Bounds for discount optimal strategies in a single server queue with controled arrivals and departures

Doolaard, Y.M.

Y.M. Doolaard

Bounds for discount optimal strategies in a single server queue with controlled arrivals and departures

Master's Thesis

Supervisor: Dr. F.M. Spieksma

August 15, 2018

Mathematical Institute, University of Leiden

Contents

Introduction

In this thesis, we will consider two versions of a discount single server system with exponentially distributed arrivals and departures on a discrete time scale.

In the first model, to be discussed in Chapter 1, we add a reward for every customer who has been served, and a fee for each time step a customer is in service. We also add arrival control. The model is described in more detail in Section 1.1. The goal is to find an optimal strategy that minimizes the expected total discounted cost (Section 1.2). To find this strategy, we use a direct, explicit algorithm called value iteration. This strategy turns out to be a threshold strategy: a given amount of customers is allowed to enter the system, and once this amount of customers is present, any newly arriving customer is refused.

Furthermore, we prove monotonicity properties of the model in Section 1.3. Lastly, the n-horizon cost function of value iteration is decomposed into a subsequent application of different operators (Section 1.4).

In the second chapter, the model of Chapter 1 is used, with the addition of departure control. We add an extra, faster server, so we have a slower Server 1 and a faster Server 2, and we can choose at any transition moment which server to use. Using Server 2 involves extra costs per time unit. A detailed description of the model can be found in Section 2.1.

In this model, we again want to find an optimal policy (Section 2.3), but now we first rewrite the explicit algorithm as a consecutive application of operators (Section 2.2). It turns out that we can split these operators into operators deciding whether or not an incoming customer should be accepted, and operators deciding which server to use. This was a surprising result, as we did not know in advance that the two decisions can be made independently from each other. Using these operators, we find that the optimal policy is a two-dimensional threshold strategy. In Section 2.4, we give a theorem on the relationship between the two thresholds.

Furthermore, we prove the convergence of the thresholds, and thus the strategy, and of the n-horizon relative cost function, by using two initial functions such that one approaches the optimal value function from below and the other from above (Section 2.5). This shows that the choice of the starting functions affects the results, which was unknown so far. Finally, we prove that such functions exist and give a numeric example to demonstrate the results graphically (Section 2.6).

Chapter 1

Discounted model with controlled arrivals

1.1 Model description

Consider a single server system with customers arriving according to an exponential distribution with mean $1/\lambda$ and service time exponentially distributed with mean $1/\mu$. Let the successive service times and the interarrival times be mutually independent. The state space is given by $S = \{0, 1, 2, \ldots\}$ and denotes the number of customers in the system.

Even though this model is a continuous time model, it will for now be treated as a discrete model with time steps of size $T \in \mathbb{R}_{>0}$. When there are $i \in S$ customers in this discretized system, we have the following probabilities p_{ij} to end up in state $j \in S$ after a time step:

$$p_{ij} = \begin{cases} \frac{\mu}{T}, & j = i-1, i \geq 1; \text{ or } j = i = 0, \\ 1 - \frac{\lambda}{T} - \frac{\mu}{T}, & j = i, i \geq 1, \\ \frac{\lambda}{T}, & j = i+1, i \geq 0, \\ 0, & \text{else.} \end{cases} \tag{1.1}$$

We would like to have the probability of staying in a state $i \geq 1$ after a time step equal to zero to have less probability transitions and thus a more simple structure. Therefore we must choose T such that $1 - \lambda/T - \mu/T = 0$. This gives $T = \lambda + \mu$. Note that $T \geq \lambda + \mu$ must always hold, because otherwise $p_{ij} < 0$ for $j = i, i \geq 1$.

Inserting $T = \lambda + \mu$ into Equation (1.1) gives:

$$p_{ij} = \begin{cases} \frac{\mu}{\lambda+\mu}, & j = i-1, i \geq 1 \text{ or } j = i = 0, \\ \frac{\lambda}{\lambda+\mu}, & j = i+1, i \geq 0, \\ 0, & \text{else.} \end{cases} \tag{1.2}$$

This is a very common model in the literature and many of its properties are already known (see [1] as an example). Therefore, in this chapter we will take a look at a slightly adapted form of this model, where we can decide whether or not to accept an incoming customer. The action space is denoted by $A = \{0, 1\}$ and thus there are two different actions $a \in A$. Action $a = 0$ means that an arriving customer is rejected, and action $a = 1$ that the customer is accepted. This action space A slightly changes the transition probabilities in Equation (1.2), yielding

$$p_{ij}(a) = \begin{cases} \frac{\mu}{\lambda+\mu}, & a \in \{0,1\} \text{ and } j = i-1, i \geq 1; \text{ or } a = 1 \text{ and } j = i = 0, \\ \frac{\lambda}{\lambda+\mu}, & a = 1 \text{ and } j = i+1, i \geq 0; \text{ or } a = 0 \text{ and } j = i, i \geq 1, \\ 1, & a = 0 \text{ and } j = i = 0, \\ 0, & \text{else.} \end{cases} \tag{1.3}$$

Equation (1.3) can be used to give matrices $P(a)$ with entries $p_{ij}(a)$:

$$P(0) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \cdots \\ 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & \\ \vdots & & & \ddots & \ddots \end{pmatrix}, \text{ and } P(1) = \begin{pmatrix} \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} & 0 & 0 & \cdots \\ \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & 0 & \\ 0 & \frac{\mu}{\lambda+\mu} & 0 & \frac{\lambda}{\lambda+\mu} & \\ 0 & 0 & \frac{\mu}{\lambda+\mu} & 0 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}.$$

Using this notation it is clear that $P(a)v_\alpha^n(i) = \sum_j p_{ij}(a)v_\alpha^n(j)$.

Next, a profit $R \in \mathbb{R}_{\geq 0}$ is added for every customer who has been served. We also add a fine per customer per unit time $b \cdot i$, where $b \in \mathbb{R}_{\geq 0}$ is a constant and $i \in S$ the number of customers in the system during the time unit under consideration.

Furthermore, we will view this model as a discounted model. Let $\rho \in \mathbb{R}_{\geq 0}$ be the rate of interest per time unit T. Then, $\alpha := 1/(1+\rho)$ is the discount factor.

This model is loosely based on *Example 8.9: Machine replacement model* [3].

1.2 Finding the minimizing strategy

The problem, as formulated in Section 1.1, is defined by both costs and profits. We choose to model it as a minimization problem. To do so, profits are considered to be negative costs. Of course, it could also be the other way around: a maximization problem where the cost are viewed as a negative profit, but this would clearly give the same results.

Let $c_i(a)$ be the direct cost per time unit T. With the above data for profit and fine, we get the following cost: $\forall i \in S$:

$$\begin{aligned} c_i(a) &= \begin{cases} b \cdot i - p_{i,(i+1)}(0) \cdot R, & a = 0, \\ b \cdot i - p_{i,(i+1)}(1) \cdot R, & a = 1, \end{cases} \\ &= \begin{cases} b \cdot i, & a = 0, \\ b \cdot i - \frac{\lambda}{\lambda+\mu} R, & a = 1. \end{cases} \end{aligned} \tag{1.4}$$

Let N be the total number of time steps taken into account. Let $v_\alpha^n(i)$ be the minimized expected discounted cost, with i customers in the system at time $(N-n)$. Paragraph 8.4 in [3] already gives a few useful statements, as it gives a way to iteratively find a sequence $v_\alpha^0(i), v_\alpha^1(i), v_\alpha^2(i), \ldots, \forall i \in S$ by applying Algorithm 1.1.

Algorithm 1.1. ***Successive approximation***

1. *Pick $v_\alpha^0(i) \in \mathbb{R}$ arbitrarily, $\forall i \in S$; let $f_\alpha^0(i) = 1$, $\forall i \in S$.*
2. *Let $v_\alpha^{n+1}(i) = \min_{a\in A}\left\{c_i(a) + \alpha P(a) v_\alpha^n(i)\right\}$, and $f_\alpha^{n+1}(i) = \arg\min_{a\in A}\left\{c_i(a) + \alpha P(a) v_\alpha^n(i)\right\}$, for $n = 0, 1, \ldots$, $i \in S$.*

Note that this algorithm satisfies the constraints in Paragraph 8.3.5 in [3] and thus is correct and converges towards the optimal values of $v_\alpha^*(i)$, $i \in S$.

In the following Subsection 1.2.1, the abovementioned theory is applied in a numerical example.

1.2.1 Numerical example

In order to achieve some understanding of this model, we take a look at a specific example of the model. Let $\lambda = 1$, $\mu = 2$, $R = 3$, $b = 1$, $\rho = 1/9$ and thus $\alpha = 1/(1 + 1/9) = 0.9$. These numbers plugged into Equation (1.3) give the following transition probabilities:

$$p_{ij}(a) = \begin{cases} \frac{2}{3}, & a \in \{0,1\} \text{ and } j = i-1, i \geq 1; \text{ or } a = 1 \text{ and } j = i = 0, \\ \frac{1}{3}, & a = 1 \text{ and } j = i+1, i \geq 0; \text{ or } a = 0 \text{ and } j = i, i \geq 1, \\ 1, & a = 0 \text{ and } j = i = 0, \\ 0, & \text{else.} \end{cases}$$

Together with Equation (1.4), the data give the following cost per time unit:

$$c_i(a) = \begin{cases} i, & a = 0, \\ i-1, & a = 1. \end{cases}$$

These values of $p_{ij}(a)$ and $c_i(a)$ can be used to iteratively determine v_α^n as well as f_α^n for $n \in S$, using Algorithm 1.1.

Assumption 1.2. *Let $v_\alpha^0(i) = \min_{a\in A}\left\{c_i(a)\right\} = c_i(1) = b \cdot i - \lambda/\left[\lambda + \mu\right] R = i - 1$, $\forall i \in S$.*

Assume Assumption 1.2 holds for the rest of this chapter, without any references.

When applying the second step of Algorithm 1.1 for the first time, we need to calculate $v_\alpha^1(i)$ separately for $i = 0$, since probabilities $p_{ij}(a)$ are regular for $i \geq 1$. For $i = 0$, the algorithm gives:

$$\begin{aligned} v_\alpha^1(0) &= \min\left\{c_0(0) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^0(0) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^0(0), c_0(1) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^0(1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^0(0)\right\} \\ &= \min\{-0.9, -1.6\} \\ &= -1.6, \end{aligned}$$

and therefore $f_\alpha^1(0) = 1$. For $i \geq 1$ we get:

$$\begin{aligned} v_\alpha^1(i) =& \min\Big\{c_i(0) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^0(i) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^0(i-1), \\ & c_i(1) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^0(i+1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^0(i-1)\Big\} \\ =& \min\{1.9i - 1.5, 1.9i - 2.2\} \\ =& 1.9i - 2.2, \end{aligned}$$

which gives $f_\alpha^1(i) = 1$ for $i \geq 1$. Note that $v_\alpha^1(i)$ is linear on $\{1, 2, \ldots\}$.

The following steps are all a repetition of step 2 from Algorithm 1.1. For $n = 1$ (where $v_\alpha^{n+1} = v_\alpha^2$ is calculated), it is again necessary to separately calculate $v_\alpha^2(0)$. Also, we need to calculate $v_\alpha^2(1)$ separately from the rest, because of its dependence on the value $v_\alpha^1(0)$ and the non-linearity of $v_\alpha^1(i)$ at $i = 0$.

$$\begin{aligned} v_\alpha^2(0) =& \min\Big\{c_0(0) + \alpha v_\alpha^1(0), c_0(1) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^1(1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^1(0)\Big\} \\ =& \min\{-1.44, -2.05\} \\ =& -2.05; \end{aligned}$$

$$\begin{aligned} v_\alpha^2(1) =& \min\Big\{c_1(0) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^1(1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^1(0), c_1(1) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^1(2) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^1(0)\Big\} \\ =& \min\{-0.05, -0.48\} \\ =& -0.48; \end{aligned}$$

$$\begin{aligned} v_\alpha^2(i) =& \min\Big\{c_i(0) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^1(i) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^1(i-1), \\ & c_i(1) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^1(i+1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^1(i-1)\Big\} \\ =& \min\{2.71i - 3.12, 2.71i - 3.55\} \\ =& 2.71i - 3.55, \qquad \text{for } i \geq 2. \end{aligned}$$

As can be seen in these calculations, for every $i \in S$, the minimizing action is $a = 1$, so that $f_\alpha^2(i) = 1$, $\forall i \in S$. Note that $v_\alpha^2(i)$ is linear on $i \in \{2, 3, \ldots\}$.

Now that we have obtained understanding of the calculation of both $v_\alpha^n(i)$ and $f_\alpha^n(i)$ using the iterative scheme of Algorithm 1.1, the program R will be used for the calculations of v_α^n and f_α^n for $n \geq 0$. With the results given above, we were able to verify the correctness of the R code, and with this code we were able to generate some extra values. The code can be found in Appendix A.

Some of the values of $v_\alpha^n(i)$ can be found in Table 1.1.

Note that the results in Table 1.1 confirm that the maximum state affected by the state 0 cost in

$v_\alpha^0(0) = -1.00$	$v_\alpha^0(1) = 0.00$	$v_\alpha^0(2) = 1.00$	$v_\alpha^0(3) = 2.00$	$v_\alpha^0(4) = 3.00$	$v_\alpha^0(5) = 4.00$
$v_\alpha^1(0) = -1.60$	$v_\alpha^1(1) = -0.30$	$v_\alpha^1(2) = 1.60$	$v_\alpha^1(3) = 3.50$	$v_\alpha^1(4) = 5.40$	$v_\alpha^1(5) = 7.30$
$v_\alpha^2(0) = -2.05$	$v_\alpha^2(1) = -0.48$	$v_\alpha^2(2) = 1.87$	$v_\alpha^2(3) = 4.58$	$v_\alpha^2(4) = 7.29$	$v_\alpha^2(5) = 10.00$
$v_\alpha^3(0) = -2.37$	$v_\alpha^3(1) = -0.67$	$v_\alpha^3(2) = 2.09$	$v_\alpha^3(3) = 5.31$	$v_\alpha^3(4) = 8.75$	$v_\alpha^3(5) = 12.19$
$v_\alpha^4(0) = -2.63$	$v_\alpha^4(1) = -0.80$	$v_\alpha^4(2) = 2.19$	$v_\alpha^4(3) = 5.84$	$v_\alpha^4(4) = 9.81$	$v_\alpha^4(5) = 13.90$
$v_\alpha^5(0) = -2.81$	$v_\alpha^5(1) = -0.92$	$v_\alpha^5(2) = 2.18$	$v_\alpha^5(3) = 6.07$	$v_\alpha^5(4) = 10.45$	$v_\alpha^5(5) = 15.06$
$v_\alpha^6(0) = -2.96$	$v_\alpha^6(1) = -1.04$	$v_\alpha^6(2) = 2.10$	$v_\alpha^6(3) = 6.13$	$v_\alpha^6(4) = 10.78$	$v_\alpha^6(5) = 15.79$
$v_\alpha^7(0) = -3.09$	$v_\alpha^7(1) = -1.15$	$v_\alpha^7(2) = 2.01$	$v_\alpha^7(3) = 6.10$	$v_\alpha^7(4) = 10.91$	$v_\alpha^7(5) = 16.48$

Table 1.1: Expected discounted costs $v_\alpha^n(i)$ in two decimals, where i is the number of customers in the queue at time $N-n$.

i iterations is state i.

The strategies $f_\alpha^n(i)$ corresponding to the values in Table 1.1 are given in Table 1.2.

$f_\alpha^0(0) = 1$	$f_\alpha^0(1) = 1$	$f_\alpha^0(2) = 1$	$f_\alpha^0(3) = 1$	$f_\alpha^0(4) = 1$	$f_\alpha^0(5) = 1$
$f_\alpha^1(0) = 1$	$f_\alpha^1(1) = 1$	$f_\alpha^1(2) = 1$	$f_\alpha^1(3) = 1$	$f_\alpha^1(4) = 1$	$f_\alpha^1(5) = 1$
$f_\alpha^2(0) = 1$	$f_\alpha^2(1) = 1$	$f_\alpha^2(2) = 1$	$f_\alpha^2(3) = 1$	$f_\alpha^2(4) = 1$	$f_\alpha^2(5) = 1$
$f_\alpha^3(0) = 1$	$f_\alpha^3(1) = 1$	$f_\alpha^3(2) = 1$	$f_\alpha^3(3) = 1$	$f_\alpha^3(4) = 1$	$f_\alpha^3(5) = 1$
$f_\alpha^4(0) = 1$	$f_\alpha^4(1) = 1$	$f_\alpha^4(2) = 1$	$f_\alpha^4(3) = 0$	$f_\alpha^4(4) = 0$	$f_\alpha^4(5) = 0$
$f_\alpha^5(0) = 1$	$f_\alpha^5(1) = 1$	$f_\alpha^5(2) = 0$	$f_\alpha^5(3) = 0$	$f_\alpha^5(4) = 0$	$f_\alpha^5(5) = 0$
$f_\alpha^6(0) = 1$	$f_\alpha^6(1) = 1$	$f_\alpha^6(2) = 0$	$f_\alpha^6(3) = 0$	$f_\alpha^6(4) = 0$	$f_\alpha^6(5) = 0$
$f_\alpha^7(0) = 1$	$f_\alpha^7(1) = 1$	$f_\alpha^7(2) = 0$	$f_\alpha^7(3) = 0$	$f_\alpha^7(4) = 0$	$f_\alpha^7(5) = 0$

Table 1.2: Strategy $f_\alpha^n(i)$ corresponding to the minimum values $v_\alpha^n(i)$ in Table 1.1, where i denotes the number of customers in the queue at time $N-n$.

We have plot the values of the expected discounted costs $v_\alpha^n(i)$ from Table 1.1 in Figure 1.1, and the thresholds that can be deduced from Table 1.2 in the Figure 1.2.

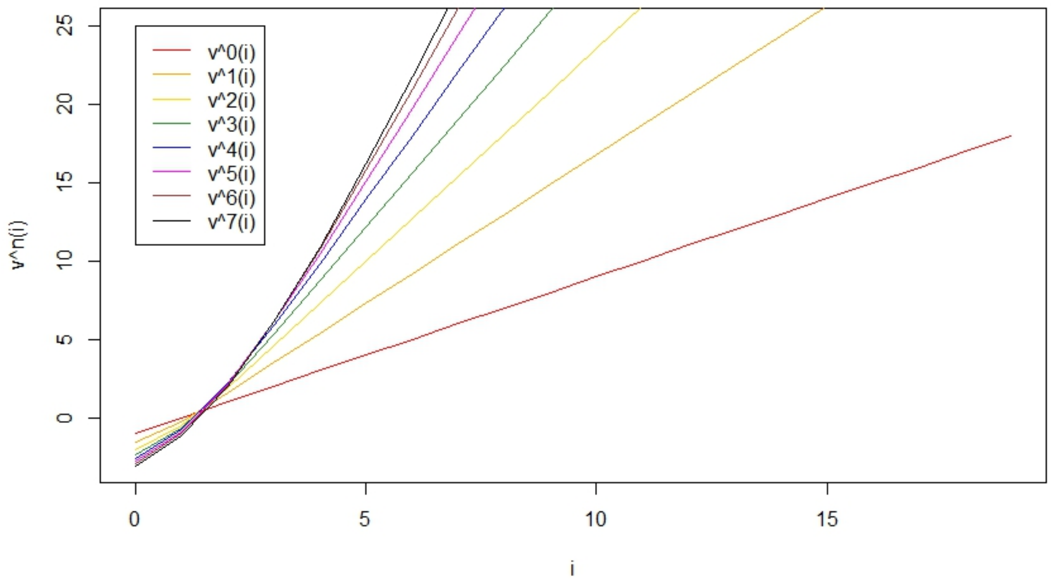


Figure 1.1: Expected discounted costs $v_\alpha^n(i)$, for several values of time step n, with parameters $\lambda = 1$, $\mu = 2$, $R = 3$, $b = 1$, $\alpha = 0.9$.

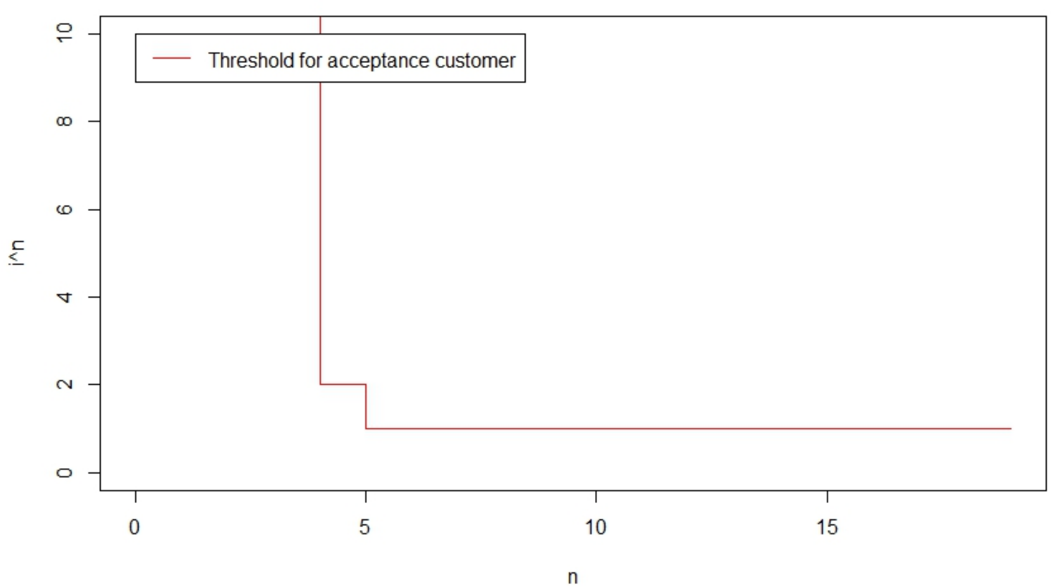


Figure 1.2: Thresholds i^n for time n, with parameters $\lambda = 1$, $\mu = 2$, $R = 3$, $b = 1$, $\alpha = 0.9$. Note that $i^n = \infty$ for $n \in \{0, \ldots, 3\}$.

1.2.2 Form of the optimal strategy

From the results of the example in Section 1.2.1, displayed in Table 1.2, we can formulate a theorem on what we would expect the optimal strategy $f_\alpha(i)$ to look like.

Theorem 1.3. *The optimal strategy $f_\alpha^n(i)$ is a threshold strategy $\forall n \in \mathbb{N}_{\geq 0}$, meaning that there exists $i^n \in S$ such that*

$$f_\alpha^n(i) = \begin{cases} 1, & i \leq i^n, \\ 0, & i > i^n. \end{cases}$$

The proof of Theorem 1.3 consists of the combination of four lemmas, which is an adaption of the roadmap given in Exercise 2.7 in [5].

Lemma 1.4. $f_\alpha^{n+1}(i) = 0$ *iff* $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$, $\forall n \geq 0$, $\forall i \in S$.

Proof. By Algorithm 1.1 holds: $v_\alpha^{n+1}(i) = \max_{a \in A} \{c_i(a) + \alpha P(a) v_\alpha^n(i)\}$, $\forall i \in S$, $n \geq 0$.

In this proof, we will distinguish the two cases $i > 0$ and $i = 0$, since for $i = 0$ the transition rates have a different structure than for $i > 0$.

<u>**Case** $i > 0$</u> The following holds for $i \in S$, $i > 0$:

$$\begin{aligned}
f_\alpha^{n+1}(i) &= \arg\min_{a \in A} \{c_i(a) + \alpha P(a) v_\alpha^n(i)\} \\
&= \arg\min \{c_i(0) + \alpha P(0) v_\alpha^n(i), c_i(1) + \alpha P(1) v_\alpha^n(i)\} \\
&= \arg\min \Bigg\{ b \cdot i + \alpha \frac{\lambda}{\lambda+\mu} v_\alpha^n(i) + \alpha \frac{\mu}{\lambda+\mu} v_\alpha^n(i-1), \\
&\qquad\qquad b \cdot i - \frac{\lambda}{\lambda+\mu} R + \alpha \frac{\lambda}{\lambda+\mu} v_\alpha^n(i+1) + \alpha \frac{\mu}{\lambda+\mu} v_\alpha^n(i-1) \Bigg\} \\
&= \arg\min \left\{ \alpha \frac{\lambda}{\lambda+\mu} v_\alpha^n(i), -\frac{\lambda}{\lambda+\mu} R + \alpha \frac{\lambda}{\lambda+\mu} v_\alpha^n(i+1) \right\} \qquad (1.5) \\
&= \arg\min \{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\},
\end{aligned}$$

where in Equation (1.5) the constant term $b \cdot i + \alpha \cdot \mu/(\lambda+\mu) v_\alpha^n(i-1)$ is left out, since it is added to both sides of the minimization term and does not affect the minimizing argument $a \in A$.

Therefore, $f_\alpha^{n+1}(i) = 0$ iff $\alpha v_\alpha^n(i) \leq -R + \alpha v_\alpha^n(i+1)$ for $i > 0$, which can be rewritten into the original form in this lemma: $f_\alpha^{n+1}(i) = 0$ iff $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$, $\forall i > 0$.

The only case left to consider is $i = 0$.

<u>**Case** $i = 0$</u> In this case, the following holds:

$$\begin{aligned}
f_\alpha^{n+1}(0) &= \arg\min_{a \in A} \{c_0(a) + \alpha P(a) v_\alpha^n(i)\} \\
&= \arg\min \left\{ \alpha v_\alpha^n(0), -\frac{\lambda}{\lambda+\mu} R + \alpha \frac{\lambda}{\lambda+\mu} v_\alpha^n(1) + \alpha \frac{\mu}{\lambda+\mu} v_\alpha^n(0) \right\}.
\end{aligned}$$

It follows that $f_\alpha^{n+1}(0) = 0$ iff $\alpha v_\alpha^n(0) \leq -\lambda/(\lambda+\mu)R + \alpha \cdot \lambda/(\lambda+\mu)v_\alpha^n(1) + \alpha \cdot \mu/(\lambda+\mu)v_\alpha^n(0)$, which results in $v_\alpha^n(1) - v_\alpha^n(0) \geq R/\alpha$.

Conclusion Now we can conclude that $\forall i \in S$: $f_\alpha^{n+1}(i) = 0$ iff $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$. $\square$

The following lemma is about some specific behaviour of $v_\alpha^n(i)$.

Lemma 1.5. *$v_\alpha^n(i)$ is a non-decreasing sequence in i, $\forall n \geq 0$.*

Proof. We prove this by induction on n.

Case $n = 0$ As in Algorithm 1.1, let

$$v_\alpha^0(i) = \min_{a\in A}\{c_i(a)\} = \min\left\{b \cdot i, b \cdot i - \frac{\lambda}{\lambda+\mu}R\right\} = b \cdot i - \frac{\lambda}{\lambda+\mu}R,$$

where the value of $c_i(a)$ is given in Equation (1.4). Now, take the difference between two succeeding values of v_α^0:

$$v_\alpha^0(i+1) - v_\alpha^0(i) = b \cdot (i+1) - \frac{\lambda}{\lambda+\mu}R - \left(b \cdot i - \frac{\lambda}{\lambda+\mu}R\right) = b \geq 0, \qquad \forall i. \tag{1.6}$$

This inequality means that v_α^0 is an increasing and thus non-decreasing sequence in i.

The general induction step is split into two parts, one for $i > 0$ and the other for $i = 0$, for the same reasons the proof of Lemma 1.9 was split into these two cases.

Case $n > 0$, $i > 0$ Assume, that v_α^n is non-decreasing, $n \geq 0$. According to Algorithm 1.1,

$$\begin{aligned}
v_\alpha^{n+1}(i) &= \min_{a\in A}\{c_i(a) + (\alpha P(a)v^n)_i\} \\
&= \min\{c_i(0) + \alpha P(0)v_\alpha^n(i), c_i(1) + \alpha P(1)v_\alpha^n(i)\} \\
&= \min\Big\{b \cdot i + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^n(i) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(i-1), \\
&\qquad b \cdot i - \frac{\lambda}{\lambda+\mu}R + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^n(i+1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(i-1)\Big\} \\
&= \frac{\lambda}{\lambda+\mu}\min\left\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\right\} + b \cdot i + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(i-1).
\end{aligned} \tag{1.7}$$

Now, take the difference between two consecutive values:

$$
\begin{aligned}
&v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i) \\
&\quad = \frac{\lambda}{\lambda+\mu} \min\big\{\alpha v_\alpha^n(i+1), -R + \alpha v_\alpha^n(i+2)\big\} + b\cdot(i+1) + \alpha\frac{\mu}{\lambda+\mu} v_\alpha^n(i) \\
&\qquad\quad - \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\big\} + b\cdot i + \alpha\frac{\mu}{\lambda+\mu} v_\alpha^n(i-1)\Big) \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^n(i+1), -R + \alpha v_\alpha^n(i+2)\big\} - \min\big\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\big\}\Big) \\
&\qquad\quad + b + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^n(i) - v_\alpha^n(i-1)\big) \qquad (1.8) \\
&\quad \geq \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^n(i+1), -R + \alpha v_\alpha^n(i+2)\big\} - \min\big\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\big\}\Big) + b. \qquad (1.9)
\end{aligned}
$$

The last inequality holds because v_α^n is non-decreasing and thus $v_\alpha^n(i) - v_\alpha^n(i-1) \geq 0$.

At this point, there are four possibly optimal strategies to consider: the action minimizing the first minimization expression in Equation (1.9) could be either zero (the first term) or one (the second term), and the same holds for the second minimization expression.

- Let $\arg\min\big\{\alpha v_\alpha^n(i+1), -R + \alpha v_\alpha^n(i+2)\big\} = 0$ and $\arg\min\big\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\big\} = 0$, which corresponds to $\big(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1)\big) = (0,0)$. Combined with Equation (1.9) this gives:

$$
v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i) \geq \frac{\lambda}{\lambda+\mu}\big(\alpha v_\alpha^n(i+1) - \alpha v_\alpha^n(i)\big) + b \geq b \geq 0,
$$

  where $v_\alpha^n(i+1) \geq v_\alpha^n(i)$ holds because v_α^n is non-decreasing.

- Let $\arg\min\big\{\alpha v_\alpha^n(i+1), -R + \alpha v_\alpha^n(i+2)\big\} = 0$ and $\arg\min\big\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\big\} = 1$, which corresponds to $\big(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1)\big) = (1,0)$. This, combined with Equation (1.9), gives:

$$
v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i) \geq \frac{\lambda}{\lambda+\mu}\big(\alpha v_\alpha^n(i+1) + R - \alpha v_\alpha^n(i+1)\big) + b = \frac{\lambda}{\lambda+\mu}R + b \geq 0.
$$

- Let $\arg\min\big\{\alpha v_\alpha^n(i+1), -R + \alpha v_\alpha^n(i+2)\big\} = 1$ and $\arg\min\big\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\big\} = 0$, which corresponds to $\big(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1)\big) = (0,1)$. This gives, together with Equation (1.9):

$$
\begin{aligned}
v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i) &\geq \frac{\lambda}{\lambda+\mu}\big(-R + \alpha v_\alpha^n(i+2) - \alpha v_\alpha^n(i)\big) + b \\
&\geq \frac{\lambda}{\lambda+\mu}\big(-R + \alpha v_\alpha^n(i+2) + R - \alpha v_\alpha^n(i+1)\big) + b \qquad (1.10) \\
&\geq 0,
\end{aligned}
$$

  where Inequality (1.10) holds because $\min\big\{\alpha v_\alpha^n(i), -R + \alpha v_\alpha^n(i+1)\big\} = \alpha v_\alpha^n(i)$. The last inequality follows from the fact that v_α^n is non-decreasing.

- Let $\arg\min\big\{\alpha v_\alpha^n(i+1), -R+\alpha v_\alpha^n(i+2)\big\} = 1$ and $\arg\min\big\{\alpha v_\alpha^n(i), -R+\alpha v_\alpha^n(i+1)\big\} = 1$, which corresponds to $\big(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1)\big) = (1,1)$. Combining this with Equation (1.9), gives:

$$v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i) \geq \frac{\lambda}{\lambda+\mu}\big(-R+\alpha v_\alpha^n(i+2) + R - \alpha v_\alpha^n(i+1)\big) + b \geq b \geq 0,$$

  which holds because v_α^n is non-decreasing.

For these four situations is shown that Equation (1.9) is always bigger than or equal to zero, and thus $v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i) \geq 0$, $\forall i > 0$.

Case $n > 0$, $i = 1$ Assume v_α^n is non-decreasing, $n \geq 0$. If v_α^{n+1} is also non-decreasing, the following inequality must hold:

$$v_\alpha^{n+1}(1) - v_\alpha^{n+1}(0) \geq 0. \tag{1.11}$$

Equation (1.7) with $i = 1$ gives an expression for $v_\alpha^{n+1}(1)$. An expression for $v_\alpha^{n+1}(0)$ can be obtained using Algorithm 1.1:

$$\begin{aligned}
v_\alpha^{n+1}(0) &= \min\left\{c_0(0) + \alpha v_\alpha^n(0), c_0(1) + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^n(1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(0)\right\} \\
&= \min\left\{b\cdot 0 + \alpha v_\alpha^n(0), b\cdot 0 - \frac{\lambda}{\lambda+\mu}R + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^n(1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(0)\right\} \\
&= \min\left\{\alpha\frac{\lambda}{\lambda+\mu}v_\alpha^n(0), -\frac{\lambda}{\lambda+\mu}R + \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^n(1)\right\} + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(0) \\
&= \frac{\lambda}{\lambda+\mu}\min\big\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\big\} + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(0).
\end{aligned} \tag{1.12}$$

We need to show that subtracting Equation (1.12) from Equation (1.7) with $i = 1$ results in Inequality (1.11).

$$\begin{aligned}
v_\alpha^{n+1}(1) - v_\alpha^{n+1}(0) = &\frac{\lambda}{\lambda+\mu}\min\big\{\alpha v_\alpha^n(1), -R+\alpha v_\alpha^n(2)\big\} + b\cdot 1 + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(0) \\
&- \left(\frac{\lambda}{\lambda+\mu}\min\big\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\big\} + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(0)\right) \\
= \frac{\lambda}{\lambda+\mu}&\Big(\min\big\{\alpha v_\alpha^n(1), -R+\alpha v_\alpha^n(2)\big\} - \min\big\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\big\}\Big) + b.
\end{aligned} \tag{1.13}$$

There are four different possibilities for the optimal strategies to consider: the action minimizing the first minimization expression in Equation (1.13) could be either zero or one, and the same holds for the second minimization expression.

- Let $\arg\min\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\} = 0$ and $\arg\min\{\alpha v_\alpha^n(1), -R+\alpha v_\alpha^n(2)\} = 0$, which corresponds to $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1)\big) = (0,0)$. These filled into Equation (1.13) gives:

$$v_\alpha^{n+1}(1) - v_\alpha^{n+1}(0) = \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^n(1) - \alpha v_\alpha^n(0)\Big) + b \geq b \geq 0,$$

  where $v_\alpha^n(1) \geq v_\alpha^n(0)$ holds by the non-decreasingness of v_α^n.

- Let $\arg\min\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\} = 1$ and $\arg\min\{\alpha v_\alpha^n(1), -R+\alpha v_\alpha^n(2)\} = 0$, which corresponds to $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1)\big) = (1,0)$. These filled into Equation (1.13) gives:

$$v_\alpha^{n+1}(1) - v_\alpha^{n+1}(0) = \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^n(1) + R - \alpha v_\alpha^n(1)\Big) + b = \frac{\lambda}{\lambda+\mu} R + b \geq 0.$$

- Let $\arg\min\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\} = 0$ and $\arg\min\{\alpha v_\alpha^n(1), -R+\alpha v_\alpha^n(2)\} = 1$, which corresponds to $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1)\big) = (0,1)$. These filled into Equation (1.13) gives:

$$\begin{aligned} v_\alpha^{n+1}(1) - v_\alpha^{n+1}(0) &= \frac{\lambda}{\lambda+\mu}\Big(-R + \alpha v_\alpha^n(2) - \alpha v_\alpha^n(0)\Big) + b \\ &\geq \frac{\lambda}{\lambda+\mu}\Big(-R + \alpha v_\alpha^n(2) + R - \alpha v_\alpha^n(1)\Big) + b \qquad (1.14) \\ &\geq 0, \end{aligned}$$

  where for Inequality (1.14) we used that $\min\big\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\big\} = \alpha v_\alpha^n(0)$ and thus $-\alpha v_\alpha^n(0) \geq R - \alpha v_\alpha^n(1)$. The last step holds because v_α^n is non-decreasing.

- Let $\arg\min\{\alpha v_\alpha^n(0), -R+\alpha v_\alpha^n(1)\} = 1$ and $\arg\min\{\alpha v_\alpha^n(1), -R+\alpha v_\alpha^n(2)\} = 1$, which corresponds to $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1)\big) = (1,1)$. These filled into Equation (1.13) gives:

$$v_\alpha^{n+1}(1) - v_\alpha^{n+1}(0) = \frac{\lambda}{\lambda+\mu}\Big(-R + \alpha v_\alpha^n(2) + R - \alpha v_\alpha^n(1)\Big) + b \geq b \geq 0,$$

  because v_α^n is non-decreasing.

In all the four possibilities of the optimal strategies, it is shown that Equation (1.11) is satisfied, and therefore v_α^{n+1} is also non-decreasing in $i = 0$.

Conclusion Combining the two induction steps, we have proven that v_α^n is a non-decreasing sequence in $i \geq 0$ for every $n \geq 0$. □

The following lemma gives a conditional statement about the form of the optimal strategy f_α^{n+1}.

Lemma 1.6. *The optimal strategy $f_\alpha^{n+1}(i)$ is a threshold strategy, if $v_\alpha^n(i)$ is convex in i, $\forall n \geq 0$.*

Proof. Let v_α^n be convex in i, i.e. $v_\alpha^n(i+2) - v_\alpha^n(i+1) \geq v_\alpha^n(i+1) - v_\alpha^n(i)$ for $i \in S$. We know that $v_\alpha^n(i)$ is non-decreasing sequence in i by Lemma 1.5.

By Lemma 1.4 we know the following:

$$f_\alpha^{n+1}(i) = 0 \Leftrightarrow v_\alpha^n(i+1) - v_\alpha^n(i) \geq \frac{R}{\alpha}.$$

Now let i^n be defined as follows:

$$i^n = \max\left\{ i \in S \cup \{\infty\} \,|\, v_\alpha^n(i+1) - v_\alpha^n(i) < \frac{R}{\alpha} \right\}.$$

Note that the convexity of $v_\alpha^n(i)$ gives that $\forall i \in \{0, 1, \ldots, i^n - 1, i^n\}$ holds $v_\alpha^n(i+1) - v_\alpha^n(i) < R/\alpha$, which is equivalent to

$$\{0, 1, \ldots, i^n - 1, i^n\} = \left\{ i | v_\alpha^n(i+1) - v_\alpha^n(i) < \frac{R}{\alpha} \right\}.$$

Lemma 1.4, combined with the given assumption that $v_\alpha^n(i)$ is non-decreasing, gives the following two possibilities for this i^n:

1. $i^n = \infty$.
   Then $f_\alpha^{n+1}(i) = 1$, $\forall i \in S$, which is a threshold strategy with threshold $i^n = \infty$.
2. $i^n \in S$.
   Then $\forall i \geq i^n$, $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$ and thus

   $$f_\alpha^{n+1}(i) = \begin{cases} 1, & 0 \leq i \leq i^n, \\ 0, & i > i^n, \end{cases}$$

   which indeed is a threshold strategy with threshold i^n.

Therefore in both cases $f_\alpha^n(i)$ is a threshold strategy as defined in Theorem 1.3. □

The following lemma is the last one needed to prove Theorem 1.3. This lemma is about the convexity of v_α^n, which was the condition used in the previous Lemma 1.6 but was not yet proven.

Lemma 1.7. *$v_\alpha^n(i)$ is convex in i, $\forall n \geq 0$.*

Proof. We will prove this by induction.

<u>**Case** $n = 0$</u> Assumption 1.2 gives an expression of $v_\alpha^0(i)$:

$$v_\alpha^0(i) = b \cdot i - \frac{\lambda}{\lambda + \mu} R.$$

This expression shows that $v_\alpha^0(i)$ is linear in i and thus convex.

The induction step is split into two parts: one for $i > 0$ and one for $i = 0$.

<u>**Case** $n > 0$**,** $i > 0$</u> Assume that v_α^n is convex for some $n \geq 0$. Then Lemma 1.6 states that the optimal strategy f_α^{n+1} is a threshold strategy.

Proving that v_α^{n+1} is convex is equivalent to showing the following:

$$v_\alpha^{n+1}(i+2) - v_\alpha^{n+1}(i+1) \geq v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i), \qquad \forall i \in S,$$

which is equivalent to proving that:

$$v_\alpha^{n+1}(i+2) - 2v_\alpha^{n+1}(i+1) + v_\alpha^{n+1}(i) \geq 0, \qquad \forall i \in S. \tag{1.15}$$

Equation (1.7) gives an expression for $v_\alpha^{n+1}(i)$ and can be used to evaluate

$$v_\alpha^{n+1}(i+2) - 2v_\alpha^{n+1}(i+1) + v_\alpha^{n+1}(i), \qquad \forall i \geq 1,$$

and see whether the outcome satisfies Inequality (1.15).

$$\begin{aligned}
&v_\alpha^{n+1}(i+2) - 2v_\alpha^{n+1}(i+1) + v_\alpha^{n+1}(i)\\
&=\frac{\lambda}{\lambda+\mu}\min\left\{\alpha v_\alpha^n(i+2), -R+\alpha v_\alpha^n(i+3)\right\} + b\cdot(i+2) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(i+1)\\
&\qquad -2\left(\frac{\lambda}{\lambda+\mu}\min\left\{\alpha v_\alpha^n(i+1), -R+\alpha v_\alpha^n(i+2)\right\} + b\cdot(i+1) + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(i)\right)\\
&\qquad +\frac{\lambda}{\lambda+\mu}\min\left\{\alpha v_\alpha^n(i), -R+\alpha v_\alpha^n(i+1)\right\} + b\cdot i + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(i-1)\\
&=\frac{\lambda}{\lambda+\mu}\min\left\{\alpha v_\alpha^n(i+2), -R+\alpha v_\alpha^n(i+3)\right\} - 2\frac{\lambda}{\lambda+\mu}\min\left\{\alpha v_\alpha^n(i+1), -R+\alpha v_\alpha^n(i+2)\right\}\\
&\qquad +\frac{\lambda}{\lambda+\mu}\min\left\{\alpha v_\alpha^n(i), -R+\alpha v_\alpha^n(i+1)\right\} + \alpha\frac{\mu}{\lambda+\mu}\left(v_\alpha^n(i+1) - 2v_\alpha^n(i) + v_\alpha^n(i-1)\right)\\
&\geq\frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha v_\alpha^n(i+2), -R+\alpha v_\alpha^n(i+3)\right\} - 2\min\left\{\alpha v_\alpha^n(i+1), -R+\alpha v_\alpha^n(i+2)\right\}\\
&\qquad +\min\left\{\alpha v_\alpha^n(i), -R+\alpha v_\alpha^n(i+1)\right\}\Big),
\end{aligned}$$

where the last inequality holds because v_α^n is convex by assumption. Also, the convexity of v_α^n combined with Lemma 1.6 gives that f_α^{n+1} is a threshold strategy, leaving four possible strategies of $\left(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1), f_\alpha^{n+1}(i+2)\right)$ which are all explored below.

- Let $\left(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1), f_\alpha^{n+1}(i+2)\right) = (0,0,0)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(i+2) - 2v_\alpha^{n+1}(i+1) + v_\alpha^{n+1}(i)\\
&\qquad\geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^n(i+2) - 2\alpha v_\alpha^n(i+1) + \alpha v_\alpha^n(i)\Big)\\
&\qquad= \alpha\frac{\lambda}{\lambda+\mu}\Big(v_\alpha^n(i+2) - 2v_\alpha^n(i+1) + v_\alpha^n(i)\Big)\\
&\qquad\geq 0,
\end{aligned}$$

  where the last inequality holds because v_α^n is convex.

- Let $\left(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1), f_\alpha^{n+1}(i+2)\right) = (1,0,0)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(i+2) - 2v_\alpha^{n+1}(i+1) + v_\alpha^{n+1}(i)\\
&\qquad\geq \frac{\lambda}{\lambda+\mu}\Big(-R + \alpha v_\alpha^n(i+3) - 2\alpha v_\alpha^n(i+1) + \alpha v_\alpha^n(i)\Big)\\
&\qquad= \frac{\lambda}{\lambda+\mu}\Big(\alpha\big[v_\alpha^n(i+3) - 2v_\alpha^n(i+1) + v_\alpha^n(i)\big] - R\Big)\\
&\qquad\geq \frac{\lambda}{\lambda+\mu}\Big(\alpha\big[v_\alpha^n(i+3) - v_\alpha^n(i+2)\big] - R\Big) \qquad (1.16)\\
&\qquad\geq 0,
\end{aligned}$$

where inequality (1.16) holds because v_α^n is convex and thus $-2v_\alpha^n(i+1)+v_\alpha^n(i) \geq -v_\alpha^n(i+2)$. The final inequality follows from $f_\alpha^{n+1}(i+2) = 0$, because that yields:

$$\alpha v_\alpha^n(i+2) \leq -R + \alpha v_\alpha^n(i+3).$$

- Let $\big(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1), f_\alpha^{n+1}(i+2)\big) = (1,1,0)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(i+2) - 2v_\alpha^{n+1}(i+1) + v_\alpha^{n+1}(i) \\
&\quad\geq \frac{\lambda}{\lambda+\mu}\Big(-R + \alpha v_\alpha^n(i+3) - 2\big(-R + \alpha v_\alpha^n(i+2)\big) + \alpha v_\alpha^n(i)\Big) \\
&\quad= \frac{\lambda}{\lambda+\mu}\Big(\alpha\big[v_\alpha^n(i+3) - 2v_\alpha^n(i+2) + v_\alpha^n(i)\big] + R\Big) \\
&\quad\geq \frac{\lambda}{\lambda+\mu}\Big(\alpha\big[-v_\alpha^n(i+1) + v_\alpha^n(i)\big] + R\Big) \qquad (1.17)\\
&\quad\geq 0,
\end{aligned}$$

where inequality (1.17) holds because v_α^n is convex and thus $v_\alpha^n(i+3)-2v_\alpha^n(i+2) \geq -v_\alpha^n(i+1)$. The last inequality follows from the fact that $f_\alpha^{n+1}(i) = 1$ and thus $\alpha v_\alpha^n(i) \geq -R+\alpha v_\alpha^n(i+1)$.

- Let $\big(f_\alpha^{n+1}(i), f_\alpha^{n+1}(i+1), f_\alpha^{n+1}(i+2)\big) = (1,1,1)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(i+2) - 2v_\alpha^{n+1}(i+1) + v_\alpha^{n+1}(i) \\
&\quad\geq \frac{\lambda}{\lambda+\mu}\Big(-R + \alpha v_\alpha^n(i+3) - 2\big(-R + \alpha v_\alpha^n(i+2)\big) - R + \alpha v_\alpha^n(i+1)\Big) \\
&\quad= \alpha\frac{\lambda}{\lambda+\mu}\Big(v_\alpha^n(i+3) - 2v_\alpha^n(i+2) + v_\alpha^n(i+1)\Big) \\
&\quad\geq 0,
\end{aligned}$$

where the final inequality is true because of the convexity of v_α^n.

Now we have shown that the convexity of v_α^n implies Inequality (1.15) for $i \geq 1$ and every possible threshold strategy f_α^{n+1}, so $v_\alpha^{n+1}(i)$ is convex for $i \geq 1$.

<u>**Case** $n > 0, i = 0$</u> It remains to prove, that

$$v_\alpha^{n+1}(2) - 2v_\alpha^{n+2}(1) + v_\alpha^{n+1}(0) \geq 0. \qquad (1.18)$$

The same induction hypothesis and assumptions hold as in the case for $n > 0, i > 0$. Equation (1.12) gives an expression for $v_\alpha^{n+1}(0)$. By Equation (1.7) we know the expressions for $v_\alpha^{n+1}(1)$ and $v_\alpha^{n+1}(2)$:

$$\begin{aligned}
v_\alpha^{n+1}(1) &= \frac{\lambda}{\lambda+\mu}\min\big\{\alpha v_\alpha^n(1), -R + \alpha v_\alpha^n(2)\big\} + b + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(0), \\
v_\alpha^{n+1}(2) &= \frac{\lambda}{\lambda+\mu}\min\big\{\alpha v_\alpha^n(2), -R + \alpha v_\alpha^n(3)\big\} + 2b + \alpha\frac{\mu}{\lambda+\mu}v_\alpha^n(1).
\end{aligned}$$

These expressions give the following equalities:

$$\begin{aligned}
&v_\alpha^{n+1}(0) - 2v_\alpha^{n+1}(1) + v_\alpha^{n+1}(2) \\
&\quad = \frac{\lambda}{\lambda+\mu} \min\big\{\alpha v_\alpha^n(0), -R + \alpha v_\alpha^n(1)\big\} + \alpha\frac{\mu}{\lambda+\mu} v_\alpha^n(0) \\
&\qquad - 2\left(\frac{\lambda}{\lambda+\mu} \min\big\{\alpha v_\alpha^n(1), -R + \alpha v_\alpha^n(2)\big\} + b + \alpha\frac{\mu}{\lambda+\mu} v_\alpha^n(0)\right) \\
&\qquad + \frac{\lambda}{\lambda+\mu} \min\big\{\alpha v_\alpha^n(2), -R + \alpha v_\alpha^n(3)\big\} + 2b + \alpha\frac{\mu}{\lambda+\mu} v_\alpha^n(1) \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^n(0), -R + \alpha v_\alpha^n(1)\big\} - 2\min\big\{\alpha v_\alpha^n(1), -R + \alpha v_\alpha^n(2)\big\} \\
&\qquad + \min\big\{\alpha v_\alpha^n(2), -R + \alpha v_\alpha^n(3)\big\}\Big) + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^n(1) - v_\alpha^n(0)\big).
\end{aligned}$$

By the convexity of v_α^n together with Lemma 1.6, we know that f_α^{n+1} is a threshold strategy, leaving us with four possibilities for strategies $f_\alpha^{n+1}(j)$, where $j = 0, 1, 2$.

- Let $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1), f_\alpha^{n+1}(2)\big) = (0, 0, 0)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(2) - 2v_\alpha^{n+1}(1) + v_\alpha^{n+1}(0) \\
&\quad = \frac{\lambda}{\lambda+\mu}\big(\alpha v_\alpha^n(2) - 2\alpha v_\alpha^n(1) + \alpha v_\alpha^n(0)\big) + \alpha\frac{\mu}{\lambda+\mu}\big(- v_\alpha^n(0) + v_\alpha^n(1)\big) \\
&\quad = \alpha\frac{\lambda}{\lambda+\mu}\big(v_\alpha^n(2) - 2v_\alpha^n(1) + v_\alpha^n(0)\big) + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^n(1) - v_\alpha^n(0)\big) \\
&\quad \geq 0,
\end{aligned}$$

  where the last inequality holds because $v_\alpha^n(2) - 2v_\alpha^n(1) + v_\alpha^n(0) \geq 0$ by the convexity of v_α^n, and $v_\alpha^n(1) - v_\alpha^n(0) \geq 0$ holds by Lemma 1.5, which states that v_α^n is a non-decreasing sequence.

- Let $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1), f_\alpha^{n+1}(2)\big) = (1, 0, 0)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(2) - 2v_\alpha^{n+1}(1) + v_\alpha^{n+1}(0) \\
&\quad = \frac{\lambda}{\lambda+\mu}\big(- R + \alpha v_\alpha^n(1) - 2\alpha v_\alpha^n(1) + \alpha v_\alpha^n(2)\big) + \alpha\frac{\mu}{\lambda+\mu}\big(- v_\alpha^n(0) + v_\alpha^n(1)\big) \\
&\quad = \frac{\lambda}{\lambda+\mu}\big(\alpha\,[v_\alpha^n(2) - v_\alpha^n(1)] - R\big) + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^n(1) - v_\alpha^n(0)\big) \\
&\quad \geq 0,
\end{aligned}$$

  where the last inequality holds because $f_\alpha^{n+1}(1) = 0$, which means that $\alpha v_\alpha^n(1) \leq -R + \alpha v_\alpha^n(2)$. Also, $v_\alpha^n(1) - v_\alpha^n(0) \geq 0$ because of Lemma 1.5.

- Let $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1), f_\alpha^{n+1}(2)\big) = (1, 1, 0)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(2) - 2v_\alpha^{n+1}(1) + v_\alpha^{n+1}(0) \\
&\quad = \frac{\lambda}{\lambda+\mu}\big(-R + \alpha v_\alpha^n(1) - 2\big[- R + \alpha v_\alpha^n(2)\big] + \alpha v_\alpha^n(2)\big) + \alpha\frac{\mu}{\lambda+\mu}\big(- v_\alpha^n(0) + v_\alpha^n(1)\big) \\
&\quad = \frac{\lambda}{\lambda+\mu}\big(\alpha\,[-v_\alpha^n(2) + v_\alpha^n(1)] + R\big) + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^n(1) - v_\alpha^n(0)\big) \\
&\quad \geq 0,
\end{aligned}$$

where the final inequality holds because $-v_\alpha^n(2) + v_\alpha^n(1) + R \geq 0$, since $f_\alpha^{n+1}(1) = 1$, and $v_\alpha^n(1) - v_\alpha^n(0) \geq 0$ because of Lemma 1.5.

- Let $\big(f_\alpha^{n+1}(0), f_\alpha^{n+1}(1), f_\alpha^{n+1}(2)\big) = (1,1,1)$. This gives the following inequalities:

$$\begin{aligned}
&v_\alpha^{n+1}(2) - 2v_\alpha^{n+1}(1) + v_\alpha^{n+1}(0) \\
&\quad= \frac{\lambda}{\lambda+\mu}\big(-R + \alpha v_\alpha^n(1) - 2\big[-R + \alpha v_\alpha^n(2)\big] - R + \alpha v_\alpha^n(3)\big) \\
&\qquad + \alpha\frac{\mu}{\lambda+\mu}\big(-v_\alpha^n(0) + v_\alpha^n(1)\big) \\
&\quad= \alpha\frac{\lambda}{\lambda+\mu}\big[v_\alpha^n(1) - 2v_\alpha^n(2) + v_\alpha^n(3)\big] + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^n(1) - v_\alpha^n(0)\big) \\
&\quad\geq 0,
\end{aligned}$$

where in the last inequality, $v_\alpha^n(1) - 2v_\alpha^n(2) + v_\alpha^n(3) \geq 0$ holds by the convexity of v_α^n, and Lemma 1.5 gives: $v_\alpha^n(1) - v_\alpha^n(0) \geq 0$.

The above calculations show that Equation (1.18) holds for all possible strategies, and therefore $v_\alpha^{n+1}(i)$ is convex in $i = 0$.

In this proof we have first shown that the convexity of $v_\alpha^n(i)$ implies that $v_\alpha^{n+1}(i)$ is also convex for $i \geq 1$. Now we have additionally proven that the convexity of $v_\alpha^n(i)$ implies Equation (1.18). Combining this with the first step of this proof, namely the convexity of $v_\alpha^0(i)$, $\forall i$, we can conclude that v_α^n is convex, $\forall n \geq 0$, $\forall i \in S$. □

Proof of Theorem 1.3.
By the lemmas stated before, we know the following:

- v_α^0 is convex in i (first part of the proof of Lemma 1.7, page 14);
- The convexity of v_α^n implies that v_α^{n+1} is also convex (Lemma 1.7);
- v_α^n being convex implies that the optimal strategy f_α^{n+1} is a threshold strategy (Lemma 1.6).

Therefore, strategy f_α^{n+1} is a threshold strategy $\forall n$, in other words, f_α^n is a threshold strategy $\forall n \in \mathbb{N}_{\geq 0}$. □

1.2.3 Convergence of the threshold

When studying the results in Table 1.2 in Section 1.2.1, we can formulate an other assumption on the convergence of the threshold in f_α^n. This hypothesis is given in the following theorem, based on Exercise 2.8 in [5].

Theorem 1.8. $(i^n)_n$ *forms a non-increasing sequence of threshold strategies, where* i^n *is the threshold of the optimal strategy in time step* n.

Before proving Theorem 1.8, first some lemmas will be stated and proven.

Lemma 1.9. *$v_\alpha^n(i+1) - v_\alpha^n(i) \geq v_\alpha^{n-1}(i+1) - v_\alpha^{n-1}(i)$ for all $i \in S$, $n \geq 1$.*

Proof. To prove this, we will use the method of induction, where the two steps will both be split into the cases $i > 0$ and $i = 0$.

Case $n = 1$, $i > 0$ In this case, we need to show that

$$v_\alpha^1(i+1) - v_\alpha^1(i) \geq v_\alpha^0(i+1) - v_\alpha^0(i), \qquad \forall i \geq 1,$$

which is equivalent to:

$$v_\alpha^1(i+1) - v_\alpha^1(i) - \left[v_\alpha^0(i+1) - v_\alpha^0(i)\right] \geq 0, \qquad \forall i \geq 1. \tag{1.19}$$

Equation (1.6) gives an expression for $v_\alpha^0(i+1) - v_\alpha^0(i)$, $\forall i$. Equation (1.8) gives an expression for $v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i)$ for $n \geq 1$. Substituting these terms into the left-hand side of Inequality (1.19) gives:

$$\begin{aligned}
&v_\alpha^1(i+1) - v_\alpha^1(i) - \left[v_\alpha^0(i+1) - v_\alpha^0(i)\right] \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha v_\alpha^0(i+1), -R + \alpha v_\alpha^0(i+2)\right\} - \min\left\{\alpha v_\alpha^0(i), -R + \alpha v_\alpha^0(i+1)\right\}\Big) \\
&\qquad\quad + b + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^0(i) - v_\alpha^0(i-1)\big) - b \\
&\quad = \frac{\lambda}{\lambda+\mu}\left(\min\left\{\alpha\left[b(i+1) - \frac{\lambda}{\lambda+\mu}R\right], -R + \alpha\left[b(i+2) - \frac{\lambda}{\lambda+\mu}R\right]\right\}\right. \\
&\qquad\quad \left. - \min\left\{\alpha\left[bi - \frac{\lambda}{\lambda+\mu}R\right], -R + \alpha\left[b(i+1) - \frac{\lambda}{\lambda+\mu}R\right]\right)\right\} + \alpha\frac{\mu}{\lambda+\mu}b \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha b, -R + 2\alpha b\right\} - \min\left\{0, -R + \alpha b\right\}\Big) + \alpha\frac{\mu}{\lambda+\mu}b.
\end{aligned} \tag{1.20}$$

From Inequality (1.19), it is clear that the expression in Equation (1.20) needs to be larger than or equal to zero for all possible optimal strategies f_α^0 and f_α^1.

By Theorem 1.3 we know that the strategy f_α^{n+1} is a threshold strategy $\forall n$. By Algorithm 1.1 we can deduce that $f_\alpha^0(i) = 1$, $\forall i \in S$, so there are no other strategies f_α^0 to be tried. Therefore only the strategy for $\big(f_\alpha^1(i), f_\alpha^1(i+1)\big)$ can be varied legitimately. Thanks to this reasoning, there are three different strategies leading to a different outcome of Equation (1.20).

- Let $\big(f_\alpha^1(i), f_\alpha^1(i+1)\big) = (0, 0)$. Then:

$$v_\alpha^1(i+1) - v_\alpha^1(i) - \left[v_\alpha^0(i+1) - v_\alpha^0(i)\right] = \frac{\lambda}{\lambda+\mu}\alpha b + \alpha\frac{\mu}{\lambda+\mu}b \geq 0,$$

  so Equation (1.19) is satisfied.

- Let $\big(f^1_\alpha(i), f^1_\alpha(i+1)\big) = (1,0)$. The following holds:

$$\begin{aligned} v^1_\alpha(i+1) - v^1_\alpha(i) - \big[v^0_\alpha(i+1) - v^0_\alpha(i)\big] &= \frac{\lambda}{\lambda+\mu}\big(\alpha b + R - \alpha b\big) + \alpha\frac{\mu}{\lambda+\mu}b \\ &= \frac{\lambda}{\lambda+\mu}R + \alpha\frac{\mu}{\lambda+\mu}b \\ &\geq 0, \end{aligned}$$

  and thus Equation (1.19) is satisfied.

- Let $\big(f^1_\alpha(i), f^1_\alpha(i+1)\big) = (1,1)$. In this case we get:

$$\begin{aligned} v^1_\alpha(i+1) - & v^1_\alpha(i) - \big[v^0_\alpha(i+1) - v^0_\alpha(i)\big] \\ &= -\frac{\lambda}{\lambda+\mu}R + 2\alpha\frac{\lambda}{\lambda+\mu}b + \frac{\lambda}{\lambda+\mu}R - \alpha\frac{\lambda}{\lambda+\mu}b + \alpha\frac{\mu}{\lambda+\mu}b \\ &= \frac{\lambda}{\lambda+\mu}b + \alpha\frac{\mu}{\lambda+\mu}b \\ &\geq b, \end{aligned}$$

  so Equation (1.19) is satisfied.

At this point we have proven that Equation (1.19) holds for all circumstances for $i \geq 1$.

Case $n = 1$, $i = 0$ To prove Lemma 1.9, we need to show the following:

$$v^1_\alpha(1) - v^1_\alpha(0) \geq v^0_\alpha(1) - v^0_\alpha(0),$$

which is equivalent to:

$$v^1_\alpha(1) - v^1_\alpha(0) - \big[v^0_\alpha(1) - v^0_\alpha(0)\big] \geq 0. \tag{1.21}$$

For the calculation of $v^0_\alpha(1) - v^0_\alpha(0)$, Equation (1.6) can be used. Equation (1.13) with $n = 0$ can be used for the calculation of $v^1_\alpha(1) - v^1_\alpha(0)$.

$$\begin{aligned} v^1_\alpha&(1) - v^1_\alpha(0) - \big[v^0_\alpha(1) - v^0_\alpha(0)\big] \\ &= \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v^0_\alpha(1), -R + \alpha v^0_\alpha(2)\big\} - \min\big\{\alpha v^0_\alpha(0), -R + \alpha v^0_\alpha(1)\big\}\Big) + b - b \\ &= \frac{\lambda}{\lambda+\mu}\left(\min\left\{\alpha\left[b - \frac{\lambda}{\lambda+\mu}R\right], -R + \alpha\left[2b - \frac{\lambda}{\lambda+\mu}R\right]\right\}\right. \\ &\qquad \left. - \min\left\{\alpha\left[-\frac{\lambda}{\lambda+\mu}R\right], -R + \alpha\left[b - \frac{\lambda}{\lambda+\mu}R\right]\right\}\right) \\ &= \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha b, -R + 2\alpha b\big\} - \min\big\{0, -R + \alpha b\big\}\Big). \end{aligned} \tag{1.22}$$

Since Equation (1.22) needs to satisfy Inequality (1.21), all possible combinations of strategies have to be distinguished to find out whether the expression is indeed greater than or equal to zero. Note that the strategy for $n = 1$ has to be a threshold strategy, so $(f^1_\alpha(0), f^1_\alpha(1)) = (0,1)$ cannot occur. Also note that the strategy for $n = 0$ is $f^0_\alpha(i) = 1$, $\forall i \in S$ by Algorithm 1.1, so no variations are possible for this strategy.

- Let $(f_\alpha^1(0), f_\alpha^1(1)) = (0, 0)$. Then Equation (1.22) becomes:

$$v_\alpha^1(1) - v_\alpha^1(0) - \big[v_\alpha^0(1) - v_\alpha^0(0)\big] = \frac{\lambda}{\lambda+\mu}\big(\alpha b - 0\big) = \frac{\lambda}{\lambda+\mu}\alpha b \geq 0,$$

  so Equation (1.21) is satisfied.

- Let $(f_\alpha^1(0), f_\alpha^1(1)) = (1, 0)$. Then Equation (1.22) becomes:

$$v_\alpha^1(1) - v_\alpha^1(0) - \big[v_\alpha^0(1) - v_\alpha^0(0)\big] = \frac{\lambda}{\lambda+\mu}\big(\alpha b + R - \alpha b\big) = \frac{\lambda}{\lambda+\mu}R \geq 0,$$

  so Equation (1.21) is satisfied.

- Let $(f_\alpha^1(0), f_\alpha^1(1)) = (1, 1)$. Then Equation 1.22 becomes:

$$v_\alpha^1(1) - v_\alpha^1(0) - \big[v_\alpha^0(1) - v_\alpha^0(0)\big] = \frac{\lambda}{\lambda+\mu}\big(-R + 2\alpha b + R - \alpha b\big) = \frac{\lambda}{\lambda+\mu}\alpha b \geq 0,$$

  so Equation (1.21) is satisfied.

In the above calculations is shown that Inequality (1.21) holds for all circumstances. This combined with the previous step in this proof shows that $v_\alpha^1(i+1) - v_\alpha^1(i) \geq v_\alpha^{n-1}(i+1) - v_\alpha^{n-1}(i)$, $\forall i \in S$.

Case $n > 1$, $i > 0$ Assume $v_\alpha^n(i+1) - v_\alpha^n(i) \geq v_\alpha^{n-1}(i+1) - v_\alpha^{n-1}(i)$ holds $\forall n \leq N$ for some $N \geq 1$. Then the induction hypothesis gives:

$$v_\alpha^N(i+1) - v_\alpha^N(i) \geq v_\alpha^{N-1}(i+1) - v_\alpha^{N-1}(i), \qquad \forall i \in S. \tag{1.23}$$

Now we need to show for $i \geq 1$, that

$$v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \big[v_\alpha^N(i+1) - v_\alpha^N(i)\big] \geq 0. \tag{1.24}$$

Equation (1.8) gives an expression for $v_\alpha^{n+1}(i+1) - v_\alpha^{n+1}(i)$ that we can insert in Equation (1.24) for $n = N$ and $n = N - 1$.

$$
\begin{aligned}
&v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \left[v_\alpha^{N}(i+1) - v_\alpha^{N}(i)\right] \\
&= \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha v_\alpha^N(i+1), -R + \alpha v_\alpha^N(i+2)\right\} - \min\left\{\alpha v_\alpha^N(i), -R + \alpha v_\alpha^N(i+1)\right\}\Big) \\
&\qquad + b + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^N(i) - v_\alpha^N(i-1)\big) - \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha v_\alpha^{N-1}(i+1), -R + \alpha v_\alpha^{N-1}(i+2)\right\} \\
&\qquad - \min\left\{\alpha v_\alpha^{N-1}(i), -R + \alpha v_\alpha^{N-1}(i+1)\right\}\Big) - b - \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^{N-1}(i) - v_\alpha^{N-1}(i-1)\big) \\
&= \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha v_\alpha^N(i+1), -R + \alpha v_\alpha^N(i+2)\right\} - \min\left\{\alpha v_\alpha^N(i), -R + \alpha v_\alpha^N(i+1)\right\} \\
&\qquad - \min\left\{\alpha v_\alpha^{N-1}(i+1), -R + \alpha v_\alpha^{N-1}(i+2)\right\} + \min\left\{\alpha v_\alpha^{N-1}(i), -R + \alpha v_\alpha^{N-1}(i+1)\right\} \Big) \\
&\qquad + \alpha\frac{\mu}{\lambda+\mu}\Big[v_\alpha^N(i) - v_\alpha^N(i-1) - \big(v_\alpha^{N-1}(i) - v_\alpha^{N-1}(i-1)\big)\Big] \\
&\geq \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha v_\alpha^N(i+1), -R + \alpha v_\alpha^N(i+2)\right\} - \min\left\{\alpha v_\alpha^N(i), -R + \alpha v_\alpha^N(i+1)\right\} \\
&\qquad - \min\left\{\alpha v_\alpha^{N-1}(i+1), -R + \alpha v_\alpha^{N-1}(i+2)\right\} + \min\left\{\alpha v_\alpha^{N-1}(i), -R + \alpha v_\alpha^{N-1}(i+1)\right\} \Big),
\end{aligned}
\tag{1.25}
$$

where the final step holds because of the induction hypothesis in Equation (1.23).

Similarly to previous proofs, all possible strategies are verified to prove that Inequality (1.24) holds in each of them. Note that according to Theorem 1.3 the optimal strategy is a threshold strategy.

However, not all combinations of threshold strategies for $f_\alpha^N(i), f_\alpha^N(i+1)$, and $f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)$, $\forall i$, appear to be valid. To prove this, we use the following two statements:

- $f_\alpha^{N+1}(i) = 0$ iff $v_\alpha^N(i+1) - v_\alpha^N(i) \geq R/\alpha$ (Lemma 1.4);
- $v_\alpha^N(i+1) - v_\alpha^N(i) \geq v_\alpha^{N-1}(i+1) - v_\alpha^{N-1}(i)$ (Equation (1.23)).

From these statements, we can deduce that $f_\alpha^N(i) = 0$ implies that $v_\alpha^{N-1}(i+1) - v_\alpha^{N-1}(i) \geq R/\alpha$. In order to satisfy Equation (1.23) also $v_\alpha^N(i+1) - v_\alpha^N(i) \geq R/\alpha$ holds, and thus $f_\alpha^{N+1}(i) = 0$. In summary, this means:

$$
f_\alpha^N(i) = 0 \qquad \Rightarrow \qquad f_\alpha^{N+1}(i) = 0. \tag{1.26}
$$

Below, the previously mentioned possible threshold strategies are given, and for each is proven that Inequality (1.24) holds. Note that the strategies made impossible by Implication (1.26) are left out.

- $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big) = (0,0)$ and $\big(f_\alpha^N(i), f_\alpha^N(i+1)\big) = (0,0)$.

In this case Equation (1.25) becomes:

$$\begin{aligned} & v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \left[v_\alpha^N(i+1) - v_\alpha^N(i)\right] \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^N(i+1) - \alpha v_\alpha^N(i) - \alpha v_\alpha^{N-1}(i+1) + \alpha v_\alpha^{N-1}(i)\Big) \\ & \geq 0, \end{aligned}$$

where the final step holds as a result of the induction hypothesis (1.23).

- $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big) = (0,0)$ and $\big(f_\alpha^N(i), f_\alpha^N(i+1)\big) = (1,0)$.
In this case Equation (1.25) becomes:

$$\begin{aligned} & v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \left[v_\alpha^N(i+1) - v_\alpha^N(i)\right] \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^N(i+1) - \alpha v_\alpha^N(i) - \alpha v_\alpha^{N-1}(i+1) - R + \alpha v_\alpha^{N-1}(i+1)\Big) \\ & = \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^N(i+1) - \alpha v_\alpha^N(i) - R\Big) \\ & \geq 0, \end{aligned}$$

where the final step holds because: $\min\big\{\alpha v_\alpha^N(i), -R + \alpha v_\alpha^N(i+1)\big\} = \alpha v_\alpha^N(i)$.

- $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big) = (1,0)$ and $\big(f_\alpha^N(i), f_\alpha^N(i+1)\big) = (1,0)$.
In this case Equation (1.25) becomes:

$$\begin{aligned} & v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \left[v_\alpha^N(i+1) - v_\alpha^N(i)\right] \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^N(i+1) + R - \alpha v_\alpha^N(i+1) - \alpha v_\alpha^{N-1}(i+1) - R + \alpha v_\alpha^{N-1}(i+1)\Big) \\ & =0. \end{aligned}$$

- $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big) = (0,0)$ and $\big(f_\alpha^N(i), f_\alpha^N(i+1)\big) = (1,1)$.
In this case Equation (1.25) becomes:

$$\begin{aligned} & v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \left[v_\alpha^N(i+1) - v_\alpha^N(i)\right] \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^N(i+1) - \alpha v_\alpha^N(i) + R - \alpha v_\alpha^{N-1}(i+2) - R + \alpha v_\alpha^{N-1}(i+1)\Big) \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^N(i+1) + R - \alpha v_\alpha^N(i+1) - \alpha v_\alpha^{N-1}(i+2) + \alpha v_\alpha^{N-1}(i+1)\Big) \qquad (1.27) \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(R - \alpha v_\alpha^{N-1}(i+2) + \alpha v_\alpha^{N-1}(i+1)\Big) \\ & \geq 0, \end{aligned}$$

where Inequality (1.27) holds because $f_\alpha^{N+1}(i) = 0$ and thus

$$-\alpha\frac{\lambda}{\lambda+\mu}v_\alpha^N(i) \geq \frac{\lambda}{\lambda+\mu}R - \alpha\frac{\lambda}{\lambda+\mu}v_\alpha^N(i+1).$$

The final inequality holds because $f_\alpha^N(i+1) = 1$ and thus $v_\alpha^{N-1}(i+2) - v_\alpha^{N-1}(i+1) \geq R/\alpha$.

- $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big) = (1,0)$ and $\big(f_\alpha^{N}(i), f_\alpha^{N}(i+1)\big) = (1,1)$.
  In this case Equation (1.25) becomes:

$$\begin{aligned} & v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \big[v_\alpha^{N}(i+1) - v_\alpha^{N}(i)\big] \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(i+1) + R - \alpha v_\alpha^{N}(i+1) + R - \alpha v_\alpha^{N-1}(i+2) - R + \alpha v_\alpha^{N-1}(i+1)\Big) \\ & = \frac{\lambda}{\lambda+\mu}\Big(R - \alpha v_\alpha^{N-1}(i+2) + \alpha v_\alpha^{N-1}(i+1)\Big) \\ & \geq 0, \end{aligned}$$

  where the final step holds by: $\min\big\{\alpha v_\alpha^{N-1}(i+1), -R + \alpha v_\alpha^{N-1}(i+2)\big\} = -R + \alpha v_\alpha^{N-1}(i+2)$.

- $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big) = (1,1)$ and $\big(f_\alpha^{N}(i), f_\alpha^{N}(i+1)\big) = (1,1)$.
  In this case Equation (1.25) becomes:

$$\begin{aligned} & v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \big[v_\alpha^{N}(i+1) - v_\alpha^{N}(i)\big] \\ & \geq \frac{\lambda}{\lambda+\mu}\Big(- R + \alpha v_\alpha^{N}(i+2) + R - \alpha v_\alpha^{N}(i+1) + R - \alpha v_\alpha^{N-1}(i+2) - R + \alpha v_\alpha^{N-1}(i+1)\Big) \\ & \geq \alpha\frac{\lambda}{\lambda+\mu}\Big(v_\alpha^{N-1}(i+2) - v_\alpha^{N-1}(i+1) - v_\alpha^{N-1}(i+2) + v_\alpha^{N-1}(i+1)\Big) \\ & = 0, \end{aligned}$$

  where the second step holds as a result of the induction hypothesis (1.23).

From the above enumeration can be concluded that for every possible combination of strategies, Inequality (1.24) holds $\forall i \geq 1$, when assuming Induction Hypothesis (1.23). The next and final step in this proof is the induction step for $i = 0$.

Case $n > 1$, $i = 0$ Assume $\exists N \geq 1$, such that $v_\alpha^n(i+1) - v_\alpha^n(i) \geq v_\alpha^{n-1}(i+1) - v_\alpha^{n-1}(i)$ holds $\forall n \leq N$, $\forall i \geq 0$. This assumption gives Inequality (1.23), which was used in the previous step of this proof as well. The inequality we need to prove in this case is:

$$v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\alpha^{N}(1) - v_\alpha^{N}(0)\big] \geq 0. \tag{1.28}$$

Substituting Equation (1.13) into Equation (1.28) gives:

$$\begin{aligned} & v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\alpha^{N}(1) - v_\alpha^{N}(0)\big] \\ & = \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^{N}(1), -R + \alpha v_\alpha^{N}(2)\big\} - \min\big\{\alpha v_\alpha^{N}(0), -R + \alpha v_\alpha^{N}(1)\big\}\Big) + b \\ & \qquad - \bigg(\frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^{N-1}(1), -R + \alpha v_\alpha^{N-1}(2)\big\} - \min\big\{\alpha v_\alpha^{N-1}(0), -R + \alpha v_\alpha^{N-1}(1)\big\}\Big) + b\bigg) \\ & = \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^{N}(1), -R + \alpha v_\alpha^{N}(2)\big\} - \min\big\{\alpha v_\alpha^{N}(0), -R + \alpha v_\alpha^{N}(1)\big\} \\ & \qquad - \min\big\{\alpha v_\alpha^{N-1}(1), -R + \alpha v_\alpha^{N-1}(2)\big\} + \min\big\{\alpha v_\alpha^{N-1}(0), -R + \alpha v_\alpha^{N-1}(1)\big\}\Big). \end{aligned} \tag{1.29}$$

Again, we need to go through all possible strategies to prove that Equation (1.28) holds in all cases. Using Implications (1.26), some combinations of strategies can be excluded, just like we did in the case with $n > 0$, $i > 0$. We will not further discuss these.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (0,0)$ and $\big(f_\alpha^{N}(0), f_\alpha^{N}(1)\big) = (0,0)$.
  In this case Equation (1.29) becomes:

$$\begin{aligned} v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) &- \big[v_\alpha^{N}(1) - v_\alpha^{N}(0)\big] \\ &= \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(1) - \alpha v_\alpha^{N}(0) - \alpha v_\alpha^{N-1}(1) + \alpha v_\alpha^{N-1}(0)\Big) \\ &\geq 0, \end{aligned}$$

  where the last inequality follows from the induction hypothesis.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (0,0)$ and $\big(f_\alpha^{N}(0), f_\alpha^{N}(1)\big) = (1,0)$.
  In this case Equation (1.29) becomes:

$$\begin{aligned} v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) &- \big[v_\alpha^{N}(1) - v_\alpha^{N}(0)\big] \\ &= \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(1) - \alpha v_\alpha^{N}(0) - \alpha v_\alpha^{N-1}(1) - R + \alpha v_\alpha^{N-1}(1)\Big) \\ &= \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(1) - v_\alpha^{N}(0) - R\Big) \\ &\geq 0, \end{aligned}$$

  where the last inequality holds by Lemma 1.4, which states that $f_\alpha^{N+1}(0) = 0$ implies that $v_\alpha^{N}(1) - v_\alpha^{N}(0) \geq R/\alpha$.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (1,0)$ and $\big(f_\alpha^{N}(0), f_\alpha^{N}(1)\big) = (1,0)$.
  In this case Equation (1.29) becomes:

$$\begin{aligned} v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) &- \big[v_\alpha^{N}(1) - v_\alpha^{N}(0)\big] \\ &= \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(1) + R - \alpha v_\alpha^{N}(1) - \alpha v_\alpha^{N-1}(1) - R + \alpha v_\alpha^{N-1}(1)\Big) \\ &= 0. \end{aligned}$$

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (0,0)$ and $\big(f_\alpha^{N}(0), f_\alpha^{N}(1)\big) = (1,1)$.
  In this case Equation (1.29) becomes:

$$\begin{aligned} v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) &- \big[v_\alpha^{N}(1) - v_\alpha^{N}(0)\big] \\ &= \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(1) - \alpha v_\alpha^{N}(0) + R - \alpha v_\alpha^{N-1}(2) - R + \alpha v_\alpha^{N-1}(1)\Big) \\ &\geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(1) - \alpha v_\alpha^{N}(0) - \alpha v_\alpha^{N-1}(1) - R + \alpha v_\alpha^{N-1}(1)\Big) \qquad (1.30) \\ &= \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N}(1) - \alpha v_\alpha^{N}(0) - R\Big) \\ &\geq 0, \end{aligned}$$

  where Inequality (1.30) holds, since $f_\alpha^{N}(1) = 1$ implies that $R - \alpha v_\alpha^{N-1}(2) \geq -\alpha v_\alpha^{N-1}(1)$. The last inequality holds by Lemma 1.4, stating that $f_\alpha^{N+1}(0) = 0$ implies $v_\alpha^{N}(1) - v_\alpha^{N}(0) \geq R/\alpha$.

- Let $\left(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\right) = (1,0)$ and $\left(f_\alpha^{N}(0), f_\alpha^{N}(1)\right) = (1,1)$.
In this case Equation (1.29) becomes:

$$\begin{aligned} v_\alpha^{N+1}(1) - & v_\alpha^{N+1}(0) - \left[v_\alpha^N(1) - v_\alpha^N(0)\right] \\ &= \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^N(1) + R - \alpha v_\alpha^N(1) + R - \alpha v_\alpha^{N-1}(2) - R + \alpha v_\alpha^{N-1}(1)\Big) \\ &= \frac{\lambda}{\lambda+\mu}\Big(R - \alpha v_\alpha^{N-1}(2) + \alpha v_\alpha^{N-1}(1)\Big) \\ &\geq 0, \end{aligned}$$

where the last inequality holds by Lemma 1.4, which states that $f_\alpha^N(1) = 1$ implies that $v_\alpha^{N-1}(2) - v_\alpha^{N-1}(1) \leq R/\alpha$.

- Let $\left(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\right) = (1,1)$ and $\left(f_\alpha^{N}(0), f_\alpha^{N}(1)\right) = (1,1)$.
In this case Equation (1.29) becomes:

$$\begin{aligned} v_\alpha^{N+1}(1) - & v_\alpha^{N+1}(0) - \left[v_\alpha^N(1) - v_\alpha^N(0)\right] \\ &= \frac{\lambda}{\lambda+\mu}\Big(-R + \alpha v_\alpha^N(2) + R - \alpha v_\alpha^N(1) + R - \alpha v_\alpha^{N-1}(2) - R + \alpha v_\alpha^{N-1}(1)\Big) \\ &\geq \frac{\lambda}{\lambda+\mu}\Big(\alpha v_\alpha^{N-1}(2) - \alpha v_\alpha^{N-1}(1) - \alpha v_\alpha^{N-1}(2) + \alpha v_\alpha^{N-1}(1)\Big) \qquad (1.31) \\ &= 0, \end{aligned}$$

where Inequality (1.31) holds by the induction hypothesis $v_\alpha^N(2) - v_\alpha^N(1) \geq v_\alpha^{N-1}(2) - v_\alpha^{N-1}(1)$.

Now we have shown that in all possible combinations of strategies Inequality (1.28) holds, when given Induction Hypothesis (1.23). These complete the Induction step and hence the proof of Lemma 1.9. □

Proof of Theorem 1.8:
For the clarity of the proof, we will first give an overview of useful results achieved earlier.

- $v_\alpha^n(i+1) - v_\alpha^n(i) \geq v_\alpha^{n-1}(i+1) - v_\alpha^{n-1}(i)$ for all $i \in S, n \geq 1$ (Lemma 1.9);
- $f_\alpha^{n+1}(i) = 0$ iff $v_\alpha^n(i+1) - v_\alpha^n(i) \geq \frac{R}{\alpha}$ (Lemma 1.4);
- The optimal strategy $f_\alpha^n(i)$ is a threshold strategy $\forall n$, meaning that $\exists i_n^* \in S$ such that $f_\alpha^n(i) = 1$, $\forall i \leq i_n^*$ and $f_\alpha(i) = 0$, $\forall i > i_n^*$ (Theorem 1.3).

Now, let $i^N \in S$ be the threshold for strategy f_α^{N+1} for some $N \geq 0$. Then, $\forall i \leq i^N$ holds: $v_\alpha^N(i+1) - v_\alpha^N(i) < \frac{R}{\alpha}$. By the fact that $v_\alpha^N(i+1) - v_\alpha^N(i) \geq v_\alpha^{N-1}(i+1) - v_\alpha^{N-1}(i)$ follows that $v_\alpha^{N-1}(i+1) - v_\alpha^{N-1}(i) < \frac{R}{\alpha}$ and thus $f_\alpha^N(i) = 1$.

Now, we may conclude that $f_\alpha^{N+1}(i) = 1 \Rightarrow f_\alpha^N(i) = 1$, which is equivalent to $f_\alpha^N(i) \neq 1 \Rightarrow f_\alpha^{N+1}(i) \neq 1$. This can be rewritten into $f_\alpha^N(i) = 0 \Rightarrow f_\alpha^{N+1}(i) = 0$ (which is equivalent to Implication (1.26)), which means that once a strategy N does not allow the ith customer to enter the queue, strategy $N+1$ will also decline any ith customer in line. Thus $(i^n)_n$ is a non-increasing sequence of threshold strategies. □

All the above theorems and lemmas are proven for cost function $c_i(a)$ from Equation (1.4). However, in these proofs we only used the fact that $c_i(a)$ is a non-decreasing and convex function in i. Therefore, we can formulate the following theorem:

Theorem 1.10. *Theorems 1.3 and 1.8 hold for every cost function $C_i(a)$ of the form*

$$C_i(a) = \begin{cases} B(i), & a = 0, \\ B(i) - p_{i,(i+1)}(1)R, & a = 1, \end{cases}$$

where B is a non-decreasing convex function and $p_{i,(i+1)}(1) = \frac{\lambda}{\lambda+\mu}$ as defined in Equation (1.3).

The proof of this theorem consists of the proofs of Theorems 1.3 and 1.8 and their corresponding lemmas using the properties of $C_i(a)$ given in Theorem 1.10 instead of $c_i(a)$ given in Equation (1.4).

1.3 Monotonicity of the model

Next to the limiting behaviour of the optimal strategy $f_\alpha^n(i)$, it is also interesting to look at the behaviour of the model for different values of α. A presumption of this behaviour is stated in the following theorem, and proven afterwards.

Theorem 1.11. *Let $\alpha < \beta$. Then: threshold $i_\alpha^n \geq i_\beta^n$, $\forall n \geq 0$.*

Before proving Theorem 1.11, we will state and prove some lemmas.

Lemma 1.12. *Let $\alpha < \beta$. Assume $v_\alpha^n(i+1) - v_\alpha^n(i) - \big[v_\beta^n(i+1) - v_\beta^n(i)\big] \leq 0$, $\forall i \in S$ for some $n \in \mathbb{N}_{\geq 0}$. Then $\alpha\big[v_\alpha^n(i+1) - v_\alpha^n(i)\big] - \beta\big[v_\beta^n(i+1) - v_\beta^n(i)\big] \leq 0$.*

Proof. Lemma 1.5 states that $v_\alpha^n(i)$ is a non-decreasing sequence in i. Therefore, in inequality

$$v_\alpha^n(i+1) - v_\alpha^n(i) \leq v_\beta^n(i+1) - v_\beta^n(i), \tag{1.32}$$

both sides are positive. Since $\alpha < \beta$, multiplying the smaller side of (1.32) with α and the larger side with β will preserve the inequality, resulting in the assertion of the lemma. □

With the help of the following lemma, we will be able to prove Theorem 1.11.

Lemma 1.13. *Let $\alpha < \beta$. Then: $v_\alpha^n(i+1) - v_\alpha^n(i) \leq v_\beta^n(i+1) - v_\beta^n(i)$.*

Proof. In this proof, we need to show that:

$$v_\alpha^n(i+1) - v_\alpha^n(i) - \big[v_\beta^n(i+1) - v_\beta^n(i)\big] \leq 0, \qquad \forall i \in S, n \geq 0. \tag{1.33}$$

We use induction to show this. The induction step is split into two parts by considering the cases $i = 0$ and $i > 0$ separately.

Case $n = 0$**,** $i \geq 0$ Equation (1.6) substituted into Equation (1.33) with $n = 0$ gives:

$$v_\alpha^0(i+1) - v_\alpha^0(i) - \left[v_\beta^0(i+1) - v_\beta^0(i)\right] = b - b = 0 \leq 0, \qquad \forall i \in S,$$

which means that Equation (1.33) is satisfied for $n = 0$, $i \geq 0$.

At this point, we can proceed with the induction step.

Induction step Assume $\exists N \geq 0$ such that Inequality (1.33) holds $\forall i \in S$, $\forall n \leq N$.

Induction step, $i > 0$ Then Equation (1.8) yields:

$$\begin{aligned}
&v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \left[v_\beta^{N+1}(i+1) - v_\beta^{N+1}(i)\right] \\
&= \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\alpha v_\alpha^N(i+1), -R + \alpha v_\alpha^N(i+2)\right\} - \min\left\{\alpha v_\alpha^N(i), -R + \alpha v_\alpha^N(i+1)\right\}\Big) \\
&\qquad + b + \alpha\frac{\mu}{\lambda+\mu}\big(v_\alpha^N(i) - v_\alpha^N(i-1)\big) \\
&\qquad - \frac{\lambda}{\lambda+\mu}\Big(\min\left\{\beta v_\beta^N(i+1), -R + \beta v_\beta^N(i+2)\right\} - \min\left\{\beta v_\beta^N(i), -R + \beta v_\beta^N(i+1)\right\}\Big) \\
&\qquad - b - \beta\frac{\mu}{\lambda+\mu}\big(v_\beta^N(i) - v_\beta^N(i-1)\big) \\
&= \frac{\lambda}{\lambda+\mu}\Big[\min\left\{\alpha v_\alpha^N(i+1), -R + \alpha v_\alpha^N(i+2)\right\} - \min\left\{\alpha v_\alpha^N(i), -R + \alpha v_\alpha^N(i+1)\right\} \\
&\qquad - \min\left\{\beta v_\beta^N(i+1), -R + \beta v_\beta^N(i+2)\right\} + \min\left\{\beta v_\beta^N(i), -R + \beta v_\beta^N(i+1)\right\}\Big] \\
&\qquad + \frac{\mu}{\lambda+\mu}\Big[\alpha\big(v_\alpha^N(i) - v_\alpha^N(i-1)\big) - \beta\big(v_\beta^N(i) - v_\beta^N(i-1)\big)\Big] \\
&\leq \frac{\lambda}{\lambda+\mu}\Big[\min\left\{\alpha v_\alpha^N(i+1), -R + \alpha v_\alpha^N(i+2)\right\} - \min\left\{\alpha v_\alpha^N(i), -R + \alpha v_\alpha^N(i+1)\right\} \\
&\qquad - \min\left\{\beta v_\beta^N(i+1), -R + \beta v_\beta^N(i+2)\right\} + \min\left\{\beta v_\beta^N(i), -R + \beta v_\beta^N(i+1)\right\}\Big],
\end{aligned} \tag{1.34}$$

where the last inequality follows from Lemma 1.12.

Now, assume that $f_\alpha^{N+1}(i) = 0$. Then, Lemma 1.4 states, that $v_\alpha^N(i+1) - v_\alpha^N(i) \geq R/\alpha$, which is equivalent to $\alpha\big(v_\alpha^N(i+1) - v_\alpha^N(i)\big) \geq R$. By Lemma 1.12, it follows that $\beta\big(v_\beta^N(i+1) - v_\beta^N(i)\big) \geq R$ and therefore $v_\beta^N(i+1) - v_\beta^N(i) \geq R/\beta$. By Lemma 1.4, this implies, that $f_\beta^{N+1}(i) = 0$. Summarising this means, that:

$$f_\alpha^{N+1}(i) = 0 \qquad \Rightarrow \qquad f_\beta^{N+1}(i) = 0. \tag{1.35}$$

For strategies $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big)$ and $\big(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\big)$, there are three possible threshold strategies. This results in nine combinations of threshold strategies. But given Implication (1.35), not every combination of strategies is feasible. Below we will only name the pairs of threshold strategies that are valid by Implication (1.35), and show that Inequality (1.33) holds.

- Let $\left(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\right) = (0,0)$ and $\left(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\right) = (0,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned} v_\alpha^{N+1}&(i+1) - v_\alpha^{N+1}(i) - \left[v_\beta^{N+1}(i+1) - v_\beta^{N+1}(i)\right] \\ &\le \frac{\lambda}{\lambda+\mu}\Big[\alpha v_\alpha^N(i+1) - \alpha v_\alpha^N(i) - \beta v_\beta^N(i+1) + \beta v_\beta^N(i)\Big] \\ &\le 0, \end{aligned}$$

  where the last inequality holds by Lemma 1.12 combined with the induction hypothesis.

- Let $\left(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\right) = (1,0)$ and $\left(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\right) = (0,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned} v_\alpha^{N+1}&(i+1) - v_\alpha^{N+1}(i) - \left[v_\beta^{N+1}(i+1) - v_\beta^{N+1}(i)\right] \\ &\le \frac{\lambda}{\lambda+\mu}\Big[\alpha v_\alpha^N(i+1) + R - \alpha v_\alpha^N(i+1) - \beta v_\beta^N(i+1) + \beta v_\beta^N(i)\Big] \\ &\le 0, \end{aligned}$$

  where the last inequality holds because $f_\beta^{N+1}(i) = 0$ and thus $\beta v_\beta^N(i) \le -R + \beta v_\beta^N(i+1)$.

- Let $\left(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\right) = (1,1)$ and $\left(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\right) = (0,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned} v_\alpha^{N+1}&(i+1) - v_\alpha^{N+1}(i) - \left[v_\beta^{N+1}(i+1) - v_\beta^{N+1}(i)\right] \\ &\le \frac{\lambda}{\lambda+\mu}\Big[-R + \alpha v_\alpha^N(i+2) + R - \alpha v_\alpha^N(i+1) - \beta v_\beta^N(i+1) + \beta v_\beta^N(i)\Big] \\ &\le \frac{\lambda}{\lambda+\mu}\Big[R - \beta v_\beta^N(i+1) + \beta v_\beta^N(i)\Big] \qquad (1.36) \\ &\le 0, \end{aligned}$$

  where Inequality (1.36) holds, as $f_\alpha^{N+1}(i+1) = 1$ implies that $\alpha v_\alpha^N(i+2) - \alpha v_\alpha^N(i+1) \le R$. The final inequality holds, because $f_\beta^N(i) = 0$, implying that $\beta v_\beta^N(i) - \beta v_\beta^N(i+1) + R \le 0$.

- Let $\left(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\right) = (1,0)$ and $\left(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\right) = (1,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned} v_\alpha^{N+1}&(i+1) - v_\alpha^{N+1}(i) - \left[v_\beta^{N+1}(i+1) - v_\beta^{N+1}(i)\right] \\ &\le \frac{\lambda}{\lambda+\mu}\Big[\alpha v_\alpha^N(i+1) + R - \alpha v_\alpha^N(i+1) - \beta v_\beta^N(i+1) - R + \beta v_\beta^N(i+1)\Big] \\ &= 0. \end{aligned}$$

- Let $\left(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\right) = (1,1)$ and $\left(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\right) = (1,0)$.

Then, the left-hand side of Equation (1.34) becomes:

$$
\begin{aligned}
&v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \big[v_\beta^{N+1}(i+1) - v_\beta^{N+1}(i)\big] \\
&\quad \le \frac{\lambda}{\lambda+\mu}\Big[-R + \alpha v_\alpha^N(i+2) + R - \alpha v_\alpha^N(i+1) - \beta v_\beta^N(i+1) - R + \beta v_\beta^N(i+1)\Big] \\
&\quad \le \frac{\lambda}{\lambda+\mu}\Big[R - \beta v_\beta^N(i+1) + \beta v_\beta^N(i)\Big] \\
&\quad \le 0,
\end{aligned}
$$

where the second inequality holds because $f_\alpha^{N+1}(i+1) = 1$ and therefore

$$
-R + \alpha v_\alpha^{N+1}(i+2) \le \alpha v_\alpha^{N+1}(i+1).
$$

- Let $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big) = (1,1)$ and $\big(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\big) = (1,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$
\begin{aligned}
&v_\alpha^{N+1}(i+1) - v_\alpha^{N+1}(i) - \big[v_\beta^{N+1}(i+1) - v_\beta^{N+1}(i)\big] \\
&\quad \le \frac{\lambda}{\lambda+\mu}\Big[-R + \alpha v_\alpha^N(i+2) + R - \alpha v_\alpha^N(i+1) + R - \beta v_\beta^N(i+2) - R + \beta v_\beta^N(i+1)\Big] \\
&\quad \le 0,
\end{aligned}
$$

  where the last inequality holds by Lemma 1.12 combined with the induction hypothesis.

So far, we have proven that, given the induction hypothesis, Inequality (1.33) holds for every possible combination of threshold strategies for $\big(f_\alpha^{N+1}(i), f_\alpha^{N+1}(i+1)\big)$ and $\big(f_\beta^{N+1}(i), f_\beta^{N+1}(i+1)\big)$ for $i > 0$. To finish this proof, we need to show that, given the induction hypothesis, Inequality (1.33) holds for $i = 0$.

Induction step, $i = 0$ Note, that we still have the assumption that $\exists N \ge 0$ such that Inequality (1.33) holds $\forall i \in S$, $\forall n \le N$. Equation (1.13) gives:

$$
\begin{aligned}
&v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\beta^{N+1}(1) - v_\beta^{N+1}(0)\big] \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^N(1), -R + \alpha v_\alpha^N(2)\big\} - \min\big\{\alpha v_\alpha^N(0), -R + \alpha v_\alpha^N(1)\big\}\Big) + b \\
&\qquad\quad - \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\beta v_\beta^N(1), -R + \beta v_\beta^N(2)\big\} - \min\big\{\beta v_\beta^N(0), -R + \beta v_\beta^N(1)\big\}\Big) - b \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big(\min\big\{\alpha v_\alpha^N(1), -R + \alpha v_\alpha^N(2)\big\} - \min\big\{\alpha v_\alpha^N(0), -R + \alpha v_\alpha^N(1)\big\} \\
&\qquad\quad - \min\big\{\beta v_\beta^N(1), -R + \beta v_\beta^N(2)\big\} + \min\big\{\beta v_\beta^N(0), -R + \beta v_\beta^N(1)\big\}\Big).
\end{aligned}
$$

With a reasoning similar to the one leading to Implication (1.35), we can state that $f_\alpha^{N+1}(0) = 0$ is equivalent to $\alpha v_\alpha^N(1) - \alpha v_\alpha^N(0) \ge R$, which implies $\beta v_\beta^N(1) - \beta v_\beta^N(0) \ge R$, which is equivalent to $f_\beta^{N+1}(0) = 0$. In summary, this means:

$$
f_\alpha^{N+1}(0) = 0 \qquad \Rightarrow \qquad f_\beta^{N+1}(0) = 0. \tag{1.37}
$$

For $i = 1$, Implication (1.35) can be used.

Just as in the case of the induction step with $i > 0$, there are nine possible combinations of threshold strategies for $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big)$. Three of these are impossible by Implication (1.37). We will check the remaining combinations below to see if Inequality (1.33) holds.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (0,0)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big) = (0,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned}
&v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\beta^{N+1}(1) - v_\beta^{N+1}(0)\big] \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big[\alpha v_\alpha^N(1) - \alpha v_\alpha^N(0) - \beta v_\beta^N(1) + \beta v_\beta^N(0)\Big] \\
&\quad \leq 0,
\end{aligned}$$

  where the last inequality holds by Lemma 1.12 combined with the induction hypothesis.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (1,0)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big) = (0,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned}
&v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\beta^{N+1}(1) - v_\beta^{N+1}(0)\big] \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big[\alpha v_\alpha^N(1) + R - \alpha v_\alpha^N(1) - \beta v_\beta^N(1) + \beta v_\beta^N(0)\Big] \\
&\quad \leq 0,
\end{aligned}$$

  where the last inequality holds because $f_\beta^{N+1}(0) = 0$.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (1,1)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big) = (0,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned}
&v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\beta^{N+1}(1) - v_\beta^{N+1}(0)\big] \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big[-R + \alpha v_\alpha^N(2) + R - \alpha v_\alpha^N(1) - \beta v_\beta^N(1) + \beta v_\beta^N(0)\Big] \\
&\quad \leq \frac{\lambda}{\lambda+\mu}\Big[R - \beta v_\beta^N(1) + \beta v_\beta^N(0)\Big] \qquad (1.38) \\
&\quad \leq 0,
\end{aligned}$$

  where Inequality (1.38) holds because $f_\alpha^{N+1}(1) = 1$. The final inequality holds because $f_\beta^N(0) = 0$.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (1,0)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big) = (1,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned}
&v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\beta^{N+1}(1) - v_\beta^{N+1}(0)\big] \\
&\quad = \frac{\lambda}{\lambda+\mu}\Big[\alpha v_\alpha^N(1) + R - \alpha v_\alpha^N(1) - \beta v_\beta^N(1) - R + \beta v_\beta^N(1)\Big] \\
&\quad = 0.
\end{aligned}$$

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (1,1)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big) = (1,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned}
&v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\beta^{N+1}(1) - v_\beta^{N+1}(0)\big]\\
&\quad= \frac{\lambda}{\lambda+\mu}\Big[- R + \alpha v_\alpha^N(2) + R - \alpha v_\alpha^N(1) - \beta v_\beta^N(1) - R + \beta v_\beta^N(1)\Big]\\
&\quad\le \frac{\lambda}{\lambda+\mu}\Big[R - \beta v_\beta^N(1) + \beta v_\beta^N(0)\Big]\\
&\quad\le 0,
\end{aligned}$$

  where the last inequality holds because $f_\alpha^{N+1}(1) = 1$.

- Let $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big) = (1,1)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big) = (1,0)$.
  Then, the left-hand side of Equation (1.34) becomes:

$$\begin{aligned}
&v_\alpha^{N+1}(1) - v_\alpha^{N+1}(0) - \big[v_\beta^{N+1}(1) - v_\beta^{N+1}(0)\big]\\
&\quad= \frac{\lambda}{\lambda+\mu}\Big[- R + \alpha v_\alpha^N(2) + R - \alpha v_\alpha^N(1) + R - \beta v_\beta^N(2) - R + \beta v_\beta^N(1)\Big]\\
&\quad\le 0,
\end{aligned}$$

  where the last inequality holds by Lemma 1.12 combined with the induction hypothesis.

By the above we have shown that, given the induction hypothesis, Equation (1.33) holds for every feasible combination of threshold strategies for $\big(f_\alpha^{N+1}(0), f_\alpha^{N+1}(1)\big)$ and $\big(f_\beta^{N+1}(0), f_\beta^{N+1}(1)\big)$.

Conclusion By this induction, we have proven Lemma 1.13 to be correct. □

Now that we have proven Lemmas 1.12 and 1.13, we can give a proof of Theorem 1.11.

Proof of Theorem 1.11.
By Lemma 1.13 we know that $v_\alpha^n(i+1) - v_\alpha^n(i) - \big[v_\beta^n(i+1) - v_\beta^n(i)\big] \le 0$, $\forall i \in S, n \ge 0$. Then Lemma 1.12 gives: $\alpha\big[v_\alpha^n(i+1) - v_\alpha^n(i)\big] - \beta\big[v_\beta^n(i+1) - v_\beta^n(i)\big] \le 0$, $\forall i \in S, n \ge 0$.

This means that if $v_\alpha^n(i+1) - v_\alpha^n(i) \ge R/\alpha$, then $v_\beta^n(i+1) - v_\beta^n(i) \ge R/\beta$, which is equivalent to the following:

$$f_\alpha^{n+1}(i) = 0 \qquad \Rightarrow \qquad f_\beta^{n+1}(i) = 0.$$

Since the strategies for α and β both are threshold and sending off customer i in the α-model implies refusing the i^{th} customer in the β-model, we can conclude that the threshold for α has to be greater than or equal to the threshold for β, $\forall n \ge 0$. Therefore, $i_\alpha^n \ge i_\beta^n$, $\forall n \ge 0$ must hold. □

1.4 Operators

An other way to approach the discounted model with controlled arrivals, is with the help of so-called operators, see [4] and [2]. This means that the calculation for $v_\alpha^{n+1}(i)$, as mentioned in

Algorithm 1.1 and explicitly given in Equation (1.7), is calculated by a few consecutive functions. From Equation (1.7), we get the following operators:

$$v_{\alpha,4}^{n+1}(i) = v_{\alpha}^{n}(i) \tag{1.39a}$$
$$v_{\alpha,3}^{n+1}(i) = T_D v_{\alpha,4}^{n+1}(i) \tag{1.39b}$$
$$v_{\alpha,2}^{n+1}(i) = T_{CA} v_{\alpha,4}^{n+1}(i) \tag{1.39c}$$
$$v_{\alpha,1}^{n+1}(i) = T_{unif}\left(v_{\alpha,3}^{n+1}(i), v_{\alpha,2}^{n+1}(i)\right) \tag{1.39d}$$
$$v_{\alpha,0}^{n+1}(i) = T_{disc} v_{\alpha,1}^{n+1}(i) \tag{1.39e}$$
$$v_{\alpha}^{n+1}(i) = v_{\alpha,0}^{n+1}(i). \tag{1.39f}$$

The function of each of the operators in Equations (1.39a) up to (1.39f) can be found in Table 1.3, where the following notation is used for any function $g : \mathbb{N}_{\geq 0} \to \mathbb{R}$:

$$g(i-1)^+ = \begin{cases} g(i-1), & i > 0, \\ g(0), & i = 0. \end{cases}$$

This notation, also implemented in v_{α}^{n}, will occur in the rest of this thesis.

Description of the operator	**The operator**		
Departure	$T_D f(i)$	$=$	$f(i-1)^+$
Controlled arrivals	$T_{CA} f(i)$	$=$	$\min\left\{f(i), -\frac{R}{\alpha} + f(i+1)\right\}$
Uniformization	$T_{unif}\big(f(i), g(i)\big)$	$=$	$\frac{\lambda}{\lambda+\mu} f(i) + \frac{\mu}{\lambda+\mu} g(i)$
Discounted costs	$T_{disc} f(i)$	$=$	$\alpha f(i) + B(i)$

Table 1.3: The functions of the operators mentioned in Equations (1.39a) up to (1.39f).

Note that the operators can be defined in different ways, although they are forced to have a structure similar to the representation given in Table 1.3.

When implementing the operators from Table 1.3 as described in Equations (1.39a) up to (1.39f), Equation (1.7) for calculating $v_{\alpha}^{n+1}(i)$ becomes:

$$v_{\alpha}^{n+1}(i) = T_{disc} T_{unif}\Big(T_{CA} v_{\alpha}^{n}(i), T_D v_{\alpha}^{n}(i)\Big). \tag{1.40}$$

Note that the operator-representation of $v_{\alpha}^{n+1}(i)$ in Equation (1.40) can be used for the proofs in this chapter. In the following chapter, where we will add controlled departure to our model, we will show how this works by proving similar theorems using the operators.

Chapter 2

Discounted model with controlled arrivals and departures

2.1 Model description

The model in this chapter is an adaptation of the original discounted model as described in Section 1.1. All variables as introduced in Chapter 1 are defined similarly, as well as the choice of accepting or refusing any incoming customer. For the fee per customer per time unit we take $C_i(a)$, $a \in A$, as described in Theorem 1.10.

The new part in this adapted model is the choice between two servers, both with exponentially distributed service time. Server 1 has mean service time $1/\mu_1$ and Server 2 has mean service time $1/\mu_2$, such that Server 2 is faster than Server 1 and thus $\mu_1 \leq \mu_2$.

For this model, we again want to calculate $v_\alpha^{n+1}(i)$ by an algorithm similar to Algorithm 1.1. Before we can give such an algorithm, we need to give the transition matrices for every possible combination of choices. Let $a \in A = \{0, 1\}$ be the choice for accepting or refusing an incoming customer, as described in Chapter 1. The choice for the server is given by $d \in D = \{1, 2\}$, with $d = 1$ representing the choice for the slower and cheaper Server 1, and $d = 2$ for the faster, more expensive Server 2. The action space in each state is therefore given by $A \times D = \big\{(0,1), (0,2), (1,1), (1,2)\big\}$.

To be able to compare the different combinations of actions, we discretize time and uniformize the probabilities. For the discretized time steps we take $T = \lambda + \mu_1 + \mu_2$, which is similar to the choice of T in Chapter 1. The uniformized transition probabilities are then given in the following matrices $P(a, d)$ with $(a, d) \in A \times D$.

$$P(0,1) = \frac{1}{\lambda+\mu_1+\mu_2} \begin{pmatrix} \lambda+\mu_1+\mu_2 & 0 & 0 & 0 & \cdots \\ \mu_1 & \lambda+\mu_2 & 0 & 0 & \cdots \\ 0 & \mu_1 & \lambda+\mu_2 & 0 & \\ 0 & 0 & \mu_1 & \lambda+\mu_2 & \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

$$P(0,2) = \frac{1}{\lambda+\mu_1+\mu_2} \begin{pmatrix} \lambda+\mu_1+\mu_2 & 0 & 0 & 0 & \cdots \\ \mu_2 & \lambda+\mu_1 & 0 & 0 & \\ 0 & \mu_2 & \lambda+\mu_1 & 0 & \\ 0 & 0 & \mu_2 & \lambda+\mu_1 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

$$P(1,1) = \frac{1}{\lambda+\mu_1+\mu_2} \begin{pmatrix} \mu_1+\mu_2 & \lambda & 0 & 0 & \cdots \\ \mu_1 & \mu_2 & \lambda & 0 & \cdots \\ 0 & \mu_1 & \mu_2 & \lambda & \\ 0 & 0 & \mu_1 & \mu_2 & \\ \vdots & & & \ddots & \ddots \end{pmatrix}$$

$$P(1,2) = \frac{1}{\lambda+\mu_1+\mu_2} \begin{pmatrix} \mu_1+\mu_2 & \lambda & 0 & 0 & \cdots \\ \mu_2 & \mu_1 & \lambda & 0 & \\ 0 & \mu_2 & \mu_1 & \lambda & \\ 0 & 0 & \mu_2 & \mu_1 & \ddots \\ \vdots & & & \ddots & \ddots \end{pmatrix}.$$

Note, that time discretization has an effect on cost function $C_i(a)$ from Theorem 1.10. For this adapted model, cost function $C_i(a)$ is calculated similarly to the cost function in Chapter 1, Equation (1.4), and is now given by

$$\begin{aligned} C_i(a) &= \begin{cases} B(i) - p_{i,(i+1)}(0,d) \cdot R, & a = 0, d \in D, \\ B(i) - p_{i,(i+1)}(1,d) \cdot R, & a = 1, d \in D, \end{cases} \\ &= \begin{cases} B(i), & a = 0, d \in D, \\ B(i) - \frac{\lambda}{\lambda+\mu_1+\mu_2} \cdot R, & a = 1, d \in D, \end{cases} \end{aligned}$$

with $B(i) : S \to \mathbb{R}$ a non-decreasing and convex function.

Assume, that the use of the faster Server 2 brings some additional cost per time unit T, say $K \in \mathbb{R}_{\geq 0}$. In other words, let $k : D \to \mathbb{R}$ such that $k(1) = 0$ and $k(2) = K$. Additionally, assume we can choose at any change of state in the process which server is to serve the customer currently in service.

Next, we will formulate an algorithm for the calculation of $v_\alpha^n(i)$ and strategy $f_\alpha^n(i)$, $\forall i \in S, n \geq 0$, where the strategy is of the form $f_\alpha^n(i) = (a,d)$, $(a,d) \in A \times D$. When we refer to one of the components of $f_\alpha^n(i)$, we use the notation $f_\alpha^n(i) = \Big(\big(f_\alpha^n(i)\big)_1, \big(f_\alpha^n(i)\big)_2 \Big)$.

Algorithm 2.1. ***Successive approximation extended model***

1. *Pick $v_\alpha^0(i) \in \mathbb{R}$ arbitrarily, $\forall i \in S$; let strategy $f_\alpha^0(i) = (1,1)$, $\forall i \in S$.*
2. *Let $v_\alpha^{n+1}(i) = \min_{(a,d)\in A\times D}\left\{C_i(a) + \alpha P(a,d) v_\alpha^n(i) + k(d)\right\}$, and thus $f_\alpha^{n+1}(i) = \arg\min_{(a,d)\in A\times D}\left\{C_i(a) + \alpha P(a,d) v_\alpha^n(i) + k(d)\right\}$, for $n = 0, 1, \ldots$, $i \in S$.*

Note that this algorithm satisfies the constraints in Paragraph 8.3.5 in [3] and thus is correct and converges towards the optimal values of $v_\alpha^*(i)$, $i \in S$.

We make the following assumption on the starting value of $v_\alpha^0(i)$, which holds until a different starting value is given.

Assumption 2.2. *Let $v_\alpha^0(i) = \min_{(a,d)\in A\times D}\{C_i(a) + k(d)\}$.*

With the help of this Algorithm 2.1, we can elaborate on the value of $v_\alpha^n(i)$. For $n = 0$, we get:

$$\begin{aligned} v_\alpha^0(i) &= \min\left\{C_i(0) + k(1), C_i(0) + k(2), C_i(1) + k(1), C_i(1) + k(2)\right\} \\ &= \min\left\{B(i), B(i) + K, B(i) - \frac{\lambda}{\lambda + \mu_1 + \mu_2}R, B(i) - \frac{\lambda}{\lambda + \mu_1 + \mu_2}R + K\right\} \\ &= B(i) - \frac{\lambda}{\lambda + \mu_1 + \mu_2}, \end{aligned} \tag{2.1}$$

which corresponds to $f_\alpha^0(i) = (1,1)$, which is the strategy from step 1 of the algorithm, and is equivalent to accepting an incoming customer and using the slower Server 1, $\forall i \in S$. This strategy is in line with Step 1 in Algorithm 2.1 and therefore the value for $v_\alpha^0(i)$ from Assumption 2.2 is valid.

Let $n \geq 0$. Then, according to Step 2 Algorithm 2.1, the following holds:

$$
\begin{aligned}
v_\alpha^{n+1}(i) &= \min_{(a,d)\in A\times D} \left\{ C_i(a) + \alpha P(a,d) v_\alpha^n(i) + k(d) \right\} \\
&= \min \Bigg\{ \frac{\lambda+\mu_2}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i) + \frac{\mu_1}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i-1)^+ + B(i), \qquad (2.2) \\
&\qquad \frac{\lambda+\mu_1}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i) + \frac{\mu_2}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i-1)^+ + B(i) + K, \\
&\qquad \frac{\lambda}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i+1) + \frac{\mu_2}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i) + \frac{\mu_1}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i-1)^+ \\
&\qquad\quad + B(i) - R\frac{\lambda}{\lambda+\mu_1+\mu_2}, \\
&\qquad \frac{\lambda}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i+1) + \frac{\mu_1}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i) + \frac{\mu_2}{\lambda+\mu_1+\mu_2} \alpha v_\alpha^n(i-1)^+ \\
&\qquad\quad + B(i) - R\frac{\lambda}{\lambda+\mu_1+\mu_2} + K \Bigg\} \\
&= \frac{1}{\lambda+\mu_1+\mu_2} \min \Bigg\{ \min \Big[(\lambda+\mu_2)\alpha v_\alpha^n(i) + \mu_1 \alpha v_\alpha^n(i-1)^+, \qquad (2.3) \\
&\qquad\quad \lambda \alpha v_\alpha^n(i+1) + \mu_2 \alpha v_\alpha^n(i) + \mu_1 \alpha v_\alpha^n(i-1)^+ - R\lambda \Big], \\
&\qquad \min \Big[(\lambda+\mu_1)\alpha v_\alpha^n(i) + \mu_2 \alpha v_\alpha^n(i-1)^+ + K, \\
&\qquad\quad \lambda \alpha v_\alpha^n(i+1) + \mu_1 \alpha v_\alpha^n(i) + \mu_2 \alpha v_\alpha^n(i-1)^+ - R\lambda + K \Big] \Bigg\} + B(i) \\
&= \frac{1}{\lambda+\mu_1+\mu_2} \min \Big\{ \lambda \min \Big[\alpha v_\alpha^n(i), \alpha v_\alpha^n(i+1) - R \Big] + \mu_2 \alpha v_\alpha^n(i) + \mu_1 \alpha v_\alpha^n(i-1)^+, \\
&\qquad \lambda \min \Big[\alpha v_\alpha^n(i), \alpha v_\alpha^n(i+1) - R \Big] + \mu_1 \alpha v_\alpha^n(i) + \mu_2 \alpha v_\alpha^n(i-1)^+ + K \Big\} + B(i) \\
&= \frac{\mu_1+\mu_2}{\lambda+\mu_1+\mu_2} \alpha \min \Bigg\{ \frac{\mu_2}{\mu_1+\mu_2} v_\alpha^n(i) + \frac{\mu_1}{\mu_1+\mu_2} v_\alpha^n(i-1)^+, \qquad (2.4) \\
&\qquad \frac{\mu_1}{\mu_1+\mu_2} v_\alpha^n(i) + \frac{\mu_2}{\mu_1+\mu_2} v_\alpha^n(i-1)^+ + \frac{K}{\alpha(\mu_1+\mu_2)} \Bigg\} \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2} \alpha \min \left[v_\alpha^n(i), v_\alpha^n(i+1) - \frac{R}{\alpha} \right] + B(i).
\end{aligned}
$$

In Equation (2.2), the order of the strategies $f_\alpha^{n+1}(i)$ in the minimization is equal to $(0,1)$, $(0,2)$, $(1,1)$ and $(1,2)$. In Equation (2.3) the order of strategies $f_\alpha^{n+1}(i)$ is different to be able to separate the minimization regarding acceptance. The order is equal to $(0,1),(1,1)$ and $(0,2),(1,2)$, where we use that $\min\{u,v,w,x\} = \min\{\min[u,w], \min[v,x]\}$ with $u,v,w,x \in \mathbb{R}$.

2.2 Operators of extended model

In Section 1.4 we showed how operators can be used to split the calculation of $v_\alpha^n(i)$ in the original model for $n \geq 0, i \in S$ into several consecutive, less complex, parts. For the extended model of this chapter, with an additional choice of a slow or a fast, more expensive server, we can do the same with slightly different operators.

$$v_{\alpha,5}^{n+1}(i) = v_{\alpha}^{n}(i) \tag{2.5a}$$
$$v_{\alpha,4}^{n+1}(i) = T_D v_{\alpha,5}^{n+1}(i) \tag{2.5b}$$
$$v_{\alpha,3}^{n+1}(i) = T_{CA} v_{\alpha,5}^{n+1}(i) \tag{2.5c}$$
$$v_{\alpha,2}^{n+1}(i) = T_{CD} v_{\alpha,4}^{n+1}(i) \tag{2.5d}$$
$$v_{\alpha,1}^{n+1}(i) = T_{unif}\left(v_{\alpha,2}^{n+1}(i), v_{\alpha,3}^{n+1}(i)\right) \tag{2.5e}$$
$$v_{\alpha,0}^{n+1}(i) = T_{disc} v_{\alpha,1}^{n+1}(i) \tag{2.5f}$$
$$v_{\alpha}^{n+1}(i) = v_{\alpha,0}^{n+1}(i). \tag{2.5g}$$

The description of each of the operators in Equations (2.5b) up to (2.5f) can be found in Table 2.1.

Description of the operator	**The operator**		
Departure	$T_D f(i)$	$=$	$f(i-1)^+$
Controlled arrivals	$T_{CA} f(i)$	$=$	$\min\left\{f(i), -\frac{R}{\alpha} + f(i+1)\right\}$
Controlled departure	$T_{CD} f(i)$	$=$	$\min\left\{\frac{\mu_2}{\mu_1+\mu_2} f(i+1) + \frac{\mu_1}{\mu_1+\mu_2} f(i),\right.$
			$\left.\frac{\mu_1}{\mu_1+\mu_2} f(i+1) + \frac{\mu_2}{\mu_1+\mu_2} f(i) + \frac{K}{\alpha(\mu_1+\mu_2)}\right\}$
Uniformization	$T_{unif}\big(f(i), g(i)\big)$	$=$	$\frac{\mu_1+\mu_2}{\lambda+\mu 1+\mu_2} f(i) + \frac{\lambda}{\lambda+\mu 1+\mu_2} g(i)$
Discounted costs	$T_{disc} f(i)$	$=$	$\alpha f(i) + B(i)$

Table 2.1: The definitions of the operators mentioned in Equations (2.5b) up to (2.5f).

Implementing the operators from Table 2.1 in Equations (2.5a) up to (2.5g), Equation (2.4) for calculating $v_{\alpha}^{n+1}(i)$ becomes:

$$v_{\alpha}^{n+1}(i) = T_{disc} T_{unif}\Bigg(T_{CD} T_D v_{\alpha}^{n}(i), T_{CA} v_{\alpha}^{n}(i)\Bigg). \tag{2.6}$$

2.3 Finding the optimal strategy

Note, that with Equations (2.4) and (2.6), we have two equivalent ways to calculate $v_{\alpha}^{n+1}(i)$ for the extended model. We can formulate a theorem on what the optimal strategy will look like. Before we give this theorem, we will state a few observations.

As Equation (2.1) shows, for $n = 0$ the optimal strategy $f_{\alpha}^{0}(i) = (1,1)$, $\forall i \in S$, meaning that every incoming customer is accepted and Server 1 is used. This being the optimal strategy means that we only need statements about the optimal strategy for $n > 0$.

Equation (2.4) shows that the minimization term from Equation (2.2) can be split into two separate minimizations on different sets: one minimizing over $a \in A$, and the other one over $d \in D$.

Theorem 2.3. *The optimal strategy $f_\alpha^n(i)$ is a two-dimensional threshold strategy, $\forall n \in \mathbb{N}_{\geq 0}$, $i \in S$, i.e. $\exists\, i_a^n, i_d^n \in S$ such that*

$$\left(f_\alpha^n(i)\right)_1 = \begin{cases} 1, & 0 \leq i \leq i_a^n, \\ 0, & i > i_a^n, \end{cases}$$
$$\left(f_\alpha^n(i)\right)_2 = \begin{cases} 1, & 0 \leq i \leq i_d^n, \\ 2, & i > i_d^n. \end{cases}$$

In order to prove Theorem 2.3, we follow the path of the proof of the similar Theorem 1.3 using multiple lemmas.

Lemma 2.4. $\left(f_\alpha^{n+1}(i)\right)_1 = 0$ *iff* $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$.

Proof. Thanks to Equation (2.4), it is clear that the only part of the calculation of $v_\alpha^{n+1}(i)$ depending on $a \in A$ is $\min\left[v_\alpha^n(i), v_\alpha^n(i+1) - R/\alpha\right]$. Choosing the first term of this expression is equivalent to choosing $a = 0$; the second term stands for choosing $a = 1$.

Since the choice of $a \in A$ is only dependent on $v_\alpha^n(i)$ and $v_\alpha^n(i+1)$, there is no need to split the case for $i = 0$, because both terms are valid for $i = 0$.

$\Rightarrow$ When $a = 0$ is the optimal choice, the following has to hold:

$$v_\alpha^n(i) \leq v_\alpha^n(i+1) - \frac{R}{\alpha},$$

which is equivalent to the expression given in this lemma.

$\Leftarrow$ Let $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$. Then $\min_{a\in A}\left\{v_\alpha^n(i), v_\alpha^n(i+1) - R/\alpha\right\} = v_\alpha^n(i)$ and thus $\arg\min_{a\in A}\left\{v_\alpha^n(i), v_\alpha^n(i+1) - R/\alpha\right\} = 0$, which is equivalent to $f_\alpha^{n+1}(i) = 0$. □

The following lemma is similar to Lemma 2.4, but provides a statement on the difference between two consecutive values of $v_\alpha^n(i)$ and the optimal strategy $d \in D$.

Lemma 2.5. $\left(f_\alpha^{n+1}(i)\right)_2 = 2$ *iff* $v_\alpha^n(i) - v_\alpha^n(i-1)^+ \geq K/[\alpha(\mu_2 - \mu_1)]$.

Proof. This proof is very similar to the proof of Lemma 2.4. The optimal strategy for the choice of $d \in D$ only depends on the first minimization expression in Equation (2.4). First, we will look at the case $i > 0$, then we prove the statement in this Lemma 2.5 for $i = 0$.

$i > 0, \Rightarrow$ When $d = 2$ is the optimal choice in state $i \in S$, the following must hold:

$$\begin{aligned} \frac{\mu_2}{\mu_1+\mu_2}v_\alpha^n(i) + \frac{\mu_1}{\mu_1+\mu_2}v_\alpha^n(i-1)^+ &\geq \frac{\mu_1}{\mu_1+\mu_2}v_\alpha^n(i) + \frac{\mu_2}{\mu_1+\mu_2}v_\alpha^n(i-1)^+ + \frac{K}{\alpha(\mu_1+\mu_2)} \\ \Leftrightarrow \quad (\mu_2-\mu_1)v_\alpha^n(i) - (\mu_2-\mu_1)v_\alpha^n(i-1)^+ &\geq \frac{K}{\alpha} \\ \Leftrightarrow \quad v_\alpha^n(i) - v_\alpha^n(i-1)^+ &\geq \frac{K}{\alpha(\mu_2-\mu_1)}. \end{aligned} \tag{2.7}$$

$i > 0, \Leftarrow$ Let $v_\alpha^n(i) - v_\alpha^n(i-1)^+ \geq K/[\alpha(\mu_2 - \mu_1)]$. Then,

$$
\begin{aligned}
& v_\alpha^n(i) - v_\alpha^n(i-1)^+ \geq \frac{K}{\alpha(\mu_2 - \mu_1)} \\
\Leftrightarrow \quad & (\mu_2 - \mu_1) v_\alpha^n(i) - (\mu_2 - \mu_1) v_\alpha^n(i-1)^+ \geq \frac{K}{\alpha} \\
\Leftrightarrow \quad & \frac{\mu_2}{\mu_1 + \mu_2} v_\alpha^n(i) + \frac{\mu_1}{\mu_1 + \mu_2} v_\alpha^n(i-1)^+ \geq \frac{\mu_1}{\mu_1 + \mu_2} v_\alpha^n(i) + \frac{\mu_2}{\mu_1 + \mu_2} v_\alpha^n(i-1)^+ + \frac{K}{\alpha(\mu_1 + \mu_2)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& \arg\min \left\{ \frac{\mu_2}{\mu_1 + \mu_2} v_\alpha^n(i) + \frac{\mu_1}{\mu_1 + \mu_2} v_\alpha^n(i-1)^+, \frac{\mu_1}{\mu_1 + \mu_2} v_\alpha^n(i) + \frac{\mu_2}{\mu_1 + \mu_2} v_\alpha^n(i-1)^+ + \frac{K}{\alpha(\mu_1 + \mu_2} \right\} \\
& \quad = 2,
\end{aligned}
$$

in other words, $f_\alpha^{n+1}(i) = 2$.

For $i = 0$, we will prove the negation of Lemma 2.5. This is necessary and sufficient.

Let $f_\alpha^{n+1}(i) = (a, 1)$. Then:

$$
\begin{aligned}
\frac{\mu_2}{\mu_1 + \mu_2} v_\alpha^n(0) + \frac{\mu_1}{\mu_1 + \mu_2} v_\alpha^n(0) &< \frac{\mu_1}{\mu_1 + \mu_2} v_\alpha^n(0) + \frac{\mu_2}{\mu_1 + \mu_2} v_\alpha^n(0) + \frac{K}{\alpha(\mu_1 + \mu_2)} \\
0 &< \frac{K}{\alpha(\mu_1 + \mu_2)},
\end{aligned}
$$

and thus the left-hand side of the negation of Lemma 2.5 is always true for $i = 0$. The negation of the right-hand side is:

$$
v_\alpha^n(0) - v_\alpha^n(0) = 0 < \frac{K}{\alpha(\mu_2 - \mu_1)},
$$

which is true by the constraints on the variables.

This concludes the proof, so the statement in Lemma 2.5 holds $\forall i \in S$. □

Additionally to the proof of Lemma 2.5, we can state that for $i = 0$, Inequality (2.7) never holds, so that the optimal strategy $\left(f_\alpha^n(0)\right)_2 = 1$, $\forall n \in \mathbb{N}_{\geq} 0$.

The following lemma analyzes some specific behaviour of $v_\alpha^n(i)$.

Lemma 2.6. $\left\{v_\alpha^n(i)\right\}_{i \in S}$ *is a non-decreasing sequence,* $\forall n \geq 0$.

Proof. We will prove this by induction on n, using the operator-representation from Section 2.2.

Case $n = 0$ In this case, $v_\alpha^0(i) = B(i) - \lambda/[\lambda + \mu_1 + \mu_2]$, cf. Equation (2.1). This gives:

$$\begin{aligned} v_\alpha^0(i+1) - v_\alpha^0(i) &= B(i+1) - \frac{\lambda}{\lambda + \mu_1 + \mu_2} - \left(B(i) - \frac{\lambda}{\lambda + \mu_1 + \mu_2} \right) \\ &= B(i+1) - B(i), \end{aligned}$$

which is non-negative, since $B(i)$ is assumed to be non-decreasing in i.

Case $n > 0$ Assume v_α^m is non-decreasing $\forall m \leq n$. To prove that v_α^{n+1} is also non-decreasing, we will go over all the steps in Equations (2.5a) up to (2.5g) and prove that each of them is non-decreasing.

- $v_{\alpha,5}^{n+1} = v_\alpha^n$ (cf. Equation (2.5a)).
  Since v_α^n is non-decreasing by assumption, $v_{\alpha,5}^{n+1}$ is as well.

- $v_{\alpha,4}^{n+1} = T_D v_{\alpha,5}^{n+1}$ (cf. Equation (2.5b)).
  Let $T_D f(i) = f(i-1)^+$ as in Table 2.1. If f is non-decreasing, then

  $$T_D f(i+1) - T_D f(i) = f(i) - f(i-1) \geq 0, \qquad \text{for } i \geq 1.$$

  The only equation left to check, is whether $T_D f(1) - T_D f(0)$ is non-negative.

  $$T_D f(1) - T_D f(0) = f(0) - f(0) \geq 0.$$

  Therefore, $T_D f(i)$ is non-decreasing in i, given that $f(i)$ is non-decreasing in i. Since $v_{\alpha,5}^{n+1}$ is non-decreasing, $v_{\alpha,4}^{n+1} = T_D v_{\alpha,5}^{n+1}$ is non-decreasing as well.

- $v_{\alpha,3}^{n+1} = T_{CA} v_{\alpha,5}^{n+1}$ (cf. Equation (2.5c)).
  Let $T_{CA} f(i) = \min\{f(i), -R/\alpha + f(i+1)\}$ as in Table 2.1. If f is a non-decreasing function in i, the first term of the minimization is non-decreasing as it equals $f(i)$. The second term, $-R/\alpha + f(i+1)$, is also non-decreasing, because the non-decreasingness of $f(i)$ implies that $f(i+1)$ is non-decreasing, and adding the constant $-R/\alpha$ does not affect this.
  The minimum of two non-decreasing functions is by definition also non-decreasing. Therefore, $T_{CA} f$ is non-decreasing, given that f is non-decreasing. Since $v_{\alpha,5}^{n+1}$ is non-decreasing, also $v_{\alpha,3}^{n+1} = T_C A v_{\alpha,5}^{n+1}$ is non-decreasing.

- $v_{\alpha,2}^{n+1} = T_{CD} v_{\alpha,4}^{n+1}$ (cf. Equation (2.5d)).
  Let

  $$\begin{aligned} T_{CD} f(i) = \min \Bigg\{ & \frac{\mu_2}{\mu_1 + \mu_2} f(i+1) + \frac{\mu_1}{\mu_1 + \mu_2} f(i), \\ & \frac{\mu_1}{\mu_1 + \mu_2} f(i+1) + \frac{\mu_2}{\mu_1 + \mu_2} f(i) + \frac{K}{\alpha(\mu_1 + \mu_2)} \Bigg\}, \end{aligned}$$

  as in Table 2.1. Since $f(i)$ is non-decreasing, so is $f(i+1)$. Let f be a non-decreasing function in i, then so is $f(i+1)$. A convex combination of two non-decreasing functions is also non-decreasing, so the first term of the minimization is non-decreasing as well. Adding a constant to a non-decreasing function does not affect non-decreasingness, so the convex combination

of the second term in the minimization with the added constant $K/[\alpha(\mu_1+\mu_2)]$ is also non-decreasing. The minimization of two non-decreasing function is again non-decreasing, so $T_{CD}f(i)$ is non-decreasing.
Earlier was shown that $v_{\alpha,4}^{n+1}$ is non-decreasing, so $v_{\alpha,2}^{n+1} = T_{CD}v_{\alpha,4}^{n+1}$ is non-decreasing as well.

- $v_{\alpha,1}^{n+1} = T_{unif}\big(v_{\alpha,2}^{n+1}, v_{\alpha,3}^{n+1}\big)$ (cf. Equation (2.5e)).
Let
$$T_{unif}\big(f(i), g(i)\big) = \frac{\mu_1+\mu_2}{\lambda+\mu_1+\mu_2} f(i) + \frac{\lambda}{\lambda+\mu_1+\mu_2} g(i),$$
as in Table 2.1. Let f, g be non-decreasing functions in i. Then this is a convex combination of two non-decreasing functions, which gives a non-decreasing function.
As $v_{\alpha,2}^{n+1}$ and $v_{\alpha,3}^{n+1}$ have been shown to be non-decreasing, also $v_{\alpha,1}^{n+1} = T_{unif}\big(v_{\alpha,2}^{n+1}, v_{\alpha,3}^{n+1}\big)$ is non-decreasing.

- $v_{\alpha,0}^{n+1} = T_{disc}v_{\alpha,1}^{n+1}$ (cf. Equation (2.5f)).
Let $T_{disc}f(i) = \alpha f(i) + B(i)$ as in Table 2.1. Let f be a non-decreasing function in i. The first part of the expression, $\alpha f(i)$, is non-decreasing, as α is positive and $f(i)$ is non-decreasing. $B(i)$ is assumed to be non-decreasing. The sum of two non-decreasing functions is again non-decreasing, and thus $T_{disc}f(i)$ is a non-decreasing function.
We have shown that $v_{\alpha,1}^{n+1}$ is a non-decreasing function, and thus $v_{\alpha,0}^{n+1} = T_{disc}v_{\alpha,1}^{n+1}$ is non-decreasing as well.

- $v_{\alpha}^{n+1} = v_{\alpha,0}^{n+1}$ (cf. Equation (2.5g)).
Since $v_{\alpha,0}^{n+1}$ is non-decreasing, v_{α}^{n+1} is non-decreasing as well.

From the enumeration above follows that non-decreasingness of v_α^n, implies that v_α^{n+1} is non-decreasing in i as well.

Conclusion Combining the two induction steps, we have proven that $v_\alpha^n(i)$ is a non-decreasing sequence in i for every $n \geq 0$. □

The following lemma concerns the convexity of $v_\alpha^n(i)$.

Lemma 2.7. *$v_\alpha^n(i)$ is convex in $i \in S$, $\forall n \in \mathbb{N}_{\geq 0}$.*

Proof. We will prove this Lemma 2.7 by induction on n using the operators-representation from Section 2.2. Note that any convex sequence $g(i)$ has the property $g(i+2) - g(i+1) \geq g(i+1) - g(i)$, $\forall i$.

Case n=0 Using Equation (2.1), we get:

$$\begin{aligned} v_\alpha^0(i+2) - &2v_\alpha^0(i+1) + v_\alpha^0(i) \\ &= B(i+2) - \frac{\lambda}{\lambda+\mu_1+\mu_2} - 2\left(B(i+1) - \frac{\lambda}{\lambda+\mu_1+\mu_2}\right) + B(i) - \frac{\lambda}{\lambda+\mu_1+\mu_2} \\ &= B(i+2) - 2B(i+1) + B(i), \end{aligned}$$

which holds when $B(i)$ is convex in i. This is true, thanks to the conditions given in Theorem 1.10.

Case $n > 0$ Assume v_α^m is convex $\forall m \leq n$. To prove that $v_\alpha^{n+1}(i)$ is also convex in i, we will go over the steps in Equations (2.5a) up to (2.5g) and prove that each of them gives a convex function in i.

- $v_{\alpha,5}^{n+1} = v_\alpha^n$ (cf. Equation (2.5a)).
  Since v_α^n is convex by assumption, $v_{\alpha,5}^{n+1}$ is convex as well.

- $v_{\alpha,4}^{n+1} = T_D v_{\alpha,5}^{n+1}$ (cf. Equation (2.5b)).
  Let $T_D f(i) = f(i-1)^+$ as in Table 2.1. Let f be a convex and non-decreasing function in i. For $i \geq 1$, $T_D f(i+2) - 2T_D f(i+1) + T_D f(i) = f(i+1) - 2f(i) + f(i-1) \geq 0$ by the convexity of f. The only equation left to check, is whether $T_D f(2) - 2T_D f(1) + T_D f(0)$ is non-negative:

$$T_D f(2) - 2T_D f(1) + T_D f(0) = f(1) - 2f(0) + f(0) = f(1) - f(0) \geq 0,$$

  where the last inequality holds because f is non-decreasing.
  Therefore, $T_D f$ is convex, given that f is convex and non-decreasing. Since $v_{\alpha,5}^{n+1}$ is convex and non-decreasing, also $v_{\alpha,4}^{n+1} = T_D v_{\alpha,5}^{n+1}$ is convex.

- $v_{\alpha,3}^{n+1} = T_{CA} v_{\alpha,5}^{n+1}$ (cf. Equation (2.5c)).
  Let

$$T_{CA} f(i) = \min\left\{ f(i), -\frac{R}{\alpha} + f(i+1) \right\},$$

  as in Table 2.1. Let f be a convex function. We need to prove the following inequality:

$$T_{CA} f(i+2) - 2T_{CA} f(i+1) + T_{CA} f(i) \geq 0.$$

  To do so, we distinguish all possible combinations of the values of the minimization expressions.

  1. Let $T_{CA} f(i) = f(i)$. Then: $f(i+1) - f(i) \geq R/\alpha$. By the convexity of f, this implies:

$$f(i+2) - f(i+1) \geq \frac{R}{\alpha}, \text{ which is equivalent to } T_{CA} f(i+1) = f(i+1); \text{ and}$$
$$f(i+3) - f(i+2) \geq \frac{R}{\alpha}, \text{ which is equivalent to } T_{CA} f(i+2) = f(i+2).$$

     This gives us:

$$\begin{aligned} T_{CA} f(i+2) - 2T_{CA} f(i+1) + T_{CA} f(i) & \\ = f(i+2) - 2f(i+1) + f(i) & \\ \geq 0, & \end{aligned}$$

     where the last inequality holds because f is convex.
  2. Let $T_{CA} f(i) = -R/\alpha + f(i+1)$.

(a) Let $T_{CA}f(i+1) = f(i+1)$. Then $f(i+2) - f(i+1) \geq R/\alpha$. By the convexity of f, this implies:

$$f(i+3) - f(i+2) \geq \frac{R}{\alpha}, \text{ which is equivalent to } T_{CA}f(i+2) = f(i+2).$$

This gives:

$$\begin{aligned}
&T_{CA}f(i+2) - 2T_{CA}f(i+1) + T_{CA}f(i) \\
&\qquad = f(i+2) - 2f(i+1) - \frac{R}{\alpha} + f(i+1) \\
&\qquad = f(i+2) - f(i+1) - \frac{R}{\alpha} \\
&\qquad \geq 0,
\end{aligned}$$

where the last inequality holds because $T_{CA}f(i+1) = f(i+1)$.

(b) Let $T_{CA}f(i+1) = -R/\alpha + f(i+2)$.

i. Let $T_{CA}f(i+2) = f(i+2)$. This gives:

$$\begin{aligned}
&T_{CA}f(i+2) - 2T_{CA}f(i+1) + T_{CA}f(i) \\
&\qquad = f(i+2) - 2\left(-\frac{R}{\alpha} + f(i+2)\right) - \frac{R}{\alpha} + f(i+1) \\
&\qquad = \frac{R}{\alpha} - f(i+2) + f(i+1) \\
&\qquad \geq 0,
\end{aligned}$$

where the last inequality holds because $T_{CA}f(i+1) = -R/\alpha + f(i+2)$.

ii. Let $T_{CA}f(i+2) = -R/\alpha + f(i+3)$. This gives:

$$\begin{aligned}
&T_{CA}f(i+2) - 2T_{CA}f(i+1) + T_{CA}f(i) \\
&\qquad = -\frac{R}{\alpha} + f(i+3) - 2\left(-\frac{R}{\alpha} + f(i+2)\right) - \frac{R}{\alpha} + f(i+1) \\
&\qquad = f(i+3) - 2f(i+2) + f(i+1) \\
&\qquad \geq 0,
\end{aligned}$$

where the last inequality holds because of the convexity of f.

We have shown that for every combination of minimizing actions, that

$$T_{CA}f(i+2) - 2T_{CA}f(i+1) + T_{CA}f(i) \geq 0, \qquad \forall i \in S.$$

Therefore, by the convexity of $v^{n+1}_{\alpha,5}$, also $v^{n+1}_{\alpha,3} = T_{CA}v^{n+1}_{\alpha,5}$ is convex.

- $v^{n+1}_{\alpha,2} = T_{CD}v^{n+1}_{\alpha,4}$ (cf. Equation (2.5d)).
  Let
  $$T_{CD}f(i) = \frac{1}{\mu_1 + \mu_2} \min\left\{\mu_2 f(i+1) + \mu_1 f(i), \mu_1 f(i+1) + \mu_2 f(i) + \frac{K}{\alpha}\right\},$$
  as in Table 2.1. Let f be a convex function in i. To show the convexity of $T_{CD}f$, we will show that $T_{CD}f(i+2) - 2T_{CD}f(i+1) + T_{CD}f(i) \geq 0$ for every possible combination of the values of the minimization expressions.

1. Let $T_{CD}f(i) = \big[\mu_2 f(i+1) + \mu_1 f(i)\big]/(\mu_1+\mu_2)$.
   (a) Let $T_{CD}f(i+1) = \big[\mu_2 f(i+2) + \mu_1 f(i+1)\big]/(\mu_1+\mu_2)$.
      i. Let $T_{CD}f(i+2) = \big[\mu_2 f(i+3) + \mu_1 f(i+2)\big]/\big(\mu_1+\mu_2\big)$. Then:

$$\begin{aligned}
&T_{CD}f(i+2) - 2T_{CD}f(i+1) + T_{CD}f(i)\\
&\quad= \frac{1}{\mu_1+\mu_2}\Big\{\mu_2 f(i+3) + \mu_1 f(i+2) - 2\big[\mu_2 f(i+2) + \mu_1 f(i+1)\big]\\
&\qquad\qquad + \mu_2 f(i+1) + \mu_1 f(i)\Big\}\\
&\quad= \frac{1}{\mu_1+\mu_2}\Big\{\mu_2\big[f(i+3) - 2f(i+2) + f(i+1)\big]\\
&\qquad\qquad + \mu_1\big[f(i+2) - 2f(i+1) + f(i)\big]\Big\}\\
&\quad\geq 0,
\end{aligned}$$

      where the last inequality holds because of the convexity of f.
      ii. Let $T_{CD}f(i+2) = \big[\mu_1 f(i+3) + \mu_2 f(i+2) + K/\alpha\big]/(\mu_1+\mu_2)$. Then:

$$\begin{aligned}
&T_{CD}f(i+2) - 2T_{CD}f(i+1) + T_{CD}f(i)\\
&\quad= \frac{1}{\mu_1+\mu_2}\Big\{ + \mu_1 f(i+3) + \mu_2 f(i+2) + \frac{K}{\alpha} - 2\big[\mu_2 f(i+2) + \mu_1 f(i+1)\big]\\
&\qquad\qquad \mu_2 f(i+1) + \mu_1 f(i)\Big\}\\
&\quad= \frac{1}{\mu_1+\mu_2}\Big\{\mu_2\big[f(i+1) - f(i+2)\big]\\
&\qquad\qquad + \mu_1\big[f(i+3) - 2f(i+1) + f(i)\big] + \frac{K}{\alpha}\Big\}\\
&\quad\geq \frac{1}{\mu_1+\mu_2}\Big\{ - \mu_2\big[f(i+2) - f(i+1)\big]\\
&\qquad\qquad + \mu_1\big[-f(i+2) + f(i+3)\big] + \frac{K}{\alpha}\Big\} \qquad (2.8)\\
&\quad\geq \frac{1}{\mu_1+\mu_2}\Big\{ - \mu_2\big[f(i+2) - f(i+1)\big]\\
&\qquad\qquad + \mu_1\big[f(i+2) - f(i+1)\big] + \frac{K}{\alpha}\Big\} \qquad (2.9)\\
&\quad\geq 0,
\end{aligned}$$

      where for Inequality (2.8) we used that $f(i) - 2f(i+1) \geq -f(i+2)$ since f is convex. In Inequality (2.9) we used that $f(i+3) - f(i+2) \geq f(i+2) - f(i+1)$, which again holds by the convexity of f. The final inequality holds because $T_{CD}f(i+1) = 1/(\mu_1+\mu_2)\big[\mu_2 f(i+2) + \mu_1 f(i+1)\big]$.
   (b) Let $T_{CD}f(i+1) = \big[\mu_1 f(i+2) + \mu_2 f(i+1) + K/\alpha\big]/(\mu_1+\mu_2)$. Then

$$\frac{1}{\mu_1+\mu_2}\Big[(\mu_2-\mu_1)\big(f(i+2) - f(i+1)\big)\Big] \geq \frac{K}{\alpha(\mu_1+\mu_2)}.$$

By the convexity of f, this implies:

$$\frac{1}{\mu_1+\mu_2}\Big[(\mu_2-\mu_1)\big(f(i+3)-f(i+2)\big)\Big] \geq \frac{K}{\alpha(\mu_1+\mu_2)},$$

which is equivalent to:

$$T_{CD}f(i+2) = \frac{1}{\mu_1+\mu_2}\left[\mu_1 f(i+3)+\mu_2 f(i+2)+\frac{K}{\alpha}\right].$$

We now get the following:

$$\begin{aligned}
&T_{CD}f(i+2) - 2T_{CD}f(i+1) + T_{CD}f(i)\\
&\quad= \frac{1}{\mu_1+\mu_2}\Bigg\{\mu_1 f(i+3)+\mu_2 f(i+2)+\frac{K}{\alpha} - 2\left[\mu_1 f(i+2)+\mu_2 f(i+1)+\frac{K}{\alpha}\right]\\
&\qquad\qquad + \mu_2 f(i+1) + \mu_1 f(i)\Bigg\}\\
&\quad= \frac{1}{\mu_1+\mu_2}\Bigg\{\mu_2\big[f(i+2)-f(i+1)\big] + \mu_1\big[f(i+3)-2f(i+2)+f(i)\big] - \frac{K}{\alpha}\Bigg\}\\
&\quad\geq \frac{1}{\mu_1+\mu_2}\Bigg\{\mu_2\big[f(i+2)-f(i+1)\big] - \mu_1\big[f(i+1)-f(i)\big] - \frac{K}{\alpha}\Bigg\} &(2.10)\\
&\quad\geq \frac{1}{\mu_1+\mu_2}\Bigg\{\mu_2\big[f(i+2)-f(i+1)\big] - \mu_1\big[f(i+2)-f(i+1)\big] - \frac{K}{\alpha}\Bigg\} &(2.11)\\
&\quad\geq 0,
\end{aligned}$$

where, in inequality (2.10), we used that $f(i+3)-2(i+2) \geq -f(i+1)$, since f is convex. In Equation (2.11) we used that $f(i+1)-f(i) \leq f(i+2)-f(i+1)$, which again holds by the convexity of f. The final inequality holds, because

$$T_{CD}f(i+1) = \frac{1}{\mu_1+\mu_2}\left[\mu_1 f(i+2)+\mu_2 f(i+1)+\frac{K}{\alpha}\right].$$

2. Let $T_{CD}f(i) = \big[\mu_1 f(i+1)+\mu_2 f(i)+K/\alpha\big]/(\mu_1+\mu_2)$. Then we know the following: $\Big[(\mu_2-\mu_1)\big(f(i+1)-f(i)\big)\Big]/(\mu_1+\mu_2) \geq K/\big[\alpha(\mu_1+\mu_2)\big]$. By the convexity of f, this implies:

$$\frac{1}{\mu_1+\mu_2}\Big[(\mu_2-\mu_1)\big(f(i+2)-f(i+1)\big)\Big] \geq \frac{1}{\mu_1+\mu_2}\cdot\frac{K}{\alpha},$$

which is equivalent to:

$$T_{CD}f(i+1) = \frac{1}{\mu_1+\mu_2}\left[\mu_1 f(i+2)+\mu_2 f(i+1)+\frac{K}{\alpha}\right].$$

Similarly it follows, that $T_{CD}f(i+2) = \big[\mu_1 f(i+3)+\mu_2 f(i+2)+K/\alpha\big]/(\mu_1+\mu_2)$. We now get the following:

$$
\begin{aligned}
&T_{CD}f(i+2) - 2T_{CD}f(i+1) + T_{CD}f(i) \\
&\quad = \frac{1}{\mu_1+\mu_2}\Big\{\mu_1 f(i+3) + \mu_2 f(i+2) + \frac{K}{\alpha} - 2\big[\mu_1 f(i+2) + \mu_2 f(i+1) + \frac{K}{\alpha}\big] \\
&\qquad\qquad + \mu_1 f(i+1) + \mu_2 f(i) + \frac{K}{\alpha}\Big\} \\
&\quad = \frac{1}{\mu_1+\mu_2}\Big\{\mu_2\big[f(i+2) - 2f(i+1) + f(i)\big] \\
&\qquad\qquad + \mu_1\big[f(i+3) - 2f(i+2) + f(i+1)\big]\Big\} \\
&\quad \geq 0,
\end{aligned}
$$

where the last inequality holds because of the convexity of f.

We have shown, that for every combination of minimizing actions,

$$T_{CD}f(i+2) - 2T_{CD}f(i+1) + T_{CD}f(i) \geq 0$$

holds. Therefore, by the convexity of $v_{\alpha,4}^{n+1}$, also $v_{\alpha,2}^{n+1} = T_{CD}v_{\alpha,4}^{n+1}$ is convex.

- $v_{\alpha,1}^{n+1} = T_{unif}\big(v_{\alpha,2}^{n+1}, v_{\alpha,3}^{n+1}\big)$ (cf. Equation (2.5e)).
  Let
  $$T_{unif}\big(f(i), g(i)\big) = \frac{\mu_1+\mu_2}{\lambda+\mu_1+\mu_2}f(i) + \frac{\lambda}{\lambda+\mu_1+\mu_2}g(i),$$
  as in Table 2.1. Let f, g be convex functions in i. This is a convex combination of two convex functions, which gives a convex function.
  Since both $v_{\alpha,2}^{n+1}$ and $v_{\alpha,3}^{n+1}$ are shown to be convex, also $v_{\alpha,1}^{n+1} = T_{unif}\big(v_{\alpha,2}^{n+1}, v_{\alpha,3}^{n+1}\big)$ is convex.

- $v_{\alpha,0}^{n+1} = T_{disc}v_{\alpha,1}^{n+1}$ (cf. Equation (2.5f)).
  Let $T_{disc}f(i) = \alpha f(i) + B(i)$ as in Table 2.1. Let f be a convex function in i. The first part of $T_{disc}f$, αf, is convex, since α is a positive constant and f is convex. $B(i)$ is assumed to be convex in i. As the sum of two convex functions is also convex, $T_{disc}f$ is a convex function.
  We have shown, that $v_{\alpha,1}^{n+1}$ is a convex function, and thus we get that $v_{\alpha,0}^{n+1} = T_{disc}v_{\alpha,1}^{n+1}$ is also convex.

- $v_{\alpha}^{n+1} = v_{\alpha,0}^{n+1}$ (cf. Equation (2.5g)).
  Since $v_{\alpha,0}^{n+1}$ is convex, v_{α}^{n+1} is convex as well.

From the enumeration above follows that the convexity of v_{α}^{n} implies that v_{α}^{n+1} is also convex.

Conclusion Combining the two induction steps, we have shown that v_{α}^{n} is convex $\forall n \geq 0$. □

Now we can give a proof of Theorem 2.3, stating that the optimal strategy $f_{\alpha}^{n}(i)$ is a two-dimensional threshold strategy.

Proof of Theorem 2.3.

Threshold for $a \in A$ Lemma 2.4 implies that:

$$\left(f_\alpha^{n+1}(i)\right)_1 = 0 \qquad \Leftrightarrow \qquad v_\alpha^n(i+1) - v_\alpha^n(i) \geq \frac{R}{\alpha}.$$

Now let $i_a^n \in S$ be defined as follows:

$$i_a^n = \max\left\{ i \in S \cup \{\infty\} \middle| v_\alpha^n(i+1) - v_\alpha^n(i) < \frac{R}{\alpha} \right\}. \tag{2.12}$$

Together with Lemma 2.6, stating that $v_\alpha^n(i)$ is non-decreasing, this gives two possibilities for i_a^n:

1. $i_a^n = \infty$.
   Then $\left(f_\alpha^{n+1}(i)\right)_1 = 1$, $\forall i \in S$.
2. $i_a^n \in S$.
   Then, $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$, $\forall i \geq i_a^n$ and thus
   $$\left(f_\alpha^{n+1}(i)\right)_1 = \begin{cases} 1, & 0 \leq i \leq i_a^n, \\ 0, & i > i_a^n. \end{cases}$$

In both cases, $\left(f_\alpha^{n+1}(i)\right)_1$ is a threshold strategy.

Threshold for $d \in D$ Lemma 2.5 implies that:

$$\left(f_\alpha^{n+1}(i)\right)_2 = 2 \qquad \Leftrightarrow \qquad v_\alpha^n(i) - v_\alpha^n(i-1)^+ \geq \frac{K}{\alpha(\mu_2 - \mu_1)}.$$

Now let $i_d^n \in S$ be defined as follows:

$$i_d^n = \max\left\{ i \in S \cup \{\infty\} \middle| v_\alpha^n(i) - v_\alpha^n(i-1)^+ < \frac{K}{\alpha(\mu_2 - \mu_1)} \right\}. \tag{2.13}$$

Together with Lemma 2.6, stating that $v_\alpha^n(i)$ is non-decreasing, this gives two possibilities for i_d^n:

1. $i_d^n = \infty$.
   Then $\left(f_\alpha^{n+1}(i)\right)_2 = 1$, $\forall i \in S$.
2. $i_d^n \in S$.
   Then, $v_\alpha^n(i) - v_\alpha^n(i-1)^+ \geq K/[\alpha(\mu_2 - \mu_1)]$, $\forall i \geq i_d^n$ and thus
   $$\left(f_\alpha^{n+1}(i)\right)_2 = \begin{cases} 1, & 0 \leq i \leq i_d^n, \\ 2, & i > i_d^n. \end{cases}$$

In both cases, $\left(f_\alpha^{n+1}(i)\right)_2$ is a threshold strategy.

Conclusion The optimal strategy $f_\alpha^n(i)$ is a two-dimensional threshold strategy, $\forall n \in \mathbb{N}_{\geq 0}$, $i \in S$, with thresholds $i_a^n, i_d^n \in S$ from Equations (2.12) and (2.13) such that:

$$\left(f_\alpha^n(i)\right)_1 = \begin{cases} 1, & 0 \leq i \leq i_a^n, \\ 0, & i > i_a^n, \end{cases}$$

$$\left(f_\alpha^n(i)\right)_2 = \begin{cases} 1, & 0 \leq i \leq i_d^n, \\ 2, & i > i_d^n. \end{cases}$$

□

2.4 Relationship between thresholds

As the optimal strategy $f_\alpha^n(i)$ is a two-dimensional threshold strategy with thresholds $i_a^n, i_d^n \in S$, where i_a^n is given by Equation (2.12) and i_d^n by Equation (2.13). In this paragraph, we will state a theorem on the relationship between i_a^n and i_d^n.

Theorem 2.8. *Let $i_a^n, i_d^n \in S$, $n \geq 0$. Then the following hold:*

- *If $i_a^n < i_d^n - 1$, then $R < K/(\mu_2 - \mu_1)$.*
- *If $i_a^n = i_d^n - 1$, then $R/\alpha, K/\big[\alpha(\mu_2 - \mu_1)\big] \in \Big(v_\alpha^n(i_d^n) - v_\alpha^n(i_d^n - 1), v_\alpha^n(i_d^n + 1) - v_\alpha^n(i_d^n)\Big]$.*
- *If $i_a^n > i_d^n - 1$, then $R > K/(\mu_2 - \mu_2)$.*

Proof.

- Let $i_a^n < i_d^n - 1$.
  In this case, we know that $v_\alpha^n(i_a^n+1) - v_\alpha^n(i_a^n) < v_\alpha^n(i_d^n) - v_\alpha^n(i_d^n - 1)^+$, by the convexity of $v_\alpha^n(i)$ (see Lemma 2.7). These terms cannot be equal to each other, because otherwise i_a^n would be larger by Equation (2.12). This is equivalent to

$$\frac{R}{\alpha} < \frac{K}{\alpha(\mu_2 - \mu_1)}.$$

- Let $i_a^n = i_d^n - 1$.
  In this case, we know that $v_\alpha^n(i_d^n) - v_\alpha^n(i_d^n - 1)^+ = v_\alpha^n(i_a^n + 1) - v_\alpha^n(i_a^n)$ and $v_\alpha^n(i_d^n + 1) - v_\alpha^n(i_d^n)$. This means, by Equation (2.12), that

$$v_\alpha^n(i_d^n) - v_\alpha^n(i_d^n - 1)^+ < \frac{R}{\alpha} \leq v_\alpha^n(i_d^n + 1) - v_\alpha^n(i_d^n).$$

  Similarly, according to Equation (2.13), holds

$$v_\alpha^n(i_d^n) - v_\alpha^n(i_d^n - 1)^+ < \frac{K}{\alpha(\mu_2 - \mu_1)} \leq v_\alpha^n(i_d^n + 1) - v_\alpha^n(i_d^n).$$

  Therefore, we can conclude that

$$\frac{R}{\alpha}, \frac{K}{\alpha(\mu_2 - \mu_1)} \in \Big(v_\alpha^n(i_d^n) - v_\alpha^n(i_d^n - 1)^+, v_\alpha^n(i_d^n + 1) - v_\alpha^n(i_d^n)\Big].$$

- Let $i_a^n > i_d^n - 1$.
  The convexity of $v_\alpha^n(i)$, gives that $v_\alpha^n(i)$ gives $v_\alpha^n(i_a^n + 1) - v_\alpha^n(i_a^n) > v_\alpha^n(i_d^n) - v_\alpha^n(i_d^n - 1)^+$. These terms cannot be equal to each other, because otherwise i_a^n would be larger by Equation (2.13). This is equivalent to

$$\frac{R}{\alpha} > \frac{K}{\alpha(\mu_2 - \mu_1)}.$$

□

Note that in practice, Server 2 will only be used when $i_a^n > i_d^n - 1$.

2.5 Convergence in the model

2.5.1 Convergence of the thresholds

As in Section 1.2.3, we are interested in the convergence of the behaviour of the threshold in f_α^n. The following theorem addresses this issue. It is similar to Theorem 1.8. Note, that i_a^n is the largest state in which a customer is accepted, and i_d^n the final state to use slow Server 1 (see Theorem 2.3).

Theorem 2.9. *Assume $v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \geq v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i)$, $\forall i \in S$. Then $\big((i_a^n, i_d^n)\big)_n$ forms a two-dimensional non-increasing sequence of threshold strategies, where i_a^n is the threshold for the acceptance of a customer in time step n and i_d^n the threshold for the choice of server, for $n \geq 1$.*

When $v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)$, $\forall i \in S$ holds, $\big((i_a^n, i_d^n)\big)_n$ forms a two-dimensional non-decreasing sequence of threshold strategies, for $n \geq 1$.

Note that in this theorem, $n \geq 1$ must hold. This is because for $n = 0$, the strategy is already defined by the first step of Algorithm 2.1: $f^0_\alpha(i) = (1,1)$, $\forall i \in S$.

Theorem 2.9 is proven with the help of the following lemma.

Lemma 2.10. *Let function $f_1, f_2, g_1, g_2 : S \to \mathbb{R}$ be convex and non-decreasing, in such a way that $f_1(i+1) - f_1(i) \geq g_1(i+1) - g_1(i)$, as well as $f_2(i+1) - f_2(i) \geq g_2(i+1) - g_2(i)$. Then for every operator T from Table 2.1, it holds that: $Tf_1(i+1) - Tf_1(i) \geq Tg_1(i+1) - Tg_1(i)$, $\forall i \in S$. For T_{unif}, which we mention separately because it has two arguments, the following must hold: $T_{unif}\big(f_1(i+1), f_2(i+1)\big) - T_{unif}\big(f_1(i), f_2(i)\big) \geq T_{unif}\big(g_1(i+1), g_2(i+1)\big) - T_{unif}\big(g_1(i), g_2(i)\big)$, $\forall i \in S$.*

Proof. Let $f_1, f_2, g_1, g_2 : S \to \mathbb{R}$ be convex and non-decreasing, with $f_1(i+1) - f_1(i) \geq g_1(i+1) - g_1(i)$ and $f_2(i+1) - f_2(i) \geq g_2(i+1) - g_2(i)$, $\forall i \in S$. Then we get the following inequalities for the operators from Table 2.1.

- Departure-operator: $T_D f(i) = f(i-1)^+$.
  When plugging in f_1 and g_1, we get for $i > 0$:

$$\begin{aligned} &T_D f_1(i+1) - T_D f_1(i) - \big[T_D g_1(i+1) - T_D g_1(i)\big] \\ &\quad= f_1(i) - f_1(i-1) - \big[g_1(i) - g_1(i-1)\big] \\ &\quad\geq 0. \end{aligned}$$

  Let $i = 0$. Then we get:

$$\begin{aligned} &T_D f_1(1) - T_D f_1(0) - \big[T_D g_1(1) - T_D g_1(0)\big] \\ &\quad= f_1(0) - f_1(0) - \big[g_1(0) - g_1(0)\big] \\ &\quad= 0. \end{aligned}$$

  Thus $T_D f_1(i+1) - T_D f_1(i) \geq T_D g_1(i+1) - T_D g_1(i)$ holds, $\forall i \in S$.

- Controlled arrivals-operator: $T_{CA}f(i) = \min\{f(i), -R/\alpha + f(i+1)\}$.
  When plugging in f_1 and g_1, we get $\forall i \in S$:

$$\begin{aligned}
&T_{CA}f_1(i+1) - T_{CA}f_1(i) - \big[T_{CA}g_1(i+1) - T_{CA}g_1(i)\big] \\
&\quad = \min\left\{f_1(i+1), -\frac{R}{\alpha} + f_1(i+2)\right\} - \min\left\{f_1(i), -\frac{R}{\alpha} + f_1(i+1)\right\} \\
&\qquad - \left[\min\left\{g_1(i+1), -\frac{R}{\alpha} + g_1(i+2)\right\} - \min\left\{g_1(i), -\frac{R}{\alpha} + g_1(i+1)\right\}\right]. \qquad (2.14)
\end{aligned}$$

  To see that Equation (2.14) is non-negative, we need to distinguish all possible combinations for the four different minimization expressions. Denote the minimizing consecutive actions by $CA\big(f_1(i)\big), CA\big(f_1(i+1)\big), CA\big(g_1(i)\big), CA\big(g_1(i+1)\big) \in \{0,1\}$.

  Let $CA\big(g_1(i)\big) = 0$. Then $g_1(i+1) - g_1(i) \geq R/\alpha$. This implies that $f_1(i+1) - f_1(i) \geq R/\alpha$, which is equivalent to $CA\big(f_1(i)\big) = 0$. Similarly, we get

$$\begin{aligned}
CA\big(g_1(i+1)\big) = 0 \quad &\Rightarrow \quad CA\big(f_1(i+1)\big) = 0, \\
CA\big(f_1(i)\big) = 1 \quad &\Rightarrow \quad CA\big(g_1(i)\big) = 1, \\
CA\big(f_1(i+1)\big) = 1 \quad &\Rightarrow \quad CA\big(g_1(i+1)\big) = 1.
\end{aligned}$$

  Now we can distinguish the possible combinations of the minimum values of the minimization expressions and show that Equation (2.14) is non-negative.

  - Let $CA\big(f_1(i)\big) = 0$; $CA\big(f_1(i+1)\big) = 0$; $CA\big(g_1(i)\big) = 0$; $CA\big(g_1(i+1)\big) = 0$. Then, Equation (2.14) becomes:

$$\begin{aligned}
&T_{CA}f_1(i+1) - T_{CA}f_1(i) - \big[T_{CA}g_1(i+1) - T_{CA}g_1(i)\big] \\
&\quad = f_1(i+1) - f_1(i) - \big[g_1(i+1) - g_1(i)\big] \\
&\quad \geq 0,
\end{aligned}$$

    by the assumptions on f_1 and g_1.
  - Let $CA\big(f_1(i)\big) = 0$; $CA\big(f_1(i+1)\big) = 0$; $CA\big(g_1(i)\big) = 1$; $CA\big(g_1(i+1)\big) = 0$. Then, Equation (2.14) becomes:

$$\begin{aligned}
&T_{CA}f_1(i+1) - T_{CA}f_1(i) - \big[T_{CA}g_1(i+1) - T_{CA}g_1(i)\big] \\
&\quad = f_1(i+1) - f_1(i) - \left[g_1(i+1) + \frac{R}{\alpha} - g_1(i+1)\right] \\
&\quad \geq 0,
\end{aligned}$$

    where we used that $CA\big(f_1(i)\big) = 0$, implying that $f_1(i+1) - f_1(i) \geq R/\alpha$.
  - Let $CA\big(f_1(i)\big) = 0$; $CA\big(f_1(i+1)\big) = 0$; $CA\big(g_1(i)\big) = 1$; $CA\big(g_1(i+1)\big) = 1$. Then, Equation (2.14) becomes:

$$\begin{aligned}
&T_{CA}f_1(i+1) - T_{CA}f_1(i) - \big[T_{CA}g_1(i+1) - T_{CA}g_1(i)\big] \\
&\quad = f_1(i+1) - f_1(i) - \left[-\frac{R}{\alpha} + g_1(i+2) + \frac{R}{\alpha} - g_1(i+1)\right] \\
&\quad \geq 0,
\end{aligned}$$

where we used that $CA\big(f_1(i)\big) = 0$ implies $f_1(i+1) - f_1(i) \geq R/\alpha$, and $CA\big(g_1(i+1)\big) = 1$ which implies that $-\big[g_1(i+2) - g_1(i+1)\big] \geq -R/\alpha$.

– Let $CA\big(f_1(i)\big) = 1$; $CA\big(f_1(i+1)\big) = 0$; $CA\big(g_1(i)\big) = 1$; $CA\big(g_1(i+1)\big) = 0$. Then, Equation (2.14) becomes:

$$\begin{aligned}T_{CA}f_1(i+1) - T_{CA}f_1(i) - &\big[T_{CA}g_1(i+1) - T_{CA}g_1(i)\big]\\ &= f_1(i+1) + \frac{R}{\alpha} - f_1(i+1) - \left[g_1(i+1) + \frac{R}{\alpha} - g_1(i+1)\right]\\ &= 0.\end{aligned}$$

– Let $CA\big(f_1(i)\big) = 1$; $CA\big(f_1(i+1)\big) = 0$; $CA\big(g_1(i)\big) = 1$; $CA\big(g_1(i+1)\big) = 1$. Then, Equation (2.14) becomes:

$$\begin{aligned}T_{CA}f_1(i+1) - T_{CA}f_1(i) - &\big[T_{CA}g_1(i+1) - T_{CA}g_1(i)\big]\\ &= f_1(i+1) + \frac{R}{\alpha} - f_1(i+1) - \left[-\frac{R}{\alpha} + g_1(i+2) + \frac{R}{\alpha} - g_1(i+1)\right]\\ &\geq 0,\end{aligned}$$

where we used that $CA\big(g_1(i+1)\big) = 1$ implies $-\big[g_1(i+2) - g_1(i+1)\big] \geq -R/\alpha$.

– Let $CA\big(f_1(i)\big) = 1$; $CA\big(f_1(i+1)\big) = 1$; $CA\big(g_1(i)\big) = 1$; $CA\big(g_1(i+1)\big) = 1$. Then, Equation (2.14) becomes:

$$\begin{aligned}T_{CA}f_1(i+1) - T_{CA}f_1(i) - &\big[T_{CA}g_1(i+1) - T_{CA}g_1(i)\big]\\ &= -\frac{R}{\alpha} + f_1(i+2) + \frac{R}{\alpha} - f_1(i+1) - \left[-\frac{R}{\alpha} + g_1(i+2) + \frac{R}{\alpha} - g_1(i+1)\right]\\ &\geq 0,\end{aligned}$$

where we used the assumption that $f_1(i+2) - f_1(i+1) \geq g_1(i+2) - g_1(i+1)$.

Now we have shown that Equation (2.14) is non-negative $\forall i \in S$, and thus that

$$T_{CA}f_1(i+1) - T_{CA}f_1(i) \geq T_{CA}g_1(i+1) - T_{CA}g_1(i).$$

- Controlled departures-operator:
  $T_{CD}f(i) = \min\big\{\mu_2 f(i+1) + \mu_1 f(i), \mu_1 f(i+1) + \mu_2 f(i) + K/\alpha\big\}/(\mu_1+\mu_2)$.
  When plugging in f_1 and g_1, we get $\forall i \in S$:

$$\begin{aligned}T_{CD}f_1(i+1) - T_{CD}f_1(i) - &\big[T_{CD}g_1(i+1) - T_{CD}g_1(i)\big]\\ = \frac{1}{\mu_1+\mu_2}\Bigg(&\min\left\{\mu_2 f_1(i+2) + \mu_1 f_1(i+1), \mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha}\right\}\\ &- \min\left\{\mu_2 f_1(i+1) + \mu_1 f_1(i), \mu_1 f_1(i+1) + \mu_2 f_1(i) + \frac{K}{\alpha}\right\}\\ &- \Bigg[\min\left\{\mu_2 g_1(i+2) + \mu_1 g_1(i+1), \mu_1 g_1(i+2) + \mu_2 g_1(i+1) + \frac{K}{\alpha}\right\}\\ &- \min\left\{\mu_2 g_1(i+1) + \mu_1 g_1(i), \mu_1 g_1(i+1) + \mu_2 g_1(i) + \frac{K}{\alpha}\right\}\Bigg]\Bigg).\end{aligned} \tag{2.15}$$

We will use a notation similar to one for the controlled arrivals-operator, where the choice of the first minimization expression is given by $CD\big(f_1(i+1)\big) \in \{1,2\}$, where 1 denotes the first term of the minimization expression being the minimum, and 2 the second term. For the other minimization expressions in Equation (2.15) a similar notation holds.

Let $CD\big(f_1(i)\big) = 1$. Then $\big(f_1(i+1) - f_1(i)\big) \leq 0$. By the definition of $f_1(i)$ and $g_1(i)$ follows that also $\big(g_1(i+1) - g_1(i)\big) \leq 0$ must hold, which is equivalent to $CD\big(g_1(i)\big) = 1$. In a similar way we get $CD\big(f_1(i+1)\big) = 1 \Rightarrow CD\big(g_1(i+1)\big) = 1$; $CD\big(g_1(i)\big) = 2 \Rightarrow CD\big(f_1(i)\big) = 2$; and $CD\big(g_1(i+1)\big) = 2 \Rightarrow CD\big(f_1(i+1)\big) = 2$.

Before investigating all combinations of the four minimizing actions in Equation (2.15), we want to point out the following:

$$f_1(i+2) - f_1(i) - \big[g_1(i+2) - g_1(i)\big] \geq f_1(i+1) - f_1(i) - \big[g_1(i+1) - g_1(i)\big] \geq 0, \quad (2.16)$$

where we used that $f_1(i+2) - g_1(i+2) \geq f_1(i+1) - g_1(i+1)$.

Now we can distinguish the possible combinations of the minimum values of the minimization expressions and show that Equation (2.15) is non-negative.

- Let $CD\big(f_1(i)\big) = 1$; $CD\big(f_1(i+1)\big) = 1$; $CD\big(g_1(i)\big) = 1$; $CD\big(g_1(i+1)\big) = 1$. Then Equation (2.15) becomes:

$$\begin{aligned}
&T_{CD}f_1(i+1) - T_{CD}f_1(i) - \big[T_{CD}g_1(i+1) - T_{CD}g_1(i)\big] \\
&\quad= \frac{1}{\mu_1+\mu_2}\Big(\mu_2 f_1(i+2) + \mu_1 f_1(i+1) - \mu_2 f_1(i+1) - \mu_1 f_1(i) \\
&\qquad\quad - \big[\mu_2 g_1(i+2) + \mu_1 g_1(i+1) - \mu_2 g_1(i+1) - \mu_1 g_1(i)\big]\Big) \\
&\quad= \frac{1}{\mu_1+\mu_2}\Big(\mu_2\big(f_1(i+2) - f_1(i+1) - \big[g_1(i+2) - g_1(i+1)\big]\big) \\
&\qquad\quad + \mu_1\big(f_1(i+1) - f_1(i) - \big[g_1(i+1) - g_1(i)\big]\big)\Big) \\
&\quad\geq 0,
\end{aligned}$$

  where we used the assumption that $f_1(j+1) - f_1(j) \geq g_1(j+1) - g_1(j)$, with $j = i$ and $j = i+1$.
- Let $CD\big(f_1(i)\big) = 1$; $CD\big(f_1(i+1)\big) = 2$; $CD\big(g_1(i)\big) = 1$; $CD\big(g_1(i+1)\big) = 1$. Then

Equation (2.15) becomes:

$$\begin{aligned}
&T_{CD}f_1(i+1) - T_{CD}f_1(i) - \big[T_{CD}g_1(i+1) - T_{CD}g_1(i)\big]\\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha} - \mu_2 f_1(i+1) - \mu_1 f_1(i)\\
&\qquad\qquad - \big[\mu_2 g_1(i+2) + \mu_1 g_1(i+1) - \mu_2 g_1(i+1) - \mu_1 g_1(i)\big]\Big)\\
&\quad \geq \frac{1}{\mu_1+\mu_2}\left(\mu_1\big(f_1(i+2) - f_1(i)\big) + \frac{K}{\alpha}\right.\\
&\qquad\qquad \left. - \left[\mu_1 g_1(i+2) + \mu_2 g_1(i+1) + \frac{K}{\alpha} - \mu_2 g_1(i+1) - \mu_1 g_1(i)\right]\right) \qquad (2.17)\\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1\big[f_1(i+2) - f_1(i) - g_1(i+2) + g_1(i)\big]\Big)\\
&\quad \geq 0,
\end{aligned}$$

where for Inequality (2.17) we used that $CD\big(g_1(i+1)\big) = 1$, which implies the following: $-\big(\mu_2 g_1(i+2) + \mu_1 g_1(i+1)\big) \geq -\big(\mu_1 g_1(i+2) + \mu_2 g_1(i+1) + K/\alpha\big)$. The final inequality holds by Inequality (2.16).

- Let $CD\big(f_1(i)\big) = 1$; $CD\big(f_1(i+1)\big) = 2$; $CD\big(g_1(i)\big) = 1$; $CD\big(g_1(i+1)\big) = 2$. Then Equation (2.15) becomes:

$$\begin{aligned}
&T_{CD}f_1(i+1) - T_{CD}f_1(i) - \big[T_{CD}g_1(i+1) - T_{CD}g_1(i)\big]\\
&\quad = \frac{1}{\mu_1+\mu_2}\left(\mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha} - \mu_2 f_1(i+1) - \mu_1 f_1(i)\right.\\
&\qquad\qquad \left. - \left[\mu_1 g_1(i+2) + \mu_2 g_1(i+1) + \frac{K}{\alpha} - \mu_2 g_1(i+1) - \mu_1 g_1(i)\right]\right)\\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1\big(f_1(i+2) - f_1(i) - \big[g_1(i+2) - g_2(i)\big]\big)\Big)\\
&\quad \geq 0,
\end{aligned}$$

where we used Inequality (2.16).

- Let $CD\big(f_1(i)\big) = 2$; $CD\big(f_1(i+1)\big) = 2$; $CD\big(g_1(i)\big) = 1$; $CD\big(g_1(i+1)\big) = 1$. Then Equation (2.15) becomes:

$$\begin{aligned}
&T_{CD}f_1(i+1) - T_{CD}f_1(i) - \big[T_{CD}g_1(i+1) - T_{CD}g_1(i)\big]\\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha} - \mu_1 f_1(i+1) - \mu_2 f_1(i) - \frac{K}{\alpha}\\
&\qquad\qquad - \big[\mu_2 g_1(i+2) + \mu_1 g_1(i+1) - \mu_2 g_1(i+1) - \mu_1 g_1(i)\big]\Big)\\
&\quad \geq \frac{1}{\mu_1+\mu_2}\left(\mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha} - \mu_2 f_1(i+1) - \mu_1 f_1(i)\right.\\
&\qquad\qquad \left. - \left[\mu_1 g_1(i+2) + \mu_2 g_1(i+1) + \frac{K}{\alpha} - \mu_2 g_1(i+1) - \mu_1 g_1(i)\right]\right) \qquad (2.18)\\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1\big[f_1(i+2) - f_1(i) - g_1(i+2) + g_1(i)\big]\Big)\\
&\quad \geq 0,
\end{aligned}$$

where for Inequality (2.18) we used that $CD\big(f_1(i)\big) = 2$, which implies the following: $-\big[\mu_1 f_1(i+1) + \mu_2 f_1(i) + K/\alpha\big] \geq -\big[\mu_2 f_1(i+1) + \mu_1 f_1(i)\big]$; and that $CD\big(g_1(i+1)\big) = 1$ implies that $-\big[\mu_2 g_1(i+2) + \mu_1 g_1(i+1)\big] \geq -\big[\mu_1 g_1(i+2) + \mu_2 g_1(i+1) + K/\alpha\big]$. The final inequality holds by Inequality (2.16).

- Let $CD\big(f_1(i)\big) = 2$; $CD\big(f_1(i+1)\big) = 2$; $CD\big(g_1(i)\big) = 1$; $CD\big(g_1(i+1)\big) = 2$. Then Equation (2.15) becomes:

$$\begin{aligned}
&T_{CD} f_1(i+1) - T_{CD} f_1(i) - \big[T_{CD} g_1(i+1) - T_{CD} g_1(i)\big] \\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha} - \mu_1 f_1(i+1) - \mu_2 f_1(i) - \frac{K}{\alpha} \\
&\qquad - \big[\mu_1 g_1(i+2) + \mu_2 g_1(i+1) + \frac{K}{\alpha} - \mu_2 g_1(i+1) - \mu_1 g_1(i)\big]\Big) \\
&\quad \geq \frac{1}{\mu_1+\mu_2}\left(\mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha} - \mu_2 f_1(i+1) - \mu_1 f_1(i)\right. \\
&\qquad \left. - \left[\mu_1 g_1(i+2) + \frac{K}{\alpha} - \mu_1 g_1(i)\right]\right) \qquad (2.19)\\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1\big[f_1(i+2) - f_1(i) - g_1(i+2) + g_1(i)\big]\Big) \\
&\quad \geq 0,
\end{aligned}$$

where for Inequality (2.19) we used that $CD\big(f_1(i)\big) = 2$, which implies the following: $-\big[\mu_1 f_1(i+1) + \mu_2 f_1(i) + K/\alpha\big] \geq -\big[\mu_2 f_1(i+1) - \mu_1 f_1(i)\big]$. The final inequality holds by Inequality (2.16).

- Let $CD\big(f_1(i)\big) = 2$; $CD\big(f_1(i+1)\big) = 2$; $CD\big(g_1(i)\big) = 2$; $CD\big(g_1(i+1)\big) = 2$. Then Equation (2.15) becomes:

$$\begin{aligned}
&T_{CD} f_1(i+1) - T_{CD} f_1(i) - \big[T_{CD} g_1(i+1) - T_{CD} g_1(i)\big] \\
&\quad = \frac{1}{\mu_1+\mu_2}\left(\mu_1 f_1(i+2) + \mu_2 f_1(i+1) + \frac{K}{\alpha} - \mu_1 f_1(i+1) - \mu_2 f_1(i) - \frac{K}{\alpha}\right. \\
&\qquad \left. - \left[\mu_1 g_1(i+2) + \mu_2 g_1(i+1) + \frac{K}{\alpha} - \mu_1 g_1(i+1) - \mu_2 g_1(i) - \frac{K}{\alpha}\right]\right) \\
&\quad = \frac{1}{\mu_1+\mu_2}\Big(\mu_1\big(f_1(i+2) - f_1(i+1) - \big[g_1(i+2) - g_1(i+1)\big]\big) \\
&\qquad + \mu_1\big(f_1(i+1) - f_1(i) - \big[g_1(i+1) - g_1(i)\big]\big)\Big) \\
&\quad \geq 0,
\end{aligned}$$

where we used the assumption that $f_1(j+1) - f_1(j) \geq g_1(j+1) - g_1(j)$, with $j = i$ and $j = i+1$.

By the above numeration we have shown that Equation (2.15) is non-negative, $\forall i \in S$, and thus that $T_{CD} f_1(i+1) - T_{CD} f_1(i) \geq T_{CD} g_1(i+1) - T_{CD} g_1(i)$.

- Uniformization-operator: $T_{unif}\big(f(i), g(i)\big) = f(i)(\mu_1+\mu_2)/(\lambda+\mu_1+\mu_2) + g(i)\lambda/(\lambda+\mu_1+\mu_2)$.

Let $f_1(i)$, $f_2(i)$, $g_1(i)$ and $g_2(i)$ as defined at the beginning of the proof. We then get:

$$\begin{aligned}
&T_{unif}\big(f_1(i+1), f_2(i+1)\big) - T_{unif}\big(f_1(i), f_2(i)\big) \\
&\qquad - \big[T_{unif}\big(g_1(i+1), g_2(i+1)\big) - T_{unif}\big(g_1(i), g_2(i)\big)\big] \\
&= \frac{1}{\lambda+\mu_1+\mu_2}\Big((\mu_1+\mu_2)f_1(i+1) + \lambda f_2(i+1) - (\mu_1+\mu_2)f_1(i) - \lambda f_2(i) \\
&\qquad - \big[(\mu_1+\mu_2)g_1(i+1) + \lambda g_2(i+1) - (\mu_1+\mu_2)g_1(i) - \lambda g_2(i)\big]\Big) \\
&= \frac{\mu_1+\mu_2}{\lambda+\mu_1+\mu_2}\Big(f_1(i+1) - f_1(i) - \big[g_1(i+1) - g_1(i)\big]\Big) \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\Big(f_2(i+1) - f_2(i) - \big[g_2(i+1) - g_2(i)\big]\Big) \\
&\geq 0,
\end{aligned}$$

where the final inequality follows from the assumptions on $f_1(i)$, $g_1(i)$, $f_2(i)$ and $g_2(i)$.

- Discounted costs-operator: $T_{disc}f(i) = \alpha f(i) + B(i)$.
  Recall that $B(i)$ is a non-decreasing, convex function in i. Let $f_1(i)$ and $g_1(i)$ have the same properties as described before. Then:

$$\begin{aligned}
&T_{disc}f_1(i+1) - T_{disc}f_1(i) - \big[T_{disc}g_1(i+1) - T_{disc}g_1(i)\big] \\
&\quad = \alpha f_1(i+1) + B(i+1) - \alpha f_1(i) - B(i) - \big[\alpha g_1(i+1) + B(i+1) - \alpha g_1(i) - B(i)\big] \\
&\quad = \alpha\Big(f_1(i+1) - f_1(i) - \big[g_1(i+1) - g_1(i)\big]\Big) \\
&\quad \geq 0,
\end{aligned}$$

where the last inequality holds by the definition of $f_1(i)$ and $g_1(i)$.

By the above enumeration we have shown that for every operator T in Table 2.1 the statement in this Lemma 2.10 holds. □

With the help of Lemma 2.10, we can give a proof of Theorem 2.9.

Proof of Theorem 2.9:
By previous lemmas we know the following:

- $\big(f_\alpha^{n+1}(i)\big)_1 = 0$ iff $v_\alpha^n(i+1) - v_\alpha^n(i) \geq R/\alpha$ (Lemma 2.4);
- $\big(f_\alpha^{n+1}(i)\big)_2 = 2$ iff $v_\alpha^n(i) - v_\alpha^n(i-1)^+ \geq K/\big[\alpha(\mu_2-\mu_1)\big]$ (Lemma 2.5);
- The optimal strategy $f_\alpha^n(i)$ is a two-dimensional threshold strategy $\forall n$ (Theorem 2.3).

Now, assume $v_{\alpha,up}^1(i+1) - v_{\alpha,up}^1(i) \geq v_{\alpha,up}^0(i+1) - v_{\alpha,up}^0(i)$. Then, using Equations (2.5a) up to

(2.5g) combined with Lemma 2.10, we get:

$$\begin{aligned}
& v^1_{\alpha,5,u}(i+1) - v^1_{\alpha,5,u}(i) = v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i) \\
& \qquad \le v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) = v^2_{\alpha,5,u}(i+1) - v^2_{\alpha,5,u}(i) \\
\Rightarrow \quad & v^1_{\alpha,4,u}(i+1) - v^1_{\alpha,4,u}(i) = T_D v^1_{\alpha,5,u}(i+1) - T_D v^1_{\alpha,5,u}(i) \\
& \qquad \le T_D v^2_{\alpha,5,u}(i+1) - T_D v^2_{\alpha,5,u}(i) = v^2_{\alpha,4,u}(i+1) - v^2_{\alpha,4,u}(i) \\
\Rightarrow \quad & v^1_{\alpha,3,u}(i+1) - v^1_{\alpha,3,u}(i) = T_{CA} v^1_{\alpha,5,u}(i+1) - T_{CA} v^1_{\alpha,5,u}(i) \\
& \qquad \le T_{CA} v^2_{\alpha,5,u}(i+1) - T_{CA} v^2_{\alpha,5,u}(i) = v^2_{\alpha,3,u}(i+1) - v^2_{\alpha,3,u}(i) \\
\Rightarrow \quad & v^1_{\alpha,2,u}(i+1) - v^1_{\alpha,2,u}(i) = T_{CD} v^1_{\alpha,4,u}(i+1) - T_{CD} v^1_{\alpha,4,u}(i) \\
& \qquad \le T_{CD} v^2_{\alpha,4,u}(i+1) - T_{CD} v^2_{\alpha,4,u}(i) = v^2_{\alpha,2,u}(i+1) - v^2_{\alpha,2,u}(i) \\
\Rightarrow \quad & v^1_{\alpha,1,u}(i+1) - v^1_{\alpha,1,u}(i) = T_{unif}\big(v^1_{\alpha,2,u}(i+1), v^1_{\alpha,3,u}(i+1)\big) - T_{unif}\big(v^1_{\alpha,2,u}(i), v^1_{\alpha,3,u}(i)\big) \\
& \qquad \le T_{unif}\big(v^2_{\alpha,2,u}(i+1), v^2_{\alpha,3,u}(i+1)\big) - T_{unif}\big(v^2_{\alpha,2,u}(i), v^2_{\alpha,3,u}(i)\big) \\
& \qquad = v^2_{\alpha,1,u}(i+1) - v^2_{\alpha,1,u}(i) \\
\Rightarrow \quad & v^1_{\alpha,0,u}(i+1) - v^1_{\alpha,0,u}(i) = T_{disc} v^1_{\alpha,1,u}(i+1) - T_{disc} v^1_{\alpha,1,u}(i) \\
& \qquad \le T_{disc} v^2_{\alpha,1,u}(i+1) - T_{disc} v^2_{\alpha,1,u}(i) = v^2_{\alpha,0,u}(i+1) - v^2_{\alpha,0,u}(i) \\
\Rightarrow \quad & v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) = v^1_{\alpha,0,u}(i+1) - v^1_{\alpha,0,u}(i) \\
& \qquad \le v^2_{\alpha,0,u}(i+1) - v^2_{\alpha,0,u}(i) = v^2_{\alpha,up}(i+1) - v^2_{\alpha,up}(i).
\end{aligned}$$

In short, this proves that

$$v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i) \le v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \Rightarrow v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \le v^2_{\alpha,up}(i+1) - v^2_{\alpha,up}(i).$$

Using induction, it can be shown similarly that

$$v^{n-1}_{\alpha,up}(i+1) - v^{n-1}_{\alpha,up}(i) \le v^n_{\alpha,up}(i+1) - v^n_{\alpha,up}(i) \Rightarrow v^n_{\alpha,up}(i+1) - v^n_{\alpha,up}(i) \le v^{n+1}_{\alpha,up}(i+1) - v^{n+1}_{\alpha,up}(i),$$

and thus:

$$v^{n-1}_{\alpha,up}(i+1) - v^{n-1}_{\alpha,up}(i) \le v^n_{\alpha,up}(i+1) - v^n_{\alpha,up}(i), \qquad \forall n \ge 0, \forall i \in S. \tag{2.20}$$

Now, let $\big(f^n_\alpha(i)\big)_1 = 0$ for some $i \in S$. Then $v^{n-1}_\alpha(i+1) - v^{n-1}_\alpha(i) \ge R/\alpha$. By Inequality (2.20), also $v^n_\alpha(i+1) - v^n_\alpha(i) \ge R/\alpha$, and thus $\big(f^{n+1}_\alpha(i)\big)_1 = 0$. This means that declining the ith customer at time n implies that the ith incoming customer is also declined at time $n+1$.

Next, let $\big(f^n_\alpha(i)\big)_2 = 2$ for some $i \in S$. Then $v^{n-1}_\alpha(i) - v^{n-1}_\alpha(i-1)^+ \ge K/\big[\alpha(\mu_2 - \mu_1)\big]$. By Inequality (2.20), also $v^n_\alpha(i) - v^n_\alpha(i-1)^+ \ge K/\big[\alpha(\mu_2 - \mu_1)\big]$, and thus $\big(f^{n+1}_\alpha(i)\big)_2 = 2$. This means that using Server 2, when there are i customers in the system at time n, implies that Server 2 is also used when there are i customers in the system at time $n+1$.

By the previous argument, we can conclude that $\big((i^n_{a,up}, i^n_{d,up})\big)_n$ forms a two-dimensional non-increasing sequence of threshold strategies for $n \ge 1$.

The argument for the other part of the proof is similar to the above, but with reversed inequalities. Therefore, we conclude also that $v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) \le v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)$, $\forall i \in S$, implies that $\big((i^n_{a,low}, i^n_{d,low})\big)_n$ forms a two-dimensional non-decreasing sequence of threshold strategies. □

2.5.2 Convergence of the strategy

In this section we aim to determine the optimal strategy by enclosing it between a lower and an upper bound. Theorem 2.9 forms the key to this enclosure.

Corollary 2.11. *Let $v^0_{\alpha,up}(i)$ be such that $v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \geq v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i)$, $\forall i \in S$, with consecutive thresholds $\big((i^n_{a,up}, i^n_{d,up})\big)_n$. Let $v^0_{\alpha,low}(i)$ be such that $v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)$, $\forall i \in S$, with consecutive thresholds $\big((i^n_{a,low}, i^n_{d,low})\big)_n$.*
Then, if $i^n_{a,up} = i^n_{a,low} := i^*_a$ for some $n \geq 1$, this implies that i^*_a is the optimal threshold and will not change in further time steps. The same holds if $i^n_{d,up} = i^n_{d,low} := i^*_d$ for some $n \geq 1$; then i^*_d is the optimal threshold and will not change in further time steps.

Proof. This corollary follows directly from Theorem 2.9, where we note that by Algorithm 2.1, $f^0_\alpha(i) = (1,1)$, $\forall i \in S$, so $i^0_a = \infty$ and $i^0_d = \infty$, independent of the starting value. Therefore, Corollary 2.11 holds for $n \geq 1$. □

2.5.3 Convergence of $v^n_\alpha(i) - v^n_\alpha(0)$

The focus of this subsection is a result from the non-decreasingness of $v^n_\alpha(i)$ from Lemma 2.7 and the results from Subsections 2.6.1 and 2.6.2.

Corollary 2.12. *Let $v^0_{\alpha,up}(i)$ be such that $v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \geq v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i)$, $\forall i \in S$, with thresholds $\big((i^n_{a,up}, i^n_{d,up})\big)_n$. Let $v^0_{\alpha,low}(i)$ be such that $v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)$, $\forall i \in S$, with thresholds $\big((i^n_{a,low}, i^n_{d,low})\big)_n$.*
Then the following equation holds:

$$v^n_{\alpha,low}(i) - v^n_{\alpha,low}(0) \geq v^*_\alpha(i) - v^*_\alpha(0) \geq v^n_{\alpha,up}(i) - v^n_{\alpha,up}(0), \qquad \forall i \in S, \forall n \geq 0.$$

Before we prove Corollary 2.12, remark that Algorithm 2.1 converges towards the optimal values of $v^*_\alpha(i)$, $i \in S$.

Proof. Using Theorem 1.8, it holds for $n \geq 0$, $\forall i \in S$:

$$\begin{aligned}
&v^0_{\alpha,low}(i) - v^0_{\alpha,low}(0) \\
&\quad = \big(v^0_{\alpha,low}(i) - v^0_{\alpha,low}(i-1)\big) + \big(v^0_{\alpha,low}(i-1) - v^0_{\alpha,low}(i-2)\big) \\
&\qquad\qquad + \ldots + \big(v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\big) \\
&\quad \geq \big(v^1_{\alpha,low}(i) - v^1_{\alpha,low}(i-1)\big) + \big(v^1_{\alpha,low}(i-1) - v^1_{\alpha,low}(i-2)\big) \\
&\qquad\qquad + \ldots + \big(v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0)\big) \\
&\quad \geq \ldots \\
&\quad \geq \big(v^n_{\alpha,low}(i) - v^n_{\alpha,low}(i-1)\big) + \big(v^n_{\alpha,low}(i-1) - v^0_{\alpha,low}(i-2)\big) \\
&\qquad\qquad + \ldots + \big(v^n_{\alpha,low}(1) - v^n_{\alpha,low}(0)\big) \\
&\quad = v^n_{\alpha,low}(i) - v^n_{\alpha,low}(0).
\end{aligned} \tag{2.21}$$

Similarly holds for $v^n_{\alpha,up}(i)$, $\forall n \geq 0$, $\forall i \in S$:

$$\begin{aligned}
v^0_{\alpha,up}(i) &- v^0_{\alpha,up}(0) \\
&= \big(v^0_{\alpha,up}(i) - v^0_{\alpha,up}(i-1)\big) + \big(v^0_{\alpha,up}(i-1) - v^0_{\alpha,up}(i-2)\big) \\
&\qquad\qquad + \ldots + \big(v^0_{\alpha,up}(1) - v^0_{\alpha,up}(0)\big) \\
&\leq \big(v^1_{\alpha,up}(i) - v^1_{\alpha,up}(i-1)\big) + \big(v^1_{\alpha,up}(i-1) - v^1_{\alpha,up}(i-2)\big) \\
&\qquad\qquad + \ldots + \big(v^1_{\alpha,up}(1) - v^1_{\alpha,up}(0)\big) \\
&\leq \ldots \\
&\leq \big(v^n_{\alpha,up}(i) - v^n_{\alpha,up}(i-1)\big) + \big(v^n_{\alpha,up}(i-1) - v^0_{\alpha,up}(i-2)\big) \\
&\qquad\qquad + \ldots + \big(v^n_{\alpha,up}(1) - v^n_{\alpha,up}(0)\big) \\
&= v^n_{\alpha,up}(i) - v^n_{\alpha,up}(0). \qquad (2.22)
\end{aligned}$$

We know, by the choice of $v^0_{\alpha,low}(i)$ and $v^0_{\alpha,up}(i)$, that $v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)$, $\forall i \in S$. This inequality combined with Equations (2.5a) up to (2.5g) and Lemma 2.10 gives for

$$\begin{aligned}
& v^1_{\alpha,5,u}(i+1) - v^1_{\alpha,5,u}(i) = v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i) \\
&\qquad \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i) = v^1_{\alpha,5,l}(i+1) - v^1_{\alpha,5,l}(i) \\
\Rightarrow \quad & v^1_{\alpha,4,u}(i+1) - v^1_{\alpha,4,u}(i) = T_D v^1_{\alpha,5,u}(i+1) - T_D v^1_{\alpha,5,u}(i) \\
&\qquad \leq T_D v^1_{\alpha,5,l}(i+1) - T_D v^1_{\alpha,5,l}(i) = v^1_{\alpha,4,l}(i+1) - v^1_{\alpha,4,l}(i) \\
\Rightarrow \quad & v^1_{\alpha,3,u}(i+1) - v^1_{\alpha,3,u}(i) = T_{CA} v^1_{\alpha,5,u}(i+1) - T_{CA} v^1_{\alpha,5,u}(i) \\
&\qquad \leq T_{CA} v^1_{\alpha,5,l}(i+1) - T_{CA} v^1_{\alpha,5,l}(i) = v^1_{\alpha,3,l}(i+1) - v^1_{\alpha,3,l}(i) \\
\Rightarrow \quad & v^1_{\alpha,2,u}(i+1) - v^1_{\alpha,2,u}(i) = T_{CD} v^1_{\alpha,4,u}(i+1) - T_{CD} v^1_{\alpha,4,u}(i) \\
&\qquad \leq T_{CD} v^1_{\alpha,4,l}(i+1) - T_{CD} v^1_{\alpha,4,l}(i) = v^1_{\alpha,2,l}(i+1) - v^1_{\alpha,2,l}(i) \\
\Rightarrow \quad & v^1_{\alpha,1,u}(i+1) - v^1_{\alpha,1,u}(i) = T_{unif}\big(v^1_{\alpha,2,u}(i+1), v^1_{\alpha,3,u}(i+1)\big) - T_{unif}\big(v^1_{\alpha,2,u}(i), v^1_{\alpha,3,u}(i)\big) \\
&\qquad \leq T_{unif}\big(v^1_{\alpha,2,l}(i+1), v^1_{\alpha,3,l}(i+1)\big) - T_{unif}\big(v^1_{\alpha,2,l}(i), v^1_{\alpha,3,l}(i)\big) \\
&\qquad = v^1_{\alpha,1,l}(i+1) - v^1_{\alpha,1,l}(i) \\
\Rightarrow \quad & v^1_{\alpha,0,u}(i+1) - v^1_{\alpha,0,u}(i) = T_{disc} v^1_{\alpha,1,u}(i+1) - T_{disc} v^1_{\alpha,1,u}(i) \\
&\qquad \leq T_{disc} v^1_{\alpha,1,l}(i+1) - T_{disc} v^1_{\alpha,1,l}(i) = v^1_{\alpha,0,l}(i+1) - v^1_{\alpha,0,l}(i) \\
\Rightarrow \quad & v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) = v^1_{\alpha,0,u}(i+1) - v^1_{\alpha,0,u}(i) \\
&\qquad \leq v^1_{\alpha,0,l}(i+1) - v^1_{\alpha,0,l}(i) = v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i).
\end{aligned}$$

In short, this proves that $v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i) \Rightarrow v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \leq v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i)$. Using induction, it can be shown similarly that:

$$\begin{aligned}
v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i), &\qquad \forall i \in S, \\
\Rightarrow& \\
v^n_{\alpha,up}(i+1) - v^n_{\alpha,up}(i) \leq v^n_{\alpha,low}(i+1) - v^n_{\alpha,low}(i), &\qquad \forall i \in S, \forall n \geq 0. \qquad (2.23)
\end{aligned}$$

Using the same technique that was used to achieve Inequalities (2.21) and (2.22), Implication (2.23) also implies

$$v^n_{\alpha,up}(i) - v^n_{\alpha,up}(0) \leq v^n_{\alpha,low}(i) - v^n_{\alpha,low}(0), \qquad \forall i \in S, \forall n \geq 0.$$

Therefore, we can conclude that

$$v^n_{\alpha,low}(i) - v^n_{\alpha,low}(0) \geq v^*_\alpha(i) - v^*_\alpha(0) \geq v^n_{\alpha,up}(i) - v^n_{\alpha,up}(0), \qquad \forall i \in S \forall n \geq 0.$$

□

2.6 Computation of $v^0_{\alpha,low}(i)$ and $v^0_{\alpha,up}(i)$

Theorem 2.9, Corollary 2.11 and Corollary 2.12 are only of value if we can find functions $v^0_{\alpha,up}(i)$ and $v^0_{\alpha,low}(i)$ such that

$$v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \geq v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i), \qquad \forall i \in S,$$

and

$$v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i), \qquad \forall i \in S,$$

both hold.

Let $v^0_{\alpha,low}(i)$ and $v^0_{\alpha,up}(i)$ be of a similar form:

$$v^0_{\alpha,low}(i) = \gamma_l (i+1)^2, \tag{2.24}$$

$$v^0_{\alpha,up}(i) = \gamma_u (i+1)^2, \tag{2.25}$$

where $\gamma_l, \gamma_u \geq 0$ to make sure that $v^0_{\alpha,low}(i), v^0_{\alpha,up}(i)$ are convex and non-decreasing in i. Assume that the fine is of the form $B(i) = bi$ with $b \in \mathbb{R}_{>0}$.

We choose for the quadratic form in Equations (2.24) and (2.25), because the fee is linear in i and the sum of many linear functions approaches a quadratic function.

2.6.1 Computation of $v^0_{\alpha,low}(i)$

Firstly, we will deduce how to choose γ_l in Equation (2.24), in order that

$$v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i), \qquad \forall i \in S. \tag{2.26}$$

To give an expression of Inequality (2.26) for $i > 0$, we will combine Equations (2.4) and (2.24).

$$\begin{aligned}
& v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{\mu_2\gamma_l(i+2)^2 + \mu_1\gamma_l(i+1)^2, \mu_1\gamma_l(i+2)^2 + \mu_2\gamma_l(i+1)^2 + \frac{K}{\alpha}\right\}\right. \\
&\qquad\qquad \left. - \min\left\{\mu_2\gamma_l(i+1)^2 + \mu_1\gamma_l i^2, \mu_1\gamma_l(i+1)^2 + \mu_2\gamma_l i^2 + \frac{K}{\alpha}\right\}\right] \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{\gamma_l(i+2)^2, \gamma_l(i+3)^2 - \frac{R}{\alpha}\right\}\right. \\
&\qquad\qquad \left. - \min\left\{\gamma_l(i+1)^2, \gamma_l(i+2)^2 - \frac{R}{\alpha}\right\}\right] + B(i+1) - B(i) \\
&\qquad - \left[\gamma_l(i+2)^2 - \gamma_l(i+1)^2\right] \\
&\quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{\mu_2\gamma_l(4i+4) + \mu_1\gamma_l(2i+1), \mu_1\gamma_l(4i+4) + \mu_2\gamma_l(2i+1) + \frac{K}{\alpha}\right\}\right. \\
&\qquad\qquad \left. - \min\left\{\mu_2\gamma_l(2i+1), \mu_1\gamma_l(2i+1) + \frac{K}{\alpha}\right\}\right] \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{\gamma_l(2i+3), \gamma_l(4i+8) - \frac{R}{\alpha}\right\}\right. \\
&\qquad\qquad \left. - \min\left\{0, \gamma_l(2i+3) - \frac{R}{\alpha}\right\}\right] + b - \gamma_l(2i+3).
\end{aligned} \tag{2.27}$$

According to Theorem 2.3, the optimal strategy is a threshold strategy. Therefore, we can distinguish the following cases for Equation (2.27).

- Let $f^1_{\alpha,low}(i) = (0,1)$ and $f^1_{\alpha,low}(i+1) = (0,1)$.
  Then Equation (2.27) becomes:

$$\begin{aligned}
& v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\Big[\mu_2\gamma_l(4i+4) + \mu_1\gamma_l(2i+1) - \mu_2\gamma_l(2i+1)\Big] \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\Big[\gamma_l(2i+3)\Big] + b - \gamma_l(2i+3) \\
&\quad \leq 0.
\end{aligned}$$

  This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \geq \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_1}. \tag{2.28}$$

- Let $f^1_{\alpha,low}(i) = (0,1)$ and $f^1_{\alpha,low}(i+1) = (0,2)$.

Then Equation (2.27) becomes:

$$
\begin{aligned}
&v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\mu_1\gamma_l(4i+4) + \mu_2\gamma_l(2i+1) + \frac{K}{\alpha} - \mu_2\gamma_l(2i+1)\right] \\
&\qquad\quad+ \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\Big[\gamma_l(2i+3)\Big] + b - \gamma_l(2i+3) \\
&\quad\leq 0.
\end{aligned}
$$

This expression is equivalent to the following constraint on γ_l:

$$
\gamma_l \geq \frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2-\mu_1)(2i+3) + 2\mu_1\big)}. \tag{2.29}
$$

- Let $f^1_{\alpha,low}(i) = (0,2)$ and $f^1_{\alpha,low}(i+1) = (0,2)$.
  Then Equation (2.27) becomes:

$$
\begin{aligned}
&v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\mu_1\gamma_l(4i+4) + \mu_2\gamma_l(2i+1) + \frac{K}{\alpha} - \mu_1\gamma_l(2i+1) - \frac{K}{\alpha}\right] \\
&\qquad\quad+ \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\Big[\gamma_l(2i+3)\Big] + b - \gamma_l(2i+3) \\
&\quad\leq 0.
\end{aligned}
$$

This expression is equivalent to the following constraint on γ_l:

$$
\gamma_l \geq \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2}. \tag{2.30}
$$

- Let $f^1_{\alpha,low}(i) = (1,1)$ and $f^1_{\alpha,low}(i+1) = (0,1)$.
  Then Equation (2.27) becomes:

$$
\begin{aligned}
&v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\Big[\mu_2\gamma_l(4i+4) + \mu_1\gamma_l(2i+1) - \mu_2\gamma_l(2i+1)\Big] \\
&\qquad\quad+ \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\gamma_l(2i+3) - \gamma_l(2i+3) + \frac{R}{\alpha}\right] + b - \gamma_l(2i+3) \\
&\quad\leq 0.
\end{aligned}
$$

This expression is equivalent to the following constraint on γ_l:

$$
\gamma_l \geq \frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_1 + \lambda(2i+3)\big)}. \tag{2.31}
$$

- Let $f^1_{\alpha,low}(i) = (1,1)$ and $f^1_{\alpha,low}(i+1) = (0,2)$.
  Then Equation (2.27) becomes:

$$
\begin{aligned}
&v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\mu_1\gamma_l(4i+4) + \mu_2\gamma_l(2i+1) + \frac{K}{\alpha} - \mu_2\gamma_l(2i+1)\right] \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\gamma_l(2i+3) - \gamma_l(2i+3) + \frac{R}{\alpha}\right] + b - \gamma_l(2i+3) \\
&\quad \leq 0.
\end{aligned}
$$

  This expression is equivalent to the following constraint on γ_l:

$$
\begin{aligned}
\gamma_l &\geq \frac{b + \frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{(2i+3) - \frac{\alpha}{\lambda+\mu_1+\mu_2}\mu_1(4i+4)} \\
&= \frac{b + \frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\lambda+\mu_2)(2i+3) - \mu_1(2i+1)\big)}.
\end{aligned} \tag{2.32}
$$

- Let $f^1_{\alpha,low}(i) = (1,2)$ and $f^1_{\alpha,low}(i+1) = (0,2)$.
  Then Equation (2.27) becomes:

$$
\begin{aligned}
&v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\mu_1\gamma_l(4i+4) + \mu_2\gamma_l(2i+1) + \frac{K}{\alpha} - \mu_1\gamma_l(2i+1) - \frac{K}{\alpha}\right] \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\gamma_l(2i+3) - \gamma_l(2i+3) + \frac{R}{\alpha}\right] + b - \gamma_l(2i+3) \\
&\quad \leq 0.
\end{aligned}
$$

  This expression is equivalent to the following constraint on γ_l:

$$
\gamma_l \geq \frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big(\lambda(2i+3) + 2\mu_2\big)}. \tag{2.33}
$$

- Let $f^1_{\alpha,low}(i) = (1,1)$ and $f^1_{\alpha,low}(i+1) = (1,1)$.
  Then Equation (2.27) becomes:

$$
\begin{aligned}
&v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\
&\quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\Big[\mu_2\gamma_l(4i+4) + \mu_1\gamma_l(2i+1) - \mu_2\gamma_l(2i+1)\Big] \\
&\qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\gamma_l(4i+8) - \frac{R}{\alpha} - \gamma_l(2i+3) + \frac{R}{\alpha}\right] + b - \gamma_l(2i+3) \\
&\quad \leq 0.
\end{aligned}
$$

  This expression is equivalent to the following constraint on γ_l:

$$
\gamma_l \geq \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_1 - 2\lambda\big)}. \tag{2.34}
$$

- Let $f^1_{\alpha,low}(i) = (1,1)$ and $f^1_{\alpha,low}(i+1) = (1,2)$.
  Then Equation (2.27) becomes:

$$\begin{aligned} & v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\ & \quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\mu_1\gamma_l(4i+4) + \mu_2\gamma_l(2i+1) + \frac{K}{\alpha} - \mu_2\gamma_l(2i+1)\right] \\ & \qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\gamma_l(4i+8) - \frac{R}{\alpha} - \gamma_l(2i+3) + \frac{R}{\alpha}\right] + b - \gamma_l(2i+3) \\ & \quad \le 0. \end{aligned}$$

  This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \ge \frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2-\mu_1)(2i+1) + 2(\mu_2-\lambda)\big)}. \tag{2.35}$$

- Let $f^1_{\alpha,low}(i) = (1,2)$ and $f^1_{\alpha,low}(i+1) = (1,2)$.
  Then Equation (2.27) becomes:

$$\begin{aligned} & v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) - \left[v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i)\right] \\ & \quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\mu_1\gamma_l(4i+4) + \mu_2\gamma_l(2i+1) + \frac{K}{\alpha} - \mu_1\gamma_l(2i+1) - \frac{K}{\alpha}\right] \\ & \qquad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\gamma_l(4i+8) - \frac{R}{\alpha} - \gamma_l(2i+3) + \frac{R}{\alpha}\right] + b - \gamma_l(2i+3) \\ & \quad \le 0. \end{aligned}$$

  This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \ge \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_2 - 2\lambda\big)}. \tag{2.36}$$

To give an expression of Inequality (2.26) for $i = 0$, we will combine Equations (2.4) and (2.24).

$$\begin{aligned}
&v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0) - \left[v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\right] \\
&= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{\mu_2\gamma_l 2^2 + \mu_1\gamma_l 1^2, \mu_1\gamma_l 2^2 + \mu_2\gamma_l 1^2 + \frac{K}{\alpha}\right\}\right. \\
&\qquad\qquad \left. - \min\left\{\mu_2\gamma_l 1^2 + \mu_1\gamma_l 1^2, \mu_1\gamma_l 1^2 + \mu_2\gamma_l 1^2 + \frac{K}{\alpha}\right\}\right] \\
&\quad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{\gamma_l(2)^2, \gamma_l(3)^2 - \frac{R}{\alpha}\right\} - \min\left\{\gamma_l(1)^2, \gamma_l(2)^2 - \frac{R}{\alpha}\right\}\right] \\
&\quad + B(1) - B(0) - \left[\gamma_l 2^2 - \gamma_l 1^2\right] \\
&= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{3\mu_2\gamma_l, 3\mu_1\gamma_l + \frac{K}{\alpha}\right\} - \min\left\{0, \frac{K}{\alpha}\right\}\right] \\
&\quad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{3\gamma_l, 8\gamma_l - \frac{R}{\alpha}\right\} - \min\left\{0, 3\gamma_l - \frac{R}{\alpha}\right\}\right] + b - 3\gamma_l \\
&= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\min\left\{3\mu_2\gamma_l, 3\mu_1\gamma_l + \frac{K}{\alpha}\right\} \\
&\quad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[\min\left\{3\gamma_l, 8\gamma_l - \frac{R}{\alpha}\right\} - \min\left\{0, 3\gamma_l - \frac{R}{\alpha}\right\}\right] + b - 3\gamma_l. \qquad (2.37)
\end{aligned}$$

Note that $K/\alpha > 0$, so $\min\{0, K/\alpha\} = 0$. This also means that for $i = 0$ and $n = 1$, the slower Server 1 is used, which is equivalent to $\left(f^1_{\alpha,low}(0)\right)_2 = 1$.

According to Theorem 2.3, both the choice for $a \in A$ and $d \in D$ are threshold strategies. Therefore, we can distinguish the following cases for Equation (2.37).

- Let $f^1_{\alpha,low}(0) = (0, 1)$ and $f^1_{\alpha,low}(1) = (0, 1)$.
  Then Equation (2.37) becomes:

$$\begin{aligned}
&v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0) - \left[v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\right] \\
&= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\Big[3\mu_2\gamma_l\Big] + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\Big[3\gamma_l\Big] + b - 3\gamma_l \\
&\leq 0.
\end{aligned}$$

  This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \geq \frac{b}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 3\mu_1}. \qquad (2.38)$$

- Let $f^1_{\alpha,low}(0) = (0, 1)$ and $f^1_{\alpha,low}(1) = (0, 2)$.
  Then Equation (2.37) becomes:

$$\begin{aligned}
&v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0) - \left[v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\right] \\
&= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[3\mu_1\gamma_l + \frac{K}{\alpha}\right] + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\Big[3\gamma_l\Big] + b - 3\gamma_l \\
&\leq 0.
\end{aligned}$$

This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \geq \frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 3\mu_2}. \tag{2.39}$$

- Let $f^1_{\alpha,low}(0) = (1,1)$ and $f^1_{\alpha,low}(1) = (0,1)$.
  Then Equation (2.37) becomes:

$$\begin{aligned} &v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0) - \left[v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\right] \\ &= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\Big[3\mu_2\gamma_l\Big] + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[3\gamma_l - 3\gamma_l + \frac{R}{\alpha}\right] + b - 3\gamma_l \\ &\leq 0. \end{aligned}$$

This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \geq \frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 3(\lambda+\mu_1)}. \tag{2.40}$$

- Let $f^1_{\alpha,low}(0) = (1,1)$ and $f^1_{\alpha,low}(1) = (0,2)$.
  Then Equation (2.37) becomes:

$$\begin{aligned} &v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0) - \left[v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\right] \\ &= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[3\mu_1\gamma_l + \frac{K}{\alpha}\right] + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[3\gamma_l - 3\gamma_l + \frac{R}{\alpha}\right] + b - 3\gamma_l \\ &\leq 0. \end{aligned}$$

This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \geq \frac{b + \frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 3(\lambda+\mu_2)}. \tag{2.41}$$

- Let $f^1_{\alpha,low}(0) = (1,1)$ and $f^1_{\alpha,low}(1) = (1,1)$.
  Then Equation (2.37) becomes:

$$\begin{aligned} &v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0) - \left[v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\right] \\ &= \frac{1}{\lambda+\mu_1+\mu_2}\alpha\Big[3\mu_2\gamma_l\Big] + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[8\gamma_l - \frac{R}{\alpha} - 3\gamma_l + \frac{R}{\alpha}\right] + b - 3\gamma_l \\ &\leq 0. \end{aligned}$$

This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \geq \frac{b}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_1 - 2\lambda)}. \tag{2.42}$$

- Let $f^1_{\alpha,low}(0) = (1,1)$ and $f^1_{\alpha,low}(1) = (1,2)$.
  Then Equation (2.37) becomes:

$$\begin{aligned}
&v^1_{\alpha,low}(1) - v^1_{\alpha,low}(0) - \left[v^0_{\alpha,low}(1) - v^0_{\alpha,low}(0)\right] \\
&\quad = \frac{1}{\lambda+\mu_1+\mu_2}\alpha\left[3\mu_1\gamma_l + \frac{K}{\alpha}\right] + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha\left[8\gamma_l - \frac{R}{\alpha} - 3\gamma_l + \frac{R}{\alpha}\right] + b - 3\gamma_l \\
&\quad \leq 0.
\end{aligned}$$

  This expression is equivalent to the following constraint on γ_l:

$$\gamma_l \geq \frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_2 - 2\lambda)}. \tag{2.43}$$

The previous enumerations can be used to determine a lower bound for γ_l. To do so, we will distinguish all possible combinations of values of i^1_a and i^1_d.

1. Let $i^1_a < i^1_d$.

   (a) Let $f^1_{\alpha,low}(0) = (0,1)$ and $i^1_d > 0$.
   Then we get, according to Equations (2.38), (2.28), (2.29) and (2.30), where we take for i in each expression the minimal possible value to gain the maximum:

$$\begin{aligned}
\gamma_l \geq \max\Bigg\{ &\frac{b}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 3\mu_1}, \frac{b}{(1-\alpha)(2\cdot 1+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_1}, \\
&\frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i^1_d+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2-\mu_1)(2i^1_d+3) + 2\mu_1\big)}, \\
&\frac{b}{(1-\alpha)\big(2(i^1_d+1)+3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2} \Bigg\},
\end{aligned}$$

   where the expression in grey will never be the maximum value.

   (b) Let $i^1_a = 0$.
   Then we get, according to Equations (2.40), (2.28), (2.29) and (2.30), where we take for i in each expression the minimal possible value to gain the maximum:

$$\begin{aligned}
\gamma_l \geq \max\Bigg\{ &\frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 3(\lambda+\mu_1)}, \frac{b}{(1-\alpha)(2\cdot 1+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_1}, \\
&\frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i^1_d+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2-\mu_1)(2i^1_d+3) + 2\mu_1\big)}, \\
&\frac{b}{(1-\alpha)\big(2(i^1_d+1)+3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2} \Bigg\},
\end{aligned}$$

   where the expression in grey will never be the maximum value.

(c) Let $i_a^1 > 0$.
Then we get, according to Equations (2.42), (2.34), (2.31), (2.28), (2.29) and (2.30), where we take for i in each expression the minimal possible value to gain the maximum:

$$\gamma_l \geq \max\left\{\frac{b}{3(1-\alpha)+\frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_1-2\lambda)}, \frac{b}{(1-\alpha)(2\cdot 1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_1-2\lambda\big)},\right.$$
$$\frac{b+\frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_a^1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_1+\lambda(2i_a^1+3)\big)},$$
$$\frac{b}{(1-\alpha)\big(2(i_a^1+1)+3\big)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_1},$$
$$\frac{b+\frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_d^1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2-\mu_1)(2i_d^1+3)+2\mu_1\big)},$$
$$\left.\frac{b}{(1-\alpha)\big(2(i_d^1+1)+3\big)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2}\right\},$$

where the grey expressions will never be the maximum value.

2. Let $i_d^1 < i_a^1$.

(a) Let $i_d^1 = 0$.
Then we get, according to Equations (2.43), (2.36), (2.33) and (2.30), where we take for i in each expression the minimal possible value to gain the maximum:

$$\gamma_l \geq \max\left\{\frac{b+\frac{K}{\lambda+\mu_1+\mu_2}}{3(1-\alpha)+\frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_2-2\lambda)}, \frac{b}{(1-\alpha)(2\cdot 1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_2-2\lambda\big)},\right.$$
$$\frac{b+\frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_a^1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(\lambda(2i_a^1+3)+2\mu_2\big)},$$
$$\left.\frac{b}{(1-\alpha)\big(2(i_a^1+1)+3\big)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2}\right\},$$

where the expression in grey will never be the maximum value.

(b) Let $i_b^1 > 0$.
Then we get, according to Equations (2.42), (2.34), (2.35), (2.36), (2.33) and (2.30),

where we take for i in each expression the minimal possible value to gain the maximum:

$$\gamma_l \geq \max\Bigg\{ \frac{b}{3(1-\alpha)+\frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_1-2\lambda)}, \frac{b}{(1-\alpha)(2\cdot 1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_1-2\lambda\big)},$$
$$\frac{b+\frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_d^1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2-\mu_1)(2i_d^1+1)+2(\mu_2-\lambda)\big)},$$
$$\frac{b}{(1-\alpha)\big(2(i_d^1+1)+3\big)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_2-2\lambda\big)},$$
$$\frac{b+\frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_a^1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(\lambda(2i_a^1+3)+2\mu_2\big)},$$
$$\frac{b}{(1-\alpha)\big(2(i_a^1+1)+3\big)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2}\Bigg\},$$

where the grey expressions will never be the maximum value.

3. Let $i_a^1 = i_d^1$.

   (a) Let $i_a^1 = i_d^1 = 0$.
   Then we get, according to Equations (2.41) and (2.30), where we take for i in each expression the minimal possible value to gain the maximum:

$$\gamma_l \geq \max\left\{ \frac{b+\frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 3(\lambda+\mu_2)}, \frac{b}{(1-\alpha)(2\cdot 1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2}\right\}.$$

   (b) Let $i_a^1 = i_d^1 > 0$.
   Then we get, according to Equations (2.42), (2.34), (2.32) and (2.30), where we take for i in each expression the minimal possible value to gain the maximum:

$$\gamma_l \geq \max\Bigg\{ \frac{b}{3(1-\alpha)+\frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_1-2\lambda)}, \frac{b}{(1-\alpha)(2\cdot 1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_1-2\lambda\big)},$$
$$\frac{b+\frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_a^1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\lambda+\mu_2-\mu_1)(2i_a^1+3)+2\mu_1\big)},$$
$$\frac{b}{(1-\alpha)\big(2(i_a^1+1)+3\big)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2}\Bigg\},$$

   where the expression in grey will never be the maximum value.

As can be seen in the enumeration above, there are seven different scenarios for the values of i_a^1 and i_d^1, which all affect the minimum value of γ_l. All different terms to be smaller than or equal to γ_l do have similarities, and by that we can give the following estimate of γ_l:

$$\gamma_l \geq \frac{b+\frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha)}, \tag{2.44}$$

since in each of the Equations (2.28) till (2.36) and (2.38) till (2.43) the numerator is either equal to or smaller than the numerator of Equation (2.44), and the denominator is greater than the denominator of Equation (2.44). Therefore, $v^0_{\alpha,low}(i)$ from Equation (2.24) becomes:

$$v^0_{\alpha,low}(i) = \gamma_l (i+1)^2, \qquad \text{where } \gamma_l \geq \frac{b + \frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha)}.$$

2.6.2 Computation of $v^0_{\alpha,up}(i)$

In order to determine $v^0_{\alpha,up}(i)$ from Equation (2.25), we must give a value of γ_u such that

$$v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \geq v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i), \qquad \forall i \in S.$$

We will determine γ_u in a similar way as in Subsection 2.6.1 for γ_l. In Equation (2.27), no property of γ_l or $v^0_{\alpha,low}(i)$ is used, and since $v^0_{\alpha,low}(i)$ and $v^0_{\alpha,up}(i)$ are of a similar form (cf. Equations (2.24) and (2.25)), γ_l can be replaced by γ_u:

$$\begin{aligned} & v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) - \left[v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i)\right] \\ &= \frac{1}{\lambda+\mu_1+\mu_2}\alpha \Bigg[\min\left\{\mu_2\gamma_u(4i+4) + \mu_1\gamma_u(2i+1), \mu_1\gamma_u(4i+4) + \mu_2\gamma_u(2i+1) + \frac{K}{\alpha}\right\} \\ &\qquad - \min\left\{\mu_2\gamma_u(2i+1), \mu_1\gamma_u(2i+1) + \frac{K}{\alpha}\right\}\Bigg] \\ &\quad + \frac{\lambda}{\lambda+\mu_1+\mu_2}\alpha \Bigg[\min\left\{\gamma_u(2i+3), \gamma_u(4i+8) - \frac{R}{\alpha}\right\} \\ &\qquad - \min\left\{0, \gamma_u(2i+3)^2 - \frac{R}{\alpha}\right\}\Bigg] + b - \gamma_u(2i+3). \end{aligned} \tag{2.45}$$

Also, for the enumeration starting on page 61, no properties of γ_l are used, except for the fact that there Equation (2.27) has to be non-positive, and in this case with γ_u we need (2.45) to be nonnegative. Therefore, this enumeration from pages 61 till 64 result in the same restrictions on γ_u as on γ_l, but with an opposite sign. Similarly, the restrictions from the enumeration starting on page 65 on γ_l are equal to the restrictions on γ_u with opposite signs.

With these similarities between the determination of γ_l and γ_u and thus $v^0_{\alpha,low}(i)$ and $v^0_{\alpha,up}(i)$, we can give yet another enumeration similar to the one starting on page 67, where we distinguish all possible combinations of values of i^1_a and i^1_d.

1. Let $i^1_a < i^1_d$.

   (a) Let $f^1_{\alpha,up}(0) = (0,1)$ and $i^1_d = 0$.
   Then we get, according to Equations (2.39) and (2.30), the following, where we take for i in each expression the maximal possible value to gain the minimum:

   $$\gamma_u \leq \min\left\{\frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 3\mu_2}, \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2}\right\},$$

   where the grey expression will never attain the minimum value.

(b) Let $f^1_{\alpha,up}(0) = (0,1)$ and $i^1_d > 0$.
Then we get, according to Equations (2.38), (2.28), (2.29) and (2.30), where we take for i in each expression the maximal possible value to gain the minimum:

$$\gamma_u \leq \min \left\{ \frac{b}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 3\mu_1}, \frac{b}{(1-\alpha)\big(2(i^1_d - 1) + 3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_1}, \right.$$
$$\frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i^1_d + 3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2 - \mu_1)(2i^1_d + 3) + 2\mu_1\big)},$$
$$\left. \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_2} \right\},$$

where the expressions in grey will never attain the minimum value.

(c) Let $i^1_a = 0$.
Then we get, according to Equations (2.40), (2.28), (2.29) and (2.30), where we take for i in each expression the maximal possible value to gain the minimum:

$$\gamma_u \leq \min \left\{ \frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 3(\lambda + \mu_1)}, \frac{b}{(1-\alpha)\big(2(i^1_d - 1) + 3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_1}, \right.$$
$$\frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i^1_d + 3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2 - \mu_1)(2i^1_d + 3) + 2\mu_1\big)},$$
$$\left. \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_2} \right\},$$

where the expressions in grey will never attain the minimum value.

(d) Let $i^1_a > 0$.
Then we get, according to Equations (2.42), (2.34), (2.31), (2.28), (2.29) and (2.30), where we take for i in each expression the maximal possible value to gain the minimum:

$$\gamma_u \leq \min \left\{ \frac{b}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_1 - 2\lambda)}, \frac{b}{(1-\alpha)\big(2(i^1_a - 1) + 3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(2\mu_1 - 2\lambda)}, \right.$$
$$\frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i^1_a + 3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big(2\mu_1 + \lambda(2i^1_a + 3)\big)},$$
$$\frac{b}{(1-\alpha)\big(2(i^1_d - 1) + 3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_1},$$
$$\frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i^1_d + 3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2 - \mu_1)(2i^1_d + 3) + 2\mu_1\big)},$$
$$\left. \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_2} \right\},$$

where the grey expressions will never attain the minimum value.

2. Let $i_d^1 < i_a^1$.

   (a) Let $i_d^1 = 0$.
   Then we get, according to Equations (2.43), (2.36), (2.33) and (2.30), where we take for i in each expression the maximal possible value to gain the minimum:

$$\gamma_u \leq \min \Bigg\{ \frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_2 - 2\lambda)}, \frac{b}{(1-\alpha)\big(2(i_a^1 - 1) + 3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(2\mu_2 - 2\lambda)}, \frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_a^1 + 3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big(\lambda(2i_a^1 + 3) + 2\mu_2\big)}, \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_2} \Bigg\},$$

   where the expressions in grey will never attain the minimum value.

   (b) Let $i_d^1 > 0$.
   Then we get, according to Equations (2.42), (2.34), (2.35), (2.36), (2.33) and (2.30), where we take for i in each expression the maximal possible value to gain the minimum:

$$\gamma_u \leq \min \Bigg\{ \frac{b}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_1 - 2\lambda)}, \frac{b}{(1-\alpha)\big(2(i_d^1 - 1) + 3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(2\mu_1 - 2\lambda)}, \frac{b + \frac{K}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_d^1 + 3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\mu_2 - \mu_1)(2i_d^1 + 1) + 2(\mu_2 - \lambda)\big)}, \frac{b}{(1-\alpha)\big(2(i_a^1 - 1) + 3\big) + \frac{\alpha}{\lambda+\mu_1+\mu_2}(2\mu_2 - 2\lambda)}, \frac{b + \frac{\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_a^1 + 3) + \frac{\alpha}{\lambda+\mu_1+\mu_2}\big(\lambda(2i_a^1 + 3) + 2\mu_2\big)}, \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_2} \Bigg\},$$

   where the grey expressions will never attain the minimum value.

3. Let $i_a^1 = i_d^1$.

   (a) Let $i_a^1 = i_d^1 = 0$.
   Then we get, according to Equations (2.41) and (2.30), where we take for i in each expression the maximal possible value to gain the minimum:

$$\gamma_u \leq \min \Bigg\{ \frac{b + \frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 3(\lambda + \mu_2)}, \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3) + \frac{\alpha}{\lambda+\mu_1+\mu_2} \cdot 2\mu_2} \Bigg\},$$

   where the expression in grey will never attain the minimum value.

(b) Let $i_a^1 = i_d^1 > 0$.
Then we get, according to Equations (2.42), (2.34), (2.32) and (2.30), where we take for i in each expression the maximal possible value to gain the minimum:

$$\gamma_u \leq \min\Bigg\{ \frac{b}{3(1-\alpha)+\frac{\alpha}{\lambda+\mu_1+\mu_2}(3\mu_1-2\lambda)}, \frac{b}{(1-\alpha)\big(2(i_a^1-1)+3\big)+\frac{\alpha}{\lambda+\mu_1+\mu_2}(2\mu_1-2\lambda)}, \frac{b+\frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{(1-\alpha)(2i_a^1+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\big((\lambda+\mu_2-\mu_1)(2i_a^1+3)+2\mu_1\big)}, \lim_{i\to\infty} \frac{b}{(1-\alpha)(2i+3)+\frac{\alpha}{\lambda+\mu_1+\mu_2}\cdot 2\mu_2} \Bigg\},$$

where the expression in grey will never attain the minimum value.

In every of the beforementioned cases for the relation between i_a^1 and i_d^1, γ_u has to be less than or equal to zero. At the introduction of γ_u in Equation (2.25) was stated that $\gamma_u \geq 0$. Therefore, the only possible value is $\gamma_u = 0$, which gives the following equation for $v^0_{\alpha,up}(i)$:

$$v^0_{\alpha,up}(i) = \gamma_u(i+1)^2 = 0, \qquad \text{with } \gamma_u = 0.$$

In Subsections 2.6.1 and 2.6.2 we have shown that the choices of

$$v^0_{\alpha,low}(i) = \gamma_l(i+1)^2, \qquad \text{with } \gamma_l \geq \frac{b+\frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha)}, \tag{2.46}$$

$$v^0_{\alpha,up}(i) = \gamma_u(i+1)^2 = 0, \qquad \text{with } \gamma_u = 0, \tag{2.47}$$

give a starting value for Theorem 2.9, Corollary 2.11 and Corollary 2.12, which makes them useful not only in theory, but also in practice.

Note that in Equations (2.24) and (2.25), the basis equations use $(i+1)^2$. We have also tried the more natural equation of i^2, but this gave us an extra constraint on the variables. The equivalent of Equation (2.42) requires $\mu_1 \geq 2\lambda$ and the equivalent of Equation (2.43) requires $\mu_2 \geq 2\lambda$. Since by definition $\mu_1 < \mu_2$, these restrictions can be summarized in $\mu_1 \geq 2\lambda$. This constraint is probably due to the lack of steepness in $i = 0$ for the function i^2, which the function we used, $(i+1)^2$, does have.

2.6.3 Numerical example

Now that we have proven that $v^0_{\alpha,low}(i)$ and $v^0_{\alpha,up}(i)$ can be chosen such that

$$v^1_{\alpha,up}(i+1) - v^1_{\alpha,up}(i) \geq v^0_{\alpha,up}(i+1) - v^0_{\alpha,up}(i), \qquad \forall i \in S,$$

and

$$v^1_{\alpha,low}(i+1) - v^1_{\alpha,low}(i) \leq v^0_{\alpha,low}(i+1) - v^0_{\alpha,low}(i), \qquad \forall i \in S,$$

both hold, we will show graphically the results of Corollary 2.11 and Corollary 2.12. The corresponding R-code can be found in Appendix B.

We give one numerical example, with the choice of parameters mostly equal to the situation in Subsection 1.2.1. We choose $\lambda = 1$, $\mu_1 = 2$, $\mu_2 = 3$, $K = 1$, $R = 3$, $b = 1$ and $\alpha = 0.9$. For the value of $v_{\alpha,up}(i)$, we follow Equation (2.47) and thus choose $\gamma_u = 0$. For the value of $v^0_{\alpha,low}(i)$, we take, according to Equation (2.46),

$$\gamma_l = \frac{b + \frac{K+\lambda R}{\lambda+\mu_1+\mu_2}}{3(1-\alpha)} = \frac{1 + \frac{1+1\cdot 3}{1+2+3}}{3(1-0.9)} = \frac{50}{9}.$$

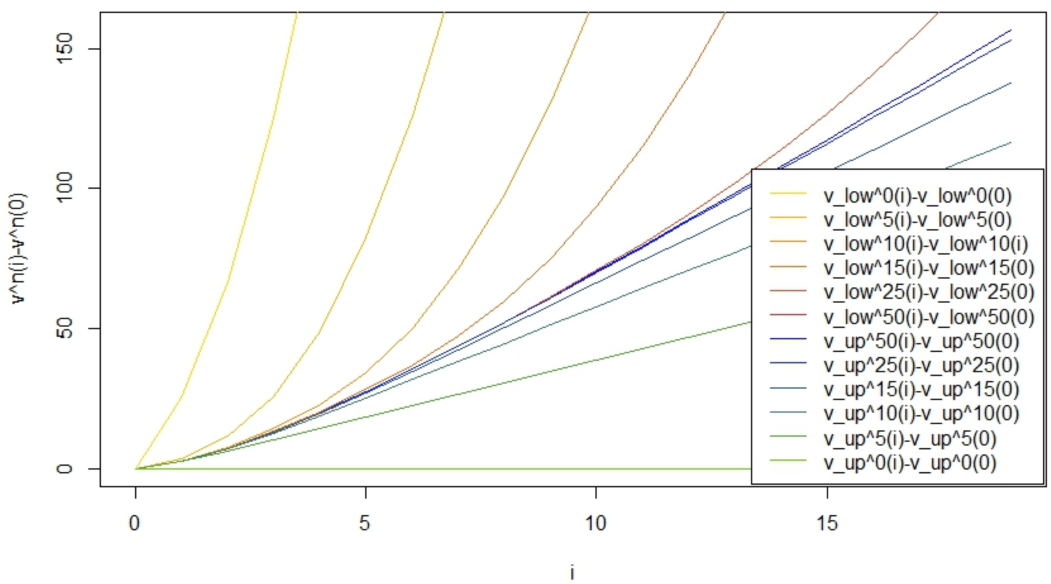
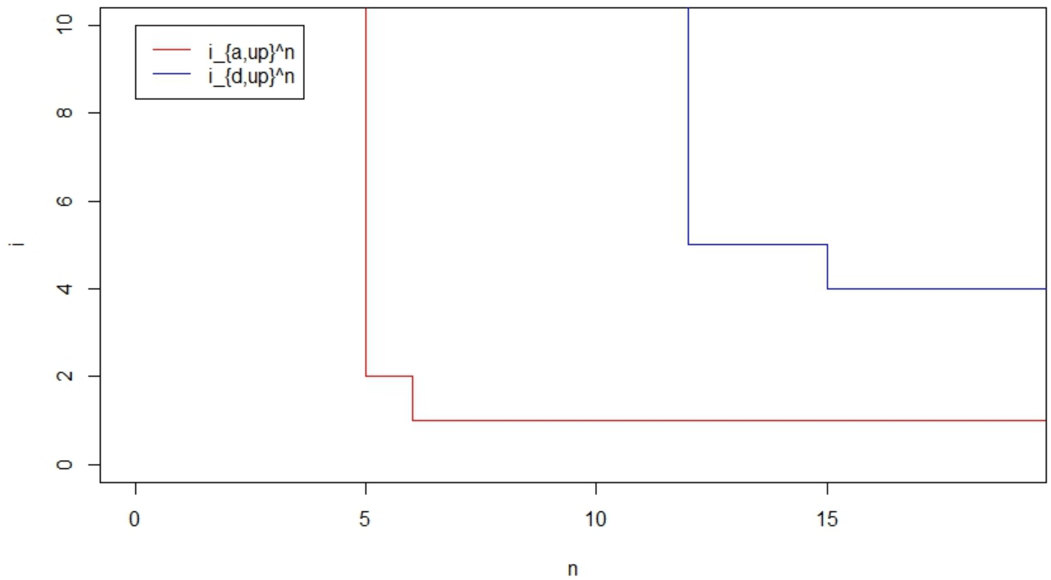


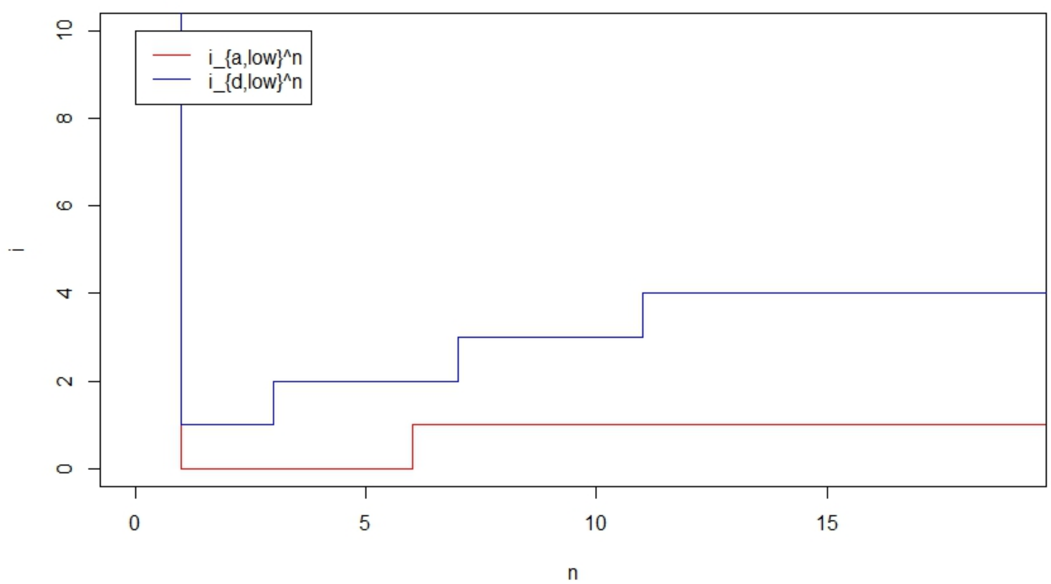
Figure 2.1: Expected reduced discount costs $v^n_{\alpha,low}(i) - v^n_{\alpha,low}(0)$ and $v^n_{\alpha,up}(i) - v^n_{\alpha,up}(0)$, for several values of time step n, with parameters $\lambda = 1$, $\mu_1 = 2$, $\mu_2 = 3$, $K = 1$, $R = 3$, $b = 1$, $\alpha = 0.9$, $\gamma_u = 0$, $\gamma_l = 50/9$.

In Figure 2.1, we have plotted the expected reduced discount costs of $v^n_{\alpha,low}(i)$ and $v^n_{\alpha,up}(i)$ for several values of n. The graph shows that for $n = 50$, the graphs of $v^{50}_{\alpha,low}(i) - v^{50}_{\alpha,low}(0)$ and $v^{50}_{\alpha,up}(i) - v^{50}_{\alpha,up}(0)$ are already that close to each other, that they are drawn at the same spot. Therefore, this graph confirms the claim of Corollary 2.12 and also shows that the convergence goes rather fast; within 50 time steps, the difference between the lower bound and the upper bound is hardly visible, and thus the range of the optimal expected discount costs decreases rapidly.

Figure 2.2(a) shows the thresholds $i^n_{a,up}$ and $i^n_{d,up}$ for different values of n. Figure 2.2(b) shows the thresholds $i^n_{a,low}$ and $i^n_{d,low}$ for different values of n. We can see, that for $n \geq 6$, $i^n_{a,low} = i^n_{a,up} = 1$, and thus $i^*_a = 1$. For $n \geq 15$, we get $i^n_{d,low} = i^n_{d,up} = 4$, and thus $i^*_d = 4$. So within 15 iterations of Algorithm 2.1, we know the optimal strategy. This validates the statement in Corollary 2.11.



(a) Thresholds $i_{a,up}^n$ and $i_{d,up}^n$ for time n. Note that $i_{a,up}^n = \infty$ for $n \in \{0, \ldots, 4\}$, and $i_{d,up}^n = \infty$ for $n \in \{0, \ldots, 11\}$.



(b) Thresholds $i_{a,low}^n$ and $i_{d,low}^n$ for time n. Note that $i_{a,low}^0 = i_{d,low}^0 = \infty$ by the definition of f_α^0 in Algorithm 2.1.

Figure 2.2: Thresholds for time n, with parameters $\lambda = 1$, $\mu_1 = 2$, $\mu_2 = 3$, $K = 1$, $R = 3$, $b = 1$, $\alpha = 0.9$, $\gamma_u = 0$, $\gamma_l = 50/9$.

Appendices

Appendix A

Discounted model: R code

```
X <- 20 #Max size queue is (X-1); queue can be 0
T <- 20 #Number of time steps
lambda <- 1 #Arrival rate
mu <- 2 #Departure rate
R <- 3 #Profit per customer served
b <- 1 #Fine per customer per time unit in queue
alpha <- 0.9 #Discount rate
plambda <- lambda/(lambda+mu) #Prob arrival
pmu <- mu/(lambda+mu) #Prob departure

i<- NULL; for (j in 1:X) {i <- c(i,j)}
#i is vector size X, values 1, ..., X
c0 <- b*(i-1) #Cost refusing customer
c1 <- b*(i-1) - plambda*R #Cost accepting customer

v <- matrix(0,X,T) #Expected discounted profit
vhelp <- array(0,dim=c(X,T,2))
#vhelp(,,1): incoming customer sent away;
#vhelp(,,2): incoming customer accepted

#Note: the code does not compile with the extra enters added in
#the following lines, but they are necessary for readability.
for (k in 1:X) {
  vhelp[1,k,1] <- c0[k] #Set the first time step
  vhelp[1,k,2] <- c1[k]
  v[1,k] <- min( vhelp[1,k,1], vhelp[1,k,2])
}
for (j in 2:T) {
  vhelp[j,1,1] <- c0[1] + alpha*v[(j-1),1] #Empty queue
  vhelp[j,1,2] <- c1[1] + alpha*plambda*v[j-1,2]
                        + alpha*pmu*v[j-1,1]
  v[j,1] <- min( vhelp[j,1,1], vhelp[j,1,2])
```

```
  for (k in 2:(X-1)) {
    vhelp[j,k,1] <- c0[k] + alpha*plambda*v[j-1,k]
                          + alpha*pmu*v[j-1,k-1]
    vhelp[j,k,2] <- c1[k] + alpha*plambda*v[j-1,k+1]
                          + alpha*pmu*v[j-1,k-1]
    v[j,k] <- min( vhelp[j,k,1], vhelp[j,k,2])
  }
  vhelp[j,X,1] <- 2*vhelp[j,X-1,1] - vhelp[j,X-2,1] #(X-1) customers
  vhelp[j,X,2] <- 2*vhelp[j,X-1,2] - vhelp[j,X-2,2]
  v[j,X] <- min( vhelp[j,X,1], vhelp[j,X,2])
}
#This only works because fee=b*i,
#so vhelp[,,1] and vhelp[,,2] are linear

f <- matrix(0,T,X) #Strategy f
for (j in 1:T) {
  for (k in 1:X) {
    if (vhelp[j,k,1] > vhelp[j,k,2]) {
      f[j,k] <- 1
    }
  }
}
#f[,]=0: refuse customer; f[,]=1: accept customer


#Threshold function
thr <- matrix(0,T,1)
for (j in 1:T) {
  for (k in 1:X) {
    if (f[j,k] == 1) {
      thr[j,1] <- k-1
    }
  }
}
```

Appendix B

Discounted model with the choice between two servers: R code

```
X <- 20 #Max size queue is (X-1); queue can be 0
T <- 51 #Number of time steps
lambda <- 1 #Arrival rate
mu1 <- 2 #Departure rate slow Server 1
mu2 <- 3 #Departure rate fast Server 2
K <- 1 #2 #Extra cost for Server 2
R <- 3 #Profit per customer served
b <- 1 #Fine per customer per time unit in queue
alpha <- 0.9 #Discount rate
plambda <- lambda/(lambda+mu1+mu2) #Prob arrival
pmu1 <- mu1/(lambda+mu1+mu2) #Prob departure from Server 1
pmu2 <- mu2/(lambda+mu1+mu2) #Prob departure from Server 2

B <- integer(X)
c0 <- integer(X)
c1 <- integer(X)
for (i in 1:X) {
  B[i] <- i-1
  c0[i] <- B[i] #Cost refusing customer
  c1[i] <- B[i] - plambda*R #Cost accepting customer
}

#Run code twice: for v_low and v_up separately
gamma <- (b+(K+lambda*R)/(lambda+mu1+mu2))/((1-alpha)*3) #v_low
#gamma <- 0 #v_up

v <- matrix(0,T,X) #Expected discount cost
f <- array(0,dim=c(T,X,2)) #Strategy
#f(,,1)=0: refuse customer; f(,,1)=1: accept customer,
#f(,,2)=1: use slow server; f(,,2)=2: use fast server.
```

```
RS <- matrix(0,T,X) #Refuse incoming customer, Slow server
RF <- matrix(0,T,X) #Refuse incoming customer, Fast server
AS <- matrix(0,T,X) #Accept incoming customer, Slow server
AF <- matrix(0,T,X) #Accept incoming customer, Fast server

for (k in 1:X) { #Set the first time step
  v[1,k] <- gamma*(k)^2 #i in 0:(X-1), k in 1:X, so k^2=(i+1)^2

  f[1,k,1] <- 1
  f[1,k,2] <- 1
}

for (j in 2:T) {
  #Left boundary, X=1 (empty queue)

  #Note: the code does not compile with the extra enters added in
  #the following lines, but they are necessary for readability.

  RS[j,1] <- alpha*v[j-1,1] + c0[1]
  RF[j,1] <- alpha*v[j-1,1] + c0[1] + K
  AS[j,1] <- plambda*alpha*v[j-1,2]
              + (pmu1+pmu2)*alpha*v[j-1,1] + c1[1]
  AF[j,1] <- plambda*alpha*v[j-1,2]
              + (pmu1+pmu2)*alpha*v[j-1,1] + c1[1] + K
  v[j,1] <- min(RS[j,1], RF[j,1], AS[j,1], AF[j,1])
  if (v[j,1] == RS[j,1] | v[j,1] == RF[j,1]) {
    f[j,1,1] <- 0
  }
  if (v[j,1] == AS[j,1] | v[j,1] == AF[j,1]) {
    f[j,1,1] <- 1
  }
  if (v[j,1] == RS[j,1] | v[j,1] == AS[j,1]) {
    f[j,1,2] <- 1
  }
  if (v[j,1] == RF[j,1] | v[j,1] == AF[j,1]) {
    f[j,1,2] <- 2
  }

  for (k in 2:(X-1)) {
    #Regular, non-boundary entries

    #Note: the code does not compile with the extra enters added in
    #the following lines, but they are necessary for readability.

    RS[j,k] <- (plambda+pmu2)*alpha*v[j-1,k]
            + pmu1*alpha*v[j-1,k-1]+ c0[k]
```

```
      RF[j,k] <- (plambda+pmu1)*alpha*v[j-1,k]
                 + pmu2*alpha*v[j-1,k-1]+ c0[k] + K
      AS[j,k] <- plambda*alpha*v[j-1,k+1]
                 + pmu2*alpha*v[j-1,k] + pmu1*alpha*v[j-1,k-1] + c1[k]
      AF[j,k] <- plambda*alpha*v[j-1,k+1]
                 + pmu1*alpha*v[j-1,k] + pmu2*alpha*v[j-1,k-1] + c1[k] + K
      v[j,k] <- min(RS[j,k], RF[j,k], AS[j,k], AF[j,k])
      if (v[j,k] == RS[j,k] | v[j,k] == RF[j,k]) {
        f[j,k,1] <- 0
      }
      if (v[j,k] == AS[j,k] | v[j,k] == AF[j,k]) {
        f[j,k,1] <- 1
      }
      if (v[j,k] == RS[j,k] | v[j,k] == AS[j,k]) {
        f[j,k,2] <- 1
      }
      if (v[j,k] == RF[j,k] | v[j,k] == AF[j,k]) {
        f[j,k,2] <- 2
      }
    }

    #Right boundary, with (X-1) customers in the system

    #This only works because fee=b*i, so RS[j,], RF[j,], AS[j,]
    #and AF[j,] are linear for every time step j.

    RS[j,X] <- 2*RS[j,X-1] - RS[j,X-2]
    RF[j,X] <- 2*RF[j,X-1] - RF[j,X-2]
    AS[j,X] <- 2*AS[j,X-1] - AS[j,X-2]
    AF[j,X] <- 2*AF[j,X-1] - AF[j,X-2]
    #This only works because fee=b*i,
    #so RS[j,], RF[j,], AS[j,] and AF[j,] are linear
    v[j,X] <- min(RS[j,X], RF[j,X], AS[j,X], AF[j,X])
    if (v[j,X] == RS[j,X] | v[j,X] == RF[j,X]) {
      f[j,X,1] <- 0
    }
    if (v[j,X] == AS[j,X] | v[j,X] == AF[j,X]) {
      f[j,X,1] <- 1
    }
    if (v[j,X] == RS[j,X] | v[j,X] == AS[j,X]) {
      f[j,X,2] <- 1
    }
    if (v[j,X] == RF[j,X] | v[j,X] == AF[j,X]) {
      f[j,X,2] <- 2
    }
}
```

```
# Threshold function
thr <- matrix(0,T,2)
for (j in 1:T) {
  for (k in 1:X) {
    if (f[j,k,1] == 1) {
      thr[j,1] <- k
    }
    if (f[j,k,2] == 1) {
      thr[j,2] <- k
    }
  }
}

#Run code twice: for v_low and v_up separately
v_low <- v
#v_up <- v

thr_low <- thr
#thr_up <- thr
```

Bibliography

[1] I. Adan and J. Resing, *Queueing Theory*, 2002.

[2] S. Bhulai, *Markov Decision Processes: monotonicity properties*, 2017.

[3] L. Kallenberg and F. Spieksma, *Besliskunde A*, 2014.

[4] G. Koole, *Monotonicity in Markov Reward and Decision Chains: Theory and Applications*, Foundations and Trends ® in Stochastic Systems, 1 (2006), pp. 1–76.

[5] F. Spieksma, *Markov Decision Processes*, 2016.