Random variables on non-separable Banach spaces

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1 Introduction

When lifting well-known results from probability theory to random variables that take values in spaces other than the real numbers, one usually considers the ones that take values in separable Banach spaces. This is done for technical reasons, in particular because a necessary condition for a random variable to be integrable is to be (a version of) a separably valued random variable. However, this does not mean that these results cannot be lifted to the setting of non-separable Banach spaces. It turns out that restricting to separable subspaces allows us to lift a lot of results from the separable case to the non-separable one. A lot of care has to be taken in this process and therefore it is necessary to keep all definitions and arguments very precise. Many of the arguments will turn out to be far more subtle and involved than may be thought at first sight. Moreover, we will rigorously prove several elementary results, that usually are assumed to hold in this more general setting, but are almost never proven.

There is not much literature on this subject (see, e.g. [1], [2], [3] and [4]) and the existing literature is not always consistent or precise in its definitions. Therefore we thought it useful to start by making a detailed overview of all the definitions and basic results necessary to prove our results later on. Chapters 2 to 7 provide some well-known elementary concepts and results from Functional Analysis and Measure Theory. In Chapters 8 to 12 we consider the standard construction of random variables and Bochner integrals in Banach spaces. When working in the non-separable setting, it turns out that the concept of independence of random variables is less straight-forward and that none of the literature rigorously builds up this concept, even in the separable setting. Therefore, we spent a lot of effort in Chapters 13 and 14 to develop this concept from scratch. Moreover, this allows us to lift one of the main results from [11] to the non-separable setting.

One of the main reasons to consider non-separable spaces is because if E is a Banach space and $T > 0$, then the Banach space

$$\mathbb{D}_E[0,T] := \{f : [0,T] \to E \mid f \text{ cadlag}\}$$

equipped with supremum norm is not separable if $E \neq \{0\}$.

In Chapter 15 it turns out that our thorough set-up allows us to find a simple proof for another main result from [11], which gives a set of conditions for a series of elements in the space $\mathbb{D}_E[0,T]$ to converge. In Chapter 16 and 17 we describe the standard set-up of conditional expectations and martingales in Banach spaces. Finally in Chapter 18, we obtain several new results on the convergence of series of random variables.

The main new results can be found in Chapter 13, in particular Theorem 13.20, Chapter 14 and Chapter 18.

2 Sets

In this section we will give some basic definitions about sets, such as the definition of a π-system in a set Ω and the definition of a σ-algebra in a set Ω. We will also give some basic properties that deal with these definitions.

Definition 2.1. Let Ω be a set. A family $\mathcal{I}$ of subsets of Ω is a **π-system** if for all $I_1, I_2 \in \mathcal{I}$ we have that $I_1 \cap I_2 \in \mathcal{I}$.

Definition 2.2. Let Ω be a set. A family $\mathcal{D}$ of subsets of Ω is a **d-system in** Ω if the following hold

1. $\Omega \in \mathcal{D}$,
2. If $E, F \in \mathcal{D}$ with $E \subseteq F$ then also $F \setminus E \in \mathcal{D}$,
3. If $E_n \in \mathcal{D}$ for all $n \in \mathbb{N}$ and $E_n \subseteq E_{n+1}$ for all $n \in \mathbb{N}$ then also $\bigcup\limits_{n=1}^{\infty} E_n \in \mathcal{D}$.

Definition 2.3. Let Ω be a non-empty set. A family $\mathscr{A}$ of subsets of Ω is an **algebra in** Ω if:

1. $\Omega \in \mathscr{A}$ or, equivalently, $\mathscr{A} \neq \emptyset$,
2. If $A, B \in \mathscr{A}$ then also $A \cup B \in \mathscr{A}$,
3. If $A \in \mathscr{A}$ then also $A^C := \Omega \setminus A \in \mathscr{A}$.

If, additionally, for $(A_n)_{n \in \mathbb{N}} \subset \mathscr{A}$ we have $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$ then $\mathscr{A}$ is called a **σ-algebra**.

Definition 2.4. Let Ω be a set and let $\mathcal{I}$ be a family of subsets of Ω. Then we define $\sigma(\mathcal{I})$ as the smallest σ-algebra in Ω that contains $\mathcal{I}$, which is the intersection of all σ-algebras in Ω that contain $\mathcal{I}$. We say that $\sigma(\mathcal{I})$ is the **σ-algebra generated by** $\mathcal{I}$.

There are several connections between the above definitions. Some of these connections will be useful and are therefore stated below.

Proposition 2.5. *[8, Proposition 1.13] Let Ω be a set. Let Σ be a family of subsets of Ω. Σ is a σ-algebra if and only if Σ is a π-system and a d-system.*

Lemma 2.6. *[8, Corollary 1.15] Let Ω be a set, let $\mathcal{I}$ be a π-system in Ω and let $\mathcal{D}$ be a d-system in Ω. If $\mathcal{I} \subseteq \mathcal{D}$ then $\sigma(\mathcal{I}) \subseteq \mathcal{D}$.*

Definition 2.7. Let Ω be a set. A family $\mathcal{S}$ of subsets of Ω is a **semiring** if it is nonempty and:

1. $\emptyset \in \mathcal{S}$,
2. If $A, B \in \mathcal{S}$ then also $A \cap B \in \mathcal{S}$,
3. If $A, B \in \mathcal{S}$ then there exist $n \in \mathbb{N}$, $C_1, ..., C_n \in \mathcal{S}$ such that $C_i \cap C_j = \emptyset$ for all $i, j \in \{1, ..., n\}$ with $i \neq j$ and such that $A \setminus B = \bigcup_{i=1}^{n} C_i$.

Remark 2.8. Note that every algebra is a semiring.

Definition 2.9. Let Ω be a set. A relation $\leq$ on Ω is called a **partial order** if:

1. $\leq$ is transitive i.e. if $x, y, z \in \Omega$ with $x \leq y$ and $y \leq z$, then also $x \leq z$;
2. $\leq$ is reflexive i.e. $x \leq x$ for all $x \in \Omega$;
3. $\leq$ is antisymmetric i.e. if $x, y \in \Omega$ with $x \leq y$ and $y \leq x$, then $x = y$.

The set Ω equipped with a partial order $\leq$ is called a **partially ordered set**.

Definition 2.10. Let T be a topological space. The **Borel σ-algebra of T**, denoted by $\mathcal{B}(T)$, is the smallest σ-algebra containing all open subsets of T. Sets of $\mathcal{B}(T)$ are called the **Borel sets of T**.

Definition 2.11. Let V be a real vector space. We define the **convex hull** of a set $A \subseteq V$ by $\text{conv}(S) := \left\{\sum_{j=1}^{k} \lambda_j x_j \mid \lambda_j \geq 0,\ \sum_{j=1}^{k} \lambda_j = 1 \text{ and } x_j \in S \text{ for all } j \in \{1, ..., k\} \text{ and for some } k \in \mathbb{N}\right\}$.

Definition 2.12. Let V be a real vector space. We define the **linear span** of a set $S \subseteq V$ by $\text{span}(S) := \left\{\sum_{j=1}^{k} \lambda_j v_j \mid \text{for some } k \in \mathbb{N},\ \lambda_j \in \mathbb{R} \text{ for } j \in \{1, ..., k\} \text{ and } v_j \in S \text{ for } j \in \{1, ..., k\}\right\}$.

3 Vector spaces and metric spaces

In this section we will introduce the notion of a Banach space and a Riesz space, and we will also give some properties of Banach spaces. After that we will define separability and discuss properties of separability, for instance subspaces of separable spaces.

Definition 3.1. Let (E, d) be a metric space. A sequence $(x_n)_{n\in\mathbb{N}} \subset E$ is a **Cauchy sequence** if for each $\epsilon > 0$ there exists an n_0 (depending on ϵ) satisfying $d(x_n, x_m) < \epsilon$ for all $n, m \geq n_0$. (Or equivalently $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.)

Definition 3.2. Let (E, d) be a metric space. (E, d) is called **complete** if every Cauchy sequence in E converges in E.

Definition 3.3. Let E be a vector space. E is called a **Banach space** if it is a normed space that is also a complete metric space under the metric induced by the norm, i.e. $d(x, y) = ||x - y||$ for $x, y \in E$.

Theorem 3.4. *[7, Theorem 2.28.(e)] Let E be a Banach space and $F \subseteq E$ a closed linear subspace of E. Then F is also a Banach space.*

Definition 3.5. Let E be a metric space and $\phi : E \to \mathbb{R}$ a function. ϕ is called **lower semi coninuous** if $\phi(x_0) \leq \liminf\limits_{x\to x_0} \phi(x)$ for all $x_0 \in E$.

Definition 3.6. Let E_1, E_2 and E_3 be normed spaces and let $\beta : E_1 \times E_2 \to E_3$ be a bilinear map. Then β is called **bounded** if for some constant $c \in \mathbb{R}$ we have $||\beta(x, y)|| \leq c||x|| \cdot ||y||$ for all $x \in E_1$ and all $y \in E_2$.

Definition 3.7. Let V be a vector space over $\mathbb{R}$ equipped with a partial order. V is called a **(partially) ordered vector space** if for all $x, y, z \in V$ and all $\lambda \in \mathbb{R}_{\geq 0}$ we have

1. If $x \leq y$ then also $x + z \leq y + z$;
2. If $x \leq y$ then also $\lambda x \leq \lambda y$.

Definition 3.8. Let V be a partially ordered vector space, $S \subseteq V$ a subset and $b \in V$ a vector. b is called an **upper (lower) bound of** S if $b \geq s$ ($b \leq s$) for all $s \in S$.
Furthermore b is called the **supremum (infimum) of** S, denoted by $\sup S$ ($\inf S$), if b is an upper (lower) bound of S and if for each upper (lower) bound a of S it holds that $a \geq b$ ($a \leq b$).

Definition 3.9. A partially ordered vector space V is called a **Riesz space** if for all $x, y \in V$ we have that $\sup\{x, y\}$ and $\inf\{x, y\}$ exist in V.
The functions $(x, y) \mapsto \sup\{x, y\}$ and $(x, y) \mapsto \inf\{x, y\}$ are called the **lattice operations** on V, and the **absolute value** of $x \in V$ is defined as $|x| = \sup\{x, -x\}$.

Definition 3.10. A **normed Riesz space** is a Riesz space V equipped with a lattice norm, i.e. a norm $||\cdot||$ such that for $x, y \in V$ with $|x| \leq |y|$ we have that $||x|| \leq ||y||$.

Definition 3.11. A complete normed Riesz space is called a **Banach lattice**.

Definition 3.12. Let V be a metric (or topological) space. V is called **separable** if it contains a countable dense subset, i.e. a countable subset $D \subseteq V$ such that $\overline{D} = V$.

Now we will consider some simple properties of separability that are often used without proof.

Theorem 3.13. *[7, Theorem 1.43] Let E be a separable metric space and $F \subseteq E$ a subset of E. Then F is also separable.*

Lemma 3.14. *Let E be a metric space and $(A_i)_{i\in\mathbb{N}} \subseteq E$ separable subsets. Then $A := \bigcup_{i=1}^{\infty} A_i$ is also separable.*

Proof. Let $(D_i)_{i\in\mathbb{N}} \subseteq E$ with $D_i \subseteq A_i$ a countable dense subset of A_i for all $i \in \mathbb{N}$. Then we have that $D := \bigcup_{i=1}^{\infty} D_i \subseteq A$ and D is countable. Now let $x \in A$. Then we have that $x \in A_i$ for at least one $i \in \mathbb{N}$ and since D_i is dense in A_i we have that there exists a sequence $(x_i)_{i\in\mathbb{N}} \subseteq D_i$ with $x_i \to x$ for $i \to \infty$. Since $D_i \subseteq D$, we thus have a sequence in D converging to x. So D is dense in A and thus A is separable. ∎

Theorem 3.15. *Let E be a metric space with metric d and $F \subseteq E$ a separable subset. Then $\overline{F}$ is also separable.*

Proof. Let $A \subseteq F$ be a countable dense subset of F and let $x \in \overline{F}$. Then we have two cases.

- If $x \in F$ then there exists a sequence $(x_i)_{i\in\mathbb{N}} \subseteq A$ with $x_i \to x$ for $i \to \infty$ since A is dense in F.
- If $x \in \overline{F} \setminus F$, then there exists a sequence $(x_i)_{i\in\mathbb{N}} \subseteq F$ with $x_i \to x$ for $i \to \infty$, since x is an element of the closure of F. Now for all $i \in \mathbb{N}$ there exists a sequence $(y_n^i)_{n\in\mathbb{N}} \subseteq A$ with $y_n^i \to x_i$ for $n \to \infty$ since $x_i \in F$ and A is dense in F. Now consider the sequence $(z_i)_{i\in\mathbb{N}}$ given by $z_i := y_{n_i}^i$ where n_i such that $d(y_{n_i}^i, x_i) < \frac{1}{i}$. Now we obviously have that $z_i \in A$ for all $i \in \mathbb{N}$ and $z_i \to x$ for $i \to \infty$.

Thus in both cases there exists a sequence in A converging to x. Thus A is also dense in $\overline{F}$, hence $\overline{F}$ is separable. ∎

Theorem 3.16. *Let E be a normed vector space and $V \subseteq E$ a separable subset of E. Then $span(V)$ is also separable in E and hence $\overline{span(V)}$ is seperable as well.*

Proof. Let $W \subseteq V$ be a countable dense subset of V. Define

$$D := \{\lambda_1 w_1 + ... + \lambda_n w_n : n \in \mathbb{N}, w_i \in W \ \forall i \in \mathbb{N}, \lambda_i \in \mathbb{Q} \ \forall i \in \mathbb{N}\}.$$

Then we know that D is countable and that $D \subseteq \text{span}(V)$. We will show that D is dense in $\text{span}(V)$.
Let $x \in \text{span}(V)$. Then we can write, for some $n \in \mathbb{N}$, $v_1, ..., v_n \in V$ and $\lambda_1, ..., \lambda_n \in \mathbb{R}\setminus\{0\}$ that $x = \sum_{i=1}^{n} \lambda_i v_i$.
For all $i \in \{1, ..., n\}$ we have that $v_i \in V$ and thus there exists a sequence $(w_m^i)_{m\in\mathbb{N}} \subseteq W$ with $w_m^i \to v_i$ for $m \to \infty$ since W is dense in V. Now consider the sequence $(y_j)_{j\in\mathbb{N}}$ defined by

$$y_j = q_1^j w_{m_j^1}^1 + ... + q_n^j w_{m_j^n}^n = \sum_{i=1}^{n} q_i^j w_{m_j^i}^i,$$

where we have, for all i and all j that

$$m_j^i \text{ satisfies } ||w_{m_j^i}^i - v_i|| < \frac{1}{|\lambda_i| \cdot 2nj}$$

and that

$$q_i^j \text{ satisfies } |q_i^j - \lambda_i| < \frac{1}{(||w^i_{m^i_j}|| + 1) \cdot 2nj}.$$

Then we obviously have that $y_j \in D$ for all $j \in \mathbb{N}$ and we have

$$||y_j - x|| = \left|\left|\sum_{i=1}^n q_i^j w^i_{m^i_j} - \sum_{i=1}^n \lambda_i v_i\right|\right| \leq \sum_{i=1}^n \left|\left|q_i^j w^i_{m^i_j} - \lambda_i v_i\right|\right| = \sum_{i=1}^n \left|\left|q_i^j w^i_{m^i_j} - \lambda_i v_i + \lambda_i w^i_{m^i_j} - \lambda_i w^i_{m^i_j}\right|\right|$$

$$\leq \sum_{i=1}^n \left(||q_i^j w^i_{m^i_j} - \lambda_i w^i_{m^i_j}|| + ||\lambda_i w^i_{m^i_j} - \lambda_i v_i|| \right) = \sum_{i=1}^n \left(|q_i^j - \lambda_i| \cdot ||w^i_{m^i_j}|| + |\lambda_i| \cdot ||w^i_{m^i_j} - v_i|| \right)$$

$$< \sum_{i=1}^n \left(\frac{||w^i_{m^i_j}||}{(||w^i_{m^i_j}|| + 1) \cdot 2nj} + \frac{|\lambda_i|}{|\lambda_i| \cdot 2nj} \right) = \sum_{i=1}^n \left(\frac{||w^i_{m^i_j}||}{(||w^i_{m^i_j}|| + 1)} \cdot \frac{1}{2nj} + \frac{1}{2nj} \right)$$

$$< \sum_{i=1}^n \left(\frac{1}{2nj} + \frac{1}{2nj} \right) = n \cdot 2 \cdot \frac{1}{2nj} = \frac{1}{j},$$

thus we obtain that $y_j \to x$ for $j \to \infty$. Thus there exists a sequence in D converging to x and thus D is dense in $\text{span}(V)$. Since D is a countable dense subset of $\text{span}(V)$, it follows that $\text{span}(V)$ is separable. With Theorem 3.15 we now obtain that $\overline{\text{span}(V)}$ is also separable since E is a normed vector space and consequently a metric space. ■

Definition 3.17. For a topological vector space E over $\mathbb{K}$ (i.e. a vector space equipped with a topology such that the vector space operations are continuous with respect to the topology, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{Q}$) the **dual space**, denoted by E^*, is the vector space of all continuous linear mappings from E to $\mathbb{K}$. For $x^* \in E^*$, the norm of x^* is defined by $||x^*|| = \sup_{x \in E: ||x|| \leq 1} |\langle x, x^* \rangle|$ where $\langle x, x^* \rangle := x^*(x)$.

Definition 3.18. Let E be a Banach space with dual space E^*.

1. A linear subspace $F \subseteq E^*$ is called **norming for a subset** $S \subseteq E$ if for all $x \in S$ we have $||x|| = \sup_{x^* \in F: ||x^*|| \leq 1} |\langle x, x^* \rangle|$.
2. A linear subspace that is norming for E is simply called **norming**.

Definition 3.19. Let E be a Banach space with dual space E^*. A linear subspace F of E^* is said to **seperate the points of a subset** S **of** E if for every pair $x, y \in S$ with $x \neq y$ there exists an $x^* \in F$ such that $\langle x, x^* \rangle \neq \langle y, x^* \rangle$.

Definition 3.20. Let E, F be two ordered vector spaces. An operator $T : F \to E$ is a **positive operator** if it is a linear operator that maps positive vectors to positive vectors, i.e. T is positive if $x \geq 0$ in F implies $T(x) \geq 0$ in E.

4 Set functions

Sometimes we will consider set functions that are more general than the standard setting of measures.

Definition 4.1. A set function $\mu : \mathcal{S} \to \mathbb{R} \cup \{\infty, -\infty\}$ on a semiring $\mathcal{S}$ is

- **monotone** if $A \subset B$ with $A, B \in \mathcal{S}$ imply $\mu(A) \leq \mu(B)$;
- **(finitely) additive** if for each finite family $\{A_1, ..., A_n\}$ of pairwise disjoint sets in $\mathcal{S}$ with $\bigcup_{i=1}^n A_i \in \mathcal{S}$ we have $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$;

- **σ-additive(or countably additive)** if for each countable family $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{S}$ with $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$ we have $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$;
- **subadditive** if $\{A_1, ..., A_n\} \subseteq \mathcal{S}$ and $\bigcup_{i=1}^{n} A_i \in \mathcal{S}$ imply $\mu(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} \mu(A_i)$;
- **σ-subadditive** if $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{S}$ and $\bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$ imply $\mu(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu(A_i)$.

Definition 4.2. A set function $\mu : \mathcal{S} \to [-\infty, \infty]$ on a semiring is

- **a signed charge** if it is additive, assumes at most one of the values $-\infty$ and ∞ and $\mu(\emptyset) = 0$;
- **a charge** if it is a signed charge that assumes only nonnegative values;
- **a signed measure** if it is σ-additive, assumes at most one of the values $-\infty$ and ∞ and $\mu(\emptyset) = 0$;
- **a measure** if it is a signed measure that assumes only nonnegative values.

5 Measure spaces

In this section we will define what a measure space is. When considering measure spaces it can be useful to widen the σ-algebra we are working with, and there is a very intuitive method to do this, and therefore we will explain this. With this expansion introduced, we can consider two different σ-algebras and thus two different measure spaces. Therefore we will, as well in this section as in later sections, have some definitions that appear similar, but have subtle differences. These subtle differences, however, can have major effects. They make proving several theorems significantly more difficult and the proofs become more subtle. For generalities on measure theory we refer to [5] and [13].

Definition 5.1. The triple (Ω, Σ, μ) with Ω a set, Σ a σ-algebra on Ω and μ a measure on Σ is called a **measure space**.

Proposition 5.2. *[8, Proposition 1.8] Let (Ω, Σ, μ) be a measure space and let $(E_n)_{n\in\mathbb{N}}$ be a sequence in Σ.*

1. *If the sequence is increasing with limit $E = \bigcup_{i=1}^{\infty} E_i$ then $\lim_{i\to\infty} \mu(E_i) = \mu(E)$.*
2. *If the sequence is decreasing with limit $E = \bigcap_{i=1}^{\infty} E_i$ and if $\mu(E_i) < \infty$ for $i \geq n$ for some $n \in \mathbb{N}$, then $\lim_{i\to\infty} \mu(E_i) = \mu(E)$.*

Definition 5.3. Let (Ω, Σ, μ) be a measure space. $N \subseteq \Omega$ is called a **μ-null set** if there exists an $A \in \Sigma$ such that $N \subseteq A$ and $\mu(A) = 0$.
Notation 5.4. The collection of all μ-null sets is denoted by $\mathcal{N}_\mu$.

Definition 5.5. Let (Ω, Σ, μ) be a measure space. (Ω, Σ, μ) is called **complete** if Σ contains all μ-null sets, i.e. $\mathcal{N}_\mu \subseteq \Sigma$. Then μ is called a **complete** measure.

Definition 5.6. Let (Ω, Σ, μ) be a measure space. Define $\Sigma_\mu := \sigma(\Sigma, \mathcal{N}_\mu)$.

Lemma 5.7. *[13, Theorem 13.B] Let (Ω, Σ, μ) be a measure space. Then $A \in \Sigma_\mu$ if and only if there exists $B \in \Sigma$ with $(A \Delta B) \in \mathcal{N}_\mu$, where $A \Delta B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.*

Definition 5.8. Let (Ω, Σ, μ) be a measure space. For $A \in \Sigma_\mu$ we define $\hat{\mu}(A) = \mu(B)$ where $B \in \Sigma$ such that $(A \Delta B) \in \mathcal{N}_\mu$.
Remark 5.9. $\hat{\mu}$ is a measure on (Ω, Σ_μ).

Definition 5.10. Let (Ω, Σ, μ) be a measure space. Then $(\Omega, \Sigma_\mu, \hat{\mu})$ is a complete measure space and it is called the **completion of** (Ω, Σ, μ).

Remark 5.11. This definition gives an alternative way to define a complete measure space: Let (Ω, Σ, μ) be a measure space. If $\Sigma = \Sigma_\mu$ then (Ω, Σ, μ) is **complete**.

Definition 5.12. Let (Ω, Σ, μ) be a measure space.

- A set $S \subseteq \Omega$ is called **measurable** if $S \in \Sigma$.
- A set $S \subseteq \Omega$ is called **μ-measurable** if $S \in \Sigma_\mu$.

Definition 5.13. Let (Ω, Σ, μ) be a measure space. (Ω, Σ, μ) is called **σ-finite** if there exist $(A_n)_{n\in\mathbb{N}}$ with $A_n \in \Sigma$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$ such that $\Omega = \bigcup_{n=1}^{\infty} A_n$.

Lemma 5.14. *[9, Lemma 2.10] Let (Ω, Σ, μ) be a measure space and $(A_n)_{n\in\mathbb{N}} \subseteq \Sigma$ such that $\sum\limits_{n=1}^{\infty} \mu(A_n) < \infty$ then $\mu\Big(\bigcap\limits_{k\geq 1} \bigcup\limits_{n\geq k} A_n\Big) = 0$.*

6 Functions to a (Banach) space

Notation 6.1. For a set Ω and a subset $A \subset \Omega$ we denote the indicator function with $\mathbb{1}_A$, i.e.
$\mathbb{1}_A(\omega) = \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}$.

Definition 6.2. Let Ω be a set, $\mathscr{A}$ an algebra on Ω, μ a charge and E a vector space. Let $\varphi : \Omega \to E$ be a function that assumes only a finite number of values in E, say $x_1, ..., x_n \in E$.
φ is called an **E-simple function** if $A_i := \varphi^{-1}(\{x_i\}) \in \mathscr{A}$ for each i, and φ is called an **E-step function** if additionally $\mu(A_i) < \infty$ for each nonzero x_i.
The **standard representation** of φ is given by $\varphi = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$.

Definition 6.3. Let (Ω, Σ, μ) be a measure space and E a Banach space. Then we define
$L_E := \{f : \Omega \to E \mid f \text{ is an } E\text{-step function}\}$. This is a vector space.

Definition 6.4. Let (Ω, Σ, μ) be a measure space and E a Banach space. Then we define
$E^\Omega := \{f \mid f : \Omega \to E \text{ a function}\}$. This is also a vector space.

Definition 6.5. Let (Ω, Σ, μ) be a measure space, E a Banach space and $f : \Omega \to E$ a function. Then the real-valued, nonnegative function $||f|| : \Omega \to \mathbb{R}$, defined by $||f||(\omega) = ||f(\omega)||$ for each $\omega \in \Omega$, is called the **norm function of f**.

The following definition will introduce a notion that will become very important later on.
Definition 6.6. Let (Ω, Σ, μ) be a measure space, E a vector space and $f, g : \Omega \to E$ be two functions with $f = g$ μ-almost everywhere. Then they are called **μ-versions** of each other.

7 Measurable functions

Definition 7.1. Let (Ω, Σ, μ) and $(Y, \mathcal{F}, \nu)$ be two measure spaces and let $f : \Omega \to Y$ be a function.

1. f is called **measurable** (or **$\Sigma\backslash\mathcal{F}$-measurable** if it is not clear which σ-algebras are used) if $f^{-1}(A) \in \Sigma$ for all $A \in \mathcal{F}$.
2. f is called **μ-measurable** (or **$\Sigma_\mu\backslash\mathcal{F}$-measurable** if it is not clear which σ-algebras are used) if f is measurable with respect to Σ_μ, i.e. $f^{-1}(A) \in \Sigma_\mu$ for all $A \in \mathcal{F}$.

Lemma 7.2. *Let (Ω, Σ, μ) be a measure space and E a Banach space equipped with the Borel σ-algebra. Let $f : \Omega \to E$ be a function. If f has a measurable μ-version, then f is μ-measurable.*

Proof. Let $g : \Omega \to E$ be a measurable μ-version of f and let $N \in \Sigma$ with $\mu(N) = 0$ be such that$f = g$ on $\Omega \backslash N$. Let A be a Borel set of E. Then $f^{-1}(A) \setminus g^{-1}(A) \subseteq N$ and $g^{-1}(A) \setminus f^{-1}(A) \subseteq N$, so $f^{-1}(A) \Delta g^{-1}(A) \subseteq N$, hence $f^{-1}(A) \Delta g^{-1}(A) \in \mathcal{N}_\mu$. Since g is measurable we have $g^{-1}(A) \in \Sigma$ and thus Lemma 5.7 yields that $f^{-1}(A) \in \Sigma_\mu$. Thus f is μ-measurable. ∎

A partial converse is given in Theorem 9.11

8 Strongly measurable functions

A major part of the analysis of measurable real functions relies on the fact that they can be apprixmated by simple functions or step functions. For functions with values in Banach spaces mere measurability is too weak for such an approximation property. In this section we will define strongly measurable functions and strongly μ-measurable functions as being a suitable limit of simple or step functions and mention some of their known properties. There are two definitions of strong μ-measurability that we can use and that appear to be different. One uses step functions and the other simple functions. However we will show that the definitions are equivalent if we assume that the measure space is σ-finite. First we will define strong measurability.

Definition 8.1. Let (Ω, Σ, μ) be a measure space, E a Banach space and $f : \Omega \to E$ be a function. We say that f is **strongly measurable** (or **strongly Σ-measurable** if it is not clear which σ-algebra is used) if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-simple functions such that $\lim_{n\to\infty} \varphi_n = f$ pointwise on Ω (i.e. for all $\omega \in \Omega$ it holds that $\lim_{n\to\infty} \varphi_n(\omega) = f(\omega)$).

Next we will define strong μ-measurability.

Definition 8.2. Let (Ω, Σ, μ) be a measure space, E a Banach space and $f : \Omega \to E$ a function. f is called **strongly μ-measurable** if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-simple functions that converge to f μ-almost everywhere.

Theorem 8.3. *Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space and $f : \Omega \to E$ a function. Then the following are equivalent:*

1. *f is strongly μ-measurable;*
2. *there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-step functions converging to f μ-almost everywhere.*

Proof.
"⇒" Let f be a function such that there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-simple functions converging to f μ-almost everywhere, and let $(\varphi_n)_{n\in\mathbb{N}}$ be such a sequence. Let $(B_n)_{n\in\mathbb{N}} \subseteq \Sigma$ with $\mu(B_n) < \infty$ for all n and such that $\Omega = \bigcup_{n=1}^{\infty} B_n$. This is possible, since (Ω, Σ, μ) is σ-finite. Write $\varphi_n = \sum_{k=1}^{N_n} \mathbb{1}_{A_k^n} x_k^n$ with $A_k^n \in \Sigma$ and $x_k^n \in E$ for all k and all $n \in \mathbb{N}$. Define $C_n = \bigcup_{k=1}^{N_1+\ldots+N_n} B_k$. Then for all n we have that $\mu(C_n) = \mu\left(\bigcup_{k=1}^{N_1+\ldots+N_n} B_k\right) \leq \sum_{k=1}^{N_1+\ldots+N_n} \mu(B_k) < \infty$ since $\mu(B_k) < \infty$ for all k and $N_1 + ... + N_n$ is finite. Now consider $\varphi_n' := \varphi_n \mathbb{1}_{C_n}$. Then we have that φ_n' is an E-step function for all n and since $\mathbb{1}_{C_n} \to \mathbb{1}_\Omega$ we obtain that $(\varphi_n')_{n\in\mathbb{N}}$ converges to f μ-almost everywhere since $(\varphi_n)_{n\in\mathbb{N}}$ converges to f μ-almost everywhere. So there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-step functions converging to f μ-almost everywhere.

"⇐" Since a sequence of E-step functions is also a sequence of E-simple functions we have that (2) implies (1).

So we have that the statements are equivalent. ∎

In the theorem above we assumed that the measure space was σ-finite, and then the two properties turned out to be equivalent. However, when we do not assume that the measure space is σ-finite, the two statements turn out to be completely different. To illustrate this, consider the following example.

Example 8.4. Let E be a Banach space and consider the measure space (Ω, Σ, μ) with $\Omega \neq \emptyset$ and μ defined by

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \\ \infty & \text{if } A \in \Sigma \setminus \{\emptyset\} \end{cases}.$$

Then obviously (Ω, Σ, μ) is not σ-finite.

Now consider the function f given by $f(\omega) = 1$ for all $\omega \in \Omega$. Then f is obviously an E-simple function, but clearly not an E-step function. Thus there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-simple functions such that $\lim_{n\to\infty} ||f(\omega) - \varphi_n(\omega)|| = 0$ for μ-almost all $\omega \in \Omega$, by taking $\varphi_n = f$ for all $n \in \mathbb{N}$. However, the only E-step function is given by $\mathbb{1}_\emptyset = 0$ since $\emptyset$ is the only element of Σ with finite measure. Moreover we have that 0 does not converge μ-almost everywhere to f. So there does not exist a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-step functions converging to f μ-almost everywhere.

In the remainder of this section we will consider some results.

Definition 8.5. For (Ω, Σ, μ) a σ-finite measure space and E a Banach space, we define $\mathcal{M}(\Omega, E) := \{f \in E^\Omega : f \text{ is strongly } \mu\text{-measurable}\}$ and $\mathcal{SM}(\Omega, E) := \{f \in E^\Omega : f \text{ is strongly measurable}\}$.

Remark 8.6. [6, Lemma 11.40] Note that $L_E \subseteq \mathcal{M}(\Omega, E) \subseteq E^\Omega$, and it is easy to verify that $\mathcal{M}(\Omega, E)$ is a vector subspace of E^Ω. Similarly, $\mathcal{SM}(\Omega, E)$ is a vector subspace of E^Ω.

Lemma 8.7. *[6, Lemma 11.39] Let (Ω, Σ, μ) be a measure space and E a Banach space. If $f : \Omega \to E$ is strongly μ-measurable, then the real function $||f||$ is μ-measurable.*

The following Proposition will be frequently used.

Proposition 8.8. *[9, Proposition 1.10] Let (Ω, Σ, μ) be a σ-finite measure space and E a Banach space. For $f : \Omega \to E$ are equivalent:*

1. *f is strongly μ-measurable;*
2. *f has a strongly measurable μ-version.*

Lemma 8.9. *Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space and $f : \Omega \to E$ a strongly μ-measurable function. Let $\widetilde{f}$ be a μ-version of f. Then $\widetilde{f}$ is also strongly μ-measurable.*

Proof. By Proposition 8.8 f has a μ-version g which is strongly measurable. By definition of μ-versions we have that g is also a μ-version of $\widetilde{f}$ and thus we obtain with Proposition 8.8 that $\widetilde{f}$ is strongly μ-measurable. ∎

9 Pettis measurable functions

In this section we will define one more notion of measurability and after that we will connect the different notions of measurabilty. This will be very useful for some later theorems, since sometimes we will be given one notion of measurability while needing another notion.

Definition 9.1. Let (Ω, Σ, μ) be a measure space, E a Banach space, F a norming subspace of E^* and $f : \Omega \to E$ a function.

1. f is called **Pettis measurable(or weakly measurable)** if $\langle f, x^* \rangle : \Omega \to \mathbb{K}$, $\langle f, x^* \rangle (\omega) := \langle f(\omega), x^* \rangle$, is measurable for all $x^* \in E^*$.
   Furthermore we say that f is **Pettis measurable on** F if $\langle f, x^* \rangle$ is measurable for all $x^* \in F$.
2. If (Ω, Σ, μ) is σ-finite then f is called **Pettis μ-measurable(or weakly μ-measurable)** if $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in E^*$.
   Furthermore we say that f is **Pettis μ-measurable on** F if $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in F$.

Definition 9.2. Let (Ω, Σ, μ) be a measure space, E a Banach space and $f : \Omega \to E$ a function.

1. f is called **separably valued** if there exists a separable subspace $F \subseteq E$ such that $f(\omega) \in F$ for all $\omega \in \Omega$.
2. If (Ω, Σ, μ) is σ-finite then f is called **μ-separably valued** if there exists a separable subspace $F \subseteq E$ such that $f(\omega) \in F$ for μ-almost all $\omega \in \Omega$.

Theorem 9.3. *[9, Theorem 1.5 and Proposition 1.8] Let (Ω, Σ, μ) be a measure space, E a Banach space, F a norming subspace of E^*. For $f : \Omega \to E$ are equivalent:*

1. *f is strongly measurable;*
2. *f is separably valued and Pettis measurable;*
3. *f is separably valued and Pettis measurable on F;*
4. *f is separably valued and measurable.*

Corollary 9.4. *Let (Ω, Σ, μ) be a measure space, E a separable Banach space and $f : \Omega \to E$ a function. f is measurable if and only if f is strongly measurable.*

Corollary 9.5. *[9, Corollary 1.6] The pointwise limit of a sequence of strongly measurable functions is strongly measurable.*

Corollary 9.6. *[9, Corollary 1.7] Let (Ω, Σ, μ) be a measure space, E and F two Banach spaces and $f : \Omega \to E$ a strongly measurable function. Let $\phi : E \to F$ a continuous function. Then $\phi \circ f$ is strongly measurable.*

Theorem 9.7. *[9, Theorem 1.11] Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space, F a norming subspace of E^*. For $f : \Omega \to E$ are equivalent:*

1. *f is strongly μ-measurable;*
2. *f is μ-separably valued and Pettis μ-measurable;*
3. *f is μ-separably valued and Pettis μ-measurable on F.*

Corollary 9.8. *[9, Corollary 1.12] The μ-almost everywhere limit of a sequence of strongly μ-measurable functions is strongly μ-measurable.*

Corollary 9.9. *[9, Corollary 1.13] Let (Ω, Σ, μ) be a σ-finite measure space, E and F two Banach spaces and $f : \Omega \to E$ a strongly μ-measurable function. Let $\phi : E \to F$ a continuous function. Then $\phi \circ f$ is strongly μ-measurable.*

Corollary 9.10. *[9, Corollary 1.14] Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space and $f, g : \Omega \to E$ two strongly μ-measurable functions. Let F be a subspace of E^* separating the points of E. If we have that $\langle f, x^* \rangle = \langle g, x^* \rangle$ μ-almost everywhere for all $x^* \in F$, then we have that $f = g$ μ-almost everywhere.*

Theorem 9.11. *Let (Ω, Σ, μ) be a σ-finite measure space and E a separable Banach space. For $f : \Omega \to E$ are equivalent:*

1. f is strongly μ-measurable;
2. f is Pettis μ-measurable;
3. f is μ-measurable.
4. f has a measurable μ-version.

Proof. Combining Theorem 9.3 and Theorem 9.7 give the equivalence of 1,2 and 3. The implication $4 \Rightarrow 3$ is contained in Lemma 7.2. Proposition 8.8 and Theorem 9.3 yield that 1 implies 4. ∎

With some more care, the implication $1 \Rightarrow 3$ of Theorem 9.11 can be proven also for non-separable Banach spaces E.

The next convenient relation seems to be missing in [9].

Lemma 9.12. *Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space and $f : \Omega \to E$ a strongly μ-measurable function. Then f is μ-measurable.*

Proof. With Proposition 8.8 there exists a function $\widetilde{f} : \Omega \to E$ such that $\widetilde{f}$ is strongly measurable and $f = \widetilde{f}$ μ-a.s., so let $\Omega^* \subseteq \Omega$ such that $\Omega^* \in \Sigma$, $\mu(\Omega^*) = 1$ and $f(\omega) = \widetilde{f}(\omega)$ for all $\omega \in \Omega^*$. Let B be a Borel set in E. Then we can write:

$$f^{-1}(B) = (f^{-1}(B) \cap \Omega^*) \cup (f^{-1}(B) \cap (\Omega \setminus \Omega^*)) \overset{(*)}{=} (\widetilde{f}^{-1}(B) \cap \Omega^*) \cup (f^{-1}(B) \cap (\Omega \setminus \Omega^*))$$

where $(*)$ follows since $f = \widetilde{f}$ on Ω^* and thus $f^{-1}(B) \cap \Omega^* = \widetilde{f}^{-1}(B) \cap \Omega^*$.

Now we have that $f^{-1}(B) \cap (\Omega \setminus \Omega^*)$ is a μ-null set, since $\mu(\Omega \setminus \Omega^*) = 0$ (because $\mu(\Omega^*) = 1$), and thus $f^{-1}(B) \cap (\Omega \setminus \Omega^*) \in \Sigma_\mu$.

Furthermore we have that $\widetilde{f}^{-1}(B) \cap \Omega^* = \widetilde{f}^{-1}(B) \setminus \big(\widetilde{f}^{-1}(B) \cap (\Omega \setminus \Omega^*)\big)$. Since $\widetilde{f}$ is strongly measurable we have with Theorem 9.3 that $\widetilde{f}$ is measurable, so we obtain that $\widetilde{f}^{-1}(B) \in \Sigma$ and thus that $\widetilde{f}^{-1}(B) \in \Sigma_\mu$. In the same way as above we get that $\widetilde{f}^{-1}(B) \cap (\Omega \setminus \Omega^*)$ is a μ-null set and thus $\widetilde{f}^{-1}(B) \cap (\Omega \setminus \Omega^*) \in \Sigma_\mu$. Since Σ_μ is a σ-algebra, we obtain that $\widetilde{f}^{-1}(B) \cap \Omega^* = \widetilde{f}^{-1}(B) \setminus \big(\widetilde{f}^{-1}(B) \cap (\Omega \setminus \Omega^*)\big) \in \Sigma_\mu$.

Thus $f^{-1}(B) = (\widetilde{f}^{-1}(B) \cap \Omega^*) \cup (f^{-1}(B) \cap (\Omega \setminus \Omega^*)) \in \Sigma_\mu$ since Σ_μ is a σ-algebra. So f is Σ_μ-measurable, i.e. f is μ-measurable. ∎

10 Bochner Integral

Definition 10.1. Let Ω be a set, $\mathscr{A}$ an algebra on Ω, μ a charge, E a vector space and φ an E-step function with standard representation $\varphi = \sum_{i=1}^n x_i \mathbb{1}_{A_i}$. The **integral of** φ is the vector $\int_\Omega \varphi \mathrm{d}\mu$ defined by $\int_\Omega \varphi \mathrm{d}\mu = \sum_{i=1}^n \mu(A_i) x_i$. This integral is independent of the representation and will later be known as the Bochner Integral. For $B \in \mathscr{A}$ we define the **integral of** φ **over** B as $\int_B \varphi \mathrm{d}\mu = \int_\Omega \varphi \mathbb{1}_B \mathrm{d}\mu$.

Theorem 10.2. *[6, Theorem 11.34] Let (Ω, Σ, μ) be a measure space and E a Banach space. For all $\varphi, \psi \in L_E$ and all $\alpha, \beta \in \mathbb{R}$ we have $\int (\alpha\varphi + \beta\psi) d\mu = \alpha \int \varphi d\mu + \beta \int \psi d\mu$. (i.e. the operator $\int \cdot d\mu$ is a linear operator from L_E to E.)*
If E is a Banach lattice then we have that L_E is a Riesz space under the pointwise lattice operations (i.e. we use the pointwise ordening on L_E, so $f \leq g$ if and only if $f(\omega) \leq g(\omega)$ for all $\omega \in \Omega$, and then $\sup\{f, g\}(\omega) = \sup\{f(\omega), g(\omega)\}$ and $\inf\{f, g\}(\omega) = \inf\{f(\omega), g(\omega)\}$ for all $\omega \in \Omega$) and the operator $\int \cdot d\mu$ is a positive operator from L_E to E.

Lemma 10.3. *[6, Lemma 11.34] Let* (Ω, Σ, μ) *be a measure space and* E *a Banach space. Let* $\varphi \in L_E$ *with standard representation* $\varphi = \sum_{i=1}^{n} x_i \mathbb{1}_{A_i}$*. Then the norm function* $||\varphi||$ *of* φ *is a real step function (i.e.* $||\varphi|| \in L_{\mathbb{R}}$*) with standard representation* $||\varphi|| = \sum_{i=1}^{n} ||x_i|| \mathbb{1}_{A_i}$*. Moreover,* $\int_\Omega ||\varphi|| d\mu = \sum_{i=1}^{n} ||x_i|| \mu(A_i)$ *and* $|| \int_\Omega \varphi d\mu|| \leq \int_\Omega ||\varphi|| d\mu$*.*

Lemma 10.4. *[6, Lemma 11.41] Let* (Ω, Σ, μ) *be a* σ*-finite measure space and* E *a Banach space. Let* $f : \Omega \to E$ *be a strongly* μ*-measurable function. Suppose that for two sequences* $(\varphi_n)_{n\in\mathbb{N}}$ *and* $(\psi_n)_{n\in\mathbb{N}}$ *of* E*-step functions the real* μ*-measurable functions* $||f - \varphi_n||$ *and* $||f - \psi_n||$ *are Lebesgue integrable for all* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \int ||f - \varphi_n|| d\mu = \lim_{n\to\infty} \int ||f - \psi_n|| d\mu = 0.$$

Then for each $A \in \Sigma$ *we have*

$$\lim_{n\to\infty} \int_A \varphi_n \, d\mu = \lim_{n\to\infty} \int_A \psi_n \, d\mu$$

where the last two limits are taken with respect to the norm topology on E*.*

Definition 10.5. Let (Ω, Σ, μ) be a σ-finite measure space and E a Banach space. Let $f : \Omega \to E$ be a strongly μ-measurable function. We say that f is **μ-Bochner integrable** if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of E-step functions such that the real μ-measurable function $||f - \varphi_n||$ is Lebesgue integrable for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \int ||f - \varphi_n|| \mathrm{d}\mu = 0$.
In this case we define for each $A \in \Sigma$ the **Bochner Integral of f over A** by $\int_A f \mathrm{d}\mu = \lim\limits_{n\to\infty} \int_A \varphi_n \mathrm{d}\mu$ where the last limit is in the norm topology of E.

Theorem 10.6. *[6, Theorem 11.43] Let* (Ω, Σ, μ) *be a* σ*-finite measure space and* E *a Banach space. Let* $f, g : \Omega \to E$ *be* μ*-Bochner integrable functions and let* $\alpha, \beta \in \mathbb{R}$*. Then* $\alpha f + \beta g$ *is* μ*-Bochner integrable and* $\int_A (\alpha f + \beta g) d\mu = \alpha \int_A f d\mu + \beta \int_A g d\mu$ *for all* $A \in \Sigma$*.*
Moreover, if E *is a Banach lattice and* $f(\omega) \leq g(\omega)$ *for* μ*-almost all* $\omega \in \Omega$ *then* $\int_A f d\mu \leq \int_A g d\mu$ *for all* $A \in \Sigma$*.*

Theorem 10.7. *[6, Theorem 11.44] Let* (Ω, Σ, μ) *be a finite measure space and* E *a Banach space. Let* $f : \Omega \to E$ *be a* μ*-measurable function. Then* f *is* μ*-Bochner integrable if and only if its norm function* $||f||$ *is Lebesgue integrable (i.e.* $\int ||f|| d\lambda < \infty$ *with* λ *the Lebesgue measure).*

Lemma 10.8. *[6, Lemma 11.45] Let* (Ω, Σ, μ) *be a* σ*-finite measure space, let* E *and* F *be two Banach spaces and let* $f : \Omega \to E$ *be a* μ*-Bochner integrable function. If* $T : E \to F$ *is a bounded operator, then the function* $Tf : \Omega \to F$*, defined by* $(Tf)(\omega) = T(f(\omega))$*, is* μ*-Bochner integrable with* $\int_\Omega Tf d\mu = T(\int_\Omega f d\mu)$*.*

Proposition 10.9. *[9, Proposition 1.16] Let* (Ω, Σ, μ) *be a* σ*-finite measure space, let* E *be a Banach space and let* $f : \Omega \to E$ *be a strongly* μ*-measurable function.* f *is* μ*-Bochner integrable if and only if* $\int_\Omega ||f|| d\mu < \infty$ *and in this case we have* $|| \int_\Omega f d\mu|| \leq \int_\Omega ||f|| d\mu$*.*

Proposition 10.10. *[9, Proposition 1.17] Let* (Ω, Σ, μ) *be a* σ*-finite measure space, let* E *be a Banach space and let* $f : \Omega \to E$ *be a* μ*-Bochner integrable function. If* $\mu(\Omega) = 1$*, then* $\int_\Omega f d\mu \in \overline{conv}\{f(\omega) : \omega \in \Omega\}$*.*

Theorem 10.11. *[9, Proposition 1.18] Let* (Ω, Σ, μ) *be a* σ*-finite measure space, let* E *be a Banach space, let* $(f_n)_{n\in\mathbb{N}}$*, with* $f_n : \Omega \to E$ *for all* $n \in \mathbb{N}$*, be a sequence of* μ*-Bochner integrable functions. Assume that there exists a function* $f : \Omega \to E$ *and a* μ*-Bochner integrable function* $g : \Omega \to \mathbb{R}$ *such that*

1. $\lim\limits_{n\to\infty} f_n = f$ *μ-almost everywhere;*
2. $||f_n|| \leq |g|$ *μ-almost everywhere.*

Then f *is* μ*-Bochner integrable and* $\lim\limits_{n\to\infty} \int_\Omega ||f_n - f|| d\mu = 0$*.*
In particular we have $\lim\limits_{n\to\infty} \int_\Omega f_n d\mu = \int_\Omega f d\mu$*.*

The following Proposition is known as Jensen's inequality.
Proposition 10.12. *Let (Ω, Σ, μ) be a σ-finite measure space with $\mu(\Omega) = 1$, E a Banach space, $f : \Omega \to E$ a Bochner integrable function and $\phi : E \to \mathbb{R}$ a convex and lower semi continuous function. If $\phi \circ f$ is intergrable, then we have $\phi\big(\int_\Omega f d\mu\big) \leq \int_\Omega \phi \circ f d\mu$.*

11 L^p-spaces

Let (Ω, Σ, μ) be a measure space and E a normed space. On E^Ω we define the equivalence relation $\sim$ by $f \sim g$ if and only if $f = g$ μ-almost everywhere. Recall the following definitions.
Definition 11.1. Let (Ω, Σ, μ) be a measure space. For $1 \leq p < \infty$ we define $L^p(\Omega) := \{f : \Omega \to \mathbb{R} \mid f$ measurable and $\int_\Omega |f|^p \mathrm{d}\mu < \infty\}/\sim$.

Definition 11.2. Let (Ω, Σ, μ) be a measure space. We define $L^\infty(\Omega) := \{f : \Omega \to \mathbb{R} \mid f$ measurable and $\exists M \geq 0$ such that $|f| \leq M$ a.s.$\}/\sim$.

Lemma 11.3. *[7, Theorem 1.61, Example 2.5] Let (Ω, Σ, μ) be a measure space and $1 \leq p < \infty$. Then $||f||_{L^p(\Omega)} := \big(\int_\Omega |f|^p d\mu\big)^{\frac{1}{p}}$ is a norm on $L^p(\Omega)$ and $L^p(\Omega)$ equipped with this norm is a Banach space.*

Lemma 11.4. *[7, Theorem 1.61, Example 2.5] Let (Ω, Σ, μ) be a measure space. Then $||f||_{L^\infty(\Omega)} := \inf\{M \geq 0 \ : \ |f| \leq M$ a.s.$\}$ is a norm on $L^\infty(\Omega)$ and $L^\infty(\Omega)$ equipped with this norm is a Banach space.*

However we will not consider $\mathbb{R}$ but general Banach spaces in this thesis. Therefore we will give the above definitions in a similar fashion but for general Banach spaces.
Definition 11.5. Let (Ω, Σ, μ) be a σ-finite measure space and E a Banach space. For $1 \leq p < \infty$ we define $L^p(\Omega; E) := \{f : \Omega \to E \mid f$ strongly μ-measurable and $\int_\Omega ||f||^p \mathrm{d}\mu < \infty\}/\sim$.

Definition 11.6. Let (Ω, Σ, μ) be a σ-finite measure space and E a Banach space. We define $L^\infty(\Omega; E) := \{f : \Omega \to E \mid f$ strongly μ-measurable and $\exists r \geq 0$ such that $\mu(\{||f|| > r\}) = 0\}/\sim$.

Lemma 11.7. *[9, p. 12] Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space and $1 \leq p < \infty$. Then $||f||_{L^p(\Omega;E)} := \big(\int_\Omega ||f||^p d\mu\big)^{\frac{1}{p}}$ is a norm on $L^p(\Omega; E)$ and $L^p(\Omega; E)$ equipped with this norm is a Banach space.*

Lemma 11.8. *[9, p. 12] Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space and $1 \leq p < \infty$. Then the set of all E-step functions is dense in $L^p(\Omega; E)$.*

Lemma 11.9. *[9, p. 12] Let (Ω, Σ, μ) be a σ-finite measure space and E a Banach space. Then $||f||_{L^\infty(\Omega;E)} := \inf\{r \geq 0 \ : \ \mu(\{||f|| > r\}) = 0\}$ is a norm on $L^\infty(\Omega; E)$ and $L^\infty(\Omega; E)$ equipped with this norm is a Banach space.*

Remark 11.10. Let (Ω, Σ, μ) be a σ-finite measure space and $1 \leq p < \infty$. Then we have that $L^p(\Omega; \mathbb{R}) = L^p(\Omega)$ and that $L^\infty(\Omega; \mathbb{R}) = L^\infty(\Omega)$. It is straightforward to prove this.

Definition 11.11. Let (Ω, Σ, μ) be a σ-finite measure space, E a Banach space and $1 \leq p \leq \infty$. Let $\mathcal{G} \subseteq \Sigma$ be a sub-σ-algebra. Then we define
$L^p(\Omega, \mathcal{G}; E) := \{f \in L^p(\Omega; E) \mid f$ has a μ-version that is strongly μ-measurable with respect to $\mathcal{G}\}$.

Definition 11.12. Let (Ω, Σ, μ) be a σ-finite measure space with $\mu(\Omega) < \infty$, E a Banach space and let $1 \leq p < \infty$. $T \subseteq L^p(\Omega; E)$ is **uniformly p-integrable** (or just **uniformly integrable** if $p = 1$) if for all $\epsilon > 0$ there exists an $r > 0$ such that

$$\sup_{f \in T} \int_\Omega \mathbb{1}_{\{||f|| > r\}} ||f||^p \mathrm{d}\mu \leq \epsilon.$$

Remark 11.13. [10, above Proposition 2.6.39] Definition 11.12 is equivalent with the statement that for all $\epsilon > 0$ there exists an $r > 0$ such that

$$\sup_{g \in \left\{ ||f||^p : f \in T \right\}} \int_{\Omega} \mathbb{1}_{\left\{ |g| > r \right\}} |g| \mathrm{d}\mu \leq \epsilon.$$

12 Random Variables

Approximation by step functions plays an essential role in probability theory. Reduction steps in proofs to step functions are even commonly referred to as the standard machinery. Banach space valued random variables are therefore defined as strongly $\mathbb{P}$-measurable maps, to keep the ability to use these approximations in proofs.

Definition 12.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space (i.e. $\mathbb{P}(\Omega) = 1$) and E a Banach space. $X : \Omega \to E$ is a **random variable** if X is strongly $\mathbb{P}$-measurable.

Definition 12.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $X : \Omega \to E$ a random variable.

- If X is $\mathbb{P}$-Bochner integrable, the **expectation of** X is defined as the Bochner integral $\mathbb{E}(X) := \int_{\Omega} X \mathrm{d}\mathbb{P}$.
- The **distribution of** X is the Borel probability measure μ_X on E defined by $\mu_X(B) := \mathbb{P}(X \in B) := \mathbb{P}(\widetilde{X} \in B) := \mathbb{P}(\{\omega \in \Omega \ : \ \widetilde{X}(\omega) \in B\})$ for $B \in \mathcal{B}(E)$ where $\widetilde{X}$ is a strongly measurable $\mathbb{P}$-version of X. (All goes well by Proposition 8.8 and Theorem 9.3.)

Definition 12.3. Let E be a Banach space. If E-valued random variables, not necessarily defined on the same probability spaces, have the same distribution they are said to be **identically distributed**.

Theorem 12.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $X : \Omega \to E$ a random variable. Denote by Y a strongly measurable $\mathbb{P}$-version of X. Define $B := \{Y(\omega) : \omega \in \Omega\}$. Then B as well as $\overline{B}$ are separable.*

Proof. With Proposition 8.8 we have that Y exists, since X is a random variable and thus strongly $\mathbb{P}$-measurable. Y is strongly measurable thus there exists a sequence of E-simple functions $(\varphi_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} \varphi_n(\omega) = Y(\omega)$ for all $\omega \in \Omega$. For every $n \in \mathbb{N}$ we have that $\{\varphi_n(\omega) : \omega \in \Omega\}$ is finite since ϕ_n is an E-simple function. Now consider $A := \bigcup_{n=1}^{\infty} \{\varphi_n(\omega) : \omega \in \Omega\}$. Then A is obviously countable and dense in $\overline{A}$, so $\overline{A}$ is separable. Now we will show that $B \subseteq \overline{A}$. Let $x \in B$. Then there exists an $\omega \in \Omega$ such that $Y(\omega) = x$, and thus $\lim_{n \to \infty} \varphi_n(\omega) = x$. By definition of A we obviously have that $\varphi_n(\omega) \in A$ for all $n \in \mathbb{N}$ and thus by definition of the closure we obtain that $x \in \overline{A}$. Thus we have that $B \subseteq \overline{A}$, and since $\overline{A}$ is a separable metric space we obtain by Theorem 3.13 that B is separable. Furthermore by Theorem 3.15 we now obtain that $\overline{B}$ is also separable. ∎

Lemma 12.5. *[9, above Proposition 2.3] Let E and F be Banach spaces and let X and Y be two identically distributed E-valued random variables defined on some, not necessarily the same, probability spaces. Let $f : E \to F$ be a Borel measurable function. Then $f(X)$ and $f(Y)$ are identically distributed.*

Proposition 12.6. *[9, Proposition 2.3] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $X : \Omega \to E$ a random variable. Then for every $\epsilon > 0$ there exists a compact set $K \subseteq E$ such that $\mathbb{P}(\{X \notin K\}) < \epsilon$.*

Definition 12.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $\mathscr{X}$ a family of E-valued random variables on Ω. $\mathscr{X}$ is called **uniformly tight** if for every $\epsilon > 0$ there exists a compact set $K \subseteq E$ such that $\mathbb{P}(\{X \notin K\}) < \epsilon$ for all $X \in \mathscr{X}$.

Lemma 12.8. *[9, Lemma 2.5] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $\mathscr{X}$ a family of E-valued random variables on Ω. If $\mathscr{X}$ is uniformly tight, then $\mathscr{X} - \mathscr{X} := \{X_1 - X_2 \ : \ X_1, X_2 \in \mathscr{X}\}$ is uniformly tight.*

Definition 12.9. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(X_n)_{n\in\mathbb{N}}$ a sequence of E-valued random variables on Ω. We say that $(X_n)_{n\in\mathbb{N}}$ **converges in probability** to the random variable $X : \Omega \to E$ if $\lim\limits_{n\to\infty} \mathbb{P}(||X_n - X|| > r) = 0$ for all $r > 0$.

The following lemma is called Chebyshev's inequality.

Lemma 12.10. *[9, p. 20] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X \in L^p(\Omega)$. Then $\mathbb{P}(|X| \geq r) \leq \frac{1}{r^p}\mathbb{E}(|X|^p)$ for all $r > 0$ and $1 \leq p < \infty$.*

Corollary 12.11. *[9, p. 20] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(X_n)_{n\in\mathbb{N}}$ a sequence of E-valued random variables on Ω. If $(X_n)_{n\in\mathbb{N}}$ converges to a random variable $X : \Omega \to E$ in $L^p(\Omega; E)$ for some $1 \leq p < \infty$, then $\lim\limits_{n\to\infty} X_n = X$ in probability.*

Definition 12.12. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(X_n)_{n\in\mathbb{N}}$ a sequence of E-valued random variables on Ω. We say that $(X_n)_{n\in\mathbb{N}}$ **converges a.s.** to some random variable $X : \Omega \to E$ if $\mathbb{P}(\omega : \lim\limits_{n\to\infty} ||X_n(\omega) - X(\omega)|| = 0) = 1$.

Proposition 12.13. *[9, Proposition 2.11] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(X_n)_{n\in\mathbb{N}}$ a sequence of E-valued random variables on Ω. If $(X_n)_{n\in\mathbb{N}}$ converges in probability, then it has a subsequence $(X_{n_k})_{k\in\mathbb{N}}$ that is a.s. convergent.*

Definition 12.14. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $X : \Omega \to E$ a random variable. X is called **symmetric** if X and $-X$ are identically distributed.

Lemma 12.15. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $X : \Omega \to E$ a symmetric random variable. Let $\widetilde{X}$ be a $\mathbb{P}$-version of X. Then $\widetilde{X}$ is also a symmetric random variable.*

Proof. By Lemma 8.9 we have that $\widetilde{X}$ is a random variable. By definition we have that X and $-X$ are identically distributed. Furthermore we have by definition that $X = \widetilde{X}$ a.s. and thus we obtain that X and $\widetilde{X}$ are identically distributed. We also have by definition of $\mathbb{P}$-versions that $-X = -\widetilde{X}$ a.s. and thus $-X$ and $-\widetilde{X}$ are identically distributed. Thus we obtain that $\widetilde{X}$ and $-\widetilde{X}$ are identically distributed, and so $\widetilde{X}$ is a symmetric random variable. ∎

Lemma 12.16. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and let $X, (X_i)_{i\in\mathbb{N}} : \Omega \to E$ be random variables. Let $\widetilde{X}$ be a strongly measurable $\mathbb{P}$-version of X and for $i \in \mathbb{N}$, let $\widetilde{X_i}$ be a strongly measurable $\mathbb{P}$-version of X_i. Then $\widetilde{X} - \sum\limits_{i=1}^{k} \widetilde{X_i}$ is a strongly measurable $\mathbb{P}$-version of $X - \sum\limits_{i=1}^{k} X_i$ for all $k \in \mathbb{N}$.*

Proof. $\widetilde{X}$ and $(\widetilde{X_i})_{i\in\mathbb{N}}$ exist by Proposition 8.8 and by definition we have that $\widetilde{X} \overset{a.s.}{=} X$ and $\widetilde{X_i} \overset{a.s.}{=} X_i$ for all $i \in \mathbb{N}$. Now let $k \in \mathbb{N}$. Then we thus have that $X - \sum\limits_{i=1}^{k} X_i \overset{a.s.}{=} \widetilde{X} - \sum\limits_{i=1}^{k} \widetilde{X_i}$ so we obtain that $\widetilde{X} - \sum\limits_{i=1}^{k} \widetilde{X_i}$ is a $\mathbb{P}$-version of $X - \sum\limits_{i=1}^{k} X_i$ for all $k \in \mathbb{N}$.

Furthermore we have that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space so with Remark 8.6 we have that $\mathcal{SM}(\Omega, E)$ is a vector subspace of E^Ω and thus $\widetilde{X} - \sum\limits_{i=1}^{k} \widetilde{X_i}$ is strongly measurable on Ω since $\widetilde{X}, \widetilde{X_i} \in \mathcal{SM}(\Omega, E)$ for all $i \in \mathbb{N}$. Hence $\widetilde{X} - \sum\limits_{i=1}^{k} \widetilde{X_i}$ is a strongly measurable $\mathbb{P}$-version of $X - \sum\limits_{i=1}^{k} X_i$ for all $k \in \mathbb{N}$. ∎

13 Independence

Many results we want to lift to the non-separable setting involve independence of random variables. As pointed out in Chapter 1, we need to build up this concept from scratch, since the literature is somewhat sloppy with the non-separable case. The main goal of this section is to lift [11, Theorem 2.4], which is described in Theorem 13.19, to the non-separable setting.

Definition 13.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{I}$ an index set. A sequence of σ-algebras $(\mathcal{F}_i)_{i\in\mathcal{I}}$ with $\mathcal{F}_i \subseteq \mathcal{F}$ for all $i \in \mathcal{I}$ is called **independent** if for every $n \in \mathbb{N}$ and for all choices of $E_i \in \mathcal{F}_{j_i}$ for $i \in \{1, ..., n\}$ and with $j_i \neq j_k$ for all $i, k \in \{1, ..., n\}$ with $i \neq k$ it holds that $\mathbb{P}(E_1 \cap ... \cap E_n) = \prod_{i=1}^{n} \mathbb{P}(E_i)$.

Definition 13.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{I}$ be an index set, and let E_i be a Banach space and $X_i : \Omega \to E_i$ a random variable for all $i \in \mathcal{I}$. We say that $(X_i)_{i\in\mathcal{I}}$ are **independent** if for all $N \in \mathbb{N}$, $i_1, ..., i_N \in \mathcal{I}$, and $B_1, ..., B_N$ Borel sets in $E_{i_1}, ..., E_{i_N}$ we have that

$$\mathbb{P}(X_{i_1} \in B_1, ..., X_{i_N} \in B_N) = \prod_{n=1}^{N} \mathbb{P}(X_{i_n} \in B_n).$$

Definition 13.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{I}$ and $\mathcal{J}$ be two index sets, and let E_i and F_j be Banach spaces and $X_i : \Omega \to E_i$ and $Y_j : \Omega \to F_j$ be random variables for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$. We say that $(X_i)_{i\in\mathcal{I}}$ is **independent** of $(Y_j)_{j\in\mathcal{J}}$ if for all $n, m \in \mathbb{N}$, $i_1, ..., i_n \in \mathcal{I}$, $j_1, ..., j_m \in \mathcal{J}$, $B_1, ..., B_n$ Borel sets in $E_{i_1}, ..., E_{i_n}$ and $C_1, ..., C_m$ Borel sets in $F_{j_1}, ..., F_{j_m}$ we have that

$$\mathbb{P}(X_{i_1} \in B_1, ..., X_{i_n} \in B_n, Y_{j_1} \in C_1, ..., Y_{j_m} \in C_m) = \mathbb{P}(X_{i_1} \in B_1, ..., X_{i_n} \in B_n)\mathbb{P}(Y_{j_1} \in C_1, ..., Y_{j_m} \in C_m).$$

The next proposition is a result that is often used, though a proof is difficult to find. It is however proven in [12], as Lemma 4.2. As the proof in [12] omits some details, we include a detailed proof here.

Proposition 13.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{I}_1, \mathcal{I}_2, ...$ be π-systems with $\mathcal{I}_i \subseteq \mathcal{F}$ for all $i \in \mathbb{N}$. Suppose that for all $m \in \mathbb{N}$ and all choices of $I_j \in \mathcal{I}_{i_j}$ for $j \in \{1, ..., m\}$ and with $i_j \neq i_k$ for all $j, k \in \{1, ..., m\}$ with $j \neq k$ it holds that $\mathbb{P}(I_1 \cap ... \cap I_m) = \prod_{j=1}^{m} \mathbb{P}(I_j)$.*
Then $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), ...$ are independent.

Proof. Without loss of generality we assume that $\Omega \in \mathcal{I}_i$ for all $i \in \mathbb{N}$. This is possible since if $\Omega \notin \mathcal{I}_i$ for some $i \in \mathbb{N}$, then $\mathcal{I}_i \cup \{\Omega\}$ is still a π-system, $\sigma(\mathcal{I}_i) = \sigma(\mathcal{I}_i \cup \{\Omega\})$ and $\mathcal{I}_1$, ...,$\mathcal{I}_{i-1}$, $\mathcal{I}_i \cup \{\Omega\}$, $\mathcal{I}_{i+1}$, ... satisfy the same property as stated in the Proposition for $\mathcal{I}_1, \mathcal{I}_2, ...$ Define

$$\mathcal{D} := \Big\{ A \in \mathcal{F} \ : \ \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B) \ \forall B \in \big\{ I_2 \cap ... \cap I_n : n \in \mathbb{N}_{\geq 2} \text{ and } I_i \in \mathcal{I}_i \ \forall i \in \{2, ..., n\} \big\} \Big\}.$$

First I will prove that $\mathcal{I}_1 \subseteq \mathcal{D}$. Let $C \in \mathcal{I}_1$. Then for all choices of $n \in \mathbb{N}$ and $B_2 \in \mathcal{I}_2, ..., B_n \in \mathcal{I}_n$ we have

$$\mathbb{P}(C \cap B_2 \cap ... \cap B_n) \overset{(*_1)}{=} \mathbb{P}(C)\mathbb{P}(B_2) \cdot ... \cdot \mathbb{P}(B_n) \overset{(*_2)}{=} \mathbb{P}(C)\mathbb{P}(B_2 \cap ... \cap B_n),$$

where $(*_1)$ and $(*_2)$ follow by the assumption. Thus for all $B \in \big\{ I_2 \cap ... \cap I_n : n \in \mathbb{N}_{\geq 2} \text{ and } I_i \in \mathcal{I}_i \ \forall i \in \{2, ..., n\} \big\}$ we have that $\mathbb{P}(C \cap B) = \mathbb{P}(C)\mathbb{P}(B)$. So we have that $C \in \mathcal{D}$ since $C \in \mathcal{F}$, and thus $\mathcal{I}_1 \subseteq \mathcal{D}$.

Next I will prove that $\mathcal{D}$ is a d-system.

- $\mathcal{F}$ is a σ-algebra, and thus we have that $\Omega \in \mathcal{F}$. For $B \in \big\{ I_2 \cap ... \cap I_n : n \in \mathbb{N}_{\geq 2} \text{ and } I_i \in \mathcal{I}_i \ \forall i \in \{2, ..., n\} \big\}$ we obviously have that $\mathbb{P}(\Omega \cap B) = \mathbb{P}(B) = 1 \cdot \mathbb{P}(B) = \mathbb{P}(\Omega)\mathbb{P}(B)$. Hence $\Omega \in \mathcal{D}$.

- Let $A, B \in \mathcal{D}$ with $A \subseteq B$ and let $C \in \big\{I_2 \cap ... \cap I_n : n \in \mathbb{N}_{\geq 2} \text{ and } I_i \in \mathcal{I}_i \ \forall i \in \{2, ..., n\}\big\}$. Then we have

$$\mathbb{P}((B \setminus A) \cap C) = \mathbb{P}((B \cap C) \setminus (A \cap C)) \overset{(*_1)}{=} \mathbb{P}(B \cap C) - \mathbb{P}(A \cap C) \overset{(*_2)}{=} \mathbb{P}(B)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(C)$$

$$= (\mathbb{P}(B) - \mathbb{P}(A))\mathbb{P}(C) \overset{(*_3)}{=} \mathbb{P}(B \setminus A)\mathbb{P}(C),$$

where $(*_1)$ holds since $A \subseteq B$ and thus $(A \cap C) \subseteq (B \cap C)$, $(*_2)$ since $A, B \in \mathcal{D}$ and $(*_3)$ since $A \subseteq B$. Thus we obtain that $B \setminus A \in \mathcal{D}$ since C was arbitrary.

- Let $A_m \in \mathcal{D}$ for $m \in \mathbb{N}$ with $A_m \subseteq A_{m+1}$ for all $m \in \mathbb{N}$. Let $B \in \big\{I_2 \cap ... \cap I_n : n \in \mathbb{N}_{\geq 2} \text{ and } I_i \in \mathcal{I}_i \ \forall i \in \{2, ..., n\}\big\}$. Then we have

$$\mathbb{P}\Big((\bigcup_{m=1}^{\infty} A_m) \cap B \Big) = \mathbb{P}\Big(\bigcup_{m=1}^{\infty} (A_m \cap B) \Big).$$

Now consider the sequence $E_N = \bigcup_{m=1}^{N} (A_m \cap B)$. Then this is an increasing sequence with limit $E = \bigcup_{N=1}^{\infty} E_N = \bigcup_{N=1}^{\infty} \bigcup_{m=1}^{N} (A_m \cap B) = \bigcup_{m=1}^{\infty} (A_m \cap B)$. With part (i) of Proposition 5.2 we get $\lim_{N \to \infty} \mathbb{P}(E_N) = \mathbb{P}(E)$, i.e. $\lim_{N \to \infty} \mathbb{P}\Big(\bigcup_{m=1}^{N} (A_m \cap B) \Big) = \mathbb{P}\Big(\bigcup_{m=1}^{\infty} (A_m \cap B) \Big)$. Thus we have

$$\mathbb{P}\Big(\bigcup_{m=1}^{\infty} (A_m \cap B) \Big) = \lim_{N \to \infty} \mathbb{P}\Big(\bigcup_{m=1}^{N} (A_m \cap B) \Big) \overset{(*_1)}{=} \lim_{N \to \infty} \mathbb{P}(A_N \cap B) \overset{(*_2)}{=} \lim_{N \to \infty} \mathbb{P}(A_N)\mathbb{P}(B)$$

$$= \mathbb{P}(B) \lim_{N \to \infty} \mathbb{P}(A_N) \overset{(*_3)}{=} \mathbb{P}(B)\mathbb{P}\Big(\bigcup_{N=1}^{\infty} A_N \Big) = \mathbb{P}\Big(\bigcup_{N=1}^{\infty} A_N \Big)\mathbb{P}(B),$$

where $(*_1)$ follows since $(A_N)_{N \in \mathbb{N}}$ is an increasing sequence and thus $\bigcup_{m=1}^{N} (A_m \cap B) = A_N \cap B$, $(*_2)$ follows since $A_N \in \mathcal{D}$ for all $N \in \mathbb{N}$ and $(*_3)$ follows with part (i) of Proposition 5.2 since $(A_N)_{N \in \mathbb{N}}$ is an increasing sequence with limit $\bigcup_{N=1}^{\infty} A_N$. This yields $\bigcup_{m=1}^{\infty} A_m \in \mathcal{D}$.

So $\mathcal{D}$ is a d-system.

Thus $\mathcal{D}$ is a d-system containing the π-system $\mathcal{I}_1$ and thus by Lemma 2.6 we have that $\sigma(\mathcal{I}_1) \subseteq \mathcal{D}$. So for all choices $A \in \sigma(\mathcal{I}_1)$ and all choices $B \in \big\{I_2 \cap ... \cap I_n : n \in \mathbb{N}_{\geq 2} \text{ and } I_i \in \mathcal{I}_i \ \forall i \in \{2, ..., n\}\big\}$ we have that $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$. Since $\Omega \in \mathcal{I}_i$ for all $i \in \mathbb{N}$ we obtain that $\sigma(\mathcal{I}_1)$, $\mathcal{I}_2$, $\mathcal{I}_3$, ... satisfy the same property as stated in the Proposition for $\mathcal{I}_1$, $\mathcal{I}_2$,..., since a σ-algebra is also a π-system by Proposition 2.5.

Now we can apply the above repeatedly to obtain that $\sigma(\mathcal{I}_1), ..., \sigma(\mathcal{I}_n), \mathcal{I}_{n+1}, ...$ satisfy the same property for each $n \in \mathbb{N}$. By definition, this yields that $\sigma(\mathcal{I}_1), \sigma(\mathcal{I}_2), ...$ are independent. ■

Next we will study independence of random variables. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space. Recall that an E-valued random variable is a strongly $\mathbb{P}$-measurable map $X : \Omega \to E$. For E-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$ it can be useful to consider the σ-algebra generated by the random variables. However the definition of this σ-algebra is not necessarily clear, since this σ-algebra has to be contained in $\mathcal{F}$. When discussing independence of random variables the definition below appears useful, but this definition only works when all the random variables are measurable. Therefore we introduce the following assumption.

Assumption A1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{I}$ an index set, $(E_i)_{i \in \mathcal{I}}$ Banach spaces and $(X_i)_{i \in \mathcal{I}} : \Omega \to E_i$ random variables. X_i is measurable for all $i \in \mathcal{I}$.

When $\mathcal{F}$ is complete, we have with Lemma 9.12 that random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ are measurable. If $\mathcal{F}$ is not complete, we can consider the completion of $\mathcal{F}$, $\mathcal{F}_{\mathbb{P}}$, and thus we can still use the next definition with respect to $(\Omega, \mathcal{F}_{\mathbb{P}}, \hat{\mathbb{P}})$.

Definition 13.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{I}$ an index set, $(E_i)_{i\in\mathcal{I}}$ Banach spaces and $(X_i)_{i\in\mathcal{I}} : \Omega \to E_i$ random variables. Assume that A1 is satisfied. Then the **σ-algebra generated by** $(X_i)_{i\in\mathcal{I}}$, denoted by $\sigma(X_i : i \in \mathcal{I})$, is the smallest σ-algebra such that X_i is measurable with respect to this σ-algebra for all $i \in \mathcal{I}$. We can write

$$\sigma(X_i : i \in \mathcal{I}) :=$$
$$\sigma\left(\left\{\{\omega \in \Omega : X_{j_i}(\omega) \in B_{j_i} \text{ for all } i \in \mathbb{N}\} : j_i \in \mathcal{I} \text{ for all } i \in \mathbb{N}, B_{j_i} \text{ a Borel set in } E_{j_i} \text{ for all } i \in \mathbb{N}\right\}\right).$$

Remark 13.6. Note that

$$\left\{\{\omega \in \Omega : X_{j_i}(\omega) \in B_{j_i} \text{ for all } i \in \mathbb{N}\} : j_i \in \mathcal{I} \text{ for all } i \in \mathbb{N}, B_{j_i} \text{ a Borel set in } E_{j_i} \text{ for all } i \in \mathbb{N}\right\}$$
$$:= \Pi(X_i : i \in \mathcal{I})$$

is a π-system and for $\mathcal{I} = \{1\}$ it is a σ-algebra.

Definition 13.5 corresponds with the definition of the generated σ-algebra for $\mathbb{R}$-valued random variables, but since the general definition of an E-valued random variable involves strong $\mathbb{P}$-measurability, we might want to consider a different construction of the generated σ-algebra than the above, namely the smallest σ-algebra such that the random variables are strongly measurable. But for this σ-algebra to be contained in $\mathcal{F}$, and thus to be well-defined, there has to be a stronger assumption on the random variables than in Definition 13.5. Under this assumption the σ-algebras turn out to be the same, and with Proposition 8.8 it seems that the assumption is not really a restriction.

Lemma 13.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{I}$ an index set, $(E_i)_{i\in\mathcal{I}}$ Banach spaces and $(X_i)_{i\in\mathcal{I}} : \Omega \to E_i$ random variables. Assume that X_i is strongly $\mathcal{F}$-measurable for all $i \in \mathcal{I}$. The smallest σ-algebra such that X_i is strongly measurable with respect to this σ-algebra for all $i \in \mathcal{I}$ equals $\sigma(X_i : i \in \mathcal{I})$.*

Proof. Denote by $\sigma\sigma(X_i : i \in \mathcal{I})$ the smallest σ-algebra such that for all $i \in \mathcal{I}$ the random variable X_i is strongly measurable with respect to this σ-algebra. Let $i \in \mathcal{I}$. Since X_i is strongly measurable with respect to $\sigma\sigma(X_j : j \in \mathcal{I})$ we have with Theorem 9.3 that X_i is measurable with respect to $\sigma\sigma(X_j : j \in \mathcal{I})$ and that X_i is separably valued. Now since X_i is measurable with respect to $\sigma(X_j : j \in \mathcal{I})$ and separably valued we have with Theorem 9.3 that X_i is strongly measurable with respect to $\sigma(X_j : j \in \mathcal{I})$. Thus we obtain that $\sigma(X_i : i \in \mathcal{I}) = \sigma\sigma(X_i : i \in \mathcal{I})$. ∎

Proposition 13.8. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{I}$ an index set. Let E_i be a Banach space and $X_i : \Omega \to E_i$ a random variable for all $i \in \mathcal{I}$. Assume that A1 is satisfied. Then we have that $(X_i)_{i\in\mathcal{I}}$ are independent if and only if the σ-algebras generated by the random variables are independent, i.e. $(\sigma(X_i))_{i\in\mathcal{I}}$ are independent σ-algebras.*

Proof. Since we assume that A1 is satisfied the generated σ-algebras are well-defined.

- Suppose that $(X_i)_{i\in\mathcal{I}}$ are independent. Let $n \in \mathbb{N}$, $j_i \in \mathcal{I}$ for $i \in \{1, ..., n\}$ with $j_i \neq j_k$ for all $i, k \in \{1, ..., n\}$ with $i \neq k$, and let $A_i \in \sigma(X_{j_i})$ for $i \in \{1, ..., n\}$. Then we have for $i \in \{1, ..., n\}$ that $A_i = \{\omega \in \Omega : X_{j_i}(\omega) \in B_i\}$ for some Borel set $B_i \subseteq E_{j_i}$. We obtain

$$\mathbb{P}(A_1 \cap ... \cap A_n) = \mathbb{P}\left(\bigcap_{i=1}^{n} \{\omega \in \Omega : X_{j_i}(\omega) \in B_i\}\right) = \mathbb{P}(X_{j_1} \in B_1, ..., X_{j_n} \in B_n)$$
$$\overset{(*)}{=} \prod_{i=1}^{n} \mathbb{P}(X_{j_i} \in B_i) = \prod_{i=1}^{n} \mathbb{P}(\{\omega \in \Omega : X_{j_i}(\omega) \in B_i\}) = \prod_{i=1}^{n} \mathbb{P}(A_i),$$

  where $(*)$ follows since $(X_i)_{i\in\mathcal{I}}$ are independent. So we obtain that $(\sigma(X_i))_{i\in\mathcal{I}}$ are independent σ-algebras.

- Now suppose that $(\sigma(X_i))_{i\in\mathcal{I}}$ are independent σ-algebras. Let $n \in \mathbb{N}$, $j_1, ..., j_n \in \mathcal{I}$ and $B_1, ..., B_n$ Borel sets in $E_{j_1}, ..., E_{j_n}$. Then we have:

$$\mathbb{P}(X_{j_1} \in B_1, ..., X_{j_n} \in B_n) = \mathbb{P}\bigg(\bigcap_{i=1}^{n}\{\omega \in \Omega : X_{j_i}(\omega) \in B_i\}\bigg)$$

$$\overset{(*)}{=} \prod_{i=1}^{n}\mathbb{P}(\{\omega \in \Omega : X_{j_i}(\omega) \in B_i\}) = \prod_{i=1}^{n}\mathbb{P}(X_{j_i} \in B_i),$$

where $(*)$ holds since $\{\omega \in \Omega : X_{j_i}(\omega) \in B_i\} \in \sigma(X_{j_i})$ for all $i \in \{1, ..., n\}$ and $(\sigma(X_i))_{i\in\mathcal{I}}$ are independent σ-algebras. Thus we have that $(X_i)_{i\in\mathcal{I}}$ are independent.

■

Proposition 13.9. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{I}$ and $\mathcal{J}$ index sets and E_i and F_j Banach spaces for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$. Let $X_i : \Omega \to E_i$ and $Y_j : \Omega \to F_j$ be random variables for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$. Assume that A1 is satisfied. Then $(X_i)_{i\in\mathcal{I}}$ and $(Y_j)_{j\in\mathcal{J}}$ are independent if and only if $\sigma(X_i : i \in \mathcal{I})$ and $\sigma(Y_j : j \in \mathcal{J})$ are independent.*

Proof. As before the generated σ-algebras are well-defined.

- Suppose that $(X_i)_{i\in\mathcal{I}}$ and $(Y_j)_{j\in\mathcal{J}}$ are independent. Let $A_1 \in \Pi(X_i : i \in \mathcal{I})$ and let $A_2 \in \Pi(Y_j : j \in \mathcal{J})$, where we use the notation of Remark 13.6. Then we have that $A_1 = \{\omega \in \Omega : X_{i_k}(\omega) \in B_k \text{ for all } k \in \mathbb{N}\}$ with for all $k \in \mathbb{N}$, B_k a Borel set in E_{i_k}. Similarly, $A_2 = \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l \text{ for all } l \in \mathbb{N}\}$ with for all $l \in \mathbb{N}$, C_l a Borel set in F_{j_l}. Then we have:

$$\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(\{\omega \in \Omega : X_{i_k}(\omega) \in B_k \text{ for all } k \in \mathbb{N}\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l \text{ for all } l \in \mathbb{N}\})$$

$$= \mathbb{P}\left(\left(\bigcap_{i=1}^{\infty}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\}\right) \cap \left(\bigcap_{i=1}^{\infty}\{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right)\right)$$

$$= \mathbb{P}\left(\bigcap_{i=1}^{\infty}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right).$$

Now consider the sequence given by $E_n = \bigcap_{i=1}^{n}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}$ for $n \in \mathbb{N}$. Then we have for all $n \in \mathbb{N}$ that $E_{n+1} = \bigcap_{i=1}^{n+1}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\} \subseteq \bigcap_{i=1}^{n}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\} = E_n$ so the sequence is decreasing. Furthermore we have that $\mathbb{P}(E_n) \le 1$ for all $n \in \mathbb{N}$ and the sequence has limit $E := \bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty}\bigcap_{i=1}^{n}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\} = \bigcap_{i=1}^{\infty}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}$. So with part (ii) of Proposition 5.2 we have that $\lim_{n\to\infty}\mathbb{P}(E_n) = \mathbb{P}(E)$. Thus we obtain:

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right) = \mathbb{P}(E) = \lim_{n\to\infty}\mathbb{P}(E_n)$$

$$= \lim_{n\to\infty}\mathbb{P}\left(\bigcap_{i=1}^{n}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \cap \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right)$$

$$= \lim_{n\to\infty}\mathbb{P}\left(\left(\bigcap_{i=1}^{n}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\}\right) \cap \left(\bigcap_{i=1}^{n}\{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right)\right)$$

$$\overset{(*_1)}{=} \lim_{n\to\infty}\mathbb{P}\left(\bigcap_{i=1}^{n}\{\omega \in \Omega : X_{i_k}(\omega) \in B_k\}\right)\mathbb{P}\left(\bigcap_{i=1}^{n}\{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right),$$

where $(*_1)$ follows since $(X_i)_{i\in\mathcal{I}}$ and $(Y_j)_{j\in\mathcal{J}}$ are independent.

Now consider the sequence given by $G_n = \bigcap_{i=1}^n \{\omega \in \Omega : X_{i_k}(\omega) \in B_k\}$ for all $n \in \mathbb{N}$ and the sequence given by $D_n = \bigcap_{i=1}^n \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}$. Then we have for all $n \in \mathbb{N}$ that $G_{n+1} = \bigcap_{i=1}^{n+1} \{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} \subseteq \bigcap_{i=1}^n \{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} = G_n$ and $D_{n+1} = \bigcap_{i=1}^{n+1} \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\} \subseteq \bigcap_{i=1}^n \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\} = D_n$ thus the sequences are decreasing. Furthermore we have that $\mathbb{P}(G_n) \leq 1$ and $\mathbb{P}(D_n) \leq 1$ for all $n \in \mathbb{N}$ and the sequences have the limits $G := \bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty \bigcap_{i=1}^n \{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} = \bigcap_{i=1}^\infty \{\omega \in \Omega : X_{i_k}(\omega) \in B_k\} = A_1$ and $D := \bigcap_{n=1}^\infty D_n = \bigcap_{n=1}^\infty \bigcap_{i=1}^n \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\} = \bigcap_{i=1}^\infty \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\} = A_2$. So with part (ii) of Proposition 5.2 we have that $\lim_{n\to\infty} \mathbb{P}(G_n) = \mathbb{P}(G) = \mathbb{P}(A_1)$ and $\lim_{n\to\infty} \mathbb{P}(D_n) = \mathbb{P}(D) = \mathbb{P}(A_2)$, thus we obtain:

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^n \{\omega \in \Omega : X_{i_k}(\omega) \in B_k\}\right) \mathbb{P}\left(\bigcap_{i=1}^n \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^n \{\omega \in \Omega : X_{i_k}(\omega) \in B_k\}\right) \cdot \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=1}^n \{\omega \in \Omega : Y_{j_l}(\omega) \in C_l\}\right) \\
&= \lim_{n\to\infty} \mathbb{P}(A_n) \cdot \lim_{n\to\infty} \mathbb{P}(D_n) = \mathbb{P}(A_1)\mathbb{P}(A_2).
\end{aligned}$$

Thus we have that $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1)\mathbb{P}(A_2)$ so now we obtain with Proposition 13.4 that $\sigma(X_i : i \in \mathcal{I})$ is independent of $\sigma(Y_j : j \in \mathcal{J})$.

- Now suppose that $\sigma(X_i : i \in \mathcal{I})$ is independent of $\sigma(Y_j : j \in \mathcal{J})$. Let $n, m \in \mathbb{N}$, $i_1, ..., i_n \in \mathcal{I}$, $j_1, ..., j_m \in \mathcal{J}$, $B_1, ..., B_n$ Borel sets of $E_{i_1}, ..., E_{i_n}$ and $C_1, ..., C_m$ Borel sets of $F_{j_1}, ..., F_{j_m}$. Then we have:

$$\begin{aligned}
&\mathbb{P}(X_{i_1} \in B_1, ..., X_{i_n} \in B_n, Y_{j_1} \in C_1, ..., Y_{j_m} \in C_m) \\
&= \mathbb{P}(\{\omega \in \Omega : X_{i_k}(\omega) \in B_i \text{ for all } k \in \{1, ..., n\}\} \cap \{\omega \in \Omega : Y_{j_k}(\omega) \in C_j \text{ for all } k \in \{1, ..., m\}\}) \\
&\overset{(*)}{=} \mathbb{P}(\{\omega \in \Omega : X_{i_k}(\omega) \in B_i \text{ for all } k \in \{1, ..., n\}\})\mathbb{P}(\{\omega \in \Omega : Y_{j_k}(\omega) \in C_j \text{ for all } k \in \{1, ..., m\}\}) \\
&= \mathbb{P}(X_{i_1} \in B_1, ..., X_{i_n} \in B_n)\mathbb{P}(Y_{j_1} \in C_1, ..., Y_{j_m} \in C_m)
\end{aligned}$$

where $(*)$ since $\{\omega \in \Omega : X_{i_k}(\omega) \in B_i$ for all $k \in \{1, ..., n\}\} \in \sigma(X_i : i \in \mathcal{I})$ and $\{\omega \in \Omega : Y_{j_k}(\omega) \in C_j$ for all $k \in \{1, ..., m\}\} \in \sigma(Y_j : j \in \mathcal{J})$ and $\sigma(X_i : i \in \mathcal{I})$ is independent of $\sigma(Y_j : j \in \mathcal{J})$. Thus we have that $(X_i)_{i\in\mathcal{I}}$ is independent of $(Y_j)_{j\in\mathcal{J}}$.

So $(X_i)_{i\in\mathcal{I}}$ and $(Y_j)_{j\in\mathcal{J}}$ are independent if and only if $\sigma(X_i : i \in \mathcal{I})$ and $\sigma(Y_j : j \in \mathcal{J})$ are independent. ■

Lemma 13.10. *[9, p. 21] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{I}$ be an index set and let E_i be a Banach space and $X_i : \Omega \to E_i$ a random variable for all $i \in \mathcal{I}$. We have that $(X_i)_{i\in\mathcal{I}}$ are independent if and only if every finite subfamily of $(X_i)_{i\in\mathcal{I}}$ is independent.*

Proposition 13.11. *[9, Proposition 2.13] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $E_1, ..., E_N$ Banach spaces and $X_1 : \Omega \to E_1$,..., $X_N : \Omega \to E_N$ random variables. $X_1, ..., X_N$ are independent if and only if $\mu_{(X_1,...,X_N)} = \mu_{X_1} \times ... \times \mu_{X_N}$, where $\mu_{(X_1,...,X_N)}$ is the distribution of the $E_1 \times ... \times E_N$-valued random variable $(X_1, ..., X_N)$ and $\mu_{X_1} \times ... \times \mu_{X_N}$ is the product measure.*

Proposition 13.12. *[9, Proposition 2.14] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E_1, E_2 Banach spaces, $(X_n)_{n\in\mathbb{N}}, X : \Omega \to E_1$ random variables and $(Y_n)_{n\in\mathbb{N}}, Y : \Omega \to E_2$ random variables. If $\lim_{n\to\infty} X_n = X$ in probability, $\lim_{n\to\infty} Y_n = Y$ in probability and if for each $n \in \mathbb{N}$ we have that X_n is independent of Y_n, then X and Y are independent.*

Lemma 13.13. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{I}$ an index set, $(E_i)_{i\in\mathcal{I}}$ Banach spaces and $(X_i)_{i\in\mathcal{I}} : \Omega \to E_i$ random variables. Let $(\widetilde{X_i})_{i\in\mathcal{I}} : \Omega \to E_i$ such that $\widetilde{X_i}$ is a $\mathbb{P}$-version of X_i for all $i \in \mathcal{I}$. Then we have that $(X_i)_{i\in\mathcal{I}}$ are independent if and only if $(\widetilde{X_i})_{i\in\mathcal{I}}$ are independent.*

Proof. By Lemma 8.9 we have that $(\widetilde{X_i})_{i\in\mathcal{I}}$ are random variables, and thus we can speak of independence of $(\widetilde{X_i})_{i\in\mathcal{I}}$.

- Suppose $(X_i)_{i\in\mathcal{I}}$ are independent. Let $N\in\mathbb{N}$, $i_1,...,i_N\in\mathcal{I}$ and $B_1,...,B_N$ Borel sets in $E_{i_1},...,E_{i_N}$. Then we have:

$$\mathbb{P}(\widetilde{X_{i_1}}\in B_1,...,\widetilde{X_{i_N}}\in B_N)\overset{(*_1)}{=}\mathbb{P}(X_{i_1}\in B_1,...,X_{i_N}\in B_N)\overset{(*_2)}{=}\prod_{n=1}^{N}\mathbb{P}(X_{i_n}\in B_n)\overset{(*_3)}{=}\prod_{n=1}^{N}\mathbb{P}(\widetilde{X_{i_n}}\in B_n)$$

  where $(*_1)$ and $(*_3)$ since we have that $X_i=\widetilde{X_i}$ a.s. for all $i\in\mathcal{I}$ and $(*_2)$ since $(X_i)_{i\in\mathcal{I}}$ are independent. So $(\widetilde{X_i})_{i\in\mathcal{I}}$ are independent random variables.

- Suppose $(\widetilde{X_i})_{i\in\mathcal{I}}$ are independent. Let $N\in\mathbb{N}$, $i_1,...,i_N\in\mathcal{I}$ and $B_1,...,B_N$ Borel sets in $E_{i_1},...,E_{i_N}$. Then we have:

$$\mathbb{P}(X_{i_1}\in B_1,...,X_{i_N}\in B_N)\overset{(*_1)}{=}\mathbb{P}(\widetilde{X_{i_1}}\in B_1,...,\widetilde{X_{i_N}}\in B_N)\overset{(*_2)}{=}\prod_{n=1}^{N}\mathbb{P}(\widetilde{X_{i_n}}\in B_n)\overset{(*_3)}{=}\prod_{n=1}^{N}\mathbb{P}(X_{i_n}\in B_n)$$

  where $(*_1)$ and $(*_3)$ hold since we have that $X_i=\widetilde{X_i}$ a.s. for all $i\in\mathcal{I}$ and $(*_2)$ since $(\widetilde{X_i})_{i\in\mathcal{I}}$ are independent. So $(X_i)_{i\in\mathcal{I}}$ are independent random variables.

So $(X_i)_{i\in\mathcal{I}}$ are independent if and only if $(\widetilde{X_i})_{i\in\mathcal{I}}$ are independent. ∎

Lemma 13.14. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, $\mathcal{I}$ and $\mathcal{J}$ index sets, $(E_i)_{i\in\mathcal{I}}$ and $(F_i)_{i\in\mathcal{J}}$ Banach spaces, and $X_i:\Omega\to E_i$ and $Y_j:\Omega\to F_j$ random variables for all $i\in\mathcal{I}$ and all $j\in\mathcal{J}$. Let $\widetilde{X_i}$ be a $\mathbb{P}$-version of X_i for $i\in\mathcal{I}$ and let $\widetilde{Y_j}$ be a $\mathbb{P}$-version of Y_j for $j\in\mathcal{J}$. Then $(X_i)_{i\in\mathcal{I}}$ and $(Y_j)_{j\in\mathcal{J}}$ are independent if and only if $(\widetilde{X_i})_{i\in\mathcal{I}}$ and $(\widetilde{Y_j})_{j\in\mathcal{J}}$ are independent.*

Proof. By Lemma 8.9 we have that $(\widetilde{X_i})_{i\in\mathcal{I}}$ and $(\widetilde{Y_j})_{j\in\mathcal{J}}$ are sequences of random variables, and thus we can speak of independence of $(\widetilde{X_i})_{i\in\mathcal{I}}$ and $(\widetilde{Y_j})_{j\in\mathcal{J}}$.

- Suppose $(X_i)_{i\in\mathcal{I}}$ is independent of $(Y_j)_{j\in\mathcal{J}}$. Let $n,m\in\mathbb{N}$, $i_1,...,i_n\in\mathcal{I}$, $j_1,...,j_m\in\mathcal{J}$, $B_1,...,B_n$ Borel sets in $E_{i_1},...,E_{i_n}$ and $C_1,...,C_m$ Borel sets in $F_{j_1},...,F_{j_m}$. Then we have:

$$\begin{aligned}&\mathbb{P}(\widetilde{X_{i_1}}\in B_1,...,\widetilde{X_{i_n}}\in B_n,\widetilde{Y_{j_1}}\in C_1,...,\widetilde{Y_{j_m}}\in C_m)\\ \overset{(*_1)}{=}\ &\mathbb{P}(X_{i_1}\in B_1,...,X_{i_n}\in B_n,Y_{j_1}\in C_1,...,Y_{j_m}\in C_m)\\ \overset{(*_2)}{=}\ &\mathbb{P}(X_{i_1}\in B_1,...,X_{i_n}\in B_n)\mathbb{P}(Y_{j_1}\in C_1,...,Y_{j_m}\in C_m)\\ \overset{(*_3)}{=}\ &\mathbb{P}(\widetilde{X_{i_1}}\in B_1,...,\widetilde{X_{i_n}}\in B_n)\mathbb{P}(\widetilde{Y_{j_1}}\in C_1,...,\widetilde{Y_{j_m}}\in C_m)\end{aligned}$$

  where $(*_1)$ and $(*_3)$ follow since we have that $X_i=\widetilde{X_i}$ a.s. for all $i\in\mathcal{I}$ and $Y_j=\widetilde{Y_j}$ a.s. for all $j\in\mathcal{J}$, and $(*_2)$ since $(X_i)_{i\in\mathcal{I}}$ is independent of $(Y_j)_{j\in\mathcal{J}}$. So $(\widetilde{X_i})_{i\in\mathcal{I}}$ is independent of $(\widetilde{Y_j})_{j\in\mathcal{J}}$.

- Suppose $(\widetilde{X_i})_{i\in\mathcal{I}}$ is independent of $(\widetilde{Y_j})_{j\in\mathcal{J}}$. Let $n,m\in\mathbb{N}$, $i_1,...,i_n\in\mathcal{I}$, $j_1,...,j_m\in\mathcal{J}$, $B_1,...,B_n$ Borel sets in $E_{i_1},...,E_{i_n}$ and $C_1,...,C_m$ Borel sets in $F_{j_1},...,F_{j_m}$. Then we have:

$$\begin{aligned}&\mathbb{P}(X_{i_1}\in B_1,...,X_{i_n}\in B_n,Y_{j_1}\in C_1,...,Y_{j_m}\in C_m)\\ \overset{(*_1)}{=}\ &\mathbb{P}(\widetilde{X_{i_1}}\in B_1,...,\widetilde{X_{i_n}}\in B_n,\widetilde{Y_{j_1}}\in C_1,...,\widetilde{Y_{j_m}}\in C_m)\\ \overset{(*_2)}{=}\ &\mathbb{P}(\widetilde{X_{i_1}}\in B_1,...,\widetilde{X_{i_n}}\in B_n)\mathbb{P}(\widetilde{Y_{j_1}}\in C_1,...,\widetilde{Y_{j_m}}\in C_m)\\ \overset{(*_3)}{=}\ &\mathbb{P}(X_{i_1}\in B_1,...,X_{i_n}\in B_n)\mathbb{P}(Y_{j_1}\in C_1,...,Y_{j_m}\in C_m)\end{aligned}$$

  where $(*_1)$ and $(*_3)$ hold since we have that $X_i=\widetilde{X_i}$ a.s. for all $i\in\mathcal{I}$ and $Y_j=\widetilde{Y_j}$ a.s. for all $j\in\mathcal{J}$, and $(*_2)$ since $(\widetilde{X_i})_{i\in\mathcal{I}}$ is independent of $(\widetilde{Y_j})_{j\in\mathcal{J}}$. So $(X_i)_{i\in\mathcal{I}}$ is independent of $(Y_j)_{j\in\mathcal{J}}$.

So $(X_i)_{i\in\mathcal{I}}$ and $(Y_j)_{j\in\mathcal{J}}$ are independent if and only if $(\widetilde{X_i})_{i\in\mathcal{I}}$ and $(\widetilde{Y_j})_{j\in\mathcal{J}}$ are independent. ∎

Proposition 13.15. *[9, Proposition 2.16] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, and $X, Y : \Omega \to E$ random variables. If X is symmetric and independent of Y, then for all $1 \le p < \infty$ we have $\mathbb{E}(||X||^p) \le \mathbb{E}(||X+Y||^p)$.*

The following theorem is called the Itô-Nisio theorem.

Theorem 13.16. *[9, Theorem 2.17] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space. Let $X_n : \Omega \to E$ for $n \ge 1$ be independent symmetric random variables. Put $S_n := \sum_{i=1}^{n} X_i$ and let $S : \Omega \to E$ be a random variable. The following are equivalent:*

1. *$\lim_{n\to\infty} \langle S_n, x^*\rangle = \langle S, x^*\rangle$ a.s. for all $x^* \in E^*$;*
2. *$\lim_{n\to\infty} \langle S_n, x^*\rangle = \langle S, x^*\rangle$ in probability for all $x^* \in E^*$;*
3. *$\lim_{n\to\infty} S_n = S$ a.s.;*
4. *$\lim_{n\to\infty} S_n = S$ in probability.*

If these hold and $\mathbb{E}(||S||^p) < \infty$ for some $1 \le p < \infty$ then $\lim_{n\to\infty} \mathbb{E}(||S_n - S||^p) = 0$.

Lemma 13.17. *[9, Lemma 2.18] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space. Let $X_1, ..., X_n : \Omega \to E$ be independent symmetric random variables. Put $S_k := \sum_{i=1}^{k} X_i$ for $k = 1, ..., n$. Then $\mathbb{P}(\max_{1\le k\le n} ||S_k|| > r) \le 2\mathbb{P}(||S_n|| > r)$ for all $r \ge 0$.*

Theorem 13.18. *[9, Theorem 3.1] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(X_n)_{n\in\mathbb{N}} : \Omega \to E$ a sequence of independent symmetric random variables. Then for all $N \in \mathbb{N}$, $a_1, ..., a_N \in \mathbb{R}$ and $1 \le p < \infty$ we have $\mathbb{E}\Big(||\sum_{i=1}^{N} a_i X_i||^p\Big) \le (\max_{1\le i\le N} |a_i|)^p \mathbb{E}\Big(||\sum_{i=1}^{N} X_i||^p\Big)$.*

The next result is the key tool in the analysis of [11].

Theorem 13.19. *[11, Theorem 2.4] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a separable Banach space and $(X_i)_{i\in\mathbb{N}} : \Omega \to E$ a sequence of symmetric, independent random variables. Suppose there exists a random variable X such that for all $k \in \mathbb{N}$ there exists a random variable Δ_k with $\Delta_k = X - \sum_{i=1}^{k} X_i$ a.s. and Δ_k is independent of $X_1, ..., X_k$. Then $S_N := \sum_{i=1}^{N} X_i$ converges with probability 1 to a random variable S.*

With all preparations in hand we can lift this result to the non-separable setting.

Theorem 13.20. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(X_i)_{i\in\mathbb{N}} : \Omega \to E$ a sequence of symmetric, independent random variables. Suppose there exists a random variable X such that for all $k \in \mathbb{N}$ there exists a random variable Δ_k with $\Delta_k = X - \sum_{i=1}^{k} X_i$ a.s. and Δ_k is independent of $X_1, ..., X_k$. Then $S_N := \sum_{i=1}^{N} X_i$ converges with probability 1 to a random variable S.*

Proof. With Proposition 8.8 we can let $(\widetilde{X_i})_{i\in\mathbb{N}}, \widetilde{X}, (\widetilde{\Delta_k})_{k\in\mathbb{N}} : \Omega \to E$ strongly measurable $\mathbb{P}$-versions of $(X_i)_{i\in\mathbb{N}}, X$ and $(\Delta_k)_{k\in\mathbb{N}}$ respectively. With Lemma 13.13 and Lemma 12.15 we obtain that $(\widetilde{X_i})_{i\in\mathbb{N}}$ is a sequence of symmetric, independent random variables, and by Lemma 13.13 we also have for every $k \in \mathbb{N}$ that $\widetilde{\Delta_k}$ is independent of $\widetilde{X_1}, ..., \widetilde{X_k}$. Furthermore we have that $\widetilde{\Delta_k} = \widetilde{X} - \sum_{i=1}^{k} \widetilde{X_i}$ a.s. for every $k \in \mathbb{N}$, since

$$\widetilde{\Delta_k} \overset{a.s.}{=} \Delta_k \overset{a.s.}{=} X - \sum_{i=1}^{k} X_i \overset{a.s.}{=} \widetilde{X} - \sum_{i=1}^{k} X_i \overset{a.s.}{=} \widetilde{X} - \sum_{i=1}^{k} \widetilde{X_i}.$$

Now consider $(D_i)_{i\in\mathbb{N}}$ where $D_i := \{\widetilde{X_i}(\omega) : \omega \in \Omega\}$, $A := \{\widetilde{X}(\omega) : \omega \in \Omega\}$ and $(C_k)_{k\in\mathbb{N}}$ where $C_k := \{\widetilde{\Delta_k}(\omega) : \omega \in \Omega\}$. Then we obviously have that $A \subseteq E$, that $D_i \subseteq E$ for all $i \in \mathbb{N}$ and that $C_k \subseteq E$ for all $k \in \mathbb{N}$. Furthermore we have by Theorem 12.4 that $\overline{A}$ is separable, $\overline{D_i}$ is separable for all $i \in \mathbb{N}$ and $\overline{C_k}$ is separable for all $k \in \mathbb{N}$.

Now define $U := \left(\bigcup_{i=1}^{\infty} \overline{D_i}\right) \bigcup \overline{A} \bigcup \left(\bigcup_{k=1}^{\infty} \overline{C_k}\right)$. Then obviously $U \subseteq E$ and we have that U is a countable union of separable sets and thus by Lemma 3.14 U is also separable.

Now consider $\widetilde{E} := \overline{\operatorname{span}(U)}$. Since E is a Banach space, and thus a normed vector space, we have with Theorem 3.16 that $\widetilde{E}$ is separable, and since it is a closed subspace of E we have with Lemma 3.4 that $\widetilde{E}$ is a Banach space. Thus $\widetilde{E}$ is a separable Banach space, and by construction we have that $(\widetilde{X_i})_{i\in\mathbb{N}}, \widetilde{X}$ and $(\widetilde{\Delta_k})_{k\in\mathbb{N}}$ are $\widetilde{E}$-valued.

Now we can apply Theorem 13.19 to obtain that $\widetilde{S_N} := \sum_{i=1}^{N} \widetilde{X_i}$ converges with probability 1 to a random variable S.

Now for all $N \in \mathbb{N}$ define $S_N := \sum_{i=1}^{N} X_i$. Then we have for all $N \in \mathbb{N}$, since $\widetilde{X_i} = X_i$ a.s. for all $i \in \mathbb{N}$, that $\sum_{i=1}^{N} \widetilde{X_i} = \sum_{i=1}^{N} X_i$ a.s. and thus that $S_N = \widetilde{S_N}$ a.s.

For all $N \in \mathbb{N}$ let $\Omega^N := \{\omega \in \Omega : S_N(\omega) = \widetilde{S_N}(\omega)\}$ and let $\Omega^C := \{\omega \in \Omega : \lim_{N\to\infty} \widetilde{S_N}(\omega) = S(\omega)\}$. Then we have for all $N \in \mathbb{N}$ that $\mathbb{P}(\Omega^N) = 1$ and $\mathbb{P}(\Omega^C) = 1$. Let $\widetilde{\Omega} := (\bigcap_{N=1}^{\infty} \Omega^N) \bigcap \Omega^C$. Then we know that $\mathbb{P}(\widetilde{\Omega}) = 1$ since $\widetilde{\Omega}$ is a countable intersection of sets with measure 1. Furthermore we have for $\omega \in \widetilde{\Omega}$ that

$$\lim_{N\to\infty} S_N(\omega) = \lim_{N\to\infty} \widetilde{S_N}(\omega) = S(\omega)$$

and thus S_N converges with probability 1 to the random variable S. ∎

14 Symmetrization

We want to extend Theorem 13.20 to more general settings. In particular, we want to apply it to sequences of random variables which are not symmetric. It turns out to be useful to consider the so-called symmetrization of a random variable. In this section we show that the process of symmetrization preserves properties like independence, so that we can apply our previous results to these symmetrizations.

Definition 14.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space.

- An identical **copy** $(\Omega', \mathcal{F}', \mathbb{P}')$ **of** $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space such that there exists a bijection $f : \Omega \to \Omega'$ satisfying
  1. for all $A \in \mathcal{F}$ we have $f(A) \in \mathcal{F}'$ and $\mathbb{P}'(f(A)) = \mathbb{P}(A)$;
  2. for all $A' \in \mathcal{F}'$ we have $f^{-1}(A') \in \mathcal{F}$ and $\mathbb{P}(f^{-1}(A')) = \mathbb{P}'(A')$.
- Let $X : \Omega \to E$ be a random variable. The **copy** X' **of** X **on** $(\Omega', \mathcal{F}', \mathbb{P}')$ is an E-valued random variable on $(\Omega', \mathcal{F}', \mathbb{P}')$ such that for all $\omega \in \Omega$ we have that $X(\omega) = X'(f(\omega))$ with f the bijection from the definition of $(\Omega', \mathcal{F}', \mathbb{P}')$. (And we thus also have for all $\omega' \in \Omega'$ that $X'(\omega') = X(f^{-1}(\omega'))$.)

Lemma 14.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let $X : \Omega \to E$ be a random variable and let X' be a copy of X on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1. Then we have the following:*

1. *X is also a random variable on $\Omega \times \Omega'$ and on $\Omega' \times \Omega$, where we view X as a map $(\omega, \omega') \mapsto X(\omega)$ and $(\omega', \omega) \mapsto X(\omega)$, respectively.*
2. *$X - X'$ is a random variable on $\Omega \times \Omega'$.*

Proof. I will first prove the first statement, and then I will prove the second statement by using the first.

1. X is a random variable so let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of E-simple functions on Ω such that $\lim_{n\to\infty} \varphi_n = X$ $\mathbb{P}$-a.s. Let $\Omega^* \subseteq \Omega$ such that $\Omega^* \in \mathcal{F}$, $\mathbb{P}(\Omega^*) = 1$ and for all $\omega \in \Omega^*$ we have that $\lim_{n\to\infty} \varphi_n(\omega) = X(\omega)$. Write $\varphi_n := \sum_{i=1}^{N_n} x_i^n \mathbb{1}_{A_i^n}$ and define $\varphi'_n := \sum_{i=1}^{N_n} x_i^n \mathbb{1}_{A_i^n \times \Omega'}$. For every $n \in \mathbb{N}$ obviously φ'_n is an E-simple function on $\Omega \times \Omega'$ and for all $(\omega, \omega') \in \Omega^* \times \Omega'$ we have $\lim_{n\to\infty} \varphi'_n(\omega, \omega') = \lim_{n\to\infty} \varphi_n(\omega) = X(\omega)$. Furthermore, we have that $\mathbb{P} \times \mathbb{P}'(\Omega^* \times \Omega') = \mathbb{P}(\Omega^*)\mathbb{P}'(\Omega') = 1$ and thus $\lim_{n\to\infty} \varphi'_n = X$ $\mathbb{P} \times \mathbb{P}'$-a.s. So X is a random variable on $\Omega \times \Omega'$. Similarly, we obtain that X is a random variable on $\Omega' \times \Omega$.
2. Since X' is a copy of X it is a random variable. By the first statement we have that X and X' are E-valued random variables on $\Omega \times \Omega'$. $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P} \times \mathbb{P}')$ is a probability space since $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ are probability spaces. Using Remark 8.6 it follows that $\mathcal{M}(\Omega \times \Omega', E)$ is a vector subspace of $E^{\Omega \times \Omega'}$ and thus $X - X'$ is an E-valued random variable on $\Omega \times \Omega'$.

∎

Lemma 14.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let $X : \Omega \to E$ be a random variable and let X' be a copy of X on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1. Let $\widetilde{X}$ be a strongly measurable $\mathbb{P}$-version of X. Then $\widetilde{X}'$ (the copy of $\widetilde{X}$ on Ω') is a strongly measurable $\mathbb{P}'$-version of X'.*

Proof. $\widetilde{X}$ exists by Proposition 8.8 and by definition we have that $\widetilde{X} = X$ $\mathbb{P}$-a.s. So let $\Omega^* \subseteq \Omega$ be such that $\Omega^* \in \mathcal{F}$, $\mathbb{P}(\Omega^*) = 1$ and for all $\omega \in \Omega^*$ we have that $\widetilde{X}(\omega) = X(\omega)$. Let $\omega \in \Omega^*$ and let f be as in Definition 14.1. Then we have

$$\widetilde{X}'(f(\omega)) = \widetilde{X}(\omega) = X(\omega) = X'(f(\omega)),$$

and thus $\widetilde{X}'$ is a $\mathbb{P}'$-version of X'. Since $\widetilde{X}$ is strongly measurable and $\widetilde{X}'$ is the copy of $\widetilde{X}$ on Ω', we get that $\widetilde{X}'$ is also strongly measurable. So $\widetilde{X}'$ is a strongly measurable $\mathbb{P}'$-version of X'. ∎

Lemma 14.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let $X : \Omega \to E$ be a random variable and let X' be a copy of X on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1. Let $\widetilde{X}$ be a strongly measurable $\mathbb{P}$-version of X and let $\widetilde{X'}$ be a strongly measurable $\mathbb{P}'$-version of X'. Then $\widetilde{X} - \widetilde{X'}$ is a strongly measurable $\mathbb{P} \times \mathbb{P}'$-version of $X - X'$.*

Proof. $\widetilde{X}$ and $\widetilde{X'}$ exist by Proposition 8.8 and by definition we have that $\widetilde{X} = X$ $\mathbb{P}$-a.s. and $\widetilde{X'} = X'$ $\mathbb{P}'$-a.s. So let $\Omega_1^* \subseteq \Omega$ such that $\Omega_1^* \in \mathcal{F}$, $\mathbb{P}(\Omega_1^*) = 1$ and $\widetilde{X}(\omega) = X(\omega)$ for all $\omega \in \Omega_1^*$. Furthermore let $\Omega_2^* \subseteq \Omega'$ such that $\Omega_2^* \in \mathcal{F}'$, $\mathbb{P}'(\Omega_2^*) = 1$ and $\widetilde{X'}(\omega') = X'(\omega')$ for all $\omega' \in \Omega_2^*$. Let $(\omega, \omega') \in \Omega_1^* \times \Omega_2^*$. Then we have $(X - X')(\omega, \omega') = X(\omega) - X'(\omega') = \widetilde{X}(\omega) - \widetilde{X'}(\omega') = (\widetilde{X} - \widetilde{X'})(\omega, \omega')$ so $\widetilde{X} - \widetilde{X'}$ is a $\mathbb{P} \times \mathbb{P}'$-version of $X - X'$.

We have that $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}', \mathbb{P} \times \mathbb{P}')$ is a probability space since $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\Omega', \mathcal{F}', \mathbb{P}')$ are probability spaces so with Remark 8.6 we have that $\mathcal{SM}(\Omega \times \Omega', E)$ is a vector subspace of $E^{\Omega \times \Omega'}$. Since $\widetilde{X}, \widetilde{X'} \in \mathcal{SM}(\Omega \times \Omega', E)$ it follows that $\widetilde{X} - \widetilde{X'}$ is strongly measurable on $\Omega \times \Omega'$. So $\widetilde{X} - \widetilde{X'}$ is a strongly measurable $\mathbb{P} \times \mathbb{P}'$-version of $X - X'$. ∎

Lemma 14.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and E a Banach space. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let $X : \Omega \to E$ be a random variable and let X' be a copy of X on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1. Then $X^s := X - X'$ is a symmetric random variable.*

Proof. X^s is well-defined and a random variable by Lemma 14.2. Let with Proposition 8.8 $\widetilde{X}$ be a strongly measurable $\mathbb{P}$-version of X. Let with Lemma 14.3 $\widetilde{X}'$ be a strongly measurable $\mathbb{P}'$-version of X'. With Lemma 14.4 we have that $\widetilde{X} - \widetilde{X}'$ is a strongly measurable $\mathbb{P} \times \mathbb{P}'$-version of $X - X'$. Let f as in Definition 14.1. Now we have for B a Borel set in E that

$$\mathbb{P} \times \mathbb{P}'(X^s \in B) = \mathbb{P} \times \mathbb{P}'((X - X') \in B) = \mathbb{P} \times \mathbb{P}'\big(\{(\omega, \omega') \in \Omega \times \Omega' : \widetilde{X}(\omega) - \widetilde{X}'(\omega') \in B\}\big)$$

$$= \mathbb{P} \times \mathbb{P}\big(\{(\omega_1, \omega_2) \in \Omega \times \Omega : \widetilde{X}(\omega_1) - \widetilde{X}'(f(\omega_2)) \in B\}\big) = \mathbb{P} \times \mathbb{P}\big(\{(\omega_1, \omega_2) \in \Omega \times \Omega : \widetilde{X}(\omega_1) - \widetilde{X}(\omega_2) \in B\}\big)$$

$$= \mathbb{P}' \times \mathbb{P}\big(\{(\omega', \omega) \in \Omega' \times \Omega : \widetilde{X}(f^{-1}(\omega')) - \widetilde{X}(\omega) \in B\}\big) = \mathbb{P}' \times \mathbb{P}\big(\{(\omega', \omega) \in \Omega' \times \Omega : \widetilde{X}'(\omega') - \widetilde{X}(\omega) \in B\}\big)$$

$$= \mathbb{P} \times \mathbb{P}'\big(\{(\omega, \omega') \in \Omega \times \Omega' : \widetilde{X}'(\omega') - \widetilde{X}(\omega) \in B\}\big) = \mathbb{P} \times \mathbb{P}'(-X^s \in B)$$

so X^s is symmetric. So $X^s := X - X'$ is a symmetric random variable. ∎

Next we show that the symmetrization of independent random variables yield independent random variables.

Proposition 14.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathcal{I}$ be an index set, let E be a Banach space and let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let $(X_j)_{j \in \mathcal{I}} : \Omega \to E$ be independent random variables and let for all $j \in \mathcal{I}$ X_j' be a copy of X_j on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1. Define for all $j \in \mathcal{I}$ $X_j^s : \Omega \times \Omega' \to E$ by $X_j^s := X_j - X_j'$. Then we have that $(X_j^s)_{j \in \mathcal{I}}$ are independent random variables.*

Proof. For all $j \in \mathcal{I}$ we have with part 2 of Lemma 14.2 that X_j^s is a random variable. Since for all $j \in \mathcal{I}$ we have that X_j' is a copy of X_j we have that $(X_j')_{j \in \mathcal{I}}$ are also independent random variables. For all $j \in \mathcal{I}$ consider X_j and X_j' as E-valued random variables on $\Omega \times \Omega'$. This is possible by part 1 of Lemma 14.2. The independence then obviously persists.

Let $j \in \mathcal{I}$ and let with Proposition 8.8 $\widetilde{X_j}$ be a strongly measurable $\mathbb{P}$-version of X_j defined on Ω and let $\widetilde{X_j'}$ be a strongly measurable $\mathbb{P}'$-version of X_j' defined on Ω'. Let $\widetilde{X_j^s}$ be the strongly measurable $\mathbb{P} \times \mathbb{P}'$-version of X_j^s as defined in Lemma 14.4, i.e. $\widetilde{X_j^s} = \widetilde{X_j} - \widetilde{X_j'}$.

By Lemma 13.13 we have that $(\widetilde{X_j})_{j \in \mathcal{I}}$ are independent random variables since $(X_j)_{j \in \mathcal{I}}$ are and that $(\widetilde{X_j'})_{j \in \mathcal{I}}$ are independent random variables since $(X_j')_{j \in \mathcal{I}}$ are. Now again consider, for all $j \in \mathcal{I}$, $\widetilde{X_j}$ and $\widetilde{X_j'}$ as E-valued random variables on $\Omega \times \Omega'$, which is possible by part 1 of Lemma 14.2. The independence then obviously persists.

For all $i \in \mathcal{I}$ we have that $\widetilde{X_i}(\omega, \omega')$, with $(\omega, \omega') \in \Omega \times \Omega'$, does not depend on ω' and thus we can rewrite for all $i \in \mathcal{I}$

$$\sigma(\widetilde{X_i}) = \Big\{\{(\omega, \omega') \in \Omega \times \Omega' : \widetilde{X_i}(\omega, \omega') \in B\} : B \text{ a Borel set in } E\Big\}$$

$$= \Big\{\{\omega \in \Omega : \widetilde{X_i}(\omega) \in B\} \times \Omega' : B \text{ a Borel set in } E\Big\}.$$

Similarly we can rewrite

$$\sigma(\widetilde{X_i'}) = \Big\{\Omega \times \{\omega' \in \Omega' : \widetilde{X_i'}(\omega') \in B\} : B \text{ a Borel set in } E\Big\}.$$

Furthermore denote with $\mathcal{F}_{\widetilde{X_i}}$ the σ-algebra generated by $\widetilde{X_i}$ with respect to $(\Omega, \mathcal{F}, \mathbb{P})$ and with $\mathcal{F}'_{\widetilde{X_i'}}$ the σ-algebra generated by $\widetilde{X_i'}$ with respect to $(\Omega', \mathcal{F}', \mathbb{P}')$. Then we have that $\sigma(\widetilde{X_i}) = \mathcal{F}_{\widetilde{X_i}} \times \Omega'$ and

$\sigma(\widetilde{X_i'}) = \Omega \times \mathcal{F}'_{\widetilde{X_i'}}$.

Now we have for all $i \in \mathcal{I}$:

$$\sigma(\widetilde{X_i^s}) = \Big\{\{(\omega,\omega') \in \Omega \times \Omega' : \widetilde{X_i^s}(\omega,\omega') \in B\} : B \text{ a Borel set in } E\Big\}$$

$$= \Big\{\{(\omega,\omega') \in \Omega \times \Omega' : \widetilde{X_i}(\omega) - \widetilde{X_i'}(\omega') \in B\} : B \text{ a Borel set in } E\Big\}.$$

Define $\mathcal{I}_i := \{A \times B : A \in \mathcal{F}_{\widetilde{X_i}}, B \in \mathcal{F}'_{\widetilde{X_i'}}\}$ for all $i \in \mathcal{I}$. Since $\widetilde{X_i}$ is strongly measurable with respect to $(\Omega, \mathcal{F}, \mathbb{P})$ we have with Theorem 9.3 that $\widetilde{X_i}$ is measurable with respect to $(\Omega, \mathcal{F}, \mathbb{P})$ and thus we obtain that $\mathcal{F}_{\widetilde{X_i}} \subseteq \mathcal{F}$. Similarly we have that $\mathcal{F}'_{\widetilde{X_i'}} \subseteq \mathcal{F}'$ so we obtain that $\mathcal{I}_i \subseteq \mathcal{F} \times \mathcal{F}'$ for all $i \in \mathcal{I}$ and thus that $\sigma(\mathcal{I}_i) \subseteq \mathcal{F} \times \mathcal{F}'$ for all $i \in \mathcal{I}$ since $\mathcal{F} \times \mathcal{F}'$ is a σ-algebra.

Now I will show that for all $i \in \mathcal{I}$ we have that $\mathcal{I}_i$ is a π-system. Let $i \in \mathcal{I}$ and let $A \times A', B \times B' \in \mathcal{I}_i$. Then we have that $(A \times A') \cap (B \times B') = (A \cap B) \times (A' \cap B')$. Since $\mathcal{F}_{\widetilde{X_i}}$ is a σ-algebra and $A, B \in \mathcal{F}_{\widetilde{X_i}}$ we have that $A \cap B \in \mathcal{F}_{\widetilde{X_i}}$. Since $\mathcal{F}'_{\widetilde{X_i'}}$ is a σ-algebra and $A', B' \in \mathcal{F}'_{\widetilde{X_i'}}$ we have that $A' \cap B' \in \mathcal{F}'_{\widetilde{X_i'}}$. Thus we obtain that $(A \times A') \cap (B \times B') \in \mathcal{I}_i$. So $\mathcal{I}_i$ is a π-system.

Next I will prove that $\sigma(\widetilde{X_i^s}) \subseteq \sigma(\mathcal{I}_i)$ for all $i \in \mathcal{I}$. Let $i \in \mathcal{I}$ and let B be a Borel set in E. Then we have that

$$\widetilde{X_i}^{-1}(B) = \{(\omega,\omega') \in \Omega \times \Omega' : \widetilde{X_i}(\omega,\omega') \in B\} = \{\omega \in \Omega : \widetilde{X_i}(\omega) \in B\} \times \Omega'.$$

Since $\{\omega \in \Omega : \widetilde{X_i}(\omega) \in B\} \in \mathcal{F}_{\widetilde{X_i}}$ and $\Omega' \in \mathcal{F}'_{\widetilde{X_i'}}$ since $\mathcal{F}'_{\widetilde{X_i'}}$ is a σ-algebra, we obtain that $\widetilde{X_i}^{-1}(B) \in \sigma(\mathcal{I}_i)$. So $\widetilde{X_i}$ is measurable with respect to $\sigma(\mathcal{I}_i)$. Similarly $\widetilde{X_i'}$ is measurable with respect to $\sigma(\mathcal{I}_i)$.
Since $\widetilde{X_i}$ and $\widetilde{X_i'}$ are strongly measurable, we have with Theorem 9.3 that they are separably valued. Since they are measurable with respect to $\sigma(\mathcal{I}_i)$ and separably valued, we have with Theorem 9.3 that they are strongly measurable with respect to $\sigma(\mathcal{I}_i)$. By Remark 8.6 we have that $\widetilde{X_i} - \widetilde{X_i'} = \widetilde{X_i^s}$ is also strongly measurable with respect to $\sigma(\mathcal{I}_i)$ and thus by Theorem 9.3 we have that $\widetilde{X_i^s}$ is measurable with respect to $\sigma(\mathcal{I}_i)$. Since $\sigma(\widetilde{X_i^s})$ is the smallest σ-algebra such that $\widetilde{X_i^s}$ is measurable we obtain that $\sigma(\widetilde{X_i^s}) \subseteq \sigma(\mathcal{I}_i)$. Thus for all $i \in \mathcal{I}$ we have that $\sigma(\widetilde{X_i^s}) \subseteq \sigma(\mathcal{I}_i)$.

Now I will prove that $(\sigma(\mathcal{I}_i))_{i \in \mathcal{I}}$ are independent σ-algebras. Let $n \in \mathbb{N}$ and let $\{\mathcal{I}_{i_1}, ..., \mathcal{I}_{i_n}\} \subseteq \{\mathcal{I}_i : i \in \mathcal{I}\}$ be a collection of π-systems and for convenience of notation say $i_1 = 1, i_2 = 2$, etc. Let $I_i \in \mathcal{I}_i$ for $i = 1, ..., n$. Then we have for all $i \in \{1, ..., n\}$ that $I_i = A_i \times B_i$ with $A_i \in \mathcal{F}_{\widetilde{X_i}}$ and $B_i \in \mathcal{F}'_{\widetilde{X_i'}}$. Then we have

$$\mathbb{P} \times \mathbb{P}'(I_1 \cap ... \cap I_n) = \mathbb{P} \times \mathbb{P}'((A_1 \times B_1) \cap ... \cap (A_n \times B_n))$$

$$= \mathbb{P} \times \mathbb{P}'((A_1 \cap ... \cap A_n) \times (B_1 \cap ... \cap B_n)) = \mathbb{P}(A_1 \cap ... \cap A_n)\mathbb{P}'(B_1 \cap ... \cap B_n).$$

Since $(\widetilde{X_i})_{i \in \mathcal{I}}$ are independent random variables we have with Proposition 13.8 that $(\mathcal{F}_{\widetilde{X_i}})_{i \in \mathcal{I}}$ are independent and thus since $A_1 \in \mathcal{F}_{\widetilde{X_1}}$,...,$A_n \in \mathcal{F}_{\widetilde{X_n}}$ we have that $\mathbb{P}(A_1 \cap ... \cap A_n) = \prod_{k=1}^n \mathbb{P}(A_k)$. Similarly we have that $\mathbb{P}'(B_1 \cap ... \cap B_n) = \prod_{k=1}^n \mathbb{P}'(B_k)$. So we obtain

$$\mathbb{P}(A_1 \cap ... \cap A_n)\mathbb{P}'(B_1 \cap ... \cap B_n) = \left(\prod_{k=1}^n \mathbb{P}(A_k)\right)\left(\prod_{k=1}^n \mathbb{P}'(B_k)\right)$$

$$= \prod_{k=1}^n \mathbb{P}(A_k)\mathbb{P}'(B_k) = \prod_{k=1}^n \mathbb{P} \times \mathbb{P}'(A_k \times B_k) = \prod_{k=1}^n \mathbb{P} \times \mathbb{P}'(I_k).$$

Thus by Proposition 13.4 we obtain that $\sigma(\mathcal{I}_1), ..., \sigma(\mathcal{I}_n)$ are independent. Since n was arbitrary and $\{\mathcal{I}_{i_1}, ..., \mathcal{I}_{i_n}\} \subseteq \{\mathcal{I}_i : i \in \mathcal{I}\}$ was an arbitrary collection we obtain that $(\sigma(\mathcal{I}_i))_{i\in\mathcal{I}}$ are independent. Since for all $i \in \mathcal{I}$ we have that $\sigma(\widetilde{X_i^s}) \subseteq \sigma(\mathcal{I}_i)$ we thus obtain that $\sigma(\widetilde{X_i^s})_{i\in\mathcal{I}}$ are independent. So by Proposition 13.8 we obtain that $(\widetilde{X_i^s})_{i\in\mathcal{I}}$ are independent and then with Lemma 13.13 we obtain that $(X_i^s)_{i\in\mathcal{I}}$ are independent. ∎

Proposition 14.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let E be a Banach space and let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let $(X_j)_{j\in\mathbb{N}} : \Omega \to E$ be independent random variables and suppose there exists a random variable $X : \Omega \to E$ such that for all $k \in \mathbb{N}$ there exists a random variable Δ_k with $\Delta_k = X - \sum_{i=1}^k X_i$ a.s. and Δ_k is independent of $X_1, ..., X_k$. Let for all $j \in \mathbb{N}$ X_j' be a copy of X_j on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1 and define for all $j \in \mathbb{N}$ $X_j^s : \Omega \times \Omega' \to E$ by $X_j^s := X_j - X_j'$.*

Then there exists a random variable $Y : \Omega \times \Omega' \to E$ such that for all $k \in \mathbb{N}$ there exists a random variable δ_k with $\delta_k = Y - \sum_{i=1}^k X_i^s$ a.s. and δ_k is independent of $X_1^s, ..., X_k^s$.

Proof. Let X' be a copy of X on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1 and let Δ_k' be a copy of Δ_k on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1 for all $k \in \mathbb{N}$. Since X', Δ_k' and X_j' are copies of X, Δ_k and X_j for all $j \in \mathbb{N}$ and $k \in \mathbb{N}$ we have that $(X_j')_{j\in\mathbb{N}}$ are also independent random variables, X' is also a random variable and for all $k \in \mathbb{N}$ we have that $\Delta_k' = X' - \sum_{i=1}^k X_i'$ a.s., Δ_k' is also a random variable and that Δ_k' is independent of $X_1', ..., X_k'$.

Define $Y := X^s := X - X'$ and define for all $k \in \mathbb{N}$ $\delta_k := \Delta_k^s := \Delta_k - \Delta_k'$. Then we have

$$\delta_k \overset{a.s.}{=} X - \sum_{i=1}^k X_i - \Big(X' - \sum_{i=1}^k X_i'\Big) = (X - X') - \sum_{i=1}^k (X_i - X_i') = X^s - \sum_{i=1}^k X_i^s.$$

We have with part 2 of Lemma 14.2 for all $j \in \mathbb{N}$ and all $k \in \mathbb{N}$ that X_j^s, Y and δ_k are random variables.

Let $j \in \mathbb{N}$ and let with Proposition 8.8 $\widetilde{X_j}$ and $\widetilde{Y} := \widetilde{X}$ be strongly measurable $\mathbb{P}$-versions of X_j and X, respectively, defined on Ω, and $\widetilde{X_j'}$ and $\widetilde{X'}$ strongly measurable $\mathbb{P}'$-versions of X_j' and X', respectively, defined on Ω'. Then let $\widetilde{X_j^s}$ and $\widetilde{X^s}$ be the strongly measurable $\mathbb{P} \times \mathbb{P}'$-versions of X_j^s respectively X^s as defined in Lemma 14.4, i.e. $\widetilde{X_j^s} = \widetilde{X_j} - \widetilde{X_j'}$ and $\widetilde{X^s} = \widetilde{X} - \widetilde{X'}$. Let $k \in \mathbb{N}$. Let $\widetilde{\Delta_k}$ be the strongly measurable $\mathbb{P}$-version of Δ_k as defined in Lemma 12.16, i.e. $\widetilde{\Delta_k} = \widetilde{X} - \sum_{j=1}^k \widetilde{X_j}$, $\widetilde{\Delta_k'}$ the strongly measurable $\mathbb{P}'$-version of Δ_k' as defined in Lemma 12.16, i.e. $\widetilde{\Delta_k'} = \widetilde{X'} - \sum_{j=1}^k \widetilde{X_j'}$ and let $\widetilde{\delta_k} := \widetilde{\Delta_k^s}$ be the strongly measurable $\mathbb{P} \times \mathbb{P}'$-version of Δ_k^s as defined in Lemma 14.4, i.e. $\widetilde{\Delta_k^s} = \widetilde{\Delta_k} - \widetilde{\Delta_k'}$.

By Lemma 13.13 we have that $(\widetilde{X_j})_{j\in\mathbb{N}}$ are independent random variables since $(X_j)_{j\in\mathbb{N}}$ are and that $(\widetilde{X_j'})_{j\in\mathbb{N}}$ are independent random variables since $(X_j')_{j\in\mathbb{N}}$ are. By Lemma 13.14 we have that $\widetilde{\Delta_k}$ is independent of $\widetilde{X_1}, ..., \widetilde{X_k}$ and that $\widetilde{\Delta_k'}$ is independent of $\widetilde{X_1'}, ..., \widetilde{X_k'}$ for all $k \in \mathbb{N}$. Now consider for all $j \in \mathbb{N}$ and all $k \in \mathbb{N}$, $\widetilde{X_j}$, $\widetilde{X_j'}$, $\widetilde{X}$, $\widetilde{X'}$, $\widetilde{\Delta_k}$ and $\widetilde{\Delta_k'}$ as E-valued random variables on $\Omega \times \Omega'$, which is possible with part 1 of Lemma 14.2. The independence then obviously persists.

Let $k \in \mathbb{N}$. Just like in the proof of Proposition 14.6 denote by $\mathcal{F}_{\widetilde{\Delta_k}}$ the σ-algebra generated by $\widetilde{\Delta_k}$ with respect to $(\Omega, \mathcal{F}, \mathbb{P})$, by $\mathcal{F}'_{\widetilde{\Delta_k'}}$ the σ-algebra generated by $\widetilde{\Delta_k'}$ with respect to $(\Omega', \mathcal{F}', \mathbb{P}')$, by $\mathcal{F}_{\widetilde{X_1}, ..., \widetilde{X_k}}$ the σ-algebra generated by $\widetilde{X_1}, ..., \widetilde{X_k}$ with respect to $(\Omega, \mathcal{F}, \mathbb{P})$ and by $\mathcal{F}'_{\widetilde{X_1'}, ..., \widetilde{X_k'}}$ the σ-algebra generated by $\widetilde{X_1'}, ..., \widetilde{X_k'}$ with respect to $(\Omega', \mathcal{F}', \mathbb{P}')$.

Define for all $k \in \mathbb{N}$ the set $\mathcal{J}_k^1 := \{A \times B : A \in \mathcal{F}_{\widetilde{\Delta_k}}, B \in \mathcal{F}'_{\widetilde{\Delta_k'}}\}$ and the set $\mathcal{J}_k^2 := \{A \times B : A \in \mathcal{F}_{\widetilde{X_1}, ..., \widetilde{X_k}}, B \in \mathcal{F}'_{\widetilde{X_1'}, ..., \widetilde{X_k'}}\}$. Identically to the proof of Proposition 14.6, we obtain that $\mathcal{J}_k^1$ and $\mathcal{J}_k^2$ are

π-systems and that $\sigma(\widetilde{\Delta_k^s}) \subseteq \sigma(\mathcal{J}_k^1) \subseteq \mathcal{F} \times \mathcal{F}'$ and $\sigma(\widetilde{X_1^s}, ..., \widetilde{X_k^s}) \subseteq \sigma(\mathcal{J}_k^2) \subseteq \mathcal{F} \times \mathcal{F}'$.

Now I will prove that $\sigma(\mathcal{J}_k^1)$ is independent of $\sigma(\mathcal{J}_k^2)$ for all $k \in \mathbb{N}$. Let $k \in \mathbb{N}$. Let $A \in \mathcal{J}_k^1$, i.e. $A = A_1 \times A_2$ with $A_1 \in \mathcal{F}_{\widetilde{\Delta_k}}$ and $A_2 \in \mathcal{F}'_{\widetilde{\Delta'_k}}$, and let $B \in \mathcal{J}_k^2$, i.e. $B = B_1 \times B_2$ with $B_1 \in \mathcal{F}_{\widetilde{X_1}, ..., \widetilde{X_k}}$ and $B_2 \in \mathcal{F}'_{\widetilde{X'_1}, ..., \widetilde{X'_k}}$. Then we have

$$\mathbb{P} \times \mathbb{P}'(A \cap B) = \mathbb{P} \times \mathbb{P}'((A_1 \times A_2) \cap (B_1 \times B_2))$$
$$= \mathbb{P} \times \mathbb{P}'((A_1 \cap B_1) \times (A_2 \cap B_2)) = \mathbb{P}(A_1 \cap B_1)\mathbb{P}'(A_2 \cap B_2).$$

Since $A_1 \in \mathcal{F}_{\widetilde{\Delta_k}}$, $B_1 \in \mathcal{F}_{\widetilde{X_1}, ..., \widetilde{X_k}}$ and $\widetilde{\Delta_k}$ is independent of $\widetilde{X_1}, ..., \widetilde{X_k}$ and thus with Proposition 13.9 $\mathcal{F}_{\widetilde{\Delta_k}}$ is independent of $\mathcal{F}_{\widetilde{X_1}, ..., \widetilde{X_k}}$ we obtain that $\mathbb{P}(A_1 \cap B_1) = \mathbb{P}(A_1)\mathbb{P}(B_1)$. Similarly we obtain that $\mathbb{P}'(A_2 \cap B_2) = \mathbb{P}'(A_2)\mathbb{P}'(B_2)$. Thus we get

$$\mathbb{P}(A_1 \cap B_1)\mathbb{P}'(A_2 \cap B_2) = \mathbb{P}(A_1)\mathbb{P}(B_1)\mathbb{P}'(A_2)\mathbb{P}'(B_2)$$
$$= \mathbb{P} \times \mathbb{P}'(A_1 \times A_2) \cdot \mathbb{P} \times \mathbb{P}'(B_1 \times B_2) = \mathbb{P} \times \mathbb{P}'(A) \cdot \mathbb{P} \times \mathbb{P}'(B).$$

So with Proposition 13.4 we obtain that $\sigma(\mathcal{J}_k^1)$ is independent of $\sigma(\mathcal{J}_k^2)$. Since $\sigma(\widetilde{\Delta_k^s}) \subseteq \sigma(\mathcal{J}_k^1)$ and $\sigma(\widetilde{X_1^s}, ..., \widetilde{X_k^s}) \subseteq \sigma(\mathcal{J}_k^2)$ we obtain that $\sigma(\widetilde{\Delta_k^s})$ is independent of $\sigma(\widetilde{X_1^s}, ..., \widetilde{X_k^s})$. So by Proposition 13.9 we obtain that $\widetilde{\Delta_k^s}$ is independent of $\widetilde{X_1^s}, ..., \widetilde{X_k^s}$. Now we obtain with Lemma 13.14 that $\Delta_k^s = \delta_k$ is independent of $X_1^s, ..., X_k^s$. ■

15 Stochastic processes

In order to formulate and prove a theorem similar to Theorem 13.20 with stochastic processes instead of random variables, we have to introduce the notion of a stochastic process.

Definition 15.1. Let E be a Banach space. A **stochastic process** with values in E with time set $[0, T]$, $T \in \mathbb{R}_{>0}$, is a family of E-valued random variables $X := (X_t)_{t \in [0,T]}$ which are all defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 15.2. Let E be Banach space and $I \subseteq \mathbb{R}_+$ an interval. A function $f : I \to E$ is called cadlag (continu à droite, limites à gauche) if for all $t \in I$:

1. If t is not equal to the left endpoint the left limit, $f(t_-) := \lim_{s \uparrow t} f(s)$, exists;
2. If t is not equal to the right endpoint the right limit, $f(t_+) := \lim_{s \downarrow t} f(s)$, exists and $f(t_+) = f(t)$.

Definition 15.3. Let E be a Banach space and let $T > 0$. Then we define

$$\mathbb{D}_E[0, T] := \{f : [0, T] \to E \mid f \text{ cadlag}\}.$$

The proof of the next Lemma is very similar to the proof that $(C[0, T], ||\cdot||_\infty)$ is a Banach space. Furthermore we refer to [14] for discussion of the space $\mathbb{D}_E[0, T]$.

Lemma 15.4. *Let E be a Banach space and let $T > 0$. Define on $\mathbb{D}_E[0, T]$ the function $|| \cdot ||_T$ such that for $f \in \mathbb{D}_E[0, T]$ we have $||f||_T := \sup_{t \in [0,T]} ||f(t)||$. Then $|| \cdot ||_T$ is a norm and $\mathbb{D}_E[0, T]$ equipped with this norm is a Banach space.*

Definition 15.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $X = (X_t)_{t \in [0,T]}$ a stochastic process. X is called a **cadlag process** if for $\mathbb{P}$-almost all $\omega \in \Omega$ we have that $t \mapsto X_t(\omega)$ is an E-valued cadlag function on $[0, T]$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(X_t)_{t\in[0,T]}$ a cadlag process. Now we would like to be able to state that $\omega \longmapsto (t \mapsto X_t(\omega))$ is a $\mathbb{D}_E[0,T]$-valued random variable, and this is also often used in literature. However the following example shows that we have to be very careful when using this, as it appeares not to be true in general.

Example 15.6. Let $E = \mathbb{R}$ and consider the probability space $([0,1], \mathcal{B}([0,1]), \lambda)$ where $\mathcal{B}([0,1])$ is the Borel σ-algebra on $[0,1]$ and λ the Lebesgue measure on $[0,1]$. Define $X(t,\omega) = \mathbb{1}_{[\omega,T]}(t)$. Clearly, X has values in $\mathbb{D}_E[0,T]$. Then for all $A \subseteq [0,1]$ with $\lambda(A) = 1$ we have $\text{range}(X|_A) = \{t \mapsto \mathbb{1}_{[\omega,T]}(t) : \omega \in A\}$. Since A is uncountable and for all $f, g \in \text{range}(X|_A)$ we have that $||f - g|| = 1$ we obtain that $\text{range}(X|_A)$ is not separable and thus X is not λ-separably valued. Now it follows with Theorem 9.7 that X is not strongly λ-measurable and thus not a $\mathbb{D}_E[0,T]$-valued random variable.

The following was first shown in [11, Theorem 3.10.1]. The proof there was very involved and encountered many technical difficulties. With our preparations, in particular Theorem 13.20, it can now be shown more easily.

Theorem 15.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let E be a Banach space and let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let for $j \in \mathbb{N}$ $(X_j(t))_{t\in[0,T]} : \Omega \to E$ be independent cadlag processes and assume that we can consider the process $X_j(t)$ as a random variable from Ω to $\mathbb{D}_E[0,T]$ for all $j \in \mathbb{N}$. Suppose there exists a random variable $X : \Omega \to \mathbb{D}_E[0,T]$ such that for every $k \in \mathbb{N}$ there exists a random variable $\Delta_k : \Omega \to \mathbb{D}_E[0,T]$ with $\Delta_k = X - \sum_{i=1}^{k} X_i$ a.s. and Δ_k is independent of $X_1, ..., X_k$. Let for all $j \in \mathbb{N}$ X'_j be a copy of X_j on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in definition 14.1 and define for all $j \in \mathbb{N}$ $X^s_j : \Omega \times \Omega' \to \mathbb{D}_E[0,T]$ by $X^s_j := X_j - X'_j$. Finally define for $n \in \mathbb{N}$ $S^s_n : \Omega \times \Omega' \to \mathbb{D}_E[0,T]$ by $S^s_n := \sum_{i=1}^{n} X^s_i$. Then there exists a random variable $S : \Omega \times \Omega' \to \mathbb{D}_E[0,T]$ such that $||S - S^s_n||_T \xrightarrow{a.s.} 0$.*

Proof. By Lemma 14.5, Proposition 14.6 and Proposition 14.7 we obtain that $(X^s_j)_{j\in\mathbb{N}}$ are independent, symmetric random variables and that there exists a random variable X^s such that for all $k \in \mathbb{N}$ there exists a random variable Δ^s_k with $\Delta^s_k = X^s - \sum_{i=1}^{k} X^s_i$ a.s. and Δ^s_k is independent of $X^s_1, ..., X^s_k$. Thus all of the conditions of Theorem 13.20 are met, so we can apply this theorem to obtain the stated result. ∎

16 Conditional Expectation

Definition 16.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra and $X \in L^1(\Omega; E)$. The random variable $\mathbb{E}(X|\mathcal{G})$ is the unique element of $L^1(\Omega, \mathcal{G}; E)$ with the property that for all $G \in \mathcal{G}$ we have $\int_G \mathbb{E}(X|\mathcal{G})\mathrm{d}\mathbb{P} = \int_G X\mathrm{d}\mathbb{P}$ and is called the **conditional expectation of** X.

The existence and uniqueness of the conditional expectation is proven in [10, Theorem 2.6.23].

In the setting of Definition 16.1 we have with Proposition 8.8 that $\mathbb{E}(X|\mathcal{G})$ has a $\mathbb{P}|_{\mathcal{G}}$-version that is strongly $\mathcal{G}$-measurable, and thus a $\mathbb{P}$-version that is strongly $\mathcal{G}$-measurable.
Note that we can not conclude with Proposition 8.8 that $\mathbb{E}(X|\mathcal{G})$ is always strongly $\mathbb{P}$-measurable with respect to $\mathcal{G}$ since a $\mathbb{P}|_{\mathcal{G}}$-version is also a $\mathbb{P}$-version but the converse is not necessarily true. However we can assume without loss of generality that $\mathbb{E}(X|\mathcal{G})$ is strongly $\mathcal{G}$-measurable, which leads to the following extension of Definition 16.1.

Continuation of Definition 16.1. *Without loss of generality we assume that $\mathbb{E}(X|\mathcal{G})$ is strongly $\mathcal{G}$-measurable.*

Next we will consider some useful properties of the conditional expectation. Many of the properties we know for $\mathbb{R}$-valued random variables are easily extended to this general case, however not even all of them are still well-defined, and thus some can not be extended.

Proposition 16.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra and $X \in L^1(\Omega; E)$. Then we have:*

1. $\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\big) = \mathbb{E}(X)$.
2. For $a, b \in \mathbb{R}$ and $Y \in L^1(\Omega; E)$ we have $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$ a.s.
3. If $X \in L^1(\Omega, \mathcal{G}; E)$ then we have $X = \mathbb{E}(X|\mathcal{G})$ a.s.
4. If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H})$ a.s. ***(the tower property)***
5. Let $\mathcal{H} \subseteq \mathcal{F}$ be another sub-σ-algebra independent of $\sigma(X, \mathcal{G})$. Then $\mathbb{E}(X|\sigma(\mathcal{G}, \mathcal{H})) = \mathbb{E}(X|\mathcal{G})$ a.s. In particular if X is independent of $\mathcal{G}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ a.s.
6. Let $\phi : E \to \mathbb{R}$ be a convex and lower semi continuous function and suppose that $\phi \circ X \in L^1(\Omega)$. Then $\phi \circ \mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(\phi \circ X|\mathcal{G})$ a.s. *(the* ***Conditional Jensen's inequality****)*

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra and $X \in L^1(\Omega; E)$. Then we have:

1. $\mathbb{E}\big(\mathbb{E}(X|\mathcal{G})\big) := \int_\Omega \mathbb{E}(X|\mathcal{G})\mathrm{d}\mathbb{P} \overset{\Omega \in \mathcal{G}}{=} \int_\Omega X\mathrm{d}\mathbb{P} := \mathbb{E}(X)$.
2. We have for all $G \in \mathcal{G}$ that
$$\int_G \mathbb{E}(aX + bY|\mathcal{G})\mathrm{d}\mathbb{P} = \int_G (aX + bY)\mathrm{d}\mathbb{P} \overset{Theorem\ 10.6}{=} a\int_G X\mathrm{d}\mathbb{P} + b\int_G Y\mathrm{d}\mathbb{P}$$
$$= a\int_G \mathbb{E}(X|\mathcal{G})\mathrm{d}\mathbb{P} + b\int_G \mathbb{E}(Y|\mathcal{G})\mathrm{d}\mathbb{P} \overset{Theorem\ 10.6}{=} \int_G \big(a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})\big)\mathrm{d}\mathbb{P}$$
so $a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$ is a version of the conditional expectation of $aX + bY$ but so is $\mathbb{E}(aX + bY|\mathcal{G})$ and thus we obtain that $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$ a.s.
3. This follows directly from Definition 16.1.
4. This proof can be found in [10, Proposition 2.6.33].
5. This proof can be found in [10, Proposition 2.6.35 and Proposition 2.6.36].
6. This proof can be found in [10, Proposition 2.6.29].

■

Proposition 16.3. *[10, Corollary 2.6.30] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra and $1 \leq p < \infty$. Suppose $X \in L^p(\Omega; E)$. Then $||\mathbb{E}(X|\mathcal{G})||^p \leq \mathbb{E}(||X||^p|\mathcal{G})$ a.s. In particular we have $\mathbb{E}(X|\mathcal{G}) \in L^p(\Omega; E)$ and $||\mathbb{E}(X|\mathcal{G})||_p \leq ||X||_p$.*

The following is known as the **conditional dominated convergence theorem**.
Theorem 16.4. *[10, Theorem 2.6.28] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\in L^1(\Omega; E)$. Suppose that $\lim\limits_{n\to\infty} X_n = X$ a.s. for some X and that there exists an $Y \in L^1(\Omega)$ such that $||X_n|| \leq Y$ a.s. Then we have*

- $\lim\limits_{n\to\infty} \mathbb{E}\big(||X_n - X||\ |\mathcal{G}\big) = 0$ *a.s.*
- $\lim\limits_{n\to\infty} \mathbb{E}(X_n|\mathcal{G}) = \mathbb{E}(X|\mathcal{G})$ *a.s.*

Proposition 16.5. *[10, Proposition 2.6.39] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, $\mathcal{I}$ an index set, $(\mathcal{G}_i)_{i\in\mathcal{I}}$ a family of sub-σ-algebras of $\mathcal{F}$ and $\mathcal{G} := \bigcap_{i\in\mathcal{I}} \mathcal{G}_i$. Let $1 \leq p < \infty$. Then for all $f \in L^p(\Omega; E)$ the family $\big\{\mathbb{E}(f|\mathcal{G}_i) : i \in \mathcal{I}\big\}$ is uniformly p-integrable.*

Proposition 16.6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E_1, E_2, E Banach spaces, $\mathcal{G} \subseteq \mathcal{F}$ a sub-σ-algebra, $\beta : E_1 \times E_2 \to E$ a bounded bilinear mapping, $X \in L^1(\Omega, \mathcal{G}; E_1)$ and $Y \in L^1(\Omega; E_2)$. Assume that $\beta(X, Y) \in L^1(\Omega; E)$. Then $\mathbb{E}\big(\beta(X, Y)|\mathcal{G}\big) = \beta\big(X, \mathbb{E}(Y|\mathcal{G})\big)$ a.s.*

Proof. This follows from [10, Proposition 2.6.31]. ■

17 Martingales

Definition 17.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $\mathcal{I}$ a partially ordered set.

- A family $(\mathcal{F}_i)_{i\in\mathcal{I}}$ of sub-σ-algebras of $\mathcal{F}$ is called a **filtration** if for $i, j \in \mathcal{I}$ with $i \leq j$ we have $\mathcal{F}_i \subseteq \mathcal{F}_j$.
- A family $(X_i)_{i\in\mathcal{I}} \subseteq L^1(\Omega; E)$ of E-valued random variables is **adapted to the filtration** $(\mathcal{F}_i)_{i\in\mathcal{I}}$ if $X_i \in L^1(\Omega, \mathcal{F}_i; E)$ for all $i \in \mathcal{I}$.
- Assume that A1 is satisfied. Then the **filtration generated by a sequence of random variables** $X = (X_i)_{i\in\mathcal{I}} \subseteq L^1(\Omega; E)$ is the filtration $(\mathcal{F}_i^X)_{i\in\mathcal{I}}$ where $\mathcal{F}_i^X := \sigma(X_j : j \leq i)$, and X is adapted to this filtration.

Definition 17.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, $\mathcal{I}$ a partially ordered set and $(\mathcal{F}_i)_{i\in\mathcal{I}}$ a filtration in $\mathcal{F}$. $(M_i)_{i\in\mathcal{I}} \subseteq L^1(\Omega; E)$ is called a **martingale with respect to** $(\mathcal{F}_i)_{i\in\mathcal{I}}$ if it is adapted to $(\mathcal{F}_i)_{i\in\mathcal{I}}$ and $\mathbb{E}(M_j|\mathcal{F}_i) = M_i$ a.s. for all $i, j \in \mathcal{I}$ with $i \leq j$. If in addition $\mathbb{E}(||M_i||^p) < \infty$ for all $i \in \mathcal{I}$ then $(M_i)_{i\in\mathcal{I}}$ is called an L^p**-martingale**.

Definition 17.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(\mathcal{F}_n)_{n\in\mathbb{N}}$ a filtration in $\mathcal{F}$. Then we define $\mathcal{F}_\infty := \sigma(\mathcal{F}_n : n \in \mathbb{N})$.

Theorem 17.4. *[10, Theorem 3.3.2] Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space and $(\mathcal{F}_i)_{i\in\mathbb{N}}$ a filtration in $\mathcal{F}$. If $X \in L^1(\Omega; E)$, then $\lim_{n\to\infty} \mathbb{E}(X \mid \mathcal{F}_n) \overset{a.s.}{=} \mathbb{E}(X \mid \mathcal{F}_\infty)$.*

18 Convergence of series of random variables

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a Banach space E and $(X_j)_{j\in\mathbb{N}} : \Omega \to E$ independent random variables. It is interesting to consider the a.s. convergence of $\sum_{i=1}^N (X_i - c_i)$ with $(c_n)_{n\in\mathbb{N}}$ functions to or elements of E. The first result that catches our attention is that, under the assumptions of Theorem 13.20, for every sequence $(c_n)_{n\in\mathbb{N}}$ in E of which the sum converges we have that $\sum_{i=1}^N (X_i - c_i)$ converges a.s. This is stated in the following Proposition.

Proposition 18.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let E be a Banach space. Let $(X_j)_{j\in\mathbb{N}} : \Omega \to E$ be symmetric independent random variables and suppose there exists a random variable $X : \Omega \to E$ such that for all $k \in \mathbb{N}$ there exists a random variable $\Delta_k : \Omega \to E$ with $\Delta_k = X - \sum_{i=1}^k X_i$ a.s. and Δ_k is independent of $X_1, ..., X_k$. Let $(c_n)_{n\in\mathbb{N}} \subseteq E$ be a sequence such that $\sum_{i=1}^N c_i$ converges. Then $\sum_{i=1}^N (X_i - c_i)$ converges a.s.*

Proof. By Theorem 13.20 we have that $\sum_{i=1}^N X_i$ converges a.s., and by assumption we have that $\sum_{i=1}^N c_n$ converges. Thus, since $\sum_{i=1}^N (X_i - c_i) = \sum_{i=1}^N X_i - \sum_{i=1}^N c_i$, we have that $\sum_{i=1}^N (X_i - c_i)$ converges a.s. ∎

The next result that is interesting to consider is what happens with the a.s. convergence of $\sum_{i=1}^N (X_i - c_i)$ when c_n would be equal to the value of the random variable X_n at a certain point $\omega_n \in \Omega$. It would be preferred if this ω is independent of n. The following Theorem deals with this situation.

Theorem 18.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let E be a Banach space. Let $(X_j)_{j\in\mathbb{N}} : \Omega \to E$ be independent random variables and suppose there exists a random variable $X : \Omega \to E$ such that for all $k \in \mathbb{N}$ there exists a random variable $\Delta_k : \Omega \to E$ with $\Delta_k = X - \sum_{i=1}^k X_i$ a.s. and Δ_k is independent of $X_1, ..., X_k$. Then there exist $(c_n)_{n\in\mathbb{N}}$ with $c_n \in E$ for all $n \in \mathbb{N}$ and such that $\sum_{i=1}^N (X_i - c_i)$ converges a.s. where for all $n \in \mathbb{N}$ $c_n = X_n(\overline{\omega})$ for some $\overline{\omega} \in \Omega$.*

Proof. Let $(\Omega', \mathcal{F}', \mathbb{P}')$ be a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ as defined in Definition 14.1. Let for all $j \in \mathbb{N}$ X'_j be a copy of X_j on $(\Omega', \mathcal{F}', \mathbb{P}')$ as defined in Definition 14.1. Define for all $j \in \mathbb{N}$ $X^s_j : \Omega \times \Omega' \to E$ by $X^s_j := X_j - X'_j$ and define for all $n \in \mathbb{N}$ $S^s_n : \Omega \times \Omega' \to E$ by $S^s_n := \sum_{i=1}^n X^s_i$. By Lemma 14.5, Proposition 14.6 and Proposition 14.7 we have that $(X^s_j)_{j\in\mathbb{N}}$ are independent, symmetric random variables and that there exists a random variable X^s such that for all $k \in \mathbb{N}$ there exists a random variable Δ^s_k with $\Delta^s_k = X^s - \sum_{i=1}^k X^s_i$ a.s. and Δ^s_k is independent of $X^s_1, ..., X^s_k$. So all of the conditions of Theorem 13.20 are met and thus we obtain with Theorem 13.20 that S^s_n converges a.s. in $\Omega \times \Omega'$ to some random variable S.

So now there exists a set $\Omega^* \subseteq \Omega \times \Omega'$ with $\Omega^* \in \mathcal{F} \times \mathcal{F}'$, $\mathbb{P} \times \mathbb{P}'(\Omega^*) = 1$ and for all $(\omega, \omega') \in \Omega^*$ we have that $\lim_{n\to\infty} S^s_n(\omega, \omega') = S(\omega, \omega')$.

Define $\Omega^*_1 := \{\omega' \in \Omega' : \exists \omega \in \Omega \text{ s.t. } (\omega, \omega') \in \Omega^*\}$ and define for all $\omega' \in \Omega'$ $\Omega^*_{\omega'} := \{\omega \in \Omega : (\omega, \omega') \in \Omega^*\}$. Note that for $\omega' \in \Omega^*_1$ we have that $\Omega^*_{\omega'} \neq \emptyset$ and for $\omega' \in \Omega' \setminus \Omega^*_1$ we have that $\Omega^*_{\omega'} = \emptyset$.

Now we have

$$1 = \mathbb{P} \times \mathbb{P}'(\Omega^*) = \int_{\Omega\times\Omega'} \mathbb{1}_{\Omega^*}(\omega, \omega') \mathrm{d}\mathbb{P} \times \mathbb{P}'(\omega, \omega') \overset{Fubini}{=} \int_{\Omega'} \int_{\Omega} \mathbb{1}_{\Omega^*}(\omega, \omega') \mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}'(\omega')$$

$$= \int_{\Omega'} \int_{\Omega} \mathbb{1}_{\Omega^*_1}(\omega')\mathbb{1}_{\Omega^*_{\omega'}}(\omega)\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}'(\omega') \overset{(*_1)}{=} \int_{\Omega'} \mathbb{1}_{\Omega^*_1}(\omega') \int_{\Omega} \mathbb{1}_{\Omega^*_{\omega'}}(\omega)\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}'(\omega')$$

$$= \int_{\Omega'} \mathbb{1}_{\Omega^*_1}(\omega') \int_{\omega\in\Omega^*_{\omega'}} \mathbb{1}\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}'(\omega') = \int_{\omega'\in\Omega^*_1} \mathbb{1} \int_{\omega\in\Omega^*_{\omega'}} \mathbb{1}\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}'(\omega') = \int_{\omega'\in\Omega^*_1} \int_{\omega\in\Omega^*_{\omega'}} \mathbb{1}\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}'(\omega')$$

where $(*_1)$ follows since $\mathbb{1}_{\Omega^*_1}(\omega')$ does not depend on ω.

Thus now we find that there exists at least one $\omega' \in \Omega^*_1$ such that $\mathbb{P}(\Omega^*_{\omega'}) = 1$, otherwise we would have $\int_{\omega'\in\Omega^*_1} \int_{\omega\in\Omega^*_{\omega'}} \mathbb{1}\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}'(\omega') < 1$. So take this ω'. Now we have for all $\omega \in \Omega^*_{\omega'}$ that $(\omega, \omega') \in \Omega^*$ and thus $\lim_{n\to\infty} S^s_n(\omega, \omega') = S(\omega, \omega')$. Define $S_n : \Omega \to E$ by $S_n := \sum_{i=1}^n X_i$ and $S'_n : \Omega' \to E$ by $S'_n := \sum_{i=1}^n X'_i$. Then we have that

$$S^s_n = \sum_{i=1}^n X^s_i = \sum_{i=1}^n (X_i - X'_i) = \sum_{i=1}^n X_i - \sum_{i=1}^n X'_i = S_n - S'_n.$$

Since $S^s_n(\omega, \omega')$ converges for all $\omega \in \Omega^*_{\omega'}$ and $\mathbb{P}(\Omega^*_{\omega'}) = 1$ we now obtain that $S_n - S'_n(\omega') : \Omega \to E$ converges a.s. Thus if we let $c'_n = X'_n(\omega')$ we have that

$$\sum_{n=1}^N (X_n - c'_n) = \sum_{n=1}^N (X_n - X'_n(\omega')) = \sum_{n=1}^N X_n - \sum_{n=1}^N X'_n(\omega') = S_N - S'_N(\omega'),$$

and thus that $\sum_{n=1}^N (X_n - X'_n(\omega'))$ converges a.s.

Since Ω' is a copy of Ω and for all $j \in \mathbb{N}$ X'_j is a copy of X_j we can let $\overline{\omega} \in \Omega$ such that $X_j(\overline{\omega}) = X'_j(\omega')$ for all $j \in \mathbb{N}$. Now let $c_n = X_n(\overline{\omega})$. Then we obtain

$$\sum_{n=1}^N (X_n - c_n) = \sum_{n=1}^N (X_n - X_n(\overline{\omega})) = \sum_{n=1}^N (X_n - X'_n(\omega')) = \sum_{n=1}^N (X_n - c'_n)$$

and thus $\sum_{n=1}^N (X_n - c_n)$ converges a.s.

So $\sum_{n=1}^N (X_n - c_n)$ converges a.s. where for all $n \in \mathbb{N}$ $c_n = X_n(\overline{\omega})$ for some $\overline{\omega} \in \Omega$. ■

The above Theorem gives the a.s. convergence for the sequence $(c_n = X_n(\omega))_{n\in\mathbb{N}}$ for some $\omega \in \Omega$ when the random variables satisfy the assumptions of Theorem 13.20. However, this is not a very useful case as the element ω' is not obtained constructively. We would prefer that $\sum_{i=1}^{N}(X_i - \mathbb{E}(X_i))$ would converge a.s. and we would also prefer to obtain similar results when the assumptions of Theorem 13.20 are not satisfied. However without the assumptions of Theorem 13.20 the desired result is not necessarily true. To illustrate this we consider the Banach space $E = \mathbb{R}$. The following two examples will show the difference between random variables to $\mathbb{R}$ that satisfy the assumptions of Theorem 13.20 and random variables that do not.

Example 18.3. Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given by $\Omega = \Big\{(a_1, a_2, ...) : a_i \in \{0,1\}\Big\}$,

$$\mathcal{F} = \sigma\Big(\big\{\{a_1 = i_1, ..., a_n = i_n\} : n \in \mathbb{N}, i_j \in \{0,1\} \forall j \in \{1, ..., n\}\big\}\Big)$$

and for $A = \{a_1 = i_1, ..., a_n = i_n\}$ with $n \in \mathbb{N}$ and $i_j \in \{0,1\}$ for all $j \in \{1, ..., n\}$ we have that $\mathbb{P}(A) = \frac{1}{2^n}$. Then let $(X_j)_{j\in\mathbb{N}} : \Omega \to \mathbb{R}$ be given by $X_i(\omega) = \omega_i$ for all $\omega \in \Omega$, where ω_i means the i-th element of the sequence ω. For all $i \in \mathbb{N}$ we have that X_i is obviously integrable since $\mathbb{E}(X_i) = \frac{1}{2}$, thus $(X_j)_{j\in\mathbb{N}}$ are random variables, and they are obviously independent. Furthermore we obviously have that $\lim_{n\to\infty}\sum_{i=1}^{n} X_i$ does not always exist and thus the assumptions of Theorem 13.20 are not satisfied. Now we have that $\sum_{i=1}^{N}(X_i - \mathbb{E}(X_i)) = \sum_{i=1}^{N}(X_i - \frac{1}{2}) = \sum_{i=1}^{N} X_i - \frac{N}{2}$ for all $N \in \mathbb{N}$ and thus $\sum_{i=1}^{N}(X_i - \mathbb{E}(X_i))$ does not converge a.s.

Example 18.4. Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ be given as in Example 18.3. Let $(X_j)_{j\in\mathbb{N}} : \Omega \to \mathbb{R}$ be given by $X_i(\omega) = \frac{1}{i^2}\omega_i$ for all $\omega \in \Omega$. For all $i \in \mathbb{N}$ we have that X_i is obviously integrable since $\mathbb{E}(X_i) = \frac{1}{i^2}\frac{1}{2}$, and thus $(X_j)_{j\in\mathbb{N}}$ are random variables, and they are obviously independent. Now we have that $\sum_{i=1}^{\infty} X_i$ is a random variable and thus with $X = \sum_{i=1}^{\infty} X_i$ the assumptions of Theorem 13.20 are satisfied. Since $\sum_{i=1}^{\infty}\mathbb{E}(X_i) = \sum_{i=1}^{\infty}\frac{1}{i^2}\frac{1}{2} = \frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{i^2} < \infty$ we have with Proposition 18.1 that $\sum_{i=1}^{n}(X_i - \mathbb{E}(X_i))$ converges a.s.

Theorem 18.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let E be a Banach space. Let $(X_j)_{j\in\mathbb{N}} : \Omega \to E$ be independent random variables with $X_j \in L^1(\Omega; E)$ for all $j \in \mathbb{N}$. Suppose there exists a random variable $X : \Omega \to E$ such that $X \in L^1(\Omega; E)$ and that for all $k \in \mathbb{N}$ there exists a random variable $\Delta_k : \Omega \to E$ with $\Delta_k = X - \sum_{i=1}^{k} X_i$ a.s. and Δ_k is independent of $X_1, ..., X_k$. Then $S_n := \sum_{i=1}^{n}(X_i - \mathbb{E}(X_i))$ converges a.s.*

Proof. Note that $\Delta_k \in L^1(\Omega; E)$ for all $k \in \mathbb{N}$ since $X \in L^1(\Omega; E)$ and $X_j \in L^1(\Omega; E)$ for all $j \in \mathbb{N}$. With Proposition 8.8, Lemma 12.16 and Lemma 13.13 we can let $(\widetilde{X_i})_{i\in\mathbb{N}}$, $\widetilde{X}$ and $(\widetilde{\Delta_k})_{k\in\mathbb{N}}$ be strongly measurable $\mathbb{P}$-versions of respectively $(X_i)_{i\in\mathbb{N}}$, X and $(\Delta_k)_{k\in\mathbb{N}}$, such that $(\widetilde{X_i})_{i\in\mathbb{N}}$ is a sequence of independent random variables and that $\widetilde{\Delta_k} = \widetilde{X} - \sum\limits_{i=1}^{k}\widetilde{X_i}$ a.s. and $\widetilde{\Delta_k}$ is independent of $\widetilde{X_1}, ..., \widetilde{X_k}$ for all $k \in \mathbb{N}$. Note that it also holds that $\widetilde{X} \in L^1(\Omega; E)$, $\widetilde{X_j} \in L^1(\Omega; E)$ and $\widetilde{\Delta_k} \in L^1(\Omega; E)$ for all $j, k \in \mathbb{N}$.

Now by Theorem 9.3 we have that A1 is satisfied and thus we can define $\mathcal{F}_k := \sigma\big(\widetilde{X_1}, ..., \widetilde{X_k}\big)$ for all $k \in \mathbb{N}$. Define $\widetilde{S_n} := \sum_{i=1}^{n}\Big(\widetilde{X_i} - \mathbb{E}\big(\widetilde{X_i}\big)\Big)$. Let $k \in \mathbb{N}$. Since $\widetilde{\Delta_k}$ is independent of $\widetilde{X_1}, ..., \widetilde{X_k}$, it follows that $\widetilde{\Delta_k}$ is independent of $\widetilde{X_1} - \mathbb{E}\big(\widetilde{X_1}\big), ..., \widetilde{X_k} - \mathbb{E}\big(\widetilde{X_k}\big)$ and thus with Proposition 16.2.5 we obtain

$$\mathbb{E}\Big(\widetilde{\Delta_k} - \mathbb{E}\big(\widetilde{\Delta_k}\big) \mid \mathcal{F}_k\Big) \overset{a.s.}{=} \mathbb{E}\Big(\widetilde{\Delta_k} - \mathbb{E}\big(\widetilde{\Delta_k}\big)\Big) \overset{a.s.}{=} \mathbb{E}\big(\widetilde{\Delta_k}\big) - \mathbb{E}\Big(\mathbb{E}\big(\widetilde{\Delta_k}\big)\Big) = 0.$$

Now we get, where all the equalities hold almost surely,

$$0 = \mathbb{E}\Big(\widetilde{\Delta_k} - \mathbb{E}\big(\widetilde{\Delta_k}\big) | \mathcal{F}_k\Big) = \mathbb{E}\left(\widetilde{X} - \sum_{i=1}^{k}\widetilde{X_i} - \mathbb{E}\left(\widetilde{X} - \sum_{i=1}^{k}\widetilde{X_i}\right) \mid \mathcal{F}_k\right) = \mathbb{E}\left(\widetilde{X} - \sum_{i=1}^{k}\widetilde{X_i} - \mathbb{E}\big(\widetilde{X}\big) + \sum_{i=1}^{k}\mathbb{E}\big(\widetilde{X_i}\big) \mid \mathcal{F}_k\right)$$

$$\overset{Proposition\ 16.2.2}{=} \mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_k\Big) - \sum_{i=1}^{k}\mathbb{E}\Big(\widetilde{X_i} - \mathbb{E}\big(\widetilde{X_i}\big) \mid \mathcal{F}_k\Big).$$

Since $\widetilde{X_i}$ is $\mathcal{F}_k$-measurable for all $i \in \{1, ..., k\}$ we have that $\widetilde{X_i} - \mathbb{E}\big(\widetilde{X_i}\big)$ is $\mathcal{F}_k$-measurable for all $i \in \{1, ..., k\}$. Thus with Theorem 9.3 we obtain that $\widetilde{X_i}$ is separably valued for all $i \in \{1, ..., k\}$ and thus again with Theorem 9.3 we have that $\widetilde{X_i}$ is strongly $\mathcal{F}_k$-measurable for all $i \in \{1, ..., k\}$. Thus now we have with Proposition 16.2.3

$$\mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_k\Big) - \sum_{i=1}^{k} \mathbb{E}\Big(\widetilde{X_i} - \mathbb{E}\big(\widetilde{X_i}\big) \mid \mathcal{F}_k\Big) \overset{a.s.}{=} \mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_k\Big) - \sum_{i=1}^{k} \mathbb{E}\Big(\widetilde{X_i} - \mathbb{E}\big(\widetilde{X_i}\big)\Big).$$

And thus we have

$$\mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_k\Big) \overset{a.s.}{=} \sum_{i=1}^{k} \mathbb{E}\Big(\widetilde{X_i} - \mathbb{E}\big(\widetilde{X_i}\big)\Big) := \widetilde{S_k}.$$

By definition we have that $(\mathcal{F}_i)_{i\in\mathbb{N}}$ is a filtration in $\mathcal{F}$. By Theorem 17.4 we obtain, since $\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \in L^1(\Omega; E)$, that

$$\lim_{k\to\infty} \mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_k\Big) = \mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_\infty\Big) \ a.s.$$

i.e. $\mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_k\Big)$ converges a.s. Now since $\mathbb{E}\Big(\widetilde{X} - \mathbb{E}\big(\widetilde{X}\big) \mid \mathcal{F}_k\Big) \overset{a.s.}{=} \widetilde{S_k}$ we have that $\widetilde{S_k}$ converges a.s. and thus since

$$\widetilde{S_k} = \sum_{i=1}^{k} \Big(\widetilde{X_i} - \mathbb{E}\big(\widetilde{X_i}\big)\Big) = \sum_{i=1}^{k} \Big(\widetilde{X_i} - \mathbb{E}\big(X_i\big)\Big) \overset{a.s.}{=} \sum_{i=1}^{k} \Big(X_i - \mathbb{E}\big(X_i\big)\Big) = S_k,$$

S_k converges a.s. ∎

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