Applications of stochastic cooperative game theory to renewable energy prediction problems

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Master's Thesis

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July 11, 2018

Mathematisch Instituut, Universiteit Leiden

Contents

1 Acknowledgements

There are several people who have contributed significantly to the creation of this thesis. In that regard, I would first like to thank Dr. Floske Spieksma. She has given me invaluable advice and guidance. Furthermore, I am grateful for the helpful suggestions from and answers to my questions by Rens Philipsen (Technical University Delft), Prof. Peter Borm (Tilburg University), and Dr. Judith Timmer (University of Twente). Last but not least, I wish to thank my parents, Ernst and Monique Muller. They have supported me throughout my long educational journey and during the process of writing this thesis in particular.

2 Abstract

In his thesis, we will first outline the theory of stochastic cooperative game theory without transfer payments, as developed by Borm, Tijs and Timmer in [6]. We will compare and contrast it with classical cooperative game theory. Subsequently, we will use this theory to model a two-player game in which owners of intermittent renewable energy sources (such as wind mills and solar panels) could cooperate in order to minimize their fine due to prediction errors of their production. We will describe two different kinds of cooperation and compare and contrast them in terms of their respective beneficiality to the players involved by computing the core. Furthermore, we will consider the case in which the prediction errors of the players are correlated. Subsequently, we compute the Shapley values of the two-player game, after which we generalize our results to the case of three participating players. Throughout this thesis, we stress the practical relevance of this work to real-world examples by pointing towards and incorporating previous experimental work on the actual prediction error distributions of renewable energy production devices.

3 Background

Cooperative game theory was introduced by von Neumann and Morgenstern in 1944 in their seminal work *Theory of Games and Economic Behavior* [16]. Their theory mathematically describes the possible benefits of cooperation to individual players. Their work and and the work of subsequent mathematicians including Lloyd Shapley, Donald B. Gillies, David Schmeidler and others did in this field [13] predominantly focused on a deterministic setting. This means that the participants of the game know beforehand exactly what the coalitional values of the game are. In other words: if all players are rational, then it is always possible to perfectly predict their modus operandi in the game.

In many real-world examples, however, benefits or drawbacks resulting from cooperation between two or more people or other entities is not-predetermined. Rather, the act of initiating cooperation is a decision with a certain amount of risk attached to it. People do not always know what the full consequences of their actions will be and how others will behave.

It wasn't until the early 70s that the mathematical theory of cooperative decision-making under risk was seriously studied. The mathematicians Abraham Charnes and Daniel Granot jointly initiated the study of stochastic cooperative game theory in the 1973 with their paper entitled *Prior solutions: extensions of convex nucleolus solutions to chance-constrained games* [8]. Their work extended the theory of cooperative games to situations in which the benefits obtained by players are random variables. As in the deterministic variant of von Neumann and Morgenstern, Charnes and Granot investigated how the randomly distributed benefits ought to be distributed among the different players.

Their theory consists of a two-stage allocation process. A perhaps slightly surprising aspect of their theory is that, although the benefits are random, a deterministic amount is allocated during each stage. During the first stage – which is *before* it is known by the players what the benefits are – payoffs are promised to the individuals. In the second stage, *after* the players are told what the realization of the randomly distributed benefits is, the promised payoffs from stage one are modified.

This two-stage process is described for several different important allocation values, including the core and the Shapley value.

The work of Charnes and Granot was ground-breaking. However – its pivotal nature notwithstanding – there are some drawbacks inherent to their approach. First of all, it does not allow for an explicit incorporation of each individual player's attitude towards risk [24]. This means that risk-averse or risk-seeking behavior of players cannot be taken into account in their model. Although that would not be a problem in reformulating deterministic cooperative games, it is a drawback because people, companies and other entities in the real world have different ways of approaching risk. To me, another downside of their approach is that the two-stage allocation process seems a bit unnatural or contrived.

For a long time after Charnes and Granot did their pioneering work, there was no alternative approach towards stochastic cooperative game theory. It took until 1999 – more than a quarter century later – until the deficiencies in their model were repaired by Jeroen Suijs, Anja de Waegenaere and Peter Borm. Their model, which they first applied to a problem in actuarial science [23], explicitly includes the preferences of the individuals. This means that any kind of behavior towards risk can be expressed by the preferences. This allows for the inclusion of varying degrees of risk-aversity and risk-propensity of the participating players.

In addition, the work by Suijs et al. does not require a two-stage process. In their model, the agents decide on the allocation before the realization of the random variables that denote the benefits or costs. As a result, the whole allocation process is more elegant.

In 2000, yet another important contribution was made to the body of work on stochastic cooperative game theory. Judith Timmer, Peter Borm, and Stef Tijs based their model [6] on the one presented in the articles by Suijs et al. The main tennets of their theory are therefore similar to the one by Suijs et al. There are some minor differences, however. Suijs et al. divide an allocation into two parts. First of all, the allocation of risk is determined. This manifests itself through a determination of which non-negative multiples of the random payoff (or cost) are divided up amongst the players. Subsequently, it is decided how to distribute the deterministic payments to and from each player. This deterministic aspect enables the agents to "insure themselves" against possibly bad stochastic outcomes. The more deterministic benefits the agents receive, the bigger the risk they are willing to bear.

A difference is that Timmer et al. allow for multiples of random payoffs, which can even be negative. Suijs et al. only allow non-negative fractions of random payoffs. This difference is subtle, however: the negative multiples or multiples larger than one are only potentially necessary in the approach of Timmer et al. when dealing with the Shapley value, as the marginal contribution of a player can be bigger than one or negative. In other aspects of their theory (including the core and the imputation set) the allocations are non-negative fractions of random payoffs, too.

Another difference between the models by Timmer et al. and Suijs et al. lies in their exclusion and inclusion respectively of deterministic benefit transfers [24]. As Timmer et al. do not allow for the transfer of deterministic payments, their model becomes a tad more simple than the model by Suijs et al.

This was one of the reasons why we chose to adopt the model by Timmer et al. Thanks to the relative simplicity of their model, it becomes easier to apply it to real-world problems and perform calculations. Furthermore, we suspect – though we can't solidify this argument – that players in the real world are more likely to cooperate within the boundaries of this model. It allows for easier and quicker computations, both analytically and computationally. It is for these reasons that the model by Timmer, Borm, and Thijs was chosen as the object of study for this thesis, and as a tool to tackle a specific problem related to the prediction of renewable energy production that will be outlined in the following chapters.

4 Introduction to stochastic cooperative game theory

4.1 Preferences

Classical cooperative game theory was largely developed in the 1950s. It involves a set of players N and a characteristic function $v : 2^N \to \mathbb{R}$ that maps all coalitions to their corresponding value. In case these values are all non-negative, we call it a *value* or *profit* game. Conversely, when these values are less than or equal to zero we call it a *cost* game. In both cases, the aim is to determine how to divide the value of the grand coalition.

In classical cooperative game theory, the payoffs coalitions receive are deterministic. There is no ambiguity involved in the payoffs and the players know exactly how much benefit all coalitions will yield for them. In [6], however, Timmer et al. developed a generalization of classical cooperative game theory. In their model, payoffs for coalitions are not deterministic anymore. Rather, the payoffs will be stochastic. We will mostly follow the model described in the work by Timmer et al., albeit with some small alterations.

In order to work with stochastic payoffs, we need to define the triple $(\Omega, \mathcal{F}, \mathbb{P})$. This is a probability space, where Ω is the outcome space, $\mathcal{F}$ is a σ-algebra in Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. Furthermore, a stochastic variable X is a measurable function that assigns a real number $X(\omega)$ to each outcome $\omega \in \Omega$. In addition, we define $\mathcal{L} := \{X \mid \mathbb{E}|X| < \infty\}$, and $\mathcal{L}_+$ is a subset of consisting of those stochastic variables in $\mathcal{L}$ that are nonnegative. So we have $\mathcal{L}_+ := \{X \in \mathcal{L} \mid \mathbb{P}(X < 0) = 0\}$.

Denote by $R(S) \in \mathcal{L}_+$ the stochastic payoff to coalition S. Let $\mathcal{R} = \{R(S) \mid S \subset N, S \neq \emptyset\}$ be the set of all of these payoffs. Each player $i \in S$ receives a multiple of the payoff of the whole coalition. Let $p \in \mathbb{R}^S$ be an allocation for coalition S. Player $i \in S$ receives $p_i R(S)$. The allocation p is *efficient* when

$$\sum_{i \in S} p_i = 1, \qquad \text{and } p_i \geq 0 \text{ for all } i.$$

Let $\Delta^*(S) := \{p \in \mathbb{R}^S \mid \sum_{i \in S} p_i = 1\}$ be the set of efficient allocations for coalition S.

An additional element of stochastic cooperative game theory (one that is not present in the classical version) is the notion of player preferences. By means of these preferences, players determine how to compare two stochastic payoffs. Let $\mathcal{A} := \{S \subset N \mid S \neq \emptyset,\ u_i(R(S)) \neq 0, \quad \forall i \in N\}$ be the set of coalitions for which the payoff is nonzero. Allocations are multiples of payoffs. Therefore, the set of all possible payoffs to players $i \in N$ is $\{pR(S) \mid p \in \mathbb{R}, i \in S \in \mathcal{A}\}$.

Players prefer some stochastic payoffs or costs over others. We use the notation $\succsim_i$ to denote a preference relation of player $i \in N$. If player i weakly prefers stochastic payoff X to Y, we write $X \succsim_i Y$. If player i strictly prefers X to Y, we write: $X \succ_i Y$. If $X \succsim_i Y$ *and* $Y \succsim_i X$ simultaneously, we write $X \sim_i Y$. This means that player i is indifferent to receiving X or Y.

In order that the stochastic ordering is in a certain sense "well-behaved", we demand that it satisfies a certain property. Note that at this point, we deviate slightly from the original formulation by Timmer et al. in [6]. We believe that they needlessly complicated matters when they tried to convey a desirable property for allocations of stochastic payments. The following is a reformulation of their original approach. Depending on the functions one chooses in this reformulation, it is possible to make this approach coincide with and yield the same results as their approach.

Property 1 *For each player $i \in N$ there exist pairs of utility functions $U_i : \mathcal{L}_+ \to \mathbb{R}$, and $u_i : \mathcal{L}_+ \to \mathbb{R}$ such that $U_i(p_i R(S)) = p_i u_i R(S)$.*

In other words, we require that the stochastic payoffs for coalitions are linear in p_i. There are many examples of preference relations that satisfy **Property 1**. Here we list three of them.

Example 1.1 Players can prefer random variables based on their expected values. So for player i we could have $X \succsim_i Y$ iff $\mathbb{E}(X) \geq \mathbb{E}(Y)$ for any payoffs X and Y. Here, $\mathbb{E}(X)$ denotes the expectation of X. If this is the case, we say that player i has *expectational preferences*. So here we have the utility functions $U_i(R(S)) = \mathbb{E}(R(S)) = u_i(R(S))$. These preferences satisfy Property 1.

Example 1.2 The following type of preference involves quantiles of random variables. For $\beta_i \in (0, 1)$, define $u^X_{\beta_i} := \sup\{t \in \mathbb{R} \mid \mathbb{P}(X \leq t) \leq \beta_i\}$. We refer to this value as the β_i-quantile of X. Assume that $0 < \beta_i < 1$ is such that $u^{R(S)}_{\beta_i} > 0$ for all $S \in \mathcal{A}$. Players can choose whichever value of β_i they like.

Furthermore, we define the utility function Q_{β_i} as follows:

$$Q_{\beta_i}(X) = \begin{cases} u^X_{\beta_i} & \text{if } \mathbb{E}(X) \geq 0 \\ u^X_{1-\beta_i} & \text{if } \mathbb{E}(X) < 0. \end{cases}$$

We say that player i has *quantile preferences* if $X \succsim_i Y$ if and only if $Q_{\beta_i}(X) \geq Q_{\beta_i}(Y)$. In this case, we define $U_i(R(S)) = Q_{\beta_i}(R(S))$. It is perhaps not that evident that all requirements of Property 1 are satisfied. We will therefore prove it.

Lemma 4.1.1 The quantile utility function satisfies Property 1.

Proof Let $q \in \mathbb{R}$. Then

$$\begin{aligned} u^{q\cdot R(S)}_{\beta_i} &= \sup\{t \in \mathbb{R} \mid \mathbb{P}(q \cdot R(S) \leq t) \leq \beta_i\} \\ &= \sup\{t \in \mathbb{R} \mid \mathbb{P}(R(S) \leq t/q) \leq \beta_i\}. \end{aligned}$$

Now, define $t' := t/q$. Then:

$$\begin{aligned} u^{q\cdot R(S)}_{\beta_i} &= \sup\{q \cdot t' \mid \mathbb{P}(R(S) \leq t') \leq \beta_i\} \\ &= q \sup\{t' \mid \mathbb{P}(R(S) \leq t') \leq \beta_i\} \\ &= q \cdot u^{R(S)}_{\beta_i}. \end{aligned}$$

This means that $Q_{\beta_i}(p_i R(S)) = p_i Q_{\beta_i}(R(S))$. So, if we define $u_i(\cdot) := Q_{\beta_i}(\cdot) =: U_i(\cdot)$, this yields a pair of functions that satisfies Property 1. $\square$

This type of preference is perhaps not as intuitively clear as the previous one. For more background and intuition behind the choice for this type of preference relation and how it relates to other notions of risk-averseness and risk-propensity, we refer to **Appendix A** (see 14.1) of this thesis. For the moment, it suffices to point out that the higher his chosen value for β_i is in a profit game, the more risk-seeking a player is (he aims to maximize his profits for the highest possible profit value there is). Conversely, lower values of β_i are associated with more risk-averse behavior. For the cost game, the situation is reversed: risk-seeking players choose a low value of β_i, whereas risk-averse players choose a high one.

Note that the value $\beta_i = \frac{1}{2}$, in particular, is important. For this value, the associated quantile is the *median* of the random variable X.

Example 1.3 The expectational preferences (see Example 1.1) are a special case of *von Neumann-Morgenstern preferences*. Player i is said to have von Neumann-Morgenstern preferences if there exists a utility function $w_i : \mathbb{R} \to \mathbb{R}$ such that $X \succsim Y$ if and only if $\mathbb{E}(w_i(X)) \geq \mathbb{E}(w_i(Y))$ for every X and Y. For expectational preferences, we may take $w_i(x) = x$. Suppose $w_i(x) = x^a$ for some $a \in \mathbb{Q}_{>0}$. Then, for $u_i(X) = \mathbb{E}(w_i(X))$, this implies that $u_i((R(S))) = \mathbb{E}(w_i(R(S))) = \mathbb{E}(R(S)^a)$, so we have $p_i u_i((R(S))) = p_i \mathbb{E}(R(S)^a)$. We require that $U_i(p_i R(S)) = p_i u_i(R(S)) = p_i \mathbb{E}(R(S)^a)$. This in turn implies that $U_i(X) = u_i(X)/p_i^{a-1}$. (Notice that the case of $a = 1$ coincides with Example 1.1).

4.2 The tracking function

In order to determine whether a player wishes to join a particular coalition, it is important that he is able to compare the payoffs of one coalition to another. In Property 1, we required that utility functions U_i with $i \in N$, satisfy the relationship $U_i(p_i R(S)) = p_i u_i(R(S))$.

As a consequence, for all players $i \in N$ there exists a unique number $\alpha_i = \alpha_i(S, T, p_i) \in \mathbb{R}$ such that $U_i(p_i R(S)) = U_i(\alpha_i R(T))$ for all $S, T \subset N$ and $p_i \in \mathbb{R}$. This number α_i is dependent on p_i, $R(S)$, $R(T)$, and U_i. In order to keep track of which α corresponds to these combinations of variables, we define the **tracking function** $\alpha_i : \mathcal{A} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$. By Property 1, the function that determines the unique value of α_i is such that $U_i(p_i R(S)) = p_i u_i(R(S)) = \alpha_i u_i(R(T)) = U_i(p_i R(T))$. We therefore find that

$$\alpha_i(S,T,p_i) = p_i \frac{u_i(R(S))}{u_i(R(T))}.$$

There is still, however, a minor problem if $u_i(R(T)) = 0$ for some coalition T. Then, our function $\alpha_i(\cdot,\cdot,\cdot)$ is not defined properly. We need to extend the domain of α_i in order to take into account the case in which $u_i(R(T)) = 0$. We do so by putting: $\alpha_i(S,T,p_i) = \infty$, if $u_i(R(T)) = 0$.

4.3 Comparison with the original formulation

As we mentioned before, Timmer et al. had quite a different approach concerning the comparison of two stochastic payoffs. In this subsection, we will first outline their approach. Afterwards, we will compare their description of how stochastic payoffs can be measured against one another with our new approach.

Instead of our Property 1, Timmer et al. proposed the following assumption regarding the way a player compares two payoffs.

Assumption *For each player $i \in N$ there exist functions $f_S^i : \mathbb{R} \to \mathbb{R}$, $S \in \mathcal{A}$, that are surjective, continuous and strictly monotone increasing, such that*

1. $f_S^i(t)R(S) \succsim_i f_T^i(t')R(T)$ *if and only if* $t \geq t'$,
2. $f_S^i(0) = 0$.

So in the approach of Timmer et al., if player i compares the payoffs $pR(S)$ and $qR(T)$ then $pR(S) \succsim_i qR(T)$ if and only if $t = (f_S^i)^{-1}(p) \geq t' = (f_T^i)^{-1}(q)$. One may interpret the function $(f_S^i)^{-1}$ as a utility function with respect to multiples of $R(S)$. The second part of the assumption, $f_S^i(0) = 0$, is a normalization condition. We can choose our $f_S^i(\cdot)$ in such a way that it satisfies the assumption above, and coincides with the examples we have given above.

For instance, suppose that the preferences of player i are such that $X \succsim_i Y$ if and only if $\mathbb{E}(X) \geq \mathbb{E}(Y)$ (as in Example 1.1 above). Then $f_S^i(t) = t/\mathbb{E}(R(S))$ ensures that $\succsim_i$ satisfies the assumption above.

Timmer et al. define the tracking function, too. As they have a different basic setup about the way stochastic payoffs are compared, the tracking function is consequentially found in a different manner, too. Again, we have a tracking function $\widetilde{\alpha}_i : \mathcal{A} \times \mathcal{A} \times \mathbb{R} \to \mathbb{R}$. By the above Assumption, we have $\widetilde{\alpha}_i(S,T,p) = f_T^i((f_S^i)^{-1}(p))$. In the case of expectational preferences (as described above), it holds that $(f_S^i)^{-1}(p) = p \cdot \mathbb{E}(R(S))$, so

$$\begin{aligned}\alpha_i(S,T,p) &= f_T^i((f_S^i)^{-1}(p)) \\ &= f_T^i(p \cdot \mathbb{E}(R(S))) \\ &= p\frac{\mathbb{E}(R(S))}{\mathbb{E}(R(T))}.\end{aligned}$$

So the tracking functions coincide in this case.

However, consider the case of the von-Neumann-Morgenstern preferences (as we described for our model in Example 1.3). If we have $u_i(x) = x^n$, then the only plausible choice for our $f_S^i(\cdot)$ would be

$$f_S^i(t) = \begin{cases} \frac{t^{1/n}}{(\mathbb{E}(u_i(R(S))))^{1/n}}, & \text{if } t \geq 0 \\ -\frac{t^{1/n}}{(\mathbb{E}(u_i(R(S))))^{1/n}}, & \text{if } t < 0. \end{cases}$$

This results in the following tracking function:

$$\alpha(S,T,p) = \begin{cases} p\frac{(\mathbb{E}(u_i(R(S))))^{1/n}}{(\mathbb{E}(u_i(R(T))))^{1/n}} & \text{if } p \geq 0 \\ -p\frac{(\mathbb{E}(u_i(R(S))))^{1/n}}{(\mathbb{E}(u_i(R(T))))^{1/n}} & \text{if } p < 0. \end{cases}$$

Let us focus on the case $p > 0$. So here we have the expectation of powers of random variables in the tracking function, just like in our reformulation. However, a difference is the fractional powers of the expectational values. These do not coincide with our tracking function in Example 1.3. We could choose different expressions of $U_i(\cdot)$ and $u_i(\cdot)$ in order to make the tracking function coincide with the one found above. However, we don't see why the additional fractional power should be included in the case von Neumann-Morgenstern preferences. As can be seen in our exposition of this type of preference in Section 8, the von Neumann-Morgenstern utility functions do not seem to require this additional fractional exponent. We therefore hold on to our own interpretation of the von Neumann-Morgenstern preferences and their corresponding tracking functions. We will use the tracking functions as described in Example 1.3 to study stochastic cooperative games in which players have different attitudes towards risk.

Furthermore, we believe our new formulation of stochastic cooperative games, and in particular the way the stochastic payoffs are compared by players according to their own preferences, is easier to understand and more intuitive than the description by Timmer et al. [6]. So, although our approach is based on the insights by Timmer et al., we will work with our definition of preference relations to apply it to different examples of the stochastic cooperative energy prediction games in the subsequent sections of this thesis.

We conclude this section by stating the following theorem. Thanks to our reformulation of Property 1 (Property 2.1 in [6]) and the resulting simplification of the tracking function, it becomes almost trivial to prove it. For the original theorem and its proof, we refer to [6], p. 5 and 6.

Theorem 4.3.1 *For all $i \in N$, the following three statements hold:*

1. $\alpha_i(S, S, h) = h$ *for all* $h \in \mathbb{R}$, $S \in \mathcal{A}$,
2. $\alpha_i(T, U, \alpha_i(S, T, p)) = \alpha_i(S, U, p)$ *for any* $p \in \mathbb{R}$ *and* $S, T, U \in \mathcal{A}$,
3. $\alpha_i(S, T, p) = p \cdot \alpha_i(S, T, 1)$ *for any* $p \in \mathbb{R}$ *and* $S, T \in \mathcal{A}$.

Proof

1. We have: $\alpha_i(S, S, h) = h\frac{u_i(R(S))}{u_i(R(S))} = h$.
2. Furthermore, the following equalities hold:

$$\begin{aligned}\alpha_i(T, U, \alpha_i(S, T, p)) &= \alpha_i\Big(T, U, p\frac{u_i(R(S))}{u_i(R(T))}\Big) \\ &= p\frac{u_i(R(S))}{u_i(R(T))} \cdot \frac{u_i(R(T))}{u_i(R(U))} \\ &= p\frac{u_i(R(S))}{u_i(R(U))}.\end{aligned}$$

3. We see that

$$\begin{aligned}\alpha_i(S, T, p) &= p\frac{u_i(R(S))}{u_i(R(T))} \\ &= p \cdot \alpha_i(S, T, 1). \qquad \square\end{aligned}$$

4.4 Relationship with classical cooperative game theory

In a sense, stochastic cooperative game theory introduced by Timmer et al. is both a generalization and a reformulation of classical cooperative game theory. Let us first focus on the former characterization. Timmer et al.'s model is a generalization because any deterministic cooperative game with N players and a characteristic function $v : 2^N \to \mathbb{R}$ can be embedded in the stochastic framework by defining $u_i(R(S)) = v(S)$. This consequentially means that $\alpha_i(S, T, p) = p\frac{v(S)}{v(T)}$. If all coalitional values are deterministic (meaning that $\mathbb{P}(R(S) = q) = 1$ for some $q \in \mathbb{R}$), then the game becomes a "regular" classical cooperative game.

The only difference between the "regular" game and this game, is that is still a reformulation within this new framework. Whereas before, we would have allocations x with $\sum_{i\in N} x_i = v(N) \neq 0$

among the players, and player i receives a share of x_i of the grand total. In the new framework, this is equivalent to player i receiving the fraction $p_i = \frac{x_i}{v(N)}$ of the total payoff $v(N)$. We could therefore say that before, the allocations were given in *absolute* terms, whereas they are now stated *relative* to the allocation for the grand coalition N. In the following section, we will give a concrete example of a game, in which it becomes clear that the two formulations of cooperative game theory are equivalent in the case of deterministic payoffs.

5 The model

In this section, we recall definitions of key concepts from Section 4. Afterwards, we describe notions such as the imputation set and the core. In Section 11.2, we describe the Shapley value too.
The tuple $(N, \mathcal{R}, \mathcal{A}, \alpha)$ gives a complete description of a stochastic cooperative game. Here, we have:

- $N = \{1, 2, \dots, n\}$ is the player set,
- $\mathcal{R} := \{R(S) \mid S \subset N, S \neq \emptyset\}$ is the set of coalitional payoffs,
- $\mathcal{A}$ is the set of coalitions with a nonzero payoff (as defined in section 4.1),
- $\alpha = (\alpha_i)_{i \in N}$ is the collection of tracking functions of all players.

We will now use this terminology to generalize notions of classical cooperative game theory in terms of their stochastic counterpart. As mentioned in the previous section, an allocation for coalition S is a vector $p \in \mathbb{R}^S$ in which each player i receives a fraction p_i of the random payoff $R(S)$.

If we demand that an individual player i gains more benefits from cooperating within the coalition S than operating on his own, the allocation $\alpha_i R(S)$ for coalition S must be 'larger' than the allocation $p_i R(\{i\}) = 1R(\{i\})$. In this case, $p_i = 1$ because an individual player – a singleton coalition – does not share the profits with any other player. Therefore, an allocation p for a coalition S will be called *individually rational* if $p_i \geq \alpha_i(\{i\}, S, 1)$ for all players $i \in S$. Let $IR(S)$ be the set of all efficient individual rational allocations for coalitions S.

When an allocation is both individually rational and efficient, we call such an allocation an *imputation*. The *imputation set* is defined as follows:

$$I(N, \alpha) = \{p \in \Delta^*(N) \mid p_i \geq \alpha_i(\{i\}, N, 1) \text{ for all players } i \in N\}.$$

Note that $I(N, \alpha) = IR(N)$. Please also observe that this is the imputation set for the profit game. For the cost game, the inequalities are reversed.

We now introduce the concept of a dominated set. This concept is related tot the notion of the core in stochastic cooperative games, which is why it is important. The set $\text{dom}(S)$ contains those allocations for coalition N that are dominated by coalition S. This means there exists an allocation $q \in \Delta^*(S)$ that is strictly preferred by all members of S:

$$\text{dom}(S) = \{p \in \mathbb{R}^N \mid \exists q \in \Delta^*(S) : \ \alpha_i(S, N, q_i) > p_i \text{ for all } i \in S\}.$$

Those allocations for N that are *not* dominated by some allocation for S belong to the *core* of the game. Therefore, we can define the core as follows:

$$C(N, \alpha) = \{p \in \Delta^*(N) \mid p \notin \text{dom}(S) \quad \forall S \subset N\}.$$

This brings us to the following simpler characterization.

Lemma 5.0.1 *It holds that*

$$C(N, \alpha) := \Big\{p \in \Delta^*(N) \mid \sum_{i \in S} \frac{p_i}{\alpha_i(S, N, 1)} \geq 1 \quad \forall S \subset N\Big\}.$$

Proof We will prove this by showing that both inclusions $C(N, \alpha) \subseteq L$ and $L \subseteq C(N, \alpha)$ hold. Define $L := \Big\{p \in \Delta^*(N) \mid \sum_{i \in S} \frac{p_i}{\alpha_i(S,N,1)} \geq 1 \quad \forall S \subset N\Big\}$.

- $L \subseteq C(N, \alpha)$
  Let $p \in L$. Then, $\sum_{i \in S} \frac{p_i}{\alpha_i(S,N,1)} \geq 1 \quad \forall S \subset N$. Suppose that $p \in \text{dom}(S)$. This means there exists a q such that $\alpha_i(S, N, q_i) > p_i \quad \forall i \in S$. By part 3 of Theorem 4.3.1, this is equivalent to $q_i \alpha_i(S, N, 1) > p_i \quad \forall i$. This in turn means that $q_i > p_i / \alpha_i(S, N, 1)$ for all i. If we sum both sides over all $i \in S$, we obtain $1 = \sum_{i \in S} q_i > \sum_{i \in S} p_i / \alpha_i(S, N, 1) \geq 1$. This is a contradiction, thus $p \notin \text{dom}(S)$.

- $C(N,\alpha) \subseteq L$
  Let $p \in C(N,\alpha)$. Suppose now that $p \notin L$. This means $\exists S \subset N$ such that $\gamma := \sum_{i\in S} p_i/\alpha_i(S,N,1) < 1$. Define q with $\epsilon = (1-\gamma)/|S|$ such that $q_i = p_i/\alpha_i(S,N,1)+\epsilon$. Then $\sum_{i\in S} q_i = 1$. As $\gamma < 1$, we have $\epsilon > 0$. But this means we have found a q such that $q_i > p_i/\alpha_i(S,N,1)$ for all $i \in S$. By part 3 of Theorem 4.3.1, this means that $p \in \text{dom}(S)$. This contradicts $p \in C(N,\alpha)$. Therefore, the assumption that $p \notin L$ was not correct, and we have proved what we needed to. Thus, $C(N,\alpha) = L$. □

We will later define the Shapley value for stochastic cooperative games. For now, the core suffices.

Notice that the above characterization of the core is related to the *profit* maximization version of the game. We will focus on the version with (positive) *costs*. Players aim to minimize these costs. In this case, the inequalities for the core are flipped.

5.1 Example for comparison with the classical model

Let us consider a simple, deterministic game to compare the new framework with the classical one. Consider the following game with characteristic function $v(\cdot)$:

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	1	2	2	4	5	5	9

The core of this game is given by:

$$C(v) = \left\{ x \in \mathbb{R}^N \middle| \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\}$$
$$= \left\{ x \in \mathbb{R}^N \middle| \sum_{i\in N} x_i = 9, x_1 \geq 1, x_2 \geq 2, x_3 \geq 2, x_1+x_2 \geq 4, x_1+x_3 \geq 5, x_2+x_3 \geq 5 \right\}.$$

We can also compute the core within the new framework when all players have deterministic preferences. So, $\alpha_i(S,T,p) = p\frac{v(S)}{v(T)}$ for $i = 1,2,3$. It is equivalent to the "previous" core and it is given by:

$$C(N,\alpha) = \left\{ p \in \Delta^*(N) \;\middle|\; p_S \notin \text{dom}(S) \text{ for all coalitions } S \in 2^N \right\}$$
$$= \left\{ p \in \Delta^*(N) \;\middle|\; \sum_{i\in S} \frac{p_i}{\alpha_i(S,N,1)} \geq 1 \text{ for all coalitions } S \in 2^N \right\}$$
$$= \left\{ p \in \Delta^*(N) \mid \frac{p_1}{1/9} \geq 1,\ \frac{p_2}{2/9} \geq 1,\ \frac{p_3}{2/9} \geq 1,\ \frac{p_1}{4/9} + \frac{p_2}{4/9} \geq 1,\ \frac{p_1}{5/9} + \frac{p_3}{5/9} \geq 1,\ \frac{p_2}{5/9} + \frac{p_3}{5/9} \geq 1 \right\}$$
$$= \left\{ p \in \Delta^*(N) \mid p_1 \geq \frac{1}{9},\ p_2 \geq \frac{2}{9},\ p_3 \geq \frac{2}{9},\ p_1+p_2 \geq \frac{4}{9},\ p_1+p_3 \geq \frac{5}{9},\ p_2+p_3 \geq \frac{5}{9} \right\}.$$

5.2 Example with stochastic payoffs

Consider a situation in which there are three musicians: a drummer, a guitarist, and a piano player. These three musicians each have the option to either give a musical performance by themselves, or work together with one or both of the other musicians. They perform on the street, so their income depends on the generosity of the people passing by. They therefore don't know in advance how much a performance will yield in terms of monetary value. However, they can gauge their approximate incomes based on past performances. Their income levels from past street concerts tells them, that the income from a performance is distributed uniformly. For each type of collaborative effort with the other musicians, this is the case, too. Their yields can be summarized as follows, with the drummer being player 1, the guitarist player 2, and the piano player player 3:

$$R(S) = \begin{cases} U[0,2] & \text{if } S = \{1\} \\ U[1,4] & \text{if } S = \{2\} \\ U[0,4] & \text{if } S = \{3\} \\ U[5,8] & \text{if } S = \{1,2\} \\ U[2,7] & \text{if } S = \{1,3\} \\ U[6,8] & \text{if } S = \{2,3\} \\ U[18,22] & \text{if } S = \{1,2,3\} \end{cases}$$

Suppose furthermore, that player 1 has expectational preferences, player 2 has quantile preferences with $\beta_2 = 0.6$, and player 3 has quantile preferences with $\beta_3 = 0.9$.
For this stochastic game, we can calculate the imputation set and the core. Note that for a uniformly distributed random variable X on $[a,b]$, we have $F(x) = P(X \leq x) = (x-a)/(b-a)$. So the quantile function is $Q_\beta(U[a,b]) = \beta \cdot (b-a) + a$. From the definition of the imputation set in Section 5, we obtain that:

$$\begin{aligned} p_1 \geq \alpha_1(\{1\}, N, 1) &= \frac{\mathbb{E}(R(\{1\}))}{\mathbb{E}(R(N))} = \frac{1/2(0+2)}{1/2(18+22)} = \frac{1}{20} \\ p_2 \geq \alpha_2(\{1\}, N, 1) &= \frac{Q_{0.6}(U[1,4])}{Q_{0.6}(U[18,22])} = \frac{0.6(4-1)+1}{0.6(22-18)+18} = \frac{2.8}{20.4} = \frac{7}{51} \\ p_3 \geq \alpha_3(\{3\}, N, 1) &= \frac{Q_{0.9}(U[0,4])}{Q_{0.9}(U[18,22])} = \frac{0.9(4-0)+0}{0.9(22-18)+18} = \frac{3.6}{21.6} = \frac{1}{6}. \end{aligned}$$

Therefore, the imputation set is

$$I(N,\alpha) = \left\{ p \in \Delta^*(N) \mid p_1 \geq \frac{1}{20}, p_2 \geq \frac{7}{51}, p_3 \geq \frac{1}{6} \right\}$$

We can also calculate the core using Lemma 5.0.1. Consider the subset $S = \{1,2\}$. We must have $\sum_{i \in S} p_i/\alpha_i(S,N,1) = p_1/\alpha_1(\{1,2\},N,1) + p_2/\alpha_2(\{1,2\},N,1) \geq 1$. We derive that

$$\begin{aligned} \alpha_1(\{1,2\}, N, 1) &= \frac{\mathbb{E}(R(\{1,2\}))}{\mathbb{E}(R(N))} = \frac{1/2(5+8)}{1/2(18+22)} = \frac{13}{40} \\ \alpha_2(\{1,2\}, N, 1) &= \frac{0.6(8-5)+5}{0.6(22-18)+18} = \frac{1}{3}. \end{aligned}$$

So for this particular subset, we obtain the inequality

$$\frac{40}{13} p_1 + 3p_2 \geq 1.$$

Similarly, we can deduce the other inequalities from the other subsets, which yields the core:

$$C(N,\alpha) = \left\{ p \in I(N,\alpha) \mid \frac{40}{13} p_1 + 3p_2 \geq 1, \frac{40}{9} p_1 + \frac{216}{65} p_3 \geq 1, \frac{17}{6} p_2 + \frac{108}{19} p_3 \geq 1 \right\},$$

as $I(N,\alpha) \subset C(N,\alpha)$.

6 An energy prediction game

In most countries in which wind mills, solar panels and other intermittent renewable energy sources generate energy and deliver that to the main grid, the owners of these renewable energy sources are required to predict how much energy their sources will produce. These predictions have to be communicated some time period ahead to the transmission system operator (TSO). After these predictions have been sent, participants in the market are then financially responsible for any deviation from the contract. Deviations of production from prediction levels need to be compensated by the producers by paying imbalance prices for the amount of energy over or underdelivered [19]. In the Netherlands, the imbalance prices fluctuate wildly and differ day by day. Furthermore, the imbalance price could differ significantly depending on there being overproduction or underproduction. In Spain, however, the imbalance prices for positive and negative imbalances are equal. To make the problem slightly more tractable, we assume the Spanish situation in which the imbalance costs for overproduction and underproduction are equal [19]. This in turn means that the amount of imbalance costs to be paid are directly proportional to the amount of energy that is less or more than the predicted amount of energy.

The distribution of the normalized forecast error (the normalized deviation of the amount of energy produced) of wind mills was the topic of investigation by Hodge et al. in [12]. (Roughly similar results are found on p. 12 of [7] and Figures 12 and 13 of [1].) The distribution is as follows:

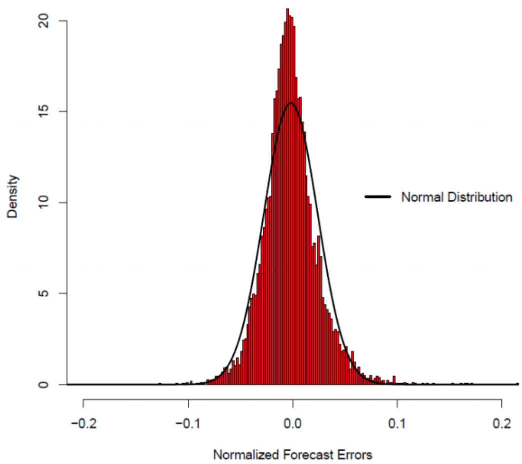

Figure 1: The normalized day-ahead prediction error distribution of a wind mill

In Section 3.2 of their article, Hodge et al. propose that the hyperbolic distribution most accurately models the distribution error. The hyperbolic distribution, however, is quite hard to work with. It is a complicated distribution involving Bessel functions. Furthermore, it is not known whether a closed form expression for the convolution of two independent hyperbolic distributions can be computed.

As can be inferred from Figure 1, the normal distribution is a rough estimate of the prediction error. Though it is not perfect, we will from now on assume that the prediction error has a normal distribution. We choose this distribution, because it is more or less accurate and because much more about this distribution is known. It is easier to handle than a hyperbolic distribution.

Thus, we assume that the normalized prediction error for a renewable energy source has a normal distribution with mean μ and standard deviation σ. The value of σ changes according to the changing time frame of the prediction. In the day-ahead case, for which the forecast error density is plotted in Figure 1 for wind mills, we have $\sigma = 0,026$. Furthermore, Hodge et al. experimentally found that $\mu = 0,002$. We consider this to be negligible compared to the value of σ and close enough to zero. Therefore, we assume that $\mu = 0$.

In the previous paragraphs, we established that the imbalance costs are proportional to the forecasting error. The larger the forecasting error, the larger the cost. If the prediction error X has a normal distribution $N(0, \sigma^2)$, Y has a so-called "half-normal" distribution, $H(0, \sigma^2)$, with probability density function

$$f_Y(y,\sigma) = \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}e^{-\frac{y^2}{2\sigma^2}}, \qquad y \geq 0.$$

The following plot [15] illustrates the difference between the probability density functions of the normal and associated half-normal distributions:

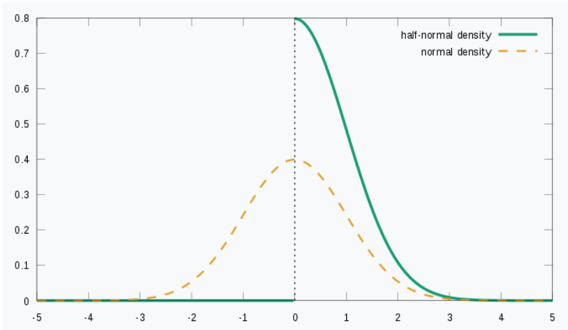

Figure 2: A standard normal distribution and the corresponding half-normal distribution

The corresponding cumulative distribution function of the half-normal distribution is

$$F_Y(y,\sigma) = \operatorname{erf}\left(\frac{y}{\sqrt{2}\sigma}\right) = \frac{2}{\sqrt{\pi}}\int_0^{y/(\sqrt{2}\sigma)} \exp(-z^2)dz, \qquad y \geq 0.$$

Let Y_1, Y_2 be independent, half-normally distributed random variables with unequal variances. So, $Y_1 \sim H(0,\sigma_1^2)$ and $Y_2 \sim H(0,\sigma_2^2)$, where $\sigma_1 \neq \sigma_2$. Then the costs for the prediction cost game for two players owning a renewable energy source are as follows:

$$R(S) = \begin{cases} Y_1 & \text{if } S = \{1\} \\ Y_2 & \text{if } S = \{2\} \\ Y_1 + Y_2 & \text{if } S = \{1,2\}. \end{cases}$$

In order to figure out the coalitional stochastic value $R(\{1,2\})$, we have to compute the convolution of two half-normal distributions with unequal variances. Furthermore, we also need the cumulative distribution function if we use quantile utility functions (which we saw in Example 1.2).

It turns out that the following result is useful for the computation of the aforementioned cumulative distribution function (which can be found in [3], albeit with a minor mistake – a sign error – in the fourth equality of the derivation).

Lemma 6.0.1

$$\int_0^z e^{-a^2x^2}\operatorname{erf}(bx)\,dx = \frac{\tan^{-1}(b/a)}{a\sqrt{\pi}} - \frac{2\sqrt{\pi}}{a}T\left(\sqrt{2}az, b/a\right),$$

where $T(x,m)$ is Owen's T-function, defined as $T(x,m) := \frac{1}{2\pi}\int_0^m \frac{e^{-(1/2)x^2(1+y^2)}}{1+y^2}dy$.

Proof Let

$$\begin{aligned} h(z) &:= \int_0^z e^{-a^2x^2}\operatorname{erf}(bx)\,dx \\ &= \frac{2}{\sqrt{\pi}}\int_0^z e^{-a^2x^2}\left(\int_0^{bx} e^{-t^2}dt\right)dx. \end{aligned}$$

We now perform the substitution $y := t/x$. This means that $dy/dt = 1/x$, so $dy = dt/x$. In other words, $t = yx$ and $dt = xdy$. Substituting these values in the previous integrals gives:

$$
\begin{aligned}
h(z) &= \frac{2}{\sqrt{\pi}} \int_0^z e^{-a^2x^2} \left(\int_0^b e^{-(yx)^2} x \, dy \right) dx \\
&= \frac{2}{\sqrt{\pi}} \int_0^b \left(\int_0^z e^{-(a^2+y^2)x^2} x dx \right) dy.
\end{aligned}
$$

For the inner integral, we make the substitution $u := x^2$. This means $du/dx = 2x$, so $dx = du/2x$. Thus:

$$
\begin{aligned}
h(z) &= \frac{2}{\sqrt{\pi}} \int_0^b \int_0^{z^2} \frac{1}{2} e^{-(a^2+y^2)u} du dy \\
&= \frac{1}{\sqrt{\pi}} \int_0^b \left[\frac{1}{-(a^2+y^2)} e^{-(a^2+y^2)u} \right]_0^{z^2} dy \\
&= \frac{1}{\sqrt{\pi}} \int_0^b \frac{1 - e^{-(a^2+y^2)z^2}}{a^2+y^2} dy \\
&= \frac{1}{a\sqrt{\pi}} \left[\tan^{-1}(b/a) - 2\pi T(\sqrt{2}az, b/a) \right]. \qquad \square
\end{aligned}
$$

We are now ready to prove the following theorem.

Theorem 6.0.2 *Let $Z = Y_1 + Y_2$, with $Y_1 \sim H(0, \sigma_1^2)$ and $Y_2 \sim H(0, \sigma_2^2)$, where Y_1 and Y_2 are independent. Let $\sigma_Z := \sqrt{\sigma_1^2 + \sigma_2^2}$. Then:*

$$
F_Z(z) = \frac{\sqrt{2}}{\sigma_Z \sqrt{\pi}} \left[\left(\frac{\sigma_Z \sqrt{2} \tan^{-1}\left(\frac{\sigma_2}{\sigma_1}\right)}{\sqrt{\pi}} - 2\sigma_Z \sqrt{2\pi} T\left(\frac{z}{\sigma_Z}, \frac{\sigma_2}{\sigma_1}\right) \right) + \left(\frac{\sigma_Z \sqrt{2} \tan^{-1}\left(\frac{\sigma_1}{\sigma_2}\right)}{\sqrt{\pi}} - 2\sigma_Z \sqrt{2\pi} T\left(\frac{z}{\sigma_Z}, \frac{\sigma_1}{\sigma_2}\right) \right) \right].
$$

Proof
Using the definition of the convolution, we start with:

$$
\begin{aligned}
f_Z(z) &= \int_0^z f_{Y_2}(z-x) \cdot f_{Y_1}(x) dx \\
&= \frac{2}{\sigma_1\sigma_2\pi} \int_0^z \exp\left(\frac{-(z-x)^2}{2\sigma_2^2} \right) \exp\left(\frac{-x^2}{2\sigma_1^2} \right) dx \\
&= \frac{2}{\sigma_1\sigma_2\pi} \int_0^z \exp\left(\frac{-(x^2 - 2xz + z^2)}{2\sigma_2^2} \right) \exp\left(\frac{-x^2}{2\sigma_1^2} \right) dx \\
&= \frac{2}{\sigma_1\sigma_2\pi} \exp\left(\frac{-z^2}{2\sigma_2^2} \right) \int_0^z \exp\left(\frac{-x^2 + 2xz}{2\sigma_2^2} \right) \exp\left(\frac{-x^2}{2\sigma_1^2} \right) dx \\
&= \frac{2}{\sigma_1\sigma_2\pi} \exp\left(\frac{-z^2}{2\sigma_2^2} \right) \int_0^z \exp\left(\frac{-(\sigma_1^2+\sigma_2^2)x^2 + 2z\sigma_1^2 x}{2\sigma_1^2\sigma_2^2} \right) dx.
\end{aligned}
$$

Using our definition of σ_Z, we obtain:

$$
f_Z(z) = \frac{2}{\sigma_1\sigma_2\pi} \exp\left(\frac{-z^2}{2\sigma_2^2} \right) \int_0^z \exp\left(\frac{-\sigma_Z^2 x^2 + 2z\sigma_1^2 x}{2\sigma_1^2\sigma_2^2} \right) dx.
$$

We now use the method of completing the square to re-write this expression in a more convenient form. This means we write the expression in the numerator of the exponent within the integral, which is of the form $ax^2 + bx$ (in our case: $a = -\sigma_Z^2$, $b = 2z\sigma_1^2$), and transform it into something of the form $a(x-h)^2 + k$. In our case, we find $h = -b/(2a) = (z\sigma_1^2)/(\sigma_Z^2)$ and $k = -(b^2)/(4a) = (z^2\sigma_1^4)/(\sigma_Z^2)$. Then:

$$f_Z(z) = \frac{2}{\sigma_1\sigma_2\pi}\exp\left(\frac{-z^2}{2\sigma_2^2}\right)\int_0^z \exp\left(\frac{-\sigma_Z^2\left(x-\frac{z\sigma_1^2}{\sigma_Z^2}\right)^2+\frac{(z\sigma_1^2)^2}{\sigma_Z^2}}{2\sigma_1^2\sigma_2^2}\right)dx$$
$$= \frac{2}{\sigma_1\sigma_2\pi}\exp\left(\frac{-z^2}{2\sigma_2^2}\right)\exp\left(\frac{z^2\sigma_1^2}{2\sigma_1^2\sigma_2^2\sigma_Z^2}\right)\int_0^z \exp\left(\frac{-\sigma_Z^2\left(x-\frac{z\sigma_1^2}{\sigma_Z^2}\right)^2}{2\sigma_1^2\sigma_2^2}\right)dx$$
$$= \frac{2}{\sigma_1\sigma_2\pi}\exp\left(\frac{-z^2(\sigma_Z^2-\sigma_1^2)}{2\sigma_2^2\sigma_Z^2}\right)\int_0^z \exp\left(\frac{-\sigma_Z^2\left(x-\frac{z\sigma_1^2}{\sigma_Z^2}\right)^2}{2\sigma_1^2\sigma_2^2}\right)dx$$
$$= \frac{2}{\sigma_1\sigma_2\pi}\exp\left(\frac{-z^2}{2\sigma_Z^2}\right)\int_0^z \exp\left(\frac{-\sigma_Z^2\left(x-\frac{z\sigma_1^2}{\sigma_Z^2}\right)^2}{2\sigma_1^2\sigma_2^2}\right)dx.$$

We multiply this last expression by $1 = \sqrt{2\pi\left(\frac{\sigma_1\sigma_2}{\sigma_Z}\right)^2}/\sqrt{2\pi\left(\frac{\sigma_1\sigma_2}{\sigma_Z}\right)^2}$, where we put the numerator outside the integral and the denominator inside [10]. This yields:

$$f_Z(z) = \frac{2}{\sigma_1\sigma_2\pi}\exp\left(\frac{-z^2}{2\sigma_Z^2}\right)\sqrt{2\pi\left(\frac{\sigma_1\sigma_2}{\sigma_Z}\right)^2}\int_0^z \frac{1}{\sqrt{2\pi\left(\frac{\sigma_1\sigma_2}{\sigma_Z}\right)^2}}\exp\left(\frac{-\left(x-\frac{z\sigma_1^2}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_Z}\right)^2}\right)dx$$
$$= \frac{2\sqrt{2}}{\sigma_Z\sqrt{\pi}}\exp\left(\frac{-z^2}{2\sigma_Z^2}\right)\int_0^z \frac{1}{\sqrt{2\pi\left(\frac{\sigma_1\sigma_2}{\sigma_Z}\right)^2}}\exp\left(\frac{-\left(x-\frac{z\sigma_1^2}{\sigma_Z^2}\right)^2}{2\left(\frac{\sigma_1\sigma_2}{\sigma_Z}\right)^2}\right)dx.$$

The expression in the integral represents the probability density function of a normal distribution with mean $z\sigma_1^2/\sigma_Z^2$ and standard deviation $\sigma_1\sigma_2/\sigma_Z$. If we define $\Phi(x, m, s)$ as the value of the cumulative normal distribution value at x with mean m and standard deviation s, this means that:

$$f_Z(z) = \frac{2\sqrt{2}}{\sigma_Z\sqrt{\pi}}\exp\left(\frac{-z^2}{2\sigma_Z^2}\right)\left[\Phi\left(z, \frac{z\sigma_1^2}{\sigma_Z^2}, \frac{\sigma_1\sigma_2}{\sigma_Z}\right) - \Phi\left(0, \frac{z\sigma_1^2}{\sigma_Z^2}, \frac{\sigma_1\sigma_2}{\sigma_Z}\right)\right].$$

In order to find the cumulative distribution function, we need to integrate the expression above from 0 to z. Let $\Phi(x, 0, 1) := \Phi(x)$. This is notation that is widely used to denote the cumulative standard normal distribution (in which $m = 0$ and $\sigma = 1$). We will use the identity $\Phi(z) = \frac{1}{2}(1 + \text{erf}(\frac{z}{\sqrt{2}}))$, where $\text{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$. This gives us:

$$F_Z(z) = \int_0^z f_Z(x)dx$$
$$= \frac{2\sqrt{2}}{\sigma_Z\sqrt{\pi}}\int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\left[\Phi\left(x, \frac{x\sigma_1^2}{\sigma_Z^2}, \frac{\sigma_1\sigma_2}{\sigma_Z}\right) - \Phi\left(0, \frac{x\sigma_1^2}{\sigma_Z^2}, \frac{\sigma_1\sigma_2}{\sigma_Z}\right)\right]dx$$
$$= \frac{2\sqrt{2}}{\sigma_Z\sqrt{\pi}}\int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\left[\frac{1}{2}\left(1+\text{erf}\left(\frac{x-\frac{x\sigma_1^2}{\sigma_Z^2}}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right)\right) - \frac{1}{2}\left(1+\text{erf}\left(\frac{-\frac{x\sigma_1^2}{\sigma_Z^2}}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right)\right)\right]dx$$
$$= \frac{\sqrt{2}}{\sigma_Z\sqrt{\pi}}\int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\left[\text{erf}\left(\frac{x-\frac{x\sigma_1^2}{\sigma_Z^2}}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right) - \text{erf}\left(\frac{-\frac{x\sigma_1^2}{\sigma_Z^2}}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right)\right]dx.$$

Splitting the above expression into the difference of two integrals yields:

$$F_Z(z) = \frac{\sqrt{2}}{\sigma_Z\sqrt{\pi}}\left[\int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(\frac{x-\frac{x\sigma_1^2}{\sigma_2^2}}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right)dx - \int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(\frac{-\frac{x\sigma_1^2}{\sigma_Z^2}}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right)dx\right]$$

$$= \frac{\sqrt{2}}{\sigma_Z\sqrt{\pi}}\left[\int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(\frac{x\left(1-\frac{\sigma_1^2}{\sigma_Z^2}\right)}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right)dx - \int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(\frac{-\frac{x\sigma_1^2}{\sigma_Z^2}}{\frac{\sigma_1\sigma_2}{\sigma_Z}\sqrt{2}}\right)dx\right]$$

$$= \frac{\sqrt{2}}{\sigma_Z\sqrt{\pi}}\left[\int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(x\frac{\sigma_2}{\sqrt{2}\sigma_Z\sigma_1}\right)dx - \int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(-x\frac{\sigma_1}{\sqrt{2}\sigma_Z\sigma_2}\right)dx\right].$$

The error function is an odd function. This means that $\operatorname{erf}(-x) = -\operatorname{erf}(x)$. Therefore, we obtain:

$$F_Z(z) = \frac{\sqrt{2}}{\sigma_Z\sqrt{\pi}}\left[\int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(x\frac{\sigma_2}{\sqrt{2}\sigma_Z\sigma_1}\right)dx + \int_0^z \exp\left(\frac{-x^2}{2\sigma_Z^2}\right)\operatorname{erf}\left(x\frac{\sigma_1}{\sqrt{2}\sigma_Z\sigma_2}\right)dx\right].$$

Now we can apply Lemma 6.0.1 to both integrals, as our expression is of the form:

$$F_Z(z) = \frac{\sqrt{2}}{\sigma_Z\sqrt{\pi}}\left[\int_0^z \exp(-a^2x^2)\operatorname{erf}(b_1x)dx + \int_0^z \exp(-a^2z^2)\operatorname{erf}(b_2x)dx\right],$$

with $a = 1/(\sigma_Z\sqrt{2})$, $b_1 = \sigma_2/(\sqrt{2}\sigma_Z\sigma_1)$, and $b_2 = \sigma_1/(\sqrt{2}\sigma_Z\sigma_2)$. This gives us the desired expression for $F_Z(\cdot)$. □

Though our result gives us insight in the cumulative distribution of the convolution of two half-normal random variables, the final expression is still a bit complicated. In order to make further calculations less cumbersome, we will consider the case in which the standard deviations of the two half-normal variables are equal to one another.

Corollary 6.0.2.1 *Let $Y_1, Y_2 \sim H(0,\sigma)$ be independent and let $Z = Y_1 + Y_2$ as before. Then $F_Z(z) = \operatorname{erf}\left(\frac{z}{2\sigma}\right)^2$.*

Proof Use the previous theorem and plug in the values $\sigma_1 = \sigma_2 = \sigma$. Then we find:

$$F_Z(z) = \frac{1}{\sigma\sqrt{\pi}}\left[\left(\frac{2\sigma\tan^{-1}(1)}{\sqrt{\pi}} - 4\sigma\sqrt{\pi}T\Big(\frac{z}{\sigma\sqrt{2}},1\Big)\right) + \left(\frac{2\sigma\tan^{-1}(1)}{\sqrt{\pi}} - 4\sigma\sqrt{\pi}T\Big(\frac{z}{\sigma\sqrt{2}},1\Big)\right)\right]$$

$$= \frac{4\tan^{-1}(1)}{\pi} - 8T\Big(\frac{z}{\sigma\sqrt{2}},1\Big)$$

$$= 1 - 8T\Big(\frac{z}{\sqrt{2}\sigma},1\Big).$$

We now use Property 2.3 of Table II of [17], which on p. 414 states that

$$T(z,1) = \frac{1}{2}\Phi(z)\big[1-\Phi(z)\big].$$

This implies that

$$F_Z(z) = 1 - 8\left[\frac{1}{2}\Phi\Big(\frac{z}{\sigma\sqrt{2}}\Big)\Big(1-\Phi\Big(\frac{z}{\sigma\sqrt{2}}\Big)\Big)\right]$$

$$= \left[2\Phi\Big(\frac{z}{\sqrt{2}\sigma}\Big) - 1\right]^2.$$

Now we use the following identity relating the cumulative normal distribution function and the error function: $\Phi(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$. Substituting this identity in the above expression gives:

$$F_Z(z) = \Big[2\Big(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\big(\frac{z}{2\sigma}\big)\Big) - 1\Big]^2$$
$$= \operatorname{erf}\Big(\frac{z}{2\sigma}\Big)^2. \qquad \square$$

Example 2

Assume that both players have quantile preferences with $\beta_1 = 0.7$ and $\beta_2 = 0.9$ and that $\sigma_1 = \sigma_2 = \sigma = 0.026$. So, $\alpha_i(S,T,p) = pu_{\beta_i}^{R(S)}/u_{\beta_i}^{R(T)}$. Then we find $I(N,\alpha) = C(N,\alpha) = \{p \in \Delta^*(N) \mid p_1 \leq 0.526, p_2 \leq 0.597\}$. We see that the core of this game is non-empty.

This calls for the question: for exactly which values of the quantiles is the core of the game non-empty? The following theorem elucidates the answer to that very question.

Proposition 6.0.1 *If two players play the energy prediction game and both have quantile preferences with $\beta_1 = \beta_2 = \beta$, then the core of this game is non-empty iff $0,626 \leq \beta \leq 1$.*

Proof From the definition of the imputation set for a two-person cost game (see Equation 5 and Lemma 5.0.1), we know that $I(N,\alpha) = \{p \in \Delta^*(N) \mid p_i \leq \alpha_i(\{i\}, N, 1) \text{ for all } i \in N\} = C(N,\alpha)$, where $N = \{1,2\}$. For quantile preferences with equal quantiles for both players, this means that the fraction $\frac{z_1}{z_2}$ (where z_1 and z_2 are the quantiles such that $\mathbb{P}(R(\{i\}) \leq z_1) = \operatorname{erf}(\frac{z_1}{\sqrt{2}\sigma}) = \beta$ (for $i = 1,2$) and $\mathbb{P}(R(\{1,2\}) \leq z_2) = \operatorname{erf}(\frac{z_2}{2\sigma})^2 = \beta$) has to be bigger than 1/2. If this is not the case, no efficient allocation anymore exists. This is because $p_1 \leq \alpha_1(\{1\}, N, 1) = \frac{z_1}{z_2}$ and $p_2 \leq \alpha_2(\{2\}, N, 1) = \frac{z_1}{z_2}$ in the core of the game. If these fractions are smaller than 1/2, the sum of p_1 and p_2 cannot be equal to 1 anymore, so the allocation cannot be efficient. (Note that for the calculation of the quantiles, we don't need to use the supremum and the $\leq$-inequality anymore. For continuous functions, equality is sufficient.)

To find out for which values of β this holds, we compute the inverse of both $g_{\{i\}}(z_1) := \operatorname{erf}(\frac{z_1}{\sqrt{2}\sigma})$ and $g_{\{1,2\}}(z_2) := \operatorname{erf}(\frac{z_2}{2\sigma})^2$. We see that $Q_\beta(R(\{i\})) = z_1 = g_{\{1\}}^{-1}(\beta) = \sqrt{2}\sigma \operatorname{erf}^{-1}(\beta)$ and $Q_\beta(R(\{1,2\})) = z_2 = g_{\{1,2\}}^{-1}(\beta) = 2\sigma \operatorname{erf}^{-1}(\sqrt{\beta})$. Therefore, $\alpha(\{i\}, N, 1) = \frac{z_1}{z_2} = \frac{\operatorname{erf}^{-1}(\beta)}{\sqrt{2}\operatorname{erf}^{-1}(\sqrt{\beta})}$ for $i = 1,2$. If we set $\frac{z_1}{z_2} = 1/2$, we find $\beta = 0,626$. We quickly see that $\frac{z_1}{z_2} > 1/2$ iff $0,626 \leq \beta \leq 1$. This completes the proof. $\square$

6.1 A variant of the game: combined predictions

Recall that in Theorem 6.0.2 we computed the convolution of two half-normal distributions. We did this in order to compare the stochastic costs associated with the grand coalition $N = \{1,2\}$ and the single-player costs associated with coalitions $\{1\}$ and $\{2\}$. It must be noted that that the computation of the two half-normal distributions implies that the two fines associated with the prediction error of both individual players are added to one another to find the total cost, after which these total costs will be distributed among the two players. For expectational preferences, cooperation in this manner does not yield any advantage to the players, as the expectation of the sum is equal to the sum of the expectations. For the quantile preferences, cooperation does yield benefits, depending on the value of β_i. We elaborate on the benefits and drawbacks of cooperation in Section 7.

Instead of having individual predictions, it could also be the case that the two players would make their predictions *together*. In this case, the predictions are added to up to form a new prediction of the two players as a single, cooperating entity. This means that we first have to compute the convolution of two normally distributed random variables. That tells us how the prediction error of the two players combined would be distributed. Taking the absolute value of that random variable generates the distribution of the costs that they would have to pay in total.

So this new game is a variation of the energy prediction game we described earlier. It is interesting to see how it differs from the previous game. Let us analyze it.

If the prediction error of player 1 is distributed as $P_1 \sim N(0, \sigma_1^2)$, then the distribution of the fine to be paid equals $Q_1 = |P_1| \sim H(0, \sigma_1^2)$. In a similar vein, the cost for player 2, who has prediction error distribution $P_2 \sim N(0, \sigma_2^2)$, is equal to $Q_2 = |P_2| \sim H(0, \sigma_2^2)$.
When the two players work together in this new game, their combined prediction error is distributed as $P_1 + P_2 \sim N(0, \sigma_1^2 + \sigma_2^2)$. This means that the fines to be paid are distributed as $Q = |P_1 + P_2| \sim H(0, \sigma_1^2 + \sigma_2^2)$.

So, the stochastic cost for the coalitions in a two-player game are as follows:

$$R(S) = \begin{cases} |P_1|, & \text{if } S = \{1\} \\ |P_2|, & \text{if } S = \{2\} \\ |P_1 + P_2| & \text{if } S = \{1, 2\}. \end{cases}$$

So the cumulative distribution errors for the fines are

$$F_{Q_i}(y, \sigma_i) = \operatorname{erf}\Big(\frac{y}{\sqrt{2}\sigma_i}\Big), \qquad y \geq 0, \quad i = 1, 2$$

and

$$F_Q(y, \sqrt{\sigma_1^2 + \sigma_2^2}) = \operatorname{erf}\left(\frac{y}{\sqrt{2(\sigma_1^2 + \sigma_2^2)}}\right), \qquad y \geq 0.$$

Let us assume players play the cost-sharing game in which both of them have quantile preferences. The question arises: for which quantile values is the core of the game non-empty? The following theorem sheds light on the matter.

Proposition 6.1.1 *If two players play the variant of the energy prediction game and both have quantile preferences with $\beta_1 = \beta_2 = \beta$, and $\sigma_1 = \sigma_2 = \sigma$, then the core of the game is non-empty iff $p_i \leq \frac{\sqrt{2}}{2} \approx 0.71$ for $i = 1, 2$.*

Proof Let z_1 and z_2 be such that $\operatorname{erf}\Big(\frac{z_1}{\sqrt{2}\sigma}\Big) = \beta$ and $\operatorname{erf}\Big(\frac{z_2}{2\sigma}\Big) = \beta$. The core is non-empty iff the fraction $\frac{z_1}{z_2}$ is bigger than 1/2, otherwise there exists no efficient allocation. We can find out for which values of β this holds by computing the inverses of $g_{\{i\}}(z_1) := \operatorname{erf}\Big(\frac{z_1}{\sqrt{2}\sigma}\Big)$ (with $i = 1, 2$) and $g_{\{1,2\}} := \operatorname{erf}\Big(\frac{z_2}{2\sigma}\Big)$. This amounts to $g_{\{i\}}^{-1}(\beta) = \sigma\sqrt{2}\operatorname{erf}^{-1}(\beta)$ and $g_{\{1,2\}}^{-1}(\beta) = 2\sigma\operatorname{erf}^{-1}(\beta)$. Then we compute $\alpha_i(\{i\}, N, 1) = \frac{g_{\{i\}}^{-1}(\beta)}{g_{\{1,2\}}^{-1}(\beta)} = \frac{\sqrt{2}\sigma\operatorname{erf}^{-1}(\beta)}{2\sigma\operatorname{erf}^{-1}(\beta)} = \frac{\sqrt{2}}{2} \approx 0.71$. Note that this fraction is bigger than 1/2. Thus, by Lemma 5.0.1, the core of the game is $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_i \leq \frac{\sqrt{2}}{2} \text{ for } i = 1, 2\}$.
□

We obtain a similar result when we consider expectational preferences.

Proposition 6.1.2 *If two players play the variant of the energy prediction game and both have expectational preferences, and $\sigma_1 = \sigma_2 = \sigma$, then the core of the game is non-empty iff $p_i \leq \frac{\sqrt{2}}{2} \approx 0.71$ for $i = 1, 2$.*

Proof As both players have expectational preferences, we have $\alpha_i(S, T, p) = p\frac{\mathbb{E}(R(S))}{\mathbb{E}(R(N))}$ for $i = 1, 2$. This means that $\alpha_i(\{i\}, N, 1) = \frac{\mathbb{E}(R(\{i\}))}{\mathbb{E}(R(N))}$. We first calculate the numerator of this fraction. We have

$$\begin{aligned} \mathbb{E}(R(\{i\})) &= \int_0^\infty y f_{Q_i}(y) dy \\ &= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\int_0^\infty y \exp\Big(-\frac{y^2}{2\sigma^2}\Big) dy. \end{aligned}$$

Implementing the substitution $u = y^2$, with $dy = du/2y$, gives

$$\begin{aligned}\mathbb{E}(R(\{i\})) &= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\int_0^\infty \exp\Big(-\frac{u}{2\sigma^2}\Big)\cdot(\frac{1}{2})du \\ &= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\Big(-\frac{u}{2\sigma^2}\Big)du \\ &= \frac{1}{\sigma}\frac{1}{\sqrt{2\pi}}\Big[-2\sigma^2\exp\Big(-\frac{u}{2\sigma^2}\Big)\Big]_0^\infty \\ &= \sigma\sqrt{\frac{2}{\pi}}.\end{aligned}$$

Now we calculate the value of the denominator.

$$\begin{aligned}\mathbb{E}(R(N)) &= \int_0^\infty y f_Q(y)dy \\ &= \frac{1}{\sigma\sqrt{\pi}}\int_0^\infty y\exp\Big(-\frac{y^2}{4\sigma^2}\Big)dy.\end{aligned}$$

Again, we use the substitution $u = y^2$, with $dy = du/2y$. It yields:

$$\begin{aligned}\mathbb{E}(R(N)) &= \frac{1}{\sigma\sqrt{\pi}}\int_0^\infty \exp\Big(-\frac{u}{4\sigma^2}\Big)\cdot(\frac{1}{2})du \\ &= \frac{1}{2\sigma\sqrt{\pi}}\Big[-4\sigma^2\exp\Big(-\frac{u}{4\sigma^2}\Big)\Big]_0^\infty \\ &= \frac{2\sigma}{\sqrt{\pi}}.\end{aligned}$$

Therefore, we find that $\alpha_i(\{i\}, N, 1) = \frac{\mathbb{E}(R(\{i\}))}{\mathbb{E}(R(N))} = \frac{\sigma\sqrt{\frac{2}{\pi}}}{\frac{2\sigma}{\sqrt{\pi}}} = \frac{\sqrt{2}}{2}$ for $i = 1, 2$. So, by Lemma 5.0.1, we have $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_i \leq \frac{\sqrt{2}}{2} \text{ for } i = 1, 2\}$. □

7 Interpretation

So far, we have mainly focused on two variants of the energy prediction game. In the first variant, each player individually predicts his energy production levels. Then the fines for deviations from the predictions are calculated, again for both players individually. Finally, the two fines are added together and the costs were distributed in a certain manner. Let us call this variant of the game **Variant A**. We described this variant of the game in Section 6, until halfway through page 16.

In the second variant, the predictions of both players are aggregated and the fines for the combined prediction errors are calculated. After that, the fines associated with these errors are distributed among the two players. We call this variant of the game **Variant B**. We analyzed this variant of the game in Section 6.1.

7.1 Analysis of variants A and B for quantile preferences

We have analyzed these variants in more detail by examining the case in which we have two players, and both players have quantile preferences. Both players have the same quantile β. Proposition 6.0.1 shows that for variant A, core of the game is non-empty iff $0,626 \leq \beta \leq 1$. This means that the players must be both quite risk-seeking to find cooperation beneficial to themselves.

Proposition 6.1.1 shows that the core of the game of Variant B is non-empty iff $p_i \leq \frac{\sqrt{2}}{2}$ for $i = 1, 2$. It is striking that the fraction of the total costs the players are willing to bear of the grand coalition is constant and independent of the value of their quantile β. So no matter which value of β they choose (so no matter how risk-averse or risk-seeking they are), they always want to cooperate under the same conditions.
So although variants A and B may be superficially similar, the requirements for cooperation between the two players differ substantially. In variant A the act of cooperating at all depends on the value of β of both players, whereas in variant B the question of cooperating or not is β-independent. Furthermore, notice that the utilitarian benefits for the players in variant A increase as the value for β increases. For $\beta = 0,626$, the core is non-empty, but just barely. The players are only willing to cooperate if they are obliged to pay *at most* half of the total costs incurred. When $\beta = 1$, however, the core of the game becomes equal to the core for the game of variant B: $p_i \leq \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ for $i = 1, 2$. This means that the more risk-averse the players are, the larger the core becomes and the more beneficial it is to players to cooperate.

Returning to variant A: for $\beta = 0,626$, the core is a singleton: only consisting of the allocation $p = (p_1, p_2) = (1/2, 1/2)$. If we propose the only allowed efficient allocation of the costs (which is $(1/2, 1/2)$, too), the players don't really benefit: they pay equally much when they play the game on their own.

When $\beta = 1$, however, we have $p_i \leq \frac{\sqrt{2}}{2} \approx 0.71$ for $i = 1, 2$. If we now propose some "fair" allocation of the costs – say $(1/2, 1/2)$ again – the players benefit substantially: their utility associated with these preferences increases by $-(0,5 - 0,71) = 0,21$ percentage points. This means the percentual increase of utilities $\frac{0,21}{0,71} \times 100\% \approx 29.3\%$, compared to when they play on their own. In other words, their relative utility increases with 29.3%. Now cooperation surely is a more attractive option than opting to play individually! Note that the actual costs do not decrease in Variant A. Only the utilities with respect to the preferences increase. Notice furthermore that for all values except $\beta = 1$, players prefer to play variant B over variant A. Thus we conclude that in general, players would prefer to play variant B over variant A of the game.

7.2 Variant B with expectational preferences

What is interesting is that under expectational preferences, the core of the game for Variant B is the same as for quantile preferences. We see this in Propositions 6.1.1 and 6.1.2. Expectational preference can be viewed as a risk-neutral preference. So for risk-neutral players, it is beneficial to cooperate when $p_i \leq \frac{\sqrt{2}}{2}$. We again stress that cooperation is much preferred over not doing so. However, this time it is not just the utilities that decrease. The actual costs decrease, too. When working together, the players with expectational preferences, too, need to pay almost 30% less than when they don't. The reasoning for this fact goes as follows: when player i ($i = 1, 2$) operates on its own, his expected payments are

$$\mathbb{E}(R(\{i\})) = \frac{\sigma\sqrt{2}}{\sqrt{\pi}}.$$

The expected costs for both players combined are

$$\mathbb{E}(R(N)) = \frac{2\sigma}{\sqrt{\pi}}.$$

When these combined costs are divided fairly among the two players, each pays only $\sigma/(\sqrt{\pi})$. Thus, the percentual decrease of costs per player is

$$\begin{aligned}
-\frac{(\frac{\sigma}{\sqrt{\pi}} - \sigma\sqrt{\frac{2}{\pi}})}{\frac{\sigma\sqrt{2}}{\sqrt{\pi}}} \times 100\% &= \frac{\frac{\sigma}{\sqrt{\pi}}(\sqrt{2}-1)}{\frac{\sigma\sqrt{2}}{\sqrt{\pi}}} \times 100\% \\
&= \frac{\sqrt{2}-1}{\sqrt{2}} \times 100\% \\
&\approx 29.3\%.
\end{aligned}$$

This is important: it shows that owners of renewable energy sources can save a substantial amount of costs associated with badly predicting the energy production levels of their wind mills or solar panels. Cooperating by sharing the costs for the prediction error makes renewable energy sources less costly to own and operate, and more profitable.

7.3 Variant B with other preferences

When looking at quantile preferences, it is often the case that the core of the game changes according to the different values of β the players choose. Picking a high value of β means the player is relatively risk-seeking, whereas players with low values of β are more risk-averse. However, we have shown that the value of β does not matter for the characterization of the core for variant B of the game. So the core does not change according to this quantification of risk-aversity or risk-seekingness.

There are, however, other ways to quantify the measure of risk the players are willing to take when playing the game. This involves so-called Neumann-Morgenstern preferences. We will analyze these preferences in the next section.

8 Neumann-Morgenstern preferences

Let us immediately define this new type of preference. We say that player i has Neumann-Morgenstern (N-M) preferences, if there exists a utility function $u_i : \mathbb{R} \to \mathbb{R}$ and a functions $w(x) = x^n$ such that $X \succsim_i Y$ if and only if $u_i(X) = \mathbb{E}(w_i(X)) \geq \mathbb{E}(w_i(Y)) = u_i(Y)$ for any X and Y. The special case in which $u_i(x) = x$, corresponds to expectational preferences.
Certain other properties of the function $w(\cdot)$ correspond to specific attitudes regarding risk. The following image [25] helps us to understand this concept:

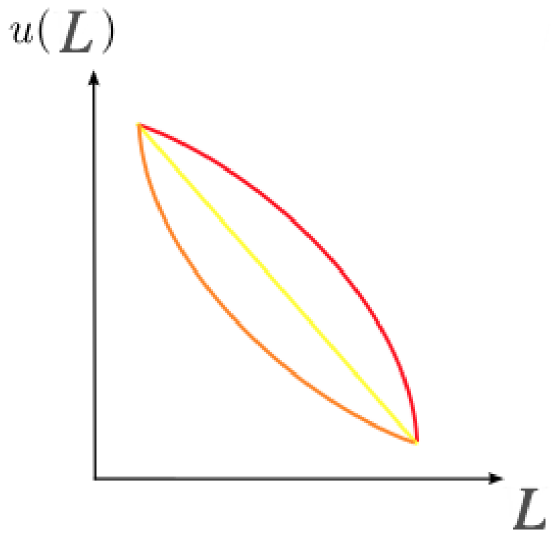

Figure 3: Decreasing linear, concave and convex utility functions for loss games

This image represents a series of utility functions. On the L-axis, we see the value of the loss, whereas the $u(L)$-axis shows the (perceived) utility of the corresponding value. The yellow graph shows the correspondence between value and utility for risk-neutral players. The utility decreases linearly with the loss. This relationship is characterized by the utility function $w(x) = ax + b$, with $a < 0$.

The orange graph shows the same relationship between loss and utility for a risk-averse player. The player's utility decreases quickly when the loss increases. This graph could correspond to the utility function $w(x) = a\sqrt{x} + b$, with $a < 0$. In general, the orange graph could be any convex decreasing function.

Finally, the red graph shows the relationship for a risk-seeking player. The risk-seeking player is relatively unphased by a little bit of extra loss. Only when the losses get very big, the player becomes wholly dissatisfied. For instance, its graph could be of the form $w(x) = ax^2 + b$, with $a < 0$. In general, it could be any concave decreasing function.

In the above discussion of the convex, concave and linear utility functions, we have mostly tried to give the reader a bit of intuition as to why certain utility functions correspond to certain types of attitudes towards risk. We have not provided any solid, analytical evidence for the relationship between convexity or concavity and the utility players derive from some stochastic game. This relationship was more formally described in an article by Michael Rothschild and Joseph Stiglitz [20]. They prove that certain characterizations for attitudes towards risk are equivalent to one another. For a short exposition on their work, we refer to Appendix B (see 14.2) of this thesis.

8.1 Application of von N-M preferences

In general, we can incorporate the von N-M (von Neumann-Morgenstern) utility functions within the framework of preference relations as laid out in Section 4.1. We defined the von N-M preferences in Section 8.1. We have already calculated the core of the variant B game, when both players have von N-M preferences with $w(x) = -x$, as this corresponds to expectational preferences.

It remains to the figured out, however, what the core looks like when the players either have convex or concave utility functions. For these situations, we take the utility functions

$$\begin{aligned} w_i^1 : \mathbb{R}_{>0} &\to \mathbb{R} \\ x &\mapsto -\sqrt{x} \end{aligned}$$

and

$$\begin{aligned} w_i^2 : \mathbb{R}_{>0} &\to \mathbb{R} \\ x &\mapsto -x^2 \end{aligned}$$

respectively. For the first type of function, we have

$$u_i^1(R(S)) = \mathbb{E}(R(S)^{1/2}),$$

and $\alpha_i(S,T,p) = p\frac{\mathbb{E}(R(S)^{1/2})}{\mathbb{E}(R(T)^{1/2})}$. For the second type of function, we have

$$u_i^2(R(S)) = \mathbb{E}(R(S)^2),$$

and it similarly holds that $\alpha_i(S,T,p) = p\frac{\mathbb{E}(R(S)^2)}{\mathbb{E}(R(T)^2)}$.

We first calculate the core of a game in which two players with identical variances of their error probability density functions have N-M preferences with $w_i^2(x) = -x^2$ $(i = 1,2)$, so they are risk-seeking. So we have to calculate the appropriate α-value $\alpha_i(\{i\}, N, 1) = \frac{\mathbb{E}(u_i^2(R(\{1\}))}{\mathbb{E}(u_i^2(R(N))}$. Notice, that if we really want to pick a proper loss utility function, it would for instance be of the form $u_i^2(x) = -x^2 + b$ for some $b > 0$ (because we start off with high utility for zero losses). In order to be able to calculate both the numerator and the denominator of this fraction, we need the following lemma, which we state without proof [4]:

Lemma 8.1.1 *The following holds:*

$$\int_0^\infty x^n e^{-ax^2} dx = \begin{cases} \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)/a^{\frac{n+1}{2}}, & \text{if } n > -1, a > 0 \\ \frac{(2k-1)!!}{2^{k+1}a^k}\sqrt{\frac{\pi}{a}}, & \text{if } n = 2k, k \in \mathbb{Z}, a > 0 \qquad (!! \text{ is the double factorial }) \\ \frac{k!}{2a^{k+1}}, & \text{if } n = 2k+1, k \in \mathbb{Z}, a > 0, \end{cases}$$

where the double factorial function is defined as $n!! := \prod_{k=0}^{\lceil \frac{n}{2} \rceil - 1} (n-2k)$ *and* $\Gamma(\cdot)$ *is the Gamma function, defined as* $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$. □

Proposition 8.1.1 *Assume that two players play variant B of the energy prediction game and both have N-M preferences with* $w_i^2(x) = -x^2$ *and* $\sigma_1 = \sigma_2 = \sigma$, *and* α *specified above. Then* $C(N,\alpha) = \{p \in \Delta^*(N) \mid p_i = \frac{1}{2} \text{ for } i = 1,2\}$.

Proof We begin with the numerator. By the Law of the Unconscious Statistician, we have:

$$\begin{aligned} \mathbb{E}(w_i^2(R(\{1\}))) &= \int_0^\infty w_i^2(y) f_{R(\{1\})}(y) dy \\ &= \int_0^\infty -y^2 \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) dy \\ &= -\frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty y^2 \exp\Big(-\frac{y^2}{2\sigma^2}\Big) dy. \end{aligned}$$

The second part of Lemma 8.1.1 with $n = 2$, $k = 1$, and $a = 1/(2\sigma^2)$ yields:

$$\begin{aligned} u_i^2 R(\{1\}) = \mathbb{E}(w_i^2(R(\{1\}))) &= -\frac{1}{\sigma}\sqrt{\frac{2}{\pi}}\Big(\frac{\sigma^3}{2}\sqrt{2\pi}\Big) \\ &= -\sigma^2. \end{aligned}$$

Doing the exact same procedure, but now with $a = 1/(4\sigma^2)$, yields

$$u_i^2(R(\{N\})) = \mathbb{E}(w_i^2(R(N)) = -2\sigma^2.$$

As $\alpha_i(\{i\}, N, 1) = \frac{u_i(R(\{i\}))}{u_i(R(N))}$, we find with Lemma 5.0.1 that $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_i = \frac{1}{2} \text{ for } i = 1, 2\}$. $\square$

So, the core is non-empty, but it contains only one allocation. Risk-seeking players don't derive much benefit from collaborating in this game. They are equally willing to cooperate as to work on their own, as the expected payment for both players together is exactly twice the expected payment for each individual player.
Next, we will take a look at the same game played by risk-averse players. We have obtained the following result.

Proposition 8.1.2 *Assume that two players play variant B of the energy prediction game and both have N-M preferences with $w_i^1(x) = -\sqrt{x}$ and $\sigma_1 = \sigma_2 = \sigma$, and α specified above, then $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_i \leq \frac{1}{2^{1/4}} \approx 0.84$ for $i = 1, 2\}$.*

Proof Again we have to calculate $\alpha_i(\{i\}, N, 1) = \frac{\mathbb{E}(w_i^1(R(\{1\})))}{\mathbb{E}(w_i^1(R(N)))}$. Using the Law of the Unconscious Statistician once more, we first compute the numerator:

$$\begin{aligned}\mathbb{E}(w_i^1(R(\{1\}))) &= \int_0^\infty w_i^1(y) f_{R(\{1\})}(y) dy \\ &= \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \int_0^\infty -\sqrt{y} \exp\Big(-\frac{y^2}{2\sigma^2}\Big) dy.\end{aligned}$$

The first part of Lemma 8.1.1, with $n = 1/2, a = 1/(2\sigma^2)$, yields the following:

$$u_i^1(R(\{1\}))) = \mathbb{E}(w_i^1(R(\{1\}))) = -\frac{1}{\sigma\sqrt{2\pi}}\Big[\Gamma\Big(\frac{3}{4}\Big)/(2\sigma^2)^{-3/4}\Big].$$

In a similar manner, we can use Lemma 8.1.1 with $n = 1/2$ and $a = 1/(4\sigma^2)$ to calculate that

$$u_i^1(R(\{N\}))) = \mathbb{E}(w_i^2(R(N))) = -\frac{1}{2\sigma\sqrt{\pi}}\Big[\Gamma\Big(\frac{3}{4}\Big)/(4\sigma^2)^{-3/4}\Big].$$

This means that

$$\alpha_i(\{i\}, N, 1) = \frac{\mathbb{E}(w_i^1(R(\{1\})))}{\mathbb{E}(w_i^1(R(N)))} = \frac{1}{2^{1/4}} \approx 0.84.$$

By Lemma 5.0.1, the proof is complete. $\square$.

There are a couple of interesting things to note at this point.

First of all, our results thus far suggest that the more risk-averse the players of the game are, the more they gain from cooperating in the grand coalition. On the other hand, the risk-seeking players are only willing to participate in the coalition if they must pay exactly half of the total costs, the players with expectational preferences (so those that are risk-neutral) are already willing to take on a larger share of the total cost: approximately 71 %. Risk-averse players (so those with a concave utility function) allow for an ever greater burden of the costs: they are willing to bear 84 %. The greater the share of the costs of the grand coalition the players are willing to take on, the bigger the benefits they can reap from cooperation.

Second of all, it seems that the quadratic utility function is the maximally risk-seeking utility function that yields a non-empty core. If the degree of the utility function is higher than two, the core will become empty. In other words: it will not be beneficial for the players anymore to cooperate, when their utility functions are of the form $u_i(x) = x^{2+\epsilon}$ for some $\epsilon > 0$. Indeed, $\alpha(\{i\}, N, 1) = \left(\frac{1}{2}\right)^{1+\epsilon}$ in this case. As the core consists of the efficient allocations p with $p_1 \leq \left(\frac{1}{2}\right)^{1+\epsilon}$ and $p_2 \leq \left(\frac{1}{2}\right)^{1+\epsilon}$, it is empty when $\epsilon > 0$.

9 Correlated prediction errors

So far, we have assumed that the random variables that describe the prediction error distributions are independent of each other. This means that, because these errors are normally distributed, we have $P_1 \sim N(0, \sigma_1^2)$ and $P_2 \sim N(0, \sigma_2^2)$, and $P_1 + P_2 = N(0, \sigma_1^2 + \sigma_2^2)$.

In practice, however, the prediction errors are not always independent of each other. When, for instance, a solar panel and a wind turbine are placed in close vicinity to each other, the prediction error of the wind turbine is correlated with the prediction error of the solar panel. This has been experimentally verified by Zhang et al. in [26]. They computed Pearson's correlation coefficient between wind and solar power forecast errors. This coefficient is defined as follows:

$$\rho := \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2}.$$

It is a well-established result (see, for instance, [2]) that the sum of two non-independent normally distributed random variables X_1 and X_2 is distributed as follows:

$$X_1 + X_2 \sim N(0, \sigma_1^2 + \sigma_2^2 + 2\text{Cov}(X_1, X_2)).$$

By the definition of Pearson's correlation coefficient, we find that

$$X_1 + X_2 \sim N(0, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2).$$

In Propositions 6.1.1 and 6.1.2, we have characterized the cores of variant B of the energy prediction game with two players with quantile and expectational preferences, respectively. We have shown that the cores for these preferences are the same.

Next, we will analyze the same game with correlated energy prediction error. We will investigate for which values of Pearson's correlation coefficient, it is beneficial for players to cooperate. The stochastic payments are defined as on p. 17, in Equation 6.1 :

$$R(S) = \begin{cases} |P_1|, & \text{if } S = \{1\} \\ |P_2|, & \text{if } S = \{2\} \\ |P_1 + P_2|, & \text{if } S = \{1, 2\}, \end{cases}$$

Here, $P_i \sim N(0, \sigma_i^2)$ for $i = 1, 2$ and $P_1 + P_2 \sim N(0, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$. This means that, when $\sigma_1 = \sigma_2 = \sigma$, it holds that

$$R(S) = \begin{cases} Q_1, & \text{if } S = \{1\} \\ Q_2, & \text{if } S = \{2\} \\ Q, & \text{if } S = \{1, 2\}, \end{cases}$$

with $Q_i \sim H(0, \sigma^2)$ for $i = 1, 2$, and $Q \sim H(0, 2\sigma^2(1 + \rho))$.

Proposition 9.0.1 *Suppose two players play variant B of the energy production prediction game, their prediction errors are correlated with Pearson's correlation coefficient ρ, and $\sigma_1 = \sigma_2 = \sigma$, and both players have expectational preferences. Then $C(N, \alpha) \neq \emptyset$, if $-1/2 \leq \rho \leq 1$.*

Proof To determine the core, we need to determine the value of $\alpha_i(\{i\}, N, 1) = \frac{\mathbb{E}(R(\{i\}))}{\mathbb{E}(R(N))}$.

We have $\mathbb{E}(R(\{i\})) = \sigma\sqrt{\frac{2}{\pi}}$ for $i = 1, 2$, and $\mathbb{E}(R(N)) = 2\sigma\sqrt{\frac{1+\rho}{\pi}}$. This means that $\alpha_i(\{i\}, N, 1) = \frac{1}{2}\sqrt{\frac{2}{1+\rho}}$. By Lemma 5.0.1, we have $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_i \leq \frac{1}{2}\sqrt{\frac{2}{1+\rho}}, \quad i = 1, 2\}$. An allocation in the core can therefore only be efficient if $\alpha_i(\{i\}, N, 1) \geq \frac{1}{2}$. So setting $\frac{1}{2}\sqrt{\frac{2}{1+\rho}} = \frac{1}{2}$ gives us an upper bound for ρ. With a bit of basic algebra, we see that $\rho = 1$.

On the other hand, an allocation in the core cannot be efficient when $p_i > 1$ for any $i \in N$. So setting $\alpha_i(\{i\}, N, 1) = \frac{1}{2}\sqrt{\frac{2}{1+\rho}} = 1$ gives us a lower bound for ρ. Again, we perform some elementary algebraic operations to find $\rho = -\frac{1}{2}$. Thus we find that the core is non-empty when $-\frac{1}{2} \leq \rho \leq 1$. $\square$

We can do a similar analysis, when both players have quantile (instead of expectational) preferences:

Proposition 9.0.2 *Suppose two players play variant B of the energy production prediction game and their prediction errors are correlated with Pearson's correlation coefficient ρ, and $\sigma_1 = \sigma_2 = \sigma$, and both players have quantile preferences with $\beta_1 = \beta_2 = \beta$. Then $C(N, \alpha) \neq \emptyset$ if $-\frac{1}{2} \leq \rho \leq 1$.*

Proof Again, we ought to calculate $\alpha_i(\{i\}, N, 1) = u_{\beta_i^{R(\{1\})}} / u_{\beta_i^{R(\{N\})}}$. The nominator and denominator of this fraction are the numbers z_1 and z_2 respectively such that $F_{X_i}(z_1, \sigma) = \text{erf}\left(\frac{z_1}{\sqrt{2}\sigma}\right) = \beta$ and $F_{X_1+X_2}(z_2, \sqrt{2}\sigma) = \text{erf}\left(\frac{z_1}{2\sigma\sqrt{1+\rho}}\right)$. We define the inverses of these functions as $g_{\{1\}}(\beta)$ and $g_{\{N\}}(\beta)$ respectively. Calculating them yields $g_{\{1\}}(\beta) = \text{erf}^{-1}(\beta)\sqrt{2}\sigma$ and $g_{\{N\}}(\beta) = \text{erf}^{-1}(\beta)2\sigma\sqrt{1+\rho}$. This means that

$$\alpha_i(\{i\}, N, 1) = \frac{u_{\beta_i^{R(\{1\})}}}{u_{\beta_i^{R(\{N\})}}} = \frac{g_{\{1\}}(\beta)}{g_{\{N\}}(\beta)} = \frac{\text{erf}^{-1}(\beta)\sqrt{2}\sigma}{\text{erf}^{-1}(\beta)2\sigma\sqrt{1+\rho}} = \frac{1}{\sqrt{2(1+\rho)}}.$$

Analogous to the previous proof, by setting $z_1/z_2 = \frac{1}{2}$, we find the upper bound for ρ such that the core is non-empty. We find that $\rho = 1$. Setting $z_1/z_2 = 1$ gives us the lower bound. It yields $\rho = -\frac{1}{2}$. This gives the desired result. $\square$

It is a perhaps curious fact that for these two vastly different stochastic preferences, the core is non-empty for the same values of the Pearson correlation coefficient.

9.1 Experimental Results

After having computed the values of the Pearson correlation coefficient for which cooperation is desirable, it is of interest to compare this to the values this coefficient actually attains in the real world in the context of renewable energy sources.

There is evidence that that there is a one-to-one correspondence between the Pearson correlation coefficient of the power output of renewable energy sources and the same coefficient of the normalized prediction error of these energy sources.

Consider the case in which there are two renewable energy sources of the same type. It has been experimentally verified that the Pearson correlation coefficient is a function of distance, resource, terrain and time scale. We first focus on the first of these dimensions. As the distance between two renewable energy sources increases, the correlation in power output (and therefore also in the normalized prediction error) generally increases. This can be inferred from Section IV.A of [11]. In other words: the more geographically diverse the energy sources are distributed, the less variable and the less uncertain their aggregate production levels are. The following table provides us with a summary of the relationship between the distance between two wind mills and the correlation coefficient, which was taken from Table 3 on p. 328 of [11].

Distance (km)	Correlation (1 hr avg)	Time scale of benefit	Electric power system benefits
1	0.9	seconds-minutes	Voltage control
10	0.9	minutes-10 minutes	Regulation
100	0.7	1 hr	Regulation Operating reserves
500	0.35-0.7	hours	Operating reserves Forecasting Scheduling
1,000	0.1-0.5	hours	Operating reserves Forecasting Scheduling
2,000	0	days	Forecasting Scheduling Reliability
10,000	-0.1	days-weeks	Reliability

Figure 4: Table summarizing the relationship between the distance and correlation coefficient of the power output of wind mills

Indeed, the correlation decreases as the distance increases. When the distance becomes very large (in some cases, when the distance gets larger than about 1.500 kilometers), the correlation can become negative. In that case, the value of the Pearson correlation coefficient decreases, but it still remains fairly low. In all observed cases, the coefficient floats between $0,9$ and $-0,1$. The data in the image above is represented in graphical form in the image below:

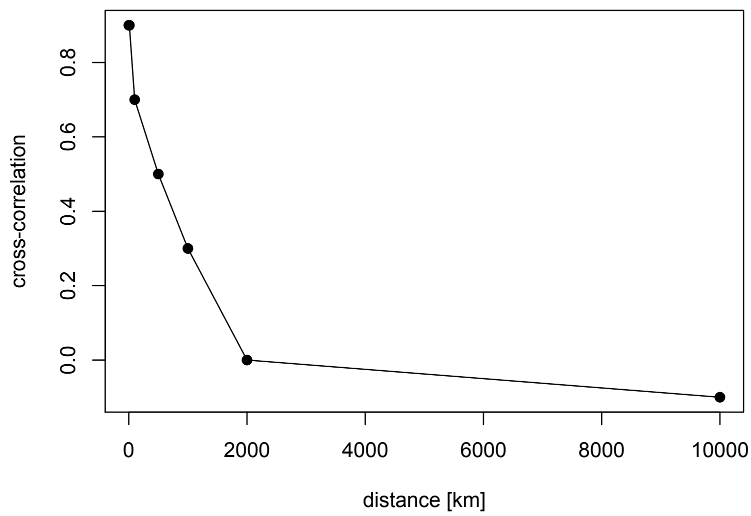

Figure 5: The distance versus Pearson's correlation coefficient

As we found in Propositions 9.0.1 and 9.0.2 of this thesis, cooperation between renewable energy sources is beneficial to both players when $\rho \in [-\frac{1}{2}, 1]$. So it is safe to assume that players can benefit from cooperation, when their distance between them is within reasonable bounds (i.e. less than 10.000 kilometers). Even when this distance bound is exceeded, the synergistic effects of cooperation will still manifest itself. Even when, for some distance, we would find $\rho < -1/2$, the core can still be non-empty and cooperation can still yield benefits to both players. Propositions 9.0.1 and 9.0.2 only gave sufficient, and not necessary, conditions for a non-empty core.

In the proof of Proposition 9.0.1, we mentioned that the core of the two-player prediction game for variant B in which both players have expectational preferences is as follows:

$$C(N,\alpha) = \left\{ p \in \Delta^*(N) \mid p_1 \leq \frac{1}{2}\sqrt{\frac{2}{1+\rho}},\ p_2 \leq \frac{1}{2}\sqrt{\frac{2}{1+\rho}} \right\}.$$

This means that the core becomes larger when the value of the correlation coefficient ρ becomes smaller. In the figure below, we plot the function $y = \frac{1}{2}\sqrt{\frac{2}{1+x}}$ to see exactly how much players benefit from cooperating. The higher the value of the function, the bigger the part of the costs the players are willing to bear to cooperate. We can see that, especially when $\rho < -0,8$, this fraction increases dramatically.

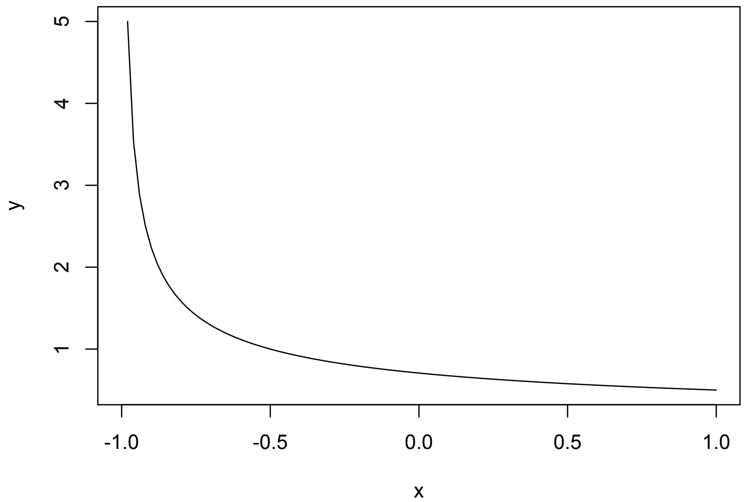

Figure 6: Correlation versus the fraction of the costs players are willing to bear to still find cooperation beneficial to themselves

In turn, this means that players gain more from cooperating when their prediction error distributions are less correlated. Taking into account the discussion of the relationship between the correlation error and the distance between renewable energy sources above, we can infer that the benefits of cooperation increase when the renewable energy sources are further away from each other (until the distance becomes more than 1.500 kilometers).

Our suspicion, that there is a one-to-one correspondence between the correlation coefficient of the power production and the correlation coefficient of the prediction error is justified by an article by Focken et al. [9]. In their research, they found out that the cross correlation (in other words: the Pearson correlation coefficient) of the prediction error decreases as the distance between wind mills in Germany increases. This is illustrated by the following image of their article, which was presented on page 10.

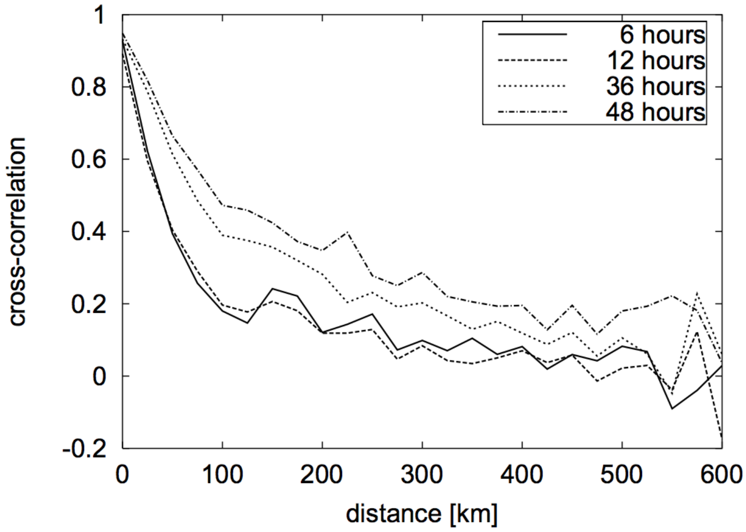

Figure 7: The relationship between the distance and the prediction error correlation coefficient between wind mills for different time frames

The different lines represent the relationship between the distance and the Pearson correlation coefficient of the prediction errors for different time frames of the predictions (as can be seen in the top right corner of the image). The cross-correlation for the prediction error seems to decrease a bit quicker than the cross-correlation for the power output, so perhaps the relationship is not entirely one-to-one. The general behavior of the graphs is similar, however. Based on the graph above, cooperation is most beneficial to the participating players when the distance between the renewable energy sources is about 600 kilometers (though it differs for the different time frames).

Similar results have been obtained for the correlation between power outputs of geographically dispersed solar panels by Mills et al. in [14]. This relationship in particular is visualized on page 10 of their paper. Again, the experimental evidence suggests that cooperation between the renewable energy sources is beneficial for both entities. Furthermore, the benefits increase as the distance increases, because the Pearson correlation coefficient decreases with increased spreading.

10 Unequal standard deviations

So far, we have mainly looked at and proven results on the case for equal standard deviations of the prediction error distributions of both players. This is not, however, very realistic. According to Zhang et al. in [26], the values of the standard deviations can change dramatically, either by altering the time frame for the predictions, or by looking at different types of renewable energy resources (such as wind mills and solar panels). In the following image, the error distributions for day-head (24 hours), four-hour-ahead and one-hour-ahead predictions are shown for a pair of wind mils and solar panels in the Western Interconnection in the United States.

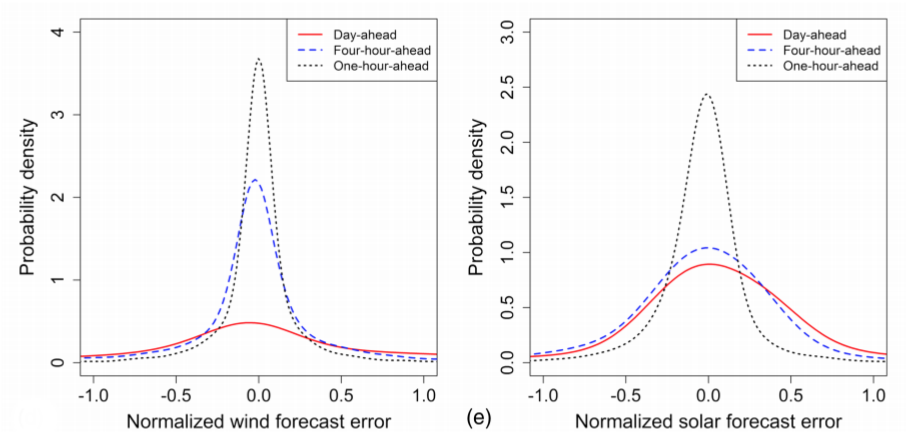

Figure 8: Normalized prediction error distributions of wind mills (left) and solar panels (right) for different time frames

We see that the standard deviation increases as the time between the moment of prediction and the moment of production increases. This was to be expected. And the longer the difference between the time at which the prediction was made and the time for which the energy output was to be predicted, the more inaccurate the prediction is (likely) going to be.

Furthermore, there are differences between the prediction errors for wind mills and solar panels. For the one-hour-ahead predictions, the prediction for wind mills is more accurate than for solar panels. However, the day-ahead predictions are not nearly as bad for solar panels as for the wind mills.

A couple of further remarks need to be made. First of all, the values of the standard deviations are not given in [26]. We could, however, approximate these values by plotting the normal distribution with $\mu = 0$ for different values of σ. Then we can compare the resulting plots with the image above to infer the values for the standard deviations that are approximately correct.

Second, the prediction errors are not normally distributed. As we already explained in the beginning of Section 6, the errors follow a hyperbolic distribution. However, due to the fact that is hard to work with this distribution, we use the normal distribution with mean $\mu = 0$.

10.1 The core and the Shapley value

As mentioned previously, so far we have focused on the case in which $\sigma_1 = \sigma_2 = \sigma$. In particular, we have proved numerous results regarding the core of the energy prediction games with these parameters. So far, we have neglected an important value of such games: the Shapley value. When the standard deviations of the distributions governing the prediction errors are equal, it is a trivial fact that the Shapley vector is $(1/N, 1/N, \ldots, 1/N)$, when N players participate in the game.

Things become more interesting, however, when the standard deviations are not equal anymore. From now on we therefore suppose that $\sigma_2 = k\sigma_1$ for some constant $k \in \mathbb{R}_{>0}$.

So, the prediction error for player 1 is $P_1 = N(0, \sigma^2)$, for player 2 it is $P_2 = N(0, k^2 \cdot \sigma^2)$, and for both players combined it is $P_3 = N(0, \sigma^2 \cdot (1 + k^2))$. The stochastic cost for the coalitions in a two-player game are therefore as follows:

$$R(S) = \begin{cases} |P_1| = Q_1 = H(0,\sigma^2), & \text{if } S = \{1\} \\ |P_2| = Q_2 = H(0,k^2\cdot\sigma^2), & \text{if } S = \{2\} \\ |P_1 + P_2| = Q = H(0,\sigma^2(1+k^2)) & \text{if } S = \{1,2\}. \end{cases}$$

We will investigate for which values of k the Shapley value is an element of the core of the game.

We note furthermore that from now on, we will focus exclusively on variant B of the game, and all players have expectational preferences. We henceforth assume this is the variant of the game the players play. This is due to the fact that convolutions of half-normal distributions are complicated. Another reason we focus on variant B of the game is that we find it more natural that the players combine their predictions of their production levels and that they derive more surplus value from cooperation.

Before we can calculate the Shapley values, we must of course first define the notion of this value in the context of stochastic cooperative game theory. Again, we rely on [6].

A bijection σ of the players in N is a function $\sigma : \{1, 2, \ldots, n\} \to N$. Each bijection sorts the players in a different order, and $\sigma(i)$ denotes which player in N is at position i of the order. We define $\Pi(N)$ as the set of all bijections of N. Furthermore, let $S_i^\sigma := \{\sigma(k) \mid k \le i\}$ be the set of the first i players in the order, according to bijection σ (where $i \in \{1, 2, \ldots, n\}$ and $S_0^\sigma := \emptyset$).

Recall that for a deterministic cooperative game with transferable utility and a characteristic function $v(\cdot)$, the marginal vector $m^\sigma(v)$ is defined as

$$m^\sigma_{\sigma(k)}(v) := v(S_k^\sigma) - v(S_{k-1}^\sigma) = v(S_k^\sigma) - \sum_{i=1}^{k-1} m^\sigma_{\sigma(i)}(v)$$

for each $k \in \{1, 2, \ldots, n\}$.

In an analogous manner, we can define marginal vectors in the context of stochastic cooperative game theory. Let $y^\sigma_{\sigma(i)}(\alpha)$ be the marginal contribution of player i in the order of bijection σ in terms of the game (N, α). As such a contribution is formulated as a multiple of the random payoff for S_i^σ. Hence, the contribution of the first player according to σ equals this player's random payoff. So we have $y^\sigma_{\sigma(1)}(\alpha) = 1$. The second player in the order is $\sigma(2)$. When he joins the first player, the coalition S_2^σ is formed. The marginal contribution of $\sigma(2)$ is the random payoff of S_2^σ minus the marginal contribution of player $\sigma(1)$. Thus,

$$y^\sigma_{\sigma(2)}(\alpha) = 1 - \alpha_{\sigma(1)}(S_1^\sigma, S_2^\sigma, y^\sigma_{\sigma(1)}(\alpha)).$$

Similarly, the marginal contribution of the third player is

$$y^\sigma_{\sigma(3)}(\alpha) = 1 - \sum_{k=1}^{2} a_{\sigma(k)}(S_k^\sigma, S_3^\sigma, y^\sigma_{\sigma(k)}(\alpha))).$$

When we extrapolate this idea, we can recursively define the marginal contribution of $\sigma(i)$ to coalition S_{i-1}^σ. We have:

$$y^\sigma_{\sigma(i)}(\alpha) = 1 - \sum_{k=1}^{i-1} a_{\sigma(k)}(S_k^\sigma, S_i^\sigma, y^\sigma_{\sigma(k)}(\alpha))$$

for all $i \in \{1, 2, \ldots, n\}$. This in turn enables us to define the marginal vector $m^\sigma(\alpha)$ by:

$$m^\sigma_{\sigma(i)}(\alpha) := \alpha_{\sigma(i)}(S_i^\sigma, N, y^\sigma_{\sigma(i)}(\alpha)),$$

for $i = 1, 2, \ldots, n$. This, finally, allows us to define the Shapley value $\phi(\alpha)$ of a stochastic cooperative game as the average of all $n!$ previously defined marginal vectors:

$$\phi(\alpha) := \frac{1}{n!} \sum_{\sigma \in \Pi(N)} m^\sigma(\alpha).$$

An important property of the Shapley value is that it is an efficient allocation for the grand coalition.

Lemma 10.1.1 $\phi(\alpha) \in \Delta^*(N)$.

Proof We prove that the marginal vector $m^\sigma(\alpha)$ is efficient when σ is the identity function from N to N. The proof for the other bijections σ goes analogously.

The following equalities hold:

$$\begin{aligned} m^\sigma_{\sigma(|N|)}(\alpha) &= m^\sigma_{|N|}(\alpha) \\ &= \alpha_{|N|}(N, N, y_{|N|}(\alpha)) \\ &= y_{|N|}(\alpha)\alpha_{|N|}(N, N, 1) \\ &= y_{|N|}(\alpha)\frac{u_{|N|}(N)}{u_{|N|}(N)} \\ &= 1 - \sum_{k=1}^{|N|-1} y^\sigma_k \frac{u_k(S_k)}{u_k(N)} \\ &= 1 - \sum_{k=1}^{|N|-1} m^\sigma_k(\alpha). \end{aligned}$$

Therefore, the marginal vector $m^\sigma(\alpha)$ is efficient. The proof for the other marginal vectors is similar. If all individual marginal vectors are efficient, then the average of these vectors is efficient, too. $\square$

10.2 Deterministic equivalent

In Subsection 4.4, we described the relationship between classical cooperative game theory and stochastic cooperative game theory. We explained that in this case, $\alpha_i(S, T, p) = p\frac{v(S)}{v(T)}$, where $v(\cdot)$ is the characteristic function of deterministic cooperative games. In this subsection, we will show that the Shapley value as laid out by Timmer et al. [6] is equivalent to the classical formulation of this value, denoted by $\Phi(v)$, e.g. cf. Peter Borm in [5]. Assume that the marginal vector $M^\sigma(v) \in \mathbb{R}^N$, for $\sigma \in \Pi(N)$, is in the classical case defined by

$$\begin{aligned} M^\sigma_{\sigma(k)}(v) &= v(\{\sigma(1), \ldots, \sigma(k-1), \sigma(k)\}) - v(\{\sigma(1), \ldots, \sigma(k-1)\}) \\ &= v(S^\sigma_k) - v(S^\sigma_{k-1}) \end{aligned}$$

for all $k \in \{1, \ldots, |N|\}$. In order to prove the equivalence (up to a normalizing factor $v(N)$) of the two characterizations of the Shapley value, we must first prove the following lemma.

Lemma 10.2.1

$$\sum_{k=1}^{i} y^\sigma_{\sigma(k)}(\alpha)v(S^\sigma_k) = v(S^\sigma_i) \qquad (*)$$

Proof We prove this lemma by means of induction. For $i = 1$, we have

$$y^\sigma_{\sigma(1)}(\alpha) \cdot v(S^\sigma_1) = 1 \cdot v(S^\sigma_1) = v(S^\sigma_1).$$

That is our base case. We now assume that $(*)$ holds for $i = 1, \ldots, l$ (this is our induction hypothesis), and show that it implies that for the case $i = l + 1$, the equality is true, too. We find the following equalities

$$\begin{aligned} \sum_{k=1}^{l+1} y^\sigma_{\sigma(k)}(\alpha)v(S^\sigma_k) &= v(S^\sigma_l) + y^\sigma_{\sigma(l+1)}(\alpha)v(S^\sigma_{l+1}) \\ &= v(S^\sigma_l) + \left(1 - \sum_{k=1}^{l} y^\alpha_{\sigma(k)} \cdot \frac{v(S^\sigma_k)}{v(S^\sigma_{l+1})}\right)v(S^\sigma_k) \\ &= v(S^\sigma_l) + v(S^\sigma_{l+1}) - \sum_{k=1}^{l} y^\sigma_{\sigma(k)}(\alpha)v(S^\sigma_k) \\ &= v(S^\sigma_l) + v(S^\sigma_{l+1}) - v(S^\sigma_l) = v(S^\sigma_{l+1}). \end{aligned}$$

When we combine the base case with the implication as shown above, we obtain the desired proof. □

We can use this lemma to prove the following assertion.

Proposition 10.2.1

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v) = v(N) \cdot \phi(\alpha).$$

In [6], Timmer et al. describe that the marginal contributions $y^\sigma_{\sigma(i)}(\alpha)$ of player i, as expressed by a multiple of the random payoff for S^σ_i, are given by the recursive expression

$$y^\sigma_{\sigma(i)}(\alpha) = 1 - \sum_{k=1}^{i-1} \alpha_{\sigma(k)}(S^\sigma_k, S^\sigma_i, y^\sigma_{\sigma(k)}(\alpha)).$$

In our case, we assign a utility $u_{\sigma(i)}(\cdot)$ to this random payoff S^σ_i, making the multiples $y^\sigma_{\sigma(i)}$ "deterministic" in a sense.

We know, from Theorem 4.3.1, that

$$\begin{aligned} \alpha_{\sigma(k)}(S, T, p) &= p \cdot \alpha_{\sigma(k)}(S, T, 1) \\ &= p\frac{v(S)}{v(T)} \qquad \forall k \leq n. \end{aligned}$$

Therefore, the proportional marginal contributions can be rewritten as

$$y^\sigma_{\sigma(i)}(\alpha) = 1 - \sum_{k=1}^{i-1} y^\sigma_{\sigma(k)}(\alpha)\frac{v(S^\sigma_k)}{v(S^\sigma_i)} \qquad \text{with } y^\sigma_{\sigma(1)} = 1.$$

This gives rise to the following expression of the marginal vectors.

$$\begin{aligned} m^\sigma_{\sigma(i)}(\alpha) &= \left(1 - \sum_{k=1}^{i-1} y^\sigma_{\sigma(k)}(\alpha)\frac{v(S^\sigma_k)}{v(S^\sigma_i)}\right) \cdot \frac{v(S^\sigma_i)}{v(N)} \\ &= \frac{v(S^\sigma_i)}{v(N)} - \sum_{k=1}^{i-1} y^\sigma_{\sigma(k)}(\alpha)\frac{v(S^\sigma_k)}{v(N)} \\ &= \frac{1}{v(N)}\Big(v(S^\sigma_i) - \sum_{k=1}^{i-1} y^\sigma_{\sigma(k)}(\alpha)v(S^\sigma_k)\Big). \end{aligned}$$

Remember that in stochastic cooperative game theory, the allocations are given as fractions, relative to the value of the grand coalition $u_i(R(N))$. In this case, we have $u_i(R(N)) = v(N)$. So if we multiply this value with the marginal vectors as given above, we obtain the absolute contributions of each player in each marginal vector. We therefore find

$$M^\sigma_{\sigma(i)}(\alpha) = v(S^\sigma_i) - \sum_{k=1}^{i-1} y^\sigma_{\sigma(k)}(\alpha)v(S^\sigma_k).$$

When we apply Lemma 10.2.1 to the second term of the right side of the equality above, we obtain

$$M^\sigma_{\sigma(i)}(\alpha) = v(S^\sigma_i) - v(S^\sigma_{i-1}).$$

This completes the proof. □

10.3 An example

Let us apply this new concept of the Shapley value to our energy prediction game. We apply it to variant B of the game. It gives us the following result.

Proposition 10.3.1 *Assume that two players play variant B of the energy prediction game, they have expectational preferences, and $\sigma_2 = k\sigma_1 = k\sigma$ for some constant $k \in \mathbb{R}_{>0}$. Then the Shapley value is given by*

$$\phi(\alpha) = \left(\frac{1}{2} + \frac{1-k}{2\sqrt{1+k^2}}, \frac{1}{2} + \frac{k-1}{2\sqrt{1+k^2}}\right).$$

Proof Let us first look at the bijection $\sigma = (1, 2)$. We immediately find $y^{\sigma}_{\sigma(1)}(\alpha) = 1$. In the proof of Proposition 6.1.2, we found $\mathbb{E}(R(\{1\})) = \mathbb{E}(Q_1) = \sigma\sqrt{\frac{2}{\pi}}$. As the denominator can be found by computing the expected value of Q (as described in Equation 10.1), we obtain

$$\begin{aligned}
y^{\sigma}_{\sigma(2)}(\alpha) &= 1 - \alpha_{\sigma(1)}(\{1\}, N, 1) \\
&= 1 - \alpha_1(\{1\}, N, 1) \\
&= 1 - \frac{\mathbb{E}(Q_1)}{\mathbb{E}(Q)} \\
&= 1 - \frac{\sigma\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{1+k^2}\sqrt{\frac{2}{\pi}}} \\
&= 1 - \frac{1}{\sqrt{1+k^2}}.
\end{aligned}$$

We can do the same thing for the other bijection: $\sigma = (2, 1)$. We quickly see that $y^{\sigma}_{\sigma(1)}(\alpha) = 1$. Notice that, in this case, $\mathbb{E}(R(\{2\})) = \mathbb{E}(Q_2) = k\sigma\sqrt{\frac{2}{\pi}}$ (cf. Proposition 6.1.2 and Equation 10.1), because $\sigma_2 = k\sigma$. Hence we compute:

$$\begin{aligned}
y^{\sigma}_{\sigma(2)}(\alpha) &= 1 - \alpha_{\sigma(1)}(\{1\}, N, 1) \\
&= 1 - \alpha_2(\{1\}, N, 1) \\
&= 1 - \frac{\mathbb{E}(Q_2)}{\mathbb{E}(Q)} \\
&= 1 - \frac{k\sigma\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{1+k^2}\sqrt{\frac{2}{\pi}}} \\
&= 1 - \frac{k}{\sqrt{1+k^2}}.
\end{aligned}$$

Thus $y^{(1,2)} = (1, 1 - \frac{1}{\sqrt{1+k^2}})$, and $y^{(2,1)} = (1 - \frac{k}{\sqrt{1+k^2}}, 1)$. We can use these marginal contributions to find the marginal vectors. For $\sigma = (1, 2)$, we have

$$\begin{aligned}
m^{\sigma}_{\sigma(1)}(\alpha) &= m^{\sigma}_1(\alpha) \\
&= \alpha_1(\{1\}, N, 1) \\
&= \frac{\mathbb{E}(Q_1)}{\mathbb{E}(Q)} \\
&= \frac{1}{\sqrt{1+k^2}},
\end{aligned}$$

and

$$\begin{aligned} m^{\sigma}_{\sigma(2)}(\alpha) &= m^{\sigma}_2(\alpha) \\ &= \alpha_2\Big(N, N, 1 - \frac{1}{\sqrt{1+k^2}}\Big) \\ &= 1 - \frac{1}{\sqrt{1+k^2}}. \end{aligned}$$

When $\sigma = (2, 1)$, we find that

$$\begin{aligned} m^{\sigma}_{\sigma(1)}(\alpha) &= m^{\sigma}_2(\alpha) \\ &= \alpha_2(\{2\}, N, 1) \\ &= \frac{\mathbb{E}(Q_2)}{\mathbb{E}(Q)} \\ &= \frac{k}{\sqrt{1+k^2}}, \end{aligned}$$

and

$$\begin{aligned} m^{\sigma}_{\sigma(2)}(\alpha) &= m^{\sigma}_1(\alpha) \\ &= \alpha_1\Big(N, N, 1 - \frac{k}{\sqrt{1+k^2}}\Big) \\ &= 1 - \frac{k}{\sqrt{1+k^2}}. \end{aligned}$$

This means that $m^{(1,2)}(\alpha) = \Big(\frac{1}{\sqrt{1+k^2}}, 1 - \frac{1}{\sqrt{1+k^2}}\Big)$ and $m^{(2,1)}(\alpha) = \Big(1 - \frac{k}{\sqrt{1+k^2}}, \frac{k}{\sqrt{1+k^2}}\Big)$. We therefore find that

$$\begin{aligned} \phi(\alpha) &= \frac{1}{2!}\big(m^{(1,2)}(\alpha) + m^{(2,1)}(\alpha)\big) \\ &= \Big(\frac{1}{2} + \frac{1-k}{2\sqrt{1+k^2}}, \frac{1}{2} + \frac{k-1}{2\sqrt{1+k^2}}\Big). \end{aligned}$$

This completes the proof. □

Let us take a closer look at this Shapley value. When $k = 1$, we see that $\phi(\alpha) = \big(\frac{1}{2}, \frac{1}{2}\big)$. This makes sense, because we have $\sigma_2 = k\sigma_1 = \sigma_1$ in this situation. This in turn implies that both players are equally (im)precise when predicting their energy production levels. Thus, it is be fair for them to pay half of the costs when cooperating.

When $k \neq 1$, when can also intuitively grasp that this Shapley value is correct. As $k > 1$ increases, the value for the first player decreases, while the value for the second player increases. This is fair because the bigger the value k, the better the prediction of player 1 is compared to the prediction of player 2 (as we have $\sigma_1 < \sigma_2$). So player 1 deserves to pay a smaller fraction of the total costs. As $k < 1$ decreases, the same reasoning applies. Analogously, player 1 should bear a larger part of the total costs as $k < 1$.

10.4 Shapley in the core

Though the Shapley value seems a fair allocation of the total costs when two players cooperate in variant B of the energy prediction game, we must also verify that this value is an element of the core of the game. In order to be able to verify this, we first compute the core of the game when the two players have unequal variances.

Proposition 10.4.1 *Assume that two players play variant B of the energy prediction game and* $\sigma_1 = k\sigma_2$ *and their prediction error distributions are uncorrelated. Then,* $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_1 \leq \frac{1}{\sqrt{1+k^2}}, p_2 \leq \frac{k}{\sqrt{1+k^2}}\} \neq \emptyset$.

Proof By Proposition 6.1.2 and Equation 10.1, we have

$$\alpha_1(\{1\}, N, 1) = \frac{\mathbb{E}(Q_1)}{\mathbb{E}(Q)} = \frac{\sigma\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{1+k^2}\sqrt{\frac{2}{\pi}}} = \frac{1}{\sqrt{1+k^2}}.$$

and

$$\alpha_2(\{2\}, N, 1) = \frac{\mathbb{E}(Q_2)}{\mathbb{E}(Q)} = \frac{k\sigma\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{1+k^2}\sqrt{\frac{2}{\pi}}} = \frac{k}{\sqrt{1+k^2}}.$$

By the definition of the core (see Lemma 5.0.1), we obtain

$$C(N, \alpha) = \left\{ p \in \Delta^*(N) \mid p_1 \leq \frac{1}{\sqrt{1+k^2}},\ p_2 \leq \frac{k}{\sqrt{1+k^2}} \right\}.$$

We now show that $C(N, \alpha) \neq \emptyset$. The core of a two-person game can only be empty whenever $\alpha_1(\{1\}, N, 1) + \alpha_2(\{2\}, N, 1) = \frac{k+1}{\sqrt{1+k^2}} < 1$ because then the allocation p cannot be efficient anymore. Setting $\frac{k+1}{\sqrt{1+k^2}} < 1$ yields $k + 1 < \sqrt{1+k^2}$. This implies $(1+k)^2 < 1 + k^2$. In other words, $2k < 0$. This contradicts $k > 0$. Therefore, $C(N, \alpha) \neq \emptyset$. □

Proposition 10.3.1 allows us to prove the following assertion:

Proposition 10.4.2 *Assume that two players play variant B of the energy prediction game. Then $\phi(\alpha) \in C(N, \alpha)$, for whichever value of $k \in \mathbb{R}_{>0}$ is chosen such that $\sigma_2 = k\sigma_1 = k\sigma$.*

Proof From Proposition 10.3.1, we know that for an efficient allocation to lie in the core, we must have $p_1 \leq \frac{1}{\sqrt{1+k^2}}$ and $p_2 \leq \frac{k}{\sqrt{1+k^2}}$. So we must verify whether $p_1 = \frac{1}{2} + \frac{1-k}{2\sqrt{1+k^2}} \leq \frac{1}{\sqrt{1+k^2}}$ and $p_2 = \frac{1}{2} + \frac{k-1}{2\sqrt{1+k^2}} \leq \frac{k}{\sqrt{1+k^2}}$ for all values of $k \in \mathbb{R}_{>0}$. We only verify this for player 2 . We have

$$\begin{aligned}
& 2k > 0 \quad (\text{as } k > 0) \\
\implies & 1 + k^2 < 1 + 2k + k^2 = (1+k)^2 \\
\implies & 4(1+k^2) < 4(1+k)^2 \\
\implies & 2\sqrt{1+k^2} \leq 2(k+1) \\
\implies & 1 \leq \frac{2(k+1)}{2\sqrt{1+k^2}} = \frac{k+1}{\sqrt{1+k^2}} \\
\implies & \frac{1}{2} \leq \frac{k+1}{2\sqrt{1+k^2}} = \frac{k}{\sqrt{1+k^2}} - \frac{(k-1)}{2\sqrt{1+k^2}} \\
\implies & \frac{1}{2} + \frac{k-1}{2\sqrt{1+k^2}} \leq \frac{k}{\sqrt{1+k^2}}
\end{aligned}$$

The last implication verifies the necessary inequality. Furthermore, the Shapley value is efficient by Lemma 10.1.1. This completes the proof. □

10.5 Correlations and unequal standard deviations

Next, we consider the case in which that error distributions are correlated. It is not much of a stretch from Proposition 10.3.1 to generalize that for this game, we obtain the following result:

Proposition 10.5.1 *When two players play variant B of the energy prediction game, where $\sigma_2 = k\sigma_1 = k\sigma$, and their prediction errors are correlated through Pearson's correlation coefficient ρ, we have*

$$\phi(\alpha) = \Big(\frac{1}{2} + \frac{1-k}{2\sqrt{1+2k\rho+k^2}},\ \frac{1}{2} + \frac{k-1}{2\sqrt{1+2k\rho+k^2}}\Big).$$

We present this result without proof, it would be a tad repetitive to repeat the calculations presented in the proof op Proposition 10.3.1. In addition, we ascertain that for this generalization, too, the Shapley value lies in the core.

It is worthwhile to quickly point out the role of ρ in the Shapley value. In the case of a positive correlation $\rho > 0$, the correlation coefficient acts as a damper of the extra (negative) costs that need to be paid by both players. This is true because it increases the denominator in both fractions for the proportion of the costs to be paid per player. However, when $\rho < 0$, the correlation coefficient starts acting as an amplifier, as it decreases the value of this denominator.

By similar calculations to those in the proof of Proposition 10.4.1, we see that

Proposition 10.5.2 *When two players play variant B of the energy prediction game, where $\sigma_2 = k\sigma_1 = k\sigma$, and their prediction errors are correlated through Pearson's correlation coefficient ρ, we have*

$$C(N,\alpha) = \Big\{p \in \Delta^*(N) \mid p_1 \leq \frac{1}{\sqrt{1+k^2+2k\rho}},\ p_2 \leq \frac{k}{\sqrt{1+k^2+2k\rho}}\Big\} \neq \emptyset.$$

We only show that $C(N,\alpha) \neq \emptyset$. The calculation of the core itself is similar to those in Proposition 10.4.1, so we don't deem it necessary to present them here again.

We note that the core of the game is empty whenever $\alpha_1(\{1\}, N, 1) + \alpha_2(\{2\}, N, 1) < 1$. In our case, this translates to $\frac{k+1}{\sqrt{1+k^2+2k\rho}} < 1$. After some basic algebraic operations, this boils down to to $2k \leq 2k\rho$, implying that $\rho > 1$. By the definition of the Pearson correlation coefficient, $\rho \in [-1, 1]$, so this is impossible. This means that, however correlated the prediction errors are, it is always in the interest of the individual participating players to cooperate. □

10.6 Real-world consequences

This implies that in the real world, it is beneficial for the owners of differing types of renewable energy sources to cooperate by jointly performing a prediction of the energy production levels of their devices. In Section 9.1 we pointed out that cooperation is often beneficial for the same types of renewable energy sources (when their standard deviations of the prediction error distributions are exactly the same, which is, admittedly, unlikely). The results above moreover show that cooperation is also beneficial when the standard deviations of the prediction errors are unequal to one another. This means that we can combine multiple technologies, like solar panels and wind mills, which differ in the accuracy with regards to their predictions of their production levels. Furthermore, we demonstrated that cooperation also yields benefits when the error distributions are correlated. The correlations have been determined on the basis of experiments, as can be seen in Section IV.B of [11] and citations therein. They have been calculated for different combinations of resources, such as wind and solar energy, and wind and wave energy. The positive ascertainment we can make is that the value of the Pearson correlation coefficient does not matter and thus that cooperation between any combination of renewable energy sources is always beneficial.

11 Three players

So far, we have only focused on the situation in which two players participate in the energy production prediction game. It is rather restrictive, however, to confine oneself to this setting. Instead, it is plausible that more than two players wish to participate to reap the benefits of cooperation. In the forthcoming (sub)sections, we will show that is possible that three players cooperate. We will compute the core and the Shapley value for this type of this game. Additionally, we prove that for the 3-player game, the Shapley value is an element of the core, too. We allow the standard deviations to be (possibly) unequal to one another, but uncorrelated. This means, that we have $\sigma_2 = k_1\sigma_1 = k_1\sigma$, and $\sigma_3 = k_2\sigma_1 = k_2\sigma$.

11.1 The core

Recall that in Lemma 5.0.1, we showed that the core amounts to

$$C(N,\alpha) = \Big\{p \in \Delta^*(N) \mid \sum_{i\in S} p_i/\alpha_i(S,N,1) \geq 1 \text{ for all } S \subset N\Big\}.$$

For the cost game, the inequality signs are flipped.
This means that for a two-player game, the core is equal to the imputation set (as defined on page 3). In that situation (which has been the case so far), we only need to consider two inequalities to determine both the imputation set and the core. In the three-player variant of the game, however, things become slightly more complicated. The core is more complicated than the imputation set. The costs are as follows:

$$R(S) = \begin{cases} |P_1| & \text{if } S = \{1\} \\ |P_2| & \text{if } S = \{2\} \\ |P_3| & \text{if } S = \{3\} \\ |P_1 + P_2| & \text{if } S = \{1,2\} \\ |P_1 + P_3| & \text{if } S = \{1,3\} \\ |P_2 + P_3| & \text{if } S = \{2,3\} \\ |P_1 + P_2 + P_3| & \text{if } S = \{1,2,3\}, \end{cases}$$

where, $P_1 \sim N(0,\sigma^2)$, $P_2 \sim N(0, k_1^2 \cdot \sigma^2)$, and $P_3 \sim N(0, k_2^2 \cdot \sigma^2)$.
We will now compute the core, in case all players have expectational preferences.

Proposition 11.1.1 *Assume that three players play variant B of the energy production prediction game. Then*

$$C(N,\alpha) = \left\{ p \in \Delta^*(N) \;\middle|\; \begin{array}{lll} p_1 & \leq & \frac{1}{\sqrt{1+k_1^2+k_2^2}} \\ p_2 & \leq & \frac{k_1}{\sqrt{1+k_1^2+k_2^2}} \\ p_3 & \leq & \frac{k_2}{\sqrt{1+k_1^2+k_2^2}} \\ p_1+p_2 & \leq & \sqrt{\frac{1+k_1^2}{1+k_1^2+k_2^2}} \\ p_1+p_3 & \leq & \sqrt{\frac{1+k_2^2}{1+k_1^2+k_2^2}} \\ p_2+p_3 & \leq & \sqrt{\frac{k_1^2+k_2^2}{1+k_1^2+k_2^2}} \end{array} \right\}.$$

Proof Using Lemma 5.0.1, we have

$$\begin{aligned} p_1 \leq \alpha_1(\{1\},N,1) &= \frac{\mathbb{E}(|P_1|)}{\mathbb{E}(|P_1+P_2+P_3|)} \\ &= \frac{\sigma\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{1+k_1^2+k_2^2}\sqrt{\frac{2}{\pi}}} \\ &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}. \end{aligned}$$

In a similar manner, we find $p_2 \leq \alpha_2(\{2\}, N, 1) = \frac{k_1}{\sqrt{1+k_1^2+k_2^2}}$ and $p_3 \leq \alpha_3(\{3\}, N, 1) = \frac{k_2}{\sqrt{1+k_1^2+k_2^2}}$. These are the inequalities associated with the singletons $\{1\}$, $\{2\}$, and $\{3\}$.

When we take into consideration the inequality associated with the set $S = \{1, 2\}$, we obtain

$$\sum_{i \in \{1,2\}} p_i/\alpha_i(\{1, 2\}, N, 1) = \frac{p_1}{\alpha_1(\{1, 2\}, N, 1)} + \frac{p_2}{\alpha_2(\{1, 2\}, N, 1)} \leq 1.$$

As both players have expectational preferences, we obtain

$$\begin{aligned}
\sum_{i \in \{1,2\}} p_i/\alpha_i(\{1, 2\}, N, 1) &= \frac{p_1}{\left(\frac{\mathbb{E}(|P_1+P_2|)}{\mathbb{E}(|P_1+P_2+P_3|}\right)} + \frac{p_2}{\left(\frac{\mathbb{E}(|P_1+P_2|)}{\mathbb{E}(|P_1+P_2+P_3|)}\right)} \\
&= p_1 \frac{\mathbb{E}(|P_1 + P_2 + P_3|)}{\mathbb{E}(|P_1 + P_2|)} + p_2 \frac{\mathbb{E}(|P_1 + P_2 + P_3|)}{\mathbb{E}(|P_1 + P_2|)} \\
&= (p_1 + p_2) \frac{\mathbb{E}(|P_1 + P_2 + P_3|)}{\mathbb{E}(|P_1 + P_2|)} \leq 1.
\end{aligned}$$

This implies that

$$\begin{aligned}
p_1 + p_2 &\leq \frac{\mathbb{E}(|P_1 + P_2|)}{\mathbb{E}(|P_1 + P_2 + P_3|)} \\
&= \frac{\sigma\sqrt{1 + k_1^2}\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{1 + k_1^2 + k_2^2}\sqrt{\frac{2}{\pi}}} \\
&= \sqrt{\frac{1 + k_1^2}{1 + k_1^2 + k_2^2}}.
\end{aligned}$$

Analogously, we find $p_1 + p_3 \leq \sqrt{\frac{1+k_2^2}{1+k_1^2+k_2^2}}$ and $p_2 + p_3 \leq \sqrt{\frac{k_1^2+k_2^2}{1+k_1^2+k_2^2}}$.
This completes the proof. □

11.2 The Shapley value

Next, we will show that the Shapley value is an element of the core. In order to verify this, we first need to compute the Shapley value for a three-player game. Recall that in Subsection 10.1, the Shapley value was defined in the context of stochastic cooperative games. We will follow the procedure for finding the Shapley value as given in that subsection. This leads us to the following proposition:

Proposition 11.2.1 *For a three-player energy production prediction game of variant B with $\sigma_2 = k_1\sigma_1 = k_1\sigma$ and $\sigma_3 = k_2\sigma_1 = k_2\sigma$, the Shapley value is given by*

$$\begin{aligned}
\phi(\alpha) = \frac{1}{6\sqrt{1 + k_1^2 + k_2^2}} \Big(& 2 + \sqrt{1 + k_1^2} + \sqrt{1 + k_2^2} + 2\sqrt{1 + k_1^2 + k_2^2} - 2\sqrt{k_1^2 + k_2^2} - k_1 - k_2, \\
& 2k_1 + \sqrt{k_1^2 + 1} + \sqrt{k_1^2 + k_2^2} + 2\sqrt{1 + k_1^2 + k_2^2} - 2\sqrt{1 + k_2^2} - 1 - k_2, \\
& 2k_2 + \sqrt{k_2^2 + 1} + \sqrt{k_2^2 + k_1^2} + 2\sqrt{1 + k_1^2 + k_2^2} - 2\sqrt{1 + k_1^2} - 1 - k_1\Big).
\end{aligned}$$

Proof We first calculate the marginal contribution vectors $y^\sigma(\alpha)$, where $y^\sigma_{\sigma(i)}$ is the marginal contribution of the ith player according to the bijection σ. We start with $\sigma = (1, 2, 3)$.
We have $y^\sigma_{\sigma(1)}(\alpha) = y^\sigma_1(\alpha) = 1$. In this case, we have $S^\sigma_1 = \{1\}$. Since $\sigma(2) = 2$, we find $S^\sigma_2 = \{1, 2\}$. Therefore,

$$
\begin{aligned}
y^\sigma_{\sigma(2)}(\alpha) &= y^\sigma_2(\alpha)\\
&= 1-\alpha_{\sigma(1)}(S^\sigma_1, S^\sigma_2, y^\sigma_{\sigma(1)}(\alpha))\\
&= 1-\alpha_1(\{1\}, N, 1)\\
&= 1-\frac{\mathbb{E}(|P_1|)}{\mathbb{E}(|P_1+P_2|)}\\
&= 1-\frac{\sigma\sqrt{\frac{2}{\pi}}}{\sigma\sqrt{1+k_1^2}\sqrt{\frac{2}{\pi}}}\\
&= 1-\frac{1}{\sqrt{1+k_1^2}}.
\end{aligned}
$$

Next, $\sigma(3)=3$ is added, and the coalition $S^\sigma_3=\{1,2,3\}=N$ is formed. Thus,

$$
\begin{aligned}
y^\sigma_{\sigma(3)} &= y^\sigma_3(\alpha)\\
&= 1-\sum_{k=1}^{2}\alpha_{\sigma(k)}(S^\sigma_k, S^\sigma_3, y^\sigma_{\sigma(k)}(\alpha))\\
&= 1-\alpha_1\big(\{1\}, N, 1\big)-\alpha_2\Big(\{1,2\}, N, 1-\frac{1}{\sqrt{1+k_1^2}}\Big)\\
&= 1-\frac{1}{\sqrt{1+k_1^2+k_2^2}}-\Big(1-\frac{1}{\sqrt{1+k_1^2}}\Big)\cdot\frac{\mathbb{E}(|P_1+P_2|)}{\mathbb{E}(|P_1+P_2+P_3|)}\\
&= 1-\frac{1}{\sqrt{1+k_1^2+k_2^2}}-\Big(1-\frac{1}{\sqrt{1+k_1^2}}\Big)\Big(\frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}\Big)\\
&= 1-\frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}.
\end{aligned}
$$

This means that our first marginal contribution vector is given by

$$
y^{(1,2,3)}(\alpha)=\Big(1, 1-\frac{1}{\sqrt{1+k_1^2}}, 1-\frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}\Big).
$$

Analogously, we can compute all of the other marginal contribution vectors for the other bijections. We state the results without going through all of the corresponding calculations (the calculation of the first marginal contribution vector is deemed sufficient). They're given by:

$$
\begin{aligned}
y^{(1,3,2)}(\alpha) &= \Big(1, 1-\frac{\sqrt{1+k_2^2}}{\sqrt{1+k_1^2+k_2^2}}, 1-\frac{1}{\sqrt{1+k_2^2}}\Big),\\
y^{(2,1,3)}(\alpha) &= \Big(1-\frac{k_1}{\sqrt{1+k_1^2}}, 1, 1-\frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}\Big),\\
y^{(2,3,1)}(\alpha) &= \Big(1-\frac{\sqrt{k_1^2+k_2^2}}{\sqrt{1+k_1^2+k_2^2}}, 1, 1-\frac{k_1}{\sqrt{k_1^2+k_2^2}}\Big),\\
y^{(3,1,2)}(\alpha) &= \Big(1-\frac{k_2}{\sqrt{1+k_2^2}}, 1-\frac{\sqrt{1+k_2^2}}{\sqrt{1+k_1^2+k_2^2}}, 1\Big),\\
y^{(3,2,1)}(\alpha) &= \Big(1-\frac{\sqrt{k_1^2+k_2^2}}{\sqrt{1+k_1^2+k_2^2}}, 1-\frac{k_2}{\sqrt{k_1^2+k_2^2}}, 1\Big).
\end{aligned}
$$

Next, we calculate the marginal vectors. We calculate the marginal vector $m^{(1,2,3)}(\alpha)$. We obtain

$$\begin{aligned} m^\sigma_{\sigma(1)}(\alpha) &= m^\sigma_1(\alpha) \\ &= \alpha_1(S^\sigma_1, N, y^\sigma_1(\alpha)) \\ &= \alpha_1(\{1\}, N, 1) \\ &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}. \end{aligned}$$

In addition,

$$\begin{aligned} m^\sigma_{\sigma(2)}(\alpha) &= m^\sigma_2(\alpha) \\ &= \alpha_2(S^\sigma_2, N, y^\sigma_2(\alpha)) \\ &= \alpha_2\big(\{1,2\}, N, 1 - \frac{1}{\sqrt{1+k_1^2}}\big) \\ &= \Big(1 - \frac{1}{\sqrt{1+k_1^2}}\Big)\Big(\frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}\Big) \\ &= \frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}} - \frac{1}{\sqrt{1+k_1^2+k_2^2}}. \end{aligned}$$

Finally, we derive:

$$\begin{aligned} m^\sigma_{\sigma(3)} &= m^\sigma_3(\alpha) \\ &= \alpha_3(S^\sigma_3, N, y^\sigma_3(\alpha)) \\ &= \alpha_3\Big(N, N, 1 - \frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}\Big) \\ &= 1 - \frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}. \end{aligned}$$

Thus, our first marginal vector is given by

$$\begin{aligned} m^{(1,2,3)} &= \Big(\frac{1}{\sqrt{1+k_1^2+k_2^2}}, \frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}} - \frac{1}{\sqrt{1+k_1^2+k_2^2}}, 1 - \frac{\sqrt{1+k_1^2}}{\sqrt{1+k_1^2+k_2^2}}\Big) \\ &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}\Big(1, \sqrt{1+k_1^2} - 1, \sqrt{1+k_1^2+k_2^2} - \sqrt{1+k_1^2}\Big). \end{aligned}$$

Performing the same calculations with the other bijections enables us to find the other marginal vectors, too. They are:

$$\begin{aligned} m^{(1,3,2)} &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}\Big(1, \sqrt{1+k_1^2+k_2^2} - \sqrt{1+k_2^2}, \sqrt{1+k_2^2} - 1\Big), \\ m^{(2,1,3)} &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}\Big(\sqrt{1+k_1^2} - k_1, k_1, \sqrt{1+k_1^2+k_2^2} - \sqrt{1+k_1^2}\Big), \\ m^{(2,3,1)} &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}\Big(\sqrt{1+k_1^2+k_2^2} - \sqrt{k_1^2+k_2^2}, k_1, \sqrt{k_1^2+k_2^2} - k_1\Big), \\ m^{(3,1,2)} &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}\Big(\sqrt{1-k_2^2} - k_2, \sqrt{1+k_1^2+k_2^2} - \sqrt{1+k_2^2}, k_2\Big), \\ m^{(3,2,1)} &= \frac{1}{\sqrt{1+k_1^2+k_2^2}}\Big(\sqrt{1+k_1^2+k_2^2} - \sqrt{k_1^2+k_2^2}, \sqrt{k_1^2+k_2^2} - k_2, k_2\Big). \end{aligned}$$

Therefore, the Shapley value if given by

$$\begin{aligned}\phi(\alpha) &= \frac{1}{n!}\sum_{\sigma\in\Pi(N)} m^{\sigma}(\alpha)\\ &= \frac{1}{3!}\big(m^{(1,2,3)}+m^{(1,3,2)}+m^{(2,1,3)}+m^{(2,3,1)}+m^{(3,1,2)}+m^{(3,2,1)}\big)\\ &= \frac{1}{6\sqrt{1+k_1^2+k_2^2}}\Big(2+\sqrt{1+k_1^2}+\sqrt{1+k_2^2}+2\sqrt{1+k_1^2+k_2^2}-2\sqrt{k_1^2+k_2^2}-k_1-k_2,\\ &\qquad 2k_1+\sqrt{k_1^2+1}+\sqrt{k_1^2+k_2^2}+2\sqrt{1+k_1^2+k_2^2}-2\sqrt{1+k_2^2}-1-k_2,\\ &\qquad 2k_2+\sqrt{k_2^2+1}+\sqrt{k_2^2+k_1^2}+2\sqrt{1+k_1^2+k_2^2}-2\sqrt{1+k_1^2}-1-k_1\Big),\end{aligned}$$

as desired. □

Now that we have characterized both the core and the Shapley value for the three-player game, we are ready to prove the following proposition:

Proposition 11.2.2 *Assume that three players play variant B of the energy production game, and $\sigma_2 = k_1\sigma_1$ and $\sigma_3 = k_2\sigma_1 = k_2\sigma$. Then $\phi(\alpha) \in C(N,\alpha)$.*

Proof First note that the Shapley value is efficient by Lemma 10.1.1. Furthermore, to prove the above statement, we need to verify that the Shapley value satisfies all six inequalities that characterize the core of this game , cf. Proposition 11.1.1. Let's start by proving the first inequality. This entails proving that

$$p_1 = \frac{2+\sqrt{1+k_1^2}+\sqrt{1+k_2^2}+2\sqrt{1+k_1^2+k_2^2}-2\sqrt{k_1^2+k_2^2}-k_1-k_2}{6\sqrt{1+k_1^2+k_2^2}} \leq \frac{1}{\sqrt{1+k_1^2+k_2^2}},$$

which – with a bit of basic algebra – is equivalent to the inequality

$$\begin{aligned}\sqrt{1+k_1^2}+\sqrt{1+k_2^2}+2\sqrt{1+k_1^2+k_2^2} &\leq 4+k_1+k_2+2\sqrt{k_1^2+k_2^2}\\ &= (1+k_1)+(1+k_2)+2(1+\sqrt{k_1^2+k_2^2}).\end{aligned}$$

We have re-arranged the right-hand side in order to clarify the validity of the inequality by applying the triangle inequality:

$$\begin{aligned}\sqrt{1+k_1^2} &\leq 1+k_1\\ \sqrt{1+k_2^2} &\leq 1+k_2\\ 2\sqrt{1+k_1^2+k_2^2} &\leq 2(1+\sqrt{k_1^2+k_2^2}).\end{aligned}$$

Adding these together gives us the verification we need.

We can follow a similar procedure the prove that the second and third inequalities characterizing the core hold. We therefore skip these and move to inequalities four, five, and six. For these inequalities, we construct another procedure. This procedure was inspired by an insight by Fedor Petrov [18].

We focus on the sixth (and last) inequality of the core. This is the inequality for $p_2 + p_3$. It boils down to proving that

$$4\sqrt{k_1^2+k_2^2+1}-4\sqrt{k_1^2+k_2^2} \leq \sqrt{k_1^2+1}-k_1+\sqrt{k_2^2+1}-k_2+2$$

is true for all $k_1, k_2 \in \mathbb{R}_{>0}$. To this end, we define $f(x) := \sqrt{x+1}-\sqrt{x}$. By computing the derivative, we see that $f'(x) = \frac{1}{2}(\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{x}}) < 0$ for all $x > 0$, which means that f is decreasing

for $x > 0$.
This means, that the following three inequalities hold:

$$f(k_1^2 + k_2^2) \leq f(k_1^2),$$
$$f(k_1^2 + k_2^2) \leq f(k_2^2),$$
$$2f(k_1^2 + k_2^2) \leq 2f(0).$$

Adding these terms together yields $4f(k_1^2 + k_2^2) \leq f(k_1^2) + f(k_2^2) + 2f(0)$. In other words,

$$4\sqrt{k_1^2 + k_2^2 + 1} - 4\sqrt{k_1^2 + k_2^2} \leq \sqrt{k_1^2 + 1} - k_1 + \sqrt{k_2^2 + 1} - k_2 + 2,$$

which is what we needed to prove.
Analogously, we can prove the fourth and fifth inequalities of the core by defining the functions $g(x) := \sqrt{x + k_2^2} - \sqrt{x}$ and $h(x) := \sqrt{x + k_1^2} - \sqrt{x}$, respectively. If we follow the same procedure as the one above, the result follows. $\square$

12 Optimal coalitions

In Section 9.1, we explained that cooperation gets increasingly beneficial between players for which the correlation coefficient of their prediction error is as small as possible. As the value of this correlation coefficient is inversely correlated with the distance between the renewable energy sources (up to about 600 kilometers), the further they are removed from one another, the more beneficial it is for them to cooperate.

It would be interesting to see just how much the players can save on their prediction error costs. This is probably best illustrated by means of an example. Consider the simplified, two-player case with $\sigma_1 = \sigma_2 = \sigma$, $\rho = 0$ and both have expectational preferences. Recall that in Proposition 6.1.2, we found that the core is $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_1 \leq \frac{\sqrt{2}}{2} \approx 0.707,\ p_2 \leq \frac{\sqrt{2}}{2} \approx 0.707\}$. In Proposition 10.4.2 we also showed that the Shapley value is an element of the core. For our case, this amounts to $\phi(\alpha) = (\frac{1}{2}, \frac{1}{2})$.

This means that the percentual decrease of the costs per player is $-\left(\frac{0{,}5-0{,}707}{0{,}707}\right) \times 100\% = 29,3\%$. This is already quite a substantial decrease of the costs!

We can also look at the three-player case. Again, we suppose $\sigma_1 = \sigma_2 = \sigma$. It follows from Proposition 11.1.1 that the core in this case is $C(N, \alpha) = \{p \in \Delta^*(N) \mid p_1 \leq \frac{1}{\sqrt{3}},\ p_2 \leq \frac{1}{\sqrt{3}},\ p_3 \leq \frac{1}{\sqrt{3}},\ p_1 + p_2 \leq \sqrt{\frac{2}{3}},\ p_1 + p_3 \leq \sqrt{\frac{2}{3}},\ p_2 + p_3 \leq \sqrt{\frac{2}{3}}\}$. So $p_1 \leq \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx 0.577$. Proposition 11.2.1 implies $\phi(\alpha) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Compared to the costs to be paid when an individual player does not cooperate with any other player, the costs decrease by $-\left(\frac{0.333-0.577}{0{,}577}\right) \times 100\% = 42.3\%$. Compared to the case in which two players cooperate, the three-player variant is $42.3\% - 29.3\% = 13$ percentage points cheaper.

These are the simple, albeit a bit unrealistic cases. In reality, the deviations of the prediction error distributions are often not equal to one another. Furthermore, the predictions are often correlated. We can calculate the benefits per owner of a renewable energy source in any specific case we wish, thanks to the analysis of on unequal standard deviations (Section 10) and correlated prediction errors (Section 9). Again, it could be instructive to look at an example.

Consider the case of renewable energy cooperatives in the Netherlands. Renewable energy cooperatives are citizen-led initiatives in which renewable energy sources are collectively purchased and the produced energy is sold. These renewable energy cooperatives are spread all across the Netherlands (and other countries). In some municipalities, there are wind energy cooperatives. In others, there a solar energy cooperatives. Some municipalities even have both. Sometimes, these cooperatives are not separate, but they have both wind mills and solar panels. Of course, there are also municipalities in which there is not a single renewable energy cooperative present.

Let's focus on the wind cooperatives first. The following map was made based on data collected by the renewable energy cooperatives research center Hier Opgewekt [21]. It shows all municipalities in which a wind energy cooperative is located.

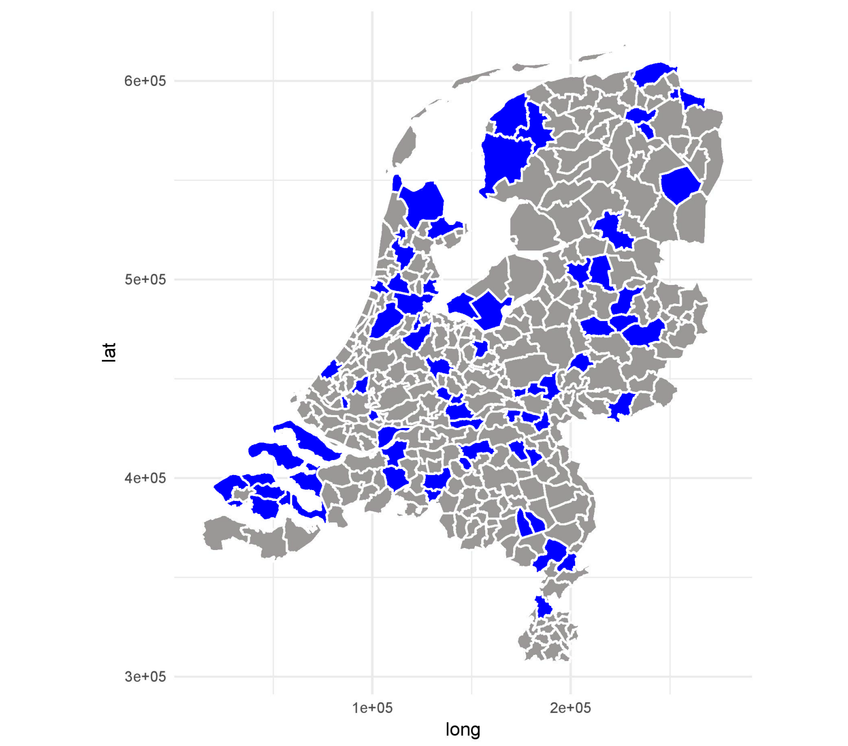

Figure 9: Municipalities in which a wind cooperative was either realised by the end of 2016, planned in 2017, or in preparation in 2017.

If, for whatever reason, only two cooperatives are allowed to cooperate, then by Section 9.1 it would be most beneficial for both players to be as far away from each other as possible. In the case of the wind energy cooperatives, it would be best if the cooperative from the municipality Veere (in the South-West corner of the province Zeeland) called "Zeeuwing/Mattenhaven", would cooperate with the cooperative from Eemsmond (in the North of the province Groningen). The latter cooperative is called "Kantens/2 EAZ". Let's call these cooperatives "Zeeuwing" and "Kantens", respectively, for convenience. As the crow flies, the distance between Veere and Eemsmond is about 290 kilometers. According the graph by Focken et al., shown in Section 9.1, the cross-correlation coefficient is about $0,1$ for the 6-hour prediction error for this distance. Furthermore, both are wind cooperatives. The prediction techniques are probably not very different, and therefore one is supposedly not much more precise than the other. We can therefore assume that $0.8 \leq k \leq 1.2$. Let's choose $k = 1.2$. In Proposition 10.5.2, we pointed out that the core in the case of unequal standard deviations and non-zero correlation coefficients is

$$C(N, \alpha) = \left\{ p \in \Delta^*(N) \mid p_1 \leq \frac{1}{\sqrt{1 + k^2 + 2k\rho}},\ p_2 \leq \frac{k}{\sqrt{1 + k^2 + 2k\rho}} \right\}.$$

Plugging in the values we deduced above for this case, we see

$$C(N, \alpha) = \{p \in \Delta^*(N) \mid p_1 \leq 0,611,\ p_2 \leq 0,733\}.$$

In Proposition 10.5.1, we also obtained an expression for the Shapley value, which is:

$$\phi(\alpha) = \left(\frac{1}{2} + \frac{1 - k}{2\sqrt{1 + 2k\rho + k^2}},\ \frac{1}{2} + \frac{k - 1}{2\sqrt{1 + 2k\rho + k^2}} \right).$$

When we again plug in the values for k and ρ we deduced above, we obtain

$$\phi(\alpha) = (0.439\ , 0.561).$$

This means that for player 1 and player 2, the percentual decrease in costs amount to $-\left(\frac{0.439-0.611}{0.611}\right) \times 100\% \approx 28.1\%$ and $-\left(\frac{0.561-0.733}{0.733}\right) \times 100\% \approx 23.5\%$ respectively. Note that the more accurate player (the one with the lower standard deviation) is rewarded a higher percentual cost decrease than the more inaccurate one.

13 Directions for further research

In this thesis, we have characterized the core and the Shapley value for two-player (which we analyzed for two variants) and three-player energy prediction games. We allowed for non-equal standard deviations and dependent prediction error distributions. We showed that for these cases and appropriate values of the correlation coefficient, the Shapley value is an element of the core. Furthermore, we characterized the core for different von-Neumann-Morgenstern preferences and for both expectational and quantile preferences, all within the framework of stochastic cooperative game theory as laid out by [6].

An obvious way to extend our research is to take into account more than three players. We strongly suspect that the core is non-empty for any amount of players that participate. Furthermore, we believe that the associated Shapley value is an element of the core.

Furthermore, different assumptions on the underlying distributions of the prediction error can be taken into account for the calculations. Throughout this thesis, we have assumed that it is a normal distribution with $\mu = 0$ and some value $\sigma > 0$. This implies that the associated cost distribution is half-normal. However, experimental data suggests that, although it is close to zero, $\mu \neq 0$. Therefore, the associated cost distribution is not half-normal. The half-normal distribution is a special case of the so-called "Folded normal distribution" when $\mu = 0$. However, when $\mu \neq 0$, we obtain different values for the expectation and the quantiles of the random variable. Thus, the core and the Shapley value change, too. We can even discard the assumption that the normalized prediction error is normally distributed at all. As we mentioned in Section 6, the hyperbolic distribution is a more accurate representation of the prediction error. Taking this fact into account would complicate matters even further, but at the same time it would make the description of the energy prediction game more accurate.

In addition, it can be verified whether other properties and concepts of stochastic cooperative game theory can be applied to the energy prediction game. In [6], Timmer et al. describe different notions of convexity. One can try to prove or disprove whether the energy prediction game is marginal convex, individual-merge convex, or coalitional-merge convex. Another property that can be checked is superadditvity.

Furthermore, the theory of stochastic cooperative game theory itself can be extended, after which it can be applied to energy prediction problems (among other things). For instance, the notion of the nucleolus has not been defined within the framework of Timmer et al. [6]. It has only been described [22] for the version of stochastic cooperative game theory as laid out by Suijs et al. [23]. Another solution concept that can be useful in this context is the Aumann-Shapley value. It can give a description of the Shapley value for energy prediction games in which a very large number of players participate. The Aumann-Shapley value can give an approximation of the "regular" Shapley value in this case, as it describes the asymptotic behaviour of the Shapley value for an infinite amount of players.

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14 Appendix

14.1 Appendix A

In Subsection 4.1, we introduced the notion of preferences for stochastic payoffs (or costs). For **Example 1.1**, it is intuitively clear why a player would prefer stochastic variable X over Y when $\mathbb{E}(X) \geq \mathbb{E}(Y)$.

For **Example 1.2**, however, things are less obvious. Below, we will explain the notion of quantile preferences in a bit more detail.

Interpretation

Let us consider the situation of a profit game. First of all, the value of β_i that player i chooses is related to the way it approaches risk. When it chooses a low value, the player is seeking to determine which random variable has the highest quantile at this low value of β_i. Since the value of β_i is low, player i can't expect to make a lot of profit at this point. But it chooses the random variable for which quantile for that value is the highest. So, in a sense, the player is maximizing the minimum value of the random variable. In other words, it is optimizing a maxmin problem. Therefore, players who choose a low value of β_i are *risk-averse* players.

On the other hand, a player could also choose a high value of β_i. In that case, the players is optimizing for the random variable with the highest maximum quantile. So the player is, in a sense, optimizing a maxmax problem. Such a player is deemed *risk-seeking*. It does not care much for the lower possible quantiles of the random variable, even if they're potentially very low.

In the cost game, the situation is reversed: high values for β_i are associated with risk-averse behavior, while low values correspond to risk-seeking.

14.2 Appendix B

The relationship between different notions of riskiness associated with different random variables was more formally established by Rothschild and Stiglitz in [20]. According to them, there are (at least) four plausible answers to the question: "When is a random variable Y 'more variable' than another random variable X ?" Two of these answers are:

1. *Every risk averter prefers X to Y.* In Section 7, we elaborated on the so-called von-Neumann-Morgenstern preferences. We explained that those players i with concave utility function $u_i(X)$ are risk-averse players. Suppose X and Y have the same mean, but a risk-averse player i prefers X to Y. In other words:

$$\mathbb{E}(u_i(X)) \geq \mathbb{E}(u_i(Y)) \quad \text{for all concave functions } u_i(\cdot).$$

   (If the utility function is convex, the inequality sign flips). If this is the case, it is reasonable to state that X is less risky than Y.

2. *Random variable Y has more weight in the tails than X.* Suppose that X and Y have density functions $f_X(\cdot)$ and $g_Y(\cdot)$ respectively, and that $g_Y(\cdot)$ was obtained from $f_X(\cdot)$ by removing some of the probability weight from the center of $g_Y(\cdot)$ and adding it to the tails of $f_X(\cdot)$ in such a way that its mean remains unchanged. Then it would be fair to say that X is less uncertain than Y.

Rothschild and Stiglitz prove in their paper that these two (and one other) notions of riskiness are equivalent to another. We will not repeat nor analyze their proof, which is **Theorem 2** of their article. We do point out, however, that the equivalence of these two characterizations provides us with a useful framework within which we can interpret the von-Neumann-Morgenstern preferences. We now see that, if a risk-averse player can choose between stochastic payoffs X and Y with the same mean, it would choose the one with the least weight in the tails. So a risk-averse player would choose stochastic payoff X, because the expected utility it derives from it is highest in that case. However, a risk-seeking player would choose Y, because there is more weight in the tails. So there is a bigger chance the risk-seeking player would make *more* profits, although there is an equally big chance it would make less profits. Equivalently, the expected utility of the risk-seeking player is more for the random variable Y than for X.