminus B)^{-n} = \{\rho \in S_0 : \rho^{+n} \in d_\alpha \setminus B\}$ is stationary in λ. Since $S^\alpha \subseteq S_0$ we have $\mathrm{cf}(g(\rho^{+n})) < \rho$ for all $\rho \in S^\alpha$, thus the function $S^\alpha \to \lambda$ given by $\rho \mapsto \mathrm{cf}(g(\rho^{+n}))$ is regressive. By Lemma 2.24 let $\eta_\alpha < \lambda$ such that $\{\rho \in S^\alpha : \mathrm{cf}(g(\rho^{+n})) \leq \eta_\alpha\}$ is λ-stationary. Then $\{\gamma \in (d'_\alpha \setminus B) : \mathrm{cf}(g(\gamma)) \leq \eta_\alpha\} \notin I$, thus $\{\gamma \in d'_\alpha : \mathrm{cf}(g(\gamma)) \leq \eta_\alpha\} \notin I^*$ and $\{\gamma \in b : f_\alpha(\gamma) < g(\gamma) \text{ and } \mathrm{cf}(g(\gamma)) \leq \eta_\alpha\} \notin I^*$. Since $\eta_\alpha < \lambda$ for all $\alpha < \lambda^{+n}$, let $\eta < \lambda$ such that $\{\alpha < \lambda^{+n} : \eta_\alpha = \eta\}$ has cardinality λ^{+n}.

Suppose $\eta_\alpha = \eta$ and $\beta < \alpha$. Then $X := \{\gamma \in b : f_\alpha(\gamma) < g(\gamma) \text{ and } \mathrm{cf}(g(\gamma)) \leq \eta\} \notin I^*$ and $[f_\beta \not< f_\alpha] \in I^*$, hence $I^* \not\ni X \cap [f_\beta < f_\alpha] \subseteq \{\gamma \in b : f_\beta(\gamma) < g(\gamma) \text{ and } \mathrm{cf}(g(\gamma)) \leq \eta\}$. Hence we could have chosen $\eta_\beta = \eta$ as well.

Thus we can assume that $c_\alpha := \{\gamma \in b : f_\alpha(\gamma) < g(\gamma) \text{ and } \mathrm{cf}(g(\gamma)) \leq \eta\} \notin I^*$, for all $\alpha < \lambda^{+n}$. Note that $\langle c_\alpha : \alpha < \lambda^{+n}\rangle$ is $\subseteq_{I^*}$-decreasing. Hence Lemma 3.49 applies and there is an ultrafilter D^* on a disjoint from I^* which contains all the c_α, and thus contains b since $b \supseteq c_0$. Then $f_\alpha <_{D^*} g$ for all $\alpha < \lambda^{+n}$. So g is an $<_{D^*}$-upper bound for $\langle f_\alpha : \alpha < \lambda^{+n}\rangle$. It is also a least upper bound:

If $h/D^* < g/D^*$, then without loss of generality $h < g$ everywhere and $h \in \prod a$. Then $h \restriction b \leq_{I^*} f_\alpha \restriction b$ for some α. Since $b \in D^*$, we have $h/D^* \leq f_\alpha/D^* < f_{\alpha+1}/D^*$, hence h is not an upper bound for $\langle f_\alpha/D^* : \alpha < \lambda^{+n}\rangle$.

Now let $e := \{\gamma \in b : \mathrm{cf}(g(\gamma)) \leq \eta\} \in D^*$. For $\gamma \in e$, let $S_\gamma \subseteq g(\gamma)$ be cofinal in $g(\gamma)$ and such that $|S_\gamma| \leq |\eta| < \lambda$. For $\gamma \in (a \setminus c)$, let $S_\gamma = \{0\}$. Note that $g \in \prod a$, hence $S_\gamma \subseteq \gamma$. On the one hand, $\prod_{\gamma\in a} S_\gamma/D^*$ cannot cofinally cut $\langle f_\alpha/D^* : \alpha < \lambda^{+n}\rangle$ by Lemma 8.3. On the other hand, it does cofinally cut: Let $\alpha < \lambda^{+n}$. For $\gamma \in c$, let $k(\gamma) \in S_\gamma$ such that $\{\gamma \in c : f_\alpha(\gamma) < k(\gamma)\} \in D^*$, this is possible since $f_\alpha <_{D^*} g$ and $e \in D^*$. Extend k to a function in $\prod_{\gamma\in a} S_\gamma$ by taking $k(\gamma) = 0$ for $\gamma \in (a \setminus e)$. Then $f_\alpha <_{D^*} k$ and $k <_{D^*} g$. Since g is a least upper bound, we must have $k <_{D^*} f_\beta$ for some $\beta < \lambda^{+n}$. But now we have shown that $\prod_{\gamma\in a} S_\gamma$ cofinally cuts $\langle f_\alpha/D^* : \alpha < \lambda^{+n}\rangle$. Thus our first assumption, that Theorem 8.2 is false, is false. This proves Theorem 8.2.

8.2 An extra assumption on generators

Most of this subsection will be concerned with proving the following lemma.

Lemma 8.7. Let a be an infinite set of regular cardinals such that $2^{|a|} < \min(a)$. Then we can choose the generators $\langle b_\lambda : \lambda \in \text{pcf}(a)\rangle$ such that $\mu \in \text{pcf}(b_\lambda)$ implies $b_\mu \subseteq b_\lambda$

So this lemma states that we can choose the generators in such a way that the relation $\mu \prec \lambda$ defined by $\mu \in b_\lambda$ is transitive: If $\mu \in b_\lambda$ and $\lambda \in b_\nu$, then $b_\lambda \subseteq b_\nu$, so $\mu \in b_\nu$ hence $\mu \prec \nu$.

Before starting on the proof, we first note that by using this lemma, we can even make an extra assumption on the generators:

Lemma 8.8. Let a be an infinite set of regular cardinals such that $2^{|a|} < \min(a)$ and suppose we have a sequence of generators $\langle b_\lambda : \lambda \in \text{pcf}(a)\rangle$ be such that $\mu \in \text{pcf}(b_\lambda)$ implies $b_\mu \subseteq b_\lambda$. Then we may arrange that $\text{pcf}(b_\lambda) = b_\lambda$ for all λ.

Proof. From $\langle b_\lambda : \lambda \in \text{pcf}(a)\rangle$ we will define a sequence $\langle b^*_\lambda : \lambda \in \text{pcf}(a)\rangle$ such that $\mu \in \text{pcf}(b^*_\lambda)$ implies $b^*_\mu \subseteq b^*_\lambda$ and $\text{pcf}(b^*_\lambda) = b^*_\lambda$.

1. If $\lambda = \min(a)$, then $b^*_\lambda := b_\lambda = \{\lambda\}$. Clearly $\text{pcf}(b^*_\lambda) = b^*_\lambda$ and $\rho \in b^*_\lambda \Rightarrow b^*_\rho \subseteq b^*_\lambda$.
2. Suppose b^*_θ has been defined for all $\theta < \lambda$. Since $\text{pcf}(\text{pcf}(b_\lambda)) = \text{pcf}(b_\lambda) \subseteq \lambda^+$, by Lemma 3.48, there exist $\theta_1, ..., \theta_n \in \text{pcf}(b_\lambda) \cap \lambda$ such that $\text{pcf}(b_\lambda) \subseteq b^*_{\theta_1} \cup ... \cup b^*_{\theta_n} \cup b_\lambda =: b^*_\lambda$. Since $b_\lambda \subseteq b^*_\lambda \in J_{<\lambda^+}$, we have that b^*_λ is indeed a generator. Now

$$\text{pcf}(b^*_\lambda) = \bigcup_{i=1}^{n} \text{pcf}(b^*_{\theta_1}) \cup \text{pcf}(b_\lambda) = \bigcup_{i=1}^{n} b^*_{\theta_i} \cup \text{pcf}(b_\lambda) = b^*_\lambda.$$

   If $\rho \in b^*_\lambda$ and for all $\eta \in b^*_\lambda$ with $\eta < \rho$ we have $b^*_\eta \subseteq b^*_\lambda$, then
   (a) $\rho \in b^*_{\theta_i}$, so $b^*_\rho \subseteq b^*_{\theta_i} \subseteq b^*_\lambda$, or
   (b) $\rho \in b_\lambda$, so $b_\rho \subseteq b_\lambda \subseteq b^*_\lambda$ and $b^*_\rho = b^*_{\eta_1} \cup ... \cup b^*_{\eta_n} \cup b_\rho$ for $\eta_1, ..., \eta_n \in \text{pcf}(b_\rho) \cap \rho \subseteq \text{pcf}(b_\lambda) \subseteq b^*_\lambda$, so $b^*_{\eta_1}, ..., b^*_{\eta_n} \subseteq b^*_\lambda$, thus $b^*_\rho \subseteq b^*_\lambda$.
   So by induction, if $\rho \in \text{pcf}(b^*_\lambda) = b^*_\lambda$ then $b^*_\rho \subseteq b^*_\lambda$.

So $\langle b^*_\lambda : \lambda \in \text{pcf}(a)\rangle$ is as desired: For all $\lambda \in \text{pcf}(a)$, b_λ generates $J_{<\lambda^+}$ over $J_{<\lambda}$, $\rho \in \text{pcf}(b^*_\lambda)$ implies $b^*_\rho \subseteq b^*_\lambda$ and $\text{pcf}(b^*_\lambda) = b^*_\lambda$. □

We now start with the proof of Lemma 8.7. Let a be an infinite set of regular cardinals such that $2^{|a|} < \min(a)$.

Let θ be a large enough regular cardinal and consider again $H(\theta)^* = (H(\theta), \in, < *)$. For any $M \prec H(\theta)^*$ let χ_M denote its characteristic function defined by $\chi_M(\mu) := \sup(M \cap \mu)$ for any regular cardinal μ. Similar to section 6, we have a notion of a nice elementary substructure.

Definition 8.9. An structure $N \prec H(\theta)^*$ is called *nice* iff

1. $|N| = 2^{|a|}$,
2. there exists a sequence $\langle N_i : i < 2^{|a|} \rangle$ such that
   (a) $N_i \prec H(\theta)^*$ for all i,
   (b) $N_i \subseteq N_{i+1}$ for all i,
   (c) $N_i = \bigcup_{j<i} N_j$ for all limit i,
   (d) $N = \bigcup_{i<2^{|a|}} N_i$,
   (e) $\langle N_j : j \leq i \rangle \in N_{i+1}$ for all i,
   (f) $2^{|a|} \subseteq N_0$,
   (g) $a \in N_0$.

Of course, similar to Lemma 6.4, for any $x \in H(\theta)^*$ there exists a nice structure $N \prec H(\theta)^*$ such that $x \in N_0$.

Lemma 8.10. Let $\lambda \in \mathrm{pcf}(a)$. Suppose $f = \langle f_i : i < \lambda \rangle$ is *persistently cofinal*, i.e. for all $h \in \prod a$ there exists $i < \lambda$ such that $h \restriction b_\lambda <_{J_{<\lambda}} f_j \restriction b_\lambda$ for all $j \geq i$, where b_λ is any generator of $J_{<\lambda^+}$ over $J_{<\lambda}$ (note that this is independent of the chosen generator, since two generators only differ on a set in $J_{<\lambda}$). Let $N \prec H(\theta)^*$ be a nice structure such that $f \in N_0$. Then $[\chi_N \restriction a \leq f_{\chi_N(\lambda)}]$ generates $J_{<\lambda^+}$ over $J_{<\lambda}$.

Proof. Let $i^* = \chi_N(\lambda)$. Since $a \in N_0$, we have $\mathrm{pcf}(a) \in N_0$. Bu $|\mathrm{pcf}(a)| \leq 2^{|a|}$ and $2^{|a|} \subseteq N_0$. So $\mathrm{pcf}(a) \subseteq N_0$, and $\lambda \in N_0$. So $J_{<\lambda^+}, J_{<\lambda} \in N_0$ and there is some generator $b_\lambda \in N_0$. For any $\alpha \in a$, if $f_{i^*}(\alpha) < \chi_N(\alpha)$, let $i_\alpha < 2^{|a|}$ be such that $f_{i^*}(\alpha) < \chi_{N_{i_\alpha}}(\alpha)$. Since $|a| < 2^{|a|}$, there is some $\bar\imath$ such that $i_\alpha < \bar\imath$ for all α, and hence

$$[f_{i^*} < \chi_N \restriction a] = [f_{i^*} < \chi_{N_{\bar\imath}} \restriction a].$$

Note that $\chi_{N_{\bar\imath}} \restriction a \in \prod a \cap N$, since $\chi_{N_i} \restriction a$ is definable in N_{i+1} for all i. So $(\chi_{N_{\bar\imath}} \restriction a) \restriction b_\lambda <_{J_{<\lambda}} f_j \restriction b_\lambda$ for some $j \in N \cap \lambda$, and thus $\chi_{N_{\bar\imath}} \restriction b_\lambda <_{J_{<\lambda}} f_{i^*} \restriction b_\lambda$, since $j < i^*$. So $[f_{i^*} \restriction b_\lambda < \chi_N \restriction b_\lambda] = [f_{i^*} \restriction b_\lambda < \chi_{N_{\bar\imath}} \restriction b_\lambda] \in J_{<\lambda}$.
Note that $\lambda \notin \mathrm{pcf}(a \setminus b_\lambda)$: If $\mathrm{cf}(\prod(a \setminus b_\lambda)/D) = \lambda$, then $\mathrm{cf}(\prod a/\hat{D}) = \lambda$ for some $\hat{D}$ which does not contain b_λ. But this contradicts $\mathrm{cf}(\prod a/\hat{D}) = \min(\lambda : b_\lambda \in \hat{D})$.
So $J_{<\lambda}(a \setminus b_\lambda) = J_{<\lambda^+}(a \setminus b_\lambda)$ and $\prod(a \setminus b_\lambda)/J_{<\lambda} = \prod(a \setminus b_\lambda)/J_{<\lambda^+}$ is λ^+-directed. So we have the following

$$\begin{aligned} H(\theta)^* &\models \exists h\, \forall i < \lambda\, (f_i \restriction (a \setminus b_\lambda) <_{J_{<\lambda}} h) \\ N &\models \exists h\, \forall i < \lambda\, (f_i \restriction (a \setminus b_\lambda) <_{J_{<\lambda}} h) \\ \text{There is an } h \in N \text{ such that } N &\models \forall i < \lambda\, (f_i \restriction (a \setminus b_\lambda) <_{J_{<\lambda}} h) \\ \text{There is an } h \in N \text{ such that } H(\theta)^* &\models \forall i < \lambda\, (f_i \restriction (a \setminus b_\lambda) <_{J_{<\lambda}} h) \end{aligned}$$

Let $h \in \prod(a \setminus b_\lambda) \cap N$ be such, then in particular $f_{i^*} \restriction (a \setminus b_\lambda) <_{J_{<\lambda}} h$. So $[\chi_N \leq f_{i^*}] \setminus b_\lambda \subseteq [f_{i^*} \geq h] \cap (a \setminus b_\lambda) \in J_{<\lambda}$. Combining this with $[\chi_N \restriction b_\lambda > f_{i^*} \restriction b_\lambda] \in J_{<\lambda}$, we get $[\chi_N \restriction a \leq f_{i^*}] =_{J_{<\lambda}} b_\lambda$ and indeed this set is generating. □

For any $M \prec H(\theta)^*$, let

$$\overline{M} = \{\gamma \in \mathbf{ON} : \sup(M \cap \gamma) = \gamma \text{ or } \gamma \in M\} = \{\sup(M \cap \gamma) : \gamma \in \mathbf{ON}\} \cup \{\gamma \in \mathbf{ON} : \gamma \in M\},$$

i.e. $\overline{M}$ is the ordinal closure of M. If $N \prec H(\theta)^*$ is nice, then

1. since $N_i \in N_{i+1}$ and the ordinal closure is definable, we have $\overline{N_i} \in N_{i+1}$.
2. since $N_i \in N_{i+1}$ and $N_i \subseteq N_{i+1}$, we have $\overline{N_i} \subseteq N_{i+1}$: Let $\gamma \in \overline{N_i}$, then $\gamma \in N_i \subseteq N_{i+1}$ or $\sup(N_i \cap \gamma) = \gamma$. In the second case, if $\delta \in N_i$ is minimal above (or equal to) γ, then $\gamma = \sup(N_i \cap \gamma) = \sup(N_i \cap \delta) \in N_{i+1}$. If $N_i \setminus \gamma = \emptyset$, then $\gamma = \sup(N_i \cap \gamma) = \sup(N_i \cap \mathbf{ON}) \in N_{i+1}$. So $\gamma \in N_{i+1}$ in all cases.

Definition 8.11. Let λ and μ be cardinals and let a be any set. A sequence $\langle f_i : i < \lambda \rangle$ of functions with domain a is called *μ-minimally obedient* iff for all $i < \lambda$ such that $\mathrm{cf}(i) = \mu$, we have

$$f_i(\alpha) = \min(\sup(f_j(\alpha) : j \in C) : C \text{ is an } i\text{-clubset of order type } \mu)$$

for all $\alpha \in a$.

Suppose $\langle f_i : i < \lambda \rangle$ is μ-minimally obedient. Suppose $i < \lambda$ satisfies $\mathrm{cf}(i) = \mu$, and for $\alpha \in a$ choose C_α i-club of order type μ such that $f_i(\alpha) = \sup(f_j(\alpha) : j \in C_\alpha)$. Consider $C := \bigcap_{\alpha \in a} C_\alpha$. If $|a|, \omega < \mu$, then C is also an i-club set of order type μ and we have

$$f_i(\alpha) = \sup(f_j(\alpha) : j \in C)$$

for all $\alpha \in a$.

Lemma 8.12. Let $\lambda \in \mathrm{pcf}(a)$. Suppose $f = \langle f_i : i < \lambda \rangle$ is $2^{|a|}$-minimally obedient. Let $N \prec H(\theta)^*$ be nice and such that $f \in N_0$. If $\gamma \in (\overline{N} \setminus N) \cap \lambda$, then there exists a γ-clubset D of order type $2^{|a|}$ such that $f_\gamma = \sup(f_j : j \in D)$ and $D \subseteq N$. In particular $f_\gamma(\alpha) \in \overline{N}$ for all $\alpha \in a$, and thus $f_\gamma \leq \chi_N \restriction a$.

Proof. We have $\gamma = \sup(N \cap \gamma)$ and thus $\langle \sup(N_i \cap \gamma) : i < 2^{|a|} \rangle$ is cofinal in γ. This sequence is also increasing: $\sup(N_i \cap \gamma) \in \overline{N_i} \subseteq N_{i+1}$ and $\sup(N_i \cap \gamma) < \gamma$, so $\sup(N_i \cap \gamma) < \sup(N_{i+1} \cap \gamma)$. Thus $\mathrm{cf}(\gamma) = 2^{|a|}$. Since $|a|, \omega < 2^{|a|}$, there exists an i-club C of order type $2^{|a|}$ such that $f_\gamma = \sup(f_i : i \in C)$. Then $D := C \cap \{\sup(N_i \cap \gamma) : i < 2^{|a|}\}$ suffices: $D \subseteq N$ and $D \subseteq C$, so $f_\gamma = \sup(f_i : i \in D)$. Since $f \in N$, $D \subseteq N$ and $a \subseteq N$, $f_\gamma(\alpha)$ is a supremum of elements in N, so an element of $\overline{N}$, for all $\alpha \in a$. ☐

Remark 8.13. In particular, if $\gamma = \chi_N(\lambda)$, then $\gamma \in (\overline{N} \setminus N) \cap \lambda$ and the lemma applies, so $f_{\chi_N(\lambda)} \leq \chi_N$.

Now for each $\lambda \in \mathrm{pcf}(a)$, let $f^\lambda = \langle f_i^\lambda : i < \lambda \rangle$ be λ-universal and $2^{|a|}$-minimally obedient. This is possible: Start with a λ-universal sequence $\langle g_i : i < \lambda \rangle$ and define $f_0^\lambda = g_0$, $f_{i+1}^\lambda > \max(f_i, g_i)$ for all i, if $\mathrm{cf}(i) = 2^{|a|}$ of course take $f_i^\lambda(\alpha) = \min(\sup(f_j^\lambda(\alpha) : j \in C) : C$ is an i-clubset of order type $2^{|a|})$ and if i is limit but $\mathrm{cf}(i) \neq 2^{|a|}$, let f_i^λ be a $<_{J_{<\lambda}}$-upper bound for $\{f_j^\lambda : j < i\}$, which exists since $\prod a/J_{<\lambda}$ is λ-directed. Then clearly $\langle f_i^\lambda : i < \lambda \rangle$ is $<_{J_{<\lambda}}$-increasing, cofinal in $\prod a/D$ for all D such that $\mathrm{cf}(\prod a/D) = \lambda$, and $2^{|a|}$-minimally obedient.

Now for $\lambda \in \mathrm{pcf}(a)$ and $\gamma < \lambda$, we will define functions $F_\gamma^\lambda \in \prod a$. First, for $\lambda \in \mathrm{pcf}(a)$ and $\alpha \in a$, define

$$\begin{aligned} F_{\lambda,\alpha} :&= \{\text{finite decreasing sequences in } a \text{ from } \lambda \text{ to } \alpha\} \\ &= \{\langle \lambda_0, ..., \lambda_n \rangle : 0 < n < \omega, \lambda_0 = \lambda, \lambda_n = \alpha, \lambda_{i+1} \in a \cap \lambda_i \text{ for all } i < n\}. \end{aligned}$$

For each sequence $\langle \lambda_0, ..., \lambda_n \rangle \in F_{\lambda,\alpha}$ and each $\gamma < \lambda$, define $\gamma_0 = \gamma$ and $\gamma_{i+1} = f^{\lambda_i}_{\gamma_i}(\lambda_{i+1})$ for $i < n$, and set $\mathrm{El}_{\langle \lambda_0,...,\lambda_n \rangle}(\gamma) := \gamma_n$. Define

$$M^{\gamma}_{\lambda,\alpha} := \{\mathrm{El}_{\langle \lambda_0,...,\lambda_n \rangle}(\gamma) : \langle \lambda_0, ..., \lambda_n \rangle \in F_{\lambda,\alpha}\}.$$

Now define $F^{\lambda}_{\gamma}(\alpha) := \max M^{\gamma}_{\lambda,\alpha}$ if this maximum exists, and otherwise $F^{\lambda}_{\gamma}(\alpha) = f^{\lambda}_{\gamma}(\alpha)$.

Then we have $f^{\lambda}_{\gamma} \leq F^{\lambda}_{\gamma}$: If $\alpha < \lambda$, then $\langle \lambda, \alpha \rangle \in F_{\lambda,\alpha}$, so $\mathrm{El}_{\langle \lambda,\alpha \rangle}(\gamma) = f^{\lambda}_{\gamma}(\alpha) \in M^{\gamma}_{\lambda,\alpha}$. If $\alpha \geq \lambda$, then $M^{\gamma}_{\lambda,\alpha} = \emptyset$. In both cases $F^{\lambda}_{\gamma}(\alpha) \geq f^{\lambda}_{\gamma}(\alpha)$.

Now let N be nice and such that $\langle f^{\lambda} : \lambda \in \mathrm{pcf}(a) \rangle \in N_0$. Since $\mathrm{pcf}(a) \subseteq N_0$, each $f^{\lambda} \in N_0$. We have $F^{\lambda}_{\chi_N(\lambda)} \leq \chi_N \restriction a$:

Let $\gamma = \chi_N(\lambda)$. If $\alpha \geq \lambda$, then $F^{\lambda}_{\gamma}(\alpha) = f^{\lambda}_{\gamma}(\alpha) \leq \chi_N(\alpha)$, where the last inequality follows from Lemma 8.12 or Remark 8.13. If $\alpha < \lambda$, then for each $\langle \lambda_0, ..., \lambda_n \rangle \in F_{\lambda,\alpha}$, again by Lemma 8.12, $f^{\lambda}_{\gamma}(\lambda_1) \in \overline{N}$, and by induction and the definition of F^{λ}_{γ}, $F^{\lambda}_{\gamma}(\alpha) \in \overline{N}$. So $F^{\lambda}_{\gamma} \leq \chi_N \restriction a$.

Since F^{λ}_{γ} is definable from f^{λ}_{γ}, and each f^{λ}_{γ} is an element of N, we have that N is nice and $\langle F^{\lambda}_i : i < \lambda \rangle \in N_0$. Also, since $f^{\lambda}_{\gamma} \leq F^{\lambda}_{\gamma}$, the sequence $\langle F^{\lambda}_i : i < \lambda \rangle$ is still persistently cofinal. Thus Lemma 8.10 applies to $\langle F^{\lambda}_i : i < \lambda \rangle$ as well, and we obtain that $[\chi_N \restriction a \leq F^{\lambda}_{\chi_N(\lambda)}]$ generates $J_{<\lambda^+}$ over $J_{<\lambda}$. Combining this with $F^{\lambda}_{\gamma} \leq \chi_N \restriction a$, we get

$$B_{\lambda} := [\chi_N \restriction a = F^{\lambda}_{\chi_N(\lambda)}] \text{ generates } J_{<\lambda^+} \text{ over } J_{<\lambda}.$$

It remains to show that if $\lambda_1 \in B_{\lambda_0}$, then $B_{\lambda_1} \subseteq B_{\lambda_0}$.

So let $\lambda_1 \in B_{\lambda_0}$ and $\alpha \in B_{\lambda_1}$. The cases $\lambda_1 = \lambda_0$ or $\alpha = \lambda_1$ is trivial, so assume $\alpha < \lambda_1 < \lambda_0$. Then

1. since $\lambda_1 \in B_{\lambda_0}$ we have $F^{\lambda_0}_{\chi_N(\lambda_0)}(\lambda_1) = \chi_N(\lambda_1)$. We proved that no sequence reaches a higher value than $\chi_N(\lambda_1)$, and also $f^{\lambda_0}_{\chi_N(\lambda_0)}(\lambda_1) \leq \chi_N(\lambda_1)$. If $f^{\lambda_0}_{\chi_N(\lambda_0)}(\lambda_1) = \chi_N(\lambda_1)$, then the sequence $\langle \lambda_0, \lambda_1 \rangle$ reaches the highest possible value, otherwise, there must be some other sequence reaching it. Thus there is a sequence $s \in F_{\lambda_0,\lambda_1}$ such that $\mathrm{El}_s(\chi_N(\lambda_0)) = \chi_N(\lambda_1)$;
2. by the same reasoning, there is a sequence $t \in F_{\lambda_1,\alpha}$ such that $\mathrm{El}_t(\chi_N(\lambda_1)) = \chi_N(\alpha)$.

But then the concatenation of these sequences is a sequence u in $F_{\lambda_0,\alpha}$ and $\mathrm{El}_u(\chi_N(\lambda_0)) = \chi_N(\alpha)$. But no seqeunce can reach a higher value, so $F^{\lambda_0}_{\chi_N(\lambda_0)}(\alpha) = \chi_N(\alpha)$ and $\alpha \in B_{\lambda_0}$. So we have proved Lemma 8.7.

8.3 Second essential lemma

We have the following setting: Let a be a set of regular cardinals such that $\min(a) > 2^{|a|}$, define $c := \mathrm{pcf}(a)$, let $d \subseteq c$ and suppose $\mu \in \mathrm{pcf}(d)$. Let $\langle b_{\lambda} : \lambda \in \mathrm{pcf}(c) \rangle$ be as in Lemma 8.7, i.e. b_{λ} generates $J_{<\lambda^+}(c)$ over $J_{<\lambda}(c)$, $\rho \in b_{\lambda}$ implies $b_{\rho} \subseteq b_{\lambda}$ and $\mathrm{pcf}(b_{\lambda}) = b_{\lambda}$.

By Theorem 3.15 we have $\mathrm{pcf}(\mathrm{pcf}(a)) = \mathrm{pcf}(a)$. If $e \subseteq \mathrm{pcf}(a)$, then $\min(e) \geq \min(a) > 2^{|a|} \geq 2^{|e|}$, so again Theorem 3.15 applies and we have $\mathrm{pcf}(\mathrm{pcf}(e)) = \mathrm{pcf}(e)$. Also note that $b_{\lambda} = \mathrm{pcf}(b_{\lambda}) \subseteq \lambda + 1$.

Lemma 8.14. There exists $\tilde{d} \subseteq d$ such that $\mathrm{pcf}(\tilde{d}) \subseteq b_{\mu}$ and $\mu \in \mathrm{pcf}(\tilde{d})$.

Proof. Since $\mu \in \mathrm{pcf}(d)$, let D be an ultrafilter on d such that $\mu = \mathrm{cf}(\prod d/D)$. Extend D to an ulrafilter $\hat{D}$ on c, then we know $\mu = \mathrm{cf}(\prod c/\hat{D}) = \min(\{\lambda : b_{\lambda} \in \hat{D}\}$ by Lemma 3.47, hence $b_{\mu} \in \hat{D}$, hence $\tilde{d} := d \cap b_\mu \in D$. Then $\mu \in \mathrm{pcf}(\tilde{d})$ since we can restrict $\hat{D}$ to $\tilde{d}$. Hence $\mu \in \mathrm{pcf}(\tilde{d}) \subseteq \mathrm{pcf}(b_\mu) = b_\mu$. □

Lemma 8.15. As in Lemma 8.14, let $\tilde{d} \subseteq d$ be such that $\mu \in \mathrm{pcf}(\tilde{d}) \subseteq b_\mu$. There exists $\hat{d} \subseteq \tilde{d}$ such that $\mu \in \mathrm{cf}(\hat{d})$ and $\mathrm{pcf}(\hat{d}) \cap \mu$ has no maximum.

Proof. Suppose $\mathrm{pcf}(d) \cap \mu$ has a maximal element μ_1. Let $d_1 := \tilde{d} \setminus b_{\mu_1}$. We will show that $\mu_1 \notin \mathrm{pcf}(d_1)$:

Let D be any ultrafilter on d_1. Extend D to an ultrafilter $\hat{D}$ on c, then $\mathrm{cf}(\prod d_1/D) = \mathrm{cf}(\prod c/\hat{D}) = \min\{\lambda : b_\lambda \in \hat{D}\}$, where the last equality is Lemma 3.47. We have $b_{\mu_1} \notin \hat{D}$, otherwise $\hat{D} \ni d_1 \cap b_{\mu_1} = \emptyset$; hence $\mathrm{cf}(\prod d_1/D) \neq \mu_1$.

Since $\mu \in \mathrm{pcf}(\tilde{d})$, let D be an ultrafilter on $\tilde{d}$ such that $\mathrm{cf}(\prod \tilde{d}/D) = \mu$. Extend D to an ultrafilter $\hat{D}$ on c, then $b_{\mu_1} \notin \hat{D}$ since $\mu_1 < \mu$, hence $d_1 \in \hat{D}$ thus $\mu \in \mathrm{pcf}(d_1)$. Since $\mathrm{pcf}(d_1) \subseteq \mathrm{pcf}(d)$, it follows that $\mathrm{pcf}(d_1) \cap \mu \subseteq \mu_1$. If also $\mathrm{pcf}(d_1) \cap \mu$ has a maximum μ_2, find in the same manner a $d_2 \subseteq d_1$ such that $\mu \in \mathrm{pcf}(d_2)$ and $\mathrm{pcf}(d_2) \cap \mu \subseteq \mu_2$. Repeat this until there is a $d_k \subseteq d$ such that $\mu \in \mathrm{pcf}(d_k)$ but $\mathrm{pcf}(d_k) \cap \mu$ has no maximum. Then $\hat{d} = d_k$ is as desired. □

Lemma 8.16. There exists $d' \subseteq d$ such that $|d'| \leq |a|$ and $\mu \in \mathrm{pcf}(d')$.

Proof. We will prove this by induction on the cardinality μ, so assume that for all $\nu < \mu$ the theorem has been shown. As in Lemma 8.14, let $\tilde{d} \subseteq d$ be such that $\mu \in \mathrm{pcf}(\tilde{d}) \subseteq b_\mu$. As in Lemma 8.15, let $\hat{d} \subseteq \tilde{d}$ be such that $\mu \in \mathrm{pcf}(\hat{d})$ and $\mathrm{pcf}(\hat{d}) \cap \mu$ has no maximum. Note that still $\mathrm{pcf}(\hat{d}) \subseteq b_\mu$. Note that $\mathrm{pcf}(\hat{d}) \cap \mu$ is infinite. Let $\langle \mu_i : i < \kappa \rangle$ be a cofinal subset of $\mathrm{pcf}(\hat{d}) \cap \mu$, where $\kappa = \mathrm{cf}(\mathrm{pcf}(\hat{d}) \cap \mu) \geq \aleph_0$. Then $\mu \in \mathrm{pcf}(\{\mu_i : i < \kappa\})$:

Let D be an ultrafilter on $\{\mu_i : i < \kappa\}$ which contains all the tails. We have

$$\mathrm{cf}(\prod\{\mu_i : i < \kappa\}/D) \in \mathrm{pcf}(\{\mu_i : i < \kappa\}) \subseteq \mathrm{pcf}(\mathrm{pcf}(\hat{d})) = \mathrm{pcf}(\hat{d}) \subseteq b_\mu = \mathrm{pcf}(b_\mu) \subseteq \mu + 1$$

so $\mathrm{cf}(\prod\{\mu_i : i < \kappa\}) < \mu^+$. On the other hand, $\mu_j < \mathrm{cf}(\prod\{\mu_i : i < \kappa\}/D)$ for all j and $\{\mu_i : i < \kappa\}$ is cofinal in $\mathrm{pcf}(\hat{d}) \cap \mu$. Thus we must have $\mathrm{cf}(\prod\{\mu_i : i < \kappa\}/D) = \mu$.

We will define an $e \subseteq \{\mu_i : i < \kappa\}$ such that $|e| \leq |a|$ and $\mu \in \mathrm{pcf}(e)$, and construct a d' from this e. Assume that $\kappa > |a|$, otherwise $e = \{\mu_1 : i < \kappa\}$ works. Let $S \subseteq \kappa$ such that $|S| = |a|$ and $a \cap \bigcup_{i<\kappa} b_{\mu_i} \subseteq \bigcup_{i \in S} b_{\mu_i}$ and let $e = \{\mu_i : i \in S\}$. Clearly $|e| \leq |a|$; it remains to show that $\mu \in \mathrm{pcf}(e)$. By Lemma 3.48, $e \subseteq b_{\delta_1} \cup ... \cup b_{\delta_k}$ for some $\delta_1, ..., \delta_k \in \mathrm{pcf}(e)$.

If $\delta_i \neq \mu$ for all i, then $\delta_i < \mu$ for all i, since $\delta_i \in \mathrm{pcf}(e) \subseteq \mathrm{pcf}(\{\mu_i : i < \kappa\}) \subseteq \mu + 1$. We will derive a contradiction. Since $|S| = |a| < \kappa$, let $j < \kappa$ be such that $\mu_j > \delta_i$ for all i and define $A := a \cap (b_{\mu_j} \setminus (b_{\delta_1} \cup ... \cup b_{\delta_k}))$. Since $\mu_j \in \mathrm{pcf}(a)$, let $\mu_j = \mathrm{cf}(\prod a/U)$ for some ultrafilter U. Extend U to an ultrafilter $\hat{U}$ on c. Since $\mu_j = \mathrm{cf}(\prod c/\hat{U}) = \min\{\lambda : b_\lambda \in \hat{U}\}$, we have $b_{\mu_j} \in \hat{U}$. Clearly $b_{\delta_1}, ..., b_{\delta_k} \notin U$, so $b_{\delta_1} \cup ... \cup b_{\delta_k} \notin U$. Since $\hat{U}$ is concentrated on a, we obtain $A \in U$. In particular $A \neq \emptyset$. On the other hand, $\mu_i \in e \subseteq b_{\delta_1} \cup ... \cup b_{\delta_k}$ for all $i \in S$, so $\bigcup_{i \in S} b_{\mu_i} \subseteq b_{\delta_1} \cup ... \cup b_{\delta_k}$ (since $\rho \in b_\lambda$ implies $b_\rho \subseteq b_\lambda$). So $a \cap b_{\mu_j} \subseteq \bigcup_{i \in S} b_{\mu_i} \subseteq b_{\delta_1} \cup ... \cup b_{\delta_k}$, which implies $A = \emptyset$, contradiction.

So $\delta_i = \mu$ for some i. So we have $e \subseteq \{\mu_i : i < \kappa\}$ such that $|e| \leq |a|$ and $\mu \in \mathrm{pcf}(e)$. Now for any $\delta \in e$ we have $\delta \in \mathrm{pcf}(\hat{d}) \cap \mu$, so in particular $\delta < \mu$, so by the induction hypothesis let $d_\delta \subseteq d$ such that $|d_\delta| \leq |a|$ and $\delta \in \mathrm{pcf}(d_\delta)$. Let $d' := \bigcup_{\delta \in e} d_\delta \subseteq d$. Then $|d'| = |e| \cdot |a| \leq |a| \cdot |a| = |a|$. For any $\delta \in e$ we have $\delta \in \mathrm{pcf}(d_\delta) \subseteq \mathrm{pcf}(d')$, hence $e \subseteq \mathrm{pcf}(d')$. So $\mu \in \mathrm{pcf}(e) \subseteq \mathrm{pcf}(\mathrm{pcf}(d')) = \mathrm{pcf}(d')$. □

8.4 Proof of the bound

In this subsection we will prove that $|\mathrm{pcf}(a)| \leq |a|^{+3}$ when a is an interval of regular cardinals such that $\min(a) > 2^{|a|}$. We start with the definition of a specific closure operation. In general, a closure operation on a set A is a map $\overline{(.)} : P(A) \to P(A)$ such that $\overline{\emptyset} = \emptyset$ and $X \subseteq \overline{X}$, $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$, $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$ and $\overline{\overline{X}} = \overline{X}$ for all $X, Y \in P(A)$.

Definition 8.17. Let a be an interval of regular cardinals such that $\min(a) > 2^{|a|}$ and such that $\min(a)$ is a successor cardinal. Let $\min(a) = \aleph_{\delta+1}$. Since $|\mathrm{pcf}(a)| \leq 2^{|a|}$ by Corollary 3.25 and $\mathrm{pcf}(a)$ is an interval of regular cardinals by Corollary 3.29, we have $\mathrm{pcf}(a) \subseteq \{\aleph_{\delta+\alpha} : 1 \leq \alpha < \left(2^{|a|}\right)^+\}$. By Theorem 3.27, $\mathrm{pcf}(a)$ contains a maximal element, and by Lemma 3.30 $\mathrm{pcf}(a)$ consists of successor cardinals. Hence $\mathrm{pcf}(a) = \{\aleph_{\delta+\alpha+1} : 0 \leq \alpha \leq \rho\}$ for some $\rho < \left(2^{|a|}\right)^+$ and we have a bijection $\mathrm{pcf}(a) \leftrightarrow \{\alpha : 0 \leq \alpha \leq \rho\}$ which will be frequently used. We define the map

$$\begin{aligned} \overline{(.)} : P(\rho+1) &\to P(\rho+1) \\ X &\mapsto \overline{X} := \{\gamma : \aleph_{\delta+\gamma+1} \in \mathrm{pcf}(\{\aleph_{\delta+\alpha+1} : \alpha \in X\})\}. \end{aligned}$$

Note that $\mathrm{pcf}(\mathrm{pcf}(a)) = \mathrm{pcf}(a)$ by Theorem 3.15, hence indeed $\bar{X} \subseteq \rho + 1$ for all X.

Definition 8.18. Let ρ be an ordinal and κ be a cardinal. Then a map $\overline{(.)} : P(\rho+1) \to P(\rho+1)$ may have the following properties:

(i) $\overline{\emptyset} = \emptyset$ and $X \subseteq \overline{X}$, $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$, $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, $\overline{\overline{X}} = \overline{X}$ for all $X, Y \in P(\rho+1)$, i.e. $\overline{(.)}$ is indeed a closure operation.

(ii) For all $X \in P(\rho+1)$, if $\gamma \in \overline{X}$ there exists some $X' \subseteq X$ such that $|X'| < \kappa$ and $\gamma \in \overline{X'}$.

(iii) $\overline{X}$ has a maximal element for all $X \in P(\rho+1)$.

(iv) If $\gamma \leq \rho$ and $\mathrm{cf}(\gamma) > \omega$ there exists a γ-club $C \subseteq \gamma$ such that $\overline{C} \subseteq \gamma + 1$.

(v) $\overline{[\kappa^{+4}, \rho]} = [\kappa^{+4}, \rho]$.

Note that it may be the case that $\kappa^{+4} > \rho$, then we have the convention that $[\kappa^{+4}, \rho] = \emptyset$. In this case (v) is already implied by (i).

Proposition 8.19. The map $\overline{(.)} : P(\rho+1) \to P(\rho+1)$ from Definition 8.17 satisfies properties (i)-(v) from Definition 8.18 for $\kappa = |a|$.

Proof. (i) $\overline{\emptyset} = \emptyset$, $X \subseteq \overline{X}$, $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$, $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$ follow from the facts that $\mathrm{pcf}(\emptyset) = \emptyset$, $a \subseteq \mathrm{pcf}(a)$, $a \subseteq b \Rightarrow \mathrm{pcf}(a) \subseteq \mathrm{pcf}(b)$ and $\mathrm{pcf}(a \cup b) = \mathrm{pcf}(a) \cup \mathrm{pcf}(b)$. It remains to show $\overline{\overline{X}} = \overline{X}$. Denote $\tilde{X} := \{\aleph_{\delta+\alpha+1} : \alpha \in X\}$. Then $\tilde{X} \subseteq \mathrm{pcf}(a)$, so $\mathrm{pcf}(\tilde{X}) \subseteq \mathrm{pcf}(\mathrm{pcf}(a)) = \mathrm{pcf}(a)$. Therefore $|\mathrm{pcf}(\tilde{X})| \leq |\mathrm{pcf}(a)| \leq 2^{|a|} < \min(a) \leq \min(\tilde{X})$. Again by Theorem 3.15 using $\min(\tilde{X}) > |\mathrm{pcf}(\tilde{X})|$ we obtain $\mathrm{pcf}(\mathrm{pcf}(\tilde{X})) = \mathrm{pcf}(\tilde{X})$. It follows that $\overline{\overline{X}} = \overline{X}$.

(ii) If $\gamma \in \overline{X}$ there there exists some $X' \subseteq X$ such that $|X'| < \kappa$ and $\gamma \in \overline{X'}$ by Lemma 8.16.

(iii) $\overline{X}$ has a maximal element: Since $\mathrm{pcf}(\{\aleph_{\delta+\eta+1} : \eta \in X\}) \subseteq \mathrm{pcf}(\{\aleph_{\delta+\eta+1} : \eta \in \rho+1\}) = \mathrm{pcf}(\mathrm{pcf}(a)) = \mathrm{pcf}(a)$, and $\mathrm{pcf}(\{\aleph_{\delta+\eta+1} : \eta \in X\})$ has a maximum, this maximum is $\aleph_{\delta+\eta_0+1}$ for some $\eta_0 \leq \rho$, and η_0 is the maximum of $\overline{X}$.

(iv) Suppose $\gamma \leq \rho$ and $\mathrm{cf}(\gamma) > \omega$. Let $\lambda := \aleph_{\delta+\gamma}$. Since λ is a limit cardinal, the proof of Lemma 3.30 shows that λ is singular. Also $\mathrm{cf}(\lambda) = \mathrm{cf}(\aleph_{\delta+\gamma}) = \mathrm{cf}(\delta+\gamma) = \mathrm{cf}(\gamma) > \omega$. Thus we have $\lambda > \mathrm{cf}(\lambda) > \omega$. Let $\Gamma \subseteq \gamma$ be γ-cofinal, be consisting of non-zero limit ordinals and such that

$\text{ot}(\Gamma) = \text{cf}(\gamma)$. Define $E := \{\aleph_{\delta+\alpha} : \alpha \in \Gamma\}$. Then, again by the proof of Lemma 3.30, E consists of singular cardinals; also $E \subseteq [\text{cf}(\lambda), \lambda)$ [since $\min(E) \geq \aleph_{\delta+1} \geq (2^{|a|})^+ > \rho \geq \gamma \geq \text{cf}(\gamma) = \text{cf}(\lambda)$] and $\text{ot}(E) = \text{cf}(\gamma) = \text{cf}(\lambda)$. As in subsection 8.1, let $c := \bigcup_{1 \leq k < \omega} E^{+k}$ and let $b_\mu(c)$ generate $J_{<\mu^+}(c)$ over $J_{<\mu}(c)$. Note that $c \subseteq a$ and $\text{pcf}(c) \subseteq \text{pcf}(a) = a$. By Theorem 8.2 for $n = 1$, let $D \subseteq \{\rho \in E : \rho^{+1} \in b_{\lambda^+}(c)\}$ be λ-club. Then $C := \{\alpha : \aleph_{\delta+\alpha} \in D\}$ is γ-club. Now $d := D^+ = \{\aleph_{\delta+\alpha+1} : \alpha \in C\} \subseteq b_{\lambda^+}$, so $\text{pcf}(d) \subseteq [\aleph_{\delta+1}, (\lambda^+)^+) = [\aleph_{\delta+1}, \aleph_{\delta+\gamma+1}]$, so $\overline{C} \subseteq [0, \gamma] = \gamma + 1$. So we have shown the existence of a $C \subseteq \gamma$ such that C is γ-club and $\overline{C} \subseteq \gamma + 1$.

(v) Since $\min(b) = \min(\text{pcf}(b))$ for all $b \subseteq a$, we have $\overline{[\alpha, \rho]} = [\alpha, \rho]$ for all α.

□

Proposition 8.20. Suppose $\rho > \kappa^{+4}$ and $\overline{(.)} : P(\rho+1) \to P(\rho+1)$ satisfies properties (i)-(v) from Definition 8.18 for κ. Define

$$\text{Cl} : P(\kappa^{+4} + 1) \to P(\kappa^{+4} + 1)$$
$$X \mapsto \text{Cl}(X) := \begin{cases} \overline{X}, & \text{if } \overline{X} \subseteq \kappa^{+4} + 1, \\ (\overline{X} \cap \kappa^{+4}) \cup \{\kappa^{+4}\}, & \text{if } \overline{X} \not\subseteq \kappa^{+4} + 1. \end{cases}$$

Then Cl satisfies properties (i)-(v) from Definition 8.18 (where $\rho = \kappa^{+4}$) for κ.

Proof. We do a simple and elaborate check of the properties.

(i) $\text{Cl}(\emptyset) = \emptyset$, $X \subseteq \text{Cl}(X)$, $X \subseteq Y \Rightarrow \text{Cl}(X) \subseteq \text{Cl}(Y)$, $\text{Cl}(X \cup Y) = \text{Cl}(X) \cup \text{Cl}(Y)$ are easy to prove. It remains to show that $\text{Cl}(\text{Cl}(X)) = \text{Cl}(X)$. If $\overline{X} \subseteq \kappa^{+4}+1$ then also $\overline{\overline{X}} = \overline{X} \subseteq \kappa^{+4}+1$, hence $\text{Cl}(\text{Cl}(X)) = \text{Cl}(\overline{X}) = \overline{\overline{X}} = \overline{X} = \text{Cl}(X)$. If $\overline{X} \not\subseteq \kappa^{+4}+1$, then $\text{Cl}(X) = (\overline{X} \cap \kappa^{+4}) \cup \{\kappa^{+4}\}$ and

$$\overline{\text{Cl}(X)} = \overline{(\overline{X} \cap \kappa^{+4}) \cup \{\kappa^{+4}\}} = \overline{\overline{X} \cap \kappa^{+4}} \cup \overline{\{\kappa^{+4}\}} \subseteq \overline{\overline{X}} \cup \overline{[\kappa^{+4}, \rho]} = \overline{X} \cup [\kappa^{+4}, \rho].$$

If $\overline{\text{Cl}(X)} \subseteq \kappa^{+4} + 1$, then $\overline{\text{Cl}(X)} \subseteq (\overline{X} \cap \kappa^{+4}) \cup \{\kappa^{+4}\} = \text{Cl}(X)$, so $\overline{\text{Cl}(X)} = \text{Cl}(X)$ and $\text{Cl}(\text{Cl}(X)) = \overline{\text{Cl}(X)} = \text{Cl}(X)$. If $\overline{\text{Cl}(X)} \not\subseteq \kappa^{+4} + 1$, then

$$\begin{aligned} \text{Cl}(\text{Cl}(X)) &= (\overline{\text{Cl}(X)} \cap \kappa^{+4}) \cup \{\kappa^{+4}\} \\ &\subseteq ((\overline{X} \cup [\kappa^{+4}, \rho]) \cap \kappa^{+4}) \cup \{\kappa^{+4}\} = (\overline{X} \cap \kappa^{+4}) \cup \{\kappa^{+4}\} = \text{Cl}(X), \end{aligned}$$

hence $\text{Cl}(\text{Cl}(X)) = \text{Cl}(X)$.

(ii) Suppose $\gamma \in \text{Cl}(X)$.
  (a) Suppose $\overline{X} \subseteq \kappa^{+4} + 1$. Then $\gamma \in \text{Cl}(X) = \overline{X}$, so there exists $X' \subseteq X$ such that $|X'| \leq \kappa$ and $\gamma \in \overline{X'} \subseteq \overline{X} \subseteq \kappa^{+4} + 1$, thus $\gamma \in \overline{X'} = \text{Cl}(X')$.
  (b) Suppose $\overline{X} \not\subseteq \kappa^{+4} + 1$. Then $\gamma \in (\overline{X} \cap \kappa^{+4}) \cup \{\kappa^{+4}\}$.
    i. Assume $\gamma \in \overline{X} \cap \kappa^{+4}$. Let $X' \subseteq X$ such that $|X'| \leq \kappa$ and $\gamma \in \overline{X}$. Then $\gamma \in \text{Cl}(X')$.
    ii. Assume $\gamma = \kappa^{+4}$. Let $\delta \in \overline{X}$ such that $\delta \notin \kappa^{+4} + 1$. Let $X' \subseteq X$ such that $|X'| \leq \kappa$ and $\delta \in \overline{X'}$. Then $\overline{X'} \not\subseteq \kappa^{+4} + 1$, hence $\text{Cl}(X') = (\overline{X'} \cap \kappa^{+4}) \cup \{\kappa^{+4}\}$. Thus $\gamma = \kappa^{+4} \in \text{Cl}(X')$.

(iii) It is obvious that $\text{Cl}(X)$ has a maximal element.

(iv) If $\gamma \leq \kappa^{+4}$ and $\text{cf}(\gamma) > \omega$, then $\gamma \leq \rho$, so let $C \subseteq \gamma$ be γ-club and such that $\overline{C} \subseteq \gamma + 1$. Then $\overline{C} \subseteq \kappa^{+4} + 1$ so $\text{Cl}(C) = \overline{C} \subseteq \gamma + 1$.

(v) We have $\overline{\{\kappa^{+4}\}} \subseteq \overline{[\kappa^{+4}, \rho]} = [\kappa^{+4}, \rho]$.
  (a) Suppose $\overline{\{\kappa^{+4}\}} \subseteq \kappa^{+4} + 1$. Then $\overline{\{\kappa^{+4}\}} = \{\kappa^{+4}\}$, hence $\mathrm{Cl}(\{\kappa^{+4}\}) = \overline{\{\kappa^{+4}\}} = \{\kappa^{+4}\}$.
  (b) Suppose $\overline{\{\kappa^{+4}\}} \nsubseteq \kappa^{+4} + 1$, then $\mathrm{Cl}(\{\kappa^{+4}\}) = (\overline{\{\kappa^{+4}\}} \cap \kappa^{+4}) \cup \{\kappa^{+4}\} = \{\kappa^{+4}\}$.
  So $\mathrm{Cl}[\kappa^{+4}, \kappa^{+4}] = [\kappa^{+4}, \kappa^{+4}]$.

□

Theorem 8.21. Let a be an interval of regular cardinals such that $\min(a) > 2^{|a|}$. Then $|\mathrm{pcf}(a)| \leq |a|^{+3}$.

Proof. Define ρ and $\overline{(.)} : P(\rho+1) \to P(\rho+1)$ as in Definition 8.17. We are done once we show that $\rho < |a|^{+4}$. So assume the contrary, i.e. $\rho \geq |a|^{+4}$. By Proposition 8.19 we know that $\overline{(.)}$ satisfies properties (i)-(v) of Definition 8.18 for ρ and $\kappa := |a|$. If $\rho > |a|^{+4}$, then Proposition 8.20 yields a map $\mathrm{Cl} : P(\kappa^{+4} + 1) \to P(\kappa^{+4} + 1)$ also satisfying (i)-(v). Thus in any case, there is a map $\overline{(.)} : P(\kappa^{+4} + 1) \to P(\kappa^{+4} + 1)$ that satisfies properties (i)-(v). Note that $|a|^{+1}$, $|a|^{+3}$ are regular uncountable cardinals and $(|a|^{+1})^{+} < |a|^{+3}$. Let $S = S(|a|^{+3}, |a|^{+}) = \{\beta < |a|^{+3} : \mathrm{cf}(\beta) = |a|^{+}\}$. By Corolarry 2.28, let $\langle S_\alpha : \alpha \in S\rangle$ be a $\diamond_{\mathrm{club}}(|a|^{+3}, |a|^{+1})$-sequence, i.e.:

1. For all $\alpha \in S$ we have $S_\alpha \subseteq \alpha$ and S_α is α-club.
2. For all $|a|^{+3}$-club C we have $\{\alpha \in S : S_\alpha \subseteq C\}$ is $|a|^{+3}$-stationary.

Now let θ be regular and large enough and let $\langle M_\beta : \beta \leq |a|^{+3}\rangle$ such that

1. $M_\beta \prec H(\theta)$ for all $\beta < |a|^{+3}$, i.e. M_β is an elementary substructure of $H(\theta)$,
2. $\beta' < \beta \Rightarrow M_{\beta'} \subseteq M_\beta$ for all $\beta', \beta < |a|^{+3}$, i.e. the sequence is $\subseteq$-increasing,
3. $M_\beta = \bigcup_{\beta' < \beta} M_{\beta'}$ for all limits $\beta \leq |a|^{+3}$, i.e. the sequence is continuous,
4. $|M_\beta| = |a|^{+3}$ for all $\beta < |a|^{+3}$.
5. $|a|^{+3} \subseteq M_0$, $\{\langle X, \overline{X}\rangle : X \subseteq |a|^{+4} + 1\} \in M_0$ and $\langle S_\alpha : \alpha \in S\rangle \in M_0$,
6. $\langle M_{\beta'} : \beta' \leq \beta\rangle \in M_{\beta+1}$ for all $\beta < |a|^{+3}$.

For $\beta \leq |a|^{+3}$, let $\gamma_\beta := \sup(M_\beta \cap |a|^{+4})$. Since $|M_\beta| = |a|^{+3}$, we have $\gamma_\beta \in |a|^{+4}$ for all $\beta \leq |a|^{+3}$. We have $\langle \gamma_\delta : \delta < \beta\rangle \in M_{\beta+1}$ for all $\beta < |a|^{+3}$:

Let $\beta < |a|^{+3}$. Since $|a|^{+3} \subseteq M_{\beta+1}$ and $\langle M_{\beta'} : \beta' \leq \beta\rangle \in M_{\beta+1}$, we have $M_{\beta'} \in M_{\beta+1}$ for all $\beta' \leq \beta$. Since $|a|^{+4} \in M_{\beta+1}$, we must have $\gamma_{\beta'} = \sup(M_{\beta'} \cap |a|^{+4}) \in M_{\beta+1}$ for all $\beta' \leq \beta$ by elementarity. Then $\langle \gamma_\delta : \delta < \beta\rangle \in M_{\beta+1}$.

For $\alpha \in S$, let $E^\beta_\alpha := \{\gamma_\delta : \delta < \beta, \delta \in S_\alpha\}$, then $E^\beta_\alpha \in M_{\beta+1}$, thus $\overline{E^\beta_\alpha} \in M_{\beta+1}$. Note that if $\overline{E^\beta_\alpha} \in M_{\beta+1}$ is bounded below $|a|^{+4}$, then $\overline{E^\beta_\alpha} \subseteq \gamma_{\beta+1}$ by elementarity.

Let $C \subseteq \gamma_{|a|^{+3}}$ be $\gamma_{|a|^{+3}}$-club and such that $\overline{C} \subseteq \gamma_{|a|^{+3}} + 1$. Both C and $\{\gamma_\beta : \beta < |a|^{+3}\}$ are closed unbounded in the ordinal $\gamma_{|a|^{+3}}$, and $\mathrm{cf}(\gamma_{|a|^{+3}}) = |a|^{+3} > \omega$. So their intersection is $\gamma_{|a|^{+3}}$-club, thus $\{\beta < |a|^{+3} : \gamma_\beta \in C\}$ is $|a|^{+3}$-club. Since $\{\alpha \in S : S_\alpha \subseteq \{\beta < |a|^{+3} : \gamma_\beta \in C\}\}$ is $|a|^{+3}$-stationary, there is at least one $\alpha \in S$ such that $S_\alpha \subseteq \{\beta < |a|^{+3} : \gamma_\beta \in C\}$. Define $S^*_\alpha := \{\gamma_\beta : \beta \in S_\alpha\}$ and note that $S^*_\alpha \subseteq C$. Since S_α is α-club, S^*_α is γ_α-club (note that α is a limit ordinal). Then $\overline{S^*_\alpha}$ has a maximum x, and $x \geq \gamma_\alpha$. Now there is some $X' \subseteq S^*_\alpha$ such that $|X'| \leq |a|$ and $x \in \overline{X'}$.

Since $X' \subseteq S^*_\alpha = \{\gamma_\beta : \beta \in S_\alpha\} \subseteq \{\gamma_\beta : \beta < \alpha\} \subseteq \gamma_\alpha$, $|X'| \leq |a|$ and $\mathrm{cf}(\alpha) = |a|^{+}$, there is some $\beta < \alpha$ such that $X' \subseteq S^*_\alpha \cap \gamma_\beta$. Therefore

$$x \in \overline{X'} \subseteq \overline{S^*_\alpha \cap \gamma_\beta} = \overline{E^\beta_\alpha} \subseteq \overline{S^*_\alpha} \subseteq \overline{C} \subseteq \gamma_{|a|^{+3}} + 1.$$

So $\overline{E_\alpha^\beta}$ is bounded below $|a|^{+4}$, so

$$x \in \overline{E_\alpha^\beta} \subseteq \gamma_{\beta+1} < \gamma_\alpha.$$

This contradicts $x \geq \gamma_\alpha$. □

So we have now shown that $|\mathrm{pcf}(a)| < |a|^{+4}$ under some reasonable assumptions on a. We can of course use this fact to prove some non-trivial bounds on cardinal exponentiation.

Theorem 8.22. Let δ be a limit ordinal. If $2^{|\delta|} < \aleph_\delta$, then

$$\aleph_\delta^{|\delta|} < \aleph_{|\delta|^{+4}}.$$

In particular, if $2^{\aleph_0} < \aleph_\omega$, then

$$\aleph_\omega^{\aleph_0} < \aleph_{\omega^{+4}}.$$

Proof. We prove the theorem under the assumption that δ is the disjoint union of $|\delta|$ many δ-cofinal subsets. This assumption is made without loss of generality, since for every ordinal δ there exists an ordinal $\delta' := \delta + |\delta| > \delta$ which has the same cardinality as δ and is the disjoint union of $|\delta|$ many δ-cofinal subsets. The proof in essentially the same as the proof of Corollary 6.11. Let $a = [(2^\delta)^+, \aleph_\delta)$. Then $\min(a)^{|a|} = ((2^{|\delta|})^+)^{|\delta|} < \aleph_\delta = \sup(a)$, and $|a|^+ \leq |\delta|^+ \leq 2^\delta < (2^{|\delta|})^+ = \min(a)$. Thus Theorem 6.1 applies and yields $\max(\mathrm{pcf}(a)) = |\prod a| = \aleph_\delta^{|\delta|}$ (in the last equality we use that δ is the disjoint union of $|\delta|$-many δ-cofinal subsets). Recall that $\mathrm{pcf}(a)$ is an interval of regular cardinals by Corollary 3.29, and that $a \subseteq \mathrm{pcf}(a)$. This lies a bound on how far $\mathrm{pcf}(a)$ can reach. In particular,

$$\aleph_\delta^{|\delta|} = \max(\mathrm{pcf}(a)) < \aleph_{\delta+|\mathrm{pcf}(a)|^+} \leq \aleph_{\delta+|a|^{+4}} \leq \aleph_{|\delta|^{+4}}.$$

□

Index

References

[1] L. Bukovský. The continuum problem and the powers of alephs. *Commentationes Mathematicae Universitatis Carolinae volume 6 issue 2 pp. 181-197*, 1965.

[2] M. Burke and M. Magidor. Shelah's pcf-theory and its applications. *Annals of Pure and Applied Logic 50, pp. 207-254*, 1990.

[3] P. Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences of the United States of America volume 50 number 6 pp. 1143-1148*, 1963.

[4] P. Cohen. The independence of the continuum hypothesis ii. *Proceedings of the National Academy of Sciences of the United States of America volume 51 number 1 pp. 105-110*, 1964.

[5] W. Easton. Powers of regular cardinals. *Annals of Mathematical Logic volume 1 number 2 pp. 139-178*, 1970.

[6] F. Galvin and A. Hajnal. Inequalities for cardinal powers. *Annals of Mathematics, Second Series volume 101 number 3 pp. 491-498*, 1975.

[7] T. Ishiu. Club guessing sequences and filters. 2003. University of California, Irvine. Dissertation.

[8] T. Jech. *Set Theory*. Springer, 1978. Third Millennium edition.

[9] J. Silver. On the singular cardinals problem. *Proceedings of the International Congress of Mathematicians volume 1 pp. 265-268*, 1975.

[10] B. van den Berg. Syllabus model theory. 2016. `staff.fnwi.uva.nl/b.vandenberg3/Onderwijs/Model%20Theory%202017/syllabus_model_theory.pdf`.