The Cup Product

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THE CUP PRODUCT

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Introduction

The main object of study in this project is the so called *cup product*, a structure in cohomology that is important in algebraic topology.

There are various cohomology theories of topological spaces: there is singular cohomology, which is obtained by dualizing the singular chain complexes of singular homology; sheaf cohomology, which involves the right derived functors of the global sections functor of sheaves on a space X; and there is Čech cohomology, where a space X is approximated by taking suitable open covers. All these theories come with the additional structure of a *cup product*.

Under certain conditions these theories yield isomorphic cohomology groups. Our goal has been to understand these isomorphisms. Additionally, we have tried to verify that they respect the cup product structure. It turns out that, after developing the right tools, there are natural proofs showing that this is indeed the case. The most important realization is that the cup product structures in the different cohomology theories are all determined by chain maps from a total complex $(\mathcal{I} \otimes \mathcal{J})^\bullet_{\mathrm{Tot}}$ of certain pure resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ to a pure resolution $\mathcal{K}^\bullet$, where the resolutions depend on the approach to cohomology that is considered (singular, sheaf, or Čech). Here a pure resolution is a resolution that remains exact upon tensoring with any sheaf $\mathcal{F}$.

The main work is in understanding pure monomorphisms in the category of sheaves, and the so called *pure injectives*, objects that have the injective property with respect to pure monomorphisms. These notions were already studied and used to define cup products in a very general setting in [8]. We study them more concretely in the category of sheaves, which also requires a detour through the category of abelian groups.

The structure of this thesis is as follows. We first introduce some theory of singular cohomology, following [1], and we introduce the cup product. As a little aside we give a nice and short proof of Borsuk-Ulam, which exploits the cup product structure of cohomology that is lacking in homology.

The main part of the thesis is section 2. We first give an introduction to sheaf cohomology, following [2]. Then we study pure monomorphisms and pure injectives in the category of sheaves, and use them to define the cup product as in [8]. We give an introduction to Čech cohomology, and we show that under certain assumptions there is a natural isomorphism between sheaf cohomology and Čech cohomology that respects the cup product structure. Finally we show how Čech cohomology is used for finding concrete cohomology rings by computing $\mathrm{H}^*(P^2, \mathbb{Z}/2\mathbb{Z})$, with P^2 the real projective plane.

In the final section of the thesis, we provide a detailed account of the construction of the isomorphism between singular cohomology and sheaf cohomology given in [2]. Using the same trick as with Čech cohomology, we show that also this isomorphism respects the cup product structure.

1 Singular cohomology

1.1 Definitions

In this section we follow [1], chapter 3.

We start with an algebraic discussion of singular cohomology groups of complexes of free abelian groups. Let C be a chain complex of free abelian groups

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

and let G be an abelian group. Define $C^n = \mathrm{Hom}(C_n, G)$, and let $\delta^n = \partial_n^* : C^{n-1} \to C^n$ be precomposition by ∂. Since $\partial^2 = 0$ we have $\delta^2 = 0$, so we get a complex $C^\bullet$

$$\cdots \xleftarrow{\delta^{n+2}} C^{n+1} \xleftarrow{\delta^{n+1}} C^n \xleftarrow{\delta^n} C^{n-1} \xleftarrow{\delta^{n-1}} \cdots$$

dual to the complex C. We define the n-th cohomology group of C with values in G to be the n-th homology group of $C^\bullet$, so $\mathrm{H}^n(C; G) = \ker \delta_{n+1} / \mathrm{im}\ \delta_n$.

We have the following relation between singular homology groups and cohomology groups with values in G.

Theorem 1.1 (Universal Coefficient Theorem for Cohomology)**.** *If a chain complex C of free abelian groups has homology groups $\mathrm{H}_n(C)$, then the cohomology groups $\mathrm{H}^n(C; G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by natural split exact sequences*

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(C), G) \to \mathrm{H}^n(C; G) \to \mathrm{Hom}(\mathrm{H}_n(C), G) \to 0 \tag{1}$$

These exact sequences are natural in the following sense: if we are given a chain map $f : C \to C'$, it induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ext}(\mathrm{H}_{n-1}(C,G) & \longrightarrow & \mathrm{H}^n(C;G) & \longrightarrow & \mathrm{Hom}(\mathrm{H}_n(C),G) & \longrightarrow & 0 \\
 & & \uparrow (f_*)^* & & \uparrow f^* & & \uparrow (f_*)^* & & \\
0 & \longrightarrow & \mathrm{Ext}(\mathrm{H}_{n-1}(C',G) & \longrightarrow & \mathrm{H}^n(C';G) & \longrightarrow & \mathrm{Hom}(\mathrm{H}_n(C'),G) & \longrightarrow & 0
\end{array}$$

Proof. See [1] paragraph 3.1. □

It should be noted that although the sequences are natural in the sense explained, the *splitting* is not natural.

Because we will not go very deep into singular cohomology in this thesis, we do not specify the maps appearing in the theorem: we will only need it for some short computations, for which the above suffices. For the same reason, we will not give the definition of the Ext groups appearing in the sequence (the interested reader can consult [1] paragraph 3.1), but we will just explain how they can be computed for finitely generated groups. This can be done using the following proposition.

Proposition 1.2. *Suppose H, H' are abelian groups. Then we have the following identities:*

- $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$.
- $\mathrm{Ext}(H, G) = 0$ *if H is free.*
- $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) \cong G/nG$*, where n is a non-zero integer.*

Now let X be a topological space. We can do the above construction with $C(X)$ the singular chain complex of X, i.e. $C_n(X)$ is the free group generated by n-simplices $\sigma : \Delta^n \to X$ in X. Since these n-simplices form a basis for $C_n(X)$, and giving a homomorphism on a free group is the same as giving values on a basis, we can view the cochain group $C^n(X; G)$ as the functions from the set of n-simplices in X to G. So for example, $C^0(X; G)$ can be viewed as the set of functions from X to G (without any continuity restrictions).

We define the n-th cohomology group of X with values in G by $\mathrm{H}^n(X; G) = \mathrm{H}^n(C(X); G)$. By theorem 1.1 these cohomology groups fit in the natural split exact sequences

$$0 \to \mathrm{Ext}(\mathrm{H}_{n-1}(X), G) \to \mathrm{H}^n(X; G) \to \mathrm{Hom}(\mathrm{H}_n(X), G) \to 0 \tag{2}$$

The following lemma provides a useful interpretation of $\mathrm{H}^1(X; G)$ for path connected spaces; we will use it to give a natural proof of Borsuk-Ulam.

Lemma 1.3. *Let X be a path connected space. Then there is a natural isomorphism* $\mathrm{H}^1(X; G) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{H}_1(X), G) = \mathrm{Hom}(\pi_1(X), G)$.

Proof. For every topological space $\mathrm{H}_0(X)$ is free, being a direct sum of $\mathbb{Z}$'s with precisely one summand per path component. Hence sequence (2) yields the first isomorphism for any space X. In case X is path connected, $\mathrm{H}_1(X)$ is naturally isomorphic to the abelianization of $\pi_1(X)$. Since G is an abelian group, this implies the last equality, since any homomorphism $\pi_1(X) \to G$ factorizes through $\mathrm{H}_1(X)$. □

Since in this thesis we will mostly concern ourselves with comparing various cohomology theories, we will not give the various tools that are available for actually computing the cohomology of a concrete space X. These tools (similar to the ones in homology) can also be found in [1] paragraph 3.1.

1.2 Cup product in singular cohomology

An interesting structure in cohmology that is not present is the *cup product.* It turns out there is a natural way of defining a product on cohomology, which carries some extra information that bare cohomology groups don't have: this allows one to distinguish spaces one couldn't from the homology or cohomology groups alone. An example of this will be given later on in the thesis, in section 2.5.

Let X be a topological space. Suppose A and B are abelian groups. Let $\phi \in C^k(X;A)$ and $\psi \in C^l(X;B)$, so ϕ is a function from the set of k-simplices in X to A and ψ is a function from the set of l-simplices in X to B. If $\sigma : \Delta^{k+l} :\to X$ is a $(k+l)$-simplex in X, its restrictions $\sigma|[v_0, \cdots, v_k]$ and $\sigma|[v_k, \cdots, v_{k+l}]$ are respectively a k and an l simplex in X. This allows us to define a map $\phi \smile \psi \in C^{k+l}(X; A \otimes B)$ in terms of ϕ and ψ:

$$(\phi \smile \psi)(\sigma) = \phi(\sigma|[v_0, \cdots, v_k]) \otimes \psi(\sigma|[v_k, \cdots, v_{k+l}]).$$

By properties of tensor products, this gives a bilinear map

$$\smile : C^k(X;A) \times C^l(X;B) \longrightarrow C^{k+l}(X; A \otimes B),$$

and by the universal property of the tensor product, this uniquely defines a map

$$\smile : C^k(X;A) \otimes C^l(X;B) \longrightarrow C^{k+l}(X; A \otimes B).$$

The two ways of considering this map, one as a bilinear map from the direct product and the other as a homomorphism from the tensor product, are equivalent. We will mostly use the second way, as it is more natural in our discussion.

The following identity is crucial and in a sense defining (to be made clear later on in the thesis) of the cup product.

Lemma 1.4. *We have* $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi$.

Proof. We write out the proof of the equivalent relation for the cup product in Čech cohomology in lemma 2.44. The reader may check that the two proofs are essentially identical. □

The equality $\delta(\phi \smile \psi) = \delta\phi \smile \psi + (-1)^k \phi \smile \delta\psi$ shows that if ϕ and ψ are cocycles (i.e. $\delta\phi = \delta\psi = 0$), then $\delta(\phi \smile \psi) = 0$, so $\phi \smile \psi$ is a cocycle. It is also clear that the product of a cocycle and a coboundary, in either order, is again a coboundary (since one of the two terms in the sum on the right hand side vanishes in that case). From this it follows that we obtain bilinear maps maps

$$\smile : \mathrm{H}^k(X;A) \otimes \mathrm{H}^l(X;B) \longrightarrow \mathrm{H}^{k+l}(X; A \otimes B)$$

called the cup product. If we take $A = B = R$ a ring, this makes the direct sum

$$\mathrm{H}^*(X;R) = \bigoplus_{k \in \mathbb{N}} \mathrm{H}^k(X;R)$$

into a graded ring, which we call the *cohomology ring* of X.

The following example is fundamental.

Theorem 1.5. *Let P^n be the n-dimensional real projective space. Then*

$$\mathrm{H}^*(P^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1})$$

with $\alpha \in \mathrm{H}^1(P^n; \mathbb{Z}/2\mathbb{Z})$.

Proof. We do the computation for $n = 2$ in section 2.5 using Čech cohomology. For a proof of the general case see theorem 3.12 in [1]. □

1.3 Proof of Borsuk-Ulam

As an application of cohomology rings, we give a natural proof of the theorem of Borsuk-Ulam using cohomology.

Theorem 1.6 (Borsuk-Ulam)**.** *Let $f : S^n \to S^n$ be an odd map, i.e. a map satisfying $f(-x) = -f(x)$. Then $\deg f$ is odd.*

We first develop some theory of covering spaces. Let X be a connected, locally path-connected and semilocally simply connected space, and choose a basepoint $x \in X$. Then X has a universal cover $u : (\tilde{X}, \tilde{x}) \to (X, x)$. In this case, there is a correspondence between the set of homomorphisms $\pi_1(X, x) \to G$ and the set of isomorphism classes of pointed G-coverings of X.

Theorem 1.7. *There is a bijective correspondence between the set of homomorphisms $\rho : \pi_1(X, x) \to G$ and the set of pointed G-coverings of X up to G-isomorphism:*

$$\mathrm{Hom}(\pi_1(X, x), G) \cong \{p : (Y, y) \to (X, x)\ G - covering\}/\sim$$

where $\sim$ indicates G-isomorphism.

In this correspondence, given a G-covering $p : (Y, y) \to (X, x)$, a homomorphism $\rho_p : \pi_1(X, x) \to G$ is constructed as follows: for $[\gamma] \in \pi_1(X, x)$ the element $\rho_p([\gamma]) \in G$ is determined by the action $\rho_p([\gamma])y = y * \gamma$, where the action on the right hand side is the monodromy action.

Suppose $p : (Y, y) \to (X, x)$ is a G-cover, and suppose we have a map $f : (Z, z) \to (X, x)$. Then the fibre product $(Y \times_X Z, (y, z))$, which fits in the the commutative diagram

$$\begin{array}{ccc} (Y \times_X Z, (y, z)) & \longrightarrow & (Y, y) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ (Z, z) & \xrightarrow{\ f\ } & (X, x) \end{array} \tag{3}$$

is a G-cover of (Z, z) via q.

Let $\rho_p : \pi_1(X, x) \to G$ denote the homomorphism corresponding to $p : (Y, y) \to (X, x)$, and similarly let $\rho_q : \pi_1(Z, z) \to G$ denote the homomorphism corresponding to $q : (Y \times_X Z, (y, z)) \to (Z, z)$.

Lemma 1.8. *We have $\rho_q = \rho_p \circ f_*$, where $f_* : \pi_1(Z, z) \to \pi_1(X, x)$ is the map induced by f.*

Proof. We have to show that for all $[\eta] \in \pi_1(Z, z)$ we have

$$(\rho_p \circ f_*)([\eta])(y, z) = (y, z) * \eta.$$

Let $\gamma = f \circ \eta$, then $\rho_p(\gamma)y = y * \gamma$, hence $(\rho_p \circ f_*)([\eta])(y, z) = (y * \gamma, z)$. But also $(y, z) * \eta = (y * \gamma, z)$ by commutativity of the diagram 3. Hence we are done. □

In this section we will denote by P^n the real projective space $\mathbb{R}P^n$ and set $G = \mathbb{Z}/2\mathbb{Z}$. We will use that S^n is a $\mathbb{Z}2/\mathbb{Z}$cover of P^n, where the action of $\mathbb{Z}/2\mathbb{Z}$ on S^n is given by $x \mapsto -x$. Let $p : S^n \to P^n$ be the corresponding quotient map.

Note that an odd map $f : S^n \to S^n$ induces a quotient map $\overline{f} : P^n \to P^n$ making the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\quad f \quad} & S^n \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ P^n & \xrightarrow{\quad \overline{f} \quad} & P^n \end{array} \tag{4}$$

commute. This diagram induces a commutative diagram of cohomology groups

$$\begin{array}{ccc} \mathrm{H}^k(S^n;\mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\quad \overline{f} \quad} & \mathrm{H}^k(S^n;\mathbb{Z}/2\mathbb{Z}) \\ {\scriptstyle p^*}\uparrow & & \uparrow{\scriptstyle p^*} \\ \mathrm{H}^k(P^n;\mathbb{Z}/2\mathbb{Z}) & \xleftarrow{\quad \overline{f}^* \quad} & \mathrm{H}^k(P^n;\mathbb{Z}/2\mathbb{Z}) \end{array} \tag{5}$$

with the arrows going in the other direction.

Proposition 1.9. *The map* $\overline{f}^* : \mathrm{H}^1(P^n;\mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1(P^n;\mathbb{Z}/2\mathbb{Z})$ *is an isomorphism.*

Proof. 1.9 Using theorem 1.7 we can interpret $\mathrm{H}^1(P^n;\mathbb{Z}/2\mathbb{Z})$ as the set of $\mathbb{Z}/2\mathbb{Z}$-coverings of P^n. Using lemma 1.8 we see that the map $\overline{f}^*$ sends the covering $S^n \to P^n$ to the fibre product $S^n \times_{P^n} P^n \to P^n$. By the universal property of the fibre product and diagram 4, we get a continuous map $S^n \to S^n \times_{P^n} P^n$. This map is bijective on fibers and by definition is a map of coverings, so it is an isomorphism. Hence $S^n \times_{P^n} P^n \to P^n$ cannot be the trivial covering $\mathbb{Z}/2\mathbb{Z} \times P^n \to P^n$, which shows $\overline{f}^*$ is non-trivial.

□

Proof. 1.6 Since $H^*(P^n;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1})$ with $\alpha \in \mathrm{H}^1(P^n;\mathbb{Z}/2\mathbb{Z})$ by 1.5. from proposition 1.9 we conclude that also

$$\mathrm{H}^n(P^n;\mathbb{Z}/2\mathbb{Z}) \xrightarrow{\overline{f}^*} \mathrm{H}^n(P^n;\mathbb{Z}/2\mathbb{Z})$$

is non-trivial. It is well known that this map is multiplication by $\deg f$ mod 2, from which we conclude the theorem. □

2 Sheaf Cohomology

2.1 Definitions

Another theory of cohomology is cohomology of sheaves on a topological space X, computed using injective resolution. The notions involved make sense for a class of categories called *abelian categories*, defined in [2] paragraph 4.2.1: they are categories with some additional structure, which for example allow us to talk about exact sequences. We will follow [2] in developing cohomology in this general setting. The category of sheaves on a topological space X has a natural structure of abelian category (as [2] shows in paragraph 4.1), so we can apply the results obtained in the more general case to define sheaf cohomology.

Definition 2.1. Let $\mathcal{C}$ be an abelian category. An object $\mathcal{I}$ in $\mathcal{C}$ is called *injective* if for every morphism $\phi : \mathcal{A} \to \mathcal{I}$ together with a monomorphism $i : \mathcal{A} \to \mathcal{B}$ there is an extension morphism $\psi : \mathcal{B} \to \mathcal{I}$, such that $\phi = \psi \circ i$.

Exercise 2.2. Show that $\mathcal{I}$ is injective if and only if the functor $\mathrm{Hom}(-, \mathcal{I}) : \mathcal{C} \to \underline{\mathbf{Ab}}$ is exact.

Definition 2.3. Let $\mathcal{C}$ be an abelian category and $\mathcal{A}$ be an object in $\mathcal{C}$. We say $j : \mathcal{A} \to \mathcal{M}^\bullet$ is a resolution if it is an exact complex in $\mathcal{C}$. A resolution $j : \mathcal{A} \to \mathcal{M}^\bullet$ is called *injective* if $\mathcal{M}^i$ is injective for all $i \geq 0$.

We say an abelian category $\mathcal{C}$ has *enough injectives* if every object $\mathcal{A}$ in $\mathcal{C}$ admits a monomorphism $j : \mathcal{A} \to \mathcal{I}^0$ into an injective object. If we have such an embedding for $\mathcal{A}$ and $\mathcal{C}$ has enough injectives, we can then embed the cokernel object $\mathrm{coker} j$ into another injective object $\mathcal{I}^1$. Continuing like this leads to the following proposition.

Proposition 2.4. *In an abelian category $\mathcal{C}$ with enough injectives, every object admits an injective resolution.*

Let $i : \mathcal{A} \to \mathcal{I}^\bullet$ be a resolution and let $j : \mathcal{B} \to \mathcal{J}^\bullet$ be an injective resolution. Suppose we are given a morphism $f : \mathcal{A} \to \mathcal{B}$. Then the defining property of injective objects allows us to extend this to a chain map from $\mathcal{A} \to \mathcal{I}^\bullet$ to $\mathcal{B} \to \mathcal{J}^\bullet$, and even in a unique way, as is made precise in the following theorem.

Theorem 2.5. *There exists a chain morphism $\phi : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ satisfying $j \circ f = f_0 \circ i$. Moreover, ϕ is unique up to chain homotopy.*

Proof. See [2] proposition 4.27. The chain map is constructed by consecutively applying the injective property of the $\mathcal{J}^i$. The same is true for the construction of the chain homotopy. □

If $\mathcal{I}$ is also injective, this leads to the following result.

Corollary 2.6. *Any two injective resolutions $\mathcal{A} \to \mathcal{I}^\bullet$ and $\mathcal{A} \to \mathcal{J}^\bullet$ are homotopy equivalent.*

Proof. Theorem 2.5 gives chain morphisms $\phi : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ and $\psi : \mathcal{J}^\bullet \to \mathcal{I}^\bullet$ extending the identity, and moreover by theorem 2.5 $\psi \circ \phi$ and $\phi \circ \psi$ must be homotopic to the identity on $\mathcal{I}^\bullet$, respectively $\mathcal{J}^\bullet$. □

Now let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories, and $F : \mathcal{C} \to \mathcal{C}'$ a left-exact functor. If $\mathcal{C}$ has enough injectives, then corollary 2.6 allows us to define the *right derived functors* with respect to F. In the case that $\mathcal{C}$ is the category of sheaves on a space X, $\mathcal{C}'$ the category of abelian groups, and F the global sections functor Γ, the right derived functors will be the cohomology groups we sought to define.

Definition 2.7. Let $\mathcal{A}$ be an object of $\mathcal{C}$, and let $\mathcal{A} \to \mathcal{I}^\bullet$ be an injective resolution. Define the i'th right derived functor $R^iF(\mathcal{A})$ with respect to F of $\mathcal{A}$ to be $R^iF(\mathcal{A}) = H^i(F(\mathcal{I}^\bullet))$.

This definition does not depend on the chosen injective resolution, as any two resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ are chain homotopy equivalent by 2.6, an thus so are the complexes $F(\mathcal{I}^\bullet)$ and $F(\mathcal{J}^\bullet)$.

Some important properties of the right derived functors are the following.

Theorem 2.8. *For objects $\mathcal{F}, \mathcal{G}, \mathcal{H}$ of $\mathcal{C}$, we have the following:*

- $R^0F(\mathcal{F}) = F(\mathcal{F})$
- *If*
$$0 \to \mathcal{F} \xrightarrow{\phi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$$
is a short exact sequence in $\mathcal{C}$, we can construct a natural long exact sequence
$$0 \to F(\mathcal{F}) \xrightarrow{\phi} F(\mathcal{G}) \xrightarrow{\psi} F(\mathcal{H}) \xrightarrow{\partial} R^1F(\mathcal{F}) \xrightarrow{\phi_*} R^1F(\mathcal{G}) \xrightarrow{\psi_*} R^1F(\mathcal{H}) \xrightarrow{\partial} \cdots$$
in $\mathcal{C}'$.
- *If $\mathcal{I}$ is injective, then $R^iF(\mathcal{I}) = 0$ for $i > 0$.*

Moreover, these three properties define the objects $R^i(F(-))$ up to canonical isomorphism.

Proof. See [2] theorem 4.28. ☐

The right derived functors can actually be calculated using acyclic resolutions, which are often easier to find than injective resolutions.

Definition 2.9. An object $\mathcal{M}$ in $\mathcal{C}$ is called F-acyclic if $R^iF(\mathcal{M}) = 0$ for all $i > 0$.

Let $\mathcal{A} \to \mathcal{M}^\bullet$ be a F-acyclic resolution of $\mathcal{A}$ (i.e. the $\mathcal{M}^i$ are F-acyclic). Proposition 2.5 gives us a chain map
$$i : \mathcal{M}^\bullet \to \mathcal{I}^\bullet$$
to an injective resolution of $\mathcal{A}$ induced by the identity, unique up to homotopy.

Proposition 2.10. *The chain map i induces an isomorphism $i_* : H^i(F(\mathcal{M}^\bullet)) \xrightarrow{\sim} R^iF(\mathcal{A})$.*

Most authors give a natural isomorphism given by a boundary map ∂ in the long exact sequence of 2.8, but they don't verify that the canonical chain map also induces an isomorphism. We verify it here.

Proof. We first follow the proof of proposition 4.32 in [2] and look at the short exact sequences

$$0 \to \mathcal{A} \to \mathcal{M}^0 \to \mathcal{B} \to 0$$

and

$$0 \to \mathcal{A} \to \mathcal{I}^0 \to \mathcal{C} \to 0,$$

where $\mathcal{B} = \operatorname{coker}(\mathcal{A} \to \mathcal{M}^0)$ and $\mathcal{C} = \operatorname{coker}(\mathcal{A} \to \mathcal{I}^0)$. Then the objects $\mathcal{B}$ and $\mathcal{C}$ admit shifted resolutions

$$0 \to \mathcal{B} \xrightarrow{d^0} \mathcal{M}^1 \to \cdots$$

and

$$0 \to \mathcal{C} \xrightarrow{d^0} \mathcal{I}^1 \to \cdots ,$$

Since F is left exact we have

$$F(\mathcal{B}) = \ker(\mathcal{M}^1 \to \mathcal{M}^2)$$

and

$$F(\mathcal{C}) = \ker(\mathcal{I}^1 \to \mathcal{I}^2).$$

This means that the long exact sequences associated to the short exact sequences give commutative diagrams

$$\begin{array}{ccccc} \mathrm{H}^1(F(\mathcal{M}^\bullet)) & \xrightarrow{=} & \operatorname{coker}(F(\mathcal{M}^0) \to F(\mathcal{B})) & \xrightarrow{\partial} & R^1F(\mathcal{A}) \\ & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{H}^1(F(\mathcal{I}^\bullet)) & \xrightarrow{=} & \operatorname{coker}(F(\mathcal{I}^0) \to F(\mathcal{C})) & \xrightarrow{\partial} & R^1F(\mathcal{A}) \end{array} \tag{6}$$

where ∂ induces an isomorphism because $\mathcal{M}^0$ and $\mathcal{I}^0$ are acyclic. We get a natural isomorphism

$$i_* : \mathrm{H}^1 F(\mathcal{M}^\bullet) \xrightarrow{\sim} R^1F(\mathcal{A}).$$

Also note i_* induces an isomorphism $i_*\colon R^1F(\mathcal{B}) \xrightarrow{\sim} R^1F(\mathcal{C})$, which will be necessary later to use induction. For $i \geq 2$ consider the short exact sequences

$$0 \to \operatorname{coker}\, d^{i-2} \xrightarrow{d^{i-1}} \mathcal{M}^i \xrightarrow{d^i} \operatorname{coker}\, d^i \to 0$$

and

$$0 \to \operatorname{coker}\, d^{i-2} \xrightarrow{d^{i-1}} \mathcal{I}^i \xrightarrow{d^i} \operatorname{coker}\, d^i \to 0.$$

Again, the two objects coker d^i admit shifted resolutions

$$0 \to \text{coker } d^i \xrightarrow{d^{i+1}} \mathcal{M}^{i+1} \to \cdots$$

$$0 \to \text{coker } d^i \xrightarrow{d^{i+1}} \mathcal{I}^{i+1} \to \cdots$$

so by left exactness of F we get commutative diagrams

$$\begin{array}{ccccc} \mathrm{H}^i(F(\mathcal{M}^\bullet)) & \xrightarrow{=} & \text{coker}(F(\mathcal{M}^i) \to F(\text{coker } d^i)) & \xrightarrow{\partial} & R^1F(\text{coker } d^{i-2}) \\ & & \downarrow i_* & & \downarrow i_* \\ \mathrm{H}^i(F(\mathcal{I}^\bullet)) & \xrightarrow{=} & \text{coker}(F(\mathcal{I}^i) \to F(\text{coker } d^i)) & \xrightarrow{\partial} & R^1F(\text{coker } d^{i-2}) \end{array} \tag{7}$$

The map i_* on the right is an isomorphism by induction and the maps ∂ are isomorphisms because $\mathcal{M}^i$ and $\mathcal{I}^i$ are acyclic. We conclude i_* induces a natural isomorphism

$$i_* \colon \mathrm{H}^i F(\mathcal{M}^\bullet) \xrightarrow{\sim} R^i F(\mathcal{A})$$

. □

We want to apply the developed machinery to the category of sheaves on a space X. This means that we have to show this category has enough injectives, i.e. for every sheaf $\mathcal{F}$ we have to construct an embedding $j : \mathcal{F} \to \mathcal{I}$ into an injective sheaf.

Lemma 2.11. *Let $\mathcal{F}$ be a sheaf on X. Define $\mathcal{F}_{God}$ by $\mathcal{F}_{God}(U) = \prod_{x\in U} \mathcal{F}_x$, with the obvious restriction morphisms. Then $\mathcal{F}_{God}$ is a sheaf on X and the canonical morphism given by $\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$ is injective.*

Proof. It is obvious that $\mathcal{F}_{\text{God}}$ is a sheaf. Suppose $\sigma \in \mathcal{F}(U)$ maps to $0 \in \prod_{x\in U} \mathcal{F}_x$. Then there exists an open cover $(U_x)_{x\in U}$, where $x \in U_x$, such that $\sigma|_{U_x} = 0$. By the unicity axiom of sheaves, $\sigma = 0$. So the map $\mathcal{F}(U) \to \prod_{x\in U} \mathcal{F}_x$ is injective for all $U \subset X$ open, so the morphism $\mathcal{F} \to \mathcal{F}_{\text{God}}$ is injective. □

It is well known that **<u>Ab</u>** has enough injectives. If we define $\mathcal{I}(U) = \prod_{x\in U} I_x$ where I_x is some injective group containing $\mathcal{F}_x$, then using the above lemma we get an injection $\mathcal{F} \to \mathcal{I}$.

Lemma 2.12. *The sheaf $\mathcal{I}$ is injective in the category of sheaves.*

Proof. We clarify the proof in [9], chapter 3 proposition 2.2. Suppose we have a morphism $\phi : \mathcal{K} \to \mathcal{I}$. For $x \in X$ and U an open containing x this defines maps

$$\phi_{U,x} : \mathcal{K}(U) \to I_x$$

after projecting, from which we obtain maps $\phi^x : \mathcal{K}_x \to I_x$ (note, these are not the maps on stalks induced by ϕ: the stalk $\mathcal{I}_x$ is not equal to I_x). To see this, suppose $f \in \mathcal{K}_x$ is represented by $\sigma, \tau \in \mathcal{K}(U)$. Then σ and τ agree when restricted to some open $V \subset U$, so $\phi_V(\sigma) = \phi_V(\tau)$, from which it follows that $\phi_{U,x}(\sigma) = \phi_{U,x}(\tau)$. Then for all opens $U \subset X$ we get commutative diagrams

$$\begin{array}{ccc} \mathcal{K}(U) & \longrightarrow & \prod_{x \in U} I_x \\ \downarrow & \nearrow & \\ \prod_{x \in U} \mathcal{K}_x & & \end{array} \tag{8}$$

Since $\mathcal{K}(U)$ injects into $\prod_{x \in U} \mathcal{K}_x$, the maps $\phi^x : \mathcal{K}_x \to \mathcal{I}_x$ completely determine the morphism $\mathcal{K} \to \mathcal{I}$. On the other hand, providing maps $\mathcal{K}_x \to I_x$ for all $x \in X$ clearly defines a morphism of sheaves $\mathcal{K} \to \mathcal{I}$ in view of the above diagram. We conclude that the functor $\mathrm{Hom}_{\underline{\mathbf{Sh_X}}}(-, \mathcal{I})$ is the composition of the direct product over all $x \in X$ of the stalk functor, with the functor $\prod_{x \in X} \mathrm{Hom}_{\underline{\mathbf{Ab}}}(-, I_x)$, which are both exact functors. So $\mathrm{Hom}_{\underline{\mathbf{Sh_X}}}(-, \mathcal{I})$ is exact and we conclude that $\mathcal{I}$ is injective by 2.2.

□

Hence we have shown that every sheaf $\mathcal{F}$ admits an embedding $\mathcal{F} \to \mathcal{I}$ into an injective sheaf $\mathcal{I}$. By 2.4, every sheaf admits an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. After all this work, we can now finally define the cohomology groups of $\mathcal{F}$.

Definition 2.13. Let $\mathcal{F}$ be a sheaf on X. Define $H^i(X, \mathcal{F}) = R^i(\Gamma(\mathcal{F})) = H^i(\Gamma(\mathcal{I}^\bullet))$, where $\mathcal{I}^\bullet$ is any injective resolution of $\mathcal{F}$ and Γ is the global sections functor.

A useful class of Γ-acyclic sheaves are the so called *flasque sheaves.*

Definition 2.14. A sheaf $\mathcal{F}$ is called *flasque* if the restriction morphisms $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ are surjective.

Proposition 2.15. *Flasque sheaves are* Γ*-acyclic.*

Proof. See [2], proposition 4.34. □

By proposition 2.10, flasque resolutions can be used to calculate the cohomology groups of a sheaf. The sheaf $\mathcal{F}_{\mathrm{God}}$ of lemma 2.11 is an example of a flasque sheaf. These particular kinds of flasque sheaves were first discovered by Godement. He used them to construct flasque resolutions of sheaves, as we will do later, and defined sheaf cohomology in terms of those resolutions. The above proposition shows this yields the same cohomology as when one works with injective resolutions.

2.2 Cup product in sheaf cohomology

In this paragraph we use the ideas in [8], where the cup product is developed in a very general setting. The notions of *pure monomorphisms* and *pure injectives* (objects that have the injective property with respect to pure monomorphisms) are central. We will investigate these notions in the category of sheaves: after taking a detour through abelian groups, we will develop criteria for morphisms of sheaves to be pure, and give an explicit pure embedding of a sheaf $\mathcal{F}$ into a pure injective $\mathcal{I}$. After that we will define the cup product for sheaves, using an approach that is a bit more general than in [8], in that we also consider pure acyclic resolutions; this might be useful if one wants to compute concrete cohomology rings. In this section $\mathcal{C}_0$ denotes either the category of sheaves on a space X or the category of abelian groups, which are both abelian categories that have a tensor product available with all the usual properties.

Definition 2.16. A double complex $K^{\bullet,\bullet}$ in $\mathcal{C}_0$ is a collection of objects $K^{p,q}$, $p, q \geq 0$, together with boundary morphisms $D_1 : K^{p,q} \to K^{p+1,q}$ and $D_2 : K^{p,q} \to K^{p,q+1}$ satisfying $D_1^2 = 0 = D_2^2$ and $D_1 \circ D_2 = D_2 \circ D_1$. The total complex $K^{\bullet}_{\mathrm{Tot}}$ associated to $\mathcal{K}^{\bullet,\bullet}$ is the collection of objects $K^n = \bigoplus_{p+q=n} K^{p,q}$, with boundary morphisms defined on $K^{p,q} \to K^{p+1,q} \oplus K^{p,q+1}$ by $D = D_1 + (-1)^p D_2$.

We first prove the following technical lemma.

Lemma 2.17. *The total complex $K^{\bullet}_{\mathrm{Tot}}$ associated to a double complex $K^{\bullet,\bullet}$ is in fact a complex. If the rows and columns of $K^{\bullet,\bullet}$ are exact, then $K^{\bullet}$ is an exact complex.*

Proof. We prove this for modules then from Mitchell's embedding theorem it follows for $\mathcal{C}$. We consider $K^n \xrightarrow{D} K^{n+1} \xrightarrow{D'} K^{n+2}$. On $K^{p,q}$, $p+q=n$, the boundary map is $D_1 + (-1)^p D_2$, so $a \in K^{p,q}$ is mapped to $D_1 a + (-1)^p D_2 a \in K^{p+1,q} \oplus K^{p,q+1}$. Applying D' yields

$$D_1^2 a + (-1)^{2p} D_2^2 a + (-1)^{p+1} D_2 D_1 a + (-1)^p D_1 D_2 a = 0$$

since $D_1^2 = D_2^2 = 0$ and $D_1 D_2 = D_2 D_1$.

We show the other inclusion if the rows and columns of $K^{\bullet,\bullet}$ are exact. Suppose an element $(a_{p,q}) \in \bigoplus_{p+q=n} K^{p,q}$ maps to 0 under D. Then we have the equations

$$D_1 a_{n,0} = 0,\ D_2 a_{p,q} = (-1)^p D_1 a_{p-1,q+1},\ D_2 a_{0,n} = 0,$$

where $p+q=n,\ 0 < p,q < n$.

We see directly that $a_{n,0} = D_1 b_{n-1,0}$ for some $b_{n,0} \in K^{n-1,0}$. It follows that $D_2 D_1 b_{n,0} = D_1 D_2 b_{n,0} = (-1)^{n-1} D_1 a_{n-1,1}$, from which it follows that $a_{n-1,1} + (-1)^n D_2 b_{n-1,1} \in \ker D_1 = \operatorname{im} D_1$. Hence $a_{n-1,1} = D_1 b_{n-1,1} + (-1)^{n-1} D_2 b_{n-1,1}$ with $b_{n-1,1} \in K^{n-2,1}$. Hence $a_{n-1,1} \in \operatorname{im} D$.

We do induction on q. Suppose by induction $a_{p,q} = D_1 b_{p-1,q} + (-1)^p D_2 b_{p,q+1}$. Then $D_1 D_2 b_{p-1,q} = D_2 D_1 b_{p-1,q} = D_2 a_{p,q} = (-1)^{p-1} D_1 a_{p-1,q+1}$, hence $D_2 b_{p-1,q} + (-1)^p a_{p-1,q+1} \in \ker D_1 = \operatorname{im} D_1$. Hence $a_{p-1,q+1} = D_1 b_{p-2,q+1} + (-1)^{p-1} D_2 b_{p-1,q}$, and $a_{p-1,q+1} \in \operatorname{im} D$.

We conclude by induction that $(a_{p,q}) \in \operatorname{im} D$. □

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X, and let $\mathcal{F} \to \mathcal{I}^{\bullet}$, $\mathcal{G} \to \mathcal{J}^{\bullet}$ be resolutions with boundary maps respectively D_1 and D_2. Then the sheaves $\mathcal{I}^p \otimes J^q$ with boundary maps $D_1 \otimes \mathrm{id}$ and $\mathrm{id} \otimes D_2$ form a double complex $(\mathcal{I} \otimes \mathcal{J})^{\bullet,\bullet}$, and we can form the total complex $(\mathcal{I}\otimes\mathcal{J})^\bullet_{\mathrm{Tot}}$. To induce the desired cup product, we would like a chain map from $\mathcal{F}\otimes\mathcal{G}\to(\mathcal{I}\otimes\mathcal{J})^\bullet_{\mathrm{Tot}}$ to an injective resolution $\mathcal{F}\otimes\mathcal{G}\to\mathcal{K}^\bullet$, but there is a problem: although $\mathcal{F}\otimes\mathcal{G}\to(\mathcal{I}\otimes\mathcal{J})^\bullet_{\mathrm{Tot}}$ forms a complex, it is not a resolution, since the rows and columns in the double complex $(\mathcal{I}\otimes\mathcal{J})^{\bullet,\bullet}$ are not exact. To overcome this difficulty, we have to work with so called *pure resolutions.*

Definition 2.18. In $\mathcal{C}_0$ an exact sequence

$$\mathcal{A}\xrightarrow{f}\mathcal{B}\xrightarrow{g}\mathcal{C}$$

is called *pure* if for every object $\mathcal{D}$ the induced sequence

$$\mathcal{A}\otimes\mathcal{D}\xrightarrow{f\otimes\mathrm{id}}\mathcal{B}\otimes\mathcal{D}\xrightarrow{g\otimes\mathrm{id}}\mathcal{C}\otimes\mathcal{D}$$

is exact.

A resolution $\mathcal{A}\to\mathcal{M}^\bullet$ is called pure if $\mathcal{A}\to\mathcal{M}^0$ is a pure monomorphism (i.e. $0\to\mathcal{A}\to\mathcal{M}^0$ is pure exact) and if $\mathcal{M}^\bullet$ is pure exact at every object.

Since the functor $-\otimes\mathcal{D}$ is right exact, we will be mostly interested in understanding pure monomorphisms. A big class of pure monomorphisms is formed by inclusions of a direct summand into a direct sum.

Lemma 2.19. *Embeddings $\mathcal{A}\to\mathcal{A}\oplus\mathcal{B}$ of the form* $(\mathrm{id},0)$ *into a direct sum are pure.*

Proof. Tensoring with $\mathcal{C}$ yields

$$(\mathcal{A}\otimes\mathcal{C})\xrightarrow{(\mathrm{id},0)}(\mathcal{A}\otimes\mathcal{C})\oplus(\mathcal{C}\otimes\mathcal{B})$$

since the tensor product distributes over direct sums. This is clearly a monomorphism.

□

In the category of abelian groups, injective objects are precisely the divisible groups. So if we have a monomorphism $\mathbb{Z}\to I$ into an injective I, tensoring with $\mathbb{Q}/\mathbb{Z}$ we get $\mathbb{Q}/\mathbb{Z}\to 0$, as $I\otimes\mathbb{Q}/\mathbb{Z}=0$. To see this, take a pure tensor $a\otimes b\in I\otimes\mathbb{Q}/\mathbb{Z}$, and note that there exists an $n\in\mathbb{Z}$ such that $nb=0$. If we let x be such that $nx=a$, then we have

$$a\otimes b=nx\otimes b=x\otimes nb=x\otimes 0=0.$$

So it is impossible to get pure resolutions of injective objects in the category of abelian groups, and hence we can also not expect to get them in the category of sheaves over X. For this reason, we will work with pure acyclic and pure injective resolutions.

Definition 2.20. An object $\mathcal{I}$ is called *pure injective* if for every map $\phi:\mathcal{A}\to\mathcal{I}$ and every pure monomorphism $i:\mathcal{A}\to\mathcal{B}$ there is an extension morphism $\psi:\mathcal{B}\to\mathcal{I}$ such that $\phi=\psi\circ i$.

We say $\mathcal{C}_0$ with $\otimes$ has enough pure injectives if every object $\mathcal{A}$ can be embedded purely in a pure injective $\mathcal{J}$.

Precisely in the same way as for injective resolutions, we can prove the following results.

Proposition 2.21. *If $\mathcal{C}_0$ has enough pure injectives, every object $\mathcal{A}$ admits a pure injective resolution $\mathcal{A} \to \mathcal{I}^\bullet$, understood as a pure resolution with the I^i pure injective objects.*

Theorem 2.22. *Suppose $\phi : \mathcal{A} \to \mathcal{B}$ is a morphism in $\mathcal{C}$, and suppose $i : \mathcal{A} \to \mathcal{I}^\bullet$ and $j : \mathcal{B} \to \mathcal{J}^\bullet$ are pure resolutions. If $\mathcal{J}^\bullet$ is pure injective, there exists a chain morphism $\phi : \mathcal{I}^\bullet \to \mathcal{J}^\bullet$ satisfying $j \circ \phi = \phi_0 \circ i$. Moreover, ϕ is unique up to chain homotopy.*

So in the same way as for injective resolutions we see that if $\mathcal{C}_0$ has enough pure injectives, any object $\mathcal{A}$ admits a pure injective resolution unique up to chain homotopy equivalence.

Our goal now is to show the category of sheaves on a space X has enough pure injectives. For this, we first turn to the category of abelian groups. Pure subgroups have apparently been studied extensively in the literature: it is a useful concept, intermediate between subgroup and direct summand. They were first studied by Prüfer, a famous group theorist who worked early in the twentieth century. For more information, see the introduction to chapter 5 in [10].

There is a slight complication in that [10] and other authors in the field of infinite abelian groups use a different definition. They say a subgroup $H \subset G$ is pure if the following holds: if the equation $nx = y$ with $y \in H$ has a solution in G, then it has a solution in H. In the following proposition, we show this is equivalent to our definition. We need a lemma from [11], which we specify to the situation of abelian groups.

Lemma 2.23. *Let M, N be modules over a commutative ring R, and suppose $\sum_{i=1}^n x_i \otimes y_i = 0$ in $M \otimes N$ with $x_i \in M$, $y_i \in N$. Then there are finitely generated submodules $M_0 \subset M$ and $N_0 \subset N$ such that $\sum_{i=1}^n x_i \otimes y_i = 0$ in $M_0 \otimes N_0$.*

Proof. See [11] corollary 2.13. □

In the following proof, the implication 2. ⇒ 1. uses ideas from the proof of proposition 2.19, iv) ⇒ iii), in [11].

Proposition 2.24. *The following are equivalent for an inclusion $H \xrightarrow{f} G$ in* **<u>Ab</u>**.

1. *For every abelian group A, the map $H \otimes A \xrightarrow{f \otimes \mathrm{id}} G \otimes A$ is injective.*

2. *For every finitely generated abelian group A, the map $H \otimes A \xrightarrow{f \otimes \mathrm{id}} G \otimes A$ is injective.*

3. *For every finitely generated abelian torsion group A, the map $H \otimes A \xrightarrow{f \otimes \mathrm{id}} G \otimes A$ is injective.*

4. *For every n, the map $H \otimes \mathbb{Z}/n\mathbb{Z} \xrightarrow{f \otimes \mathrm{id}} G \otimes \mathbb{Z}/n\mathbb{Z}$ is injective.*

5. *If the equation $nx = y$ with $y \in H$ has a solution in G, then it has a solution in H.*

Proof. The implications 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. are trivial. Statement 4. says that the induced map $H/nH \to G/nG$ is injective, i.e if $y \in H$ maps to nG, then it is contained in nH, which is precisely 5. We conclude 4. $\iff$ 5.

4. $\Rightarrow$ 3. A finitely generated torsion group is of the form

$$A = \bigoplus_{i=1}^{k} \mathbb{Z}/n_i\mathbb{Z}$$

by the structure theorem of finitely generated abelian groups, so the map $H \otimes A \xrightarrow{f\otimes\mathrm{id}} G \otimes A$ becomes

$$\bigoplus_{i=1}^{k} H/n_iH \to \bigoplus_{i=1}^{k} G/n_iG$$

with the induced map on the components. The component maps are injective by 4. hence it is injective.

3. $\Rightarrow$ 2. We can write H as

$$A = \mathbb{Z}^r \oplus A_{\mathrm{tor}}$$

with A_{tor} the torsion part of A by the structure theorem of finitely generated abelian groups. Then the map $H \otimes A \xrightarrow{f\otimes\mathrm{id}} G \otimes A$ becomes

$$H^r \oplus (H \otimes A_{\mathrm{tor}}) \to G^r \oplus (G \otimes A_{\mathrm{tor}}).$$

The map $H^r \to G^r$ is clearly injective and the map $H \otimes A_{\mathrm{tor}} \to G \otimes A_{\mathrm{tor}}$ is injective by 3.

2. $\Rightarrow$ 1. Suppose $\sum_{i=1}^{k} g_i \otimes a_i \in \ker f \otimes \mathrm{id}$, i.e. $\sum_{i=1}^{k} f(g_i) \otimes a_i = 0$. Let A' be the group generated by the a_i. Then by lemma 2.23 there exists a finitely generated subgroup $A_0 \subset A$ containing A' such that $\sum_{i=1}^{k} f(g_i) \otimes a_i = 0$ considered as an element of $G \otimes A_0$. Hence $\sum_{i=1}^{k} g_i \otimes a_i$ considered as an element of $H \otimes A_0$ is in the kernel of $H \otimes A_0 \xrightarrow{f\otimes\mathrm{id}} G \otimes A_0$. By 2. it follows that $\sum_{i=1}^{k} g_i \otimes a_i = 0$ in $H \otimes A_0$, so it is also equal to zero in $H \otimes A$. □

Now that we know we can use phrasing 5. from the previous proposition as a definition, we can use the results from the literature. To understand them a little, we will explain the concept of *cocyclic group*. We follow [10], section 1.3.

A cyclic group A can be characterised by considering morphisms into it: A is cyclic if there exists an element $a \in A$ such that a morphism $\phi : B \to A$ is surjective if and only if $a \in \mathrm{im}\ \phi$ (take a any generator of A). We define cocyclic groups by the dual concept: a group A is called cocyclic if there exists an element $a \in A$ such that a morphism $\phi : A \to B$ is injective if and only if $a \notin \ker \phi$. This element a is sometimes called a *cogenerator*. As all subgroups in A can be obtained as kernels of quotient maps, this means that a is contained in all nonzero subgroups of A, which in turn implies A has a smallest nonzero subgroup. If on the other hand A has a smallest nonzero subgroup, any element in this smallest subgroup is a cogenerator.

Example 2.25. Let p be a prime, and consider $\mathbb{Z}/p^k\mathbb{Z}$. The only nonzero subgroups are of the form $\langle p^j \rangle \cong \mathbb{Z}/p^{k-j}\mathbb{Z}$ where $0 \le j \le k-1$, and they form a chain:

$$\langle p^{k-1} \rangle \subset \langle p^{k-2} \rangle \subset \cdots \subset \mathbb{Z}/p^k\mathbb{Z}$$

It follows that cyclic groups of prime power order are examples of cocyclic groups. It is also immediately clear that cyclic groups of order mn where $m, n > 1$ and $\gcd(m, n) = 1$ are not cocyclic, because the Chinese remainder theorem yields an isomorphism

$$\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z},$$

so the subgroups $\mathbb{Z}/m\mathbb{Z} \oplus 0$ and $0 \oplus \mathbb{Z}/n\mathbb{Z}$ have trivial intersection.

Example 2.26. Let p be a prime. Consider the so called *Prüfer group* denoted by $\mathbb{Z}(p^\infty)$. It can be identified with the group of the p^n-th roots of unity in $\mathbb{C}$, where n varies over $\mathbb{N}$, or equivalently with $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, i. e. the subgroup of of fractions in $\mathbb{Q}/\mathbb{Z}$ with denominator a power of p. Within $\mathbb{Z}(p^\infty)$ the p^k-th roots of unity, where k is now fixed, form a subgroup of order p^k isomorphic to $\mathbb{Z}/p^k\mathbb{Z}$. These are easily seen to be the only proper subgroups, so the complete list of nonzero proper subgroups form a chain

$$\left\langle \frac{1}{p} \right\rangle \subset \left\langle \frac{1}{p^2} \right\rangle \subset \cdots$$

It follows that $\mathbb{Z}(p^\infty)$ is another example of a cocyclic group.

It turns out these are the only examples of cocyclic groups.

Theorem 2.27. *If A is a cocyclic group, then $A \cong \mathbb{Z}/p^k\mathbb{Z}$ or $A \cong \mathbb{Z}(p^\infty)$.*

Proof. If a is a cogenerator, then $\langle a \rangle$ cannot be infinite and equal to $\mathbb{Z}$, since $\mathbb{Z}$ has many proper subgroups. Hence it has to be finite and have prime order, otherwise it also has proper subgroups. The rest of the proof consists of showing that A can have at most one subgroup of order p^n, and that this group must be cyclic, from which the claim follows. See [10] theorem 3.1. □

In [10], pure injectives are classified in terms of cocyclic groups. The proof uses a quite a bit of theory of infinite abelian groups, so we won't include it here.

Theorem 2.28. *An abelian group is pure injective if and only if it is a direct summand of a direct product of cocyclic groups.*

Proof. See [10] theorem 30.4. □

Theorem 2.29. *Every abelian group can be purely embedded in a direct product of cocyclic groups.*

Proof. See [10] lemma 30.3. □

Combining the last two theorems implies that $\underline{\mathbf{Ab}}$ has enough pure injectives.

Theorem 2.30. *The category of abelian groups $\underline{\mathbf{Ab}}$ has enough pure injectives.*

Now we can turn to the category of sheaves on a space X. Since the question of whether a morphism of sheaves is a monomorphism can be decided on stalks, purity of a monomorphism can also be decided on stalks. Also purity can be decided by looking at the corresponding morphisms on sections. To show this, we first need the following lemma.

Lemma 2.31. *We have $(\mathcal{F} \otimes \mathcal{G})_x \cong \mathcal{F}_x \otimes \mathcal{G}_x$ for all $x \in X$.*

Proof. Since sheafification preserves stalks, we should compute the stalks of the presheaf with sections $\mathcal{F}(U) \otimes \mathcal{G}(U)$. But since the functor $\mathcal{F} \otimes -$ is a left adjoint and thus commutes with direct limits, it follows that this presheaf has stalks $\mathcal{F}_x \otimes \mathcal{G}_x$. □

Lemma 2.32. *A monomorphism of sheaves $\mathcal{F} \to \mathcal{G}$ is pure if and only if the monomorphism $\mathcal{F}_x \to \mathcal{G}_x$ is pure in $\underline{\mathbf{Ab}}$ for all $x \in X$, if and only if the monomorphism $\mathcal{F}(U) \to \mathcal{G}(U)$ is pure in $\underline{\mathbf{Ab}}$ for all $U \subset X$.*

Proof. We prove the first equivalence; the proof of the second is similar.

Suppose $\mathcal{F} \to \mathcal{G}$ is pure. Let G be an arbitrary abelian group. Then

$$\mathcal{F} \otimes \underline{G}_X \to \mathcal{G} \otimes \underline{G}_X$$

is a monomorphism, with $\underline{G}_X$ the constant sheaf with values in G. Taking stalks at x and applying 2.31 we see that $\mathcal{F}_x \otimes G \to \mathcal{G}_x \otimes G$ is injective, so $\mathcal{F}_x \to \mathcal{G}_x$ is pure.

Suppose $\mathcal{F}_x \to \mathcal{G}_x$ is pure for all x. Let $\mathcal{H}$ be a sheaf and consider

$$\mathcal{F} \otimes \mathcal{H} \to \mathcal{G} \otimes \mathcal{H}.$$

Taking stalks at x and applying 2.31 we get $\mathcal{F}_x \otimes \mathcal{H}_x \to \mathcal{G}_x \otimes \mathcal{H}_x$, which is injective by purity of $\mathcal{F}_x \to \mathcal{G}_x$. Hence $\mathcal{F} \otimes \mathcal{H} \to \mathcal{G} \otimes \mathcal{H}$ is a monomorphism and $\mathcal{F} \to \mathcal{G}$ is pure. □

With these tools in hand, we can start constructing a pure embedding into a pure injective sheaf for every sheaf $\mathcal{F}$. We use the same strategy as before when constructing an embedding into an injective sheaf: we embed $\mathcal{F}$ into $\mathcal{F}_{\text{God}}$ and then embed $\mathcal{F}_{\text{God}}$ into the sheaf $\mathcal{I}$ given by

$$\mathcal{I}(U) = \prod_{x \in U} \mathcal{I}_x$$

where $\mathcal{I}_x$ is a pure injective group purely containing $\mathcal{F}_x$. By the same argument as in 2.12, $\mathcal{I}$ is pure injective.

Proposition 2.33. *The canonical embedding $i : \mathcal{F} \to \mathcal{F}_{God}$ is pure.*

Proof. We show $\mathcal{F} \to \mathcal{F}_{\text{God}}$ induces a map of the form 2.19 into a direct summand on stalks, from which the result follows by 2.19 and 2.32.

Let $x \in X$ and consider the injection $i_x : \mathcal{F}_x \to (\mathcal{F}_{\text{God}})_x$. Let $\pi : \prod_{x \in X} \mathcal{F}_x \to \mathcal{F}_x$ be the projection map onto $\mathcal{F}_x$. Since $\mathcal{F}_{\text{God}}$ is flasque, the map $\mathcal{F}_{\text{God}}(X) \to (\mathcal{F}_{\text{God}})_x$ is surjective for all $x \in X$. To see this, note that every $\sigma_x \in (\mathcal{F}_{\text{God}})_x$

is represented by some $\sigma \in \mathcal{F}_{\mathrm{God}}(U)$ with $x \in U$, and the map $\mathcal{F}_{\mathrm{God}}(X) \to (\mathcal{F}_{\mathrm{God}})_x$ factorises as $\mathcal{F}_{\mathrm{God}}(X) \to \mathcal{F}_{\mathrm{God}}(U) \to (\mathcal{F}_{\mathrm{God}})_x$. So the diagram

$$\begin{array}{ccc} \mathcal{F}_{\mathrm{God}}(X) & \xrightarrow{=} & \prod_{x\in X} \mathcal{F}_x \\ \big\downarrow & & \big\downarrow \pi \\ (\mathcal{F}_{\mathrm{God}})_x & \xrightarrow{q} & \mathcal{F}_x \end{array} \tag{9}$$

defines a surjection $q : (\mathcal{F}_{\mathrm{God}})_x \to \mathcal{F}_x$. It is well defined, since if two sections σ and τ in $\mathcal{F}_{\mathrm{God}}(X)$ represent $f \in (\mathcal{F}_{\mathrm{God}})_x$, then they agree on some open neighbourhood of x and we must have $\pi(\sigma) = \pi(\tau)$. It is surjective because π is. Also, if $\sigma \in \mathcal{F}(U)$ represents $\sigma_x \in \mathcal{F}_x$, then $\pi \circ i(\sigma) = \sigma_x$, so $q \circ i_x(\sigma_x) = \sigma_x$. So q defines a retraction and i_x is split.

□

Definition 2.34. Let $\mathcal{F}$ be a sheaf on X. The canonical Godement resolution is obtained by first taking the injection $\mathcal{F} \to \mathcal{F}_{\mathrm{God}}$ of lemma 2.11, then embedding $\mathcal{F}_{\mathrm{God}}/\mathcal{F}$ into $(\mathcal{F}_{\mathrm{God}}/\mathcal{F})_{\mathrm{God}}$ to obtain an exact sequence $0 \to \mathcal{F} \to \mathcal{F}_{\mathrm{God}} \to (\mathcal{F}_{\mathrm{God}}/\mathcal{F})_{\mathrm{God}}$. We continue like this indefinitely to get a resolution. By 2.33 this is a pure acyclic resolution.

The last thing to show is that $\mathcal{F}_{\mathrm{God}} \to \mathcal{I}$ is pure. For this it is useful that by 2.24, purity of a morphism of abelian groups can be decided by tensoring with $\mathbb{Z}/n\mathbb{Z}$. This is an example of a group of finite presentation, which are groups A fitting in an exact sequence

$$\mathbb{Z}^m \to \mathbb{Z}^k \to A \to 0.$$

So these are finitely generated groups for which there exists a presentation of which the kernel is also finitely generated. Obviously for $\mathbb{Z}/n\mathbb{Z}$ we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Let $\{A_i\}_{i\in I}$ be a family of abelian groups and let B be an abelian group. There is a canonical morphism

$$(\prod_{i\in I} A_i) \otimes B \to \prod_{i\in I}(A_i \otimes B)$$

given by $(a_i)_{i\in I} \otimes b \to (a_i \otimes b)_{i\in I}$. In general this fails to be an isomorphism, but we have the following.

Proposition 2.35. *If B is of finite presentation, the canonical map*

$$(\prod_{i\in I} A_i) \otimes B \to \prod_{i\in I}(A_i \otimes B)$$

is an isomorphism.

Proof. See [12]. □

From this we conclude:

Corollary 2.36. *If $A_i \to B_i$ is pure for $i \in I$, then $\prod_{i\in I} A_i \to \prod_{i\in I} B_i$ is pure.*

Proof. We only have to tensor with $\mathbb{Z}/n\mathbb{Z}$ to check this by 2.24, and in this case the tensor product distributes over the direct product by the previous proposition. We conclude by purity of the maps $A_i \to B_i$. $\square$

Proposition 2.37. *The morphism $\mathcal{F}_{\text{God}} \to \mathcal{I}$ is pure.*

Proof. For $U \subset X$ the morphism $\mathcal{F}_{\text{God}}(U) \to \mathcal{I}(U)$ is a product of pure monomorphisms:

$$\prod_{x\in U} \mathcal{F}_x \to \prod_{x\in U} I_x.$$

By the previous corollary this is a pure monomorphism and we conclude by 2.32.

$\square$

Combining all of this we see:

Theorem 2.38. *The category of sheaves on X has enough pure injectives.*

Of course all this work would be futile if pure injective resolutions didn't compute the cohomology of a sheaf $\mathcal{F}$. Fortunately we have the following.

Proposition 2.39. *Pure injectives are flasque.*

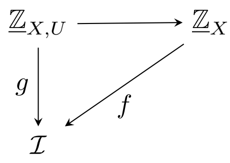
Proof. We follow the proof in [6], which proves that injectives are flasque, but the same proof shows pure injectives are flasque. Note that to prove that a pure injective $\mathcal{I}$ is flasque, we only have to show the restriction maps $\mathcal{F}(X) \to \mathcal{F}(U)$ are surjective for $U \subset X$.

So suppose $\mathcal{I}$ is pure injective, and let $U \subset X$. Define $\underline{\mathbb{Z}}_{X,U}(V)$ to be $\underline{\mathbb{Z}}_X(V)$ when $V \subset U$ and 0 otherwise. Then we have a natural exact sequence

$$0 \to \underline{\mathbb{Z}}_{X,U} \to \underline{\mathbb{Z}}_X.$$

This is in fact a pure exact sequence, since on sections it is the identity if $V \subset U$ and 0 otherwise.

Note that a morphism $f : \underline{\mathbb{Z}}_X \to \mathcal{I}$ is determined by picking an image $f(1) \in \mathcal{I}(X)$ (one can see this by using that the constant sheaf is the sheafification of the constant preseheaf), and in the same way a morphism $g : \underline{\mathbb{Z}}_{X,U} \to \mathcal{I}$ is determined by picking an image $g(1) \in \mathcal{I}(U)$. So let $g : \underline{\mathbb{Z}}_{X,U} \to \mathcal{I}$ be the morphism given by $g(1) = \sigma \in \mathcal{I}(U)$. Let f be the morphism that makes the diagram



(10)

commute, which exists by the pure injective property of $\mathcal{I}$. Then it follows that $\rho_{X,U}(f(1)) = g(1) = \sigma$, so $\rho_{X,U}$ is surjective.

□

We are now in a position to construct the cup product. Let $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X. Let $\mathcal{F} \to \mathcal{A}^\bullet$ and $\mathcal{G} \to \mathcal{B}^\bullet$ be pure acyclic resolutions. Let $\mathcal{F} \otimes \mathcal{G} \to \mathcal{K}^\bullet$ be a pure injective resolution. Then we get a complex

$$\mathcal{F} \otimes \mathcal{G} \to (\mathcal{A} \otimes \mathcal{B})^\bullet_{\mathrm{Tot}},$$

and by purity this is actually a resolution: by 2.17 $(\mathcal{A} \otimes \mathcal{B})^\bullet_{\mathrm{Tot}}$ is an exact complex in degree $n > 0$, and by using purity twice we see that

$$\mathcal{F} \otimes \mathcal{G} \to \mathcal{F} \otimes \mathcal{B}^0 \to \mathcal{A}^0 \otimes \mathcal{B}^0$$

is a monomorphism, and that the complex is exact in degree 0. It is in fact a pure resolution, by associativity of the tensor product and the fact that it distributes over direct sums. Hence we get a chain morphism

$$(\mathcal{A} \otimes \mathcal{B})^\bullet_{\mathrm{Tot}} \to \mathcal{K}^\bullet$$

extending the identity on $\mathcal{F} \otimes \mathcal{G}$, which is unique up to chain homotopy. Note that the tensor product $\mathcal{A}^k \otimes \mathcal{B}^l$ is obtained by taking the sheafification of the presheaf with sections $\mathcal{A}^k(U) \otimes \mathcal{B}^l(U)$. So there is a canonical map

$$\Gamma(\mathcal{A}^k) \otimes \Gamma(\mathcal{B}^l) \to \Gamma(\mathcal{A}^k \otimes \mathcal{B}^l).$$

Finally we get maps

$$\Gamma(\mathcal{A}^k) \otimes \Gamma(\mathcal{B}^l) \to \Gamma(\mathcal{K}^{k+l}),$$

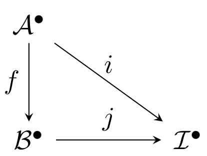
since taking global sections commutes with taking finite direct sums. Taking cohomology we get maps

$$\mathrm{H}^k(X, \mathcal{F}) \otimes \mathrm{H}^l(X, \mathcal{G}) \to \mathrm{H}^{k+l}(X, \mathcal{F} \otimes \mathcal{G}),$$

which we call the cup product.

Lemma 2.40. *Suppose we have acyclic resolutions $\mathcal{F} \to \mathcal{A}^\bullet$ and $\mathcal{F} \to \mathcal{B}^\bullet$ and a chain map $f : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ extending the identity. Then f induces an isomorphism on cohomology.*

Proof. Choose an injective resolution $\mathcal{I}$ of $\mathcal{F}$. By 2.10, chain maps $i : \mathcal{A}^\bullet \to \mathcal{I}^\bullet$ and $j : \mathcal{B}^\bullet \to \mathcal{I}^\bullet$ extending the identity on $\mathcal{F}$ induce isomorphisms on cohomology. Choose such maps and consider the diagram

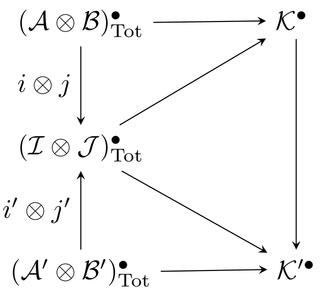


(11)

Since i and $j \circ f$ both extend $\mathcal{A}^\bullet$ to $\mathcal{I}^\bullet$, this diagram commutes up to homotopy. Hence f induces an isomorphism on cohomology. □

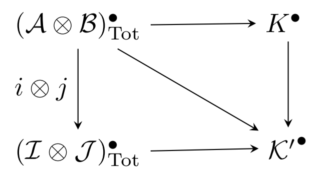
Proposition 2.41. *The cup product does not depend on the choice of pure acyclic resolutions of $\mathcal{F}$ and $\mathcal{G}$, nor on the choice of pure injective resolution of $\mathcal{F} \otimes \mathcal{G}$.*

Proof. Choose pure acyclic resolutions $\mathcal{F} \to \mathcal{A}^\bullet$, $\mathcal{F} \to \mathcal{A}'^\bullet$, $\mathcal{G} \to \mathcal{B}^\bullet$, $\mathcal{G} \to \mathcal{B}'^\bullet$, and take pure injective resolutions $(\mathcal{F} \otimes \mathcal{G}) \to \mathcal{K}^\bullet$, $(\mathcal{F} \otimes \mathcal{G}) \to \mathcal{K}'^\bullet$. Choose chain maps $i : \mathcal{A}^\bullet \to \mathcal{I}^\bullet$ and $j : \mathcal{B} \to \mathcal{J}^\bullet$ to pure injective resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ of $\mathcal{F}$ and $\mathcal{G}$, extending the identities on $\mathcal{F}$ and $\mathcal{G}$. Consider the diagram



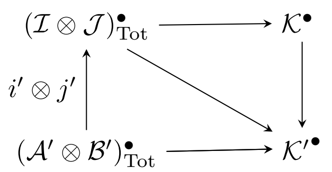
(12)

Note that the maps $i \otimes j$ and $i' \otimes j'$ are a priori maps on presheaves, but by the universal property of sheafification, they also induce maps on the corresponding sheaves. The maps $(\mathcal{I} \otimes \mathcal{J})^\bullet_{\text{Tot}}$ to $\mathcal{K}^\bullet$, $\mathcal{K}'^\bullet$ are the up to homotopy unique chain maps existing since $(\mathcal{I} \otimes \mathcal{J})^\bullet_{\text{Tot}}$ is a pure resolution. For the same reason, we have maps $(\mathcal{A} \otimes \mathcal{B})^\bullet_{\text{Tot}} \to \mathcal{K}^\bullet$ and $(\mathcal{A}' \otimes \mathcal{B}')^\bullet_{\text{Tot}} \to \mathcal{K}'^\bullet$. The diagram



(13)

commutes up to homotopy since all directions extend $(\mathcal{A} \otimes \mathcal{B})^\bullet_{\text{Tot}}$ to $\mathcal{K}'^\bullet$. Consider the diagram



(14)

The lower triangle commutes up to homotopy since both directions extend $(\mathcal{A}' \otimes \mathcal{B}')^\bullet_{\text{Tot}}$ to $\mathcal{K}'^\bullet$, while the upper triangle commutes up to homotopy since both directions extend $(\mathcal{I} \otimes \mathcal{J})^\bullet_{\text{Tot}}$ to $\mathcal{K}'^\bullet$. So the whole diagram commutes up to homotopy.

We conclude that the whole diagram 12 commutes up to homotopy. Taking global sections, we get the commutative diagram

$$
\begin{array}{ccccc}
\Gamma(\mathcal{A}^k)\otimes\Gamma(\mathcal{B}^l) & \longrightarrow & \Gamma(\mathcal{A}^k\otimes\mathcal{B}^l) & \longrightarrow & \Gamma(\mathcal{K}^{k+l}) \\
\Big\downarrow{\scriptstyle i\otimes j} & & {\scriptstyle i\otimes j}\Big\downarrow & \nearrow & \\
\Gamma(\mathcal{I}^k)\otimes\Gamma(\mathcal{J}^l) & \longrightarrow & \Gamma(\mathcal{I}^k\otimes\mathcal{J}^l) & & \\
\Big\uparrow{\scriptstyle i'\otimes j'} & & {\scriptstyle i'\otimes j'}\Big\uparrow & \searrow & \\
\Gamma(\mathcal{A}'^k)\otimes\Gamma(\mathcal{B}'^l) & \longrightarrow & \Gamma(\mathcal{A}'^k\otimes\mathcal{B}'^l) & \longrightarrow & \Gamma(\mathcal{K}'^{k+l})
\end{array}
\tag{15}
$$

Since by lemma 2.40 the morphisms i, i', j, j' induce natural isomorphisms on cohomology, finally we conclude, taking cohomology, that

$$
\begin{array}{ccc}
\mathrm{H}^k(\Gamma(\mathcal{A}^\bullet))\otimes\mathrm{H}^l(\Gamma(\mathcal{B}^\bullet)) & \longrightarrow & \mathrm{H}^{k+l}(\Gamma(\mathcal{K}^\bullet)) \\
\sim\Big\downarrow & & \sim\Big\downarrow \\
\mathrm{H}^k(\Gamma(\mathcal{A}'^\bullet))\otimes\mathrm{H}^l(\Gamma(\mathcal{B}'^\bullet)) & \longrightarrow & \mathrm{H}^{k+l}(\Gamma(\mathcal{K}'^\bullet))
\end{array}
\tag{16}
$$

commutes.

□

2.3 Čech Cohomology

In this section we introduce Čech cohomology, a computational tool for sheaf cohomology. We introduce the Čech resolution, which is much easier to handle than the other resolutions computing cohomology given so far, and show that under certain conditions, it computes sheaf cohomology. We also show, in a way that avoids using spectral sequences like [8] does, under these conditions there is a natural isomorphism between Čech and sheaf cohomology that respects the cup product. We follow [2] section 4.1.3 for the definitions.

Let $\mathcal{F}$ be a sheaf on X and $\{U_i\}_{i\in T}$ a finite open covering; T is given a well-ordering. For a set $I \subset T$ define

$$U_I = \bigcap_{i\in I} U_i.$$

Define for each set $I \subset T$ the sheaf $\mathcal{F}_I$ on X by

$$\mathcal{F}_I(U) = \mathcal{F}(U \cap U_I).$$

which is the extension by 0 of the restriction of $\mathcal{F}$ to U_I. Then we define sheaves $\mathcal{F}^k$ by

$$\mathcal{F}^k(U) = \bigoplus_{|I|=k+1} \mathcal{F}_I(U).$$

For a subset $I \subset T$, $I = \{i_0, \cdots, i_k\}$ (where $i_0 < i_1 < \cdots < i_k$), let $I_s = I \backslash \{i_s\}$. Let $\sigma = (\sigma_I)_I \in \mathcal{F}^k(U)$, where I ranges over subsets $I \subset T$ of cardinality $k+1$ and $\sigma_I \in \mathcal{F}(U \cap U_I)$. We define boundary maps $d : \mathcal{F}^k \to \mathcal{F}^{k+1}$ by

$$(d\sigma)_J = \sum_{s=0}^{k+1} (-1)^s \sigma_{J_s}|_{U \cap U_J},$$

where J ranges over the subsets $J \subset T$ of cardinality $k+2$. Also define $j : \mathcal{F} \to \mathcal{F}^0$ by

$$d\sigma = (\sigma|_{U \cap U_i})_{i \in \mathbb{N}}$$

for $\sigma \in \mathcal{F}(U)$. It is easy to show this turns $j : \mathcal{F} \to \mathcal{F}^\bullet$ into a complex.

Proposition 2.42. *The complex $j : \mathcal{F} \to \mathcal{F}^\bullet$ is in fact a pure resolution of $\mathcal{F}$.*

Proof. The fact that $j : \mathcal{F} \to \mathcal{F}^\bullet$ is a resolution is well known, see [2] proposition 4.17.

We prove purity. Note that we have

$$(\mathcal{F}^k)_x = \bigoplus_{|I|=k+1,\ x \in U_I} \mathcal{F}_x$$

since taking stalks commutes with direct sums. Hence the map j induces

$$\mathcal{F}_x \to \bigoplus_{x \in U_I} \mathcal{F}_x$$

given by the identity on each component, which is clearly a split monomorphism. For $k \geq 1$ consider

$$(\mathcal{F}^{k-1})_x \xrightarrow{d_x} (\mathcal{F}^k)_x \xrightarrow{d_x} (\mathcal{F}^{k+1})_x.$$

We tensor with a group G to get

$$\bigoplus_{|I|=k,\ x \in U_I} \mathcal{F}_x \otimes G \xrightarrow{d_x \otimes \mathrm{id}} \bigoplus_{|J|=k+1,\ x \in U_J} \mathcal{F}_x \otimes G \xrightarrow{d_x \otimes \mathrm{id}} \bigoplus_{|K|=k+2,\ x \in U_K} \mathcal{F}_x \otimes G.$$

Note that since d is an alternating sum, we have in fact $(d_x \otimes \mathrm{id})(a \otimes b) = (d_x(a) \otimes b) = d_x(a \otimes b)$, where the last d_x is the morphism induced on stalks by the boundary morphism of the Čech complex with values in the constant sheaf $\mathcal{F}_x \otimes \underline{G}$. We know this is exact since this complex is a resolution, so by 2.32 we are done. ∎

We call the above resolution the *Čech resolution* of $\mathcal{F}$.

By applying the global sections functor to $\mathcal{F}^\bullet$, we get a complex $\Gamma(\mathcal{F}^\bullet)$ given by

$$\Gamma(\mathcal{F}^k) = \bigoplus_{|I|=k+1} \mathcal{F}(U_I).$$

The *Čech cohomology groups* $\check{\mathrm{H}}^k(X, \mathcal{F})$ of $\mathcal{F}$ with respect to the covering $\{U_i\}_{i \in T}$ are defined to be the cohomology groups of this complex.

Given an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$, by 2.5 the identity on $\mathcal{F}$ induces a chain map $i : \mathcal{F}^\bullet \to \mathcal{I}^\bullet$, unique up to chain homotopy. This defines a canonical map

$$i_* : \check{\mathrm{H}}^k(X, \mathcal{F}) \to \mathrm{H}^k(X, \mathcal{F})$$

for all k. It is an important theorem that i_* is an isomorphism under some conditions on the open cover $\{U_i\}_{i\in\mathbb{N}}$.

Theorem 2.43. *Suppose $\mathcal{F}$ is a sheaf on X. Suppose $\{U_i\}_{i\in T}$ form an open cover of X such that*

$$\mathrm{H}^k(U_I, \mathcal{F}|_{U_I}) = 0$$

for $k > 0$ and all I (we say $\{U_i\}_{i\in\mathbb{N}}$ is a good open cover with respect to $\mathcal{F}$). Then the canonical map i_* is an isomorphism:

$$i_* : \check{\mathrm{H}}^k(X, \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^k(X, \mathcal{F})$$

Proof. See [6] for a proof avoiding spectral sequences. □

2.4 Cup product in Čech cohomology

Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Let $\{U_i\}$ be a good open cover of X with respect to $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{F}\otimes\mathcal{G}$. Let $\mathcal{F}^\bullet$, $\mathcal{G}^\bullet$ and $(\mathcal{F}\otimes\mathcal{G})^\bullet$ be the Čech resolutions of the sheaves $\mathcal{F}, \mathcal{G}, \mathcal{F}\otimes\mathcal{G}$. Consider the morphism

$$\smile : \mathcal{F}^k(U) \otimes \mathcal{G}^l(U) \to (\mathcal{F}\otimes\mathcal{G})^{k+l}(U)$$

where if $K = \{i_0, \cdots i_{k+l}\}$ and $I = \{i_0, \cdots, i_k\}$, $J = \{i_k, \cdots, i_{k+l}\}$ then

$$\smile (\sigma \otimes \tau)_K = \sigma_I \otimes \tau_J.$$

We have the following identity, reminiscent of the one in singular cohomology.

Lemma 2.44. *We have $d \smile (\sigma\otimes\tau) = \smile (d\sigma\otimes\tau) + (-1)^k \smile (\sigma\otimes d\tau)$ for $\sigma\otimes\tau \in \mathcal{F}^k(U)\otimes\mathcal{G}^l(U)$.*

Proof. We have

$$\smile (d\sigma\otimes\tau)_K = \sum_{s=0}^{k+1} (-1)^s \sigma_{i_0,\cdots,\hat{i_s},\cdots,i_{k+1}}|_{i_0,\cdots,i_{k+1}} \otimes \tau_{i_{k+1},\cdots,i_{k+l+1}}$$

and

$$(-1)^k \smile (\sigma\otimes d\tau)_K = \sum_{s=k}^{k+l+1} (-1)^s \sigma_{i_0,\cdots,i_k} \otimes \tau_{i_k,\cdots,\hat{i_s},\cdots,i_{k+l+1}}|_{i_k,\cdots,i_{k+l+1}}$$

where $K = \{i_0, \cdots, i_{k+l+1}\}$. When adding these expressions the last term of the first sum cancels the first term of the last, and we're left with $d \smile (\sigma\otimes\tau)$. □

From this it follows that we get a cup product on Čech cohomology. Moreover, note that by the universal property of sheafification $\smile$ induces maps of sheaves $\mathcal{F}^k \otimes \mathcal{G}^l \xrightarrow{\smile} (\mathcal{F} \otimes \mathcal{G})^{k+l}$. By the universal property of direct sums we get maps $(\mathcal{F} \otimes \mathcal{G})^{k+l}_{\mathrm{Tot}} \xrightarrow{\smile} (\mathcal{F} \otimes \mathcal{G})^{k+l}$. The lemma now implies that the diagrams

$$\begin{array}{ccc} (\mathcal{F} \otimes \mathcal{G})^{k+l}_{\mathrm{Tot}} & \xrightarrow{d} & (\mathcal{F} \otimes \mathcal{G})^{k+l+1}_{\mathrm{Tot}} \\ \smile \downarrow & & \downarrow \smile \\ (\mathcal{F} \otimes \mathcal{G})^{k+l} & \xrightarrow{d} & (\mathcal{F} \otimes \mathcal{G})^{k+l+1} \end{array} \tag{17}$$

commute, since sheafification preserves commutative diagram (it is a functor). This means precisely that the collection of these maps yields a chain map:

$$(\mathcal{F} \otimes \mathcal{G})^{\bullet}_{\mathrm{Tot}} \xrightarrow{\smile} (\mathcal{F} \otimes \mathcal{G})^{\bullet}.$$

This chain map contains the information of all the cup product maps in Čech cohomology.

We would like to be able to use Čech cohomology to compute cohomology rings, and for this we want a natural isomorphism between Čech and sheaf cohomology that respects the cup product. Let $\mathcal{F} \to \mathcal{F}^{\bullet}$ be the Čech resolution, and $\mathcal{F} \to \mathcal{I}^{\bullet}$ be a pure injective resolution. Then by purity of the Čech resolution, there is a chain map $i : \mathcal{F}^{\bullet} \to \mathcal{I}^{\bullet}$, unique up to chain homotopy. This map is a natural candidate.

Lemma 2.45. *The map i induces an isomorphism on cohomology.*

Proof. Choose an injective resolution $\mathcal{J}$ of $\mathcal{F}$. Choose chain maps $f : \mathcal{F}^{\bullet} \to \mathcal{J}^{\bullet}$ and $g : \mathcal{I}^{\bullet} \to \mathcal{J}^{\bullet}$. Now f induces an isomorphism on cohomology by 2.43 and $\mathcal{I}$ is flasque, so g also induces an isomorphism. Consider the diagram

$$\begin{array}{ccc} \mathcal{F}^{\bullet} & & \\ i \downarrow & \searrow^{f} & \\ \mathcal{I}^{\bullet} & \xrightarrow{g} & \mathcal{J}^{\bullet} \end{array} . \tag{18}$$

Since f and $g \circ i$ both extend $\mathcal{F}^{\bullet}$ to $\mathcal{J}^{\bullet}$, i induces an isomorphism. □

Theorem 2.46. *The natural isomorphism*

$$i_* : \check{\mathrm{H}}^k(X, \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^k(X, \mathcal{F})$$

induced by the chain map $i : \mathcal{F}^{\bullet} \to \mathcal{I}^{\bullet}$ respects the cup product.

Proof. Let $i : \mathcal{F}^{\bullet} \to \mathcal{I}^{\bullet}$ and $j : \mathcal{G}^{\bullet} \to \mathcal{J}^{\bullet}$ be chain maps of the Čech resolutions of $\mathcal{F}, \mathcal{G}$ to pure injective resolutions, extending the identity. Pick a chain map from $(\mathcal{F} \otimes \mathcal{G})^{\bullet}$ to a pure injective resolution $\mathcal{K}^{\bullet}$ (which is possible since it is pure), and do the same for $(\mathcal{I} \otimes \mathcal{J})^{\bullet}$. Consider the diagram

$$\begin{array}{ccc}
(\mathcal{F}\otimes\mathcal{G})^{\bullet}_{\mathrm{Tot}} & \xrightarrow{\smile} & (\mathcal{F}\otimes\mathcal{G})^{\bullet} \\
{\scriptstyle i\otimes j}\downarrow & & \downarrow \\
(\mathcal{I}\otimes\mathcal{J})^{\bullet}_{\mathrm{Tot}} & \xrightarrow{\smile} & \mathcal{K}^{\bullet}
\end{array} \tag{19}$$

Here the lower $\smile$ is the up to homotopy unique chain map inducing the cup product in sheaf cohomology.

As was discussed above, the upper $\smile$ is a chain map. It follows that all directions in the diagram extend the pure resolution $(\mathcal{F}\otimes\mathcal{G})^{\bullet}_{\mathrm{Tot}}$ to the pure injective resolution $\mathcal{K}^{\bullet}$, so the diagram commutes up to homotopy. Taking global sections, we obtain commutative diagrams

$$\begin{array}{ccccc}
\Gamma(\mathcal{F}^k)\otimes\Gamma(\mathcal{G}^l) & \longrightarrow & \Gamma(\mathcal{F}^k\otimes\mathcal{G}^l) & \xrightarrow{\smile} & \Gamma((\mathcal{F}\otimes\mathcal{G})^{k+l}) \\
{\scriptstyle i\otimes j}\downarrow & & {\scriptstyle i\otimes j}\downarrow & & \downarrow \\
\Gamma(\mathcal{I}^k)\otimes\Gamma(\mathcal{J}^l) & \longrightarrow & \Gamma(\mathcal{I}^k\otimes\mathcal{J}^l) & \xrightarrow{p} & \Gamma(\mathcal{K}^{k+l})
\end{array} \tag{20}$$

As i and j and also the vertical map on the right induce natural isomorphisms on cohomology by 2.45, passing to cohomology we get the commutative diagram

$$\begin{array}{ccc}
\check{\mathrm{H}}^k(X,\mathcal{F})\otimes\check{\mathrm{H}}^l(X,\mathcal{G}) & \xrightarrow{\smile} & \check{\mathrm{H}}^{k+l}(X,\mathcal{F}\otimes\mathcal{G}) \\
{\scriptstyle\sim}\downarrow & & {\scriptstyle\sim}\downarrow \\
\mathrm{H}^k(X,\mathcal{F})\otimes\mathrm{H}^l(X,\mathcal{G}) & \xrightarrow{p_*} & \mathrm{H}^{k+l}(X,\mathcal{F}\otimes\mathcal{G})
\end{array} \tag{21}$$

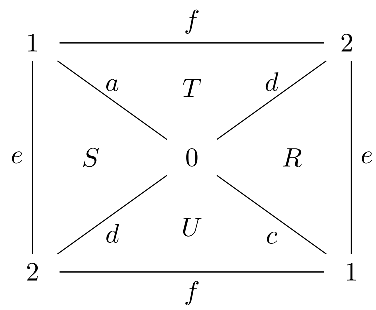
as desired. □

2.5 Some computations

In this section, we will do some explicit computations of cohomology rings using Čech cohomology. We will give an example of two spaces that have the same cohomology groups, but a different cup product, whence they cannot be homotopy equivalent.

We first compute the cohomology of the real projective plane $X = \mathbb{R}P^2$, which we will denote by P^2. We construct an open cover of S^2. Choose an equator and cover it by four open bands of the same size. Then cover the open hemisperes perpendicular to this equator by two opens, in such a way that these opens don't meet. This yields a cover $\{U_i\}_{i\in I}$ with $I = \{0,1,2,3,4,5\}$. This is a good cover with respect to $\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}/2\mathbb{Z}}$ of S^2 in the sense of 2.43 since al intersections of opens in this set are empty or contractible. Note that the $\mathbb{Z}/2\mathbb{Z} = \langle\rho\rangle$ action on S^2 permutes these opens in such a way that $\rho U_i \cap U_i = \emptyset$ for all $i \in I$, so taking the images of the opens in this cover under the covering map p yields a good open cover $\{V_i\}_{i\in I}$ with respect to $\mathbb{Z}$ and $\underline{\mathbb{Z}/2\mathbb{Z}}$ of P^2, where now $I = \{0,1,2\}$: while the non-empty intersections are not contractible, they consist of finitely many contractible components.

We will compute the cohomology with coefficients in the constant sheaves $\underline{\mathbb{Z}}_X$ and $\underline{\mathbb{Z}/2\mathbb{Z}}_X$. There are 6 components in the twofold intersections in this open cover which we call a, b, c, d, e, f and 4 different components in the threefold intersection which we call S, T, R, U. The following diagram illustrates this:



If we take coefficients in $\underline{\mathbb{Z}_X}$, complex of the global sections of the Čech complex is

$$\mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^6 \xrightarrow{B} \mathbb{Z}^4 \to 0,$$

with dual bases of the basis $\{0,1,2\}$, $\{a,b,c,d,e,f\}$, $\{S,T,R,U\}$. Writing out the boundary maps, we see that with respect to these bases we have

$$A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{pmatrix}$$

As expected we have

$$\check{\mathrm{H}}^0(P^2, \underline{Z}_X) = \ker A = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rangle \cong \mathbb{Z}$$

Moreover we have

$$\mathrm{im}\ A = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \rangle = \ker B$$

so $\check{\mathrm{H}}^1(P^2, \underline{\mathbb{Z}})_X = 0$.

Now consider the surjective homomorphism $\mathbb{Z}^4 \to \mathbb{Z}/2\mathbb{Z}$ given by

$$(x_i)_{i=0}^4 \mapsto \sum_{i=0}^4 x_i (\mathrm{mod} 2)$$

It is easily seen that im B (the subgroup generated by the columns of the matrix B) is the kernel of this homomorphism. Hence by the isomorphism theorem we conclude

$$\check{\mathrm{H}}^2(P^2, \underline{\mathbb{Z}}) = \mathbb{Z}^4/\mathrm{im}\ B \cong \mathbb{Z}/2\mathbb{Z}.$$

Because $\check{\mathrm{H}}^1(P^2, \underline{\mathbb{Z}}) = 0$, the cup product structure is not very interesting: it is just multiplication within $\mathbb{Z} = \check{\mathrm{H}}^0(P^2, \underline{\mathbb{Z}})$, and trivial on $\mathbb{Z}/2\mathbb{Z} = \check{\mathrm{H}}^2(P^2, \underline{\mathbb{Z}})$, i.e.

$$\check{\mathrm{H}}^*(P^2, \underline{\mathbb{Z}}) \cong \mathbb{Z}[\alpha]/(\alpha^2, 2\alpha),$$

with α the generator of $\check{\mathrm{H}}^2(P^2, \underline{\mathbb{Z}})$.

For this reason we now take coefficients in $\mathbb{Z}/2\mathbb{Z}$, because in this case we get a more interesting structure. The groups now become $\mathbb{Z}/2\mathbb{Z}$ modules and the morphisms $\mathbb{Z}/2\mathbb{Z}$ module morphisms: the coefficients of the matrices lie in $\mathbb{Z}/2\mathbb{Z}$. The exact same arguments as above apply to show that

$$\check{\mathrm{H}}^0(P^2, \underline{\mathbb{Z}/2\mathbb{Z}}) \cong \mathbb{Z}/2\mathbb{Z}, \check{\mathrm{H}}^2(P^2, \underline{\mathbb{Z}/2\mathbb{Z}}) \cong \mathbb{Z}/2\mathbb{Z}.$$

However, we have that in this case

$$\mathrm{im}\ A = \langle \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \rangle, \ker B = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \rangle.$$

Note that

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

so

$$\check{\mathrm{H}}^1(P^n, \underline{\mathbb{Z}/2\mathbb{Z}}) \cong \mathbb{Z}/2\mathbb{Z} \cong \langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rangle.$$

Taking the product of the generator with itself we get

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}^2 = \begin{pmatrix} 1 \cdot 1 \\ 1 \cdot 0 \\ 0 \cdot 1 \\ 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which is the generator of $\check{\mathrm{H}}^2(P^n, \underline{\mathbb{Z}/2\mathbb{Z}})$. Hence the product is not trivial, and we have

$$\check{\mathrm{H}}^*(P^n, \underline{\mathbb{Z}/2\mathbb{Z}}) \cong (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^3)$$

where α is the generator of $\check{\mathrm{H}}^1(P^n, \underline{\mathbb{Z}/2\mathbb{Z}})$.

Now take $X = S^2 \vee S^1$. Note that

$$\check{\mathrm{H}}^*(S^2 \vee S^1, \underline{\mathbb{Z}/2\mathbb{Z}}) \cong \check{\mathrm{H}}^*(S^2, \underline{\mathbb{Z}/2\mathbb{Z}}) \oplus \check{\mathrm{H}}^*(S^1, \underline{\mathbb{Z}/2\mathbb{Z}})$$

which shows that the cup product structure is trivial, since it is trivial on $\check{\mathrm{H}}^*(S^2, \underline{\mathbb{Z}/2\mathbb{Z}})$ and $\check{\mathrm{H}}^*(S^1, \underline{\mathbb{Z}/2\mathbb{Z}})$. This means that while $S^2 \vee S^1$ and P^2 have the same $\mathbb{Z}/2\mathbb{Z}$-cohomology groups, the cup product allows us to distinguish them.

3 Comparison sheaf cohomology and singular cohomology

In this section, we follow [2] to compare singular cohomology with sheaf cohomology, but we provide much more details. We also show this isomorphism respects the cup product. We start with a definition of a property of topological spaces that turns out to be crucial in the proof.

Definition 3.1. Let X be a topological space. An open cover $\{U_i\}_{i\in I}$ is called *locally finite* if for every $x \in X$ there is an open U containing x such that

$$\#\{i \in I : U \cap U_i \neq \emptyset\} < \infty.$$

A topological space X is called *paracompact* if every open cover $\{U_i\}_{i\in I}$ has a locally finite refinement. Finally, a topological space X is called *hereditarily paracompact* if every open subset $U \subset X$ is paracompact.

This condition admittedly looks a bit strange at first glance. However, the class of hereditarily paracompact spaces contains the class of manifolds, since manifolds are paracompact and every open subset of a manifold is itself a manifold.

Theorem 3.2. *Let X be a locally contractible, hereditarily paracompact space. Then we have a natural isomorphism*

$$\mathrm{H}^k(X;\mathbb{Z}) \to \mathrm{H}^k(X, \underline{\mathbb{Z}}_X)$$

with singular cohomology on the left hand side and sheaf cohomology on the right hand side.

In [2] the paracompactness assumption is omitted. Then, however, there is a problem in the proof which cannot be fixed. The theorem is true without the paracompactness assumption, see [4], but a different and much more complicated approach is needed. We give the proof for the theorem with the paracompactness assumption, and comment on what goes wrong in the proof without this assumption.

Consider the chain complex of presheaves given by the singular chain groups $C^k_{\mathrm{sing}}(U;\mathbb{Z})$ for $U \subset X$. Since $C^k_{\mathrm{sing}}(U;\mathbb{Z})$ consists of functions from n-simplices to $\mathbb{Z}$, it is a flasque presheaf. Let C^k denote the sheafification of C^k_{sing}.

We construct the isomorphism as follows. We first show $C^\bullet$ is a resolution of $\underline{\mathbb{Z}}_X$. Then we show it is in fact a flasque resolution, so it computes the sheaf cohomology of $\underline{\mathbb{Z}}_X$. Finally we show the sheafification map induces an isomorphism on cohomology.

Proposition 3.3. *The complex $\underline{\mathbb{Z}}_X \to C^\bullet$ is a resolution.*

Proof. For $x \in X$ and $k \geq 1$, consider the sequence

$$(C^{k-1}_{\mathrm{sing}})_x \xrightarrow{\partial^{k-1}_x} (C^k_{\mathrm{sing}})_x \xrightarrow{\partial^k_x} (C^{k+1}_{\mathrm{sing}})_x.$$

Since the C^k form a complex, $\partial_x^2 = 0$. Suppose $\sigma_x \in \ker \partial^k_x$. Then since x has a basis of neighbourhoods consisting of contractible spaces, we can represent

σ_x by $\sigma \in C^k_{\text{sing}}(U)$ with U contractible. Since contractible spaces have trivial singular cohomology in dimension $k > 0$, the sequence

$$(C^{k-1}_{\text{sing}})(U) \xrightarrow{\partial^{k-1}} (C^k_{\text{sing}})(U) \xrightarrow{\partial^k} (C^{k+1}_{\text{sing}})(U)$$

is exact. By replacing U with a smaller contractible open containing x if necessary, we can assume $\partial^k \sigma = 0$, so $\sigma \in \text{im } \partial^{k-1}$. Taking stalks, we see that $\sigma_x \in \text{im } \partial^{k-1}$, hence the sequence of stalks is exact. Since sheafification preserves stalks, we conclude that the complex $C^\bullet$ is exact at C^k for $k \geq 1$.

It remains to show that $0 \to \underline{\mathbb{Z}}_X \xrightarrow{j} C^0 \xrightarrow{\partial} C^1$ is exact. We proceed like in the previous case: we show that $0 \to \underline{\mathbb{Z}}_X(U) \xrightarrow{j} C^0_{\text{sing}}(U) \xrightarrow{\partial} C^1_{\text{sing}}(U)$ is exact for U contractible, we conclude that the sequence is exact at stalks, hence the conclusion follows for the sequence of sheaves.

So it remains to show that

$$0 \to \underline{\mathbb{Z}}_X(U) \xrightarrow{j} C^0_{\text{sing}}(U) \xrightarrow{\partial} C^1_{\text{sing}}(U)$$

is exact for U contractible. It is clear that j is injective: it is the map which embeds constant functions on U (since U is contractible) in the space of arbitrary functions $U \to \mathbb{Z}$. Clearly constant maps map to 0 under ∂, so $\partial \circ j = 0$. Suppose moreover that for $\sigma \in C^0_{\text{sing}}(U)$ we have $\partial\sigma = 0$. Since U is contractible and thus path connected, there exists a path between every two points $x_0, x_1 \in U$, i.e. there exists a 1-simplex Δ with boundary points x_0, x_1. So if $\partial(\sigma) = 0$ then $\partial(\sigma)(\Delta) = 0$, hence $\sigma(x_0) = \sigma(x_1)$. This shows that σ is constant and we conclude $\sigma \in \text{im } j$. □

To show $\underline{\mathbb{Z}}_X \to C^\bullet$ is a flasque resolution, we need the following lemma. Note that a presheaf $\mathcal{F}$ is a sheaf if and only if it satisfies the following axioms:

- (Unicity) Let $U \subset X$ be an open with an open covering $\{U_i\}_{i \in I}$. Let $\sigma \in \mathcal{F}(U)$. Then $\sigma|_{U_i} = 0$ for all $i \in I$ implies $\sigma = 0$.
- (Glueing) Let $U \subset X$ be an open with an open covering $\{U_i\}_{i \in I}$. Suppose there are sections $\sigma_i \in \mathcal{F}(U_i)$ such that for all $i, j \in I$ we have $\sigma_i|_{U_i \cap U_j} = \sigma_j|_{U_i \cap U_j}$. Then there exists $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ for each i.

Lemma 3.4. *Suppose X is hereditarily paracompact and suppose $\mathcal{F}$ is a presheaf on X which satisfies the glueing axiom. Then the maps*

$$\mathcal{F}(U) \to \mathcal{F}^+(U)$$

induced by sheafification are surjective.

Proof. See [3] proposition 1.13. □

Remark 3.5. The assumption that X is hereditarily paracompact (or some other assumption) is necessary here: see [4] or [5] for counterexamples when this assumption is omitted.

In [3], theorem 3.2 is stated without the paracompactness assumption, but in the proof lemma 3.4 is used; maybe this is the source of the confusion in [2].

We continue with the proof. Denote by $C^k_{\text{sing}}(U)_0$ the set of cochains that restricted to some open cover are 0, i.e.

$$C^k_{\text{sing}}(U)_0 = \{\phi \in C^k_{\text{sing}}(U) | \exists \text{ open cover } \mathcal{U} \text{ with } \phi|_V = 0 \text{ for all } V \in \mathcal{U}\}.$$

This is the kernel of the map $C^k_{\text{sing}}(U) \to C^k(U)$ induced by sheafification. So by 3.4 we have the following corollary.

Corollary 3.6. *We have natural isomorphisms*

$$C^k_{\text{sing}}(U)/C^k_{\text{sing}}(U)_0 \cong C^k(U).$$

From this it follows immediately that C^k is flasque as the quotient of a flasque presheaf. So $\mathbb{Z}_X \to \mathcal{C}^\bullet$ computes the cohomology of $\mathbb{Z}_X$. The only thing left to show is that the map

$$C^k_{\text{sing}}(X) \to C^k_{\text{sing}}(X)/C^k_{\text{sing}}(X)_0$$

induces an isomorphism on cohomology.

Let $\mathcal{U}$ be an open covering of X. Define $C^{\mathcal{U}}_k(X)$ to be the subgroup generated by k-simplices σ in X with image contained in some $V \in \mathcal{U}$. Then proposition 2.21 in [1] says that the map

$$i : C^{\mathcal{U}}_k(X) \to C_k(X)$$

induced by the inclusion is a chain homotopy equivalence. The proof uses the technique of barycentric subdivision. Hence the dual map

$$i^* : C^k_{\text{sing}}(X) \to (C^{\mathcal{U}}_{\text{sing}})^k(X)$$

is also a chain homotopy equivalence. To see this, take duals in the relation

$$i\rho - \text{id} = \partial P - P\partial$$

with the map ρ of 2.21, [1], to obtain

$$\rho^* i^* - \text{id} = \partial^* P^* - P^* \partial^*$$

so P^* is a chain homotopy between $\rho^* i^*$ and id. In the same way $i^* \rho^*$ is chain homotopic to id.

The kernel of i^* is exactly $(C^{\mathcal{U}}_{\text{sing}})^k(X)_0$, the set of cochains that restricted to $\mathcal{U}$ are 0. Taking cohomology in the sequence

$$(C^{\mathcal{U}}_{\text{sing}})^k(X)_0 \to C^k_{\text{sing}}(X) \to (C^{\mathcal{U}}_{\text{sing}})^k(X)$$

yields $\text{H}^k((C^{\mathcal{U}}_{\text{sing}})(X)_0) = 0$ for all k.

Note that $\{(C^{\mathcal{U}}_{\text{sing}})^k(X)_0\}_{\mathcal{U}}$ is a directed system partially ordered by inclusion (there is an inclusion $(C^{\mathcal{U}}_{\text{sing}})^k(X)_0 \to (C^{\mathcal{V}}_{\text{sing}})^k(X)_0$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$). It follows that the direct limit of this system is equal to the union of the system, which is equal to $C^k_{\text{sing}}(X)_0$. Since (co)homology commutes with taking direct limits, we conclude

$$\mathrm{H}^k((C_{\text{sing}})(X)_0) = 0.$$

for all k. Now taking the long exact sequence of singular cohomology associated to the short exact sequence of complexes

$$0 \to (C_{\text{sing}})^k(X)_0 \to C^k_{\text{sing}}(X) \to C^k_{\text{sing}}(X)/C^k_{\text{sing}}(X)_0 \to 0$$

shows that the quotient map induces an isomorphism on cohomology, which concludes the proof.

Remark 3.7. Note that we could have used any abelian value group G instead of $\mathbb{Z}$ and the same proof would have worked.

We turn to the cup product. Let A be an abelian group. Let $C_{\text{sing}}(A)^\bullet$ be the complex of presheaves with sections the singular cochains, and $C(A)^\bullet$ the associated complex of sheaves, where we take values in A. To prove the natural isomorphism between sheaf cohomology and singular cohomology respects the cup product, we take the same approach as with Čech cohomology: we prove that $\underline{A}_X \to C(A)^\bullet$ is a pure resolution, after which we deduce the desired compatibility in more or less the same way.

Proposition 3.8. *The resolution $\underline{A}_X \to C(A)^\bullet$ is pure.*

Proof. The group $C^k_{\text{sing}}(U; A)$ consists of maps of the free abelian group $C_k(U)$ generated by the k-simplices in X into A. Since giving a homomorphism from a free abelian group into another group is equivalent to giving images on a basis, we can interpret it as a product

$$C^k_{\text{sing}}(U; A) \cong \prod_\sigma A$$

where σ ranges over the k-simplices in U Tensoring with a group B of finite presentation now gives isomorphisms

$$C^k_{\text{sing}}(U; A) \otimes B \cong (\prod_\sigma A) \otimes B \cong \prod_\sigma (A \otimes B) \cong C^k_{\text{sing}}(U; A \otimes B)$$

where the middle isomorphism is due to 2.35. This isomorphism respects boundary maps and restrictions, so we get a chain isomorphism between $(C_{\text{sing}}(A) \otimes B)^\bullet$ and $\underline{A}_X \otimes B \to C_{\text{sing}}(A \otimes B)^\bullet$. Arguing like in the previous section, the second complex is exact on contractibles, so under our assumptions it is exact at stalks. So the same holds for the first complex, from which it follows that the complex of stalks associated to the complex $C_{\text{sing}}(A)^\bullet$ is pure. But this is also the complex of stalks associated to the complex $C(A)^\bullet$. Hence $C(A)^\bullet$ is itself pure by 2.32.

Also the embedding $\underline{A}_X \to C^0(A)$ is pure: on presheaves, for U contractible it is the embedding

$$A \to \prod_{x \in U} A$$

given by $\prod_{x \in U} \mathrm{id}$ which is clearly split. Again, it is pure at stalks, from which our claim follows. □

We need the following lemma to show the cup product on presheaves induces a map on the corresponding sheaves.

Lemma 3.9. *The sheafification of the presheaf with sections* $C^k_{\text{sing}}(U;A) \otimes C^l_{\text{sing}}(U;B)$ *is* $C^k(A) \otimes C^l(B)$.

Proof. We use the description of sheafification in terms of compatible stalks, namely: the sheafification of a presheaf $\mathcal{F}$ is given by the sheaf $\mathcal{F}^\#$ with sections

$$\mathcal{F}^\#(U) = \{(s_u) \in \prod_{u \in U} \mathcal{F}_x \text{ such that } *\}$$

where * is the following property: for every $u \in U$ there exists an open V with $u \in V \subset U$ and a section $\sigma \in \mathcal{F}(V)$ such that $\sigma_u = s_u$, with σ_u the image of σ in $\mathcal{F}_u$. See also [13].

Since sheafification preserves stalks and taking stalks commutes with tensor products, the description this gives for $(C^k(A) \otimes C^l(B))(U)$ is the subset of

$$\prod_{x \in U} (C^k_{\text{sing}})_x \otimes (C^l_{\text{sing}})_x$$

such that for every $u \in U$ there exists an open V with $u \in V \subset U$ and a section $\sigma \in C^k(U;A) \otimes C^l(U;B)$ such that $\sigma_u = s_u$. But sections in $C^k(U;A)$ and $C^l(U;B)$ are locally equal determined by the presheaves $C^k_{\text{sing}}(A)$ and $C^l_{\text{sing}}(B)$, so this is the same as requiring that for every $u \in U$ there exists an open V with $u \in V \subset U$ and a section $\sigma \in C^k_{\text{sing}}(U;A) \otimes C^l_{\text{sing}}(U;B)$ such that $\sigma_u = s_u$. This is the description of the sheafification of the presheaf with sections $C^k_{\text{sing}}(U;A) \otimes C^l_{\text{sing}}(U;B)$. □

Let A and B be abelian groups. Note that the maps on singular cohomology

$$\smile : C^k_{\text{sing}}(U;A) \otimes C^l_{\text{sing}}(U;B) \longrightarrow C^{k+l}_{\text{sing}}(U;A \otimes B)$$

which induce the cup product, by the universal property of sheafification and lemma 3.9 induce maps of sheaves

$$\smile : C^k(A) \otimes C^l(B) \to C^{k+l}(A \otimes B)$$

and the cup product is completely determined by these maps. By the universal property of direct sums we obtain maps $(C(A) \otimes C(B))^{k+l}_{\text{Tot}} \to C^{k+l}(A \otimes B)$ and the same way as in Čech cohomology, lemma 1.4 implies that these form a chain map

$$(C(A) \otimes C(B))^\bullet_{\text{Tot}} \xrightarrow{\smile} C(A \otimes B)^\bullet,$$

which contains the information of all the cup product maps in singular cohomology.

By the argument in lemma 2.45 the up to homotopy equivalence unique chain maps $i_A : C(A)^\bullet \to \mathcal{I}^\bullet_A$ and $i_B : C(B)^\bullet \to \mathcal{I}^\bullet_B$ to pure injective resolutions of $\underline{A}_X$ and $\underline{B}_X$ induce natural isomorphisms on cohomology. Let $\mathcal{K}^\bullet$ be a pure injective resolution of $\underline{A}_X \otimes \underline{B}_X$, which by the proof of lemma 3.9 is equal to $(\underline{A \otimes B})_X$. The diagram

$$\begin{array}{ccc} (C(A)\otimes C(B))^\bullet_{\mathrm{Tot}} & \xrightarrow{\smile} & C(A\otimes B)^\bullet \\ \downarrow{\scriptstyle i_A\otimes i_B} & & \downarrow \\ (\mathcal{I}_A\otimes\mathcal{I}_B)^\bullet_{\mathrm{Tot}} & \xrightarrow{\smile} & \mathcal{K}^\bullet \end{array} \tag{22}$$

where the arrows are the canonical ones, commutes since all directions extend the pure resolution $(C(A)\otimes C(B))^\bullet_{\mathrm{Tot}}$ to the pure injective resolution $\mathcal{K}^\bullet$. Since the vertical maps induce natural isomorphisms on cohomology, the horizontal maps induce the cup product on respectively Čech and sheaf cohomology, and the sheafification map $C^k_{\mathrm{sing}}(A)\to C^k(A)$ induces an isomorphism on cohomology, we conclude that the diagrams

$$\begin{array}{ccc} \mathrm{H}^k(X;A)\otimes\mathrm{H}^l(X;B) & \xrightarrow{\smile} & \mathrm{H}^{k+l}(X;A\otimes B) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \mathrm{H}^k(X,\underline{A}_X)\otimes\mathrm{H}^l(X,\underline{B}_X) & \xrightarrow{\smile} & \mathrm{H}^{k+l}(X,(\underline{A\otimes B})_X) \end{array} \tag{23}$$

commute.

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