Heights in Arakelov Geometry

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Contents

Introduction

In this thesis we aim to do two things, in the first three sections we develop some arithmetic intersection theory in the style of Gillet-Soulé. When doing intersection theory one uses Chow's moving lemma to move divisors to rational equivalent ones so that they intersect properly. When doing intersection theory over fields the intersection numbers you get this way by taking degrees only depend on the rational equivalence class of a divisor, however in case of $\operatorname{Spec} \mathbb{Z}$ the degree of a non-zero rational function is non-zero. This is remedied by in addition to the intersection theory over $\operatorname{Spec} \mathbb{Z}$, considering an analogous theory on the complex points. Here we consider smooth hermitian line bundles and green currents associated to divisors. For (green) currents there is a $*$-product which satisfies properties analogous to the product in ordinary intersection theory. We have tried to present the results in a way that showcases the similarities and the results we use in arithmetic intersection theory boil down to similar statements holding for both the intersection product and the $*$-product.

The other thing we are interested in is heights. In diophantine geometry heights are used to control the number of rational points, they are used for finiteness statements or describing distributions of infinitely many points for example. First we use the arithmetic intersection theory from section 3 to define a global height for arithmetic varieties. Next in section 4 we work with limits of models in the style of Zhang to accomplish a number of things. First by considering p-adic norms the treatment of the finite primes and the infinite prime become more similar. Second by considering limits of models we enlarge the norms and intersection numbers available to us, for example metrics at infinity don't have to be smooth anymore. We define local heights for each prime p and show that these converge under some assumptions on the line bundles. We can decompose the global height as a sum of local heights, the global height also converges under some assumptions. We also consider metrics associated to an algebraic dynamical system, i.e. we have a surjective morphism $f : X \to X$ of a smooth integral projective variety over $\mathbb{Q}$ such that $f^*L \cong L^{\otimes d}$ for some line bundle L and some $d > 0$. By a limit argument we obtain a metric on L that is invariant under f^*. In section 5 we apply this when X is an abelian variety, f is multiplication by $n > 1$ and L is a symmetric line bundle. The height obtained from the invariant metric in this case is the Neron-Tate height and we prove some of its elementary properties.

1 Intersection theory

We define an arithmetic variety X to be a normal integral scheme that is projective and flat over $\operatorname{Spec}(\mathbb{Z})$ such that the generic fiber is smooth over $\mathbb{Q}$. We denote the generic point of X by η or η_X depending on context.

In this section we develop intersection theory on excellent schemes. Though we won't apply it in its full generality. We will apply it to arithmetic varieties in the third section, but also to varieties over fields and $\mathbb{Z}_p$. We mostly focus on the geometric aspects and will refer to other sources for the commutative algebra running behind the scenes.

1.1 Intersection theory on excellent schemes

Let X be an integral excellent scheme and let $R(X) = \mathcal{O}_{X,\eta}$ be the field of rational functions on X. We let the group of codimension p cycles $Z^p(X)$ be the free abelian group generated by the set $\{\overline{\{x\}}\}_{x\in X^{(p)}}$, where $X^{(p)}$ is the set of codimension p points x, i.e. $\dim \mathcal{O}_{X,x} = p$. Similarly we define the group of l-cycles $Z_l(X)$, the free abelian group generated by dimension l points, i.e. $\dim \overline{\{x\}} = l$. If $l + p = \dim X$ then these coincide because arithmetic varieties are excellent. For a codimension p cycle $Z = \sum_x n_x\overline{\{x\}}$ in $Z^p(X)$ we define the support to be

$$\operatorname{Supp} Z = \bigcup_{n_x\neq 0} \overline{\{x\}}.$$

Now let L be a line bundle on X and s a non-zero rational section of L, i.e. $0 \neq s \in L_\eta$. We define the support of (L, s) and the divisor associated to (L, s) as follows. We set

$$\operatorname{Supp}(L, s) := \{x \in X | s_x \in \mathfrak{m}_x L_x\},$$

$$\operatorname{div}(L, s) := \sum_{x\in X^{(1)}} \operatorname{ord}_x(s)\overline{\{x\}} (\in Z^1(X)).$$

Here ord_x is defined as follows: let b be a local basis for L around x, then $s_x = fb$ for some rational function f. Then f can be written as a ratio $\frac{g}{h}$ of two regular functions. We set the order of vanishing of a regular function g at x as the length $l_{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}/(g))$ of $\mathcal{O}_{X,x}$, and we set $\operatorname{ord}_x(s) = \operatorname{ord}_x(g) - \operatorname{ord}_x(h)$. For the many good properties of the length we refer to appendix A of [2]. For us it is important that ord_x is $R(X)^*$-linear on non-zero rational sections and that it agrees with the discrete valuation at a codimension 1 point if X is normal.

Finally $\operatorname{Supp}(\operatorname{div}(L, s))$ is defined as above since $\operatorname{div}(L, s) \in Z^1(X)$. The set $\operatorname{Supp}(L, s)$ is closed and it always contains $\operatorname{Supp}(\operatorname{div}(L, s))$ in general, when X is normal both notions coincide[1].

Let $Z = \sum_{i=1}^r n_i Z_i$ be a codimension p-cycle. We say that Z and (L, s) meet properly if $\operatorname{Supp}(L, s) \cap \operatorname{Supp} Z \cap X^{(p)} = \emptyset$. Under this assumption we can define their intersection cycle as follows: We let γ_i be the generic point of Z_i. Since $\gamma_i \notin \operatorname{Supp}(L, s)$ we see that s is a unit at γ_i, i.e. $s_{\gamma_i} \notin \mathfrak{m}_{\gamma_i} L_{\gamma_i}$. Therefore $s|_{Z_i}$

[1] prop 7.2.14(b) in [7]

gives a non-zero rational section of $L|_{Z_i}$ and we define their intersection product as

$$(L, s) \cdot Z = \sum_{i=1}^{r} n_i \mathrm{div}(L|_{Z_i}, s|_{Z_i}) \in Z^{p+1}(X).$$

Since the order of a function at a point is linear, we have $\mathrm{div}(L \otimes L', s \otimes s') = \mathrm{div}(L, s) + \mathrm{div}(L', s')$. Hence $(L \otimes L', s \otimes s') \cdot Z = (L, s) \cdot Z + (L', s') \cdot Z$. So our product is linear in (L, s), and clearly it is also linear in Z, this is of course under the assumption everything intersects properly.

Proposition 1.1. *Suppose (L_1, s_1) and (L_2, s_2) are line bundles on X with rational sections and let Z be a cycle, such that they all intersect properly, then the product is commutative in the following sense*

$$(L_1, s_1) \cdot ((L_2, s_2) \cdot Z) = (L_2, s_2) \cdot ((L_1, s_1) \cdot Z).$$

Proof. We may assume Z is integral by linearity. Then we may assume $Z = X$, since writing $W = \mathrm{div}(L_j|_Z, s_j|_Z)$ we have

$$\mathrm{div}(L_i|_W, s_i|_W) = \mathrm{div}((L_i|_Z)|_W, (s_i|_Z)|_W).$$

Then we need to check that

$$(L_1, s_1) \cdot \mathrm{div}(L_2, s_2)$$

and

$$(L_2, s_2) \cdot \mathrm{div}(L_1, s_1)$$

have the same multiplicities at codimension 2 (inside Z) points of $\mathrm{Supp}(L_1, s_1) \cap \mathrm{Supp}(L_2, s_2)$. This is again a statement about lengths and we refer to appendix A of [2] or chapter 1.3 of [9]. □

Next we define the push-forward of cycles. Let $f : X \to Y$ be a proper morphism of excellent schemes and let Γ be an irreducible closed subscheme of X of dimension l. Then $f(\Gamma)$ is an irreducible closed subscheme of Y of dimension at most l. We set

$$f_*(\Gamma) = \begin{cases} [R(\Gamma) : R(f(\Gamma))] f(\Gamma), & \text{if } \dim f(\Gamma) = l \\ 0, & \text{otherwise} \end{cases},$$

and extend linearly to get a homomorphism $f_* : Z_l(X) \to Z_l(Y)$, clearly this is functorial by basic properties of field extensions.

Lemma 1.2. *Let $f : X \to Y$ be a proper morphism of integral excellent schemes, let $0 \neq \phi \in R(X)$, then $f_*(div(\phi)) = div(N_{R(X)/R(Y)}(\phi))$ if $\dim(X) = \dim(Y)$ and 0 otherwise.*

Proof. First assume $\dim(X) = \dim(Y)$, let $y \in Y$ be a codimension 1 point, since X is integral and f surjective we see that every $x \in f^{-1}(y)$ is a codimension 1 point of X. Since we only need to check the multiplicity at codimension 1 points we may assume $Y = \operatorname{Spec} \mathcal{O}_{Y,y}$ which is the spectrum of a 1-dimensional local integral domain, and

$$X = \operatorname{Spec} B = \operatorname{Spec} \prod_{x_i \in f^{-1}(y)} B_i$$

where $B_i = \mathcal{O}_{X,x_i}$. Note that B is finite over A, the lemma then follows from [9] 1.12.

If $\dim(Y) < \dim(X) - 2$ the result is 0 by definition of the push-forward. So suppose $\dim(Y) = \dim(X) - 1$, then we may assume Y to be irreducible. Let X_η be the generic fiber of $X \to Y$, note that for $Z \subset X$ irreducible $f_* Z$ is 0 unless Z is the closure of a point in the generic fiber, in which case $f_* Z = [R(Z) : R(Y)]Y$. Hence we may replace X by X_η and Y by $\operatorname{Spec} R(Y)$, it then follows from [9] 1.18. □

Now let (L, s) be a line bundle together with a rational section on Y and suppose that $f(\eta_X) \notin \operatorname{Supp}(L, s)$. Then we can consider (f^*L, f^*s) on X, where f^*s is the image of s under $L_{f(\eta_X)} \to f^*(L)_{\eta_X}$. If Z is an l-dimensional cycle on X meeting (f^*L, f^*s) properly, then also f_*Z meets (L, s) properly.

Proposition 1.3. *In the situation above we have the following projection formula:*

$$f_*((f^*L, f^*s) \cdot Z) = (L, s) \cdot f_*Z.$$

Proof. Let $x \in X$, then locally at $f(x) \in Y$ we can write $s = a\omega$ where ω is locally a basis of L and a is a non-zero rational function. Let f_x^* be the local homomorphism $O_{Y,f(x)} \to O_{X,x}$, then we note that $f_x^*(a)$ is a unit at x if and only if a is a unit at $f(x)$. It follows that $\operatorname{Supp}(f^*L, f^*s) = f^{-1}\operatorname{Supp}(L, s)$.

Now by linearity we may assume Z is integral. By the above we see that (L, s) and $f_*(Z)$ again meet properly. Now note that

$$(f^*L, f^*s) \cdot Z = \operatorname{div}(f^*L|_Z, f^*s|_Z) = \operatorname{div}(g^*(L|_{f(Z)}), g^*(s|_{f(Z)})),$$

where $g = f|_Z : Z \to f(Z)$. Taking pushforwards and using that the problem is local on the image we reduce to the case where $X = Z$ and $f : X \to Y$ is surjective, where it is left to show

$$f_*\operatorname{div}(f^*L, f^*s) = [R(X) : R(Y)]\operatorname{div}(L, s).$$

This follows from lemma 1 since locally s is in the image of $R(Y) \subset R(X)$, so that $N_{X/Y}(s) = s^{[R(X):R(Y)]}$. □

We let $\operatorname{Rat}^p(X) \subset Z^p(X)$ be the subgroup generated by rational divisors, that is cycles of the form $\operatorname{div}(\phi)$ where ϕ is a rational function on an integral subscheme Y of codimension $p - 1$ on X. We also denote such a cycle as (ϕ/Y). Then we define the Chow groups $CH^p(X)$ of X to be $Z^p(X)/\operatorname{Rat}^p(X)$, similarly we define $CH_p(X)(= CH^{\dim(X)-p}(X))$.

As another corollary of lemma 1 we see that f_* is compatible with rational equivalence hence we get a well-defined pushforward on chow groups $f_* : CH_p(X) \to CH_p(Y)$.

Lemma 1.4. *Let R be a Noetherian integral domain and X projective integral over R. Let L be a line bundle on X and $x_1, ..., x_r \in X$ points (of codimension at least 1). Then there exists a non-zero rational section s of L such that*

$$Supp(L, s) \cap \{x_1, ..., x_r\} = \emptyset.$$

Proof. Note that we may assume the x_i are closed points by picking points y_i in the closure of each x_i and noting that s not a unit at y_i implies that s is not a unit at x_i. Let $m_1, ..., m_r$ be the maximal ideal sheaves corresponding to the x_i and $I = m_1 \cdots m_r$. Consider the short exact sequence

$$0 \to IL \to L \to L \otimes \mathcal{O}_X/I \cong \bigoplus_{i=1}^{r} L \otimes \kappa(x_i) \to 0.$$

Let $\mathcal{O}(1)$ be very ample, then by [5] III.5.2 we have for $n \gg 0$

$$H^1(X, IL(n)) = 0.$$

From the long exact sequence in cohomology we find that

$$H^0(X, L(n)) \to H^0(X, \bigoplus_{i=1}^{r} L(n) \otimes \kappa(x_i))$$

is surjective. So there exists a global section s' of $L(n)$ with $s'(x_i) \neq 0$ for all i, and for $L = \mathcal{O}_X$ we get a global section t of $\mathcal{O}(n)$ with $t(x_i) \neq 0$ for all i. Hence $s = s'/t$ is a non-zero rational section of L with $s(x_i) \neq 0$ for all i. □

We now consider the intersection product on the Chow groups. Let L be a line bundle and $\alpha \in CH^p$. Let $Z \in Z^p$ represent α. Then by lemma 2 when X is integral and projective over a Noetherian integral domain, e.g. when X is an arithmetic variety, there is a rational section s of L such that (L, s) and Z intersect properly, we define

$$c_1(L) \cdot \alpha = (L, s) \cdot Z \in CH^{p+1}(X)$$

Theorem 1.5. *$c_1(L) \cdot \alpha$ does not depend on the choice of Z or s. Hence it defines a morphism $c_1(L) : CH^* \to CH^{*+1}$.*

Proof. Note that if s' is another rational section of L, then s and s' differ by some rational function ϕ. We find by linearity that

$$(L, s) \cdot Z - (L, s') \cdot Z = \text{div}(\phi|_Z) \in \text{Rat}^*(X).$$

To show independence of Z, let $(\mathcal{O}_Y, \phi_Y)$ be a rational divisor. We may assume Y is integral. Then by commutativity (proposition 1.1) we have

$$(L, s) \cdot \text{div}(\mathcal{O}_Y, \phi) = (\mathcal{O}_Y, \phi_Y) \cdot \text{div}(L|_Y, s|_Y).$$

We need to show the latter is rational. Writing $\text{div}(L|_Y, s|_Y) = \sum_i n_i Z_i$ we find

$$(L, s) \cdot \text{div}(\mathcal{O}_Y, \phi) = \sum_i n_i \text{div}(\mathcal{O}_{Z_i}, \phi|_{Z_i}) \in \text{Rat}^*(X).$$

□

From proposition 1.1 it follows that the order of the L_i does not matter, we have

$$c_1(L_1) \circ ... \circ c_1(L_d) = c_1(L_{\sigma(1)}) \circ ... \circ c_1(L_{\sigma(d)})$$

for any permutation σ, therefore we have a well-defined morphism

$$c_1(L_1, ..., L_d) : CH^* \to CH^{*+d},$$

depending only on $L_1, .., L_d$. More generally we get a well-defined morphism

$$P(L_1, ..., L_d) : CH^* \to CH^{*+d}$$

for P a homogeneous polynomial of degree d, depending only on $L_1, ..., L_d$. Then as a corollary of proposition 1.3 we have

$$f_* P(f^* L_1, ..., f^* L_d)(Z) = P(L_1, ..., L_d)(f_*(Z)).$$

Let X be projective over R, closed point $x \in X$ lies over some closed point $r \in R$ and we define $\deg(Z) = \sum_i n_i [k(x_i) : k(r_i)]$ for $Z = \sum_i n_i x_i$ a 0-cycle. Let $f : X \to Y$ be a proper morphism of schemes over R. Then from the definition of f_* it follows that $\deg f_*(Z) = \deg(Z)$. When $R = \operatorname{Spec} k$ is the spectrum of a field, note that $CH_0(\operatorname{Spec} k) \cong \mathbb{Z}$. Then the structure morphism $X \to S$ induces $CH_0(X) \to \mathbb{Z}$ which, as seen from the definition, agrees with deg.

When $R = \operatorname{Spec} \mathbb{Z}$ even in the simplest case of $X = \operatorname{Spec} \mathbb{Z}$ the degree of a non-zero rational function, which is just a rational number, is non-zero. Hence in general deg only gives a homomorphism on the cycle groups and not on the Chow groups. Note however that we do have $CH^0(\operatorname{Spec} \mathbb{Z}) \cong \mathbb{Z}$ with 1 corresponding the class of $\operatorname{Spec} \mathbb{Z}$, so we have a degree for 1-cycles that does respect rational equivalence by setting

$$\deg(Z) = \pi_*(Z) \in CH_1(\operatorname{Spec} \mathbb{Z}) \cong \mathbb{Z}$$

for a 1-cycle Z, where $\pi : X \to \operatorname{Spec} \mathbb{Z}$ is the structure morphism.

Note that $\pi_* Z$ is non-zero only when Z is flat over $\operatorname{Spec} \mathbb{Z}$. In this case $Z = \sum_i n_i \{\overline{p}_i\} + V$ with p_i closed points of the generic fiber, which is a variety over $\mathbb{Q}$, and V a sum of vertical divisors. Then $\deg(Z) = \sum_i n_i$ equals the degree of $\sum_i n_i p_i \in CH_0(X_{\mathbb{Q}})$. Hence we are essentially counting closed points of the generic fiber, as in intersection theory of varieties over $\mathbb{Q}$.

This notion of degree allows us to define the degree of a $p+1$-cycle on with respect to line bundles $L_1, ..., L_p$ on an arithmetic variety, as

$$\deg(c_1(L_1, ..., L_p)(Z)),$$

and it satisfies the projection formula

$$\deg(c_1(f^* L_1, ..., f^* L_p)(Z)) = \deg(c_1(L_1, ..., L_p)(f_* Z)).$$

Now let $v(\mathbb{Q})$ be the set of places of $\mathbb{Q}$ and let $|f|_p = p^{-\operatorname{ord}_p(f)}$ be the usual p-adic norm, and $|f|_\infty$ the usual absolute value on $\mathbb{Q} \subset \mathbb{R}$, then we have the product formula

$$\prod_{p \in v(\mathbb{Q})} |f|_p = 1$$

or equivalently if we set $\log(\infty) = 1$

$$\sum_{p \in v(\mathbb{Q})} \mathrm{ord}_p(f) \log(p) = 0,$$

for all $f \in \mathbb{Q}^* = \mathrm{Rat}(\mathrm{Spec}\,\mathbb{Z})^*$. For finite places $-\log|f|_p = \mathrm{ord}_p(f)\log(p)$ is nothing but the degree of f at p multiplied by $\log(p)$. This suggests that if we take in account what happens "at infinity" we might still obtain an intersection theory compatible with degrees.

2 Infinite parts

To take into account what happens "at infinity" line bundles are replaced by metrized line bundles, and we attach (green) currents to cycles. In this section we discuss some of the theory of these metrized line bundles and currents. In the next section we will use them to define arithmetic chow groups and deduce analogues of all the statements in the first section ending with an arithmetic degree that is compatible with rational equivalence.

2.1 Metrized line bundles

Let X be an arithmetic variety and consider the $\mathbb{C}$-points of X denoted $X(\mathbb{C})$. This can be viewed as a complex manifold since the fiber over $\mathbb{Q}$ is assumed to be smooth. For a point $x \in X(\mathbb{C})$ let p_x be the corresponding scheme theoretic point of X and denote the local ring homomorphism $\mathcal{O}_{X,p_x} \to \mathbb{C}$ with ϕ_x. A locally free coherent sheaf E on X gives rise to a locally free coherent sheaf $E_{\mathbb{C}}$ on $X(\mathbb{C})$, with fibers $E(x) = E_{p_x} \otimes_{\mathcal{O}_{X,p_X}} \mathbb{C}$ where the map $\mathcal{O}_{X,p_x} \to \mathbb{C}$ is ϕ_x. A hermitian metric h on E is a collection of hermitian metrics $\{h_x\}_{x \in X(\mathbb{C})}$ with h_x a hermitian metric on $E(x)$, h is said to be C^∞ or smooth/continuous if locally for sections $s, s' \in E_{\mathbb{C}}(U)$ the function $h_x(s(x), s'(x))$ is smooth/continuous on U. A pair $\overline{E} = (E, h)$ with h a (smooth/continuous) hermitian metric is called a (smooth/continuous) hermitian locally free coherent sheaf on X.

On $\overline{\mathcal{O}}_X$ we have the constant section 1. This induces a canonical metric by setting $h_x(1(x), 1(x)) = 1$. We denote the pair $(\mathcal{O}_X, h) = \overline{\mathcal{O}}_X^{can}$.

For two metrized line bundles (E, h) and (E', h') we naturally get a metric on $E \otimes E'$ by setting

$$(h \otimes h')_x((s \otimes s')(x), (t \otimes t')(x)) = h_x(s(x), t(x)) \cdot h'(s'(x), t'(x)),$$

where s, t are local sections of E at x and s', t' of E'.

We can also consider the pull-back of a metrized line bundle. We get the induced metric on f^*E by setting

$$f^*h_x(f^*s(x), f^*s'(x)) = h_{f(x)}(s(f(x)), s'(f(x))).$$

2.2 Green currents

Now let X be a d-dimensional complex manifold. Let $A^{p,q}(X)$ and $A_c^{p,q}(X)$ be the spaces of smooth (p,q) forms on X and smooth (p,q) forms with compact support respectively. We let $D^{p,q}(X)$ be the space of linear maps $A_c^{d-p,d-q}(X) \to \mathbb{C}$ (continuous with respect to some topology on A_c though we will not care about this here), these are called currents.

We give some important examples of currents:

1. let $\omega \in A^{p,q}(X)$, then we get $[\omega] \in D^{p,q}(X)$ defined by
$$[\omega](\eta) = \int_X \omega \wedge \eta, \eta \in A_c^{d-p,d-q}(X).$$
This gives an identification of $A^{p,q}(X)$ as a subgroup $D^{p,q}(X)$.
2. Let Y be a codimension p complex submanifold of X, then we get a current $\delta_Y \in D^{p,p}(X)$ defined by
$$\delta_Y(\eta) = \int_Y \eta,$$
for a codimension p cycle $Y = \sum_{n_i} Y_i$ we define $\delta_Y = \sum_i n_i \delta_{Y_i}$.

If T is a (p,q)-current and ω a (p',q')-form we get a $(p+p',q+q')$ current $\omega \wedge T$ defined by
$$\omega \wedge T(\eta) = T(\omega \wedge \eta).$$
By abuse of notation we also denote it $[\omega] \wedge T$ sometimes. With this definition we have $\omega \wedge [\eta] = [\eta \wedge \omega] = (-1)^{|\omega||\eta|}\eta \wedge [\omega]$. Note in particular that if one of ω and η has even degree then the order doesn't matter.

The differentials $\partial, \overline{\partial}$ induce maps $\partial : D^{p,q}(X) \to D^{p+1,q}(X), \overline{\partial} : D^{p,q} \to D^{p,q+1}(X)$ respectively by setting
$$\partial(T)(\eta) = (-1)^{p+q+1}T(\partial(\eta)), \overline{\partial}(T)(\eta) = (-1)^{p+q+1}T(\overline{\partial}(\eta)),$$
for $\eta \in A_c^{d-p,d-q}(X)$. The sign here is chosen so that $[d\omega] = d[\omega]$ for ω a smooth form, which can be seen from Stokes' theorem.

Letting $f : X \to Y$ be a proper morphism of complex manifolds we can define a pushforward $f_* : D^{p,q}(X) \to D^{p-d+\dim Y, p-d+\dim Y}(Y)$ by setting
$$f_*(T)(\eta) = T(f^*(\eta)).$$
The pushforward of currents commutes with both ∂ and $\overline{\partial}$ because pull-back does on forms.

Note that we can view example 2 as a special case of example 1. Indeed we can consider [1] on Y, then $\delta_Y = i_*[1]$ where $i : Y \to X$ is the inclusion.

We set $d^c = \frac{1}{4\pi i}(\partial - \overline{\partial})$, then $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$. Let Y be a codimension p cycle on X. Then $g \in D^{p-1,p-1}(X)$ is called a green current for Y if there is some $\omega \in A^{p,p}(X)$ such that
$$dd^c(g) + \delta_Y = [\omega].$$

Theorem 2.1. *(Poincaré-Lelong formula) Let (L, h) be a smooth hermitian line bundle on a complex variety X and let s be a non-zero section of L. Then we have an equality of currents*

$$dd^c[-\log(h(s,s))] + \delta_{div(s)} = \frac{-i}{2\pi}[\partial\overline{\partial}\log(h(s,s))].$$

In other words $[-\log(h(s,s))]$ is a green current for $\mathrm{div}(s)$, we also denote it $[\overline{L}, s]$.

We define the first Chern class of a line bundle $c_1(L,h)$ to be the form $\frac{-i}{2\pi}[\partial\overline{\partial}\log(h(s,s))]$. Note that this doesn't depend on s, since $\partial\overline{\partial}\log|\phi| = 0$ for ϕ a rational function, which also shows that $c_1(\mathcal{O}_X^{can}) = 0$. It is clear that c_1 is linear in (L,h) and $d(c_1) = 0$, hence we get a homomorphism to the de Rham cohomology of X

$$c_1 : \widehat{\mathrm{Pic}}(X) \to H^2_{dR}(X),$$

where $\widehat{\mathrm{Pic}}(X)$ is the group of isomorphism classes of metrized line bundles with the operation $\otimes$ defined above. Further by definition of the pull-back metric it is clear that c_1 is compatible with pull-back (of forms).

2.3 Complex analogs

We work here with algebraic projective complex varieties. Cycles, morphisms and (sections of) line bundles are all assumed to be algebraic.

First we define the $*$ product on green currents. Let $(\overline{L}, s)$ a smooth hermitian line bundle and let g_Z be a green current for a cycle Z on X such that (L, s) and Z intersect properly. We define

$$[\overline{L}, s] * g_Z = [\overline{L}, s] \wedge \delta_Z + c_1(\overline{L}) \wedge g_Z.$$

This is again a green current for $(L, s) \cdot Z$, indeed we may assume Z is integral. Then by Poincaré-Lelong on Z we have

$$dd^c[-\log(h(s,s))] \wedge \delta_Z = c_1(\overline{L}) \wedge \delta_Z - \delta_{\mathrm{div}(s|_Z)},$$

hence we find

$$\begin{aligned} dd^c([\overline{L}, s] * g_Z) + \delta_{(L,s)\cdot Z} &= c_1(\overline{L}) \wedge \delta_Z + dd^c(c_1(\overline{L}) \wedge g_Z) \\ &= c_1(\overline{L}) \wedge \delta_Z + c_1(\overline{L}) \wedge dd^c(g_Z) \\ &= c_1(\overline{L}) \wedge \omega_Z, \end{aligned}$$

where $\omega_Z = dd^c(g_Z) + \delta_Z$. As we shall now see the $*$-product satisfies similar properties to the intersection product defined in the first section.

Lemma 2.2. *Suppose $f : X \to Y$ is a morphism and $Z \subset X$ a cycle. Let g_Z be a green current for Z and $\omega_Z = dd^c g_z + \delta_Z$. If $f_*[\omega_Z] = [\alpha_Z]$ for some smooth form α_Z then $f_* g_Z$ is a green current for $f_* Z$.*

Proof. We have

$$dd^c f_* g_Z = f_* dd^c g_Z = f_*[\omega_Z] - f_*\delta_Z = f_*[\omega_Z] - \delta_{f_* Z}.$$

□

Now let $f : X \to Y$ be a proper morphism. Let $\overline{L} = (L, h)$ a hermitian line bundle on Y, s a non-zero rational section of L intersecting a cycle Z properly. Let g_Z, ω_Z as in the lemma.

Proposition 2.3. *In the above situation we have a projection formula*

$$f_*([f^*\overline{L}, f^*s] * g_Z) = [\overline{L}, s] * f_*g_Z.$$

Proof. By linearity we may assume that Z is integral. First note we have the equality

$$f_*(c_1(f^*\overline{L}) \wedge g_Z) = c_1(\overline{L}) \wedge f_*g_Z.$$

Indeed if we let ω be a smooth form we have by definition

$$\begin{aligned} f_*(c_1(f^*\overline{L}) \wedge g_Z)(\omega) &= c_1(f^*\overline{L}) \wedge g_Z(f^*\omega) \\ &= g_Z(c_1(f^*\overline{L}) \wedge f^*\omega)) \\ &= g_Z(f^*(c_1(L) \wedge \omega)) \\ &= c_1(L) \wedge f_*g_Z(\omega). \end{aligned}$$

It remains to show that

$$f_*([-f^*(\log h(s,s))] \wedge \delta_Z) = [-\log h(s,s)] \wedge \delta_{f_*Z}.$$

We may assume $X = Z$ is irreducible. Let again ω be a smooth form then we have

$$\begin{aligned} f_*([-f^*(\log h(s,s))] \wedge \delta_Z)(\omega) &= \int_Z -f^*(\log h(s,s)\omega) \\ &= \deg(Z/f(Z)) \int_{f(Z)} -\log h(s,s)\omega \\ &= \int_{f_*Z} -\log h(s,s)\omega \\ &= [-\log h(s,s)] \wedge \delta_{f_*Z}(\omega). \end{aligned}$$

Where in the second equality we use that f is generically finite étale. ☐

Proposition 2.4. *Let $(\overline{L}_1, s_1), (\overline{L}_2, s_2)$ be smooth hermitian line bundles with non-zero rational sections and let g_Z be a greens function for a cycle Z, suppose that (L_i, s_i) and Z intersect properly. Then the $*$ product is commutative in the following sense*[2]

$$[\overline{L}_1, s_1] * ([\overline{L}_2, s_2] * g_Z) = [\overline{L}_2, s_2] * ([\overline{L}_1, s_1] * g_Z),$$

up to exact forms.

Proof. By definition we have to show equality

$$[\overline{L}_1, s_1]*\delta_{[\overline{L}_2,s_2]\cdot Z} + c_1(\overline{L}_1)*([\overline{L}_2, s_2]*\delta_Z) = [\overline{L}_2, s_2]*\delta_{[\overline{L}_1,s_1]\cdot Z} + c_1(\overline{L}_2)*([\overline{L}_1, s_1]*\delta_Z)$$

[2] Note that $d[\alpha] = [d\alpha]$ holds for smooth forms whereas $\log h(s,s)$ has logarithmic singularities along $\mathrm{div}(s)$. Therefore we do need to prove something instead of it being a simple calculation using the definitions.

up to exact forms. By definition of δ and linearity we may assume $Z = X$.[3] It then follows from the following precise statement. $\square$

Theorem 2.5. *Let $(\overline{L}_1, s_1)$, $(\overline{L}_2, s_2)$ smooth hermitian line bundles such that (L_1, s_1) and (L_2, s_2) intersect properly, then we have*

$$[\overline{L}_1, s_1] \wedge \delta_{div(L_2,s_2)} + c_1(\overline{L}_1) \wedge [\overline{L}_2, s_2] - [\overline{L}_2, s_2] \wedge \delta_{div(L_1,s_1)} - c_1(\overline{L}_2) \wedge [\overline{L}_1, s_1]$$
$$= \frac{i}{2\pi}(\overline{\partial}([\overline{L}_1, s_1] \wedge \partial[\overline{L}_2, s_2]) + \partial([\overline{L}_2, s_2] \wedge \overline{\partial}[\overline{L}_1, s_1])).$$

Proof. The idea is to use Hironaka's resolution of singularities to get a sequence of blow-ups such that the support of the pull-backs of $\mathrm{div}(s_1), \mathrm{div}(s_2)$ and the exceptional locus is a normal crossings divisor. Using a partition of unity the problem is then reduced to a computation on $\mathbb{C}^n$, see [9] 5.3 for more details. $\square$

2.4 Local height

Let $(\overline{L}_0, s_0), ..., (\overline{L}_k, s_k)$ smooth hermitian line bundles with rational sections and let Z be a cycle, and suppose all the $\mathrm{div}(s_i)$ and Z intersect properly. Note that if Z is prime then on Z the 0 function can be interpreted as a green current for Z with $\omega_Z = 1$. And we have

$$[\overline{L}_0, s_0] * \cdots * [\overline{L}_k, s_k] \wedge \delta_Z = i_*([i^*\overline{L}_0, i^*s_0] * \cdots * [i^*\overline{L}_k, i^*s_k] * 0),$$

where $i : Z \to X$ is the inclusion.

Now assume $k = \dim Z$, then we define the local height of Z with respect to the $(\overline{L}_i, s_i)$ as

$$\langle(\overline{L}_0, s_0), ..., (\overline{L}_k, s_k)|Z\rangle_\infty = [\overline{L}_0, s_0] * \cdots * [\overline{L}_k, s_k] \wedge \delta_Z(\frac{1}{2}).$$

Note that by commutativity the local height of Z only depends on the $(\overline{L}_i, s_i)$ and not on their order. For this note that

$$[\overline{L}_i, s_i] * (g_Z + \partial A + \overline{\partial} B) = [\overline{L}_i, s_i] * g_Z + c_1(\overline{L}_i) \wedge \partial A + c_1(\overline{L}_i) \wedge \overline{\partial} B,$$

but

$$c_1(\overline{L}_i) \wedge \partial A + c_1(\overline{L}_i) \wedge \overline{\partial} B = \partial(c_1(\overline{L}_i) \wedge A) + \overline{\partial}(c_1(\overline{L}_i) \wedge B),$$

since $c_1(\overline{L}_i)$ is a closed form. Now the right hand side vanishes because the intersection of all the $\mathrm{div}(s_i)$ with Z is 0-dimensional.

Remark 2.6. From the definition it follows that for g_Z a green current for Z we have

$$[\overline{L}_0, s_0] * \cdots * [\overline{L}_k, s_k] * g_Z = [\overline{L}_0, s_0] * \cdots * [\overline{L}_k, s_k] \wedge \delta_Z + c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_k) \wedge g_Z.$$

[3] Here and possibly in other places we use Hironaka's resolution of singularities so that we may assume Z is smooth.

Then since $[\overline{L}_k, s_k] * g_Z$ is a green current for $Y = (L_k, s_k) \cdot Z$, if we plug this in the above equation things cancel and we find

$$\begin{aligned}\langle(\overline{L}_0, s_0), ..., (\overline{L}_k, s_k)|Z\rangle_\infty &= \langle(\overline{L}_0, s_0), ..., (\overline{L}_{k-1}, s_{k-1})|Y\rangle_\infty + c_1(\overline{L}_0) \wedge \cdots \wedge c_1(\overline{L}_{k-1}) \wedge s_k \wedge \delta_Z(\frac{1}{2}) \\ &= \langle(\overline{L}_0, s_0), ..., (\overline{L}_{k-1}, s_{k-1})|Y\rangle_\infty - \frac{1}{2}\int_Z \log h_k(s_k, s_k) c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_{k-1}).\end{aligned}$$

This allows us to compute the local height inductively, combining this with the projection formula from proposition 2.3 allows us to deduce a projection formula for local heights.

Proposition 2.7. *Let $f : X \to Y$ be a (proper smooth) morphism, $(\overline{L}_0, s_0), ..., (\overline{L}_k, s_k)$ smooth hermitian line bundles on Y and Z a k-cycle on X. Assume $f(Z)$ intersects the (L_i, s_i) properly, then we have*

$$\langle(f^*\overline{L}_0, f^*s_0), ..., (f^*\overline{L}_k, f^*s_k)|Z\rangle_\infty = \langle(\overline{L}_0, s_0), ..., (\overline{L}_k, s_k)|f_*Z\rangle_\infty.$$

Proof. We use induction on k, for $k = 1$ this is just the projection formula from before (proposition 2.3). Let $k > 1$ and assume it holds for $k-1$, we may assume $Z = X$ is prime and let $W = (L_k, s_k) \cdot Z$. Then by the induction formula we have

$$\begin{aligned}&\langle(f^*\overline{L}_0, f^*s_0), ..., (f^*\overline{L}_k, f^*s_k)|Z\rangle_\infty \\ &= \langle(f^*\overline{L}_0, f^*s_0), ..., (f^*\overline{L}_{k-1}, f^*s_{k-1})|W\rangle_\infty - \frac{1}{2}\int_Z \log f^*h_k(s_k, s_k) c_1(f^*\overline{L}_1) \wedge \cdots \wedge c_1(f^*\overline{L}_{k-1}).\end{aligned}$$

Using that c_1 commutes with f^*, induction on the dimension of W and the transformation formula for integrals shows that the right hand side satisfies the projection formula hence is equal to

$$\langle(\overline{L}_0, s_0), ..., (\overline{L}_{k-1}, s_{k-1})|f_*W\rangle_\infty - \frac{1}{2}\int_{f_*Z} \log h_k(s_k, s_k) c_1(\overline{L}_1) \wedge \cdots \wedge c_1(\overline{L}_{k-1}).$$

Using the induction formula again shows the proposition. □

3 Arithmetic intersection theory.

3.1 Arithmetic intersection theory

Let X again be an arithmetic variety. We define an arithmetic cycle of codimension p to be a pair (Z, T) where Z is a cycle of codimension p and T is a $(p-1, p-1)$ current on $X(\mathbb{C})$. We say that (Z, T) is of green type if T is a green current for $Z(\mathbb{C})$. The group of arithmetic cycles of codimension p, respectively those of green type are denoted $\widehat{Z}^p_D(X)$ and $\widehat{Z}^p(X)$ respectively. Similarly we can define arithmetic cycles of dimension l and if $p + l = \dim X$ we have

$$\widehat{Z}^p_D(X) = \widehat{Z}_{D,l}(X), \widehat{Z}^p(X) = \widehat{Z}_l(X).$$

We give some important examples:

1. Let $\overline{L} = (L, h)$ be a smooth hermitian line bundle on X and s a non-zero rational section of L. Then by Poincaré-Lelong $(\mathrm{div}(s), -\log h(s,s))$ is of green type, denoted by $\widehat{div}(L, s)$.

2. Let Y be an integral subscheme of codimension $p-1$ on X and ϕ a non-zero rational function on Y. Then we can define

$$[-\log|\phi|^2]_{Y(\mathbb{C})}(\eta) = \int_{Y(\mathbb{C})} (-\log|\phi|^2)\eta.$$

   By Poincaré-Lelong the cycle $(\mathrm{div}(\mathcal{O}_Y, \phi), [-\log|\phi|^2]_{Y(\mathbb{C})})$ is of green type, it is denoted by $\widehat{(\phi/Y)}$.

3. If u, v are currents of type $(p-2, p-1)$ and $(p-1, p-2)$ respectively then $(0, \partial(u) + \overline{\partial}(v))$ is a p-dimensional cycle of green type.

Let $\widehat{\mathrm{Rat}}^p(X)$ be the group generated by cycles in examples 2 and 3. Then we define arithmetic chow groups/arithmetic chow groups of green type respectively as follows

$$\widehat{CH}^p_D(X) = \widehat{Z}^p_D(X)/\widehat{\mathrm{Rat}}^p(X), \widehat{CH}^p(X) = \widehat{Z}^p(X)/\widehat{\mathrm{Rat}}^p(X).$$

For a morphism $f : X \to Y$ of arithmetic varieties we can define a pushforward of arithmetic cycles by

$$f_*(Z, T) = (f_*Z, f_*T),$$

this induces a homomorphism $f_* : \widehat{CH}_{D,l}(X) \to \widehat{CH}_{D,l}(Y)$ as we see in the following lemma.

Lemma 3.1. *Let $f : X \to Y$ be a proper morphism and let Z be a d-dimensional subvariety of X. Then for ϕ a non-zero rational function on Z and η a smooth (d, d) form on Y with compact support we have*

$$\int_Z \log|\phi|^2 f^*\eta = \int_{f(Z)} \log|N_{R(Z)/R(f(Z))}(\phi)|^2\eta$$

if $\dim(f(Z)) = d$ *and 0 otherwise. In other words we have*

$$f_*[\log|\phi|^2] \wedge \delta_Z = [\log|N_{R(Z)/R(f(Z))}(\phi)|^2] \wedge \delta_{f_*Z}.$$

Proof. First if $\dim(Z) > \dim(f(Z))$ then η is 0 on $f(Z)$. If $\dim(Z) = \dim(f(Z))$, then we may assume $X = Z$ is irreducible. Then we can consider a dense smooth open subset $U \subset f(Z)$ such that $f|_U : f^{-1}(U) \to U$ is finite étale and $f^{-1}(U)$ does not intersect the support of ϕ. By finite étale the change of coordinates formula implies that

$$\int_{f^{-1}(U)} f^*\omega = \deg(Z/f(Z)) \int_U \omega,$$

for any form ω. Since $f^{-1}(U)$ does not intersect the support of ϕ we have $N_{R(Z)/R(f(Z))}(\phi) = \phi^{\deg(Z/f(Z))}$, so that taking log multiplies by $\deg(Z/f(Z))$. □

Now let $\overline{L} = (L, h)$ be a smooth hermitian line bundle and s a non-zero rational section of L. Let (Z, T) be an arithmetic cycle of codimension p such that Z and (L, s) meet properly. Then we define the arithmetic intersection cycle as follows

$$(\overline{L}, s) \cdot (Z, T) = ((L, s) \cdot Z, [\overline{L}, s] * T).$$

From proposition 1.1 we see that this is linear in $(\overline{L}, s)$ and (Z, T).

We again have a projection formula. Let $f : X \to Y$ be a proper morphism of arithmetic varieties. Let $\overline{L} = (L, h)$ a hermitian line bundle on Y, s a non-zero rational section of L and assume that $f(\eta_X) \notin \mathrm{Supp}(L, s)$. Let $f^* s$ as in proposition 2. Let (Z, T) be an arithmetic cycle on X such that $(f^* L, f^* s)$ and Z meet properly.

Proposition 3.2. *In the above situation we get an arithmetic projection formula*

$$f_*((f^* \overline{L}, f^* s) \cdot (Z, T)) = (\overline{L}, s) \cdot f_*(Z, T).$$

Proof. This follows from propositions 1.3 and 2.3. □

We now consider the intersection product on the arithmetic Chow groups. Let $\overline{L}$ be a hermitian line bundle and $\alpha \in \widehat{CH}^p_D$. Let $(Z, T) \in \widehat{Z}_p$ represent α. Then there is a rational section s of L such that (L, s) and Z intersect properly and we define

$$\widehat{c}_1(L) \cdot \alpha = (\overline{L}, s) \cdot (Z, T).$$

Theorem 3.3. $\widehat{c}_1(\overline{L}) \cdot \alpha$ *does not depend on the choice of* (Z, T) *or* s*. Hence it defines a morphism* $\widehat{c}_1(\overline{L}) : \widehat{CH}^*_D \to \widehat{CH}^{*+1}_D$.

Proof. First we show independence of s, so let s' be another section such that the supports intersect properly. Then we have $(\overline{L}, s') = (\overline{L} \otimes \overline{\mathcal{O}}^{can}_X, s \otimes \phi)$ for some rational function ϕ. By linearity then

$$(\overline{L}, s) \cdot (Z, T) - (\overline{L}, s') \cdot (Z, t) = (\overline{\mathcal{O}}^{can}_X, \phi) \cdot (Z, T) = (\widehat{\phi/Z})$$

since $c_1(\overline{\mathcal{O}}^{can}_X) = 0$.

So far we have morphism $c(\overline{L}) : \widehat{Z}^* \to \widehat{CH}^{*+1}_D$, and we need to show that $\widehat{\mathrm{Rat}}^* \subset \ker c(\overline{L})$. First consider $(0, \partial A + \overline{\partial} B) \in \widehat{\mathrm{Rat}}^*$. Then since $c_1(\overline{L})$ is a closed form we have (perhaps up to sign) equality

$$(\overline{L}, s) \cdot (0, \partial A + \overline{\partial} B) = (0, \partial(c_1(\overline{L}) \wedge A) + \overline{\partial}(c_1(\overline{L}) \wedge B)) \in \widehat{\mathrm{Rat}}^{*+1}.$$

Next we consider $\widehat{\phi/Y}$, we may assume Y is integral by linearity. Then if Y is vertical we are done by the first section. So suppose Y is horizontal. First assume it is generically smooth. By 5.19(commutatitivity up to exact forms at least for generically smooth Y) and the projection formula we have

$$\begin{aligned}(\overline{L}, s) \cdot (\widehat{\phi/Y}) &= j_*((j^* \overline{L}, j^* s) \cdot \widehat{\mathrm{div}}(\overline{O}^{can}_Y, \phi)) \\ &= (\phi/\mathrm{div}(s|_Y))^\wedge = 0,\end{aligned}$$

inside CH_D.

The general case follows by resolution of singularities, see [9] 5.20 for more details. □

Now let $d = \dim X$ and let $(Z,T) \in \widehat{Z}_{D,0}(X)$. Write $Z = \sum_{x \in X_{(0)}} n_x x$ then we define the arithmetic degree

$$\widehat{\deg}(Z,T) = \sum_{x \in X_{(0)}} n_x \log \#\kappa(x) + \frac{1}{2} T(1).$$

Theorem 3.4. *The arithmetic degree has the following properties:*

1. *for $f : X \to Y$ a morphism of arithmetic varieties we have $\widehat{\deg}(Z,T) = \widehat{\deg} f_*(Z,T)$.*
2. *for $\alpha \in \widehat{Rat}_0(X)$ the degree is zero, hence it induces $\widehat{\deg} : \widehat{CH}_{D,0}(X) \to \mathbb{R}$.*

Proof. For x a closed point we have

$$f_* x = [\kappa(x) : \kappa(f(x))] f(x),$$

so that

$$\widehat{\deg}(f_* x, 0) = [\kappa(x) : \kappa(f(x))] \log \#\kappa(f(x)) = \log \#\kappa(x).$$

For currents T we have $f_* T(1) = T(f^*(1)) = T(1)$, hence 1. follows.

By 1. it suffices to show 2. for $X = \operatorname{Spec} \mathbb{Z}$. Here it is just the product formula, that is

$$\sum_{p \in v(\mathbb{Q})} \log |f|_p = 0$$

for $f \in \mathbb{Q}$. Note that $|f|_\infty^2 = h_{can}(f,f)$, hence the factor $\frac{1}{2}$ in the definition. $\square$

Remark 3.5. Note that if $z = \sum_i n_i P_i$ is a cycle of closed points contained in one of the fibres X_p, then $\widehat{\deg}(Z,0) = \deg(Z)\log(p)$, where on the right hand side we view Z as a cycle on the fibre X_p viewed as a scheme over $\mathbb{F}_p$ and the degree is defined as in section 1. In particular if Z is a q-cycle contained in X_p and $\overline{L}_1, ..., \overline{L}_q$ is a collection of smooth hermitian line bundles then

$$\langle \overline{L}_1, ..., \overline{L}_q | Z \rangle = \deg(c_1(L_1) \cdots c_1(L_q) \cdot Z) \log(p).$$

3.2 Global and local heights

Now let Z be a q-cycle of X, and let $\overline{L}_1, ..., \overline{L}_q$ be a collection of smooth hermitian line bundles. We define the height of Z with respect to $\overline{L}_1, ..., \overline{L}_q$ as

$$\langle \overline{L}_1, ..., \overline{L}_q | Z \rangle := \widehat{\deg}(\widehat{c}_1(\overline{L}_1) \cdots \widehat{c}_1(\overline{L}_q) \cdot (Z,0)).$$

For Z a $q+1$-cycle of X we define it as in section 1.[4] From the projection formula and theorem 2.1 we obtain a projection formula for heights:

$$\langle f^*\overline{L}_1, ..., f^*\overline{L}_q | Z \rangle = \langle \overline{L}_1, ..., \overline{L}_q | f_* Z \rangle.$$

[4] note that $CH^0(\operatorname{Spec} \mathbb{Z}) \cong \widehat{CH}^0(\operatorname{Spec} \mathbb{Z}) \cong \mathbb{Z}$

Now let $s_1, ..., s_q$ be rational sections of $L_1, ..., L_q$ used to compute the height. Note that by theorem 3.4(i) the height decomposes as a sum

$$\langle \overline{L}_1, ..., \overline{L}_q | Z \rangle = \sum_p \langle (L_1, s_1), ..., (L_q, s_q) | Z \rangle_p + \langle (\overline{L}_1, s_1), ..., (\overline{L}_q, s_q) | Z \rangle_\infty,$$

where p runs over the primes of $\mathbb{Z}$. Here we say that $\langle . \rangle_p$ are the local heights at p with respect to the $\overline{L}_i, s_i$. Note that $\langle . \rangle_\infty$ is the same local height at infinity as in section 2.

Remark 3.6. Now write $\mathrm{div}(s_q|_Z) = H + \sum_p V_p$, where H is the horizontal part of $\mathrm{div}(s_q|_Z)$ and V_p is the part supported on the fiber over p. By linearity the height decomposes as a sum of the heights of H and the V_p. By remark 3.5 the heights of V_p only give a contribution at p calculated by intersection theory in the fiber over p. Hence from the induction formula for the local height at infinity (remark 2.6) and the linear independence of $\log(p)$ over $\mathbb{Q}$ it follows that the local heights at p verify the following induction formula:

$$\langle (L_1, s_q), ..., (L_q, s_q) | Z \rangle_p = \langle (L_1|_Z, s_1|_Z), ..., (L_{q-1}|_Z, s_{q-1}|_Z) | H \rangle_p + c_1(L_1|_Z) \cdots c_1(L_{q-1}|_Z) V_p \log(p).$$

Note the similarities with the induction formula at infinity, where the c_1 term is given by an integral over Z. In the next section we will introduce p-adically metrized line bundles. As it turns out the c_1 term can then be interpreted as integrating a certain measure denoted $c_1(L_1) \cdots c_1(L_{q-1}) \log ||s_q|| \delta_Z$ over a suitable analytic space defined over $\mathbb{Q}_p$, this is done by considering the Berkovich analytic space as explained in [1].

4 Models and adelic metrics

In the preceeding sections we gave the infinite place/prime special treatment in a sense, by considering metrized line bundles on the complex points. Whereas we grouped the finite places/primes together via intersection theory over $\mathrm{Spec}\,\mathbb{Z}$. We note however that the definition of metrized line bundles still makes sense for the finite places if we consider the $\mathbb{Q}_p$ points $X(\mathbb{Q}_p)$ instead of $\mathbb{C}$-points, and instead of hermitian metrics we consider p-adic norms. Note that for these definitions we only need to consider the generic fiber of X. For arithmetic varieties there is a natural way to define p-adic metrics, and it is closely related to the intersection theory of the fibers over p.

4.1 Model metrics

Let $|.|_p$ be the standard p-adic norm on $\mathbb{Q}$, and let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the p-adic norm. Let X be a smooth projective variety over $\mathbb{Q}_p$ and let L be a line bundle. A p-adic metric on L is a collection of p-adic norms on each fiber $L(x)$, $x \in X(\mathbb{Q}_p)$. We can put continuity assumptions on the metrics but this does not really matter for us right now as we will only consider model metrics. As in the complex case there are natural norms on pull-backs and tensor products. By putting $f^*||f^*s||(x) = ||s||(f(x))$ and $||s \otimes t||(x) = ||s||(x) \cdot ||t||(x)$.

Suppose $\widetilde{X}$ is projective, integral and flat over $\operatorname{Spec}\mathbb{Z}_p$, and suppose the generic fiber is isomorphic to X. Let $\widetilde{L}$ be a line bundle on $\widetilde{X}$ and suppose $\widetilde{L}|_X \cong L^{\otimes e}$ for some e. These data define a model $(\widetilde{X}, \widetilde{L}, e)$ for (X, L). To such a model there is a natural way to assign a p-adic metric on L. Namely let $x \in X(\mathbb{Q}_p)$, then since $\widetilde{X}$ is projective over $\operatorname{Spec}(\mathbb{Z}_p)$, by the valuative criterion of properness there is a unique extension $\widetilde{x} \in \widetilde{X}(\mathbb{Z}_p)$ of x, i.e. we have a commutative diagram

$$\begin{array}{cccc} x: & \operatorname{Spec}\mathbb{Q}_p & \to & X \\ & \downarrow & & \downarrow, \\ \widetilde{x}: & \operatorname{Spec}\mathbb{Z}_p & \to & \widetilde{X} \end{array}$$

hence $x^*L^{\otimes e} = \widetilde{x}^*\widetilde{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. Whenever we have a 1-dimensional $\mathbb{Q}_p$ vector space V with a $\mathbb{Z}_p$ submodule W of rank 1, there is a natural p-adic norm on V. Namely by setting $||x||_W = \inf_{r \in \mathbb{Q}_p}\{|r| : x \in rW\}$ for $x \in V$. Since $\widetilde{L}$ is a model for $L^{\otimes e}$ we put $||s||_{\widetilde{L}}(x) = ||s^{\otimes e}||^{1/e}_{\widetilde{x}^*\widetilde{L}}$, we refer to such a metric as a model metric.

Remark 4.1. Assume $e = 1$, then $\log ||s||_{\widetilde{L}}(x)$ is nothing but the order of vanishing of s at the closed point of $\widetilde{x}$ multiplied by a factor of $\log(p)$, i.e. where the zariski closure of x inside $\widetilde{X}$ intersects the closed fiber. Compare for example when $\widetilde{X} = \operatorname{Spec}\mathbb{Z}_p$ and $s \in \mathbb{Q}_p$, then $||s||(\operatorname{Spec}\mathbb{Q}_p) = |s|_p$. If s is supported on the closed fiber and $\widetilde{X}$ is regular, then each $\widetilde{x}$ intersects a unique irreducible component of the closed fiber so that $\log ||s||$ is locally constant and gives the multiplicities of the irreducible components. This allows us to evaluate $\log ||s||$ on the generic points of irreducible components. In general one may always blow-up to get enough sections that intersect a unique irreducible component.

Proposition 4.2. *Let $(\widetilde{X}, \widetilde{L}, e)$ be a model for (X, L). Suppose we have another model $\widetilde{X}'$ for X together with a birational morphism $\varphi : \widetilde{X}' \to \widetilde{X}$ which restricts to an isomorphism over the generic fiber, then $(\widetilde{X}', \varphi^*\widetilde{L}', e)$ is another model for (X, L) inducing the same metric.*

Proof. Note that φ is projective since it is a morphism of projective schemes over $\operatorname{Spec}(\mathbb{Z}_p)$.[5] Let $x \in \widetilde{X}(\mathbb{Q}_p)$ and let $\widetilde{x} \in \widetilde{X}(\mathbb{Z}_p)$ be the unique section extending x. Because φ is an isomorphism over the generic fiber there is a 1-1 correspondence between $\widetilde{X}(\mathbb{Q}_p)$ and $\widetilde{X}'(\mathbb{Q}_p)$. Let $x' \in \widetilde{X}'(\mathbb{Q}_p)$ be the point corresponding to x. Then we have the following commutative diagram

$$\begin{array}{cccc} x': & \operatorname{Spec}(\mathbb{Q}_p) & \to & \widetilde{X}' \\ & \downarrow & & \downarrow \varphi. \\ \widetilde{x}: & \operatorname{Spec}(\mathbb{Z}_p) & \to & \widetilde{X} \end{array}$$

Since φ is projective $\widetilde{x}$ lifts to a unique section $\widetilde{x}' \in \widetilde{X}'(\mathbb{Z}_p)$ extending x' and we have equality $\widetilde{x}^*\widetilde{L} \cong \widetilde{x}'^*\varphi^*\widetilde{L}$. ☐

Now suppose $\overline{L}_0, ..., \overline{L}_k$ is a collection of line bundles with model metrics, and suppose $(\widetilde{X}_i, \widetilde{L}_i, e_i)$ are models inducing the metrics. Consider the product

[5] [7] corollary 3.3.32

$X_{0,...,k} = \widetilde{X}_0 \times_{\operatorname{Spec}\mathbb{Z}_p} \cdots \times_{\operatorname{Spec}\mathbb{Z}_p} \widetilde{X}_k$ and set $\widetilde{X}$ to be the Zariski closure of $\Delta(X)$ inside $X_{0,...,k}$, where Δ is the diagonal map of X into the generic fiber $X \times_{\operatorname{Spec}\mathbb{Q}_p} \cdots \times_{\operatorname{Spec}\mathbb{Q}_p} X$ of $X_{0,...,k}$. Let ι be the inclusion of $\widetilde{X}$ into $X_{0,...,k}$, and let p_i be the projection onto $\widetilde{X}_i$. Now note that $p_i \circ \iota$ restricts to $p_i \circ \Delta = id_X$ on the generic fiber, hence we can apply the proposition to see that $\iota^* p_i^* \widetilde{L}_i$ induces the same metric as $\widetilde{L}_i$. So we find that $\widetilde{X}$ is a common model for all the metrics.

As another consequence we can and will assume from now on that our models are normal. When we have a collection of line bundles with model metrics they have common normal model to which we can apply the intersection theory of section 1.

Now assume again $(\overline{L}_0, s_0), ..., (\overline{L}_k, s_k)$ is a collection of line bundles with model metrics and rational sections, suppose $Z \subset X$ is a k-cycle and that the (L_i, s_i) and Z all intersect properly. Let $(\widetilde{X}, \widetilde{L}_i, e_i)$ be a normal common model for all the model metrics and let $\widetilde{Z}$ be the zariski closure of Z inside $\widetilde{X}$. We define the local height at p of Z with respect to the bundles as

$$\langle (\overline{L}_0, s_0), ..., (\overline{L}_k, s_k) | Z \rangle_p := \frac{1}{e_0 \cdots e_k} \deg_p((\widetilde{L}_0, s_0^{\otimes e_0}) \cdots (\widetilde{L}_k, s_k^{\otimes e_k}) \cdot \widetilde{Z}),$$

where for the right hand side the sections $s_i^{\otimes e_i}$ are viewed as rational sections of $\widetilde{L}_i$. This intersection product is contained in the closed fiber and $\deg_p$ is the degree of this intersection product viewed as a scheme over $\operatorname{Spec}\mathbb{F}_p$ multiplied by a factor of $\log(p)$.

Note that the local height does not depend on the models. Indeed suppose $(\widetilde{X}', \widetilde{L}'_0, ..., \widetilde{L}'_k, e'_0, ..., e'_k)$ is another model for the metrics, then again using the diagonal construction there is a model $\widetilde{X}''$ dominating both $\widetilde{X}'$ and $\widetilde{X}$, which is an isomorphism on the generic fiber X. The right hand side in the definition of the local height being defined in terms of intersection theory satisfies the projection formula. Therefore we can directly compare the local heights defined by the $\widetilde{L}'_i$ and the $\widetilde{L}_i$, a telescoping argument then finishes the proof cf. theorem 4.5.

4.2 Limits

Let $||.||$ be a p-adic metric for L on X. We say $||.||$ is approximated by a sequence of models $(\widetilde{X}_i, \widetilde{L}_i, e_i)$ if $\log ||.||/||.||_{\widetilde{L}_i}$ converges uniformly to zero on $X(\mathbb{Q}_p)$.

Consider the following example, suppose L has a p-adic model metric $||.||_1$. Suppose we have a surjective morphism $f : X \to X$ together with an isomorphism $\varphi : f^*L \to L^{\otimes d}$ for some integer $d > 1$. Consider the metrics inductively defined as $||s||_n = (\varphi^* f^* ||s^{\otimes d}||_{n-1})^{\frac{1}{d}}$, where $\varphi^*||s|| = ||\varphi(s)||$.

Theorem 4.3. *The limit $||.||_0 = \lim_{n\to\infty} ||.||_n$ exists and is the unique metric on L satisfying $\varphi^* f^* ||.||_0^{\frac{1}{d}} = ||.||_0$.*

Proof. Put $h = \log ||.||_1/||.||_2$, note that h is bounded on $X(\mathbb{Q}_p)$ because $||.||_1$ and $||.||_2$ are model metrics. We can put them on a common model and consider the rational section s extending 1. Then $\log \frac{||s||_1}{||s||_2}(x)$ is the order of vanishing at the closed point of $\tilde{x}$, up to a factor $\log(p)$, so this is bounded over $X(\mathbb{Q}_p)$. Then note that $\log \frac{||.||_1}{||.||_2}(x)$ does not depend on $0 \neq s \in L(x)$, indeed if $0 \neq s'$ is a second section then $s' = fs$ for some $f \in \mathbb{Q}_p$ and we have

$$\log \frac{||s'||_1(x)}{||s'||_2(x)} = \log \frac{||s||_1(x)|f|_p}{||s||_2(x)|f|_p} = \log \frac{||s||_1(x)}{||s||_2(x)}.$$

Now note that

$$\begin{aligned} \log ||.||_n &= \frac{1}{d}\varphi^* f^* \log ||.||_{n-1} \\ \vdots &= (\frac{1}{d}\varphi^* f^*)^{n-2} \log ||.||_2 \\ &= (\frac{1}{d}\varphi^* f^*)^{n-2}(h + \log ||.||_1) \\ &= (\frac{1}{d}\varphi^* f^*)^{n-2} h + \log ||.||_{n-1}. \end{aligned}$$

Therefore $\log ||.||_n = \log ||.||_1 + \sum_{k=0}^{n-2} (\frac{1}{d}\varphi^* f^*)^k h$, since h is bounded we see that

$$\sup_{x \in X(\mathbb{Q}_p)} |(\frac{1}{d}\varphi^* f^* h| \leq \frac{1}{d^k} \sup_{x \in X(\mathbb{Q}_p)} |h|,$$

so that $h_\infty := \sum_{k=0}^{\infty} (\frac{1}{d}\varphi^* f^*)^k h$ exists and we see that $||.||_n$ converges to $||.||_0 = ||.||_1 e^{h_\infty}$.

By definition it is clear that $\varphi^* f^* ||.||_0^{\frac{1}{d}}$, now if $||.||'$ also satisfies the equation, then $\frac{||.||_0}{||.||'}$ satisfies it at well. Then we find that

$$\sup_{x \in X(\mathbb{Q}_p)} |\log \frac{||.||_0}{||.||'}| = \sup_{x \in X(\mathbb{Q}_p)} |\frac{1}{d}\varphi^* f^* \log \frac{||.||_0}{||.||'}| \leq \frac{1}{d} \sup_{x \in X(\mathbb{Q}_p)} |\log \frac{||.||_0}{||.||'}|,$$

therefore the supremum is 0 so that $||.||_0 = ||.||'$. □

Note that the theorem still holds for all p (including $p = \infty$) when one starts with a bounded metric $||.||_1$, which in case of finite p just means that $\log \frac{||.||_1}{||.||}$ is bounded for some model metric $||.||$, and in case of $p = \infty$ just means that $||.||_1$ is bounded on the compact complex manifold $X(\mathbb{C})$ for all rational sections s of L.

We want to extend the local height to metrics that are approximated by models. For the limit of heights to exist we need some positivity criteria on the line bundles. Recall that a line bundle L on a variety X is nef if $c_1(L)C \geq 0$ for all irreducible curves $C \subset X$. By Kleiman's theorem (e.g. [6] section 1.4.B) if $L_1, .., L_{\dim X}$ is a collection of nef line bundles then $c_1(L_1) \cdots c_1(L_{\dim X}) \geq 0$.

Let $(\widetilde{X}, \widetilde{L})$ be a model, we say it is vertically nef/ample if $\widetilde{L}$ is nef/ample when restricted to the closed fiber. Note that vertically ample implies vertically nef and that the pull-back of nef/ample line bundles is again nef/ample (the nef case follows from the projection formula).

A p-adic line bundle $\overline{L}$ on X is said to be semipositive if it can be approximated by vertically nef models. We say a p-adic line bundle $\overline{L}$ is integrable if it can be written as the difference of two semipositive line bundles. We make a final remark before extending the local height to integrable line bundles.

Remark 4.4. Suppose $\widetilde{X}$ is a model for X, and denote X_p the closed fiber. Let $Z \subset X$ be an irreducible d dimensional subvariety of X and let $L_1, ..., L_d$ be line bundles on X. Assume the line bundles extend to line bundles on $\widetilde{X}$ which by abuse of notation we will also call $L_1, ..., L_d$. Let $\widetilde{Z}$ be the Zariski closure of Z and let Z_p be the closed fiber of $\widetilde{Z}$. Then the intersection numbers $c_1(L_1) \cdots c_1(L_d) Z, c_1(L_1|_{X_p}) \cdots c_1(L_d|_{X_p}) Z_p$ and $c_1(L_1) \cdots c_1(L_d)\widetilde{Z}$ agree. By linearity these equalities extend to all d-cycles. We refer to [2] 20.3 for details, as we have not discussed intersections of general cycles.

Theorem 4.5. *Let $(\overline{L}_0, s_0), ..., (\overline{L}_d, s_d)$ be semipositive p-adic line bundles with s_i rational sections intersecting a d-cycle $Z \subset X$ properly. Suppose $(\widetilde{X}_i, \widetilde{L}_{i,0}, ..., \widetilde{L}_{i,d}, e_{i,1}, ..., e_{i,d})$ are vertically nef models for $L_j^{e_{i,j}}$ approximating $||.||_1, ..., ||.||_d$. Then the limit*

$$\langle (\overline{L}_0, s_0), ..., (\overline{L}_d, s_d) | Z \rangle_p := \lim_{i \to \infty} \langle (\widetilde{L}_{i,0}, s_0), ..., (\widetilde{L}_{i,d}, s_d) | Z \rangle_p$$

exists.

Proof. Let $\varepsilon > 0$ and fix $m, n \gg 0$. Let $(\widetilde{X}, \widetilde{L}_0, \widetilde{L}'_0, ..., \widetilde{L}_d, \widetilde{L}'_d, e_0 = e'_0, ..., e_d = e'_d)$ be a common model such that $(\widetilde{L}_i, e_i), (\widetilde{L}'_i, e'_i)$ induce the metrics on L_i given by the models $\widetilde{X}_m$ and $\widetilde{X}_n$ respectively. Further we may assume m and n are big enough so that

$$\left| \log \frac{||\cdot||_{\widetilde{L}_i}}{||\cdot||_{\widetilde{L}'_i}} \right| < \varepsilon. (*)$$

We need to compare

$$\frac{1}{e_0 \cdots e_n} (c_1(\widetilde{L}_0) \cdots c_1(\widetilde{L}_d) - c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_d))$$

with respect to the sections $s_0, ..., s_d$. We expand the brackets as a telescoping sum:

$$\begin{aligned} c_1(\widetilde{L}_0) \cdots c_1(\widetilde{L}_d) - c_1(\widetilde{L}'_1) \cdots c_1(\widetilde{L}'_d) &= c_1(\widetilde{L}_0) \cdots c_1(\widetilde{L}_d) - c_1(\widetilde{L}'_0) c_1(\widetilde{L}_2) \cdots c_1(\widetilde{L}_d) \\ &+ c_1(\widetilde{L}'_0) c_1(\widetilde{L}_2) \cdots c_1(\widetilde{L}_d) - c_1(\widetilde{L}'_0) c_1(\widetilde{L}'_2) c_1(\widetilde{L}_3) \cdots c_1(\widetilde{L}_d) \\ &\vdots \\ &+ c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_{d-1}) c_1(\widetilde{L}_d) - c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_d) \\ &= \sum_{k=1}^{d} c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_{k-1}) c_1(\widetilde{L}'_k \otimes \widetilde{L}_k^{-1}) c_1(\widetilde{L}_k) \cdots c_1(\widetilde{L}_d). \end{aligned}$$

Let $\widetilde{s}_k$ be the section extending $s_k \otimes s_k^{-1} = 1$. Note that it has support on the closed fiber $\widetilde{X}_p$. By (*) we have

$$p^{-\varepsilon e_0 \cdots e_d} \leq ||\widetilde{s}_k||(x) \leq p^{\varepsilon e_0 \cdots e_d}$$

for each $x \in X(\mathbb{Q}_p)$. It then follows from remark 4.1 that the divisors

$$D_{1,k} = [\varepsilon e_1 \cdots e_d] \widetilde{X}_p + \operatorname{div}(\widetilde{s}_k)$$

and

$$D_{2,k} = [\varepsilon e_1 \cdots e_d]\widetilde{X}_p - \mathrm{div}(\widetilde{s}_k)$$

are both effective, where [.] is the integer part of a real number. Using the vertical nefness we see that

$$c_1(\widetilde{L}'_0|_{D_{i,k}}) \cdots c_1(\widetilde{L}'_{k-1}|_{D_{i,k}})c_1(\widetilde{L}_{k+1}|_{D_{i,k}}) \cdots c_1(\widetilde{L}_d|_{D_{i,k}}) \geq 0,$$

for $i = 1, 2$ and $k = 0, ..., d$. Therefore the absolute value of

$$\begin{aligned} I_k &= c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_{k-1})c_1(\widetilde{L}'_k \otimes \widetilde{L}_k^{-1})c_1(\widetilde{L}_k) \cdots c_1(\widetilde{L}_d) \\ &= c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_{k-1})c_1(\widetilde{L}_{k+1}) \cdots c_1(\widetilde{L}_d)\mathrm{div}(\widetilde{s}_k) \end{aligned}$$

is bounded by

$$\varepsilon e_0 \cdots e_d c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_{k-1})c_1(\widetilde{L}_{k+1}) \cdots c_1(\widetilde{L}_d)\widetilde{X}_p,$$

which, by remark 4.4, equals

$$\varepsilon e_0 \cdots e_d c_1(L_0) \cdots c_1(L_{k-1})c_1(L_{k+1}) \cdots c_1(L_d).$$

Now using the triangle inequality on the telescoping sum we see that

$$\begin{aligned} &\frac{1}{e_0 \cdots e_d} |\deg_p(c_1(\widetilde{L}_0) \cdots c_1(\widetilde{L}_d) - c_1(\widetilde{L}'_0) \cdots c_1(\widetilde{L}'_d))| \\ &\qquad \leq \sum_{k=0}^{d} \frac{|I_k| \log(p)}{e_0 \cdots e_d} \\ &\qquad \leq \varepsilon \sum_{k=0}^{d} c_1(L_0) \cdots c_1(L_{k-1})c_1(L_{k+1}) \cdots c_1(L_d) \log(p). \end{aligned}$$

□

Note that the limit does not depend on the approximation used, indeed if $\{\widetilde{X}'_{i,j}\}_{i>0, j=0,...,k}$ also approximate the line bundles, then apply the theorem to the alternating sequences $\{\widetilde{X}_{1,j}, \widetilde{X}'_{1,j}, ...\}_{j=0,...,k}$.

We quickly mention that for $p = \infty$ there are also notions of positivity for smooth hermitian line bundles, and thus we get a notion of integrability. The proof of 4.5 for the local height at infinity is similar, the induction formula (remark 2.6) implies that the contribution of I_k is given by

$$I_k = \int_{X(\mathbb{C})} \log ||\widetilde{s}_k||^{-1} c_1(L_0) \wedge \cdots \wedge c_1(L_{k-1}) \wedge c_1(L_{k+1}) \wedge \cdots \wedge c_1(L_d).$$

Then use positivity plus the fact that $\log ||\widetilde{s}_k||^{-1} \leq \varepsilon$. Note that an integrable hermitian line bundle is not necessarily smooth anymore.

From theorem 4.5 we obtain a local height for integrable line bundles, note that by definition of the local height given in terms of intersection theory it inherits the following properties:

1. The local height is multilinear and symmetric in the $(\overline{L}_i, s_i)$ and linear in Z,

2. it satisfies an induction formula,

3. suppose $f : X \to Y$ is a proper morphism of $\mathbb{Q}_p$ varieties, then the local height satisfies the projection formula:

$$\langle (f^*\overline{L}_0, f^*s_0), ..., (f^*\overline{L}_k, f^*s_k)|Z\rangle_p = \langle (\overline{L}_0, s_0), ..., (\overline{L}_k, s_k)|f_*Z\rangle_p.$$

For 2 note that H (see remark 3.6) only depends on the section s_q, the local height with respect to H converges by the theorem. Therefore the vertical part also converges to a constant depending only on $s_0, ..., s_q$.

For 3 note that given a model $\widetilde{Y}$ of Y, there exists a model $\widetilde{X}$ of X and a proper morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ extending f. Indeed given any model $\widetilde{X}$ of X, then f determines a rational map $\widetilde{X} \to \widetilde{Y}$ and we can eliminate the indeterminacy. It is easy to see that the pull-back metric is induced by $\widetilde{X}$, since $x \in X(\mathbb{Q}_p)$ gives $f \circ x \in Y(\mathbb{Q}_p)$ and $\widetilde{x} \in \widetilde{X}(\mathbb{Z}_p)$ gives $\widetilde{f} \circ \widetilde{X} \in \widetilde{Y}(\mathbb{Z}_p)$. And we have $x^*f^*L = (f \circ x)^*L$, $\widetilde{x}^*\widetilde{f}^*L = (\widetilde{f} \circ \widetilde{x})^*L$, further as we noted before the pull-back of a nef/ample line bundle is again nef/ample.

4.3 Adelic metrics

Now assume X is a smooth projective variety over $\mathbb{Q}$. An adelic metrized line bundle on X is a collection of metrics $\{||.||_p\}_{p \in v(\mathbb{Q})}$ such that:

- for each finite p, $||.||_p$ is a p-adic metric for $L_{\mathbb{Q}_p}$ on $X_{\mathbb{Q}_p}$,

- $||.||_\infty$ is a hermitian metric,

- for every local section s of L, $||s||_p = 1$ for almost all p.

We say the adelic metrized line is integrable if all the metrics $||.||_p$ are integrable.

For example if $\widetilde{X}$ is an arithmetic variety with generic fiber X together with a smooth hermitian line bundle $\overline{L}$, then $(\widetilde{X} \times \operatorname{Spec} \mathbb{Z}_p, L \otimes \mathbb{Z}_p)$ are models for $L_{\mathbb{Q}_p}$, and the adelic metric on $L_{\mathbb{Q}}$ is integrable. An adelic metric coming from an arithmetic model will be called an adelic model metric.

Now let $\overline{L}_0, ..., \overline{L}_d$ be integrable adelic metrized line bundles on X and let Z be a d-cycle. Let $s_0, ..., s_d$ be rational sections of $L_0, ..., L_d$ such that they intersect Z properly. By theorem 4.4 the local heights exist and we define the global height to be the sum of the local heights

$$\langle \overline{L}_0, ..., \overline{L}_d|Z\rangle := \sum_{p \in v(\mathbb{Q})} \langle (\overline{L}_0, s_0), ..., (\overline{L}_d, s_d)|Z_{\mathbb{Q}_p}\rangle_p.$$

Note that by assumption the right hand side is a finite summation, and if the line bundles have adelic model metrics then we recover the height from section 3. Further note that it inherits properties 1 (multilinearity) and 3 (projection formula) from the local heights.

Proposition 4.6. *The global height does not depend on the chosen rational sections.*

Proof. Note that if s_0' is another rational section then $s_0' = fs_0$ for some rational function f, and $||fs_0'||_p = |f|_p||s_0'||$, where $|f|_p$ is the canonical p-adic metric on $\mathcal{O}_X$ and the canonical metric at ∞ as defined in section 2. By linearity it thus suffices to show that

$$\langle \mathcal{O}_X^{can}, \overline{L}_1, ..., \overline{L}_d | Z \rangle = 0.$$

We continue to use $f, s_1, ..., s_d$, first note that since $c_1(\mathcal{O}_X) = 0$ the vertical term vanishes in the induction formula for each p. Therefore we may assume $d = 0$ and $Z \in X(\mathbb{Q})$, applying the induction formula once more we have (see remark 4.1)

$$\langle (\mathcal{O}_X^{can}, f) | Z \rangle_p = -\log |f|_p.$$

Then by the product formula

$$\sum_{p \in v(\mathbb{Q}_p)} \log |f|_p = 0.$$

□

Suppose $\{\widetilde{X}_n^{(p)}\}_{n \geq 1}$ give model metrics approaching the p-adic metrics. Then we get adelic metrics for each n and can take the global height. In general there is no reason for these global heights to converge, even though the local ones do, i.e. we can't just switch the summation over p and the limit over n. For this reason one may put extra assumptions on the adelic metrics, for example that all but finitely many of the metrics are model metrics. Or if they are approachable by a sequence of adelic model metrics, then for almost all p the metric does not depend on n.

Continuing with the example in theorem 4.2 if we consider $f : X \to X$ surjective where this time X is a smooth projective variety over $\mathbb{Q}$, such that $\varphi : L^d \to f^*L$ is an isomorphism for some $d > 1$, if we further assume $(\widetilde{X}, \widetilde{L}, e)$ is an arithmetic model inducing an adelic metric $||.||$. The morphism f and the isomorphism φ extend over an open set U of $\operatorname{Spec} \mathbb{Z}$, then by normalizing the n-fold composition of this extension we obtain a morphism $\widetilde{f}_n : \widetilde{X}_n \to \widetilde{X}$ extending n-fold composition $f \circ \cdots \circ f$. Note that for $p \in U$ we have $||.||_p = (\varphi^* f^* ||.||_p)^{\frac{1}{d}}$. Now set $||.||_{n,p}$ the p-adic metric induced by the model $(\widetilde{X}_n, \widetilde{f}_n^* \widetilde{L}, nde)$. Then for $p \in U$ we have $||.||_{n,p} = ||.||_p$, and by theorem 4.2 for all p the metric $||.||_{n,p}$ converges to some metric $||.||_{0,p}$. Hence we get an adelic metric satisfying $\varphi^* f^* ||.||_0 = ||.||_0$.

Now let $\overline{L}$ be an integrable ample line bundle on X and $Z \subset X$ a closed subvariety, we define the normalized height of Z with respect to L as

$$h_{\overline{L}}(Z) := \frac{\langle \overline{L}, ..., \overline{L} | Z \rangle}{(\dim Z + 1) \deg c_1(L)^{\dim Z} Z}.$$

Now suppose f is as above and $\overline{L}$ has the adelic metric $||.||_0$, then since L is ample this metric is integrable. Further note that f is finite of degree $d^{\dim X}$.

Proposition 4.7. *In the situation above we have the equality*

$$h_{\overline{L}}(f_* Z) = d h_{\overline{L}}(Z).$$

Proof. Note that the numerator and denominator in the definition of the normalized height both satisfy the projection formula therefore we have

$$h_{\overline{L}}(f_*Z) = h_{f^*\overline{L}}(Z).$$

By definition of $||.||_0$ we have $f^*\overline{L} \cong \overline{L}^d$, therefore by multilinearity (note the global height contains $\dim Z + 1$ factors of L) we find that

$$h_{\overline{L}}(f_*Z) = \frac{d^{\dim Z+1}}{d^{\dim Z}} h_{\overline{L}}(Z) = dh_{\overline{L}}(Z).$$

□

Note that even though the local metrics from theorem 4.2 depend on the isomorphism φ up to a constant rational fucntion, by the product formula this dependence disappears so that the global height only depends on φ.

5 Neron-Tate heights

Recall that an abelian variety A over a field k is a connected projective group variety over $\operatorname{Spec} k$, where a group variety over k is a group object $(X, m, i, 0)$ in the category of varieties over $\operatorname{Spec} k$. Here $m : X \times_{\operatorname{Spec} k} X \to X$ is the multiplication, $i : X \to X$ is the inverse and $0 \in X(k)$ is the neutral element. For the basic theory of abelian varieties we use here we refer to [8].

5.1 Neron-Tate height

Let A be an abelian variety over $\mathbb{Q}$. Let $[1] = \mathrm{id}$, $[n] = m \circ (\Delta, [n-1])$ if $n > 1$, $[n] = i \circ [-n]$ if $n < 0$, i.e. $[n]$ is the multiplication by n map that sends a closed point a to $a + ... + a = na$.

Let L be a line bundle on A, then we say L is symmetric if $[-1]^*L \cong L$ and anti-symmetric if $[-1]^*L \cong L^{-1}$. By the theorem of the cube we have

$$[n]^*L \cong L^{\otimes \frac{n^2+n}{2}} \otimes [-1]^*L^{\otimes \frac{n^2-n}{2}}.$$

In particular $[n]^*L \cong L^{\otimes n^2}$ if L is symmetric and $[n]^*L \cong L^{\otimes n}$ if L is anti-symmetric.

We can always write $L^{\otimes 2}$ as the sum of a symmetric and anti-symmetric line bundle, namely

$$L_{sym} := L \otimes [-1]^*L$$

is symmetric and

$$L_{asym} := L \otimes [-1]^*L^{-1}$$

is anti-symmetric.

Now let $\widetilde{A}$ be a model for A and suppose L is an ample line bundle. Applying proposition 4.7 with $f = [n]$ and the limit metrics on L_{sym} or L_{asym} respectively shows that

$$h_{\overline{L}_{sym}}([n]_*Z) = n^2 h_{\overline{L}_{sym}}(Z),$$

$$h_{\overline{L}_{asym}}([n]_*Z) = nh_{\overline{L}_{asym}}(Z).$$

Further note that since $[nm] = [n][m]$ the heights

$$h_{\overline{L}_{sym}}, h_{\overline{L}_{asym}}$$

do not depend on $n > 1$.

Now assume L is symmetric and suppose Z is a rational point, then $h_{\overline{L}}$ is called the Neron-Tate height of A with respect to $\overline{L}$. We observe the following properties of the Neron-Tate height, we have by definition

$$h_{\overline{L}}(Z) = \widehat{c}_1(\overline{L})Z,$$

it immediately follows that this is linear in $\overline{L}$, further by the projection formula it follows that $h_{f^*\overline{L}} = h_{\overline{L}} \circ f$.

Now consider the morphisms $\sigma, \delta, \pi_1, \pi_2 : A \times A \to A$, where σ is the multiplication $(P, Q) \mapsto P + Q$, $\delta(P, Q) = P - Q$, and π_i are the projections. By the seesaw principle we have

$$\sigma^* L \otimes \delta^* L \cong \pi_1^* L^{\otimes 2} \otimes \pi_2^* L^{\otimes 2}.$$

Therefore by considering the point (P, Q) on $A \times A$ we see by the projection formula that

$$h_{\overline{L}}(P + Q) + h_{\overline{L}}(P - Q) = 2h_{\overline{L}}(P) + 2h_{\overline{L}}(Q).$$

This equality implies that

$$\langle P, Q \rangle := \frac{1}{2}(h_{\overline{L}}(P + Q) - h_{\overline{L}}(P) - h_{\overline{L}}(Q))$$

is a quadratic form on $A \times A$.

5.2 Conjectures and theorems related to heights

We state some conjectures and theorems related to the Neron-Tate heights, and the more general heights coming from theorem 4.3.

Consider X, f, L as in the end of section 4. Suppose $y \in X(\mathbb{Q})$ has a finite orbit $\{y, f(y), ...\}$ then clearly $h_{\overline{L}}(y) = 0$. One can wonder if the converse also holds, suppose the height is zero, must the orbit then also be finite. This turns out to be true, and for abelian varieties this means that a point is torsion if and only if the height is zero. Naturally one is led to wonder if the same thing is true if Y is an effective cycle of positive dimension.

Suppose C is a smooth curve of genus at least 2 over $\mathbb{Q}$, then C can be embedded into its jacobian variety J which is an abelian variety. Fix such an embedding and consider the Neron-Tate height on J with respect to a symmetric ample divisor as before. Then the Manin-Mumford conjecture states that C can only contain finitely many torsion points of J, more generally the Bogomolov conjecture states that there is some $\varepsilon > 0$ such that only finitely many points have height bounded by ε. These have both been proved by Raynaud, and Ullmo and Zhang respectively.

The Northcott property for a height function is, if there only finitely many points of bounded height and degree, e.g. only finitely many rational points. Abelian varieties satisfy the Northcott property with respect to the Neron-Tate height.

Consider again an abelian variety A over $\mathbb{Q}$, the Mordell-Weil theorem states that $A(\mathbb{Q})$ is finitely generated. Assuming $A(\mathbb{Q})/mA(\mathbb{Q})$ is finite for some $m \geq 2$ makes it possible for an easy proof using the Neron-Tate height:
Let $P_1, ..., P_k$ be representatives of $A(\mathbb{Q})/mA(\mathbb{Q})$, let $h_{\overline{L}}$ be a Neron-Tate height on A and let $C = \max_{i=1,...,k} h_{\overline{L}}(P_i)$. Then the set

$$S = \{x \in A(\mathbb{Q}) | h_{\overline{L}}(x) \leq C\}$$

is finite by the Northcott property. Now consider a point Q not in $\langle S \rangle$, by Northcott we may assume $h_{\overline{L}}(Q)$ is minimal, and write $Q = mQ' + P_i$ for some i and $Q' \notin \langle S \rangle$. Then we have

$$m^2 h_{\overline{L}}(Q') = h_{\overline{L}}(mQ') = h_{\overline{L}}(Q - P_i) \leq 2h_{\overline{L}}(Q) + 2h_{\overline{L}}(P_i) < 4h_{\overline{L}}(Q),$$

we find that $h_{\overline{L}}(Q') < h_{\overline{L}}(Q)$, therefore $Q' \in S$ which is a contradiction.

Some final remarks: the intersection theory developed in the first 3 sections holds in much more generality. For example one can intersect arbitrary cycles, consider the pull-back of cycles a long a flat morphism and there are product formulas relating the intersection theory on a product of two spaces to both spaces.

Almost all results over $\mathbb{Q}$ translate without much effort to number fields K, where one has to replace $\operatorname{Spec} \mathbb{Z}$ with $\operatorname{Spec} \mathcal{O}_K$ and take good care of the places. For example the model metrics still work by considering K_v in place of $\mathbb{Q}_p$, and the conjectures/theorems above are valid for number fields.

In arithmetic intersection theory there are a plethora of arithmetic analogues of theorems that hold in ordinary intersection theory. For example there is an arithmetic Riemann-Roch theorem. Or if $\overline{L}$ is a smooth hermitian line bundle we can consider the number of small sections

$$h^0(\overline{L}) := \#\{s \in H^0(X, L) | ||s||_{sup} \leq 1\}.$$

Then there is an arithmetic Nakai-Moishezon criterion stating when large powers of $\overline{L}$ are generated by small sections, and for example there also is an arithmetic Hilber-Szamuel formula giving the asymptotics of $h^0(\overline{L}^n)$.

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