Applications of toric geometry to convex polytopes and matroids

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Introduction

Toric geometry is a branch of mathematics in the intersection of algebraic geometry and geometric combinatorics. Algebraic geometry is often regarded as a rather tough and abstract discipline, whereas combinatorics would be much easier accessible, and this might be true to some extent. In toric geometry, however, there is a surprisingly fruitful interaction between these two fields of study. An important aim of this thesis is to give the reader some taste of this interaction, and show how algebraic geometry can be made serviceable for combinatorial purposes.

The starting point for any introduction in toric geometry is the notion of a *(polyhedral) cone*. In § 1, we construct an affine variety X_σ from such a cone σ; i.e. we associate an object in algebraic geometry to a combinatorial object. The variety X_σ is called *toric*, since it contains a torus as an open and dense subset, which, considered as an algebraic group, acts on X_σ.

If one starts with a 'special' set of cones, called a *fan*, then the affine toric varieties associated to its cones can be glued together into a new, not necessarily affine, variety. The variety thus obtained is again toric. This construction is described in § 2. It turns out that certain properties of fans can be nicely related to properties of the associated toric varieties, which provides us with a 'dictionary' between the language of cones and fans on the one hand, and the language of toric varieties on the other hand.

The theory of cones, fans and toric varieties, as developed in § 1 and § 2, opens up the possibility of applying tools from algebraic geometry to combinatorial problems. Two such applications will be described in § 5 and § 6. However, in order to be able do so, somewhat more theory will be needed. In § 3 we discuss divisors on algebraic varieties, together with some intersection theory. In § 4 the notions of *matroids* and *Bergman fans* are introduced. These two sections are by no means meant to be a full-fledged introduction in the topics discussed, not even when restricted to the context of toric geometry. The material treated in them is mainly selected for its relevance in § 5 and § 6, although it is certainly valuable on its own as well.

Our first application of toric geometry, which is discussed in § 5, concerns f-vectors and h-vectors of simplicial convex polytopes. In 1971 McMullen conjectured that a certain property of a vector h of integers is necessary and sufficient for h to be the h-vector of a simplicial convex polytope. We will present Stanley's proof in [27] of the necessity of this condition, which uses toric geometry, together with some tools from algebraic geometry and algebraic topology.

§ 6 treats an entirely different application, namely to the theory of matroids. The Heron-Rota-Welsh conjecture states that the characteristic polynomial of a matroid M is log-concave. It was proved by Adiprasito, Huh and Katz in 2015. Instead of looking at this proof for the general case, we will present the proof by Huh and Katz in [18] for the special case that M is a representable matroid, since the latter proof involves an especially nice application of toric geometry.

Finally, let us make some remarks about the literature on toric geometry. A standard introduction to toric geometry is Fulton's book [10], and another important source is [9], by Ewald, who has a slightly broader scope than toric geometry. Besides [10], the main sources for § 1, § 2 and § 3 are [5] and

[25]. The article [3], by Baker, gives a concise introduction in matroid theory, and it discusses recent developments concerning the study of characteristic polynomials of matroids and their relation to algebraic geometry. It is our main reference for § 4 and it motivates § 6. Katz' article [18], which is the main source for § 6, also provides a nice introduction to the interaction between matroid theory and algebraic geometry. Regarding § 5, McMullen first stated his conjecture in [21]. Stanley's proof of the necessity of McMullen's condition, which we follow in § 5, can be found in [27], and a proof of its sufficiency, by Billera and Lee, in [4].

1 Cones and affine toric varieties

The goal of this section is to give a construction of affine toric varieties. We will roughly follow the approach of [10, §§1.1–1.3] and [5, §§1–2]. To a polyhedral lattice cone σ, we associate its dual cone $\check{\sigma}$ (§ 1.1). Intersecting $\check{\sigma}$ with the lattice concerned, we obtain a finitely generated monoid S_σ and, subsequently, a finitely generated $\mathbb{C}$-algebra R_σ (§ 1.3). In the final step this gives rise to a complex affine variety X_σ (§ 1.5).

1.1 Cones and dual cones

Definition 1.1. Let $v_1, \ldots, v_r$ be vectors in some finite-dimensional $\mathbb{R}$-vector space V. A set of the form

$$\sigma = \{\lambda_1 v_1 + \ldots + \lambda_r v_r : \lambda_i \in \mathbb{R}_{\geq 0}\}$$

is called a *(polyhedral) cone* in V. The vectors $v_1, \ldots, v_r$ are called *generators* of σ. The *dimension* of σ, denoted by $\dim \sigma$, is the dimension of the linear span $\operatorname{span}(\sigma) \subseteq V$ of σ. A cone of dimension 1 is called a *ray*.

Notation 1.2. For a finite set of vectors $S = \{v_1, \ldots, v_r\} \subseteq V$, let

$$\mathcal{C}(S) = \mathcal{C}(v_1, \ldots, v_r) = \{\lambda_1 v_1 + \ldots + \lambda_r v_r : \lambda_i \in \mathbb{R}_{\geq 0}\}$$

denote the cone generated by $v_1, \ldots, v_r$. Furthermore, if an infinite subset $T \subseteq V$ has a finite subset $T' \subseteq T$ such that

$$\left\{ \sum_{v \in T} \lambda_v v : \lambda_v \in \mathbb{R}_{\geq 0} \text{ all but finitely many } 0 \right\} = \mathcal{C}(T'),$$

then we write $\mathcal{C}(T) = \mathcal{C}(T')$.

If we use the notation $\mathcal{C}(T)$ for some infinite set of vectors T, it will always be clear how to choose a finite subset $T' \subseteq T$ as above, even though T' is not mentioned.

Remark 1.3. Since, by convention, the empty sum is zero, we have $\mathcal{C}(\varnothing) = \{0\}$.

If V is an $\mathbb{R}$-vector space and V^* its dual, let $\langle \cdot, \cdot \rangle_d : V^* \times V \to \mathbb{R}$ denote the usual duality pairing.

Definition 1.4. The *dual cone* associated to a cone $\sigma \subseteq V$ is

$$\check{\sigma} = \{u \in V^* : \langle u, v \rangle_d \geq 0 \text{ for all } v \in \sigma\}.$$

For simplicity we will often just identify an n-dimensional $\mathbb{R}$-vector space V with $\mathbb{R}^n$ and assume that σ is contained in $\mathbb{R}^n$. Furthermore, we will identify $V^* \cong (\mathbb{R}^n)^*$ with $\mathbb{R}^n$ via the isomorphism $e_i^* \mapsto e_i$, where $e_1^*, \ldots, e_n^* \in (\mathbb{R}^n)^*$ is the basis of $(\mathbb{R}^n)^*$ dual to the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Under this identification, the duality pairing $\langle \cdot, \cdot \rangle_d : V^* \times V \to \mathbb{R}$ corresponds to the dot product $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, and we can pretend that $\check{\sigma}$ is a subset of $\mathbb{R}^n$.

Remark 1.5. It is not immediately clear that the dual cone $\check{\sigma} \subseteq \mathbb{R}^n$ of a cone $\sigma \subseteq \mathbb{R}^n$ is also a cone, but, as a matter of fact, this is the case. A proof will be given in Theorem 1.17.

Example 1.6. The dual cone of $\sigma = \mathcal{C}(e_1, e_1 + e_2) \subseteq \mathbb{R}^2$ is $\check{\sigma} = \mathcal{C}(e_2, e_1 - e_2)$.

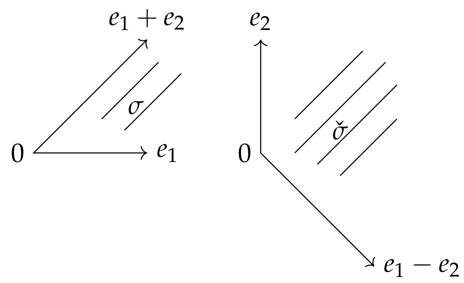

Figure 1: The cone σ and its dual cone $\check{\sigma}$.

The following lemma will be useful in what follows.

Lemma 1.7. *Let $\sigma \subseteq \mathbb{R}^n$ be a cone and $v \in \mathbb{R}^n \setminus \sigma$. Then there is some $u \in \check{\sigma}$ such that $\langle v, u \rangle < 0$.*

Proof. See [6, Cor. 3], where this result is deduced from a more general theorem about convex polytopes. $\square$

We have the following direct consequence (cf. [10, (1), p. 9] and [6, Cor. 4]).

Theorem 1.8 (Duality Theorem)**.** *For a cone $\sigma \subseteq \mathbb{R}^n$, we have $\check{\check{\sigma}} = \sigma$.*

Proof. We clearly have $\sigma \subseteq \check{\check{\sigma}}$. If $\sigma = \mathbb{R}^n$, there is nothing to prove. Otherwise, let $v \in \mathbb{R}^n \setminus \sigma$. By Lemma 1.7 there is some $u \in \check{\sigma}$ such that $\langle v, u \rangle < 0$; hence, $v \notin \check{\check{\sigma}}$. It follows that $\check{\check{\sigma}} \subseteq \sigma$ and $\check{\check{\sigma}} = \sigma$. $\square$

1.2 Faces of cones

Definition 1.9. A *face* of a cone σ is an intersection
$$\tau = \sigma \cap u^\perp = \{v \in \sigma : \langle v, u \rangle = 0\}$$
for some vector $u \in \check{\sigma}$. We call a face τ of σ *proper* if $\tau \neq \sigma$.

Example 1.10. The faces of $\sigma = \mathcal{C}(e_1, e_1 + e_2)$ (see Example 1.6) are
$$\begin{aligned}
\tau_1 &= \sigma \cap 0^\perp = \sigma; \\
\tau_2 &= \sigma \cap e_2^\perp = \mathcal{C}(e_1); \\
\tau_3 &= \sigma \cap (e_1 - e_2)^\perp = \mathcal{C}(e_1 + e_2); \\
\tau_4 &= \sigma \cap v^\perp = \{0\};
\end{aligned}$$
for any $v \in \check{\sigma} \setminus (\mathcal{C}(e_2) \cup \mathcal{C}(e_1 - e_2))$.

The next two propositions give some basic properties of faces (cf. [10, pp. 9–10], [5, Property 1.2] and [6, Prop. 8, Thm. 9]).

Proposition 1.11. *Let $\sigma = \mathcal{C}(v_1, \ldots, v_r) \subseteq \mathbb{R}^n$ be a cone. Then we have the following:*

(1) *Let $\tau = \sigma \cap u^\perp$ with $u \in \check{\sigma}$ be a face of σ. Then τ is the cone generated by those v_i such that $\langle u, v_i \rangle = 0$.*
(2) *An intersection of faces of σ is a face of σ.*
(3) *A face of a face of σ is a face of σ.*

Proof.

(1) Without loss of generality we may assume that there is $k \leq r$ such that $\langle u, v_i \rangle = 0$ if and only if $i \leq k$. The inclusion $\mathcal{C}(v_1, \ldots, v_k) \subseteq \tau$ is clear. Conversely, let $v \in \tau$ and write $v = \lambda_1 v_1 + \ldots + \lambda_r v_r$ with $\lambda_i \in \mathbb{R}_{\geq 0}$. We find
$$0 = \langle u, v \rangle = \langle u, \lambda_1 v_1 + \ldots + \lambda_r v_r \rangle = \sum_{i=1}^{k} \lambda_i \langle u, v_i \rangle + \sum_{i=k+1}^{r} \lambda_i \langle u, v_i \rangle.$$
Notice that $\langle u, v_i \rangle$ equals 0 if $i \leq k$ and is positive if $k < i \leq r$, since u is contained in $\check{\sigma}$. Hence, we have $\lambda_{k+1} = \ldots = \lambda_r = 0$ and therefore $v \in \mathcal{C}(v_1, \ldots, v_k)$. It follows that $\tau = \mathcal{C}(v_1, \ldots, v_k)$.
(2) See [6, Prop. 8 (2)].
(3) See [6, Prop. 8 (3)].

□

It follows from Proposition 1.11 (1) that a cone has only finitely many faces. More precisely, we have the following corollary.

Corollary 1.12. *A cone $\mathcal{C}(v_1, \ldots, v_r)$ has at most 2^r faces.*

Proof. By Proposition 1.11 (1), every face of $\mathcal{C}(v_1, \ldots, v_r)$ is of the form $\mathcal{C}(S)$ for some subset $S \subseteq \{v_1, \ldots, v_r\}$. □

Notation 1.13. For cones $\sigma, \tau \subseteq \mathbb{R}^n$, we write $\tau \leq \sigma$ if τ is a face of σ and $\tau < \sigma$ if τ is a proper face of σ.

Remark 1.14. By Proposition 1.11, '$\leq$' is a partial ordering on the set of cones.

Definition 1.15. A *facet* of a cone σ is a face $\tau < \sigma$ with $\dim \tau = \dim \sigma - 1$.

Let $\sigma \subseteq \mathbb{R}^n$ be a cone with $\operatorname{span}(\sigma) = \mathbb{R}^n$ and $\tau < \sigma$ a facet. Since τ has dimension $n - 1$, there is a vector $u_\tau \in \check{\sigma}$, unique up to multiplication by a positive number, such that $\tau = \sigma \cap u_\tau^\perp$ (cf. [10, p. 11]). Let
$$H_\tau = \{v \in \mathbb{R}^n : \langle u_\tau, v \rangle \geq 0\}$$
denote the half-space defined by u_τ.

Proposition 1.16. *Let $\sigma = \mathcal{C}(v_1, \ldots, v_r) \subseteq \mathbb{R}^n$ be a cone. Then we have the following:*

(1) *Every proper face of σ is contained in a facet of σ.*
(2) *Every proper face of σ is the intersection of the facets it is contained in.*
(3) *If $\sigma \neq \mathbb{R}^n = \operatorname{span}(\sigma)$, then we have $\sigma = \bigcap_{\tau < \sigma \text{ facet}} H_\tau$.*

Proof.

(1) See [6, Prop. 8 (4)].
(2) See [6, Prop. 8 (4)].
(3) See [6, Thm. 9].

□

At this point we are able to prove that a dual cone is actually a cone (see [10, (9), p. 11]).

Theorem 1.17 (Farkas' Theorem)**.** *The dual cone $\check{\sigma}$ of a cone $\sigma \subseteq \mathbb{R}^n$ is a cone.*

Proof. We may assume that $\operatorname{span}(\sigma) = \mathbb{R}^n$. If $\operatorname{span}(\sigma) = \sigma$, then we have $\check{\sigma} = \{0\} = \mathcal{C}(0)$. Now assume that $\operatorname{span}(\sigma) \neq \sigma$. We claim that $\check{\sigma} = \mathcal{C}(U)$, where $U = \{u_\tau : \tau < \sigma \text{ facet}\}$ (which is a finite set by Corollary 1.12). Let $u \in \check{\sigma}$ and suppose that $u \notin \mathcal{C}(U)$. By Lemma 1.7 there is some $u' \in \mathbb{R}^n$ such that $\langle u, u' \rangle < 0$ and $\langle u_\tau, u' \rangle \geq 0$ for all facets $\tau < \sigma$. But now Proposition 1.16 (3) yields $u' \in \sigma$, which is a contradiction. It follows that $u \in \mathcal{C}(U)$ and $\check{\sigma} \subseteq \mathcal{C}(U)$. Conversely, let $u \in \mathcal{C}(U)$ and $v \in \sigma$. It follows from Proposition 1.16 (3) that $\langle u', v \rangle \geq 0$ for all $u' \in U$. Hence, we find $\langle u, v \rangle \geq 0$ and $u \in \check{\sigma}$. This proves the inclusion $\mathcal{C}(U) \subseteq \check{\sigma}$. □

Remark 1.18. The proof of Theorem 1.17 shows that the dual cone of a cone σ equals $\check{\sigma} = \mathcal{C}(\{u_\tau : \tau < \sigma \text{ facet}\})$.

Example 1.19. Returning to Example 1.6, we see that the facets of $\sigma = \mathcal{C}(e_1, e_1 + e_2)$ are $\tau_2 = \mathcal{C}(e_1)$ and $\tau_3 = \mathcal{C}(e_1 + e_2)$. The dual cone $\check{\sigma}$ is generated by $u_{\tau_2} = e_2$ and $u_{\tau_3} = e_1 - e_2$ and we can write

$$\sigma = H_{\tau_2} \cap H_{\tau_3} = \{v \in \mathbb{R}^2 : \langle e_2, v \rangle \geq 0,\ \langle e_1 - e_2, v \rangle \geq 0\}.$$

1.3 Lattices and monoids

Definition 1.20. A *lattice* is a finitely generated torsion-free abelian group.

Remark 1.21. By the fundamental theorem of finitely generated abelian groups, lattices are precisely the abelian groups isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{Z}_{\geq 0}$. In particular, every lattice has a $\mathbb{Z}$-basis. If N is a lattice, the unique integer $n \in \mathbb{Z}_{\geq 0}$ such that $N \cong \mathbb{Z}^n$ is called the *rank* of N.

Definition 1.22. Let N be a lattice. An element $v \in N$ is called a *primitive lattice vector* if for all $m \in \mathbb{Z}$ and $w \in N$ with $v = mw$, we have $m = \pm 1$.

Let $N \cong \mathbb{Z}^n$ be a lattice and let $N^* = \operatorname{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$ denote its dual lattice. We define the $\mathbb{R}$-vector spaces $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$ and $N^*_\mathbb{R} = N^* \otimes_\mathbb{Z} \mathbb{R} \cong \mathbb{R}^n$. There are natural inclusions $N \hookrightarrow N_\mathbb{R}$ and $N^* \hookrightarrow N^*_\mathbb{R}$ of abelian groups, both given by $x \mapsto x \otimes 1$. Therefore, N and N^* can be viewed as subgroups of $N_{\mathbb{R}}$ and $N^*_{\mathbb{R}}$, respectively.

Definition 1.23. A *monoid* is a triple $(S, \circ, e)$, where S is a set, $\circ : S \times S \to S$ is a binary operation on S, written as $(x, y) \mapsto x \circ y$, and e is an element of S, such that for all $x, y, z \in S$,

(i) $x \circ (y \circ z) = (x \circ y) \circ z$;
(ii) $e \circ x = x \circ e = x$.

The element e is called the *identity element* of S.

Every monoid $(S, \circ, e)$ is assumed to be commutative; i.e. we have $x \circ y = y \circ x$ for all $x, y \in S$. We will often just talk about 'the monoid S', the monoid operation $\circ$ and the identity element e being understood. Furthermore, $x \circ y$ will often be written as xy.

Definition 1.24. Let S_1 and S_2 be monoids. A map $f : S_1 \to S_2$ is a *morphism of monoids* if we have $f(xy) = f(x)f(y)$ for all $x, y \in S_1$.

Definition 1.25. Let S be a monoid. For a subset $T \subseteq S$, the *span* of T is

$$\langle T \rangle = \{t_1 \cdots t_r : r \in \mathbb{Z}_{\geq 0}, t_1, \ldots, t_r \in T\},$$

where, by convention, the empty product equals e. If $S', T \subseteq S$ satisfy $S' = \langle T \rangle$, we say that S' is *generated* by T, or that T is a set of *generators* for S'. We call S' *finitely generated* if there is a finite subset $T \subseteq S$ such that $S' = \langle T \rangle$. For a finite subset $\{s_1, \ldots, s_r\} \subseteq S$ we also write $\langle s_1, \ldots, s_r \rangle = \langle \{s_1, \ldots, s_r\} \rangle$.

Definition 1.26. A *submonoid* of a monoid $(S, \circ, e)$ is a monoid $(S', \circ, e)$, where $S' \subseteq S$ is a subset.

Notice that for a subset T of a monoid S, the span $\langle T \rangle$ is the smallest submonoid of S containing T.

Remark 1.27. If $\sigma \subseteq N_{\mathbb{R}}$ is a cone, then $(\sigma \cap N, +, 0)$ is a monoid, where '$+$' is the addition on N and 0 is the identity element of N.

Definition 1.28. A *lattice cone (with respect to the lattice N)* is a cone $\sigma \subseteq N_{\mathbb{R}}$ such that there are vectors $v_1, \ldots, v_r \in N$ with $\sigma = \mathcal{C}(v_1, \ldots, v_r)$.

Remark 1.29. If $\sigma \subseteq N_{\mathbb{R}}$ is a lattice cone with respect to N, then $\check{\sigma} \subseteq N^*_{\mathbb{R}}$ is a lattice cone with respect to N^* (cf. [5, Property 1.1]).

Proposition 1.30 (Gordon's Lemma)**.** *If σ is a lattice cone with respect to the lattice N, then $\sigma \cap N$ is a finitely generated monoid.*

Proof. See [5, Lem. 1.3]. □

Now we can make the next step in our construction of an affine variety from a cone σ, by defining the finitely generated monoid S_σ.

Notation 1.31. If σ is a lattice cone with respect to N, then we write $S_\sigma = \check{\sigma} \cap N^*$.

Remark 1.32. It follows by Remark 1.29 and Proposition 1.30 that S_σ is indeed a finitely generated monoid.

Often we will just identify the lattices N and N^* with $\mathbb{Z}^n$ for some $n \in \mathbb{Z}_{\geq 0}$, and the $\mathbb{R}$-vector spaces $N_\mathbb{R}$ and $N^*_\mathbb{R}$ with $\mathbb{R}^n$. Obviously, the embeddings $N \hookrightarrow N_\mathbb{R}$ and $N^* \hookrightarrow N^*_\mathbb{R}$ are identified with the inclusion $\mathbb{Z}^n \subseteq \mathbb{R}^n$.

Example 1.33. Take $N = N^* = \mathbb{Z}^2$. Consider the lattice cone $\sigma = \mathcal{C}(e_1 + e_2, -e_1 + 2e_2) \subseteq \mathbb{R}^2$ and its dual cone $\check{\sigma} = \mathcal{C}(-e_1 + e_2, 2e_1 + e_2) \subseteq \mathbb{R}^2$. The figure below shows the finitely generated monoids $\sigma \cap N$ and $S_\sigma = \check{\sigma} \cap N^*$. Notice that the generators $-e_1 + e_2$ and $2e_1 + e_2$ of the cone $\check{\sigma}$ do not generate the monoid S_σ. In order to obtain a set of generators for S_σ, we should add the vectors e_2 and $e_1 + e_2$.

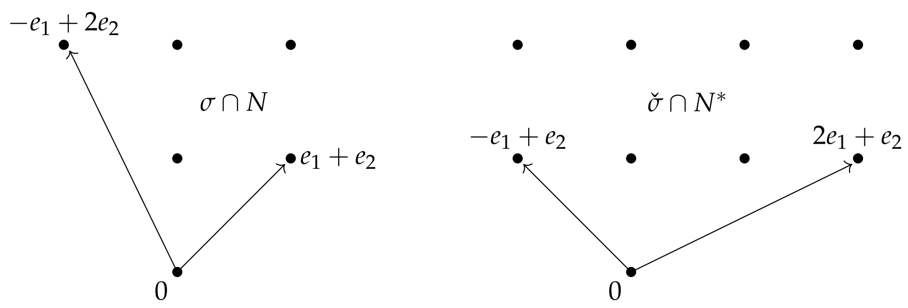

Figure 2: The finitely generated monoids $\sigma \cap N$ and $S_\sigma = \check{\sigma} \cap N^*$ (denoted by the dots).

To a monoid S one can associate the monoid algebra $\mathbb{C}[S]$, which is the $\mathbb{C}$-algebra generated by the elements χ^x for $x \in S$. For $x, y \in S$ we have $\chi^x \cdot \chi^y = \chi^{x \circ y}$, where $\circ$ is the monoid operation.

Notation 1.34. For a lattice cone σ we define $R_\sigma = \mathbb{C}[S_\sigma]$.

Let σ be a lattice cone with respect to the lattice N. Under the identification of N and N^* with $\mathbb{Z}^n$ (and therefore, $S_\sigma \subseteq \mathbb{Z}^n$), there is the following alternative characterization of R_σ. Let $\mathbb{C}[x, x^{-1}] = \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ denote the ring of Laurent polynomials in n variables. We write $x^k = x_1^{k_1} \cdots x_n^{k_n}$, where $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. Define the *support* of a Laurent polynomial $f = \sum_{k \in \mathbb{Z}^n} a_k x^k \in \mathbb{C}[x, x^{-1}]$ by

$$\operatorname{supp}(f) = \{k \in \mathbb{Z}^n : a_k \neq 0\}.$$

Then the monoid algebra $R_\sigma = \mathbb{C}[S_\sigma]$ can be identified with the $\mathbb{C}$-algebra

$$\left\{f \in \mathbb{C}[x, x^{-1}] : \operatorname{supp}(f) \subseteq S_\sigma\right\} \subseteq \mathbb{C}[x, x^{-1}].$$

1.4 Algebraic varieties

In the next subsection, the algebraic geometry comes into play. So let us now recall a few important notions (cf. [8, §4.3]; the definitions of k-spaces and their morphisms can be found here as well).

An *affine variety* over an algebraically closed field k is a k-space isomorphic to $(X, \mathcal{O}_X)$, where

$$X = Z(I) = \{a \in k^n : f(a) = 0 \text{ for all } f \in I\} \subseteq k^n$$

is the zero set of some ideal $I \subseteq k[x_1, \ldots, x_n]$, supplied with the Zariski topology, and $\mathcal{O}_X$ is its sheaf of regular functions. For $U \subseteq X$ open, $\mathcal{O}_X(U) \subseteq \mathrm{Map}(U, k)$ is the subring of regular functions on U, where a function $U \to k$ is *regular* if it can locally be written as g/h with $g, h \in k[x_1, \ldots, x_n]$.

An *algebraic variety* is a k-space $(X, \mathcal{O}_X)$ of which every element has an open neighborhood $U \subseteq X$ such that $(U, \mathcal{O}_X|_U)$ is an affine variety (here $\mathcal{O}_X|_U$ is defined by $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$ for every $V \subseteq U$ open). A *morphism* between the varieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is just a morphism of k-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$. (Notice that this notion of algebraic varieties is very general: we do not require them to be irreducible or even quasi-projective.) The category of algebraic varieties thus obtained, admits finite products; the product of two algebraic varieties $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ is written as $(X \times Y, \mathcal{O}_{X \times Y})$.

An algebraic variety is called *projective* if it is isomorphic to a closed subvariety of $\mathbb{P}^n(k)$ for some n, and it is called *quasi-projective* if it is isomorphic to an open subvariety of a projective variety.

Furthermore, an algebraic variety X is called *irreducible* if it is irreducible as a topological space; i.e. if it is non-empty and if for all closed subsets $Z_1, Z_2 \subseteq X$ with $X = Z_1 \cup Z_2$, we have $Z_1 = X$ or $Z_2 = X$.

The *dimension* $\dim(X)$ of an irreducible variety X is the largest integer m (if it exists) such that there is a chain of closed irreducible subvarieties $Y_0 \subsetneq \ldots \subsetneq Y_{m-1} \subsetneq Y_m = X$. If such an integer m does not exist, we set $\dim(X) = \infty$. The *codimension* of a subvariety Y of an irreducible variety X is $\mathrm{codim}(Y) = \dim(X) - \dim(Y)$.

An algebraic variety over $\mathbb{C}$ is called a *complex variety*. The complex affine n-space $\mathbb{C}^n$ will be denoted by $\mathbb{A}^n$ and the complex projective n-space $\mathbb{P}^n(\mathbb{C})$ by $\mathbb{P}^n$.

1.5 Affine toric varieties

Let σ be a lattice cone with respect to a lattice N. Since S_σ is a finitely generated monoid, R_σ is a finitely generated $\mathbb{C}$-algebra. So we have $R_\sigma \cong \mathbb{C}[y_1, \ldots, y_m]/I_\sigma$ for some $m \in \mathbb{Z}_{\geq 0}$ and some ideal $I_\sigma \subseteq \mathbb{C}[y_1, \ldots, y_m]$. Now we associate to the cone σ, the complex affine variety

$$X_\sigma = Z(I_\sigma) = \{a \in \mathbb{A}^m : f(a) = 0 \text{ for all } f \in I_\sigma\} \subseteq \mathbb{A}^m.$$

Remark 1.35. Since R_σ is an integral domain, the ideal I_σ is prime and the variety X_σ is irreducible.

Alternatively, X_σ can be defined as $\mathrm{MaxSpec}(R_\sigma)$: the set of maximum ideals of R_σ, supplied with the Zariski topology (cf. [10, §1.3] and [5, §2.3]). This definition of X_σ is somewhat neater, since it does not depend on a choice of coordinates. However, in practice it will be more convenient to work with the original definition using the ideal I_σ.

Example 1.36. Reconsider Example 1.33. Since the monoid S_σ is generated by $-e_1 + e_2,\ e_2,\ e_1 + e_2$ and $2e_1 + e_2$, we have

$$R_\sigma = \mathbb{C}[x_2/x_1, x_2, x_1x_2, x_1^2x_2] = \mathbb{C}[y_1, y_2, y_3, y_4]/I_\sigma,$$

where $I_\sigma = (y_1y_3 - y_2^2, y_2y_4 - y_3^2)$. The affine variety associated to the cone σ is $X_\sigma = Z(I_\sigma)$.

We now give a general recipe for determining the affine variety X_σ (cf. [5, p. 11]), which is suggested by the previous example. Let $v_1, \dots, v_m$ be generators for the monoid $S_\sigma \subseteq \mathbb{R}^n$. Then, R_σ is generated by $x^{v_1}, \dots, x^{v_m}$ as a $\mathbb{C}$-algebra. Relations

$$\sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i' v_i, \quad \lambda_i, \lambda_i' \in \mathbb{Z}_{\geq 0}$$

between the generators of S_σ induce relations

$$\prod_{i=1}^m (x^{v_i})^{\lambda_i} = \prod_{i=1}^m (x^{v_i})^{\lambda_i'}, \quad \lambda_i, \lambda_i' \in \mathbb{Z}_{\geq 0}$$

between the generators of R_σ. The latter relations generate the ideal $I_\sigma \subseteq \mathbb{C}[y_1, \dots, y_m]$, where $y_i = x^{v_i}$, and we have $X_\sigma = Z(I_\sigma)$.

It turns out that if the lattice cone σ is *strongly convex*, then the affine variety X_σ is *toric*. Let us define these two notions. For a cone σ we write $-\sigma = \{-v : v \in \sigma\}$.

Definition 1.37. A cone σ is called *strongly convex* if $\sigma \cap -\sigma = \{0\}$.

Proposition 1.38. *Let $\sigma \subseteq V$ be a cone. Then σ is strongly convex if and only if* $\dim \check{\sigma} = \dim_{\mathbb{R}} V$.

Proof. Since the subspaces $U_1 = \operatorname{span}(\sigma \cap -\sigma)$ and $U_2 = \operatorname{span}(\check{\sigma})$ of V are complementary, we have $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} U_1 + \dim_{\mathbb{R}} U_2$. Now the result follows, since we have $\dim_{\mathbb{R}} U_1 = 0$ if and only if σ is strongly convex. $\square$

Example 1.39. The cone $\sigma_1 = \mathcal{C}(e_1, e_2) \subseteq \mathbb{R}^2$ is strongly convex, whereas the cone $\sigma_2 = \mathcal{C}(e_1, -e_1, e_2) \subseteq \mathbb{R}^2$ is not. Accordingly, we have

$$\dim \check{\sigma}_1 = \dim \mathcal{C}(e_1, e_2) = 2, \quad \dim \check{\sigma}_2 = \dim \mathcal{C}(e_2) \neq 2.$$

Definition 1.40. An *algebraic group* is an algebraic variety X, supplied with a group structure, such that the group multiplication $\cdot : X \times X \to X$ and the inversion operation $(\cdot)^{-1} : X \to X$ are morphisms of varieties.

Definition 1.41. The *n-dimensional torus* is the algebraic group $\mathbb{T}^n = (\mathbb{C}^*)^n$.

By convention, the 0-dimensional torus $\mathbb{T}^0$ is the trivial group $\{0\}$.

Definition 1.42. A *toric variety* is a complex algebraic variety X together with an open and dense embedding $i : \mathbb{T}^n \hookrightarrow X$ and a group action $\varphi : \mathbb{T}^n \times X \to X$, such that

(i) $\varphi : \mathbb{T}^n \times X \to X$ is a morphism of varieties;
(ii) the restriction $\mathbb{T}^n \times i(\mathbb{T}^n) \to X$ of φ is given by $(t, i(t')) \mapsto i(t \cdot t')$, where '$\cdot$' is the multiplication on $\mathbb{T}^n$.

The group action $\varphi : \mathbb{T}^n \times X \to X$ is called the *torus action*.

So, identifying the torus $\mathbb{T}^n$ with its image $i(\mathbb{T}^n)$, we could say that the restriction of the torus action to the torus itself 'is' the multiplication on the torus.

Example 1.43. The parabola $X = Z(x^2 - y) \subseteq \mathbb{A}^2$ is an affine toric variety: we have an open and dense embedding $i : \mathbb{T}^1 \hookrightarrow X$, given by $t \mapsto (t, t^2)$. The toric action $\mathbb{T}^1 \times X \to X$ is given by $(t, (x, y)) \mapsto (tx, t^2 y)$.

Example 1.44. More generally, we will show that the affine variety $X = Z(f) \subseteq \mathbb{A}^2$, where $f = ax^n + by^m \in \mathbb{C}[x, y]$ is an irreducible polynomial, is toric. Indeed, choose $\alpha, \beta \in \mathbb{C}$, not both 0, such that $a\alpha^n + b\beta^m = 0$. Then we have

$$a(\alpha t^m)^n + b(\beta t^n)^m = (a\alpha^n + b\beta^m)t^{mn} = 0$$

for all $t \in \mathbb{T}^1$. Since the polynomial f is irreducible, n and m are not both even. Therefore, $t \mapsto (\alpha t^m, \beta t^n)$ defines an open embedding $i : \mathbb{T}^1 \hookrightarrow X$. Since the polynomial f is irreducible, the variety $X = Z(f)$ is irreducible as well; therefore, i is dense.

Furthermore, the torus action $\varphi : \mathbb{T}^1 \times X \to X$ is given by $(t, (x, y)) \mapsto (t^m x, t^n y)$. For $t, t' \in \mathbb{T}^1$ we find

$$i(tt') = (\alpha(tt')^m, \beta(tt')^n) = \varphi(t, (\alpha t'^m, \beta t'^n)) = \varphi(t, i(t')).$$

Now we have the following important theorem. For $t = (t_1, \ldots, t_n) \in \mathbb{T}^n$ and $v = (v^1, \ldots, v^n) \in \mathbb{Z}^n$ we write $t^v = t_1^{v^1} \cdots t_n^{v^n}$.

Theorem 1.45. *Let σ be a strongly convex lattice cone in $N_{\mathbb{R}}$ with $\dim_{\mathbb{R}} N_{\mathbb{R}} = n$ and let $v_1, \ldots, v_m$ be generators for the monoid S_σ. Then there is an open and dense embedding*

$$i : \mathbb{T}^n \hookrightarrow X_\sigma,\ t \mapsto (t^{v_1}, \ldots, t^{v_m})$$

and there is a torus action

$$\varphi : \mathbb{T}^n \times X_\sigma \to X_\sigma,\ (t, (a_1, \ldots, a_m)) \mapsto (t^{v_1} a_1, \ldots, t^{v_m} a_m).$$

Proof. We may assume that $N = \mathbb{Z}^n$, $N_{\mathbb{R}} = \mathbb{R}^n$ and $v_1, \ldots, v_m \in \mathbb{Z}^n$. Write $v_i = (v_i^1, \ldots, v_i^n)$ for $i = 1, \ldots, m$. We have $X_\sigma = Z(I_\sigma) \subseteq \mathbb{A}^m$ for some ideal $I_\sigma \subseteq \mathbb{C}[y_1, \ldots, y_m]$, generated by the k relations

$$\prod_{i=1}^{m} (x^{v_i})^{\lambda_{i,j}} = \prod_{i=1}^{m} (x^{v_i})^{\lambda'_{i,j}}, \quad j = 1, \ldots, k,\ \lambda_{i,j}, \lambda'_{i,j} \in \mathbb{Z}_{\geq 0},$$

where $y_i = x^{v_i}$. It follows immediately from these relations that the map $i : \mathbb{T}^n \hookrightarrow X_\sigma$ is well-defined. For the proof that i is an open and dense embedding, we refer to [5, Prop. 2.2] (this proof implicitly uses that $\text{span}(\check{\sigma}) =$

$\mathbb{R}^n$, which follows from Proposition 1.38). It remains to be shown that φ is a torus action.

First we check that φ is a well-defined map. Let $(t,(a_1,\dots,a_m)) \in \mathbb{T}^n \times X_\sigma$ and $j \in \{1,\dots,k\}$. Since $i(t) = (t^{v_1},\dots,t^{v_m})$ and $(a_1,\dots,a_m)$ are contained in X_σ, we find that

$$\prod_{i=1}^m (t^{v_i})^{\lambda_{i,j}} - \prod_{i=1}^m (t^{v_i})^{\lambda'_{i,j}} = 0 = \prod_{i=1}^m a_i^{\lambda_{i,j}} - \prod_{i=1}^m a_i^{\lambda'_{i,j}}$$

and therefore,

$$\prod_{i=1}^m (t^{v_i}a_i)^{\lambda_{i,j}} - \prod_{i=1}^m (t^{v_i}a_i)^{\lambda'_{i,j}} = \prod_{i=1}^m (t^{v_i})^{\lambda_{i,j}} \cdot \prod_{i=1}^m a_i^{\lambda_{i,j}} - \prod_{i=1}^m (t^{v_i})^{\lambda'_{i,j}} \cdot \prod_{i=1}^m a_i^{\lambda'_{i,j}} = 0.$$

Hence, we have $(t^{v_1}a_1,\dots,t^{v_m}a_m) \in X_\sigma$, which shows that φ is well-defined.

Since the composition of φ with every coordinate projection is a regular function, φ is a morphism of varieties. Furthermore, it is easily verified that φ is a group action. Finally, for $t,t' \in \mathbb{T}^n$ we have

$$\varphi(t,i(t')) = \varphi(t,(t'^{v_1},\dots,t'^{v_m})) = ((tt')^{v_1},\dots,(tt')^{v_m}) = i(tt').$$

Hence, $\varphi : \mathbb{T}^n \times X_\sigma \to X_\sigma$ is a torus action. $\square$

As a direct consequence we have the following.

Corollary 1.46. *If σ is a strongly convex lattice cone in $N_\mathbb{R}$, then X_σ is an affine toric variety with* $\dim X_\sigma = \dim_\mathbb{R} N_\mathbb{R}$. $\square$

Notation 1.47. We denote the image of the torus embedding in X_σ by $\mathbb{T}_\sigma$.

As the following example shows, the dimension statement of Corollary 1.46 need not be true for a cone which is not strongly convex.

Example 1.48. Consider the lattice cone $\sigma = \mathcal{C}(e_1,-e_1,e_2) \subseteq \mathbb{R}^2$, which is not strongly convex. We have $S_\sigma = \langle e_2 \rangle$ and $R_\sigma = \mathbb{C}[x_2]$. So the variety $X_\sigma \cong \mathbb{A}^1$ has dimension $1 \neq \dim_\mathbb{R} \mathbb{R}^2$.

1.6 Faces and their induced inclusions

In the remaining part of this section, we will study the inclusions induced by a face $\tau \leq \sigma$ of a cone σ.

Proposition 1.49. *Let σ be a cone and let $\tau = \sigma \cap u^\perp$ be a face of σ, where $u \in \check{\sigma}$.*

(1) *If $\check{\sigma} = \mathcal{C}(S)$, then $\check{\tau} = \mathcal{C}(S \cup \{-u\})$.*
(2) *Suppose that σ is a lattice cone and that $T \subseteq S_\sigma$ is a subset with $u \in T$ and $\langle T \rangle = S_\sigma$. Then we have $S_\tau = \langle T \cup \{-u\} \rangle$.*

Proof.

(1) See [5, Property. 1.3].
(2) We may pretend that both σ and $\check{\sigma}$ are lattice cones with respect to the lattice $\mathbb{Z}^n$. Also, we may assume that $T = \{t_1,\dots,t_r\} \subseteq \mathbb{Z}^n$ is a finite set with $t_1 = u$ (otherwise, we can replace T by a finite subset $T' \subseteq T$

which still generates S_σ and contains u). Then we have $\mathcal{C}(T) = \check{\sigma}$. Let $x \in S_\tau = \check{\tau} \cap \mathbb{Z}^n$. By part (1) we can write

$$x = \lambda_1 t_1 + \ldots + \lambda_r t_r - \lambda u = (\lambda_1 - \lambda)t_1 + \lambda_2 t_2 + \ldots + \lambda_r t_r$$

with $\lambda_1, \ldots, \lambda_r, \lambda \in \mathbb{R}_{\geq 0}$. Choose $a \in \mathbb{Z}_{\geq 0}$ such that $\lambda_1 - \lambda + a \geq 0$. Since $x + at_1$ is contained in $S_\sigma = \check{\sigma} \cap \mathbb{Z}^n$, we can write

$$x + at_1 = \mu_1 t_1 + \ldots + \mu_r t_r$$

with $\mu_1, \ldots, \mu_r \in \mathbb{Z}_{\geq 0}$. This yields

$$x = \mu_1 t_1 + \ldots + \mu_r t_r - au \in \langle T \cup \{-u\} \rangle$$

and $S_\tau \subseteq \langle T \cup \{-u\} \rangle$, which finishes the proof, since the reverse inclusion is clear.

□

Example 1.50. In Examples 1.33 and 1.36 we studied the strongly convex lattice cone $\sigma \subseteq \mathbb{R}^2$ with

$$\begin{aligned}
\sigma &= \mathcal{C}(e_1 + e_2, -e_1 + 2e_2);\\
\check{\sigma} &= \mathcal{C}(-e_1 + e_2, 2e_1 + e_2);\\
S_\sigma &= \langle -e_1 + e_2, e_2, e_1 + e_2, 2e_1 + e_2 \rangle;\\
R_\sigma &= \mathbb{C}[x_2/x_1, x_2, x_1x_2, x_1^2x_2] = \mathbb{C}[y_1, y_2, y_3, y_4]/(y_1y_3 - y_2^2, y_2y_4 - y_3^2);\\
X_\sigma &= Z(y_1y_3 - y_2^2, y_2y_4 - y_3^2).
\end{aligned}$$

Now consider the face $\tau = \sigma \cap u^\perp$ of σ, where $u = -e_1 + e_2 \in \check{\sigma}$. Notice that $\check{\tau}$ can be obtained from $\check{\sigma}$ by adding the vector $-u$ as a new generator; the same holds for S_τ and S_σ (cf. Proposition 1.49). Accordingly, we add the generator x_1/x_2 to R_σ in order to obtain R_τ; i.e. $R_\tau = R_\sigma[y_1^{-1}]$ is the localization of R_σ at the element $y_1 = x_2/x_1 \in R_\sigma$. This yields

$$\begin{aligned}
\tau &= \mathcal{C}(e_1 + e_2);\\
\check{\tau} &= \mathcal{C}(-e_1 + e_2, 2e_1 + e_2, e_1 - e_2);\\
S_\tau &= \langle -e_1 + e_2, e_2, e_1 + e_2, 2e_1 + e_2, e_1 - e_2 \rangle;\\
R_\tau &= \mathbb{C}[x_2/x_1, x_2, x_1x_2, x_1^2x_2, x_1/x_2]\\
&= \mathbb{C}[y_1, y_2, y_3, y_4, y_5]/(y_1y_3 - y_2^2, y_2y_4 - y_3^2, y_1y_5 - 1);\\
X_\tau &= Z(y_1y_3 - y_2^2, y_2y_4 - y_3^2, y_1y_5 - 1);
\end{aligned}$$

and there are embeddings

$$\begin{aligned}
\tau &\hookrightarrow \sigma, \quad v \mapsto v;\\
\check{\tau} &\hookleftarrow \check{\sigma}, \quad v \leftarrow\!\shortmid v;\\
S_\tau &\hookleftarrow S_\sigma, \quad v \leftarrow\!\shortmid v;\\
R_\tau &\hookleftarrow R_\sigma, \quad [f] \leftarrow\!\shortmid [f];\\
X_\tau &\hookrightarrow X_\sigma, \quad (a_1, a_2, a_3, a_4, a_5) \mapsto (a_1, a_2, a_3, a_4).
\end{aligned}$$

By the embedding $X_\tau \hookrightarrow X_\sigma$, which is the projection on the first four coordinates, X_τ can be identified with the open subvariety $X_\sigma \setminus Z(y_1)$ of X_σ.

Embeddings as in the previous example exist in general.

Theorem 1.51. *Let σ be a strongly convex lattice cone and $\tau < \sigma$ a proper face. Then there are embeddings*

$$\begin{aligned} \tau &\hookrightarrow \sigma; \\ \check{\tau} &\hookleftarrow \check{\sigma}; \\ S_\tau &\hookleftarrow S_\sigma; \\ R_\tau &\hookleftarrow R_\sigma; \\ X_\tau &\hookrightarrow X_\sigma; \end{aligned}$$

such that the embedding $X_\tau \hookrightarrow X_\sigma$ induces an isomorphism $X_\tau \cong X_\sigma \setminus Z(y_1)$, where y_1 is one of the coordinates of X_σ.

Proof. The embedding $\tau \hookrightarrow \sigma$ is clear. By Proposition 1.49 we have embeddings $\check{\sigma} \hookrightarrow \check{\tau}$ and $S_\sigma \hookrightarrow S_\tau$ (notice that we can always choose T and u as in Proposition 1.49). The embedding $S_\sigma \hookrightarrow S_\tau$ induces, consecutively, embeddings $R_\sigma \hookrightarrow R_\tau$ and $X_\tau \hookrightarrow X_\sigma$ (for more details, see [5, p. 17]). $\square$

Example 1.52. Every lattice cone $\sigma \subseteq \mathbb{R}^n$ has the zero cone $\{0\} \subseteq \mathbb{R}^n$ as a face. Accordingly, we have an inclusion $\mathbb{T}_\sigma = (\mathbb{C}^*)^n = X_{\{0\}} \subseteq X_\sigma$ of toric varieties (notice that $S_{\{0\}} = \langle e_1, \ldots, e_n, -e_1, \ldots, -e_n \rangle$ and therefore $X_{\{0\}} = (\mathbb{C}^*)^n$).

2 Fans and general toric varieties

In the previous section it was shown how a cone σ gives rise to an affine toric variety X_σ. In this section we will show how a *fan* Σ, which is a set of cones satisfying certain conditions, gives rise to a, not necessarily affine, toric variety X_Σ.

After having introduced the notion of a *fan* in § 2.1, we will describe the construction of X_Σ (§ 2.2) and discuss some of its properties. For instance, the so called *orbit-cone correspondence* is crucial for understanding the structure of the toric variety X_Σ (see § 2.4). There will be special attention for how properties of a fan Σ relate to properties of the associated toric variety X_Σ (§ 2.5). The section will be concluded by a discussion of polytopal fans (§ 2.6).

2.1 Fans

Let us introduce the notion of a *fan* (cf. [10, §1.4], [5, §3.1] and [25, §3.1]).

Definition 2.1. A *fan* in $N_\mathbb{R}$ is a finite non-empty set Σ of strongly convex lattice cones in $N_\mathbb{R}$ such that

(i) every face $\tau \leq \sigma$ of a cone $\sigma \in \Sigma$ is itself contained in Σ;
(ii) for all cones $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

Definition 2.2. Let Σ be a fan and $S = \{\sigma_1, \ldots, \sigma_r\}$ a set of strongly convex lattice cones. We say that Σ is *generated* by S if

$$\Sigma = \{\tau : \tau \leq \sigma_i \text{ for some } i = 1, \ldots, r\}.$$

In this case we write $\Sigma = \mathcal{F}(S) = \mathcal{F}(\sigma_1, \ldots, \sigma_r)$.

Definition 2.3. The *dimension* of a fan Σ is $\dim \Sigma = \max_{\sigma \in \Sigma} \dim \sigma$. We call a cone $\sigma \in \Sigma$ *maximal* if $\dim \sigma = \dim \Sigma$.

Example 2.4. This figure shows the fan $\mathcal{F}(\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1 = \mathcal{C}(e_1, e_1 + e_2)$, $\sigma_2 = \mathcal{C}(e_1, -e_2)$ and $\sigma_3 = \mathcal{C}(-e_2, -2e_1 - e_2)$.

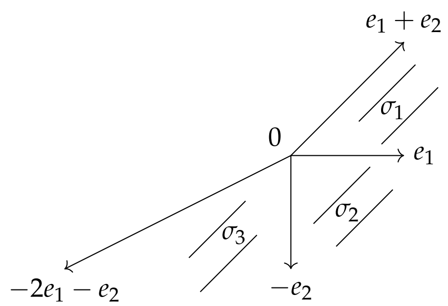

Figure 3: The fan $\mathcal{F}(\sigma_1, \sigma_2, \sigma_3)$.

2.2 The toric variety associated to a fan

In § 1 we saw how a cone σ gives rise to an affine toric variety X_σ. Given a fan Σ, the affine toric varieties induced by its cones can be glued together into a new, not necessarily affine, toric variety X_Σ (cf. [10, §1.4]). Let Σ be a fan and take the disjoint union $X'_\Sigma = \bigsqcup\{X_\sigma : \sigma \in \Sigma\}$. For distinct cones $\sigma_1, \sigma_2 \in \Sigma$, the affine toric variety $X_{\sigma_1 \cap \sigma_2}$, induced by the face $\sigma_1 \cap \sigma_2 \in \Sigma$ of σ_1 and σ_2, can be considered as an open subvariety of both X_{σ_1} and X_{σ_2} (see Theorem 1.51); say we have embeddings $\varphi_1 : X_{\sigma_1 \cap \sigma_2} \hookrightarrow X_{\sigma_1}$ and $\varphi_2 : X_{\sigma_1 \cap \sigma_2} \hookrightarrow X_{\sigma_2}$. Now for all distinct cones $\sigma_1, \sigma_2 \in \Sigma$, we glue X_{σ_1} and X_{σ_2} along $X_{\sigma_1 \cap \sigma_2}$ by identifying the images of φ_1 and φ_2, thus obtaining the variety X_Σ from X'_Σ.

Remark 2.5. Since the variety X_Σ is obtained by gluing irreducible varieties (see Remark 1.35) along non-empty open subvarieties, X_Σ is itself irreducible.

Theorem 2.6. *If Σ is a fan in $N_\mathbb{R}$, then X_Σ is a toric variety with* $\dim X_\Sigma = \dim_\mathbb{R} N_\mathbb{R}$. □

Proof. For every $\sigma \in \Sigma$, the torus $\mathbb{T}_\sigma \subseteq X_\sigma$ can be identified with the open and dense subvariety $X_{\{0\}} \subseteq X_\sigma$, associated to the zero cone $\{0\} \leq \sigma$ (see Example 1.52). Since for all distinct cones $\sigma_1, \sigma_2 \in \Sigma$, the toric variety $X_{\{0\}}$ is contained in $X_{\sigma_1 \cap \sigma_2}$, all tori $\mathbb{T}_\sigma \subseteq X_\sigma$ are identified in X_Σ with the open and dense subvariety $X_{\{0\}} \subseteq X_\Sigma$. Furthermore, this identification is compatible with the torus action. The dimension statement follows from Corollary 1.46. □

Notation 2.7. We denote the image $X_{\{0\}} \subseteq X_\Sigma$ of the torus embedding by $\mathbb{T}_\Sigma$.

Let us look at some examples of toric varieties associated to fans.

Example 2.8. Take $\sigma_0 = \mathcal{C}(e_1) \subseteq \mathbb{R}^1$, $\sigma_1 = \mathcal{C}(-e_1) \subseteq \mathbb{R}^1$ and $\Sigma = \mathcal{F}(\sigma_0, \sigma_1)$. We will show that the toric variety X_Σ is (isomorphic to) the complex projective line $\mathbb{P}^1$.

We have $S_{\sigma_0} = \langle e_1 \rangle$ and therefore $R_{\sigma_0} = \mathbb{C}[x]$ and $X_{\sigma_0} = \mathbb{A}^1_x$. (By subscripts as in $\mathbb{A}^1_x$ we denote the coordinates that are used, which will be relevant for the gluing.) Also we have $S_{\sigma_1} = \langle -e_1 \rangle$, and therefore $R_{\sigma_1} = \mathbb{C}[x^{-1}]$ and $X_{\sigma_1} = \mathbb{A}^1_{x^{-1}}$. For the joint face $\tau = \sigma_0 \cap \sigma_1 = \{0\}$ of σ_0 and σ_1, we have $S_\tau = \langle e_1, -e_1 \rangle$, $R_\tau = \mathbb{C}[x, x^{-1}]$ and $X_\tau = \mathbb{A}^1_x \setminus \{0\}$.

Furthermore, there are embeddings $\varphi_0 : X_\tau \hookrightarrow X_{\sigma_0}, x \mapsto x$ and $\varphi_1 : X_\tau \hookrightarrow X_{\sigma_1}, x \mapsto x^{-1}$. By identifying $x \in X_{\sigma_0} \setminus \{0\}$ with $x^{-1} \in X_{\sigma_1} \setminus \{0\}$, we obtain X_Σ, which is just the complex projective line $\mathbb{P}^1_{(t_0:t_1)}$, where $x = t_1/t_0$. By the embeddings $X_{\sigma_0} \hookrightarrow \mathbb{P}^1, t_1 \mapsto (1 : t_1)$ and $X_{\sigma_1} \hookrightarrow \mathbb{P}^1, t_0 \mapsto (t_0 : 1)$, the varieties X_{σ_0} and X_{σ_1} can be identified with the coordinate charts $U_0 = \{(t_0 : t_1) \in \mathbb{P}^1 : t_0 \neq 0\}$ of $\mathbb{P}^1$ and $U_1 = \{(t_0 : t_1) \in \mathbb{P}^1 : t_1 \neq 0\}$, respectively.

More generally, we have the following (cf. [5, Example 3.4]).

Example 2.9. Set $I = \{0, \dots, n\}$ and define, for $i \in I$, $\sigma_i = \mathcal{C}(\{e_j : j \in I \setminus \{i\}\})$, where $e_1, \ldots, e_n$ are the standard basis vectors of $\mathbb{R}^n$ and $e_0 = -(e_1 + \ldots + e_n)$. Then, using the suggestive notation $\Sigma_{\mathbb{P}^n} = \mathcal{F}(\sigma_0, \ldots, \sigma_n)$, we have $X_{\Sigma_{\mathbb{P}^n}} \cong \mathbb{P}^n$. The affine varieties $X_{\sigma_0}, \ldots, X_{\sigma_n}$ can be identified with the coordinate charts $U_0, \ldots, U_n$ of $\mathbb{P}^n$, respectively.

Example 2.10. Consider the strongly convex lattice cones $\sigma_1 = \mathcal{C}(e_1, e_2)$ and $\sigma_2 = \mathcal{C}(-e_1, e_2)$.

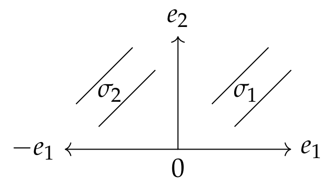

Figure 4: The cones σ_1 and σ_2.

We have

$$
\begin{aligned}
S_{\sigma_1} &= \langle e_1, e_2 \rangle, \quad R_{\sigma_1} = \mathbb{C}[x_1, x_2], \quad X_{\sigma_1} = \mathbb{A}^2_{(x_1, x_2)}; \\
S_{\sigma_2} &= \langle -e_1, e_2 \rangle, \quad R_{\sigma_2} = \mathbb{C}[x_1^{-1}, x_2], \quad X_{\sigma_2} = \mathbb{A}^2_{(x_1^{-1}, x_2)}.
\end{aligned}
$$

The gluing of X_{σ_1} and X_{σ_2} yields $X_{\mathcal{F}(\sigma_1, \sigma_2)} = \mathbb{P}^1_{(t_0 : t_1)} \times \mathbb{A}^1_{x_2}$, where $x_1 = t_0/t_1$.

Definition 2.11. An algebraic variety X is called *separated* if the diagonal $\Delta_X = \{(x, x) : x \in X\}$ is closed in $X \times X$.

For a fan Σ, the toric variety X_Σ turns out to be separated, which follows from the next two lemmas.

Lemma 2.12. *Let Σ be a fan and $\sigma, \tau \in \Sigma$. The diagonal map $X_{\sigma \cap \tau} \to X_\sigma \times X_\tau$, given by $x \mapsto (x, x)$, is a closed embedding; i.e. it is an embedding which maps closed sets to closed sets.*

Proof. See [10, p. 21]. $\square$

Lemma 2.13. *Let T be a topological space and $\{U_i : i \in I\}$ an open cover of T. A subset $S \subseteq T$ is closed if and only if $S \cap U_i$ is closed in U_i for all $i \in I$.*

Proof. Suppose that $S \cap U_i$ is closed in U_i for all $i \in I$. Then, for all $i \in I$ we have $U_i \setminus S = U_i \cap V_i$ for some open $V_i \subseteq T$. It follows that $T \setminus S = (\bigcup_{i \in I} U_i) \setminus S = \bigcup_{i \in I} (U_i \cap V_i)$ is open and S is closed. The reverse implication is immediately clear. $\square$

Proposition 2.14. *For a fan Σ, the toric variety X_Σ is separated.*

Proof. Since $\{X_\sigma : \sigma \in \Sigma\}$ is an open cover of the variety X_Σ, we know that $\{X_\sigma \times X_\tau : \sigma, \tau \in \Sigma\}$ is an open cover of the product variety $X_\Sigma \times X_\Sigma$. By

Lemma 2.12, for any $\sigma, \tau \in \Sigma$, the intersection $\Delta_{X_\Sigma} \cap (X_\sigma \times X_\tau)$ is closed in $X_\sigma \times X_\tau$. It follows by Lemma 2.13 that Δ_{X_Σ} is closed in $X_\Sigma \times X_\Sigma$. $\square$

2.3 Hirzebruch surfaces

An interesting example of a toric variety is the *Hirzebruch surface* $\mathcal{H}_q$ (cf. [10, pp. 7–8], [5, Example 3.7] and [25, Example 3.6]), named after the German mathematician Friedrich Hirzebruch (1927–2012).

Let $q \in \mathbb{Z}_{\geq 0}$ and consider the fan $\Sigma_q = \mathcal{F}(\sigma_1, \sigma_2, \tau_q, \xi_q)$, where

$$\sigma_1 = \mathcal{C}(e_1, e_2),\ \sigma_2 = \mathcal{C}(-e_1, e_2),\ \tau_q = \mathcal{C}(-e_1, qe_1 - e_2),\ \xi_q = \mathcal{C}(e_1, qe_1 - e_2).$$

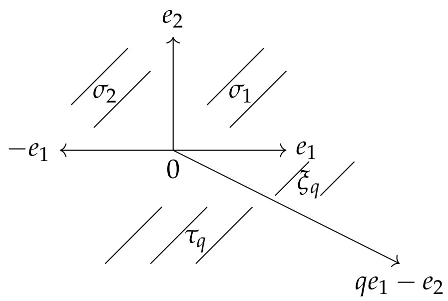

Figure 5: The fan Σ_q.

The *Hirzebruch surface* $\mathcal{H}_q$ is defined as $\mathcal{H}_q = X_{\Sigma_q}$. In order to obtain an explicit description of $\mathcal{H}_q$, we will perform the gluing construction. We have

$$\begin{aligned}
S_{\sigma_1} &= \langle e_1, e_2 \rangle, & R_{\sigma_1} &= \mathbb{C}[x_1, x_2], & X_{\sigma_1} &= \mathbb{A}^2_{(x_1, x_2)};\\
S_{\sigma_2} &= \langle -e_1, e_2 \rangle, & R_{\sigma_2} &= \mathbb{C}[x_1^{-1}, x_2], & X_{\sigma_2} &= \mathbb{A}^2_{(x_1^{-1}, x_2)};\\
S_{\tau_q} &= \langle -e_1 - qe_2, -e_2 \rangle, & R_{\tau_q} &= \mathbb{C}[x_1^{-1}x_2^{-q}, x_2^{-1}], & X_{\tau_q} &= \mathbb{A}^2_{(x_1^{-1}x_2^{-q}, x_2^{-1})};\\
S_{\xi_q} &= \langle e_1 + qe_2, -e_2 \rangle, & R_{\tau_q} &= \mathbb{C}[x_1 x_2^q, x_2^{-1}], & X_{\tau_q} &= \mathbb{A}^2_{(x_1 x_2^q, x_2^{-1})}.
\end{aligned}$$

As in Example 2.10, the gluing of X_{σ_1} and X_{σ_2} yields $X_{\mathcal{F}(\sigma_1,\sigma_2)} = \mathbb{P}^1_{(t_0:t_1)} \times \mathbb{A}^1_{x_2}$, where $x_1 = t_0/t_1$. Similarly, the gluing of X_{τ_q} and X_{ξ_q} yields $X_{\mathcal{F}(\tau_q,\xi_q)} = \mathbb{P}^1_{(s_0:s_1)} \times \mathbb{A}^1_{x_2^{-1}}$, where $x_1 x_2^q = s_0/s_1$. Now we glue $X_{\mathcal{F}(\sigma_1,\sigma_2)}$ and $X_{\mathcal{F}(\tau_q,\xi_q)}$ along $\mathbb{P}^1 \times (\mathbb{A}^1 \setminus \{0\})$ by the identification

$$\begin{aligned}
\psi : \mathbb{P}^1_{(t_0:t_1)} \times (\mathbb{A}^1_{x_2} \setminus \{0\}) &\to \mathbb{P}^1_{(s_0:s_1)} \times \left(\mathbb{A}^1_{x_2^{-1}} \setminus \{0\}\right)\\
((a:b), c) &\mapsto ((ac^q : b), c^{-1}).
\end{aligned}$$

We claim that the obtained result $\mathcal{H}_q$ equals

$$X = \{((u_0 : u_1 : u_2), (v_0 : v_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 : u_0 v_0^q = u_1 v_1^q\}.$$

Indeed, there are embeddings

$$\begin{aligned}\varphi_1 : \mathbb{P}^1_{(t_0:t_1)} \times \mathbb{A}^1_{x_2} &\hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1, \quad ((a:b),c) \mapsto ((a:ac^q:b),(c:1));\\ \varphi_2 : \mathbb{P}^1_{(s_0:s_1)} \times \mathbb{A}^1_{x_2^{-1}} &\hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1, ((d:e),f) \mapsto ((df^q:d:e),(1:f));\end{aligned}$$

which satisfy $\varphi_1|_{\mathbb{P}^1\times(\mathbb{A}^1\setminus\{0\})} = \varphi_2 \circ \psi$ and $\mathrm{im}(\varphi_1) \cup \mathrm{im}(\varphi_2) = X$.

Example 2.15. We have

$$\mathcal{H}_0 = \{((u_0:u_1:u_2),(v_0:v_1)) \in \mathbb{P}^2 \times \mathbb{P}^1 : u_0v_0^0 = u_1v_1^0\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Definition 2.16. A *fiber bundle* is data (E, B, F, π), where E, B and F are algebraic varieties and $\pi : E \to B$ is a surjective morphism, such that for every element $b \in B$ there is an open subvariety $U \subseteq B$ containing b and an isomorphism $\psi : \pi^{-1}(U) \to U \times F$ satisfying $\pi_1 \circ \psi = \pi|_{\pi^{-1}(U)}$, where $\pi_1 : U \times F \to U$ is the projection on the first coordinate. A fiber bundle (E, B, F, π) is called an *F-bundle over B*.

Proposition 2.17. *Let $q \in \mathbb{Z}_{\geq 0}$. The Hirzebruch surface $\mathcal{H}_q$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.*

Proof. Let $\pi : \mathcal{H}_q \to \mathbb{P}^1$ be the projection on the second coordinate. We claim that $(\mathcal{H}_q, \mathbb{P}^1, \mathbb{P}^1, \pi)$ is a fiber bundle. Clearly, π is a surjective morphism. Let U_0 and U_1 be the coordinate charts of $\mathbb{P}^1$. Notice that we have

$$\begin{aligned}\pi^{-1}(U_0) &= \{((df^q:d:e),(1:f)) \in \mathbb{P}^2 \times \mathbb{P}^1 : ((d:e),f) \in \mathbb{P}^1 \times \mathbb{C}\};\\ \pi^{-1}(U_1) &= \{((a:ac^q:b),(c:1)) \in \mathbb{P}^2 \times \mathbb{P}^1 : ((a:b),c) \in \mathbb{P}^1 \times \mathbb{C}\}.\end{aligned}$$

So we have isomorphisms

$$\begin{aligned}\psi_0 : \pi^{-1}(U_0) &\xrightarrow{\sim} U_0 \times \mathbb{P}^1, ((df^q:d:e),(1:f)) \mapsto ((1:f),(d:e));\\ \psi_1 : \pi^{-1}(U_1) &\xrightarrow{\sim} U_1 \times \mathbb{P}^1, ((a:ac^q:b),(c:1)) \mapsto ((c:1),(a:b));\end{aligned}$$

satisfying the required condition. Since U_0 and U_1 form an open cover of $\mathbb{P}^1$, this finishes the proof. □

2.4 The orbit-cone correspondence

Notice that for a lattice cone σ we have a one-to-one correspondence (cf. [5, §4.2])

$$\begin{array}{ccccccc} X_\sigma & \leftrightarrow & \mathrm{MaxSpec}(R_\sigma) & \leftrightarrow & \mathrm{Hom}_{\mathbb{C}\text{-alg}}(R_\sigma, \mathbb{C}) & \leftrightarrow & \mathrm{Hom}_{\mathrm{Mon}}(S_\sigma, (\mathbb{C}, \cdot)) \\ & & \mathcal{M}_a & \mapsto & (f \mapsto f(a)) & & \\ & & \ker\varphi & \leftarrow\!\shortmid & \varphi, & & \end{array}$$

where $\mathcal{M}_a \subseteq R_\sigma$ is the maximal ideal corresponding to an element $a \in X_\sigma$.

Definition 2.18. Let σ be a lattice cone with $S_\sigma = \langle v_1, \ldots, v_m \rangle$. To each face $\tau \leq \sigma$ we associate the *distinguished point* $d_{\tau\leq\sigma} \in X_\sigma$, which is the element in X_σ corresponding to the morphism $\varphi_{\tau\leq\sigma} : S_\sigma \to \mathbb{C}$ given by

$$v_i \mapsto \begin{cases} 1, & \text{if } v_i \in \tau^\perp \\ 0, & \text{if } v_i \notin \tau^\perp. \end{cases}$$

Notice that the morphism $\varphi_{\tau\leq\sigma}$ and the distinguished point $d_{\tau\leq\sigma}$, associated to a face $\tau \leq \sigma$, are dependent on a choice of generators for S_σ.

Definition 2.19. Let σ be a lattice cone and $\tau \leq \sigma$ a face. The *orbit of* τ *in* X_σ, denoted by $O_\sigma(\tau)$, is the orbit of $d_{\tau\leq\sigma} \in X_\sigma$ under the torus action.

Example 2.20. Consider the cone $\sigma = \mathcal{C}(e_1, e_1 + e_2)$ (cf. Examples 1.6 and 1.19). We have $S_\sigma = \langle v_1, v_2\rangle$, where $v_1 = e_2$ and $v_2 = e_1 - e_2$, and the toric variety associated to σ is $X_\sigma = \mathbb{A}^1_{x_2} \times \mathbb{A}^1_{x_1x_2^{-1}}$. The morphisms $\varphi_{\tau\leq\sigma}$, distinguished points $d_{\tau\leq\sigma}$ and orbits $O_\sigma(\tau)$ corresponding to every face τ of σ are as follows:

τ	$\varphi_{\tau\leq\sigma}$	$d_{\tau\leq\sigma}$	$O_\sigma(\tau)$
$\mathcal{C}(0)$	$v_1 \mapsto 1, v_2 \mapsto 1$	$(1,1)$	$(\mathbb{A}^1_{x_2} \setminus \{0\}) \times \left(\mathbb{A}^1_{x_1x_2^{-1}} \setminus \{0\}\right)$
$\mathcal{C}(e_1)$	$v_1 \mapsto 1, v_2 \mapsto 0$	$(1,0)$	$(\mathbb{A}^1_{x_2} \setminus \{0\}) \times \{0\}_{x_1x_2^{-1}}$
$\mathcal{C}(e_1 + e_2)$	$v_1 \mapsto 0, v_2 \mapsto 1$	$(0,1)$	$\{0\}_{x_2} \times \left(\mathbb{A}^1_{x_1x_2^{-1}} \setminus \{0\}\right)$
σ	$v_1 \mapsto 0, v_2 \mapsto 0$	$(0,0)$	$\{(0,0)\}$

Remark 2.21.

(1) Let σ be a cone and $\tau \leq \sigma$ a face. Via the inclusion $X_\tau \hookrightarrow X_\sigma$, we can identify the distinguished point $d_{\tau\leq\sigma} \in X_\sigma$ with the distinguished point $d_{\tau\leq\tau} \in X_\tau$, and the orbit $O_\sigma(\tau) \subseteq X_\sigma$ with the orbit $O(\tau) = O_\tau(\tau) \subseteq X_\tau$.
(2) For a fan Σ and a cone $\sigma \in \Sigma$ we have an inclusion $X_\sigma \subseteq X_\Sigma$. Hence, every orbit $O(\sigma) \subseteq X_\sigma$ can be considered as the orbit of $d_{\sigma\leq\sigma}$ in X_Σ under the torus action.

We have the following correspondence between cones in Σ and orbits in X_Σ.

Theorem 2.22 (Orbit-cone correspondence)**.** *Let* Σ *be a fan in* $N_\mathbb{R}$.

(1) *The map* $\Sigma \to \{O(\sigma) : \sigma \in \Sigma\}$, *given by* $\sigma \mapsto O(\sigma)$, *is a bijection.*
(2) *For all* $\sigma \in \Sigma$ *we have* $X_\sigma = \bigsqcup_{\tau\leq\sigma} O(\tau)$.
(3) *For all* $\sigma \in \Sigma$ *we have* $O(\sigma) \cong \mathbb{T}^{\dim X_\Sigma - \dim\sigma}$.

Proof.

(1) See [25, Lem. 3.17 and Prop. 3.19 (1)].
(2) See [25, Prop. 3.19 (3)].
(3) See [25, p. 26 and Prop. 3.19 (2)] for the proof that $O(\sigma) \cong \mathbb{T}^{\dim_\mathbb{R} N_\mathbb{R} - \dim\sigma}$. The result follows because of the identity $\dim X_\Sigma = \dim_\mathbb{R} N_\mathbb{R}$ (Theorem 2.6).

$\square$

We have the following immediate consequence.

Corollary 2.23. *If* Σ *is a fan, then* $O(\{0\}) = \mathbb{T}_\Sigma$.

Example 2.24. Reconsider Example 2.20 and set

$$\Sigma = \{\mathcal{C}(0), \mathcal{C}(e_1), \mathcal{C}(e_1 + e_2), \sigma\}.$$

By Remark 2.21 (1), the first and fourth column of the table in Example 2.20 correspond to the bijection

$$\Sigma \to \{O(\tau) : \tau \in \Sigma\}, \quad \tau \mapsto O(\tau).$$

Furthermore, we see that for every cone $\tau \in \Sigma$, the affine toric variety X_τ is the disjoint union of the orbits corresponding to its faces. For instance, we have

$$\begin{aligned}
X_\sigma &= \mathbb{A}^2_{(x_2, x_1 x_2^{-1})} = \bigsqcup_{\tau \leq \sigma} O_\sigma(\tau) = \bigsqcup_{\tau \leq \sigma} O(\tau); \\
X_{\mathcal{C}(e_1)} &= \left(\mathbb{A}^1_{x_2} \setminus \{0\}\right) \times \mathbb{A}^1_{x_1 x_2^{-1}} \\
&= \left(\left(\mathbb{A}^1_{x_2} \setminus \{0\}\right) \times \{0\}_{x_1 x_2^{-1}}\right) \sqcup \left(\left(\mathbb{A}^1_{x_2} \setminus \{0\}\right) \times \left(\mathbb{A}^1_{x_1 x_2^{-1}} \setminus \{0\}\right)\right) \\
&= \bigsqcup_{\tau \leq \mathcal{C}(e_1)} O_\sigma(\tau) = \bigsqcup_{\tau \leq \mathcal{C}(e_1)} O(\tau).
\end{aligned}$$

Finally, notice that

$$O(\mathcal{C}(0)) \cong \mathbb{T}^2,\ O(\mathcal{C}(e_1)) \cong O(\mathcal{C}(e_1 + e_2)) \cong \mathbb{T}^1,\ O(\sigma) \cong \mathbb{T}^0.$$

Notation 2.25. If Σ is a fan and $\sigma \in \Sigma$ a cone, then $V(\sigma) = \overline{O(\sigma)}$ denotes the topological closure of the orbit of σ in X_Σ.

Proposition 2.26. *For a fan Σ and a cone $\tau \in \Sigma$, we have $V(\tau) = \bigsqcup_{\tau \leq \sigma} O(\sigma)$.*

Proof. See [25, Prop. 3.28]. □

Corollary 2.27. *For a fan Σ we have $X_\Sigma = \bigsqcup_{\sigma \in \Sigma} O(\sigma)$.*

Proof. By Corollary 2.23 and Proposition 2.26 we have

$$X_\Sigma = \overline{\mathbb{T}_\Sigma} = \overline{O(\{0\})} = V(\{0\}) = \bigsqcup_{\sigma \in \Sigma} O(\sigma).$$

□

2.5 Properties of fans and their associated toric varieties

One could ask how properties of a fan Σ relate to properties of its associated toric variety X_Σ. We give three important connections between them, which are mentioned in [5, §3.4].

Definition 2.28. A fan Σ in $N_\mathbb{R}$ is *complete* if $\bigcup \Sigma = N_\mathbb{R}$.

Definition 2.29. A separated variety X is *complete* if for any variety Y, the projection $X \times Y \to Y$ is closed.

Remark 2.30. By Proposition 2.14, the requirement for a complete variety to be separated, is no restriction for toric varieties X_Σ.

Remark 2.31. A separated complex algebraic variety is complete if and only if it is compact as an analytic variety. Accordingly, some authors use the term 'compact' instead of 'complete' in this context (e.g. [10, §2.4], [5, Thm. 3.2.1], [9, §VI, Thm. 9.1]; by contrast, [25, §3.2.1] uses the term 'complete').

Proposition 2.32. *A fan Σ is complete if and only if the toric variety X_Σ is complete.*

Proof. A proof can be found in [9, §VI, Thm. 9.1], which uses the fact that a variety is complete if and only every sequence of points has an accumulation point in the complex topology. $\square$

Example 2.33. Let $\sigma_0, \sigma_1 \subseteq \mathbb{R}^1$ be the cones from Example 2.8. The fans $\mathcal{F}(\sigma_0)$ and $\mathcal{F}(\sigma_1)$ are not complete, and neither are the associated toric varieties $X_{\sigma_0} = \mathbb{A}^1_x$ and $X_{\sigma_1} = \mathbb{A}^1_{x^{-1}}$. However, the fan $\mathcal{F}(\sigma_0, \sigma_1)$ is complete, and so is the associated toric variety $X_{\mathcal{F}(\sigma_0,\sigma_1)} = \mathbb{P}^1$. A similar observation can be made for $\mathbb{P}^n$ (cf. Example 2.9).

We turn to the second connection between properties of a fan Σ and its associated toric variety X_Σ.

Definition 2.34. A lattice cone $\sigma \subseteq N_\mathbb{R}$ is *regular* if there is a basis $\{v_1, \dots, v_n\}$ for N such that $\sigma = \mathcal{C}(v_1, \dots, v_r)$ for some $0 \leq r \leq n$. A fan is *regular* if all its cones are regular.

Proposition 2.35. *A fan Σ is regular if and only if X_Σ is smooth.*

Proof. In order to prove that the toric variety X_Σ is smooth if Σ is regular, it suffices to show that for a regular lattice cone σ, the affine toric variety X_σ is smooth. Without loss of generality we may assume that $\sigma = \mathcal{C}(e_1, \dots, e_r) \subseteq \mathbb{R}^n$. Then we have $S_\sigma = \langle e_1, \dots, e_n, -e_{r+1}, \dots, -e_n \rangle$, and it follows that $X_\sigma = \mathbb{A}^r \times (\mathbb{A}^1 \setminus \{0\})^{n-r}$ is smooth.

For the reverse implication we refer to [5, Prop 4.5]. $\square$

Example 2.36. Notice that the cone $\sigma = \mathcal{C}(e_1 + e_2, -e_1 + 2e_2) \subseteq \mathbb{R}^2$ of Examples 1.33 and 1.36 is not regular: $\{e_1 + e_2, -e_1 + 2e_2\}$ is not a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, and it is impossible to add a third vector which is $\mathbb{Z}$-linearly independent of the first two vectors. Indeed, the toric variety $X_\sigma = Z(y_1y_3 - y_2^2, y_2y_4 - y_3^2) \subseteq \mathbb{A}^4$ is not smooth at the origin, as $(y_1y_3 - y_2^2, y_2y_4 - y_3^2) \subseteq \mathbb{C}[y_1, y_2, y_3, y_4]$ is a prime ideal and all partial derivatives of $y_1y_3 - y_2^2$ and $y_2y_4 - y_3^2$ vanish at 0.

For the third relation between properties of Σ and X_Σ, we need a number of definitions.

Definition 2.37. A *convex polytope* $P \subseteq \mathbb{R}^n$ is the convex hull $\mathrm{conv}(X)$ of a finite set $X \subseteq \mathbb{R}^n$. A *proper face* of P is a non-empty intersection $P \cap H$, where $H \subseteq \mathbb{R}^n$ is an affine hyperplane that contains no points of the interior of P. A subset of P is called a *face* of P if it is a proper face of P or equals one of $\varnothing, P$.

Definition 2.38. A fan Σ is *polytopal* if there is a convex polytope P which contains 0 in its interior such that

$$\Sigma = \{\mathcal{C}(F) : F \text{ is a proper face of } P\} \cup \{\{0\}\}.$$

Remark 2.39.

(1) Every polytopal fan is complete.
(2) Every complete fan in $\mathbb{R}^2$ is polytopal.

It is not true that every complete fan is polytopal, as the following counterexample shows (cf. [10, pp. 25–26] and [5, Rem. 3.1.2]).

Example 2.40. Consider the fan $\Sigma = \mathcal{F}(\{\sigma_{i,\alpha} : i = 1,2,3,\ \alpha = \pm 1\})$, where

$$\sigma_{i,\alpha} = \mathcal{C}(\{(v_1, v_2, v_3) \in \{\pm 1\}^3 : v_i = \alpha\}).$$

If $C \subseteq \mathbb{R}^3$ is the cube with vertices $(\pm 1, \pm 1, \pm 1)$, then we have

$$\Sigma = \{\mathcal{C}(F) : F \text{ is a proper face of } C\} \cup \{\{0\}\}.$$

Hence, Σ is both polytopal and complete.

Now, for $i = 1,2,3$ and $\alpha = \pm 1$, let $\sigma'_{i,\alpha}$ be the cone with the same generators as $\sigma_{i,\alpha}$, except that $(1,1,1)$ is replaced by $(1,2,3)$. The fan $\Sigma' = \mathcal{F}(\{\sigma'_{i,\alpha} : i = 1,2,3,\ \alpha = \pm 1\})$ is still complete, but not polytopal: if there were a polytope $P \subseteq \mathbb{R}^3$ such that

$$\Sigma' = \{\mathcal{C}(F) : F \text{ is a proper face of } P\} \cup \{\{0\}\},$$

then each of the vertices $x_1, \ldots, x_8$ of P would be contained in a different ray $\mathcal{C}(v)$ for

$$v \in \Big(\{\pm 1\}^3 \setminus \{(1,1,1)\}\Big) \cup \{(1,2,3)\}.$$

However, it can be checked that it is impossible to choose $x_1, \ldots, x_8$ in such a way that for every cone $\sigma'_{i,\alpha}$, the four corresponding vertices lie in the same affine plane (for more details we refer to [10, p. 26 and p. 134, note 17]). Therefore, the fan Σ' is not polytopal.

The following two definitions are from [9, §VI, Def. 6.3; VII, Def. 3.5].

Definition 2.41. Let X_Σ and $X_{\Sigma'}$ be toric varieties with torus embeddings $i : \mathbb{T}_\Sigma \hookrightarrow X_\Sigma$ and $i' : \mathbb{T}_{\Sigma'} \hookrightarrow X_{\Sigma'}$, respectively. A map $f : X_\Sigma \to X_{\Sigma'}$ is called *equivariant* if there is a morphism of algebraic groups $\alpha : \mathbb{T}_\Sigma \to \mathbb{T}_{\Sigma'}$ such that $f \circ i = i' \circ \alpha$ and $f(t \cdot x) = \alpha(t) \cdot f(x)$ for all $x \in X_\Sigma$ and $t \in \mathbb{T}_\Sigma$.

Definition 2.42. A complete toric variety X_Σ is called *equivariantly projective* if there is an equivariant embedding $j : X_\Sigma \hookrightarrow \mathbb{P}^r$ such that $j(X_\Sigma)$ is closed in $\mathbb{P}^r$.

Proposition 2.43. *A fan Σ is polytopal if and only if X_Σ is equivariantly projective.*

Proof. A proof can be found in [9, §VII, Thm. 3.11]. $\square$

Example 2.44. Propositions 2.32 and 2.43 enable us to give a toric variety which is complete but not equivariantly projective: take $X_{\Sigma'}$ with Σ' as in Example 2.40.

Example 2.45. For all $q \in \mathbb{Z}_{\geq 0}$, the fan Σ_q from § 2.3 is complete, regular and polytopal; hence, by Propositions 2.32, 2.35 and 2.43, the Hirzebruch surface $\mathcal{H}_q$ is complete, smooth and equivariantly projective.

2.6 Fans associated to convex polytopes

In this subsection we will associate a polytopal fan to a convex polytope whose vertices lie in some lattice.

Definition 2.46. Let $P \subseteq \mathbb{R}^n$ be a convex polytope. The *polar polytope* of P is

$$P^\circ = \{v \in \mathbb{R}^n : \langle u, v \rangle \geq -1 \text{ for all } u \in P\}.$$

Definition 2.47. Let $P \subseteq \mathbb{R}^n$ be a convex polytope. The *dimension* of a face F of P is

$$\dim(F) = \begin{cases} \dim_{\mathbb{R}}(\operatorname{span}(F')), & F \neq \varnothing \\ -1, & F = \varnothing, \end{cases}$$

where F' is a translation of F such that F' contains 0. A face of dimension k is called a *k-face*. A *vertex* is a 0-face, an *edge* is a 1-face and a *facet* is a $(\dim(P) - 1)$-face. The convex polytope $P \subseteq \mathbb{R}^n$ is called *full-dimensional* if $\dim(P) = n$.

Definition 2.48. A convex polytope $P \subseteq \mathbb{R}^n$ is a *convex lattice polytope with respect to the lattice N* if its vertices are contained in the lattice N.

Notation 2.49. For a convex polytope P, let $\operatorname{int}(P)$ denote its interior.

Proposition 2.50. *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional convex polytope with* $0 \in \operatorname{int}(P)$.

(1) *P° is a full-dimensional convex polytope in $\mathbb{R}^n$, satisfying $(P^\circ)^\circ = P$.*
(2) *If F is a face of P, then the set $F^* = \{v \in P^\circ : \langle u, v \rangle = -1$ for all $u \in F\}$ is a face of P°, satisfying* $\dim(F) + \dim(F^*) = \dim(P) - 1$.
(3) *The assignment $F \mapsto F^*$ provides a one-to-one order-reversing correspondence between the faces of P and the faces of P°.*
(4) *If P is a convex lattice polytope with respect to the lattice N, then its polar polytope P° is a lattice polytope with respect to the dual lattice N^*.*

Proof. See [10, p. 24]. □

Example 2.51. Consider the square $S = [-1, 1]^2 \subseteq \mathbb{R}^2$ and its polar polytope S°.

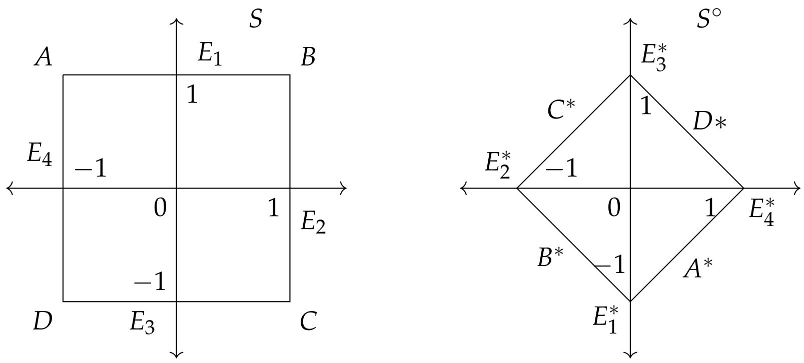

Figure 6: The square S and its polar polytope S°.

Edges in S correspond to vertices in S° and vice versa. Also, the empty face of S corresponds to S° and the empty face of S° corresponds to S.

To a full-dimensional convex lattice polytope $P \subseteq \mathbb{R}^n$ (with respect to some lattice $N \subseteq \mathbb{R}^n$) with $0 \in \operatorname{int}(P)$, we associate a polytopal fan Σ_P in $\mathbb{R}^n$ as follows (cf. [5, §4.4]). For each proper face F of P we define

$$\sigma_F = \{v \in \mathbb{R}^n : \langle u - u', v\rangle \geq 0 \text{ for all } u \in P,\ u' \in F\}.$$

It follows from the definitions that σ_F is the cone

$$\mathcal{C}(F^*) = \mathcal{C}(\{v \in \mathbb{R}^n : \langle u, v\rangle \geq -1 = \langle u', v\rangle \text{ for all } u \in P,\ u' \in F\}),$$

which is a lattice cone because of Proposition 2.50 (2), (4). Furthermore, it is clear from the definition that σ_F is strongly convex.

Now the fan Σ_P is defined as

$$\Sigma_P = \{\sigma_F :\ F \text{ is a proper face of } P\} \cup \{\{0\}\}.$$

Proposition 2.52. *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional convex lattice polytope with $0 \in \operatorname{int}(P)$.*

(1) *Σ_P is a fan in $\mathbb{R}^n$.*
(2) *We have*
$$\Sigma_P = \{\mathcal{C}(F) : F \textit{ is a proper face of } P^\circ\} \cup \{\{0\}\}.$$
(3) *We have*
$$\Sigma_{P^\circ} = \{\mathcal{C}(F) : F \textit{ is a proper face of } P\} \cup \{\{0\}\}.$$

Proof. For (1) and (2), see [10, p. 26]. (3) follows from (2) and Proposition 2.50 (1), (4) (notice that $0 \in \operatorname{int}(P^\circ)$, since we have $0 \in \operatorname{int}(P)$). □

Corollary 2.53. *For a full-dimensional convex lattice polytope P with $0 \in \operatorname{int}(P)$, the fan Σ_P is polytopal and the associated toric variety X_{Σ_P} is equivariantly projective.*

Proof. It follows from $0 \in \operatorname{int}(P)$ that $0 \in \operatorname{int}(P^\circ)$. Therefore, the fan Σ_P is polytopal by Proposition 2.52 (2). Furthermore, by Proposition 2.43, the toric variety X_{Σ_P} is equivariantly projective. □

Example 2.54. Reconsider Example 2.51. Since 0 is contained in the interior of S, we have

$$\begin{aligned}\Sigma_S &= \{\mathcal{C}(F) : F \in \{A^*, B^*, C^*, D^*, E_1^*, E_2^*, E_3^*, E_4^*\}\} \cup \{\{0\}\} \\ &= \mathcal{F}(\mathcal{C}(A^*), \mathcal{C}(B^*), \mathcal{C}(C^*), \mathcal{C}(D^*)) \\ &= \mathcal{F}(\mathcal{C}(e_1, -e_2), \mathcal{C}(-e_1, -e_2), \mathcal{C}(-e_1, e_2), \mathcal{C}(e_1, e_2)),\end{aligned}$$

which equals the fan Σ_0 from § 2.3. It follows that $X_{\Sigma_S} = \mathcal{H}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$ (see Example 2.15).

Example 2.55. Let $P \subseteq \mathbb{R}^3$ be the octahedron with vertices $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$. The polar polytope of P is the cube C with vertices $(\pm 1, \pm 1, \pm 1)$ (see [10, p. 27]). Therefore, the polytopal fan Σ from Example 2.40 equals Σ_P.

3 Divisors and intersection theory

This section is concerned with divisors and intersection theory on toric varieties. After having introduced divisors in a general setting in § 3.1, we apply them in the toric context in § 3.2. In § 3.6 the notion of *toric Cartier divisors* is linked to so-called *piecewise linear functions*. The subsections § 3.3, § 3.4 and § 3.5 in between are concerned with intersection theory and the connection between *Chow cohomology groups* of toric varieties and *Minkowski weights*.

3.1 Divisors

Let us introduce the notion of *divisors* on algebraic varieties (cf. [5, §5.1] and [25, §4.1]).

Definition 3.1. Let X be an irreducible variety. A *prime divisor* of X is an irreducible closed subvariety of X of codimension 1. By $\operatorname{Div}(X)$ we denote the free abelian group generated by the prime divisors of X; its elements are called *(Weil) divisors* on X.

Weil divisors are written as formal sums $D = \sum_Z n_Z Z$, where Z ranges over the prime divisors of X, and the coefficients n_Z are integers of which only finitely many are non-zero.

From now on, the variety X is assumed to be *normal* (i.e. the local ring at every point in X is an integrally closed domain) and connected. (In particular this means that X is irreducible). In the context of toric varieties this is no restriction, since every toric variety X_Σ is normal (see [5, Prop. 4.4]) and connected. For a prime divisor Z of X, there is a valuation map $v_Z : K(X)^\times \to \mathbb{Z}$, where $K(X)$ is the field of rational functions on X.

Remark 3.2. For a rational function $f \in K(X)^\times$, there are only finitely many prime divisors Z such that $v_Z(f)$ is non-zero.

By this remark we can associate a Weil divisor to a rational function $f \in K(X)^\times$ as follows.

Definition 3.3. The *divisor* of a rational function $f \in K(X)^\times$ is

$$\operatorname{div}(f) = \sum_Z v_Z(f) Z \in \operatorname{Div}(X),$$

where Z ranges over the prime divisors of X. A Weil divisor on X is called *principal* if it is of the form $\operatorname{div}(f)$ for some $f \in K(X)^\times$. The set of principal divisors on X, is denoted by $\operatorname{Div}_0(X)$.

Definition 3.4. Let $D = \sum_Z n_Z Z$ be a Weil divisor on X and $U \subseteq X$ a non-empty open subvariety. The *restriction* of D to U is defined as $D|_U = \sum_{Z\cap U \neq \varnothing} n_Z(Z \cap U)$.

Remark 3.5. If $U \subseteq X$ is a non-empty open subvariety and $Z \subseteq X$ a prime divisor of X with $Z \cap U \neq \varnothing$, then $Z \cap U$ is a prime divisor of U. Therefore, the restriction $D|_U$ of a Weil divisor D on X to an open subvariety U is a Weil divisor on U.

Definition 3.6. A Weil divisor $D \in \mathrm{Div}(X)$ is *Cartier* if it is locally principal; i.e. if there is a cover $X = \bigcup_{i=1}^k U_i$ of open subvarieties of X such that $D|_{U_i} \in \mathrm{Div}_0(U_i)$ for $i = 1, \ldots, k$. The set of Cartier divisors on X is denoted by $\mathrm{CDiv}(X)$.

Remark 3.7. $\mathrm{CDiv}(X)$ is a subgroup of $\mathrm{Div}(X)$, and $\mathrm{Div}_0(X)$ is a subgroup of $\mathrm{CDiv}(X)$. Hence, we have inclusions

$$\mathrm{Div}_0(X) \subseteq \mathrm{CDiv}(X) \subseteq \mathrm{Div}(X),$$

which are no equalities in general.

3.2 Toric divisors

In this subsection we will apply the theory of divisors to toric varieties (cf. [5, §5.1] and [25, §4.2]).

If Σ is a fan, the torus $\mathbb{T}_\Sigma$ acts on divisors $D = \sum_Z n_Z Z \in \mathrm{Div}(X_\Sigma)$ as follows: for $t \in \mathbb{T}_\Sigma$ we set $t \cdot D = \sum_Z n_Z (t \cdot Z)$, where $t \cdot Z = \{tz : z \in Z\}$. Notice that $t \cdot Z$ is a prime divisor on X_Σ if Z is, as $x \mapsto t \cdot x$ defines an isomorphism of varieties $X_\Sigma \xrightarrow{\sim} X_\Sigma$. Hence, for $D \in \mathrm{Div}(X_\Sigma)$ and $t \in \mathbb{T}_\Sigma$, we have $t \cdot D \in \mathrm{Div}(X_\Sigma)$.

Definition 3.8. Let Σ be a fan. A divisor $D = \sum_Z n_Z Z \in \mathrm{Div}(X_\Sigma)$ is *toric* if it is $\mathbb{T}_\Sigma$-invariant; i.e. if $t \cdot D = D$ for all $t \in \mathbb{T}_\Sigma$. The set of toric Weil divisors on X_Σ is denoted by $\mathrm{Div}_\mathbb{T}(X_\Sigma)$ and the set of toric Cartier divisors on X_Σ is denoted by $\mathrm{CDiv}_\mathbb{T}(X_\Sigma)$.

Lemma 3.9. *Let Σ be a fan and $D = \sum_Z n_Z Z \in \mathrm{Div}(X_\Sigma)$ a divisor. Then D is toric if and only if every prime divisor Z of X_Σ with $n_Z \neq 0$ is toric.*

Proof. Assume that D is toric. Let $k \in \mathbb{Z}_{\geq 1}$ be the largest integer such that there are $t \in \mathbb{T}_\Sigma$ and a prime divisor Z of X_Σ with $n_Z \neq 0$ and $t^i \cdot Z \neq Z$ for $i = 1, \ldots, k-1$ and $t^k \cdot Z = Z$ (which exists since D is toric and n_Z is non-zero for only finitely many prime divisors Z). Suppose that $k > 1$. Take $t' \in \mathbb{T}_\Sigma$ such that $t'^k = t$. Then for $i = 1, \ldots, k-1$, it would follow from $t'^i \cdot Z = Z$ that

$$t^i \cdot Z = (t'^k)^i \cdot Z = (t'^i)^k \cdot Z = Z,$$

which is a contradiction; hence, we have $t'^i \cdot Z \neq Z$. We also have $t'^k \cdot Z = t \cdot Z \neq Z$. However, this contradicts the maximality of k. We conclude that $k = 1$, which proves the 'only if' part. The reverse implication is immediately clear. $\square$

Remark 3.10. It follows from Lemma 3.9 that $\mathrm{Div}_\mathbb{T}(X_\Sigma)$ is the subgroup of $\mathrm{Div}(X_\Sigma)$ generated by the toric prime divisors of X_Σ.

Lemma 3.11. *Let Σ be a fan and $\rho \in \Sigma$ a ray. Then $V(\rho)$ is a toric prime divisor on X_Σ.*

Proof. By Theorem 2.22 (3), the orbit $O(\rho) \subseteq X_\Sigma$ has codimension 1; hence, its closure $V(\rho)$ has codimension 1 as well. Furthermore, $V(\rho)$, being the closure of the irreducible subvariety $O(\rho) \cong \mathbb{T}^{\dim X_\Sigma - 1}$ (see Theorem 2.22 (3)), is itself irreducible. Hence, $V(\rho)$ is a prime divisor of X_Σ. Finally, since $V(\rho)$ is a union of orbits by Proposition 2.26, it is toric. □

A divisor $V(\rho)$, where ρ is a ray, is called a *ray divisor*.

Example 3.12. The ray divisors of X_σ from Examples 2.20 and 2.24 are

$$V(e_1) = \mathbb{A}^1_{x_2} \times \{0\}_{x_1 x_2^{-1}} = Z(y_2);$$
$$V(e_1 + e_2) = \{0\}_{x_2} \times \mathbb{A}^1_{x_1 x_2^{-1}} = Z(y_1);$$

where $(y_1, y_2) = (x_2, x_1 x_2^{-1})$ are the coordinates of X_σ. The ray divisors are clearly $\mathbb{T}_\sigma$-invariant. We will show that they are the only toric prime divisors.

Suppose that Z is a toric prime divisor of $X_\sigma = \mathbb{A}^2_{(y_1,y_2)}$. Then we can write $Z = Z(f)$ for a non-constant irreducible polynomial $f \in \mathbb{C}[y_1, y_2]$ (see [15, Prop. 1.13]). Write

$$f(y_1, y_2) = g(y_1, y_2) y_2 + h(y_1).$$

The following fact, which is the case since Z is $\mathbb{T}_\sigma$-invariant, will be used multiple times: if $(a, b) \in \mathbb{C}^2$ is a root of f, so is $(\lambda a, \mu b)$ for all $\lambda, \mu \in \mathbb{C}^*$. We distinguish between two cases:

(1) $h = c$ is constant;
(2) h is non-constant.

Suppose that (1) is the case. First assume that $c \neq 0$. Let $(a, b) \in \mathbb{C}^2$ be a root of f. It follows from $f(a, 0) = h(a) = c \neq 0$ that $b \neq 0$. But that means that for all $\lambda \in \mathbb{C}^*$ we have

$$0 = f(a, \lambda) = g(a, \lambda)\lambda + c,$$

which is a contradiction. Now assume that $c = 0$. Since $f = g y_2$ is irreducible, g must be constant; so we have $f = c' y_2$ for some $c' \in \mathbb{C}^*$.

Suppose that (2) is the case. Let $a \in \mathbb{C}$ be a root of h. If $a \neq 0$, then it would follow from $f(a, 0) = h(a) = 0$ that $h(\lambda) = f(\lambda, 0) = 0$ for all $\lambda \in \mathbb{C}^*$, which is a contradiction; hence, we find $a = 0$ and we can write $h(y_1) = c'' y_1^k$ for some $c'' \in \mathbb{C}^*$ and $k \in \mathbb{Z}_{\geq 1}$. Let $(a_1, a_2) \neq (0, 0)$ be a root of f (which must exist, since otherwise $Z = Z(f)$ would not have codimension 1). It would follow from $a_1, a_2 \neq 0$ that (λ, μ) is a root of f for all $\lambda, \mu \in \mathbb{C}^*$, which is a contradiction. If $a_1 \neq 0$ and $a_2 = 0$, then $0 = f(a_1, a_2) = 0 + c'' a_1^k$ is also a contradiction. It follows that $a_1 = 0$ and $a_2 \neq 0$. So for all $\lambda \in \mathbb{C}^*$ we have $g(0, \lambda)\lambda = f(0, \lambda) = 0$ and therefore $g(0, \lambda) = 0$. But that means that $y_1 \mid g$. Since we also have $y_1 \mid h$ and f is irreducible, it follows that $g = 0$. We conclude that $f = h = c'' y_1^k$ and, since f is irreducible, $f = c'' y_1$.

Hence, in case (1) we have $Z = Z(y_2) = V(e_1)$ and in case (2) we have $Z = Z(y_1) = V(e_1 + e_2)$.

In the previous example, it was proved that the only toric prime divisors were the ray divisors. This holds in general, as the following proposition shows.

Proposition 3.13. *Let Σ be a fan and $\rho_1, \ldots, \rho_k$ the rays in Σ. Then we have* $\mathrm{Div}_{\mathbb{T}}(X_\Sigma) = \bigoplus_{i=1}^k \mathbb{Z}V(\rho_i)$.

Proof. The inclusion $\mathrm{Div}_{\mathbb{T}}(X_\Sigma) \supseteq \bigoplus_{i=1}^k \mathbb{Z}V(\rho_i)$ follows immediately from Lemmas 3.9 and 3.11. Conversely, let $D = \sum_{i=1}^r n_i Z_i \in \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$, where $n_1, \ldots, n_r$ are non-zero integers. We proceed as in [25, Prop. 4.11]. Let $j \in \{1, \ldots, r\}$. The prime divisor Z_j is $\mathbb{T}_\Sigma$-invariant by Lemma 3.9 and is therefore a union of $\mathbb{T}_\Sigma$-orbits. By Theorem 2.22 (3), Z_j is disjoint with the orbit $O(\{0\}) = \mathbb{T}_\Sigma$ (see Corollary 2.23), as Z_j has codimension 1. It follows by Proposition 2.26 that

$$Z_j \subseteq X_\Sigma \setminus \mathbb{T}_\Sigma = \bigcup_{\sigma \in \Sigma \setminus \{\{0\}\}} O(\sigma) = \bigcup_{i=1}^k \bigcup_{\rho_i \leq \sigma} O(\sigma) = \bigcup_{i=1}^k V(\rho_i).$$

Since $Z_j, V(\rho_1), \ldots, V(\rho_k)$ are all prime divisors of X_Σ, we conclude that Z_j equals one of the $V(\rho_i)$. Hence, we find $D \in \bigoplus_{i=1}^k \mathbb{Z}V(\rho_i)$ and $\mathrm{Div}_{\mathbb{T}}(X_\Sigma) \subseteq \bigoplus_{i=1}^k \mathbb{Z}V(\rho_i)$. □

The following result characterizes the Cartier divisors in $\mathrm{Div}_{\mathbb{T}}(X_\Sigma)$.

Proposition 3.14. *Let Σ be a fan in $N_{\mathbb{R}}$ and $\rho_1, \ldots, \rho_k$ the rays in Σ. For $i = 1, \ldots, k$, let $v_i \in \rho_i$ be a primitive lattice vector. If $D = \sum_{i=1}^k n_i V(\rho_i) \in \mathrm{Div}_{\mathbb{T}}(X_\Sigma)$ is a toric divisor on X_Σ, then the following are equivalent:*

(1) *D is a Cartier divisor.*
(2) *For each cone $\sigma \in \Sigma$, we have $D|_{X_\sigma} \in \mathrm{Div}_0(X_\sigma)$.*
(3) *For each maximal cone $\sigma \in \Sigma$, there is an element $m_\sigma \in \mathrm{Hom}(N, \mathbb{Z})$ such that $m_\sigma(v_i) = n_i$ if $\rho_i \leq \sigma$.*

Proof. See [25, Prop 4.20]. □

The following example generalizes [25, Example 4.24].

Example 3.15. Let p and q be coprime positive integers. Consider the primitive lattice vectors

$$v_1 = e_2,\ v_2 = pe_1 + qe_2,\ v_3 = e_1 \in \mathbb{Z}^2$$

and the fan $\Sigma = \mathcal{F}(\sigma_{12}, \sigma_{23})$, where $\sigma_{12} = \mathcal{C}(v_1, v_2)$ and $\sigma_{23} = \mathcal{C}(v_2, v_3)$. The rays of Σ are $\rho_i = \mathcal{C}(v_i)$ for $i = 1, 2, 3$.

We will determine under which conditions the toric divisor $D = n_1 V(\rho_1) + n_2 V(\rho_2) + n_3 V(\rho_3)$ is a Cartier divisor. The divisor D is Cartier if and only if there are morphisms $m_{\sigma_{12}}, m_{\sigma_{23}} \in \mathrm{Hom}(\mathbb{Z}^2, \mathbb{Z})$ such that (3) of Proposition 3.14 holds; i.e. if and only if there are vectors

$$v_{12} = (a_{12}, b_{12}),\ v_{23} = (a_{23}, b_{23}) \in \mathbb{Z}^2$$

such that

$$\begin{aligned}
n_1 &= \langle v_{12}, v_1 \rangle = b_{12}; \\
n_2 &= \langle v_{12}, v_2 \rangle = pa_{12} + qb_{12}; \\
n_2 &= \langle v_{23}, v_2 \rangle = pa_{23} + qb_{23}; \\
n_3 &= \langle v_{23}, v_3 \rangle = a_{23}.
\end{aligned}$$

Such vectors v_{12} and v_{23} exist if and only if

$$n_2 \equiv qn_1 \pmod{p}, \; n_2 \equiv pn_3 \pmod{q}.$$

Since p and q are coprime, we conclude by the Chinese remainder theorem that D is Cartier if and only if $n_2 \equiv qn_1 + pn_3 \pmod{pq}$.

3.3 Chow groups

In this subsection we will briefly discuss some intersection theory and introduce the notion of *Chow (cohomology) groups* (see [15, Appendix A, §§1–2] and [18, §10.2]). In the next subsections this will be applied in the context of toric varieties. The relevance of these topics will become clear in § 5 and § 6.

Let X be a normal quasi-projective n-dimensional algebraic variety.

Definition 3.16. Let $k \in \mathbb{Z}$. The *cycle group* $Z_k(X)$ is the group of formal sums $\sum_Z n_Z Z$ with integer coefficients, where Z ranges over the closed irreducible k-dimensional subvarieties of X. Its elements are called *cycles*.

Definition 3.17. A cycle $\sum_Z n_Z Z$ is called *rationally equivalent to* 0 if there are closed irreducible subvarieties W_i of X and rational functions $f_i \in K(W_i)$ for $i = 1, \ldots, l$ such that $\sum_Z n_Z Z = \sum_{i=1}^{l} \operatorname{div}(f_i)$. Let $k \in \mathbb{Z}$. The *Chow group* $A_k(X)$ is the quotient of $Z_k(X)$ by the subgroup of cycles rationally equivalent to 0. The kth *Chow cohomology group* is defined as $A^k(X) = A_{n-k}(X)$. Furthermore, we set $A^*(X) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A^k(X)$.

Strictly speaking, the elements of $A_k(X)$ are cycle classes, but we will often refer to them by picking out one of its members.

The elements of the Chow group $A_0(X)$ can be seen as formal sums of points in X. If X is complete, then there is a degree map

$$\deg : A_0(X) \to \mathbb{Z}, \quad \sum_P n_P P \mapsto \sum_P n_P.$$

Now assume that X is smooth. Then on the Chow groups there is an *intersection product*

$$\cdot : A_k(X) \times A_{k'}(X) \to A_{k+k'-n}(X).$$

The intersection product $\cdot : A^k(X) \times A^{k'}(X) \to A^{k+k'}(X)$ on the Chow cohomology groups makes $A^*(X) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} A^k(X)$ into a commutative graded ring, which is called the *Chow ring*.

Definition 3.18. Let X be a smooth complete variety. A divisor $D \in \operatorname{Div}(X)$, or the associated Chow cohomology class $c = [D] \in A^1(X)$, is called *numerically effective* or *nef* if for every curve $C \in A_1(X)$ on X we have $\deg(c \cdot [C]) \geq 0$.

3.4 Intersection theory on toric varieties

In this subsection, some intersection theory on toric varieties will be discussed. The results, including some of the proofs, can also be found in [10, §5.1].

Notation 3.19. For an n-dimensional fan Σ and $0 \leq k \leq n$, let $\Sigma^{(k)}$ denote the set of $n-k$-dimensional cones in Σ.

Theorem 3.20. *Let Σ be an n-dimensional fan. For $k = 0, \ldots, n$, the Chow group $A_k(X_\Sigma)$ is generated by the set $\{[V(\sigma)] : \sigma \in \Sigma^{(k)}\}$.*

Proof. See [10, p. 96]. $\square$

Because of this result, it suffices to study the behavior of the orbit closures of the cones in a fan Σ, if one wants to understand the intersection theory on the toric variety X_Σ. We will not explore this in full generality, but instead focus on the results needed for later purposes. Let Σ be an n-dimensional regular complete fan in $N_\mathbb{R}$ with rays $\rho_1, \ldots, \rho_k$ and primitive lattice vectors $v_1, \ldots, v_k \in N$ with $v_i \in \rho_i$ for $i = 1, \ldots, k$.

Remark 3.21. For a cone $\sigma \in \Sigma$, the orbit closure $V(\sigma) \subseteq X_\Sigma$ is irreducible and has codimension $\dim(\sigma)$ (see Theorem 2.22 (3); cf. the proof of Lemma 3.11); i.e. we have $[V(\sigma)] \in A^{\dim(\sigma)}(X_\Sigma)$.

Proposition 3.22. *For a cone $\sigma \in \Sigma$ and a ray $\rho_i \not\leq \sigma$ we have*

$$[V(\rho_i)] \cdot [V(\sigma)] = \begin{cases} [V(\mathcal{C}(\rho_i \cup \sigma))], & \textit{if } \mathcal{C}(\rho_i \cup \sigma) \in \Sigma \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. In the case $\mathcal{C}(\rho_i \cup \sigma) \in \Sigma$, the varieties $V(\rho_i)$ and $V(\sigma)$ meet transversally with intersection $V(\mathcal{C}(\rho_i \cup \sigma))$, and in the case $\mathcal{C}(\rho_i \cup \sigma) \notin \Sigma$, the varieties $V(\rho_i)$ and $V(\sigma)$ are disjoint. Thus the result follows (cf. [10, p. 98]). $\square$

In the case $\rho_i \leq \sigma$, things are somewhat more difficult. Let $\sigma \in \Sigma$ be an $n-1$-dimensional cone. Then there are unique n-dimensional cones $\sigma', \sigma'' \in \Sigma$ such that $\sigma < \sigma', \sigma''$. Let $v'_\sigma, v''_\sigma \in \{v_1, \ldots, v_k\}$ be the primitive lattice vectors contained in the rays of σ' and σ'' which are not contained in σ, respectively. There are unique integers $\alpha_i(\sigma)$, for $i \in \{1, \ldots, k\}$ with $\rho_i \leq \sigma$, such that

$$v'_\sigma + v''_\sigma = \sum_{\substack{i \in \{1,\ldots,k\} \\ \rho_i \leq \sigma}} \alpha_i(\sigma) v_i.$$

(cf. [10, p. 99 (exercise a)]).

As the following example shows, the coefficients $\alpha_i(\sigma)$ need not be positive, but can also be zero or negative.

Example 3.23. Consider the regular complete fan Σ_q in $\mathbb{R}^2$, associated to the Hirzebruch surface $\mathcal{H}_q$ (see Figure 5). It has rays $\rho_i = \mathcal{C}(v_i)$, for $i = 1, \ldots, 4$, where

$$v_1 = e_1, \quad v_2 = qe_1 - e_2, \quad v_3 = -e_1, \quad v_4 = e_2.$$

We have

$$\begin{aligned} v'_{\rho_1} + v''_{\rho_1} &= v_4 + v_2 = qv_1; \\ v'_{\rho_2} + v''_{\rho_2} &= v_1 + v_3 = 0 \cdot v_2; \\ v'_{\rho_3} + v''_{\rho_3} &= v_2 + v_4 = -qv_3; \\ v'_{\rho_4} + v''_{\rho_4} &= v_3 + v_1 = 0 \cdot v_4; \end{aligned}$$

and therefore,

$$\alpha_1(\rho_1) = q, \ \alpha_2(\rho_2) = 0, \ \alpha_3(\rho_3) = -q, \ \alpha_4(\rho_4) = 0.$$

Proposition 3.24. *For an $n-1$-dimensional cone $\sigma \in \Sigma$ and a ray $\rho_i \leq \sigma$ we have*

$$[V(\rho_i)] \cdot [V(\sigma)] = c'_i(\sigma)[V(\sigma')] + c''_i(\sigma)[V(\sigma'')],$$

where $c'_i(\sigma), c''_i(\sigma)$ are integers the sum of which is $-\alpha_i(\sigma)$.

Proof. See [10, p. 99 (exercise b)]. □

Corollary 3.25. *For an $n-1$-dimensional cone $\sigma \in \Sigma$ and a ray $\rho_i \in \Sigma$ we have*

$$\deg([V(\rho_i)] \cdot [V(\sigma)]) = \begin{cases} 1, & \text{if } \rho_i \in \{\mathcal{C}(v'_\sigma), \mathcal{C}(v''_\sigma)\} \\ -\alpha_i(\sigma), & \text{if } \rho_i \leq \sigma \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The third case follows from Proposition 3.22, and so does the first case: $[V(\mathcal{C}(v'_\sigma))] \cdot [V(\sigma)] = [V(\sigma')]$ and $[V(\mathcal{C}(v''_\sigma))] \cdot [V(\sigma)] = [V(\sigma'')]$. The second case follows from Proposition 3.24. □

3.5 Chow cohomology groups and Minkowski weights

The Chow cohomology groups of the toric variety associated to a complete regular fan are closely related to the so-called *Minkowski weights* on that fan (see [18, §10.2] and [11]). In order to define this notion, we first introduce the following notation. Let N be a lattice and Σ a fan in $N_\mathbb{R}$ of dimension n.

Notation 3.26. For $\sigma \in \Sigma$, let N_σ denote the sublattice of N generated by $\sigma \cap N$.

Notice that for cones $\tau \in \Sigma^{(k+1)}$ and $\sigma \in \Sigma^{(k)}$ with $\tau < \sigma$, the quotient lattice N_σ/N_τ has rank 1. Let $v_{\sigma/\tau} \in N_\sigma/N_\tau$ be a generator of it.

Definition 3.27. A map $c : \Sigma^{(k)} \to \mathbb{Z}$ is called a *Minkowski weight of codimension k* on the fan Σ if, for every $\tau \in \Sigma^{(k+1)}$, it satisfies the balancing condition

$$\sum_{\substack{\sigma \in \Sigma^{(k)} \\ \tau < \sigma}} c(\sigma) v_{\sigma/\tau} = 0$$

in N/N_τ.

Remark 3.28. The Minkowski weights of codimension k on the fan Σ form a subgroup $\mathcal{M}^k(\Sigma)$ of the abelian group $\mathrm{Map}(\Sigma^{(k)}, \mathbb{Z})$, where addition is defined pointwise.

Let $k \in \{0, \ldots, n\}$ and assume that the fan Σ is complete and regular. In this case the group $\mathcal{M}^k(\Sigma)$ turns out to be isomorphic to the Chow cohomology group $A^k(X_\Sigma)$. We will construct an embedding $A^k(X_\Sigma) \hookrightarrow \mathrm{Map}(\Sigma^{(k)}, \mathbb{Z})$, the image of which is $\mathcal{M}^k(\Sigma)$.

For a topological space X and a non-negative integer i, one can consider the ith *cohomology group* $H^i(X, \mathbb{Z})$ with coefficients in $\mathbb{Z}$.

Theorem 3.29. *For $i \in \mathbb{Z}_{\geq 0}$ we have $H^{2i}(X_\Sigma, \mathbb{Z}) \cong A^i(X_\Sigma)$ and $H^{2i+1}(X_\Sigma, \mathbb{Z}) \cong 0$.*

Proof. See [7, Thm. 10.8]. ☐

Notice that the toric variety X_Σ is complete and smooth. So, as a consequence of this theorem, we know by Poincaré duality that the pairing

$$A^k(X_\Sigma) \times A_k(X_\Sigma) \xrightarrow{\cdot} A_0(X_\Sigma) \xrightarrow{\deg} \mathbb{Z}$$

is perfect; i.e. we have an isomorphism

$$A^k(X_\Sigma) \xrightarrow{\sim} \mathrm{Hom}(A_k(X_\Sigma), \mathbb{Z}),\ d \mapsto (d' \mapsto \deg(d \cdot d')).$$

Furthermore, the morphism of groups

$$\bigoplus_{\sigma \in \Sigma^{(k)}} \mathbb{Z}\sigma \to A_k(X_\Sigma),\ \sigma \mapsto [V(\sigma)],$$

which is surjective by Theorem 3.20, induces an injection

$$\mathrm{Hom}(A_k(X_\Sigma), \mathbb{Z}) \hookrightarrow \mathrm{Hom}\left(\bigoplus_{\sigma \in \Sigma^{(k)}} \mathbb{Z}\sigma, \mathbb{Z}\right) \cong \mathrm{Map}(\Sigma^{(k)}, \mathbb{Z}).$$

By composition this yields an injection

$$\gamma : A^k(X_\Sigma) \hookrightarrow \mathrm{Map}(\Sigma^{(k)}, \mathbb{Z}),\ d \mapsto (\sigma \mapsto \deg(d \cdot [V(\sigma)])).$$

Theorem 3.30. *The image $\gamma(A^k(X_\Sigma)) \subseteq \mathrm{Map}(\Sigma^{(k)}, \mathbb{Z})$ of γ equals $\mathcal{M}^k(\Sigma)$.*

Proof. See [11, Thm. 3.1]). ☐

Example 3.31. Consider the fan $\Sigma = \Sigma_{\mathbb{P}^2}$ from Example 2.9. We have $\Sigma^{(1)} = \{\rho_0, \rho_1, \rho_2\}$, where $\rho_i = \mathcal{C}(e_i)$ for $i = 0, 1, 2$. A map $c : \Sigma^{(k)} \to \mathbb{Z}$ is a Minkowski weight if and only if it satisfies the balancing condition

$$(0,0) = \sum_{i=0}^{2} c(\rho_i) e_i = (c(\rho_1) - c(\rho_0), c(\rho_2) - c(\rho_0)),$$

which is the case if and only if $c(\rho_0) = c(\rho_1) = c(\rho_2)$. Furthermore, the Chow cohomology group $A^1(X_\Sigma) = A^1(\mathbb{P}^2) \cong \mathbb{Z}$ is generated by $[V(\rho_0)] = [V(\rho_1)] = [V(\rho_2)]$.

Since the primitive lattice vectors $e_0 \in \rho_0$, $e_1 \in \rho_1$ and $e_2 \in \rho_2$ satisfy $e_0 + e_1 + e_2 = 0$, we have $\alpha_j(\rho_j) = -1$ for $j = 0, 1, 2$, where $\alpha_j(\rho_j)$ is as defined in § 3.4. Hence, for an element $d = \lambda[V(\rho_0)] \in A^1(X_\Sigma)$ we find, by Corollary 3.25,

$$\gamma(d)(\rho_j) = \deg\left(\lambda[V(\rho_0)] \cdot [V(\rho_j)]\right) = \lambda, \quad j = 0, 1, 2,$$

which shows that $\gamma(d)$ is a Minkowski weight.

Conversely, if $c \in \mathcal{M}^1(\Sigma)$ is a Minkowski weight, then we have, for $j = 0, 1, 2$,

$$\gamma(c(\rho_0)[V(\rho_0)])(\rho_j) = \deg\left(c(\rho_0)[V(\rho_0)] \cdot [V(\rho_j)]\right) = c(\rho_0) = c(\rho_j)$$

and therefore $\gamma(c(\rho_0)[V(\rho_0)]) = c$.

This shows that γ induces an isomorphism $A^1(X_\Sigma) \xrightarrow{\sim} \mathcal{M}^1(\Sigma)$.

3.6 Toric Cartier divisors and piecewise linear functions

There is a close connection between the group $\mathrm{CDiv}_{\mathbb{T}}(X_\Sigma)$ of toric Cartier divisors on a toric variety X_Σ and the so-called *piecewise linear functions* on the fan Σ (see [5, §5.2] and [18, p. 55]). The following definition is from [5, Def. 4.4] (cf. [18, Def. 10.10]).

Definition 3.32. Let N be a lattice and Σ a fan on $N_{\mathbb{R}}$. A *piecewise linear function* on Σ is a map $\psi : \bigcup \Sigma \to \mathbb{R}$ such that

(i) $\psi(N \cap \bigcup \Sigma) \subseteq \mathbb{Z}$;
(ii) for all $\sigma \in \Sigma$ and $v, w \in \sigma$, $\psi(v + w) = \psi(v) + \psi(w)$;
(iii) for all $v \in \bigcup \Sigma$ and $\lambda \in \mathbb{R}_{\geq 0}$, $\psi(\lambda v) = \lambda \psi(v)$.

The set of piecewise linear functions on Σ is denoted by $\mathrm{PLF}(\Sigma)$.

Alternatively, a piecewise linear function on a fan Σ on $N_{\mathbb{R}}$ may be defined as a map $\psi : N \to \mathbb{Z}$ such that for every maximal cone $\sigma \in \Sigma$, there is $m_\sigma \in \mathrm{Hom}(N, \mathbb{Z})$ such that $\psi|_{\sigma \cap N} = m_\sigma|_{\sigma \cap N}$. Since every element $x \in \bigcup \Sigma$ can be written as $x = \lambda v + \mu w$, where $v, w \in \sigma \cap N$ for some $\sigma \in \Sigma$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$, such a map $\psi : N \to \mathbb{Z}$ can be uniquely extended to a map $\tilde{\psi} : \bigcup \Sigma \to \mathbb{R}$, satisfying the requirements of Definition 3.32, by setting $\tilde{\psi}(\lambda v + \mu w) = \lambda \psi(v) + \mu \psi(w)$.

Notice that $\mathrm{PLF}(\Sigma)$ is an abelian group under pointwise addition.

We have the following important result (see [5, Lem. 5.3]).

Proposition 3.33. *Let N be a lattice and Σ a fan on $N_{\mathbb{R}}$. Let $\rho_1, \ldots, \rho_k$ be the rays of Σ and, for $i = 1, \ldots, k$, let $v_i \in \rho_i$ be a primitive lattice vector. Then the map*

$$\varphi_{\mathrm{Div}} : \mathrm{PLF}(\Sigma) \to \mathrm{CDiv}_{\mathbb{T}}(X_\Sigma), \quad \psi \mapsto -\sum_{i=1}^{k} \psi(v_i) V(\rho_i)$$

is an isomorphism of abelian groups.

Proof. φ_{Div} is well-defined and surjective by Proposition 3.14. Furthermore, if $\psi_1, \psi_2 \in \mathrm{PLF}(\Sigma)$ agree on $v_1, \ldots, v_k$, then $\psi_1 = \psi_2$; hence, φ_{Div} is injective as well. $\square$

Definition 3.34. Let X be a convex subset of a real vector space. A map $\psi : X \to \mathbb{R}$ is called *convex* if for all $v, w \in X$ and $\lambda \in [0,1]$,

$$\psi(\lambda v + (1-\lambda)w) \geq \lambda\psi(v) + (1-\lambda)\psi(w).$$

Let N be a lattice and Σ an n-dimensional polytopal regular fan on $N_{\mathbb{R}}$. Let $\sigma_1, \ldots, \sigma_r$ be the $n-1$-dimensional cones and $\rho_1, \ldots, \rho_k$ the rays of Σ, containing the primitive lattice vectors $v_1, \ldots, v_k$, respectively. Furthermore, let $\psi \in \mathrm{PLF}(\Sigma)$ be a piecewise linear function and $\varphi_{\mathrm{Div}}(\psi) = -\sum_{i=1}^k \psi(v_i)V(\rho_i) \in \mathrm{CDiv}_{\mathbb{T}}(X_\Sigma)$ the corresponding Cartier divisor on X_Σ.

Lemma 3.35. *For any curve C in X_Σ, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}_{\geq 0}$ such that $[C] = \sum_{j=1}^r \lambda_j [V(\sigma_j)]$.*

Proof. By Theorem 3.20, the classes $[V(\sigma_1)], \ldots, [V(\sigma_r)]$ generate the Chow group $A_1(X_\Sigma)$. Hence, there are $\lambda_1, \ldots, \lambda_r \in \mathbb{Z}$ such that $[C] = \sum_{j=1}^r \lambda_j [V(\sigma_j)]$. That these coefficients can in fact be chosen non-negative, is proved in [26, Prop. 1.6]. □

Proposition 3.36. *If the piecewise linear function $\psi \in \mathrm{PLF}(\Sigma)$ is convex, then the Cartier divisor $\varphi_{\mathrm{Div}}(\psi) = -\sum_{i=1}^k \psi(v_i)V(\rho_i) \in \mathrm{CDiv}_{\mathbb{T}}(X_\Sigma)$ is nef.*

Proof. The divisor $\varphi_{\mathrm{Div}}(\psi)$ is nef if and only if $\deg([\varphi_{\mathrm{Div}}(\psi)] \cdot [C]) \geq 0$ for every curve C on X_Σ. By Lemma 3.35, this is the case if and only if

$$-\sum_{i=1}^k \psi(v_i) \deg([V(\rho_i)] \cdot [V(\sigma_j)]) \geq 0, \quad j = 1, \ldots, r,$$

which, by Corollary 3.25, is equivalent to

$$\psi(v'_{\sigma_j}) + \psi(v''_{\sigma_j}) \leq \sum_{i \in I_j} \alpha_i(\sigma_j)\psi(v_i), \quad j = 1, \ldots, r,$$

where we use the same notation as in § 3.4 and

$$I_j = \{i \in \{1, \ldots, k\} : \rho_i \leq \sigma_j\}, \quad j = 1, \ldots, r.$$

Let $j \in \{1, \ldots, r\}$ and set

$$I_j^+ = \{i \in I_j : \alpha_i(\sigma_j) \geq 0\},\ I_j^- = \{i \in I_j : \alpha_i(\sigma_j) < 0\}.$$

If ψ is convex, then we find, using that $v_i \in \sigma_j$ for all $i \in I_j^+ \cup I_j^- = I_j$,

$$\begin{aligned}\sum_{i \in I_j^+} \alpha_i(\sigma_j)\psi(v_i) = \psi\left(\sum_{i \in I_j^+} \alpha_i(\sigma_j)v_i\right) &= \psi\left(v'_{\sigma_j} + v''_{\sigma_j} + \sum_{i \in I_j^-} -\alpha_i(\sigma_j)v_i\right) \\ &\geq \psi(v'_{\sigma_j}) + \psi(v''_{\sigma_j}) + \psi\left(\sum_{i \in I_j^-} -\alpha_i(\sigma_j)v_i\right) \\ &= \psi(v'_{\sigma_j}) + \psi(v''_{\sigma_j}) - \sum_{i \in I_j^-} \alpha_i(\sigma_j)\psi(v_i)\end{aligned}$$

and it follows that $\varphi_{\mathrm{Div}}(\psi)$ is nef. □

4 Matroids and Bergman fans

In § 2.6 we have associated a fan Σ_P to a full-dimensional convex lattice polytope P. As we will see in this section, it is also possible to associate a fan Σ_M to a matroid M; the fan Σ_M is called the *Bergman fan* associated to M.

In § 4.1 the notion of a matroid will be introduced, and § 4.2 is concerned with Bergman fans; we will be roughly following [3, §2]. In § 4.3 the relation between polytopal and matroidal fans will be discussed. It turns out that there is only a very small class of fans that are both derivable from a polytope and from a matroid, namely the fans of the form Σ_{Q_n}, where Q_n is a certain convex polytope. In § 4.4 it is shown that Q_n is combinatorially equivalent to the permutohedron P_n of order n.

4.1 Matroids

Matroids were introduced by the American mathematician Hassler Whitney (1907–1989) and were meant to generalize the notion of linear independence in linear algebra. They can be characterized in many 'cryptomorphic' ways; i.e. there are many definitions of matroids, which are equivalent in a non-trivial fashion. We will discuss three of these characterizations.

Definition 4.1 (Independence axioms)**.** A *matroid* M is a pair $(E, \mathscr{I})$, where E is a finite set (called the *ground set* of M) and $\mathscr{I} \subseteq \mathcal{P}(E)$ is a family of subsets of E (called the *independent sets* of M), satisfying the following properties:

- (I1) $\varnothing \in \mathscr{I}$;
- (I2) If $I_1 \in \mathscr{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathscr{I}$;
- (I3) If $I_1, I_2 \in \mathscr{I}$ with $|I_1| < |I_2|$, then there exists $x \in I_2 \setminus I_1$ such that $I_1 \cup \{x\} \in \mathscr{I}$.

Example 4.2. Let $0 \leq k \leq n$ be integers, and set $E = \{0, \dots, n-1\}$ and $\mathscr{I} = \{X \subseteq E : |X| \leq k\}$. Then $U_{k,n} = (E, \mathscr{I})$ is a matroid. Matroids of this form are called *uniform matroids*.

Example 4.3. Consider the vector space k^n, for some field k, and let $E \subseteq k^n$ be a finite subset. If $\mathscr{I}$ is the family of linearly independent subsets of E, then $(E, \mathscr{I})$ is a matroid. (Obviously, k^n can be replaced by a general vector space V over k as well.)

More generally, let A be a $n \times m$ matrix with entries in a field k, and take $E = \{1, \dots, m\}$, where the elements of E correspond to the columns of A. Let $\mathscr{I}$ be the family of subsets of E whose elements correspond to linearly independent columns of A. Then $(E, \mathscr{I})$ is a matroid. (Alternatively, this can been be viewed as the matroid associated to a finite multiset of vectors from the vector space k^n.) Matroids of this form are said to be *representable over k*. A matroid is called *representable* if it is representable over some field k.

Example 4.4. Let $G = (V, E)$ be an undirected graph and $\mathscr{I}$ the family of subsets of E which do not contain any cycles. We claim that $M_G = (E, \mathscr{I})$ is a matroid. The axioms (I1) and (I2) clearly hold. In order to verify axiom (I3), let $I_1, I_2 \in \mathscr{I}$ with $|I_1| < |I_2|$. For $i = 1, 2$, consider the subgraph (V, I_i), which has $|V| - |I_i|$ connected components. Since (V, I_2) has fewer connected components than (V, I_1), there is an edge $e \in I_2$ such that its end points are in different connected components of (V, I_1). Then e is not contained in I_1, and $I_1 \cup \{e\}$ is independent. Hence, axiom (I3) holds. Matroids of this form are called *graphic matroids*.

Example 4.5. Let $M = (E, \mathscr{I})$ be a matroid and $X \subseteq E$ a subset. Then $(X, \mathscr{I}_X)$, where $\mathscr{I}_X = \{I \in \mathscr{I} : I \subseteq X\}$, is a matroid. It is called a *submatroid* of M.

Definition 4.6. Let $M = (E, \mathscr{I})$ be a matroid. The *rank* of M is $\mathrm{rk}(M) = \max\{|I| : I \in \mathscr{I}\}$. An independent set $I \in \mathscr{I}$ is called a *basis* of M if $|I| = \mathrm{rk}(M)$. The *rank* of a subset $X \subseteq E$ is $\mathrm{rk}_M(X) = \mathrm{rk}((X, \mathscr{I}_X))$. An independent set $I \in \mathscr{I}$ is called a *basis* of a subset $X \subseteq E$ if $I \subseteq X$ and $|I| = \mathrm{rk}_M(X)$.

Definition 4.7. Let $M = (E, \mathscr{I})$ be a matroid. An element $e \in E$ is a *loop* if it belongs to no basis of M, and it is a *coloop* if it belongs to every basis of M.

Example 4.8. Let $G = (V, E)$ be a graph and $e \in E$ an edge. Then e is a loop of the matroid M_G if and only if e is a loop of the graph G (i.e. an edge from a vertex to itself). Furthermore, e is a coloop of the matroid M_G if and only if e does not belong to any cycle in the graph G.

Matroids can also be characterized by means of a closure operation.

Definition 4.9 (Closure axioms)**.** A *matroid* M is a pair (E, cl), where E is a finite set and $\mathrm{cl} : \mathcal{P}(E) \to \mathcal{P}(E)$ is a map, satisfying the following properties:

(C1) If $X \subseteq E$, then $X \subseteq \mathrm{cl}(X)$;
(C2) If $X \subseteq Y \subseteq E$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$;
(C3) If $X \subseteq E$, then $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$;
(C4) If $X \subseteq E$ and $x, y \in E$ such that $y \in \mathrm{cl}(X \cup \{x\}) \setminus \mathrm{cl}(X)$, then $x \in \mathrm{cl}(X \cup \{y\})$.

For a subset $X \subseteq E$, the set $\mathrm{cl}(X) \subseteq E$ is called the *closure* of X. A subset $X \subseteq E$ is called *closed* if $\mathrm{cl}(X) = X$.

The relation between the characterizations given by Definitions 4.1 and 4.9 is as follows. From a matroid $M = (E, \mathscr{I})$ in the sense of Definition 4.1, we obtain a matroid (E, cl) in the sense of Definition 4.9 by defining

$$\mathrm{cl}(X) = \{x \in E : \mathrm{rk}_M(X) = \mathrm{rk}_M(X \cup \{x\})\}\,.$$

Conversely, from a matroid $M = (E, \mathrm{cl})$ in the sense of Definition 4.9, we obtain a matroid $(E, \mathscr{I})$ in the sense of Definition 4.1 by defining

$$\mathscr{I} = \{X \in \mathcal{P}(E) : x \notin \mathrm{cl}(X \setminus \{x\}) \text{ for all } x \in X\}\,.$$

A third characterization of matroids uses the notion of *flats*.

Definition 4.10 (Flat axioms)**.** A *matroid* M is a pair $(E, \mathscr{F})$, where E is a finite set and $\mathscr{F} \subseteq \mathcal{P}(E)$ is family of subsets of E (called the *flats* of M), satisfying the following properties:

(F1) $E \in \mathscr{F}$;

(F2) If $F_1, F_2 \in \mathscr{F}$, then $F_1 \cap F_2 \in \mathscr{F}$;

(F3) Let $F \in \mathscr{F}$ and let $Y_F \subseteq \mathscr{F}$ be the set of flats F' that *cover* F; i.e. $F \subsetneq F'$ and there is no $F'' \in \mathscr{F}$ such that $F \subsetneq F'' \subsetneq F'$. Then the set $\{F' \setminus F : F' \in Y_F\}$ partitions $E \setminus F$.

The characterizations of Definitions 4.9 and 4.10 relate to each other as follows. From a matroid $M = (E, \mathrm{cl})$ in the sense of Definition 4.9, we obtain a matroid $(E, \mathscr{F})$ in the sense of Definition 4.10 by defining

$$\mathscr{F} = \{X \in \mathcal{P}(E) : \mathrm{cl}(X) = X\};$$

i.e. the flats are the closed subsets of E. Conversely, if $M = (E, \mathscr{F})$ is a matroid in the sense of Definition 4.10, then we obtain a matroid (E, cl) in the sense of Definition 4.9 by defining

$$\mathrm{cl}(X) = \bigcap \{F \in \mathscr{F} : F \supseteq X\}.$$

Example 4.11. Suppose that $M = (E, \mathscr{I})$ is the representable matroid associated to a finite subset E of a vector space V. For a subset $X \subseteq E$, we have $\mathrm{cl}(X) = \mathrm{span}(X) \cap E$. Consequently, a subset $X \subseteq E$ is a flat if and only if $X = \mathrm{span}(X) \cap E$.

In the following example, which is taken from [18, Example 4.3], a representable matroid is associated to a linear subspace $V \subseteq k^{n+1}$. This construction will become important in § 6.

Example 4.12. Consider the vector space k^{n+1} with basis $e_0, \ldots, e_n$, where k is a field. Let $V \subseteq k^{n+1}$ be a linear subspace. Its inclusion map $i : V \hookrightarrow k^{n+1}$ induces a surjection $i^* : (k^{n+1})^* \to V^*$ of the dual vector spaces. Let $M = M_V = (E, \mathscr{I})$ be the representable matroid induced by the image $\{i^*(e_0^*), \ldots, i^*(e_n^*)\} \subseteq V^*$ (considered as a multiset) of the dual basis of $e_0, \ldots, e_n$ under i^*. We write $E = \{0, \ldots, n\}$, where $j \in E$ is associated with $i^*(e_j^*)$.

For a subset $X \subseteq E$ we define the following subspace of V:

$$V_X = \{(x_0, \ldots, x_n) \in V : x_j = 0 \text{ for all } j \in X\}.$$

Notice that we have

$$\mathrm{rk}_M(X) = \dim_k(V) - \dim_k(V_X).$$

This means in particular that $\mathrm{rk}(M) = \dim_k(V)$.

We give four direct consequences. First, a subset $X \subseteq E$ is a flat if and only if $\mathrm{rk}_M(X \cup \{e\}) > \mathrm{rk}_M(X)$ for all $e \in E \setminus X$, which is the case if and only if $V_Y \subsetneq V_X$ for all subsets $Y \subseteq E$ with $Y \supsetneq X$. Second, an element $j \in E$ is a loop if and only if $\dim_k(V) - \dim_k(V_{\{j\}}) = \mathrm{rk}_M(\{j\}) = 0$, which is the case if and only if V is contained in the coordinate hyperplane $x_j = 0$. Third, a subset $X \subseteq E$ is a basis for E if and only if

$$\dim_k(V_X) = \dim_k(V) - \mathrm{rk}_M(X) = \mathrm{rk}(M) - \mathrm{rk}(M) = 0,$$

which is the case if and only if $V_X = \{0\}$. Finally, an element $j \in E$ is a coloop if and only if $V_X = \{0\}$ implies that $j \in X$, which is the case if and only if $e_j \in V$.

4.2 Bergman fans

Definition 4.13. Let $M = (E, \mathscr{F})$ be a matroid. A flat F of M is called *proper* if $F \neq \varnothing, E$. Let $k \in \mathbb{Z}_{\geq 0}$. A *k-step flag* of M is a k-tuple $F_\bullet = (F_1, \ldots, F_k)$, where $F_1 \subsetneq \ldots \subsetneq F_k$ are proper flats of M. For flags $F_\bullet = (F_1, \ldots, F_k)$ and $G_\bullet = (G_1, \ldots, G_l)$ of M, we say that $G_\bullet$ *refines* $F_\bullet$ if $\{F_1, \ldots, F_k\} \subseteq \{G_1, \ldots, G_l\}$; in this case we write $F_\bullet \leq G_\bullet$.

Remark 4.14. For a matroid M, '$\leq$' defines a partial ordering on the set of flags of M.

Let $n \in \mathbb{Z}_{\geq 1}$ and let $M = (E, \mathscr{F})$ be a matroid on the ground set $E = \{0, 1, \ldots, n\}$. We describe the construction of the *Bergman fan* $\Sigma_M \subseteq \mathbb{R}^n$ associated to M. For a subset $S \subseteq E$, let $e_S = \sum_{i \in S} e_i \in \mathbb{R}^n$, where $e_0 = -(e_1 + \ldots + e_n)$. For a flag $F_\bullet = (F_1, \ldots, F_k)$ of M we define the lattice cone $\sigma_{F_\bullet} = \mathcal{C}(e_{F_1}, \ldots, e_{F_k}) \subseteq \mathbb{R}^n$. Now the *Bergman fan* associated to the matroid M is

$$\Sigma_M = \{\sigma_{F_\bullet} : F_\bullet \text{ is a flag of } M\}.$$

Remark 4.15. The assignment $F_\bullet \mapsto \sigma_{F_\bullet}$ defines an isomorphism of posets $\{\text{flags of M}\} \xrightarrow{\sim} \Sigma_M$.

Proposition 4.16. *The Bergman fan Σ_M associated to the matroid $M = (E, \mathscr{I})$ is a fan in $\mathbb{R}^n$.*

Proof. First, the Bergman fan Σ_M is non-empty: it contains the zero cone $\sigma_{()} = \mathcal{C}(\varnothing) = \{0\}$, associated to the 0-step flag $()$ of M.

Second, every face τ of a cone $\sigma_{G_\bullet} \in \Sigma$ is of the form $\tau = \sigma_{F_\bullet}$ for some flag $F_\bullet \leq G_\bullet$. Hence, Σ_M contains the faces of all its elements.

Third, for cones $\sigma_{F_\bullet}, \sigma_{F'_\bullet} \in \Sigma_M$, where $F_\bullet = (F_1, \ldots, F_k)$ and $F'_\bullet = (F'_1, \ldots, F'_l)$, we have $\sigma_{F_\bullet} \cap \sigma_{F'_\bullet} = \sigma_{\sigma_{F''_\bullet}}$, where the flag $F''_\bullet = (F''_1, \ldots, F''_m)$ consists of the flats $\{F''_1, \ldots, F''_m\} = \{F_1, \ldots, F_k\} \cap \{F'_1, \ldots, F'_l\}$. Clearly, the intersection $\sigma_{F_\bullet} \cap \sigma_{F'_\bullet}$ is a face of both $\sigma_{F_\bullet}$ and $\sigma_{F'_\bullet}$.

It remains to be shown that the cone $\sigma_{F_\bullet}$, associated to a flag $F_\bullet = (F_1, \ldots, F_k)$ with $k \in \mathbb{Z}_{\geq 1}$, is strongly convex. Let $\alpha \in E \setminus F_k$. If $\alpha = 0$, then we have $\sigma_{F_\bullet} \subseteq (\mathbb{R}_{\geq 0})^n$, which shows that $\sigma_{F_\bullet}$ is strongly convex. Now suppose that $\alpha \neq 0$ and let $x = (x_1, \ldots, x_n) \in \sigma_{F_\bullet}$. We can write $x = (\lambda_1, \ldots, \lambda_n) - (\lambda, \ldots, \lambda)$ with $\lambda_1, \ldots, \lambda_n, \lambda \in \mathbb{R}_{\geq 0}$ and $\lambda_\alpha = 0$. So we either have $x_\alpha = 0$ and $x_1, \ldots, x_n \geq 0$, or $x_\alpha < 0$. Now assume that $-x \in \sigma_{F_\bullet}$. Similarly, we either have $x_\alpha = 0$ and $x_1, \ldots, x_n \leq 0$, or $x_\alpha > 0$. It follows that $x = 0$. Hence, $\sigma_{F_\bullet}$ is strongly convex. $\square$

Example 4.17. Consider the matroid $M = (E, \mathscr{I})$ with $E = \{0, 1, 2, 3\}$ and

$$\mathscr{I} = \{\varnothing, \{0\}, \{1\}, \{2\}, \{3\}, \{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{1,3\}\}.$$

The proper flats of M are $\{0\}$ and $\{1\}$. Hence, M has three flags: $F_1 = (), F_2 = (\{0\})$ and $F_3 = (\{1\})$. Therefore, we have $\Sigma_M = \{\sigma_{F_1}, \sigma_{F_2}, \sigma_{F_3}\}$, where

$$\sigma_{F_1} = \mathcal{C}(\varnothing) = \{0\}, \ \sigma_{F_2} = \mathcal{C}(-(e_1 + e_2 + e_3)), \ \sigma_{F_3} = \mathcal{C}(e_1) \subseteq \mathbb{R}^3.$$

Notice that the 0-step flag gives rise to a 0-dimensional cone and that the 1-step flags give rise to 1-dimensional cones.

The final observation of the previous example holds in general.

Proposition 4.18. *Let M be a matroid and $F_\bullet = (F_1, \ldots, F_k)$ a k-step flag of M, where $k \in \mathbb{Z}_{\geq 0}$. Then we have $\dim \sigma_{F_\bullet} = k$.*

Proof. For $i = 1, \ldots, k$ we have $F_i \setminus \bigcup_{j=1}^{i-1} F_j \neq \varnothing$; hence, e_{F_i} is not contained in $\operatorname{span}(e_{F_1}, \ldots, e_{F_{i-1}})$. It follows that $e_{F_1}, \ldots, e_{F_k}$ are linearly independent, and we conclude that $\dim \sigma_{F_\bullet} = k$. □

The following example is from [3, Example 2.14].

Example 4.19. Let $n \in \mathbb{Z}_{\geq 2}$ and consider the uniform matroid $U_{n,n}$ on the ground set $E_n = \{0, \ldots, n-1\}$. (Notice that $U_{n,n}$ is the matroid M_G associated to any acyclic graph G with n edges.) Since every subset of E_n is a flat of $U_{n,n}$, the Bergman fan of $U_{n,n}$ is

$$\Sigma_{U_{n,n}} = \mathcal{F}\left(\left\{\mathcal{C}\left(\left\{\sum_{i=0}^{k} e_{\rho(i)} : k = 0, \ldots, n-2\right\}\right) : \rho \text{ is a permutation of } E_n\right\}\right),$$

where $e_0 = -(e_1 + \ldots + e_{n-1})$. The fan is polytopal: we have $\Sigma_{U_{n,n}} = \Sigma_{Q_n}$ for the convex lattice polytope $Q_n \subseteq \mathbb{R}^{n-1}$ whose polar polytope $Q_n^\circ \subseteq \mathbb{R}^{n-1}$ is the convex hull of

$$V_n = \left\{\sum_{i=0}^{k} e_{\rho(i)} : k = 0, \ldots, n-2,\ \rho \text{ is a permutation of } E_n\right\} \subseteq \mathbb{R}^{n-1}.$$

The elements of V_n are the vertices of Q_n°.

Let us work out the previous example for the case $n = 3$.

Example 4.20. For the matroid $U_{3,3}$ on the ground set $E_3 = \{0, 1, 2\}$ we have

$$\Sigma_{U_{3,3}} = \mathcal{F}(\{\sigma_{ij} : i, j = 0, 1, 2 \text{ and } i \neq j\}),$$

where $\sigma_{ij} = \mathcal{C}(e_i, e_i + e_j)$ for $i, j = 0, 1, 2$ with $i \neq j$. (Recall that $e_0 = -e_1 - e_2$.) We have $\Sigma_{U_{3,3}} = \Sigma_{Q_3}$, where the polar polytope $Q_3^\circ \subseteq \mathbb{R}^2$ is the convex hull of the set

$$V_3 = \{e_0, e_1, e_2, e_0 + e_1, e_0 + e_2, e_1 + e_2\} \subseteq \mathbb{R}^2.$$

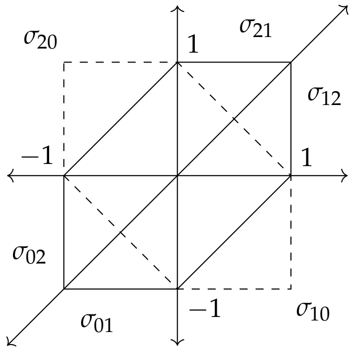

Figure 7: The convex polytopes Q_3 (dashed edges) and Q_3° (solid edges), and the cones σ_{ij} for $i, j = 0, 1, 2$ with $i \neq j$.

In § 6.4 the toric variety $X_{\Sigma_{Q_n}}$ will be discussed. It turns out that it can be obtained from the projective space $\mathbb{P}^{n-1}$ by a sequence of blow-ups.

4.3 Polytopal fans versus matroidal fans

In Example 4.19 we discussed a fan both derivable from a convex polytope and a matroid. This raises the question: how do fans derived from convex lattice polytopes (as described in § 2.6) relate to fans derived from matroids (as described in § 4.2)? Is one of the constructions a special case of the other? It will become clear that this is not the case. In fact, as this subsection shows, the class of fans simultaneously derivable from a convex lattice polytope and from a matroid, turns out to be very small: it only includes the fans $\Sigma_{U_{n,n}}$ from Example 4.19.

In what follows, the same notation is used as in Example 4.19.

Proposition 4.21. *Let $n \in \mathbb{Z}_{\geq 2}$ and let M be a matroid on the ground set $E_n = \{0, \dots, n-1\}$ such that Σ_M is a complete fan. Then we have $M = U_{n,n}$ and $\Sigma_M = \Sigma_{Q_n}$.*

Proof. First of all, notice that M has no loops: since the fan Σ_M is complete, it contains the rays $\mathcal{C}(e_0), \dots, \mathcal{C}(e_{n-1})$, which implies that the subsets $\{0\}, \dots, \{n-1\} \subseteq E_n$ are all flats. Furthermore, since Σ_M is a complete fan in $\mathbb{R}^{n-1}$, it has dimension $n-1$. So by Proposition 4.18 we know that M contains an $n-1$-step flag $(F_1, \dots, F_{n-1})$. Writing $F_n = E_n$, we have $|F_i| = i$ for $i = 1, \dots, n$. Since M has no loops, it follows inductively that F_i is an independent set for $i = 1, \dots, n$. In particular the ground set E_n is independent. Hence, we find $M = U_{n,n}$ and, by Example 4.19, $\Sigma_M = \Sigma_{Q_n}$. □

Corollary 4.22. *Let $n \in \mathbb{Z}_{\geq 2}$. Let $P \subseteq \mathbb{R}^{n-1}$ be a full-dimensional convex lattice polytope and $M = (E, \mathscr{I})$ a matroid such that $\Sigma_M = \Sigma_P$. Then we have $M = U_{n,n}$ and $\Sigma_M = \Sigma_P = \Sigma_{Q_n}$.*

Proof. Since the fan $\Sigma_M = \Sigma_P$ in $\mathbb{R}^{n-1}$ is polytopal, it is complete, and we write $E = \{0, \dots, n-1\}$. The result follows from Proposition 4.21. □

4.4 The permutohedron

According to [3, Example 2.14], the fan $\Sigma_{U_{n,n}} = \Sigma_{Q_n}$ is the fan associated to the permutohedron of order n. In this section we will discuss the permutohedron P_n of order n and its relation to the convex polytope Q_n.

Definition 4.23. Let $n \in \mathbb{Z}_{\geq 1}$. The *permutohedron* $P_n \subseteq \mathbb{R}^n$ of order n is the convex hull of the points

$$\{x_\rho : \rho \text{ is a permutation of } \{0, \ldots, n-1\}\},$$

where $x_\rho = (\rho(0), \ldots, \rho(n-1))$.

Remark 4.24. The smallest affine subspace containing P_n is the affine hyperplane $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = n(n-1)/2\}$. Therefore, the permutohedron $P_n \subseteq \mathbb{R}^n$ has dimension $n-1$.

The faces of a permutohedron can be described by using the notion of a *strict weak ordering*.

Definition 4.25. A *strict weak ordering* on a finite set X is a linearly ordered partition $W = \{W_1, \ldots, W_k\} \subseteq \mathcal{P}(X) \setminus \{\varnothing\}$ of X. If $W_1 < \ldots < W_k$ and $W_i = \{w_{i,1}, \ldots, w_{i,n_i}\}$ for $i = 1, \ldots, k$, then we use the notation $W_1|\ldots|W_k$ or $w_{1,1}, \ldots, w_{1,n_1}|\ldots|w_{k,1}, \ldots, w_{k,n_k}$. For strict weak orderings $W = W_1|\ldots|W_k$ and $W' = W'_1|\ldots|W'_{k'}$ on X with $k \leq k'$ we write $W' \preccurlyeq W$ if there are integers $1 = n_1 < \ldots < n_{k+1} = k'+1$ such that $W_i = \bigcup_{j=n_i}^{n_{i+1}-1} W'_j$ for $i = 1, \ldots, k$.

Remark 4.26. If X is a finite set, then '$\preccurlyeq$' is a partial order on the set of strict weak orderings of X.

Let $n \in \mathbb{Z}_{\geq 2}$. A vertex $x_\rho = (\rho(0), \ldots, \rho(n-1))$ of the permutohedron P_n corresponds to the strict weak ordering $\rho^{-1}(0)|\ldots|\rho^{-1}(n-1)$ on the set $\{0, \ldots, n-1\}$.

Two vertices x_ρ and $x_{\rho'}$ of the permutohedron P_n are connected by an edge if and only if x_ρ can be obtained from $x_{\rho'}$ by swapping two coordinates whose values differ by 1. This is the case if and only if there is $\alpha \in \{0, \ldots, n-2\}$ such that

$$\rho^{-1}(\alpha) = \rho'^{-1}(\alpha+1),\ \rho^{-1}(\alpha+1) = \rho'^{-1}(\alpha),\ \rho^{-1}(j) = \rho'^{-1}(j) \text{ for } j \neq \alpha, \alpha+1.$$

If this is the case, then the edge connecting x_ρ and x'_ρ corresponds to the strict weak ordering

$$\rho^{-1}(0)|\ldots|\rho^{-1}(\alpha-1)|\rho^{-1}(\alpha), \rho^{-1}(\alpha+1)|\rho^{-1}(\alpha+2)|\ldots|\rho^{-1}(n-1).$$

In general we have the following.

Remark 4.27. Let $n \in \mathbb{Z}_{\geq 2}$ and $d \in \{0, \ldots, n-1\}$. Then there is a one-to-one correspondence between the d-faces F of P_n and the strict weak orderings $W^F = W_1^F|\ldots|W_{n-d}^F$ on $\{0, \ldots, n-1\}$. A vertex x_ρ of P_n is contained in a d-face F if and only if one can obtain $W_1^F|\ldots|W_{n-d}^F$ by inserting $n-d-1$ vertical lines in the sequence $\rho^{-1}(0), \ldots, \rho^{-1}(n-1)$, or, more formally, if and only if

$$W_i^F = \{\rho^{-1}(m_{i-1}), \dots, \rho^{-1}(m_i - 1)\}, \quad i = 1, \dots, n-d,$$

where $m_i = \sum_{j=1}^{i} |W_j^F|$ for $i = 0, \dots, n-d$. For two non-empty faces F, F' of P_n we have $F' \subseteq F$ if and only if $W^{F'} \preccurlyeq W^F$; i.e. if and only if $W^{F'}$ can be obtained by inserting some extra vertical lines in W^F.

Example 4.28. The permutohedron $P_3 \subseteq \mathbb{R}^3$ of order 3 is the hexagon with vertices

$$(0,1,2), (0,2,1), (1,0,2), (1,2,0), (2,0,1), (2,1,0),$$

which correspond to the strict weak orderings

$$0|1|2, \quad 0|2|1, \quad 1|0|2, \quad 2|0|1, \quad 1|2|0, \quad 2|1|0,$$

respectively. The edges of P_3 correspond to the strict weak orderings

$$0|1,2; \quad 1|0,2; \quad 2|0,1; \quad 0,1|2; \quad 0,2|1; \quad 1,2|0.$$

For $a, b, c \in \{0,1,2\}$ distinct, the edge corresponding to $a|b,c$ contains the vertices corresponding to $a|b|c$ and $a|c|b$, and the edge corresponding to $a,b|c$ contains the vertices corresponding to $a|b|c$ and $b|a|c$. The strict weak ordering $1,2,3$ corresponds to the permutohedron P_3 itself.

Definition 4.29. Two convex polytopes P and Q are called *combinatorially equivalent* if there exists an isomorphism of posets

$$(\{\text{faces of } P\}, \subseteq) \xrightarrow{\sim} (\{\text{faces of } Q\}, \subseteq).$$

Example 4.30. Reconsider Examples 4.20 and 4.28. The polytopes P_3 and Q_3 are both hexagons. Therefore, they are combinatorially equivalent.

More generally, we have the following.

Proposition 4.31. *Let $n \in \mathbb{Z}_{\geq 2}$. The convex polytopes P_n and Q_n are combinatorially equivalent.*

Proof. By Proposition 2.50 (3) it suffices to show that there is a bijection

$$\varphi : \{\text{proper faces of } P_n\} \to \{\text{proper faces of } Q_n^\circ\}$$

such that for proper faces F, F' of P_n we have $F' \subseteq F$ if and only if $\varphi(F) \subseteq \varphi(F')$. Let $\mathcal{W}_n$ be the set of strict weak orderings $W = W_1|\dots|W_k$ on $E_n = \{0, \dots, n-1\}$ with $k \in \{2, \dots, n\}$. By Remark 4.27 it suffices to give a bijection

$$\psi : \mathcal{W}_n \to \{\text{proper faces of } Q_n^\circ\}$$

such that for $W, W' \in \mathcal{W}_n$ we have $W' \preccurlyeq W$ if and only if $\psi(W) \subseteq \psi(W')$.

For a strict weak ordering $W = W_1|\dots|W_k \in \mathcal{W}_n$ we define the set

$$Y_W = \left\{ \sum_{i=1}^{l} \sum_{a \in W_i} e_a : l = 1, \dots, k-1 \right\} \subseteq \mathbb{R}^{n-1}.$$

Then the assignment $W \mapsto \operatorname{conv}(Y_W)$ defines a bijection

$$\mathcal{W}_n \to \{\operatorname{conv}(Y_W) : W \in \mathcal{W}_n\} = \{\text{proper faces of } Q_n^\circ\}.$$

For $W, W' \in \mathcal{W}_n$ we have $W' \preccurlyeq W$ if and only if $Y_W \subseteq Y_{W'}$ if and only if $\operatorname{conv}(Y_W) \subseteq \operatorname{conv}(Y_{W'})$. $\square$

One might be inclined to think that full-dimensional convex lattice polytopes P and P' which are combinatorially equivalent, give rise to isomorphic toric varieties X_{Σ_P} and $X_{\Sigma_{P'}}$. However, this need not be the case, as the following example shows.

Example 4.32. Consider the convex lattice polytope Q_3', the polar polytope of which is displayed in the figure below. We have

$$\Sigma_{Q_3'} = \mathcal{F}(\{\sigma_{ij}' : i, j = 0, 1, 2 \text{ and } i \neq j\}),$$

where

$$\sigma_{01}' = \mathcal{C}(-e_2, -e_1 - e_2),\ \sigma_{02}' = \mathcal{C}(-e_1, -e_1 - e_2),\ \sigma_{10}' = \mathcal{C}(e_1, -e_2),$$
$$\sigma_{12}' = \mathcal{C}(e_1, 2e_1 + e_2),\ \sigma_{20}' = \mathcal{C}(-e_1, e_2),\ \sigma_{21}' = \mathcal{C}(2e_1 + e_2, e_2).$$

Notice that Q_3' is combinatorially equivalent to the convex lattice polytope Q_3 from Example 4.20 (and they are both combinatorially equivalent to the permutohedron P_3).

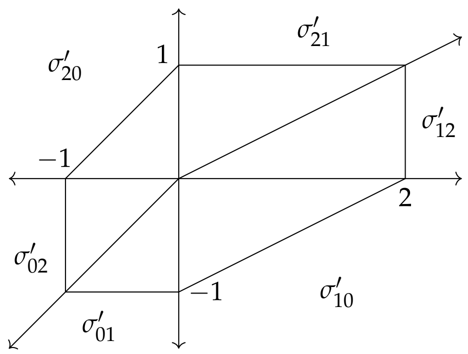

Figure 8: The polytope $Q_3'^{\circ}$ and the cones σ_{ij}' for $i, j = 0, 1, 2$ with $i \neq j$.

The fan Σ_{Q_3} is regular. By contrast, the fan $\Sigma_{Q_3'}$ is not regular, since the lattice cone σ_{21}' is not regular. It follows by Proposition 2.35 that the toric variety $X_{\Sigma_{Q_3}}$ is smooth, whereas $X_{\Sigma_{Q_3'}}$ is not. In particular, $X_{\Sigma_{Q_3}}$ and $X_{\Sigma_{Q_3'}}$ are not isomorphic.

As the previous example shows, one must be cautious when talking about 'the associated fan' or 'the associated toric variety' of the permutohedron $P_n \subseteq \mathbb{R}^n$, since these depend on the choice of a combinatorially equivalent convex lattice polytope in $\mathbb{R}^{n-1}$.

5 McMullen's conjecture

In this section we discuss an application of toric geometry to the theory of simplicial convex polytopes (cf. [5, §8.1]). After having introduced some terminology in § 5.1, we will move to McMullen's conjecture, which gives a necessary and sufficient condition for a vector of integers to be the h-vector of a simplicial convex polytope (§ 5.2). The ultimate goal of this section is to present Stanley's proof of the necessity of this condition (§ 5.5).

5.1 f-vectors and h-vectors

This subsection is concerned with *f-vectors* and *h-vectors* of convex polytopes. Before we come to these notions, however, we will give the definitions of *simplicial* and *simple* convex polytopes.

Definition 5.1. Let $k \in \mathbb{Z}_{\geq 0}$. A convex polytope P is called a *k-simplex* if $\dim P = k$ and $P = \operatorname{conv}(v_0, \dots, v_k)$, where $v_0, \dots, v_k$ are the vertices of P.

Example 5.2. A 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle and a 3-simplex is a tetrahedron.

Definition 5.3. A convex polytope is called *simplicial* if its facets are simplices.

Definition 5.4. An n-dimensional convex polytope is called *simple* if every vertex is contained in exactly n facets.

Remark 5.5. By Proposition 2.50 (3), a full-dimensional convex polytope P with $0 \in \operatorname{int}(P)$ is simplicial if and only if its polar polytope P° is simple.

Example 5.6. The convex polytope $Q_n^\circ \subseteq \mathbb{R}^{n-1}$ from Example 4.19 is simplicial. Therefore, its polar polytope $Q_n \subseteq \mathbb{R}^{n-1}$ is simple. The permutohedron $P_n \subseteq \mathbb{R}^n$ of order n, being combinatorially equivalent to Q_n, is simple as well.

Definition 5.7. Let P be a convex polytope of dimension n. For $d = -1, 0, \dots, n$, let $f_d = f_d(P)$ denote the number of d-faces of P. Then $f(P) = (f_0, \dots, f_{n-1})$ is called the *f-vector* of P. Furthermore, $f_P(x) = \sum_{i=0}^n f_{n-i-1} x^i$ is called the *f-polynomial* of P.

Remark 5.8. Let P be a convex polytope of dimension n. By Proposition 2.50 we have $f_d(P) = f_{n-d-1}(P^\circ)$ for $d = -1, 0, \dots, n$.

Example 5.9. Let $n \in \mathbb{Z}_{\geq 1}$ and $d \in \{0, \dots, n-1\}$. The number $f_d(P_n)$ of d-faces of the permutohedron P_n equals the number of strict weak orderings $W_1 | \dots | W_{n-d}$ on $\{0, \dots, n-1\}$ (see Remark 4.27). It follows that $f_d(P_n) = (n-d)! \cdot S(n, n-d)$, where

$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

is the number of ways to partition a set of n elements into k non-empty parts, which is called a *Stirling number of the second kind* (see [12, §6.1]).

For a simplicial convex polytope, we define its *h-vector*, which turns out to be very convenient for encoding its combinatorial information.

Definition 5.10. Let P be a simplicial convex polytope of dimension n. For $d = 0, \dots, n$ we set

$$h_d = h_d(P) = \sum_{j=d}^{n} (-1)^{j-d} \binom{j}{d} f_{n-j-1}(P).$$

Then $h(P) = (h_0, \dots, h_n)$ is called the *h-vector* of P. Furthermore, $h_P(x) = \sum_{i=0}^{n} h_i x^i$ is called the *h-polynomial* of P.

Proposition 5.11. *The f-polynomial and h-polynomial of a simplicial convex polytope P of dimension n satisfy the relation $h_P(x) = f_P(x-1)$.*

Proof. We find

$$\begin{aligned} h_P(x) &= \sum_{i=0}^{n} x^i \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{i} f_{n-j-1} = \sum_{j=0}^{n} f_{n-j-1} \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} x^i \\ &= \sum_{j=0}^{n} f_{n-j-1} (x-1)^j = f_P(x-1). \end{aligned}$$

□

Corollary 5.12. *Let P be a simplicial convex polytope of dimension n. For $d = 0, \dots, n$ we have $f_{d-1}(P) = \sum_{j=0}^{d} \binom{n-j}{n-d} h_{n-j}(P)$.*

Proof. We have

$$\begin{aligned} \sum_{i=0}^{n} f_{n-i-1} x^i = f_P(x) = h_P(x+1) &= \sum_{k=0}^{n} h_k (x+1)^k = \sum_{k=0}^{n} h_k \sum_{i=0}^{k} \binom{k}{i} x^i \\ &= \sum_{i=0}^{n} x^i \sum_{k=i}^{n} \binom{k}{i} h_k \end{aligned}$$

and therefore, for $i = 0, \dots, n$,

$$f_{n-i-1} = \sum_{k=i}^{n} \binom{k}{i} h_k = \sum_{j=0}^{n-i} \binom{n-j}{n-(n-i)} h_{n-j}.$$

The result follows by substituting $d = n - i$. □

The following is a well-known result for simplicial convex polytopes.

Theorem 5.13 (Dehn-Sommerville equations)**.** *The h-vector $h(P) = (h_0, \dots, h_n)$ of a simplicial convex polytope P of dimension n satisfies the relations*

$$h_i = h_{n-i}, \quad i = 0, \dots, n.$$

Proof. See [9, III.3], where the result is proved in the more general setting of cell complexes. □

Remark 5.14. Let P be a simplicial convex polytope of dimension n. By the Dehn-Sommerville equations, we have, for $d = 0, \dots, n$,

$$h_d = h_{n-d} = \sum_{i=n-d}^{n} (-1)^{i-(n-d)} \binom{i}{n-d} f_{n-i-1} = \sum_{j=0}^{d} (-1)^{j-d} \binom{n-j}{n-d} f_{j-1}.$$

Example 5.15. Let us compute the h-vector $(h_0, \dots, h_{n-1})$ of the simplicial convex polytope $Q_n^\circ \subseteq \mathbb{R}^{n-1}$, which is closely related to P_n (see Proposition 4.31). Notice that for $i = 1, \dots, n$ we have

$$f_{i-2}(Q_n^\circ) = f_{n-i}(Q_n) = f_{n-i}(P_n) = i!S(n,i)$$

(see Remark 5.8 and Example 5.9). So for $d = 0, \dots, n-1$ we find

$$\begin{aligned} h_d(Q_n^\circ) &= \sum_{j=d}^{n-1} (-1)^{j-d} \binom{j}{d} f_{n-j-2}(Q_n^\circ) = \sum_{i=1}^{n-d} (-1)^{n-d-i} \binom{n-i}{d} f_{i-2}(Q_n^\circ) \\ &= \sum_{i=1}^{n-d} (-1)^{n-d-i} \binom{n-i}{d} i!S(n,i) = A(n,d). \end{aligned}$$

where the *Eulerian number* $A(n,k)$ is the number of permutations g of $\{1, \dots, n\}$ such that $|\{i \in \{1, \dots, n-1\} : g(i) < g(i+1)\}| = k$. The final identity is well-known from the literature (see [24, p. 632, 26.14.7]). The Dehn-Sommerville equations for Q_n° are given by the identities $A(n,d) = A(n, n-d-1)$ for $d = 0, \dots, n-1$, which follow immediately from the definition of Eulerian numbers.

Example 5.16. Reconsider Example 2.9. Notice that $\Sigma_{\mathbb{P}^n} = \Sigma_{R_n^\circ}$, where $R_n = \mathrm{conv}(e_0, \dots, e_n)$ is the n-simplex in $\mathbb{R}^n$ with vertices $e_0, \dots, e_n$. Let us determine the f-vector $(f_0, \dots, f_{n-1})$ and h-vector $(h_0, \dots, h_n)$ of R_n. For $j = 0, \dots, n$, there is a one-to-one correspondence between the j-element subsets of $\{e_0, \dots, e_n\}$ and the $j-1$-faces of R_n; hence, $f_{j-1} = \binom{n+1}{j}$. So for $d = 0, \dots, n$ we find, using the identity of Remark 5.14,

$$\begin{aligned} h_d &= \sum_{j=0}^{d} \binom{n-j}{n-d} (-1)^{d-j} f_{j-1} \\ &= \sum_{j=0}^{d} (-1)^{(n-j)-(n-d)} \binom{-(n-d+1)}{(n-j)-(n-d)} (-1)^{d-j} \binom{n+1}{j} \\ &= \sum_{j=0}^{d} \binom{d-n-1}{d-j} \binom{n+1}{j} = \binom{(d-n-1)+(n+1)}{d} = \binom{d}{d} = 1, \end{aligned}$$

where we have used the rule of 'negating the upper index' and the Chu-Vandermonde identity (see [19, §1.2.6 (19), (21)]).

5.2 McMullen's condition

In 1971 McMullen conjectured in [21] a necessary and sufficient condition for a vector $h = (h_0, \ldots, h_n) \in \mathbb{Z}^{n+1}$ to be the h-vector of a simplicial convex polytope. In 1980 sufficiency was proved by Billera and Lee (see [4]) and necessity by Stanley (see [27]). We will present Stanley's proof in § 5.5. It uses toric geometry in combination with a deep result from algebraic topology and algebraic geometry, called the *hard Lefschetz theorem*.

In order to state McMullen's condition, we use the notion of an *M-vector*, following [27]. Let us define this notion. For $k, i \in \mathbb{Z}_{>0}$ there are unique integers $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$ such that

$$k = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}.$$

(In order to prove this, proceed by induction on k and choose n_i such that $\binom{n_i}{i} \leq k < \binom{n_i+1}{i}$.) Now set

$$k^{\langle i \rangle} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \ldots + \binom{n_j+1}{j+1}$$

and $0^{\langle i \rangle} = 0$.

Definition 5.17. A vector $(k_0, \ldots, k_m) \in \mathbb{Z}^{m+1}$ is called an *M-vector* if $k_0 = 1$, $k_i \geq 0$ for $i = 1, \ldots, m$ and $k_{i+1} \leq k_i^{\langle i \rangle}$ for $i = 1, \ldots, m-1$.

To every vector $f = (f_0, \ldots, f_{n-1}) \in \mathbb{Z}^n$, not necessarily an f-vector, we associate a vector $h = (h_0, \ldots, h_n)$, as defined in Definition 5.10. Now McMullen's conjecture, proved by Billera, Lee and Stanley, can be stated as follows.

Theorem 5.18. *Let $f = (f_0, \ldots, f_{n-1}) \in \mathbb{Z}^n$ be a vector. Then f is an f-vector of a simplicial convex polytope if and only if the corresponding vector $h = (h_0, \ldots, h_n)$ satisfies the Dehn-Sommerville equations and the vector*

$$(h_0, h_1 - h_0, h_2 - h_1, \ldots, h_{\lfloor n/2 \rfloor} - h_{\lfloor n/2 \rfloor - 1})$$

is an M-vector.

This theorem has a number of interesting corollaries (e.g. see [10, pp. 129–130]). One immediate consequence is (part of) the so-called *generalized lower bound conjecture*, first proposed by McMullen and Walkup in 1971 (see [22]).

Corollary 5.19 (Generalized lower bound conjecture)**.** *The h-vector $(h_0, \ldots, h_n)$ of an n-dimensional simplicial convex polytope satisfies* $1 = h_0 \leq h_1 \leq \ldots \leq h_{\lfloor n/2 \rfloor}$.

From this result, the *lower bound conjecture* can be deduced.

Corollary 5.20 (Lower bound conjecture)**.** *The f-vector $(f_0, \ldots, f_{n-1})$ of an n-*

dimensional simplicial convex polytope satisfies

$$f_i \geq \binom{n}{i} f_0 - \binom{n+1}{i+1} i, \qquad i = 0, \ldots, n-2;$$
$$f_{n-1} \geq (n-1) f_0 - (n+1)(n-2).$$

Proof. Let $i \in \{0, \ldots, n-1\}$. We find, using the identity of Corollary 5.12,

$$\begin{aligned} f_i &= \sum_{j=0}^{i+1} \binom{n-j}{n-i-1} h_{n-j} = \sum_{j=0}^{i+1} \left(\binom{n-j+1}{n-i} - \binom{n-j}{n-i} \right) h_j \\ &= \underbrace{\binom{n}{n-i}(h_1 - h_0) + \binom{n+1}{n-i} h_0}_{A_i} + \underbrace{\sum_{j=1}^{i} \binom{n-j}{n-i}(h_{j+1} - h_j)}_{R_i}. \end{aligned}$$

Expanding A_i and R_i separately yields

$$\begin{aligned} A_i &= \binom{n}{i} \left(-\binom{n}{n-1} f_{-1} + \binom{n-1}{n-1} f_0 - \binom{n}{n} f_{-1} \right) + \binom{n+1}{n-i} \binom{n}{n} f_{-1} \\ &= \binom{n}{i} (f_0 - (n+1)) + \binom{n+1}{i+1} = \binom{n}{i} f_0 - \binom{n+1}{i+1} i \end{aligned}$$

and

$$\begin{aligned} R_i &= \sum_{j=1}^{\lfloor n/2 \rfloor - 1} \binom{n-j}{n-i} (h_{j+1} - h_j) + \sum_{k=\lfloor n/2 \rfloor}^{n-1} \binom{n-k}{n-i} (h_{k+1} - h_k) \\ &= \sum_{j=1}^{\lfloor n/2 \rfloor - 1} \binom{n-j}{n-i} (h_{j+1} - h_j) - \sum_{k=\lfloor n/2 \rfloor}^{n-1} \binom{n-k}{n-i} (h_{n-k} - h_{n-k-1}) \\ &= \sum_{j=1}^{\lfloor n/2 \rfloor - 1} \binom{n-j}{n-i} (h_{j+1} - h_j) - \sum_{j=0}^{n - \lfloor n/2 \rfloor - 1} \binom{j+1}{n-i} (h_{j+1} - h_j) \\ &= \underbrace{\sum_{j=1}^{\lfloor n/2 \rfloor - 1} \left(\binom{n-j}{n-i} - \binom{j+1}{n-i} \right) (h_{j+1} - h_j)}_{R_i'} - \underbrace{\binom{1}{n-i} (h_1 - h_0)}_{R_i''}, \end{aligned}$$

where we have used that $h_{n - \lfloor n/2 \rfloor} - h_{n - \lfloor n/2 \rfloor - 1} = 0$ if n is odd. Since we have

$$j + 1 \leq \lfloor n/2 \rfloor \leq n - (\lfloor n/2 \rfloor - 1) \leq n - j, \quad j = 1, \ldots, \lfloor n/2 \rfloor - 1,$$

it follows by Corollary 5.19 that $R_i' \geq 0$. If $i \in \{0, \ldots, n-2\}$, then we have $R_i'' = 0$, so that $R_i \geq 0$ and $f_i \geq A_i$. Furthermore, we find

$$\begin{aligned} f_{n-1} &\geq A_{n-1} - \binom{1}{n-(n-1)} (h_1 - h_0) \\ &= n f_0 - (n+1)(n-1) - (f_0 - (n+1)) = (n-1) f_0 - (n+1)(n-2). \end{aligned}$$

□

Example 5.21. Let us consider the case $n = 2$ in Theorem 5.18. First of all, notice that every polygon is simplicial. Therefore, Theorem 5.18 states that a vector $f = (f_0, f_1) \in \mathbb{Z}^2$ is an f-vector if and only if $f_{-1} = h_2 = h_0 =$

$f_{-1} - f_0 + f_1$ holds and $(h_0, h_1 - h_0) = (1, (f_0 - 2) - 1)$ is an M-vector. This is the case if and only if $f_1 = f_0 \geq 3$. This expresses the fact that every convex polygon has as many vertices as edges and that this number must be at least 3. Conversely, for any $m \in \mathbb{Z}_{\geq 3}$ there exists a convex polygon with m vertices and m edges.

Example 5.22. Let us consider the case $n = 3$ in Theorem 5.18. Let $f = (f_0, f_1, f_2) \in \mathbb{Z}^3$. The vector associated to f is

$$(h_0, h_1, h_2, h_3) = (f_2 - f_1 + f_0 - 1, f_1 - 2f_0 + 3, f_0 - 3, 1).$$

Then f is an f-vector of a simplicial convex polyhedron if and only if

$$\begin{aligned} 1 = h_3 = h_0 &= f_2 - f_1 + f_0 - 1; \\ f_0 - 3 = h_2 = h_1 &= f_1 - 2f_0 + 3; \\ 0 \leq h_1 - h_0 = h_2 - h_3 &= f_0 - 4; \end{aligned}$$

which is the case if and only if

$$f_2 - f_1 + f_0 = 2; \tag{1}$$

$$2f_1 = 3f_2; \tag{2}$$

$$f_0 \geq 4. \tag{3}$$

Suppose that $f = f(P)$ is the f-vector of a convex polyhedron $P \subseteq \mathbb{R}^3$. Then P clearly satisfies (3), and it also satisfies (1), which is Euler's polyhedron formula. If P is furthermore simplicial, then (2) is satisfied as well: every edge belongs to two faces and every face has three edges. Conversely, Theorem 5.18 also tells us that if f satisfies (1)-(3), then there is some simplicial convex polyhedron P such that $f = f(P)$.

5.3 Simplicial cones and orbifolds

Before in § 5.5 Stanley's proof of the necessity of McMullen's condition will be presented, some preliminary work has to be done. In this subsection the concepts of *simplicial* cones and *orbifolds* are introduced, and in § 5.4 the relation between *Betti numbers* and h-vectors is discussed.

Definition 5.23. A cone $\sigma \subseteq \mathbb{R}^n$ is called *simplicial* if there are linearly independent vectors $v_1, \ldots, v_r \in \mathbb{R}^n$ such that $\sigma = \mathcal{C}(v_1, \ldots, v_r)$. A fan is called *simplicial* if all its cones are simplicial.

Every simplicial cone is strongly convex. Furthermore, every strongly convex cone $\sigma \subseteq \mathbb{R}^2$ is simplicial. Indeed, choose a generator for each ray of σ. Then, since σ is strongly convex, these generators are linearly independent and generate σ. It also follows that every fan in $\mathbb{R}^2$ is simplicial. However, as the following example shows, strongly convex cones in $\mathbb{R}^n$ with $n > 2$ need not be simplicial.

Example 5.24. The cone $\mathcal{C}((1,1,1), (1,1,-1), (1,-1,1), (1,-1,-1)) \subseteq \mathbb{R}^3$, is strongly convex but not simplicial.

The following remark explains why the term *simplicial* is applied to both convex polytopes on the one hand, and cones and fans on the other hand.

Remark 5.25. Let $P \subseteq \mathbb{R}^n$ be a simplicial full-dimensional convex lattice polytope with $0 \in \mathrm{int}(P)$. By Proposition 2.52 (3) we have

$$\Sigma_{P^\circ} = \{\mathcal{C}(F) : F \text{ is a proper face of } P\} \cup \{\{0\}\}.$$

Since the facets of P are simplices, the maximal cones of Σ_{P° are simplicial and so is the fan Σ_{P°.

The following definition is taken from [30, Def. 1.1] (although we use the term *orbifold* instead of *V-manifold*).

Definition 5.26. A complex algebraic variety X of dimension n is called an *orbifold* if X, considered as a complex analytic variety, has an open covering $\{U_i : i \in I\}$ such that for each $i \in I$ we have $U_i \cong Z_i/G_i$, where $Z_i \subseteq \mathbb{C}^n$ is an open ball and G_i is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$.

Proposition 5.27. *A fan Σ is simplicial if and only if X_Σ is an orbifold.*

Proof. See [16, p. 121]. □

Example 5.28. Consider the strongly convex lattice cone $\sigma = \mathcal{C}(2e_1 + e_2, e_2) \subseteq \mathbb{R}^2$. Notice that σ is simplicial but not regular. By Proposition 2.35 and Proposition 5.27, the toric variety X_σ is a non-smooth orbifold. Let us try to understand why this is the case.

We have $S_\sigma = \langle e_1, e_2, 2e_2 - e_1 \rangle$ and

$$R_\sigma = \mathbb{C}[x_1, x_2, x_2^2 x_1^{-1}] \cong \mathbb{C}[y_1, y_2, y_3]/(y_1 y_3 - y_2^2).$$

The toric variety $X_\sigma = \{(y_1, y_2, y_3) \in \mathbb{A}^3 : y_1 y_3 = y_2^2\}$ is not smooth at the origin, as $(y_1 y_3 - y_2^2) \subseteq \mathbb{C}[y_1, y_2, y_3]$ is a prime ideal and all partial derivatives of $y_1 y_3 - y_2^2$ vanish at 0.

Notice that $\mathbb{A}^2$, considered as an analytic variety, is isomorphic to an open ball in $\mathbb{C}^2$. Now consider the subgroup

$$G = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}(2, \mathbb{C}),$$

which acts on $\mathbb{A}^2$ by left-multiplication. Then we have an isomorphism

$$\mathbb{A}^2/G \xrightarrow{\sim} X_\sigma, \quad \overline{(u, v)} \mapsto (u^2, uv, v^2).$$

Hence, X_σ is an orbifold.

5.4 Betti numbers and h-vectors

Notation 5.29. Let Σ be an n-dimensional fan. For $i = 0, \dots, n$ we set

$$\begin{aligned} a_i &= a_i(\Sigma) = |\{\sigma \in \Sigma : \dim \sigma = i\}|; \\ t_i &= t_i(\Sigma) = |\{O(\sigma) \subseteq X_\Sigma : \sigma \in \Sigma,\ O(\sigma) \cong \mathbb{T}^i\}|. \end{aligned}$$

Proposition 5.30. *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional convex lattice polytope with $0 \in \mathrm{int}(P)$. For $i = 0, \dots, n$ we have $t_{n-i}(\Sigma_{P^\circ}) = a_i(\Sigma_{P^\circ}) = f_{i-1}(P)$.*

Proof. Since we have $\dim X_{\Sigma_{P^\circ}} = n$ (Theorem 2.6), the first identity follows by Theorem 2.22 (3). Since we have

$$\Sigma_{P^\circ} = \{\mathcal{C}(F) : F \text{ is a proper face of } P\} \cup \{\{0\}\}$$

(see Proposition 2.52 (3)), the second identity is clear as well. $\square$

For a topological space X, a field k of characteristic 0 and a non-negative integer i, one can consider the ith *cohomology group* $H^i(X,k)$ of X with coefficients in k. There is a *cup product*

$$H^i(X,k) \times H^j(X,k) \to H^{i+j}(X,k),$$

which makes the direct sum of k-vector spaces $H^*(X,k) = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} H^i(X,k)$ into a skew-commutative graded k-algebra. The dimension of $H^i(X,\mathbb{Q})$ as a $\mathbb{Q}$-vector space is called the ith *Betti number* β_i of X.

The next proposition gives a relation between the h-vector of a simplicial convex polytope P and the Betti numbers of the toric variety $X_{\Sigma_{P^\circ}}$. For a complete variety X, let $A^i(X)_\mathbb{Q} = A^i(X) \otimes_\mathbb{Z} \mathbb{Q}$ denote its ith Chow cohomology group with coefficients in $\mathbb{Q}$, and set $A^*(X)_\mathbb{Q} = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} A^i(X)_\mathbb{Q}$. In general, it is not clear whether the abelian group $A^*(X)$ can be supplied with a ring structure, since X need not be smooth (cf. § 3.3). However, as the following result shows, in some cases the $\mathbb{Q}$-vector space $A^*(X)_\mathbb{Q}$ can be supplied with a ring structure, even if X is not smooth (cf. Theorem 3.29).

Proposition 5.31. *Let $P \subseteq \mathbb{R}^n$ be a full-dimensional simplicial convex lattice polytope with $0 \in \operatorname{int}(P)$. Let $(f_0,\dots,f_{n-1})$ and $(h_0,\dots,h_n)$ be the f-vector and the h-vector of P, respectively, and set $\Sigma = \Sigma_{P^\circ}$. Then we have the following:*

(1) *There are isomorphisms $A^i(X_\Sigma)_\mathbb{Q} \xrightarrow{\sim} H^{2i}(X_\Sigma,\mathbb{Q})$ and $H^{2i+1}(X_\Sigma,\mathbb{Q}) \cong 0$ of $\mathbb{Q}$-vector spaces for $i \in \mathbb{Z}_{\geq 0}$, yielding an isomorphism $A^*(X_\Sigma)_\mathbb{Q} \xrightarrow{\sim} H^*(X_\Sigma,\mathbb{Q})$ of $\mathbb{Q}$-vector spaces, which makes $A^*(X_\Sigma)_\mathbb{Q}$ into a commutative graded $\mathbb{Q}$-algebra, which is generated by $A^1(X_\Sigma)_\mathbb{Q}$.*
(2) *$\beta_{2i+1} = 0$ for $i = 0,\dots,n$.*
(3) *$\beta_{2i} = \dim_\mathbb{Q} A^i(X_\Sigma)_\mathbb{Q} = h_i$ for $i = 0,\dots,n$.*

Proof. Since, the polytope P is simplicial, the fan Σ is simplicial as well (see Remark 5.25). Furthermore, Σ is polytopal and therefore complete. It follows by [7, Thm. 10.8 and Rem. 10.9] that (1) and (2) are the case (for the fact that $A^*(X_\Sigma)_\mathbb{Q}$ is generated by $A^1(X_\Sigma)_\mathbb{Q}$, see [10, p. 106]) and that for $i = 0,\dots,n$,

$$\dim_\mathbb{Q} A^i(X_\Sigma)_\mathbb{Q} = \sum_{j=i}^{n} \binom{j}{i} (-1)^{j-i} a_{n-j}(\Sigma).$$

Now (3) follows by Proposition 5.30. $\square$

We now sketch an alternative proof of Proposition 5.31 (3) in case Σ is regular, which uses the partition of X_Σ into torus-orbits. The reason for only giving a proof sketch, is that we need to use the notion of *toric schemes* over a commutative ring (in our case $\mathbb{Z}$), which generalizes the notion of toric varieties. We will not elaborate on this; for an introduction to toric schemes, see [13].

Sketch of an alternative proof of Proposition 5.31 (3) in case Σ is regular. First, notice that since the fan Σ is regular and polytopal, the toric variety X_Σ is smooth and projective (see Propositions 2.35 and 2.43). For $i = 0, \dots, n$ we have

$$t_i = t_i(\Sigma) = f_{n-i-1}(P) = f_{n-i-1}$$

by Proposition 5.30.

Now consider X_Σ as a scheme over $\mathbb{Z}$. Its reduction modulo a prime number is smooth for all but finitely many primes. So we can choose a prime p such that X_Σ is smooth (and projective) over $\mathbb{F}_{p^m}$ for all $m \in \mathbb{Z}_{\geq 1}$. Then for all $m \in \mathbb{Z}_{\geq 1}$, the $\mathbb{F}_{p^m}$-rational points of X_Σ can be counted by counting the $\mathbb{F}_{p^m}$-rational points in every torus-orbit (since by Corollary 2.27 X_Σ is the disjoint union of its torus-orbits):

$$|X_\Sigma(\mathbb{F}_{p^m})| = \sum_{i=0}^{n} t_i \cdot |\mathbb{T}^i(\mathbb{F}_{p^m})| = \sum_{i=0}^{n} f_{n-i-1} \cdot (p^m - 1)^i = f_P(p^m - 1) = h_P(p^m),$$

where we have used Proposition 5.11. As a consequence of the Weil conjectures, we find that $h_i = \beta_{2i}$ for $i = 0, \dots, n$ (see [23, Rem. 2.27]). □

Example 5.32. Let $\Sigma = \Sigma_{\mathbb{P}^n}$. Looking back at Examples 2.9 and 5.16, we find the following Betti numbers of $\mathbb{P}^n = X_\Sigma$: $(\beta_0, \dots, \beta_{2n+1}) = (1, 0, 1, 0, \dots, 1, 0)$.

Let us determine the torus-orbits of the smooth toric variety $\mathbb{P}^n = X_\Sigma$. There is a one-to-one correspondence between the non-empty subsets of $I = \{0, \dots, n\}$ and Σ, given by $J \mapsto \bigcap_{j \in J} \sigma_j = \sigma_J$. For a non-empty subset $J \subseteq I$ we have $\dim \sigma_J = n + 1 - |J|$ and

$$O(\sigma_J) = \{(a_0, \dots, a_n) \in \mathbb{P}^n : a_j \neq 0 \Leftrightarrow j \in J\} \cong \mathbb{T}^{|J|-1}.$$

Thus we recover the identity $t_i(\Sigma) = f_{n-i-1}(R_n)$ from Proposition 5.30 (where R_n is as in Example 5.16) as follows: for $i = 0, \dots, n$, the number $t_i(\Sigma)$ of torus embeddings $\mathbb{T}^i \hookrightarrow \mathbb{P}^n$ equals the number of $i+1$-element subsets of $I = \{0, \dots, n\}$, which is

$$\binom{n+1}{i+1} = \binom{n+1}{n-i} = f_{n-i-1}(R_n).$$

5.5 Stanley's proof of the necessity of McMullen's condition

The following characterization of M-vectors will be used in the proof of the necessity of McMullen's condition.

Lemma 5.33. *Let $k = (k_0, \dots, k_d) \in \mathbb{Z}^{d+1}$. Then k is an M-vector if and only if there exist a field R_0 and a commutative graded R_0-algebra $\bigoplus_{i=0}^{d} R_i$, generated by R_1, such that $\dim_{R_0} R_i = k_i$ for $i = 0, \dots, d$.*

Proof. See [28, §2] for a discussion of this result and references to the proof. □

The following result, called the *hard Lefschetz theorem*, was first stated in 1924 by Lefschetz in [20], although his 'proof' was not entirely rigorous. We use the version given in [29]; a discussion of this theorem and references for its proof can be found there, as well as a few combinatorial applications of it.

Theorem 5.34 (Hard Lefschetz theorem)**.** *Let X be an irreducible smooth projective complex variety of dimension n and k a field of characteristic* 0*. Then there is an element $\omega \in H^2(X,k)$ such that for $i = 0,\dots,n$, the map $H^i(X,k) \to H^{2n-i}(X,k)$, given by multiplication by ω^{n-i}, is an isomorphism of k-vector spaces.*

Remark 5.35. By [30, Theorem 1.13], the hard Lefschetz theorem also holds when X is taken to be an irreducible projective orbifold (cf. [3, Example 1.5]).

We can now give Stanley's proof.

Proof of the necessity of McMullen's condition. Let $P \subseteq \mathbb{R}^n$ be a full-dimensional simplicial convex polytope with f-vector $(f_0,\dots,f_{n-1})$ and h-vector $(h_0,\dots,h_n)$. Without loss of generality we may assume that the vertices of P are contained in the lattice $\mathbb{Z}^n$ and that $0 \in \mathbb{R}^n$ is contained in the interior of P. Set $\Sigma = \Sigma_{P^\circ}$.

By Proposition 5.31 (3), it follows from Poincaré duality for projective orbifolds that $(h_0,\dots,h_n)$ satisfies the Dehn-Sommerville equations. It remains to be shown that $(h_0, h_1 - h_0, h_2 - h_1, \dots, h_{\lfloor n/2 \rfloor} - h_{\lfloor n/2 \rfloor - 1})$ is an M-vector.

Set $A^i = A^i(X_\Sigma)_\mathbb{Q} \cong H^{2i}(X_\Sigma,\mathbb{Q})$ for $i \in \mathbb{Z}_{\geq 0}$. By Proposition 5.31 we know that $A = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} A^i$ is a commutative graded $\mathbb{Q}$-algebra, generated by A^1, which satisfies $h_i = \dim_\mathbb{Q} A^i$ for $i = 0,\dots,n$.

Since the convex polytope P is simplicial, the fan $\Sigma = \Sigma_{P^\circ}$ is simplicial as well (see Remark 5.25) and X_Σ is an orbifold (see Proposition 5.27). Furthermore, X_Σ is irreducible (see Remark 2.5) and, since Σ is polytopal, X_Σ is projective (see Proposition 2.43). It follows by Remark 5.35 that hard Lefschetz holds for X_Σ: there is some element $\omega \in A^1$ such that for $i = 0,\dots,\lfloor n/2 \rfloor$, the map $A^i \to A^{n-i}$, given by multiplication by ω^{n-2i}, is a bijection; in particular, the map $\varphi_i : A^i \to A^{i+1}$, given by multiplication by ω, is injective.

Let $I \subseteq A$ be the ideal generated by ω and $A^{\lfloor n/2 \rfloor + 1}$, and set $R_i = A^i/(A^i \cap I)$ for $i \in \mathbb{Z}_{\geq 0}$. Since A is generated by A^1, for $i \geq \lfloor n/2 \rfloor + 1$ we have $A^i \subseteq I$ and $R_i = 0$. Since I is a homogeneous ideal,

$$R = A/I = \bigoplus_{i\in\mathbb{Z}_{\geq 0}} R_i \cong \bigoplus_{i=0}^{\lfloor n/2 \rfloor} R_i$$

is a commutative graded R_0-algebra, generated by R_1. We have $R_0 = A^0/(A^0 \cap I) = \mathbb{Q}/(0) = \mathbb{Q}$.

Notice that for $i = 1,\dots,\lfloor n/2 \rfloor$ we have

$$h_{i-1} = \dim_\mathbb{Q} A^{i-1} = \dim_\mathbb{Q} \operatorname{im}(\varphi_{i-1}) + \dim_\mathbb{Q} \ker(\varphi_{i-1}) = \dim_\mathbb{Q}(A^i \cap I)$$

(since φ_{i-1} is injective) and therefore,

$$\dim_\mathbb{Q} R_i = \dim_\mathbb{Q} A^i - \dim_\mathbb{Q}(A^i \cap I) = h_i - h_{i-1}.$$

Finally, notice that $\dim_\mathbb{Q} R_0 = \dim_\mathbb{Q} \mathbb{Q} = 1 = h_0$. It follows by Lemma 5.33 that $(h_0, h_1 - h_0, h_2 - h_1, \dots, h_{\lfloor n/2 \rfloor} - h_{\lfloor n/2 \rfloor - 1})$ is an M-vector, which finishes the proof. □

6 The Heron-Rota-Welsh conjecture

The second application of toric geometry we will discuss, is concerned with characteristic polynomals of matroids (see § 6.1). The ultimate goal of this section is to present a proof of a special case of the Heron-Rota-Welsh conjecture. The general conjecture, which was proved by Adiprasito, Huh and Katz in 2015, states that the characteristic polynomial of any matroid M is log-concave (see § 6.2). We will present a proof for the special case that M is a representable matroid. In the course of the proof we will need many different parts of the material treated before. For instance, the permutohedral variety $X_{\Sigma_{Q_{n+1}}}$ turns out to be of great importance (see § 6.4), and the theory of piecewise linear functions and Minkowski weights will be used (see § 6.5 and § 6.6).

6.1 The characteristic polynomial of a matroid

Recall that for a graph $G = (V, E)$, the map $\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ which assigns to each non-negative integer q the number of proper q-colorings of the vertices of G, is given by a polynomial $\chi_G(t)$, which is called the *chromatic polynomial* of G. This subsection discusses a generalization of the chromatic polynomial for matroids, called the *characteristic polynomial* (see [3, §3] and [18, §7]).

Some important properties of the chromatic polynomial of a graph are given by the following proposition. For a graph $G = (V, E)$ and an edge $e \in E$, let $G \backslash e$ denote the graph G with the edge e removed, and G/e the graph G with the endpoints of e contracted to one vertex.

Proposition 6.1. *Let $G = (V, E)$ be a graph. Then we have the following:*

(1) *If G contains a loop, then $\chi_G(t) = 0$.*
(2) *If G consists of two vertices connected by one edge, then $\chi_G(t) = t(t-1)$.*
(3) *If G can be written as the direct sum $G = G_1 \oplus G_2$ of two graphs G_1 and G_2, then $\chi_G(t) = \chi_{G_1}(t)\chi_{G_2}(t)$.*
(4) *If $e = \{x, y\} \in E$ is an edge, then $\chi_G(t) = \chi_{G\backslash e}(t) - \chi_{G/e}(t)$.*

Furthermore, the polynomial $\chi_G(t)$ is uniquely characterized by these properties.

Proof. (1), (2) and (3) are clear. For (4), note that $\chi_{G\backslash e}(q)$ is the number of q-colorings of G such that the only adjacent vertices permitted to have the same color are x and y, and that $\chi_{G/e}(q)$ is the number of q-colorings of G such that x and y have the same color, but no other adjacent vertices have.

For the final statement, consider a family of polynomials $(f_G)_G$ in $\mathbb{Z}[t]$, where G ranges over all graphs, which satisfies the properties (1)-(4). It is an easy exercise to prove that $(f_G)_G = (\chi_G)_G$ by induction on the number of vertices. □

Definition 6.2. Let $M = (E, \mathscr{I})$ be a matroid. The *characteristic polynomial* of M is

$$\chi_M(t) = \sum_{A \subseteq E} (-1)^{|A|} t^{\mathrm{rk}(M) - \mathrm{rk}_M(A)}.$$

We now introduce the *Möbius function* of a matroid M. If M is loopless, this function can be used to give an alternative expression for its characteristic polynomial (cf. [18, §7.3]).

Definition 6.3. Let $M = (E, \mathscr{F})$ be a matroid. The *Möbius function* $\mu : \mathscr{F}^2 \to \mathbb{Z}$ is defined recursively by

$$\mu(F, F) = 1;$$

$$\mu(F', F) = \begin{cases} -\sum_{F' \subseteq F'' \subsetneq F} \mu(F', F''), & \text{if } F' \subsetneq F \\ 0, & \text{if } F' \not\subseteq F. \end{cases}$$

Remark 6.4. Let M be a loopless matroid and let $i \in \{0, \ldots, \mathrm{rk}(M)\}$. For a rank i flat F of M we have $|\mu(\varnothing, F)| = (-1)^i \mu(\varnothing, F) \geq 1$ (this follows from [18, Lem. 7.11]).

Proposition 6.5. *The characteristic polynomial of a loopless matroid $M = (E, \mathscr{F})$ satisfies*

$$\chi_M(t) = \sum_{F \in \mathscr{F}} \mu(\varnothing, F) t^{\mathrm{rk}(M) - \mathrm{rk}_M(F)}.$$

Proof. See [18, Thm. 7.12]. □

The following definition generalizes the notions of deletion and contraction of an edge in a graph.

Definition 6.6. Let $M = (E, \mathscr{I})$ be a matroid and $A \subseteq E$ a subset. The *deletion* of A is the submatroid $M \backslash A = (E \setminus A, \mathscr{I}_{E \setminus A})$ of M. The *contraction* of M by A is the matroid

$$M/A = (E \setminus A, \{X \subseteq E \setminus A : X \cup B \in \mathscr{I}\}),$$

where $B \subseteq A$ is a basis for the submatroid $(A, \mathscr{I}_A)$. In the case $A = \{e\}$, we also write $M \backslash A = M \backslash e$ and $M/A = M/e$.

Remark 6.7. For a graph $G = (V, E)$ and an edge $e \in E$ we have $M_{G \backslash e} = M_G \backslash e$ and $M_{G/e} = M_G/e$.

We have the following analogue of Proposition 6.1 for characteristic polynomials of matroids (see [3, Thm. 3.5]; cf. Example 4.8).

Proposition 6.8. *Let $M = (E, \mathscr{I})$ be a matroid. Then we have the following:*

(1) *If M contains a loop, then $\chi_M(t) = 0$.*
(2) *If M consists of a single coloop, then $\chi_M(t) = t - 1$.*
(3) *If M can be written as the direct sum $M = M_1 \oplus M_2$ of two matroids M_1 and M_2, then $\chi_M(t) = \chi_{M_1}(t)\chi_{M_2}(t)$.*
(4) *If $e \in E$ is not a coloop, then $\chi_M(t) = \chi_{M \backslash e}(t) - \chi_{M/e}(t)$.*

Furthermore, the polynomial $\chi_M(t)$ is uniquely characterized by these properties.

Proof. (1), (2) and (3) are clear.

For (4), notice that for any subset $A \subseteq E \setminus \{e\}$ we have

$$\mathrm{rk}_{M/e}(A) = \mathrm{rk}_M(A \cup \{e\}) - \mathrm{rk}_M(\{e\}).$$

Furthermore, since e is not a coloop, we have $\mathrm{rk}(M \backslash e) = \mathrm{rk}(M)$. It follows that

$$\begin{aligned}
\chi_{M\backslash e}(t) - \chi_{M/e}(t) &= \sum_{A \subseteq E\backslash\{e\}} (-1)^{|A|} \left(t^{\mathrm{rk}(M\backslash e) - \mathrm{rk}_{M\backslash e}(A)} - t^{\mathrm{rk}(M/e) - \mathrm{rk}_{M/e}(A)} \right) \\
&= \sum_{A \subseteq E\backslash\{e\}} (-1)^{|A|} \left(t^{\mathrm{rk}(M) - \mathrm{rk}_M(A)} - t^{\mathrm{rk}(M) - \mathrm{rk}_M(A \cup \{e\})} \right) \\
&= \sum_{A \subseteq E} (-1)^{|A|} t^{\mathrm{rk}(M) - \mathrm{rk}_M(A)} = \chi_M(t).
\end{aligned}$$

The final statement follows in the same way as the final statement of Proposition 6.1. □

The following corollary shows that the notion of a characteristic polynomial of a matroid generalizes the notion of a chromatic polynomial of a graph.

Corollary 6.9. *For a graph G with k connected components we have $\chi_G(t) = t^k \chi_{M_G}(t)$.*

Proof. We know that $\chi_G(t)$ satisfies the properties (1)-(4) of Proposition 6.1. So by Example 4.8 and Remark 6.7, $\chi_G(t)/t^k$ satisfies the properties (1)-(4) of Proposition 6.8. Since $\chi_{M_G}(t)$ is uniquely characterized by these properties, we find $\chi_{M_G}(t) = \chi_G(t)/t^k$. □

Proposition 6.10. *For a matroid $M = (E, \mathscr{I})$ with $|E| \geq 1$ we have $\chi_M(1) = 0$.*

Proof. We compute

$$\chi_M(1) = \sum_{A \subseteq E} (-1)^{|A|} = \sum_{i=0}^{|E|} \binom{|E|}{i} (-1)^i = (1-1)^{|E|} = 0.$$

□

Definition 6.11. Let $M = (E, \mathscr{I})$ be a matroid with $|E| \geq 1$. The *reduced characteristic polynomial* of M is $\overline{\chi}_M(t) = \chi_M(t)/(t-1)$.

Remark 6.12. By Proposition 6.10, the reduced characteristic polynomial of a matroid $M = (E, \mathscr{I})$ with $|E| \geq 1$ is a polynomial in $\mathbb{Z}[t]$.

The coefficients of the reduced characteristic polynomial of a loopless matroid can be expressed in terms of the Möbius function of the matroid, as follows.

Proposition 6.13. *Let $M = (E, \mathscr{F})$ be a rank $d+1$ loopless matroid with $|E| \geq 1$, and write $\overline{\chi}_M(t) = \sum_{i=0}^{d} (-1)^i \mu^i t^{d-i}$ for its reduced characteristic polynomial. For*

$e \in E$ and $r = 0, \ldots, d$ we have

$$\mu^r = \sum_{\substack{F \in \mathscr{F}_r \\ F \not\ni e}} |\mu(\varnothing, F)| = \sum_{\substack{F \in \mathscr{F}_{r+1} \\ F \ni e}} |\mu(\varnothing, F)| \geq 1,$$

where $\mathscr{F}_r \subseteq \mathscr{F}$ denotes the subset of rank r flats of M.

Proof. It is proved in [18, Lem. 7.15] that

$$\mu^r = (-1)^r \sum_{\substack{F \in \mathscr{F}_r \\ F \not\ni e}} \mu(\varnothing, F) = (-1)^{r+1} \sum_{\substack{F \in \mathscr{F}_{r+1} \\ F \ni e}} \mu(\varnothing, F).$$

The result now follows from Remark 6.4. □

6.2 Statement of the main result and overview of the proof

Definition 6.14. A sequence of real numbers $a_0, \ldots, a_n$ is called *log-concave* if $a_i^2 \geq |a_{i-1}a_{i+1}|$ for $i = 1, \ldots, n-1$. A polynomial $f(t) = \sum_{i=0}^n a_i t^{n-i} \in \mathbb{R}[t]$ is called *log-concave* if the sequence of coefficients $a_0, \ldots, a_n$ is log-concave.

Example 6.15. The sequence of binomial coefficients $\binom{n}{0}, \ldots, \binom{n}{n}$, for some positive integer n, is log-concave. Indeed, for $k = 1, \ldots, n-1$ we have

$$\frac{\binom{n}{k}^2}{\binom{n}{k-1}\binom{n}{k+1}} = \frac{(k-1)!(n-k+1)!(k+1)!(n-k-1)!}{k!^2(n-k)!^2} = \frac{(n-k+1)(k+1)}{k(n-k)} \geq 1.$$

The following theorem is known as the *Heron-Rota-Welsh conjecture* and was proven in 2015 by Adiprasito, Huh and Katz (see [1]).

Theorem 6.16. *Let M be a matroid. The characteristic polynomial $\chi_M(t)$ is log-concave.*

The goal of this section is to present the proof of this theorem in the special case that M is a representable matroid, which can be found in [18].

If M is a matroid of rank $d+1$, we write $\chi_M(t) = \sum_{i=0}^{d+1}(-1)^i \mu_i t^{d+1-i}$ and $\overline{\chi}_M(t) = \sum_{i=0}^{d}(-1)^i \mu^i t^{d-i}$ for its characteristic polynomial and its reduced characteristic polynomial, respectively.

Lemma 6.17. *Let M be a matroid. If $\overline{\chi}_M(t)$ is log-concave, then so is $\chi_M(t)$.*

Proof. If M has a loop, then by Proposition 6.8 (1) there is nothing to prove. So assume that M is loopless. Setting $\mu^{-1} = \mu^{d+1} = 0$, we have

$$\sum_{i=0}^{d+1}(-1)^i \mu_i t^{d+1-i} = (t-1)\sum_{i=0}^{d}(-1)^i \mu^i t^{d-i} = \sum_{i=0}^{d+1}(-1)^i(\mu^i + \mu^{i-1})t^{d+1-i}.$$

Assume that $\overline{\chi}_M(t)$ is log-concave. Then the sequence $\mu^{-1}, \mu^0, \ldots, \mu^{d+1}$ is log-concave as well. Recall, by Proposition 6.13, that the numbers $\mu^0, \ldots, \mu^d$ are positive. Let $j \in \{1, \ldots, d\}$. It follows from $(\mu^j \mu^{j-1})^2 \geq \mu^{j-2}\mu^{j-1}\mu^j\mu^{j+1}$

and $\mu^j, \mu^{j-1} > 0$ that $\mu^j \mu^{j-1} \geq \mu^{j-2}\mu^{j+1}$. Therefore, we find

$$\begin{aligned}\mu_j^2 &= (\mu^j + \mu^{j-1})^2 = (\mu^j)^2 + (\mu^{j-1})^2 + 2\mu^j\mu^{j-1} \\ &\geq \mu^{j-1}\mu^{j+1} + \mu^{j-2}\mu^j + \mu^j\mu^{j-1} + \mu^{j-2}\mu^{j+1} \\ &= (\mu^{j-1} + \mu^{j-2})(\mu^{j+1} + \mu^j) = \mu_{j-1}\mu_{j+1}.\end{aligned}$$

Since $\mu_{j-1} = \mu^{j-1} + \mu^{j-2}$ and $\mu_{j+1} = \mu^{j+1} + \mu^j$ are non-negative, it follows that $\chi_M(t)$ is log-concave. □

Let $M = (E, \mathscr{I})$ be a representable matroid of rank $d+1$ on the ground set $E = \{0, \ldots, n\}$. We can associate a linear subspace $V \subseteq k^{n+1}$ to M as in Example 4.12. By Lemma 6.17, in order to prove that the characteristic polynomial $\chi_M(t)$ is log-concave, it will be sufficient to show that the reduced characteristic polynomial $\overline{\chi}_M(t)$ is.

Having introduced some general theory about rational maps and blow-ups in § 6.3, we will see in § 6.4 that the permutohedral variety $X_{\Sigma_{Q_{n+1}}}$, associated to the fan $\Sigma_{Q_{n+1}}$ introduced in § 4.2, can be obtained from the projective space $\mathbb{P}^n$ by a sequence of blow-ups. It turns out that the proper transform $\widetilde{\mathbb{P}(V)} \subseteq X_{\Sigma_{Q_{n+1}}}$ of the projectivization $\mathbb{P}(V) \subseteq \mathbb{P}^n$ of the subspace $V \subseteq \mathbb{C}^{n+1}$ is of great importance.

In § 6.5 we will construct nef divisors α and β on $\widetilde{\mathbb{P}(V)}$, in such a way that the absolute values of the coefficients of $\overline{\chi}_M(t)$ are the numbers

$$\deg\left(\alpha^{d-r}\beta^r \cdot \left[\widetilde{\mathbb{P}(V)}\right]\right), \quad r = 0, \ldots, d.$$

The proof of this relation between α, β and $\overline{\chi}_M(t)$ will be sketched in § 6.6 and § 6.7. An important ingredient is the Minkowski weight $\Delta_M : \Sigma^{(n-d)} \to \mathbb{Z}$ associated to the Chow cohomology class $\left[\widetilde{\mathbb{P}(V)}\right]$.

The final step will be to apply the Khovanskii-Teissier inequality (Theorem 6.30), which tells us that the above-mentioned sequence of degrees is log-concave.

6.3 Rational maps and blow-ups

In this subsection the notions of *rational maps* and *blow-ups* are introduced. For a more extensive discussion we refer to [14, §7].

Definition 6.18. Let X_1, X_2 be quasi-projective varieties. A *rational map* $f : X_1 \dashrightarrow X_2$ is an equivalence class of pairs (f_U, U), in which $U \subseteq X_1$ is an open and dense subvariety and $f_U : U \to X_2$ a morphism, where (f_U, U) and $(f'_{U'}, U')$ are equivalent if $f_U|_{U \cap U'} = f'_{U'}|_{U \cap U'}$. A rational map $f : X_1 \dashrightarrow X_2$ with $f = [(f_U, U)]$ is said to be *defined* on U. The union of all subvarieties on which f is defined, is called the *domain* of f. Its complement in X_1 is called the *locus of indeterminacy*.

Definition 6.19. Let $f : X_1 \dashrightarrow X_2$ and $g : X_2 \dashrightarrow X_3$ be rational maps, and assume that there are open and dense subsets $U \subseteq X_1$ and $V \subseteq X_2$ with $f = [(f_U, U)]$ and $g = [(g_V, V)]$ such that $U' = f_U^{-1}(V)$ is dense in X_1.

Then the *composition* $g \circ f : X_1 \dashrightarrow X_3$ is defined as the equivalence class of $(g_V \circ f_{U'}, U')$.

It should be noticed that, in general, it might not be possible to compose two rational maps $f : X_1 \dashrightarrow X_2$ and $g : X_2 \dashrightarrow X_3$.

Definition 6.20. A rational map $f : X_1 \dashrightarrow X_2$ is called *birational* if there is a rational map $g : X_2 \dashrightarrow X_1$ such that the compositions $g \circ f$ and $f \circ g$ exist and are equal to $[(\mathrm{id}_{X_1}, X_1)]$ and $[(\mathrm{id}_{X_2}, X_2)]$, respectively. In this case, g is called the *inverse* of f.

Example 6.21. The *Cremona transformation* is the rational map $\mathrm{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ which is defined by

$$(a_0 : \ldots : a_n) \mapsto \left(\prod_{i \neq 0} a_i : \prod_{i \neq 1} a_i : \ldots : \prod_{i \neq n} a_i \right),$$

often suggestively written as $(a_0 : \ldots : a_n) \mapsto (a_0^{-1} : \ldots : a_n^{-1})$. Its locus of indeterminacy is

$$L = \{(a_0 : \ldots : a_n) \in \mathbb{P}^n : a_i = a_j = 0 \text{ for some } 0 \leq i < j \leq n\}.$$

Notice that Crem is birational: the morphism $\mathrm{Crem}_{\mathbb{P}^n \setminus L} : \mathbb{P}^n \setminus L \to \mathbb{P}^n \setminus L$ is its own inverse.

We want to define blow-ups of projective varieties. In order to do so, we will first define blow-ups of affine varieties.

Definition 6.22. Let $X \subseteq \mathbb{A}^n$ be an affine variety and $Y \subseteq X$ a closed subvariety defined by an ideal $(f_1, \ldots, f_k)$ of the coordinate ring of X. Then the *blow-up of X along Y* is the subvariety

$$\mathrm{Bl}_Y(X) = \{(a, (b_1 : \ldots : b_k)) \in X \times \mathbb{P}^{k-1} : b_j f_i(a) = b_i f_j(a),\ 1 \leq i, j \leq k\}$$

of the product variety $X \times \mathbb{P}^{k-1}$, together with the projection $\pi : \mathrm{Bl}_Y(X) \to X$. If $Z \subseteq X$ is a closed subvariety with $Z \nsubseteq Y$, then the closure of $\pi^{-1}(Z \setminus Y)$ in $\mathrm{Bl}_Y(X)$ is called the *proper transform* of Z.

Example 6.23. Let $Y = Z(x_1, \ldots, x_k) \subseteq \mathbb{A}^n$ be a coordinate plane of codimension k. The blow-up $\mathrm{Bl}_Y(\mathbb{A}^n)$ of $\mathbb{A}^n$ at Y is

$$\{((a_1, \ldots, a_n), (b_1 : \ldots : b_k)) \in \mathbb{A}^n \times \mathbb{P}^{k-1} : a_i b_j = a_j b_i,\ 1 \leq i, j \leq k\}.$$

Now let $X \subseteq \mathbb{P}^n$ be a projective variety and $Y \subseteq X$ a closed subvariety, defined by a homogeneous ideal which is generated by k homogeneous polynomials in the homogeneous coordinate ring of X. For $i = 0, \ldots, n$, the closed subvariety $Y \cap U_i$ of the affine variety $X \cap U_i$, where $U_i \subseteq \mathbb{P}^n$ is the ith standard affine open, is given by an ideal generated by k polynomials; hence, we can consider the blow-up $\mathrm{Bl}_{Y \cap U_i}(X \cap U_i) \subseteq (X \cap U_i) \times \mathbb{P}^{k-1}$. Now the *blow-up of X along Y* is defined as the variety $\mathrm{Bl}_Y(X) \subseteq X \times \mathbb{P}^{k-1}$, obtained by gluing the blow-ups $\mathrm{Bl}_{Y \cap U_i}(X \cap U_i)$, together with the projection $\pi : \mathrm{Bl}_Y(X) \to X$. *Proper transforms* are defined as in the affine case.

Remark 6.24. Let $X \subseteq \mathbb{P}^n$ be a projective variety and $Y \subseteq X$ a closed subvariety as above. By the Segre embedding $\mathbb{P}^n \times \mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{(n+1)k-1}$, the blow-up $\mathrm{Bl}_Y(X) \subseteq \mathbb{P}^n \times \mathbb{P}^{k-1}$ of X along Y can be considered as a closed subvariety of $\mathbb{P}^{(n+1)k-1}$. Hence, the variety $\mathrm{Bl}_Y(X)$ is projective.

The following fundamental result connects the notions of rational maps and blow-ups (see [14, Thm. 7.21]).

Theorem 6.25. *Let $f : X_0 \dashrightarrow \mathbb{P}^n$ be a rational map, where X_0 is an affine or a projective variety. Then there are blow-ups $\pi_i : X_{i+1} = \mathrm{Bl}_{Y_i}(X_i) \to X_i$, for $i = 0, \ldots, k$, and a morphism $g : X_{k+1} \to \mathbb{P}^n$ such that $g = f \circ \pi_0 \circ \ldots \circ \pi_k$.*

In this situation, the morphism g is said to *resolve the locus of indeterminacy* of the rational map f. (One can also define blow-ups for general quasi-projective varieties. The result is still true if X_0 is allowed to be a general quasi-projective variety.)

6.4 The permutohedral variety

The permutohedral variety $X_{\Sigma_{Q_{n+1}}}$, associated to the fan $\Sigma_{Q_{n+1}}$ introduced in § 4.2, turns out to be important in the proof of the log-concavity of the characteristic polynomial. As will become clear in this subsection, it can be obtained from the projective space $\mathbb{P}^n$ by a sequence of blow-ups.

Let us first look at the case $n = 2$.

Example 6.26. Reconsider Example 4.20. We will compute the toric variety $X_{\Sigma_{Q_3}}$. First, notice that we have

$$\sigma_{12} \cup \sigma_{21} = \sigma_0, \quad \sigma_{02} \cup \sigma_{20} = \sigma_1, \quad \sigma_{01} \cup \sigma_{10} = \sigma_2,$$

where the cones

$$\sigma_0 = \mathcal{C}(e_1, e_2), \quad \sigma_1 = \mathcal{C}(e_0, e_2), \quad \sigma_2 = \mathcal{C}(e_0, e_1)$$

are defined as in Example 2.9 (for $n = 2$). We say that the fan $\Sigma_{Q_3} = \mathcal{F}(\sigma_{12}, \sigma_{21}, \sigma_{02}, \sigma_{20}, \sigma_{01}, \sigma_{10})$ *refines* the fan $\Sigma_{\mathbb{P}^2} = \mathcal{F}(\sigma_0, \sigma_1, \sigma_2)$. Recall that we have $X_{\Sigma_{\mathbb{P}^2}} = \mathbb{P}^2$ with coordinates $(s_0 : s_1 : s_2)$, where

$$\begin{aligned}
X_{\sigma_0} &\cong U_0, \text{ with affine coordinates } (s_1/s_0, s_2/s_0) = (x_1, x_2);\\
X_{\sigma_1} &\cong U_1, \text{ with affine coordinates } (s_0/s_1, s_2/s_1) = (x_1^{-1}, x_1^{-1}x_2);\\
X_{\sigma_2} &\cong U_2, \text{ with affine coordinates } (s_0/s_2, s_1/s_2) = (x_2^{-1}, x_1x_2^{-1});
\end{aligned}$$

(see Example 2.9 and, more extensively, [5, pp. 16–18]).

Gluing the toric varieties $X_{\sigma_{21}} = \mathbb{A}^2_{(x_1, x_1^{-1}x_2)}$ and $X_{\sigma_{12}} = \mathbb{A}^2_{(x_2, x_1x_2^{-1})}$, yields

$$X_{\mathcal{F}(\sigma_{21}, \sigma_{12})} = \left\{ ((a_1, a_2), (b_1 : b_2)) \in \mathbb{A}^2_{(x_1, x_2)} \times \mathbb{P}^1_{(t_1 : t_2)} : a_1 b_2 = a_2 b_1 \right\},$$

where $t_1/t_2 = x_1x_2^{-1}$. We have embeddings

$$\begin{aligned}
X_{\sigma_{21}} &\hookrightarrow X_{\mathcal{F}(\sigma_{21}, \sigma_{12})}, \quad (u, v) \mapsto ((u, uv), (1 : v));\\
X_{\sigma_{12}} &\hookrightarrow X_{\mathcal{F}(\sigma_{21}, \sigma_{12})}, \quad (u, v) \mapsto ((uv, u), (v : 1));
\end{aligned}$$

the images of which form an open cover of $X_{\mathcal{F}(\sigma_{21},\sigma_{12})}$. Notice that $X_{\mathcal{F}(\sigma_{21},\sigma_{12})}$ is the blow-up of $\mathbb{A}^2_{(x_1,x_2)} \cong U_0$ at the origin.

Similarly, $X_{\mathcal{F}(\sigma_{02},\sigma_{20})}$ is the blow-up of $\mathbb{A}^2_{(x_1^{-1},x_1^{-1}x_2)} \cong U_1$ at the origin, and $X_{\mathcal{F}(\sigma_{01},\sigma_{10})}$ is the blow-up of $\mathbb{A}^2_{(x_2^{-1},x_1x_2^{-1})} \cong U_2$ at the origin. Hence, the toric variety $X_{\Sigma_{Q_3}}$, which can be obtained by gluing $X_{\mathcal{F}(\sigma_{21},\sigma_{12})}$, $X_{\mathcal{F}(\sigma_{02},\sigma_{20})}$ and $X_{\mathcal{F}(\sigma_{01},\sigma_{10})}$, is the blow-up of $\mathbb{P}^2$ at the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$:

$$X_{\Sigma_{Q_3}} = \Big\{ \big((s_0:s_1:s_2),(t_1:t_2),(t_1':t_2'),(t_1'':t_2'')\big) \in \mathbb{P}^2 \times (\mathbb{P}^1)^3 : \\ s_1t_2 = s_2t_1,\ s_0t_2' = s_2t_1',\ s_0t_2'' = s_1t_1'' \Big\}.$$

For general $n \in \mathbb{Z}_{\geq 1}$, the fan $\Sigma_{Q_{n+1}}$ refines the fan $\Sigma_{\mathbb{P}^n}$, and we have the following.

Proposition 6.27. *Let $n \in \mathbb{Z}_{\geq 1}$. The toric variety $X_{\Sigma_{Q_{n+1}}}$ can be obtained by the successive blow-ups*

$$X_{\Sigma_{Q_{n+1}}} = Y_{n-1} \to \ldots \to Y_1 \to Y_0 = \mathbb{P}^n,$$

where $Y_{i+1} \to Y_i$ is the blow-up along the proper transforms of the coordinate planes of dimension i in $\mathbb{P}^n$.

Proof. See [17, Prop. 4.3.13] (cf. [18, pp. 59–60]). □

6.5 Preliminary work: construction of nef divisors α and β

This subsection follows [18, §11.2], without giving every detail. Let $\pi_1 : X_{\Sigma_{Q_{n+1}}} \to \mathbb{P}^n$ denote the composition of blow-ups from Proposition 6.27. Consider the fan

$$-\Sigma_{\mathbb{P}^n} = \{-\sigma : \sigma \in \Sigma_{\mathbb{P}^n}\},$$

where $\Sigma_{\mathbb{P}^n}$ is as in Example 2.9. The toric variety $X_{-\Sigma_{\mathbb{P}^n}}$ is isomorphic to $\mathbb{P}^n$, just like $X_{\Sigma_{\mathbb{P}^n}}$, but its torus action is given by $t \circ_- (a_0 : \ldots : a_n) = t^{-1} \circ_+ (a_0 : \ldots : a_n)$, where '$\circ_+$' denotes the torus action on $X_{\Sigma_{\mathbb{P}^n}}$.

In the same way as the fan $\Sigma_{Q_{n+1}}$ refines the fan $\Sigma_{\mathbb{P}^n}$ and gives rise to a blow-up $\pi_1 : X_{\Sigma_{Q_{n+1}}} \to X_{\Sigma_{\mathbb{P}^n}} \cong \mathbb{P}^n$ (see Example 6.26 and Proposition 6.27), the fan $\Sigma_{Q_{n+1}}$ refines the fan $-\Sigma_{\mathbb{P}^n}$ and gives rise to a blow-up $\pi_2 : X_{\Sigma_{Q_{n+1}}} \to X_{-\Sigma_{\mathbb{P}^n}} \cong \mathbb{P}^n$. It turns out that π_2 resolves the locus of indeterminacy of the Cremona transformation $\mathrm{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ (see Example 6.21). In fact, the birational map $\mathrm{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ extends to an automorphism $\widetilde{\mathrm{Crem}} : X_{\Sigma_{Q_{n+1}}} \to X_{\Sigma_{Q_{n+1}}}$ that makes the following diagram commute:

$$\begin{tikzcd} X_{\Sigma_{Q_{n+1}}} \arrow[r, "\widetilde{\mathrm{Crem}}"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_2"] & X_{\Sigma_{Q_{n+1}}} \arrow[d, "\pi_1"] \\ \mathbb{P}^n \arrow[r, dashed, "\mathrm{Crem}"'] & \mathbb{P}^n. \end{tikzcd}$$

Now, let $V \subseteq \mathbb{C}^{n+1}$ be a $d+1$-dimensional subspace which is not contained in any hyperplane $x_i = 0$. Its projectivization $\mathbb{P}(V) \subseteq \mathbb{P}^n$ is not contained in the blow-up locus of $\pi_1 : X_{\Sigma_{Q_{n+1}}} \to \mathbb{P}^n$; hence, we can consider its proper transform $\widetilde{\mathbb{P}(V)} \subseteq X_{\Sigma_{Q_{n+1}}}$.

Lemma 6.28. *$\widetilde{\mathbb{P}(V)}$ is a d-dimensional complete irreducible variety.*

Proof. Since the subspace $V \subseteq \mathbb{C}^{n+1}$ has dimension $d+1$, its projectivization $\mathbb{P}(V) \subseteq \mathbb{P}^n$ has dimension d. Furthermore, $\mathbb{P}(V)$ is complete, and it is irreducible since V is irreducible. Now the result follows by noting that dimension, completeness and irreducibility are preserved under taking the proper transform. $\square$

Consider the piecewise linear function

$$a : \bigcup \Sigma_{\mathbb{P}^n} = \mathbb{R}^n \to \mathbb{R}, \quad (x_1, \dots, x_n) \mapsto \min\{0, x_1, \dots, x_n\}$$

on the fan $\Sigma_{\mathbb{P}^n}$, and notice that a is convex: we have

$$\begin{aligned} &\min\{0, \lambda v_1 + (1-\lambda) w_1, \dots, \lambda v_n + (1-\lambda) w_n\} \\ &\geq \lambda \min\{0, v_1, \dots, v_n\} + (1-\lambda) \min\{0, w_1, \dots, w_n\} \end{aligned}$$

for all $(v_1, \dots, v_n), (w_1, \dots, w_n) \in \mathbb{R}^n$ and $\lambda \in [0,1]$.

By Proposition 3.36, the Cartier divisor $\varphi_{\mathrm{Div}}(a)$ on $X_{\Sigma_{\mathbb{P}^n}} \cong \mathbb{P}^n$ is nef. As a consequence of the so-called *projection formula* (see [15, A4, p. 426]), the pullback of a nef divisor along a proper morphism is nef. Therefore, the pullback $\alpha = \pi_1^* \varphi_{\mathrm{Div}}(a)$ of $\varphi_{\mathrm{Div}}(a)$ along the proper morphism $\pi_1 : X_{\Sigma_{Q_{n+1}}} \to \mathbb{P}^n$ is a nef Cartier divisor on $X_{\Sigma_{Q_{n+1}}}$.

Similarly, the convex piecewise linear function

$$b : \bigcup -\Sigma_{\mathbb{P}^n} = \mathbb{R}^n \to \mathbb{R}, \quad (x_1, \dots, x_n) \mapsto \min\{0, -x_1, \dots, -x_n\}$$

on the fan $-\Sigma_{\mathbb{P}^n}$ gives rise to the nef Cartier divisor $\beta = \pi_2^* \varphi_{\mathrm{Div}}(b)$ on $X_{\Sigma_{Q_{n+1}}}$.

Remark 6.29. Since $\widetilde{\mathbb{P}(V)}$ is a closed subvariety of $X_{\Sigma_{Q_{n+1}}}$, the nef Cartier divisors α and β on $X_{\Sigma_{Q_{n+1}}}$ can also be considered as nef Cartier divisors on $\widetilde{\mathbb{P}(V)}$, by pullback along the inclusion $\widetilde{\mathbb{P}(V)} \hookrightarrow X_{\Sigma_{Q_{n+1}}}$.

6.6 The Minkowski weight Δ_M

Let $M = (E, \mathscr{I})$ be a representable matroid of rank $d+1$ on the ground set $E = \{0, \dots, n\}$. Choose a $d+1$-dimensional subspace $V \subseteq k^{n+1}$ such that M is the matroid M_V of Example 4.12.

We will prove that the characteristic polynomial $\chi_M(t)$ of M is log-concave, following the proof in [18, §§11–12] (cf. [3, 4.8]). For simplicity we will assume that $k = \mathbb{C}$, although this is not strictly necessary. By Proposition 6.8 (1) we may assume that M has no loops (otherwise there is nothing to prove). Then we know from Example 4.12 that V is not contained in any coordinate hyperplane $x_j = 0$ of $\mathbb{C}^{n+1}$.

The desired result will ultimately follow by applying the following theorem with the nef divisors α and β as defined in § 6.5 and $X = \widetilde{\mathbb{P}(V)}$.

Theorem 6.30 (Khovanskii-Teissier inequality)**.** *Let X be a d-dimensional complete irreducible variety, and let α, β be nef divisors on X. Then the sequence*

$$\deg\left(\alpha^{d-0}\beta^0 \cdot [X]\right), \ldots, \deg\left(\alpha^{d-d}\beta^d \cdot [X]\right)$$

is log-concave.

Proof. See [18, Thm. 12.2]. □

Write $\Sigma = \Sigma_{Q_{n+1}} = \Sigma_{U_{n+1,n+1}}$ and define the map

$$\Delta_M : \Sigma^{(n-d)} \to \mathbb{Z}, \quad \sigma_{F_\bullet} \mapsto \begin{cases} 1, & \text{if } F_\bullet \text{ is a flag of } M \\ 0, & \text{otherwise.} \end{cases}$$

We have the following result (see [18, Lem. 11.11, 11.12], where only the first lemma, stating that Δ_M is a Minkowski weight, is proved). We will sketch the proof and then give an example which clarifies the non-rigorous part of the proof sketch.

Proposition 6.31. *The map Δ_M is a Minkowski weight. Furthermore, its associated Chow cohomology class $\gamma^{-1}(\Delta_M) \in A^{n-d}(X_\Sigma)$ is $\left[\widetilde{\mathbb{P}(V)}\right]$.*

Proof sketch. Let $\sigma_{F_\bullet} \in \Sigma^{(n-d)}$, where $F_\bullet = (F_1, \ldots, F_d)$ is a d-step flag of the matroid $U_{n+1,n+1}$ (cf. Proposition 4.18). By Theorem 3.30 it suffices to show that $\deg\left(\left[\widetilde{\mathbb{P}(V)}\right] \cdot [V(\sigma_{F_\bullet})]\right)$ equals 1 if $F_\bullet$ is a flag of M, and 0 otherwise. By Example 4.12 we know that $F_\bullet$ is a flag of M if and only if for all $i = 1, \ldots, d$,

$$\forall e \in E \setminus F_i \ \exists (x_0, \ldots, x_n) \in V : x_j = 0 \text{ for } j \in F_i \text{ and } x_e \neq 0.$$

Since V is a linear subspace, this is equivalent to

$$\exists (x_0, \ldots, x_n) \in V : x_j = 0 \text{ if and only if } j \in F_i. \tag{4}$$

By Proposition 3.22 we have

$$[V(\sigma_{F_\bullet})] = [V(\mathcal{C}(e_{F_1}, \ldots, e_{F_d}))] = [V(\mathcal{C}(e_{F_1}))] \cdots [V(\mathcal{C}(e_{F_d}))],$$

which implies that it is sufficient to show that

$$\deg\left(\left[\widetilde{\mathbb{P}(V)}\right] \cdot [V(\mathcal{C}(e_{F_1}))] \cdots [V(\mathcal{C}(e_{F_d}))]\right) = \begin{cases} 1, & \text{if (4) holds for } i = 1, \ldots, d \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, it can be shown that the subvarieties $\widetilde{\mathbb{P}(V)}, V(\mathcal{C}(e_{F_1})), \ldots, V(\mathcal{C}(e_{F_d})) \subseteq X_\Sigma$ intersect transversally and that we have

$$\widetilde{\mathbb{P}(V)} \cap V(\mathcal{C}(e_{F_1})) \cap \ldots \cap V(\mathcal{C}(e_{F_d})) = \begin{cases} 1, & \text{if (4) holds for } i = 1, \ldots, d \\ 0, & \text{otherwise.} \end{cases}$$

□

Example 6.32. In Example 6.26 we computed the toric variety $X_{\Sigma_{Q_3}}$ explicitly.

Now we will study the subvariety $\widetilde{\mathbb{P}(V)} \subseteq X_{\Sigma_{Q_3}}$, for the hyperplane

$$V = \{(x_0, x_1, x_2) \in \mathbb{C}^3 : x_0 + x_1 = 0\} \subseteq \mathbb{C}^3,$$

and the intersections of $\widetilde{\mathbb{P}(V)}$ with the orbit closures of the six rays

$$\begin{aligned} \rho_0 &= \mathcal{C}(e_0),\ \rho_1 = \mathcal{C}(e_1),\ \rho_2 = \mathcal{C}(e_2), \\ \rho_{01} &= \mathcal{C}(e_0 + e_1),\ \rho_{02} = \mathcal{C}(e_0 + e_2),\ \rho_{12} = \mathcal{C}(e_1 + e_2) \end{aligned}$$

of Σ_{Q_3}.

The hyperplane $V \subseteq \mathbb{C}^3$ gives rise to the matroid $M = M_V = (E, \mathscr{I})$, where the ground set $E = \{0, 1, 2\}$ is identified with $\{i^*(e_0^*), i^*(e_1^*), i^*(e_2^*)\} \subseteq V^*$ (see Example 4.12). The independent sets and flats of M are

$$\mathscr{I} = \{\varnothing, \{0\}, \{1\}, \{2\}, \{0,2\}, \{1,2\}\},\ \mathscr{F} = \{\varnothing, \{2\}, \{0,1\}, \{0,1,2\}\},$$

respectively. Notice that a subset $S \subseteq E$ is a flat of M if and only if

$$\exists (x_0, x_1, x_2) \in V : x_j = 0 \text{ if and only if } j \in S.$$

Furthermore, we have

$$\begin{aligned} \mathbb{P}(V) &= \{(x_0 : x_1 : x_2) \in \mathbb{P}^2 : x_0 + x_1 = 0\}; \\ \widetilde{\mathbb{P}(V)} &= \overline{\pi_1^{-1}(\mathbb{P}(V) \setminus \{(0:0:1)\})} = \pi_1^{-1}(\mathbb{P}(V) \setminus \{(0:0:1)\}) \cup \{P\}; \end{aligned}$$

where $\pi_1 : X_{\Sigma_{Q_3}} \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at the points $p_0 = (1:0:0)$, $p_1 = (0:1:0)$ and $p_2 = (0:0:1)$ (see Example 6.26) and

$$P = (p_2, (0:1), (0:1), (1:-1)) \in \pi_1^{-1}(p_2).$$

We will now compute the orbit closures of the six rays in Σ_{Q_3}. The orbit closure of a cone $\sigma \in \Sigma_{\mathbb{P}^2}$ will be denoted by $V_{\mathbb{P}^2}(\sigma)$ and its proper transform in $X_{\Sigma_{Q_3}}$ by $\widetilde{V_{\mathbb{P}^2}(\sigma)}$. It follows from

$$V_{\mathbb{P}^2}(\rho_0) = \{(0 : x_1 : x_2) \in \mathbb{P}^2 : x_1, x_2 \in \mathbb{C}\}$$

(see [18, Example 10.2]) that

$$\begin{aligned} V(\rho_0) = \widetilde{V_{\mathbb{P}^2}(\rho_0)} &= \overline{\pi_1^{-1}(\{(0 : x_1 : x_2) \in \mathbb{P}^2 : x_1, x_2 \in \mathbb{C}\} \setminus \{p_1, p_2\})} \\ &= \pi_1^{-1}(\{(0 : x_1 : x_2) \in \mathbb{P}^2 : x_1, x_2 \neq 0\}) \cup \{P_0, Q_0\} \end{aligned}$$

for some points $P_0 \in \pi_1^{-1}(p_2)$ and $Q_0 \in \pi_1^{-1}(p_1)$. Similarly, we have

$$\begin{aligned} V(\rho_1) &= \pi_1^{-1}(\{(x_0 : 0 : x_2) \in \mathbb{P}^2 : x_0, x_2 \neq 0\}) \cup \{P_1, Q_1\}; \\ V(\rho_2) &= \pi_1^{-1}(\{(x_0 : x_1 : 0) \in \mathbb{P}^2 : x_0, x_1 \neq 0\}) \cup \{P_2, Q_2\}; \end{aligned}$$

where

$$\begin{aligned} P_0 &= (p_2, (0:1), (0:1), (0:1)),\ Q_0 = (p_1, (1:0), (0:1), (0:1)); \\ P_1 &= (p_2, (0:1), (0:1), (1:0)),\ Q_1 = (p_0, (0:1), (1:0), (1:0)); \\ P_2 &= (p_1, (1:0), (1:0), (0:1)),\ Q_2 = (p_0, (1:0), (1:0), (1:0)). \end{aligned}$$

Furthermore, we have

$$V(\rho_{01}) = \pi_1^{-1}(p_2),\ V(\rho_{02}) = \pi_1^{-1}(p_1),\ V(\rho_{12}) = \pi_1^{-1}(p_0)$$

(see [18, p. 52]).

Observe that for the six non-empty proper subsets $\varnothing \subsetneq S \subsetneq E$ (i.e. for the proper flats S of $U_{3,3}$; the corresponding cones $\mathcal{C}(e_S)$ are the rays of Σ_{Q_3}) we have

$$\left|\widetilde{\mathbb{P}(V)} \cap V(\mathcal{C}(e_S))\right| = \begin{cases} 1, & \text{if } S \text{ is a flat of } M \\ 0, & \text{otherwise.} \end{cases}$$

The following proposition (cf. [18, Cor. 11.16]), a proof sketch of which will be given in the next subsection, is the last ingredient needed for proving the log-concavity of $\chi_M(t)$.

Proposition 6.33. *The Chow cohomology class $\gamma^{-1}(\Delta_M) \in A^{n-d}(X_\Sigma)$, associated to the Minkowski weight Δ_M, satisfies $\mu^r = \deg\left(\alpha^{d-r}\beta^r \cdot \gamma^{-1}(\Delta_M)\right)$ for $r = 0, \dots, d$.*

Proof that $\chi_M(t)$ is log-concave. By Proposition 6.31 and Proposition 6.33 we know that $\mu^r = \deg\left(\alpha^{d-r}\beta^r \cdot \left[\widetilde{\mathbb{P}(V)}\right]\right)$ for $r = 0, \dots, d$. So it follows from Lemma 6.28, Remark 6.29 and Theorem 6.30 that $\overline{\chi}_M(t)$ is log-concave. Hence, by Lemma 6.17, $\chi_M(t)$ is log-concave as well. □

Example 6.34. Reconsider Example 6.32. We will work out the identities

$$\mu^r = \deg\left(\alpha^{1-r}\beta^r \cdot \left[\widetilde{\mathbb{P}(V)}\right]\right), \quad r = 0, 1,$$

for the matroid $M = M_V = (E, \mathscr{I})$ on the ground set $E = \{0, 1, 2\}$, where

$$V = \{(x_0, x_1, x_2) \in \mathbb{C}^3 : x_0 + x_1 = 0\} \subseteq \mathbb{C}^3.$$

Notice that M has rank $d + 1 = 2$.

The characteristic polynomial of M is

$$\begin{aligned} \chi_M(t) &= \sum_{A \subseteq E} (-1)^{|A|} t^{\mathrm{rk}(M) - \mathrm{rk}_M(A)} \\ &= (-1)^0 t^{2-0} + (-1)^1 \cdot 3t^{2-1} + (-1)^2 (t^{2-1} + 2t^{2-2}) + (-1)^3 t^{2-2} \\ &= t^2 - 2t + 1. \end{aligned}$$

Therefore, the reduced characteristic polynomial of M is

$$\overline{\chi}_M(t) = \sum_{i=0}^{1} (-1)^i \mu^i t^{1-i} = t - 1;$$

i.e. we have $\mu^0 = \mu^1 = 1$.

The nef divisors on $X_{\Sigma_{Q_3}}$, associated to the piecewise linear functions

$$\begin{aligned} a &: \mathbb{R}^2 \to \mathbb{R}, \ (x_1, x_2) \mapsto \min\{0, x_1, x_2\}; \\ b &: \mathbb{R}^2 \to \mathbb{R}, \ (x_1, x_2) \mapsto \min\{0, -x_1, -x_2\}; \end{aligned}$$

on $\Sigma_{\mathbb{P}^2}$, are

$$\begin{aligned}\alpha = \pi_1^* \varphi_{\mathrm{Div}}(a) &= -\sum_{0\le i\le 2} a(e_i)[V(\rho_i)] - \sum_{0\le i<j\le 2} a(e_i+e_j)[V(\rho_{ij})]\\ &= [V(\rho_0)] + [V(\rho_{01})] + [V(\rho_{02})];\\ \beta = \pi_2^* \varphi_{\mathrm{Div}}(b) &= -\sum_{0\le i\le 2} b(e_i)[V(\rho_i)] - \sum_{0\le i<j\le 2} b(e_i+e_j)[V(\rho_{ij})]\\ &= [V(\rho_1)] + [V(\rho_2)] + [V(\rho_{12})].\end{aligned}$$

Furthermore, since the subsets $\{2\}, \{0,1\} \subseteq E$ are flats of M and the subsets $\{0\}, \{1\}, \{0,2\}, \{1,2\} \subseteq E$ are not, we have the following Minkowski weight:

$$\Delta_M : \Sigma^{(1)} \to \mathbb{Z}, \quad \rho \mapsto \begin{cases} 1, & \text{if } \rho \in \{\rho_2, \rho_{01}\} \\ 0, & \text{if } \rho \in \{\rho_0, \rho_1, \rho_{02}, \rho_{12}\}. \end{cases}$$

As we have seen in Example 6.32, the Chow cohomology class $\left[\widetilde{\mathbb{P}(V)}\right]$, associated to Δ_M, satisfies

$$\left|\widetilde{\mathbb{P}(V)} \cap V(\rho)\right| = \begin{cases} 1, & \text{if } \rho \in \{\rho_2, \rho_{01}\} \\ 0, & \text{if } \rho \in \{\rho_0, \rho_1, \rho_{02}, \rho_{12}\}. \end{cases}$$

Hence, since $\widetilde{\mathbb{P}(V)}$ intersects the orbit closures of $X_{\Sigma_{Q_3}}$ transversally, we indeed find

$$\mu^0 = \deg\left(\alpha \cdot \left[\widetilde{\mathbb{P}(V)}\right]\right) = \deg\left(([V(\rho_0)] + [V(\rho_{01})] + [V(\rho_{02})]) \cdot \left[\widetilde{\mathbb{P}(V)}\right]\right) = 1;$$
$$\mu^1 = \deg\left(\beta \cdot \left[\widetilde{\mathbb{P}(V)}\right]\right) = \deg\left(([V(\rho_1)] + [V(\rho_2)] + [V(\rho_{12})]) \cdot \left[\widetilde{\mathbb{P}(V)}\right]\right) = 1.$$

6.7 Truncations of matroids

In order to sketch the proof of Proposition 6.33, we need to introduce the notion of *truncations* of matroids, following [18, §11.4]. Let M be as in the previous subsection.

Definition 6.35. For $r_2 \in \{1, \dots, d+1\}$ the r_2-*truncation* of $M = (E, \mathscr{I})$ is the matroid $\mathrm{Trunc}^{r_2}(M) = (E, \mathscr{I}_{r_2})$, where $\mathscr{I}_{r_2} = \{X \in \mathscr{I} : |X| \le r_2\}$.

The notion of truncated matroids can be generalized as follows.

Definition 6.36. Let $r_1, r_2 \in \{1, \dots, d+1\}$ with $r_1 \le r_2$. The (r_1, r_2)-*truncation* of $M = (E, \mathscr{I})$ is the map $\Delta_{M[r_1,r_2]} : \Sigma^{(n+r_1-r_2-1)} \to \mathbb{Z}$, defined by

$$\sigma_{(F_{r_1},\dots,F_{r_2})} \mapsto \begin{cases} |\mu(\varnothing, F_{r_1})|, & \text{if each } F_i \text{ is a flat of } M \text{ of rank } i \\ 0, & \text{otherwise.} \end{cases}$$

As the following remark shows, the notion of an (r_1, r_2)-truncation is indeed a generalization of the notion of an r_2-truncation.

Remark 6.37. Let $r_1, r_2, j \in \{1, \dots, d\}$ with $r_1 \le r_2 \le j$.

(1) We have $\Delta_{M[1,r_2]} = \Delta_{\mathrm{Trunc}^{r_2+1}(M)}$. In particular, $\Delta_{M[1,r_2]}$ is a Minkowski weight.

(2) We have $\Delta_{M[r_1,r_2]} = \Delta_{\mathrm{Trunc}^{j+1}(M)[r_1,r_2]}$.

The following proposition (cf. [18, Prop. 11.15]) not only shows that $\Delta_{M[r_1,r_2]}$ is a Minkowski weight for general r_1, but it also gives a number of identities from which Proposition 6.33 easily follows.

Proposition 6.38. *Let $r_1, r_2, r \in \{1, \dots, d\}$ with $r_1 \leq r_2$.*

(1) *The map $\Delta_{M[r_1,r_2]} : \Sigma^{(n+r_1-r_2-1)} \to \mathbb{Z}$ is a Minkowski weight on Σ, satisfying $\alpha^{d-r_2}\beta^{r_1-1} \cdot \gamma^{-1}(\Delta_M) = \gamma^{-1}(\Delta_{M[r_1,r_2]})$.*
(2) *We have $\mu^{r-1} = \deg(\alpha \cdot \gamma^{-1}(\Delta_{M[r,r]}))$.*
(3) *We have $\mu^r = \deg(\beta \cdot \gamma^{-1}(\Delta_{M[r,r]}))$.*

Proof of Proposition 6.33. Let $r \in \{1, \dots, d\}$. By Proposition 6.38 we find

$$\begin{aligned}
\mu^{r-1} &= \deg(\alpha \cdot \gamma^{-1}(\Delta_{M[r,r]})) = \deg(\alpha \cdot (\alpha^{d-r}\beta^{r-1} \cdot \gamma^{-1}(\Delta_M))) \\
&= \deg\left(\alpha^{d-(r-1)}\beta^{r-1} \cdot \gamma^{-1}(\Delta_M)\right); \\
\mu^{r} &= \deg(\beta \cdot \gamma^{-1}(\Delta_{M[r,r]})) = \deg(\beta \cdot (\alpha^{d-r}\beta^{r-1} \cdot \gamma^{-1}(\Delta_M))) \\
&= \deg\left(\alpha^{d-r}\beta^{r} \cdot \gamma^{-1}(\Delta_M)\right).
\end{aligned}$$

□

Proof of Proposition 6.38 (1). By abuse of notation we will identify a Minkowski weight c with the corresponding Chow cohomology class $\gamma^{-1}(c)$. Let $j \in \{2, \dots, d\}$. It is shown in [18, Lem. 12.4-12.5] that for a rank $j+1$ matroid $M' = (E, \mathscr{I}')$ on the ground set $E = \{0, \dots, n\}$, we have

$$\alpha \cdot \Delta_{M'} = \Delta_{M'[1,j-1]}; \tag{5}$$

$$\beta \cdot \Delta_{M'[i,j]} = \Delta_{M'[i+1,j]}, \quad 1 \leq i < j. \tag{6}$$

Notice that it follows inductively from (6) that $\Delta_{M'[i,j]}$ is a Minkowski weight for $1 \leq i < j$.

By (5), (6) and Remark 6.37 we find that for all $i, j \in \{1, \dots, d\}$ with $i < j$,

$$\begin{aligned}
\alpha \cdot \Delta_{M[1,j]} &= \alpha \cdot \Delta_{\mathrm{Trunc}^{j+1}(M)} = \Delta_{\mathrm{Trunc}^{j+1}(M)[1,j-1]} = \Delta_{M[1,j-1]}; \\
\beta \cdot \Delta_{M[i,j]} &= \beta \cdot \Delta_{\mathrm{Trunc}^{j+1}(M)[i,j]} = \Delta_{\mathrm{Trunc}^{j+1}(M)[i+1,j]} = \Delta_{M[i+1,j]}.
\end{aligned}$$

The desired result is now obtained by repeatedly applying these identities and using that $\Delta_M = \Delta_{\mathrm{Trunc}^{d+1}(M)} = \Delta_{M[1,d]}$. □

Proof of Proposition 6.38 (2), (3). We will follow the proof in [18, Lem. 12.6] and fill in the details that are left out there. The following identity is from [18, p. 55] (a proof can be found in [2, Construction 3.3]). Let $c \in \mathcal{M}^i(\Sigma)$ be a Minkowski weight. For $\sigma \in \Sigma^{(i)}$ and $\tau \in \Sigma^{(i+1)}$ with $\tau < \sigma$, let $u_{\sigma/\tau} \in N_\sigma$ be a representative of a generator $v_{\sigma/\tau}$ of N_σ / N_τ. Let $d \in \mathrm{PLF}(\Sigma)$ be a piecewise linear function and δ the corresponding Cartier divisor on X_Σ (as in § 6.5).

Then we have

$$\gamma(\delta \cdot \gamma^{-1}(c))(\tau) = d\left(\sum_{\substack{\sigma\in\Sigma^{(i)}\\ \tau<\sigma}} c(\sigma)u_{\sigma/\tau}\right) - \sum_{\substack{\sigma\in\Sigma^{(i)}\\ \tau<\sigma}} c(\sigma)d(u_{\sigma/\tau}).$$

Let $(d,\delta) \in \{(a,\alpha),(b,\beta)\}$. Applying this identity with $i = n-1$, $\tau = \{0\}$ and $c = \Delta_{M[r,r]}$ gives

$$\begin{aligned}
\deg(\delta \cdot \gamma^{-1}(\Delta_{M[r,r]})) &= \deg(\delta \cdot \gamma^{-1}(\Delta_{M[r,r]}) \cdot [V(\{0\})]) \\
&= \gamma(\delta \cdot \gamma^{-1}(\Delta_{M[r,r]}))(\{0\}) \\
&= d\left(\sum_{\sigma\in\Sigma^{(n-1)}} \Delta_{M[r,r]}(\sigma)u_{\sigma/\tau}\right) - \sum_{\sigma\in\Sigma^{(n-1)}} \Delta_{M[r,r]}(\sigma)d(u_{\sigma/\tau}) \\
&= d\left(\sum_{F\in\mathscr{F}_r} |\mu(\varnothing,F)|e_F\right) - \sum_{F\in\mathscr{F}_r} |\mu(\varnothing,F)|d(e_F),
\end{aligned}$$

where $\mathscr{F}_r$ is the set of rank r flats of M. Let $F \in \mathscr{F}_r$. It follows from $0 < r < d+1$ that $F \neq \varnothing, E$. So if $0 \in F$, then $e_F \in \{0,-1\}^n \setminus \{(0,\dots,0)\}$, and if $0 \notin F$, then $e_F \in \{0,1\}^n \setminus \{(0,\dots,0)\}$. Set

$$A_\omega = \sum_{\substack{F\in\mathscr{F}_r\\ \omega}} |\mu(\varnothing,F)|,$$

where F ranges over the flats in $\mathscr{F}_r$ that satisfy condition(s) ω. Then we find by Proposition 6.13,

$$\begin{aligned}
\deg(\alpha \cdot \gamma^{-1}(\Delta_{M[r,r]})) &= a\left(\sum_{F\in\mathscr{F}_r} |\mu(\varnothing,F)|e_F\right) - \sum_{F\in\mathscr{F}_r} |\mu(\varnothing,F)|a(e_F) \\
&= \min_{e\in E}\left(A_{F\not\ni 0, F\ni e} - A_{F\ni 0, F\not\ni e}\right) + A_{F\ni 0} \\
&= \min_{e\in E}\left(A_{F\ni e}\right) = \min_{e\in E}(\mu^{r-1}) = \mu^{r-1}; \\
\deg(\beta \cdot \gamma^{-1}(\Delta_{M[r,r]})) &= b\left(\sum_{F\in\mathscr{F}_r} |\mu(\varnothing,F)|e_F\right) - \sum_{F\in\mathscr{F}_r} |\mu(\varnothing,F)|b(e_F) \\
&= \min_{e\in E}\left(A_{F\ni 0, F\not\ni e} - A_{F\not\ni 0, F\ni e}\right) + A_{F\not\ni 0} \\
&= \min_{e\in E}\left(A_{F\not\ni e}\right) = \min_{e\in E}(\mu^{r}) = \mu^{r}.
\end{aligned}$$

□

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