Analysis of some entrance probabilities for killed birth-death processes
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Master's Thesis

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1 Introduction

The introduction in the paper of Ellens et al., from now on notated as [1], describes analytical techniques to analyze metrics that are related to random fluctuations between consecutive observations. In this thesis, we will take a closer look at the probability that the process, starting at level n, reaches level $n+1$ before an independent exponentially distributed time T expires. Also the probability that the process, starting at level n, reaches level $n-1$ before an exponentially distributed time T expires, is examined.

These exponentially distributed times T are studied in [1] to determine the distribution of the maximum of a birth-death process over a deterministic interval with given initial and terminal conditions. The probabilities to reach a higher, respectively lower, state before an exponentially distributed time T expires, both satisfy recursive formulas that look interesting to analyze. The interesting part was that numerical results in Excel seemed to imply that these probabilities do not always converge. Therefore, in this thesis we will analyze their limiting behaviour.

The remainder of this thesis is as follows. First we will describe the model of the birth-death process that we will use throughout this thesis. This chapter will also give the basic formulas we got from [1] and we introduce the basic concepts and formulas. Also an example will be given for which we will consider the relation between two sequences of probabilities.

After that, we have enough background to analyze the recursive formulas given in Section 2.1 of [1]. Chapter 3 will give observations of the graph of the recursive formula for the probability that the process arrives at a state $n+1$ before time T given that the process starts in state n, this probability will be denoted by p_n. The observations will be based on examples in Excel of the $M|M|c$-model, with $c \in \{1, 2, \ldots, 5\}$, and $M|M|\infty$-model. The remainder of the chapter will give counterexamples and proofs of observations.

Chapter 4 and Chapter 5 will state the main results of this thesis. These main results are limiting results for the probabilities p_n and q_n, where q_n is the probability that the process X arrives at state $n-1$ before time T given that the process starts in state n. Chapter 5 also introduces the concept of stochastic monotonicity. This will be needed in the proof of the main result of this chapter.

Chapter 6 summarizes the results of Sections 2.2 and 2.3 of [1] and will give a motivation to the choice of considering a killed birth-death process.

This thesis concludes with Chapter 7. This chapter will discuss what we originally planned to research in this thesis and what we actually researched.

2 The model

In this chapter we will describe the model that will be used in this thesis.

For notational purposes, define $\mathbb{N} := \{1, 2, 3, ...\}$ and $\mathbb{N}_0 := \{0, 1, 2, 3, ...\}$.

Definition 2.1. Let the stochastic process $X = (X_t)_{t\geq 0}$ be a birth-death process, where $X_t \in \mathbb{N}_0$ denotes the state of the process at time t. The birth rate in state n is $\lambda_n > 0$, and the death rate in state n is $\mu_n > 0$, for $n \in \mathbb{N}_0$, where $\mu_0 = 0$.

Remark 2.2. The stochastic process X is a continuous time Markov chain on a discrete state space.

Since we will prove properties of these functions in later chapters of this thesis, we will give the steps of the derivation of these recursive formulas in section 2.1 of [1] in this chapter.

Given an initial state X_0, we consider an independent, exponentially distributed time T till expiration. Given a level $m \in \mathbb{N}_0$, we are interested in the probability that the process X equals the level m. The goal of this chapter is to give the derivation of the probability that the maximum of the stochastic process X is equal to m over the interval $[0, T]$, of exponential length, given that the process starts in state i.

Remark 2.3. Since we consider the process X on the stochastic interval $[0, T]$, we call $\left(X_t^T\right)_{t\geq 0}$ a killed birth-death process, where $X_t^T := X_{t\wedge T}$. If necessary, we will model the killed birth-death process as a birth-death process with an extra state ∞, which will be an absorbing state, i.e., the process will never leave this state once it has arrived there. We will use this extra state ∞ in Chapter 5.

Before we can prove results for the process X, first some notation and definitions need to be introduced.

Notation 2.4.

1. Use m to denote a level. Then, m is a state in the birth-death process $(X_t)_{t\geq 0}$ about which we want to know several probabilities. (For example, we could be interested in the probability that the birth-death process $(X_t)_{t\geq 0}$ is in a state larger than m.)
2. Let $t \geq 0$. Let $\bar{X}_t := \sup_{s\in[0,t]} X_s$ denote the running time maximum associated with process X.
3. Let $t \geq 0$. Let $\hat{X}_t := \inf_{s\in[0,t]} X_s$ denote the running time minimum associated with process X.
4. Let T be an exponentially distributed random variable with rate $\tau > 0$, independent of process X.

Definition 2.5. Let $n \in \mathbb{N}_0$.

1. Let $p_n := \mathbb{P}\left(\bar{X}_T \geq n+1 \mid X_0 = n\right)$ be the probability that the process X, starting at level n, reaches level $n+1$ before time T.
2. Let $\bar{p}_n := 1 - p_n = \mathbb{P}\left(\bar{X}_T = n \mid X_0 = n\right)$ be the probability that the process X, starting at level n, does not reach level $n+1$ before time T.
3. Let $q_n := \mathbb{P}\left(\hat{X}_T \leq n-1 \mid X_0 = n\right)$ be the probability that the process X, starting at level n, reaches level $n-1$ before time T.
4. Let $i \in \mathbb{N}_0$. Let $r_{m,i} := \mathbb{P}\left(\bar{X}_T = m \mid X_0 = i\right)$ be the probability that the maximum of process X equals m over the interval $[0,T]$ (of random duration), given that it starts in state i.

Definition 2.6.

1. Define $p^+ := \limsup_{n\to\infty} p_n$ and $p^- := \liminf_{n\to\infty} p_n$.
2. Define $q^+ := \limsup_{n\to\infty} q_n$ and $q^- := \liminf_{n\to\infty} q_n$.
3. Define $\alpha^* := \lim_{n\to\infty} \frac{\lambda_n}{\mu_n}$ and $\beta^* := \lim_{n\to\infty} \frac{\tau}{\mu_n}$, provided these limits exist.

Definition 2.7. Define the probability that the maximum of the birth-death process $(X_t)_{t\geq 0}$ does not exceed level m, given its initial state i and its final value j at a deterministic time t by $q_{m,i,j,t} := \mathbb{P}\left(\bar{X}_t \leq m \mid X_0 = i, X_t = j\right)$.

To analyze $q_{m,i,j,t}$, [1] first considers the maximum over an exponential interval given the initial state of the process. The authors deduce two recursive formulas describing the probability that the maximum and minimum of the process X, starting at level n, reaches level $n+1$ and level $n-1$, respectively, before an exponential time T expires. These probabilities are given in parts 1 and 3 of Definition 2.5 as $(p_n)_{n\in\mathbb{N}_0}$ and $(q_n)_{n\in\mathbb{N}_0}$.

2.1 Formulas

The process X is a birth-death process, hence it satisfies the memoryless property. For a birth-death process, at rate λ_n a birth occurs when the process is in state n, and at rate μ_n a death. In this model the process is only considered until the exponential time T expires. Therefore, the process moves from state n to another state (where reaching time T stops the process) at rate $\lambda_n + \mu_n + \tau$. We can model this by extending the state space by an extra state, ∞, to which the process jumps to with rate τ. However, in this chapter we do not yet need to consider the extended state space. Hence the probability to reach state $n+1$ equals $\frac{\lambda_n}{\lambda_n+\tau+\mu_n}$ and the probability to reach state $n-1$ equals $\frac{\mu_n}{\lambda_n+\tau+\mu_n}$ (for $n-1 \in \mathbb{N}_0$).

Since $T \stackrel{d}{=} \exp(\tau)$, with $\tau > 0$, it holds that T satisfies the memoryless property. This means that after a jump of the birth-death process on time $t < T$, the remaining time $T - t$ is again exponentially distributed with rate τ.

Using the memoryless property of the birth-death process and the memoryless property of T gives that

$$p_0 = \frac{\lambda_0}{\lambda_0 + \tau}$$

and for $n \in \mathbb{N}$ it holds that

$$p_n = \frac{\lambda_n}{\lambda_n + \tau + \mu_n} + \frac{\mu_n}{\lambda_n + \tau + \mu_n} p_{n-1} p_n.$$

Analogously, it holds for all $n \in \mathbb{N}_0$ that

$$q_n = \frac{\mu_n}{\lambda_n + \tau + \mu_n} + \frac{\lambda_n}{\lambda_n + \tau + \mu_n} q_{n+1} q_n.$$

For all $i > m$ it holds that

$$r_{m,i} = \mathbb{P}(\bar{X}_T = m \mid X_0 = i) = 0,$$

because in this case $\bar{X}_T = \sup_{s\in[0,T]} \geq i > m$. In the next lemma it will be proven that $r_{m,i} = (p_i \cdot \ldots \cdot p_{m-1}) \cdot \bar{p}_m$ for all $i \leq m$.

Lemma 2.8. *For all $i \leq m$, it holds that*

$$r_{m,i} = (p_i \cdot \ldots \cdot p_{m-1}) \cdot \bar{p}_m. \tag{1}$$

Proof. Throughout this proof, we use the memoryless property of the random variable T.

Let $i = m$. Then,

$$r_{m,i} = r_{m,m} = \mathbb{P}(\bar{X}_T = m \mid X_0 = m) = \bar{p}_m.$$

Let $i = m - 1$. Then,

$$r_{m,i} = r_{m,m-1} = \mathbb{P}(\bar{X}_T = m \mid X_0 = m-1) = \mathbb{P}(\bar{X}_T \geq m-1 \mid X_0 = m-1) \cdot \mathbb{P}(\bar{X}_T = m \mid X_0 = m),$$

since to arrive in state m, there must be a birth before time T when the process is in state $m-1$, and, moreover, no birth may occur in state m before time T. In other words,

$$r_{m,i} = p_{m-1} \cdot \bar{p}_m.$$

Let $i < m - 1$. Since we consider a birth-death process, to reach state $m - 1$ while starting in state i before time T, the process must first reach the states $i+1, \ldots, m-2$ before time T, before being able to jump to state $m - 1$. Thus,

$$\begin{aligned} r_{m,i} &= \mathbb{P}(\bar{X}_T = m \mid X_0 = i) \\ &= \mathbb{P}(\bar{X}_T \geq m - 1 \mid X_0 = i) \cdot \mathbb{P}(\bar{X}_T = m \mid X_0 = m). \end{aligned}$$

Recursively, we get that

$$\begin{aligned} r_{m,i} &= \prod_{n=i}^{m-2} \left[\mathbb{P}(\bar{X}_T \geq n \mid X_0 = n)\right] \cdot p_{m-1} \cdot \bar{p}_m \\ &= (p_i \cdot \ldots \cdot p_{m-1}) \cdot \bar{p}_m. \end{aligned}$$

Hence, for all $i \leq m$ Eqn. (1) holds. □

Lemma 2.9. *It holds that*

$$p_n = \begin{cases} \frac{\lambda_0}{\lambda_0+\tau}, & n = 0, \\ \frac{\lambda_n}{\lambda_n+\tau+\mu_n(1-p_{n-1})}, & n \geq 1. \end{cases} \tag{2}$$

Proof. For $n = 0$ it holds that $p_0 = \mathbb{P}(\bar{X}_T \geq 1 \mid X_0 = 0) = \frac{\lambda_0}{\lambda_0+t}$, because $\mu_0 = 0$.

Let $n \geq 1$. Then, $p_n = \frac{\lambda_n}{\lambda_n+\tau+\mu_n} + \frac{\mu_n}{\lambda_n+\tau+\mu_n} p_{n-1}p_n$. Rewriting this equation gives

$$\begin{aligned} &(\lambda_n + \tau + \mu_n)p_n = \lambda_n + \mu_n p_{n-1} p_n \\ &\iff (\lambda_n + \tau + \mu_n - \mu_n p_{n-1})p_n = \lambda_n \\ &\iff (\lambda_n + \tau + \mu_n(1 - p_{n-1}))p_n = \lambda_n \\ &\iff p_n = \frac{\lambda_n}{\lambda_n + \tau + \mu_n(1 - p_{n-1})}, \end{aligned}$$

which we needed to show. □

Lemma 2.10. *It holds that*

$$q_n = \begin{cases} 0, & n = 0, \\ \frac{\mu_n}{\lambda_n+\tau+\mu_n(1-q_{n+1})}, & n \geq 1. \end{cases} \tag{3}$$

Proof. For $n = 0$ it holds that $q_0 = \mathbb{P}(\hat{X}_T \leq -1 \mid X_0 = 0) = 0$, since $X_t \geq 0$ for all $t \geq 0$.

Let $n \geq 1$. It holds that $q_n = \frac{\mu_n}{\lambda_n+\tau+\mu_n} + \frac{\lambda_n}{\lambda_n+\tau+\mu_n} q_{n+1}q_n$. Rewriting this equation gives

$$\begin{aligned} &(\lambda_n + \tau + \mu_n)q_n = \mu_n + \lambda_n q_{n+1} q_n \\ &\iff (\lambda_n + \tau + \mu_n - \lambda_n q_{n+1})q_n = \mu_n \\ &\iff (\mu_n + \tau + \lambda_n(1 - q_{n+1}))q_n = \mu_n \\ &\iff q_n = \frac{\mu_n}{\mu_n + \tau + \lambda_n(1 - q_{n+1})}, \end{aligned}$$

which we needed to show. □

2.2 Examples

In this section we consider for two models the relation between the sequence $(p_n)_{n\in\mathbb{N}_0}$ and the sequence $(q_n)_{n\in\mathbb{N}_0}$. Both models have a finite state space, $S = \{0, 1, \ldots, N\}$. Again, let T be an exponentially distributed random variable with rate τ, independent of X.

Let the parameters $\lambda_n > 0$, for $n \in \{0, 1, \ldots, N - 1\}$, $\lambda_N = 0$, $\mu_0 = 0$ and $\mu_n > 0$, for $n \in \{1, 2, \ldots, N\}$ be given.

Consider the birth-death process $X^1 = (X^1_t)_{t\geq 0}$ on the state space S with birth- and death-parameters given by $\lambda^1_n = \lambda_n > 0$, for all $n \in \{0, 1, \ldots, N-1\}$, $\lambda^1_N = 0$, $\mu^1_0 = 0$ and $\mu^1_n = \mu_n > 0$, for all $n \in \{1, 2, \ldots, N\}$. Furthermore, consider the birth-death process $X^2 = (X^2_t)_{t\geq 0}$ on the state space S with swapped birth and death parameters given by $\lambda^2_n = \mu_n > 0$, for all $n \in \{0, 1, \ldots, N - 1\}$, $\lambda^2_N = 0$, $\mu^2_0 = 0$ and $\mu^2_n = \lambda_n > 0$, for all $n \in \{1, 2, \ldots, N\}$.

Then it holds that

$$p_0^1 = \frac{\lambda_n}{\lambda_n + \tau} \text{ and } p_0^2 = \frac{\mu_n}{\mu_n + \tau}.$$

For $n \in \{1, \dots, N\}$ it holds that

$$p_n^1 = \frac{\lambda_n}{\lambda_n + \tau + \mu_n(1 - p_{n-1}^1)} \text{ and } p_n^2 = \frac{\mu_n}{\mu_n + \tau + \lambda_n(1 - p_{n-1}^2)}.$$

Furthermore it holds that

$$q_N^1 = \frac{\mu_n}{\mu_n + \tau} \text{ and } q_N^2 = \frac{\lambda_n}{\lambda_n + \tau}.$$

For $n \in \{0, 1, \dots, N-1\}$ it holds that

$$q_n^1 = \frac{\mu_n}{\mu_n + \tau + \lambda_n(1 - q_{n+1}^1)} \text{ and } q_n^2 = \frac{\lambda_n}{\lambda_n + \tau + \mu_n(1 - q_{n+1}^2)}.$$

It holds that $p_0^1 = q_N^2$ and that $p_0^2 = q_N^1$. Using induction to n it follows for all $n \in \{0, 1, \dots, N\}$ that $p_n^1 = q_{N-n}^2$ and $p_n^2 = q_{N-n}^1$.

If $\lambda_n = \lambda$, for all $n \in \{0, 1, \dots, N-1\}$, and $\mu_n = \lambda$, for all $n \in \{1, 2, \dots, N\}$, then it follows that the processes X^1 and X^2 are equal. Hence, $p_n^1 = q_{N-n}^1$ for all $n \in \{0, 1, \dots, N\}$. Note that $|S| = N + 1$ is even, if N is off. Vice versa, if $|S|$ is odd, then N is even. In the latter case it holds that $p_n^1 = q_n^1$, $n = \frac{N}{2}$

If $\lambda_n = \mu_{N-n}$, for all $n \in \{0, 1, \dots, N-1\}$, and $\lambda_N = 0 = \mu_0$, then it follows that $p_n^1 = q_{N-n}^1$ for all $n \in \{0, 1, \dots, N\}$.

3 Observations and results for p_n and x_n

When studying the paper [1] we became interested in the structural properties of the formula for the probabilities p_n, $n \in \mathbb{N}_0$. Therefore, we decided to numerically analyze six $M|M|c$-models, $c \in \{1, \ldots, 5, \infty\}$, in Excel to get a better understanding of the behaviour of p_n as a function of n. For the $M|M|c$-model it holds that $\lambda_n = \lambda$ for all $n \in \mathbb{N}$ and

$$\mu_n = \begin{cases} n\mu & \text{if } n < c, \\ c\mu & \text{if } n \geq c. \end{cases}$$

Notation 3.1. For $n \in \mathbb{N}_0$, $c \in \mathbb{N} \cup \{\infty\}$, we use the superscript c for the investigated probabilities in the $M|M|c$-model. Moreover, we use the superscript $*$ to indicate the corresponding limits as n tends to ∞. For example, p_n^c is the probability that the process X, starting at level n, reaches level $n+1$ before time T in the $M|M|c$-model, μ_n^c is the departure rate of the $M|M|c$-model in state n, and $p^{*,c} := \lim_{n\to\infty} p_n^c$.

For all the examples we considered, we saw that the graph of p_n as a function of n is non-increasing and that it has a limiting value. See, for example, Figures 1 and 2 below. Moreover, we saw that, when plotting the graphs of p_n for different values of c in one plot, that the graph of p_n was monotonic. Furthermore, we noticed that the graphs had a short steep part in the graph of p_n and for the remainder a relatively flat one. If the graph has points before this steep part of the graph, then the graph changes from relatively flat to steep and then back to relatively flat.

Figures 1 and 2 show two examples on realizations of the graphs of p_n. The plots gave rise to a number of conjectures that we have tried to prove. These conjectures concern the characterization of this steep part of the graph, the limiting value of the graph of p_n and some monotonic properties depending on the number of servers c, the birth-rates λ, the death-rates μ, and the rate τ.

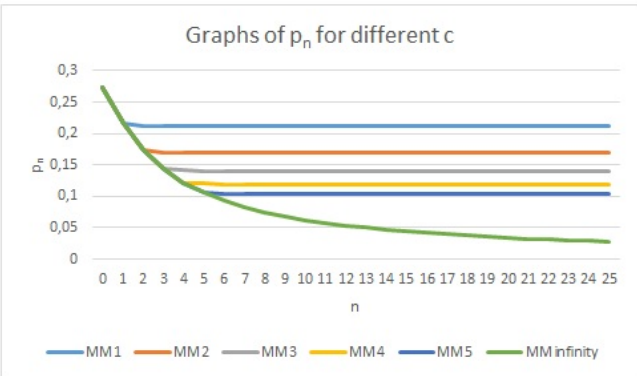

Figure 1: The graphs of p_n for $n \in \{0, 1, \ldots, 25\}$ for $M|M|c$, $c \in \{1, \ldots, 5, \infty\}$, with $\mu = 1$, $\lambda_n = \frac{3}{4}$ for all $n \in \mathbb{N}_0$, and $\tau = 2$.

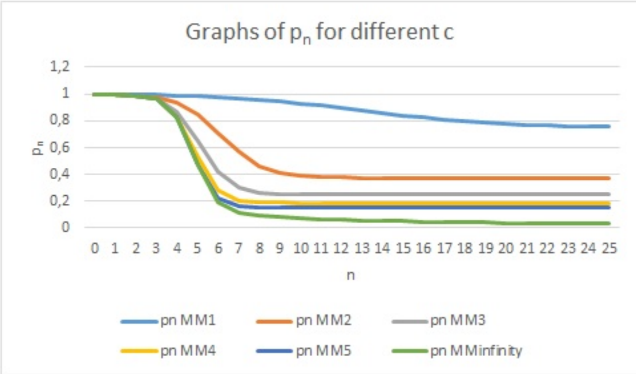

Figure 2: The graphs of p_n for $n \in \{0, 1, \ldots, 25\}$ for $M|M|c$, $c \in \{1, \ldots, 5, \infty\}$, with $\mu = 1$, $\lambda_n = \frac{3}{4}$ for all $n \in \mathbb{N}_0$, and $\tau = \frac{1}{1000}$.

3.1 Observations based on Excel

This section will first focus on observations of the graph of p_n for the $M|M|c$-model, $c \in \{1, 2, \ldots, 5, \infty\}$ We will describe the examples we considered in Excel and the observations we made using these examples. Then, we focus on the slope of the graph of p_n.

3.1.1 Observations of the graph of p_n

In Excel we plotted the graph of p_n for the $M|M|c$-model, $c \in \{1, \ldots, 5, \infty\}$ for $n = 0, 1, \ldots, 100$ and for various values of the parameters λ, μ or τ. The observations below are based on the graphs in these different Excel sheets. Before presenting the observations, we will first discuss what examples we used to base these observations on.

First we fixed λ and μ to study the effects of τ. We came up with the 17 situations in Table 1 because we wanted to consider the influence of τ on the $M|M|c$-model, with $c \in \{1, 2, \ldots, 5\}$. Therefore, we chose $\lambda = 0.75$ and $\mu = 1$, that way we made sure that $\lambda < c\mu$ and hence the queue length of this queueing system will not explode (see Chapters 3 and 4 in [4]). Then we wanted to see what would happen if τ became small. Therefore we considered the situations given in Table 1 of the Appendix.

After researching the influence of τ on the system, we wanted to see what would happen if we would change the ratio λ/μ. This made us choose $\mu = 1$ and considered for four situations of τ what would happen if we would decrease λ (with $\lambda < \mu$). The parameter combinations we considered here can be found in in the Tables 2, 3, 4 and 5 of the Appendix.

Consider the $M|M|c$-model, $c \in \{1, \ldots, 5, \infty\}$. The conjectures we made using the different examples are stated in Conjecture 3.2.

Conjecture 3.2.

1. the sequence $(p_n^c)_{n \in \mathbb{N}_0}$ is non-increasing in n.
2. the sequence $(p_n^c)_{n \in \mathbb{N}_0}$ has a limit.

3. $p_n^c = p_n^{c+1}$ for all $n \in \mathbb{N}_0$ with $n \leq c$ and $p_n^c \geq p_n^{c+1}$ for all $n > c$. It looks like $p_n^c = p_n^\infty$ for all $n \in \mathbb{N}_0$ with $n \leq c$ and $p_n^c \geq p_n^\infty$ for all $n > c$. So, it seems that the graphs of p_n (for the same $n \in \mathbb{N}_0$) are non-increasing as c is increasing.
4. $\lim_{n\to\infty} p_n^c \geq \lim_{n\to\infty} p_n^{c+1}$. This is the limit version of Conjecture 3.2.3.
5. $\lim_{c\to\infty} p^{*,c} = 0$.
6. $p_n^{\tau_1} \geq p_n^{\tau_2}$ for $\tau_1, \tau_2 > 0$ such that $\tau_1 < \tau_2$, where τ_1 and τ_2 denote two distinct extinction rates.

A counterexample for Conjecture 3.2.1 can be given. However, Lemma 3.5 shows the validity of Conjecture 3.2.1 for the $M|M|c$-model, $c \in \mathbb{N}_0 \cup \{\infty\}$.

Consider a birth-death process with $\lambda_n = 0.5$ for all $n \in \mathbb{N}_0$, $\tau = 1.5$ and

$$\mu_n = \begin{cases} 0 & \text{if } n = 0, \\ n^2 & \text{if } n \text{ is even, } n \neq 0, \\ \ln(n) + 1 & \text{if } n \text{ is odd.} \end{cases} \tag{4}$$

Then it holds for all $n \in \mathbb{N}$ that $\mu_n > 0$ (so $\frac{\lambda_n}{\mu_n}$ is well-defined for all $n \in \mathbb{N}$) and it holds that $\mu_n \to \infty$ as $n \to \infty$. Therefore, $\alpha^* := \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 0$ and $\beta^* := \lim_{n\to\infty} \frac{\tau}{\mu_n} = 0$. The graph op p_n shows that it does not hold that the sequence $(p_n)_{n\in\mathbb{N}_0}$ is non-increasing in n. Hence, this is not a generic property of birth-death processes. The corresponding graph can be found in Figure 3. Moreover, this example also seems to give a counterexample of Conjecture 3.2.2. However, as we will see in Chapter 4, also for this example $\lim_{n\to\infty} p_n$ exists.

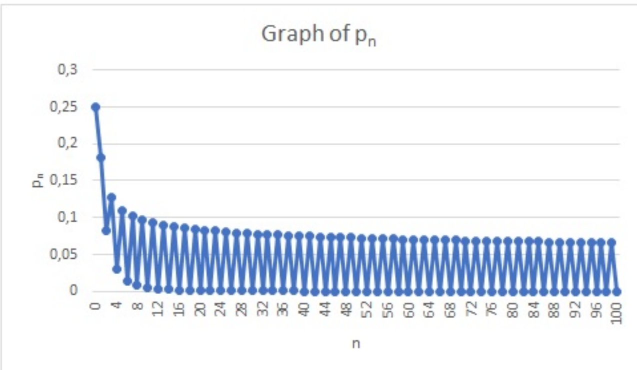

Figure 3: The graph of p_n for $n \in \{0, 1, \ldots, 100\}$ with μ_n as in Eqn. (4), $\lambda_n = \frac{1}{2}$ for all $n \in \mathbb{N}_0$, and $\tau = \frac{3}{2}$.

Moreover, for the $M|M|c$-model, $c \in \mathbb{N} \cup \{\infty\}$, we can prove that the sequence $(p_n^c)_{n\in\mathbb{N}_0}$ is non-increasing in n. This will be proven in Lemma 3.5. Conjecture 3.2.2 will be proven under some mild conditions in Chapter 4. The other conjectures will be proven for the $M|M|c$-model in this chapter.

3.1.2 Observations of the graph of x_n

Definition 3.3. Define the slope of the graph as the difference between two consecutive probabilities p_n, i.e.,

$$x_n := p_n - p_{n+1}$$

for all $n \in \mathbb{N}_0$.

In this subsection we will study the graph of x_n. We will study two aspects: monotonicity and the behaviour around its maximum. The latter is motivated by our interest in the part of the graph p_n^c, where it is steepest.

First, we considered the same examples as in Section 3.1.1 for the slope of the graph of p_n^c. In this way, we can take into account the observations made in Section 3.1.1. However, we also considered an example with $\lambda = 1$, $\mu = 0.5$ and $\tau = 0.01$, and an example with $\lambda = 0.75$, $\mu = 0.5$ and $\tau = 0.01$. This is related to the question of the monotonicity of the graph of x_n^c.

Recall Notation 3.1.

The conjectures we made using the different examples are given in Conjecture 3.4.

Conjecture 3.4.

1. for the $M|M|c$-model, $c \in \mathbb{N} \cup \{\infty\}$, the graph of x_n^c is not always monotone in c.
2. $\max_{n \in \mathbb{N}_0} x_n^c$ is non-decreasing as a function of c.
3. $n^* := \arg\max_{n \in \mathbb{N}_0} x_n^c$ denotes the point where the graph of p_n^c has the steepest descent, the steepest descent is at $p_{n^*}^c$ or $p_{n^*+1}^c$.
4. $\arg\max_{n \in \mathbb{N}_0} x_n^c$ is non-increasing as c decreases.

The goal of these observations is to come up with a heuristic method to evaluate the top of the graph of x_n^c.

The graphs of x_n^c in Figure 4 are an example showing that the graphs of x_n^c can intersect for different values of c. Hence, this example shows that monotonicity for x_n^c cannot be proven for the $M|M|c$-models, $c \in \mathbb{N} \cup \{\infty\}$.

3.1.3 Heuristic for x_n

For notational convenience, we leave the superscript c out of the notation.

In this subsection we will describe our attempts to find a heuristic method to predict $n^* = \arg\max_{n \in \mathbb{N}_0} x_n$ for $M|M|c$-models, $c \in \mathbb{N} \cup \{\infty\}$.

Excel calculations for the slope of p_n for $M|M|c$, $c \in \mathbb{N} \cup \{\infty\}$.
In this subsection we will analyze some Excel sheets we built. These Excel sheets calculate the point $n^* = \arg\max_{n \in \mathbb{N}_0} x_n$.

In these Excel sheets multiple examples are considered and can be found in Table 6 in the Appendix. Note that these Excel sheets could easily be used to calculate n^* for other parameters for $M|M|c$, with $c \in \{1, 2, 3, 4, 5\}$, and $M|M|\infty$. This can be done easily by just changing the respective values for λ, μ or τ.

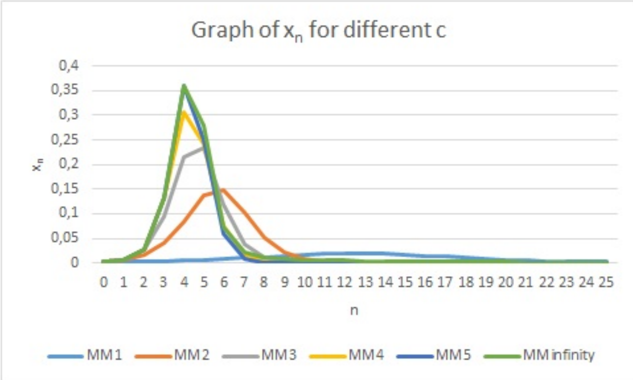

Figure 4: The graphs of x_n for $n \in \{0, 1, \dots, 25\}$ for $M|M|c$, $c \in \{1, \dots, 5, \infty\}$, with $\mu = 1$, $\lambda_n = \frac{3}{4}$ for all $n \in \mathbb{N}_0$, and $\tau = \frac{1}{1000}$.

First, the values for p_n are calculated in these Excel files. These values are then used to calculate the values for x_n. Finally, n^* is calculated by a formula that compares the calculated maximum with the Excel cells containing the values of x_n and this formula returns how many cells are above the number we sought including the cell that contains this maximum. Since we start counting from $n = 0$ we have to subtract 1 to get the correct value for n^*.

Excel observations for the slope of p_n for $M|M|c$, $c \in \mathbb{N} \cup \{\infty\}$.
This subsection will contain some observations based on the results of the examples.

1. n^* is non-decreasing as λ increases. This observation is based on situations $1 - 5$.
2. n^* is non-decreasing as μ increases. This observation is based on situations $6 - 20$.
3. n^* is non-increasing as τ increases. This observation is based on situations $21 - 34$.
4. Increasing the number of servers c does not always lead to n^* non-decreasing or non-increasing. An example is as follows. Let $\lambda = 1$, $\mu = 0.5$ and $\tau = 0.01$. For $M|M|4$ and $M|M|\infty$ it can be calculated that $n^* = 6$, while for $M|M|5$, $n^* = 5$.

The conclusion is that since increasing the number of servers does not always lead to n^* non-decreasing or non-increasing, this did not succeed. Therefore, this could be a topic of further research.

3.2 Monotonicity and convergence results for the $M|M|c$-model, with $c \in \mathbb{N} \cup \{\infty\}$

Consider the $M|M|c$-model, $c \in \mathbb{N} \cup \{\infty\}$. Let $\tau > 0$ be given. Assume that $\mu > 0$ and that $\lambda < c\mu$.

Recall the definition of p_n, Eqn. (2).

Next, we will show that the sequence $(p_n)_{n \in \mathbb{N}}$ is a non-increasing sequence for the $M|M|c$-model, $c \in \mathbb{N} \cup \{\infty\}$. Then, an easy consequence is that the sequence $(p_n)_{n \in \mathbb{N}}$ has a limit.

Lemma 3.5. *Let $\lambda, \mu \geq 0$ and let $\tau > 0$. For the $M|M|c$-model, $c \in \mathbb{N} \cup \{\infty\}$, it holds that*

1. *$p_n = 0$, $n \in \mathbb{N}_0$, if $\lambda = 0$ and $\mu \geq 0$. Moreover, the limit exists and $p^* = \lim_{n\to\infty} p_n = 0$.*
2. *$p_n = \frac{\lambda}{\lambda+\tau}$, $n \in \mathbb{N}_0$, if $\lambda > 0$ and $\mu = 0$. Moreover, the limit exists and $p^* = \lim_{n\to\infty} p_n = \frac{\lambda}{\lambda+\tau}$.*
3. *$p_n > p_{n+1}$, $n \in \mathbb{N}_0$, if $\lambda > 0$ and $\mu > 0$. Moreover, $p^* = \lim_{n\to\infty} p_n$ exists.*

Proof. 1. Let $\lambda = 0$ and $\mu \geq 0$. Then, by Eqn. (2), it immediately follows that $p_n = 0$ for all $n \in \mathbb{N}_0$. Hence, the sequence $(p_n)_{n\in\mathbb{N}_0}$ has a limit, namely $\lim_{n\to\infty} p_n = 0$.

2. Let $\lambda > 0$ and $\mu = 0$. Then, it follows immediately for all $n \in \mathbb{N}_0$ that $p_n = \frac{\lambda}{\lambda+\tau}$. So, then it holds for all $n \in \mathbb{N}_0$ that $p_n = p_{n+1}$. Thus, the sequence $(p_n)_{n\in\mathbb{N}_0}$ is constant, and hence, it has a limit, namely $\lim_{n\to\infty} p_n = \frac{\lambda}{\lambda+\tau}$.

3. Let $\lambda, \mu > 0$. Note that since $\lambda > 0$ and $\tau > 0$ it holds that $p_0 = \frac{\lambda}{\lambda+\tau} \in (0, 1)$.

   It suffices to consider two situations, namely $0 \leq n < c$, and $n \geq c$. Note that for $c = \infty$ only the first case has to be considered.

   - Suppose that $0 \leq n < c$. It holds that

   $$\begin{aligned} p_0 &= \frac{\lambda}{\lambda+\tau} \\ &\overset{(*)}{>} \frac{\lambda}{\lambda+\tau+\mu(1-p_0)} \\ &= p_1, \end{aligned}$$

     where (*) holds since $\mu > 0$ and $p_0 \in (0, 1)$.

     Since $\mu > 0$, it then follows that $\mu(1 - p_0) \leq \mu(1 - p_1)$. Moreover, since $\lambda \geq 0$ and $\tau > 0$, it therefore holds that $\lambda + \tau + \mu(1 - p_0) \leq \lambda + \tau + \mu(1 - p_1)$. Hence, $\frac{\lambda}{\lambda+\tau+\mu(1-p_0)} \geq \frac{\lambda}{\lambda+\tau+\mu(1-p_1)}$. Therefore, we have that

   $$\begin{aligned} p_1 &= \frac{\lambda}{\lambda+\tau+\mu(1-p_0)} \\ &\geq \frac{\lambda}{\lambda+\tau+\mu(1-p_1)} \\ &> \frac{\lambda}{\lambda+\tau+2\mu(1-p_1)} \\ &= p_2. \end{aligned}$$

     So, it holds that $p_1 > p_2$. Analogously, $p_n > p_{n+1}$ for all $2 \leq n < c$. So, for all $0 \leq n < c$ we have that $p_n > p_{n+1}$.

   - Suppose that $n \geq c$. Above, we already have proven that $p_{c-1} > p_c$. Since $\mu > 0$, it therefore follows that $c\mu(1 - p_{c-1}) < c\mu(1 - p_c)$. Moreover, since $\lambda > 0$ and $\tau > 0$, we also have that $\lambda + \tau + c\mu(1 - p_{c-1}) < \lambda + \tau + c\mu(1 - p_c)$. Therefore,

   $$\begin{aligned} p_c &= \frac{\lambda}{\lambda+\tau+c\mu(1-p_{c-1})} \\ &> \frac{\lambda}{\lambda+\tau+c\mu(1-p_c)} \\ &= p_{c+1}. \end{aligned}$$

So, it holds that $p_c > p_{c+1}$. Similarly we get for all $n \geq c$ that $p_n > p_{n+1}$, thus it holds for all $n \geq c$ that p_n is strictly decreasing.

Thus, we have proven for all $n \in \mathbb{N}_0$ that $p_n > p_{n+1}$.

For all $n \in \mathbb{N}_0$ it holds that p_n is a probability, hence $p_n \in [0,1]$ for all $n \in \mathbb{N}_0$. Thus the sequence $(p_n)_{n\in\mathbb{N}_0}$ is bounded. Then, by the Monotone Convergence Theorem, it follows that the sequence $(p_n)_{n\in\mathbb{N}_0}$ has a limit.

□

Remark 3.6. In Theorem 4.1 we will derive an expression for p^*, when $\lambda, \mu > 0$.

Remark 3.7. If we would have that $0 = \mu_0 \leq \mu_n \leq \mu_{n+1}$ and $\lambda_n = \lambda \geq 0$ for all $n \in \mathbb{N}_0$, then it would follow analogously that the sequence of $(p_n)_{n\in\mathbb{N}_0}$ is non-increasing. It is even strictly decreasing if $\lambda > 0$ and $\mu_n > 0$ for all $n \in \mathbb{N}$.

Remark 3.8. To be able to apply the Monotone Convergence Theorem we need a non-increasing or a non-decreasing sequence $(p_n)_{n\in\mathbb{N}_0}$. For a general birth-death process we do not necessarily have this property. Therefore we cannot use the Monotone Convergence Theorem to prove in general that the sequence $(p_n)_{n\in\mathbb{N}_0}$ has a limit.

Recall Notation 3.1.

Lemma 3.9. *Let $\lambda, \mu, \tau > 0$. Let $c \in \mathbb{N}$. Then,*

- $p^c_n = p^{c+1}_n$ *for all* $n \leq c$,
- $p^c_n > p^{c+1}_n$ *for all* $n > c$.

Proof. It holds that $\mu^c_n = n\mu$ for all $n \leq c$ and $\mu^c_n = c\mu$ for all $n \geq c$.

We use induction on n to prove the statement of the lemma.

Let $n = 0$. Then it holds that $\lambda^c_0 = \lambda = \lambda^{c+1}_0$. Hence, $p^c_0 = \frac{\lambda}{\lambda+\tau} = p^{c+1}_0$.

Suppose that $p^c_{n-1} = p^{c+1}_{n-1}$, $n < c$, then

$$p^c_n = \frac{\lambda}{\lambda + \tau + n\mu(1 - p^c_{n-1})} = \frac{\lambda}{\lambda + \tau + n\mu(1 - p^{c+1}_{n-1})} = p^{c+1}_n.$$

Next, let $n = c$. By assumption $p^c_{n-1} = p^{c+1}_{n-1}$. It holds that $\mu^c_n = c\mu = \mu^{c+1}_n$. Hence,

$$p^c_n = p^{c+1}_n.$$

Let $n = c+1$. By assumption $p^c_{n-1} = p^{c+1}_{n-1}$. It holds that $\mu^c_n = c\mu$ and $\mu^{c+1}_n = (c+1)\mu$. It holds that $c\mu < (c+1)\mu$, and thus $\mu^c_n < \mu^{c+1}_n$. Then,

$$\lambda+\tau+\mu^c_n(1-p^c_{n-1}) = \lambda+\tau+c\mu(1-p^{c+1}_{n-1}) < \lambda+\tau+(c+1)\mu(1-p^{c+1}_{n-1}) = \lambda+\tau+\mu^{c+1}_n(1-p^{c+1}_{n-1}).$$

Hence,

$$p^c_n = \frac{\lambda}{\lambda + \tau + \mu^c_n(1 - p^c_{n-1})} > \frac{\lambda}{\lambda + \tau + \mu^{c+1}_n(1 - p^{c+1}_{n-1})} = p^{c+1}_n.$$

Finally, assume that $p_n^c > p_n^{c+1}$, with $n > c+1$. It holds that $\mu_n^c = c\mu < (c+1)\mu = \mu_n^{c+1}$. Then it follows that $\mu_n^c(1-p_{n-1}^c) < \mu_n^{c+1}(1-p_{n-1}^{c+1})$, since $\mu_n^c < \mu_n^{c+1}$ and since $1-p_{n-1}^c < 1-p_{n-1}^{c+1}$. This gives that

$$p_n^c = \frac{\lambda}{\lambda+\tau+\mu_n^c(1-p_{n-1}^c)} > \frac{\lambda}{\lambda+\tau+\mu_n^{c+1}(1-p_{n-1}^{c+1})} = p_n^{c+1}.$$

So for all $n \in \mathbb{N}_0$ with $n \leq c$ it holds that $p_n^c = p_n^{c+1}$ and for all $n > c$ it holds that $p_n^c > p_n^{c+1}$. ☐

Corollary 3.10. *Let $c_1, c_2 \in \mathbb{N}$ such that $c_1 < c_2$. Then,*

- $p_n^{c_1} = p_n^{c_2}$ *for all* $n \leq c_1$,
- $p_n^{c_1} > p_n^{c_2}$ *for all* $n > c_1$.

Proof. Using Lemma 3.9 iteratively gives the statement we need to show. ☐

Lemma 3.11. *Let $c \in \mathbb{N}$. Then,*

- $p_n^c = p_n^\infty$ *for all* $n \leq c$,
- $p_n^c > p_n^\infty$ *for all* $n > c$,
- $\lim_{c\to\infty} p_n^c = p_n^\infty$.

Proof. Let $c_1 \in \mathbb{N}$.

Let $n \leq c_1$. Then, $p_n^{c_1} = p_n^{c_2}$ for all $c_2 > c_1$. Also, $p_n^{c_2} = p_n^\infty$, since the birth- and death-rates are equal for $M|M|c_2$ and $M|M|\infty$ for the states $\{0, 1, \ldots, n\}$.

Now, let $n > c_1$. Fix $c_2 > n$. By Corollary 3.10 we get $p_n^{c_1} > p_n^{c_2}$, and, similarly to the previous case, we get $p_n^{c_2} = p_n^\infty$.

Then it holds for all $n \in \mathbb{N}_0$ with $n \leq c_1$ that $p_n^{c_1} = p_n^{c_2} = p_n^\infty$ and for all $n > c_1$ it holds that $p_n^{c_1} > p_n^{c_2} = p_n^\infty$, which concludes the first part of the proof.

Since for all $n \leq c$ it holds that $p_n^c = p_n^\infty$, it immediately follows that $\lim_{c\to\infty} p_n^c = p_n^\infty$. ☐

3.3 The slope of p_n

This subsection will again focus on the slope of the graph of p_n, in other words, we will focus on the maximum of the graph of x_n. The question is, whether this graph has a unique maximum. We will give a result about conditions when the graph of x_n is increasing and decreasing. First we will focus on the $M|M|\infty$-model. Then we study the $M|M|c$-model, $c \in \mathbb{N}$. From now on, we omit the superscript as it will be clear which model will be considered.

Recall Definition 3.3, i.e., $x_n := p_n - p_{n+1}$, $n \in \mathbb{N}_0$.

3.3.1 Slope of the $M|M|\infty$-model

Consider the $M|M|\infty$-model. Let $\lambda, \mu, \tau > 0$. Then, Eqn. (7) becomes

$$p_n = \begin{cases} \frac{\lambda}{\lambda+\tau} & \text{if } n = 0, \\ \frac{\lambda}{\lambda+\tau+n\mu(1-p_{n-1})} & \text{if } n \in \mathbb{N}. \end{cases}$$

Lemma 3.12. *Consider the $M|M|\infty$-model. Let $\lambda, \mu, \tau > 0$. Then,*

$$x_n = \begin{cases} \lambda\mu \cdot \frac{1-p_0}{(\lambda+\tau)(\lambda+\tau+\mu(1-p_0))}, & n = 0, \\ \lambda\mu \cdot \frac{n(p_{n-1}-p_n)+1-p_n}{(\lambda+\tau+n\mu(1-p_{n-1}))(\lambda+\tau+(n+1)\mu(1-p_n))}, & n \geq 1. \end{cases}$$

Proof. Let $n = 0$. Then,

$$\begin{aligned} x_0 &= p_0 - p_1 \\ &= \frac{\lambda}{\lambda+\tau} - \frac{\lambda}{\lambda+\tau+\mu(1-p_0)} \\ &= \lambda \cdot \frac{\lambda+\tau+\mu(1-p_0)-(\lambda+\tau)}{(\lambda+\tau)(\lambda+\tau+\mu(1-p_0))} \\ &= \lambda\mu \cdot \frac{1-p_0}{(\lambda+\tau)(\lambda+\tau+\mu(1-p_0))}. \end{aligned} \tag{5}$$

Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} x_n &= p_n - p_{n+1} \\ &= \frac{\lambda}{\lambda+\tau+n\mu(1-p_{n-1})} - \frac{\lambda}{\lambda+\tau+(n+1)\mu(1-p_n)} \\ &= \lambda \cdot \frac{\lambda+\tau+(n+1)\mu(1-p_n)-(\lambda+\tau+n\mu(1-p_{n-1}))}{(\lambda+\tau+n\mu(1-p_{n-1}))(\lambda+\tau+(n+1)\mu(1-p_n))} \\ &= \lambda \cdot \frac{(n+1)\mu(1-p_n)-n\mu(1-p_{n-1})}{(\lambda+\tau+n\mu(1-p_{n-1}))(\lambda+\tau+(n+1)\mu(1-p_n))} \\ &= \lambda\mu \cdot \frac{n(p_{n-1}-p_n)+1-p_n}{(\lambda+\tau+n\mu(1-p_{n-1}))(\lambda+\tau+(n+1)\mu(1-p_n))}. \end{aligned} \tag{6}$$

□

Lemma 3.13. *Consider the $M|M|\infty$-model with $\lambda, \mu, \tau > 0$. Let $n \in \mathbb{N}$ be such that $x_{n-1} \geq \frac{n+2}{n} x_n$. Then it holds that $x_n \geq x_{n+1}$.*

Proof. Assume that $n \in \mathbb{N}$ is such that $x_{n-1} \geq \frac{n+2}{n} x_n$. Rewriting Eqn. (6) gives that

$$\begin{aligned} x_n &= \lambda\mu \cdot \frac{n(p_{n-1}-p_n)+1-p_n}{(\lambda+\tau+n\mu(1-p_{n-1}))(\lambda+\tau+(n+1)\mu(1-p_n))} \\ &\geq \lambda\mu \cdot \frac{n(p_{n-1}-p_n)+1-p_n}{(\lambda+\tau+n\mu(1-p_{n+1}))(\lambda+\tau+(n+1)\mu(1-p_n))} \\ &\geq \lambda\mu \cdot \frac{n(p_{n-1}-p_n)+1-p_n}{(\lambda+\tau+(n+2)\mu(1-p_{n+1}))(\lambda+\tau+(n+1)\mu(1-p_n))}, \end{aligned}$$

where the first inequality holds by Lemma 3.5 (i.e., $p_{n-1} > p_{n+1}$).

To prove that $x_n \geq x_{n+1}$ it remains to be proven that

$$n(p_{n-1}-p_n)+1-p_n \geq (n+1)(p_n-p_{n+1})+1-p_{n+1}.$$

At the contrary, assume that $n(p_{n-1}-p_n)+1-p_n < (n+1)(p_n-p_{n+1})+1-p_{n+1}$. Then,

$$n(p_{n-1}-p_n)+1-p_n-(n+1)(p_n-p_{n+1})-1+p_{n+1} < 0.$$

This implies that

$$\begin{aligned} &np_{n-1} - np_n + 1 - p_n - np_n + np_{n+1} - p_n + p_{n+1} - 1 + p_{n+1} < 0 \\ \Rightarrow\ & np_{n-1} - 2(n+1)p_n + (n+2)p_{n+1} < 0 \\ \Rightarrow\ & n(p_{n-1} - p_n) + (n+2)(p_{n+1} - p_n) < 0 \\ \Rightarrow\ & nx_{n-1} - (n+2)x_n < 0 \\ \Rightarrow\ & nx_{n-1} < (n+2)x_n, \end{aligned}$$

which yields a contradiction with the assumption that $x_{n-1} \geq \frac{n+2}{n} x_n$. Hence,

$$n(p_{n-1} - p_n) + 1 - p_n \geq (n+1)(p_n - p_{n+1}) + 1 - p_{n+1}.$$

This gives

$$\begin{aligned} x_n &\geq \lambda\mu \cdot \frac{n(p_{n-1} - p_n) + 1 - p_n}{(\lambda + \tau + (n+2)\mu(1 - p_{n+1}))(\lambda + \tau + (n+1)\mu(1 - p_n))} \\ &\geq (n+1)(p_n - p_{n+1}) + 1 - p_{n+1} \\ &= x_{n+1}, \end{aligned}$$

thus completing the proof of the lemma. □

3.3.2 Slope of the $M|M|c$-model, with $c \in \mathbb{N}$

Consider the $M|M|c$-model, with $c \in \mathbb{N}$. Again, let $\lambda, \mu, \tau > 0$. Therefore, Eqn. (7) becomes

$$p_n = \begin{cases} \frac{\lambda}{\lambda+\tau}, & n = 0, \\ \frac{\lambda}{\lambda+\tau+n\mu(1-p_{n-1})}, & 1 \leq n \leq c, \\ \frac{\lambda}{\lambda+\tau+c\mu(1-p_{n-1})}, & n > c. \end{cases}$$

Lemma 3.14. *For the $M|M|c$-model, with $c \in \mathbb{N}$, it holds that*

$$x_n = \begin{cases} \lambda\mu \cdot \frac{1-p_0}{(\lambda+\tau)(\lambda+\tau+\mu(1-p_0))}, & n = 0, \\ \lambda\mu \cdot \frac{n(p_{n-1}-p_n)+1-p_n}{(\lambda+\tau+n\mu(1-p_{n-1}))(\lambda+\tau+(n+1)\mu(1-p_n))}, & 1 \leq n < c, \\ \lambda c\mu \cdot \frac{p_{n-1}-p_n}{(\lambda+\tau+c\mu(1-p_{n-1}))(\lambda+\tau+c\mu(1-p_n))}, & n \geq c. \end{cases}$$

Proof. Consider $n = 0$. Then, analogously to deriving Eqn. (5), we get that

$$x_0 = \lambda\mu \cdot \frac{1 - p_0}{(\lambda + \tau)(\lambda + \tau + \mu(1 - p_0))}.$$

Let $n \in \mathbb{N}$ such that $n < c$. Then, analogously to deriving Eqn. (6), we get that

$$x_n = \lambda\mu \cdot \frac{n(p_{n-1} - p_n) + 1 - p_n}{(\lambda + \tau + n\mu(1 - p_{n-1}))(\lambda + \tau + (n+1)\mu(1 - p_n))}.$$

Now, let $n \in \mathbb{N}$ such that $n \geq c$. Then,

$$\begin{aligned}
x_n &= p_n - p_{n+1} \\
&= \frac{\lambda}{\lambda + \tau + c\mu(1 - p_{n-1})} - \frac{\lambda}{\lambda + \tau + c\mu(1 - p_n)} \\
&= \lambda \cdot \frac{\lambda + \tau + c\mu(1 - p_n) - (\lambda + \tau + c\mu(1 - p_{n-1}))}{(\lambda + \tau + c\mu(1 - p_{n-1}))(\lambda + \tau + c\mu(1 - p_n))} \\
&= \lambda c\mu \cdot \frac{p_{n-1} - p_n}{(\lambda + \tau + c\mu(1 - p_{n-1}))(\lambda + \tau + c\mu(1 - p_n))}.
\end{aligned}$$

□

Lemma 3.15. *Consider the $M|M|c$-model, with $c \in \mathbb{N}$. Let $\lambda, \mu, \tau > 0$. If $1 \leq n < c$ and $x_{n-1} \geq \frac{n+2}{n} x_n$, then $x_n \geq x_{n+1}$. If $n \geq c$ and $x_{n-1} \geq x_n$, then $x_n \geq x_{n+1}$.*

Proof. Let $n < c$ and $x_{n-1} \geq \frac{n+2}{n} x_n$, then, analogously to the proof of part 1 of Lemma 3.13, it follows that $x_n \geq x_{n+1}$.

Now, let $n \geq c$. Assume that $x_{n-1} \geq x_n$. Then it holds that

$$\begin{aligned}
x_n - x_{n+1} = \lambda c\mu &\cdot \frac{p_{n-1} - p_n}{(\lambda + \tau + c\mu(1 - p_{n-1}))(\lambda + \tau + c\mu(1 - p_n))} \\
&- \lambda c\mu \cdot \frac{p_n - p_{n+1}}{(\lambda + \tau + c\mu(1 - p_n))(\lambda + \tau + c\mu(1 - p_{n+1}))} \\
\geq \lambda c\mu &\cdot \frac{x_{n-1}}{(\lambda + \tau + c\mu(1 - p_{n+1}))(\lambda + \tau + c\mu(1 - p_n))} \\
&- \lambda c\mu \cdot \frac{x_n}{(\lambda + \tau + c\mu(1 - p_n))(\lambda + \tau + c\mu(1 - p_{n+1}))} \\
\geq 0&,
\end{aligned}$$

where the first inequality holds by Lemma 3.5 (i.e., $p_{n-1} > p_{n+1}$), and where the second inequality holds by the assumption that $x_{n-1} \geq x_n$. This concludes the proof of the lemma. □

4 Convergence of p_n for a general birth-death process

In this chapter we will prove that for a birth-death process the limit $\lim_{n\to\infty} p_n$ exists under mild conditions. Before we prove this theorem, we will give an example to show an application of the theorem.

One of the mild conditions is that the limit of the ratio between the birth and death parameters must exist. The other mild condition is that the limit of the ratio between the exponential rate τ and the death parameters must exist.

Theorem 4.1. *Suppose that $\alpha^* = \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} \geq 0$ and $\beta^* = \lim_{n\to\infty} \frac{\tau}{\mu_n} \geq 0$ both exist. Let p_n, $n \in \mathbb{N}_0$, be given by*

$$p_n = \begin{cases} \frac{\lambda_0}{\lambda_0+\tau} & \text{if } n = 0, \\ \frac{\lambda_n}{\lambda_n+\tau+\mu_n(1-p_{n-1})} & \text{if } n \in \mathbb{N}. \end{cases} \tag{7}$$

1. *Suppose $\alpha^* = 0$ and $\beta^* > 0$. Then, $\lim_{n\to\infty} p_n = p* = 0$.*
2. *Suppose $\alpha^* > 0$ and $\beta^* \geq 0$. Then, $\lim_{n\to\infty} p_n = p^*$ exists with p^* the smallest solution of the equation*
$$x^2 - (\alpha^* + \beta^* + 1)\,x + \alpha^* = 0, \tag{8}$$
in other words
$$p^* = \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2}.$$
3. *Suppose $\alpha^* = \beta^* = 0$. Then, $\lim_{n\to\infty} p_n = p^* = 0$.*

The proof for parts 2 and 3 of the theorem requires more work than the proof for part 1. Therefore we will first prove part 1 of the theorem.

Proof of Theorem 4.1 part 1. Let $\alpha^* = 0$ and let $\beta^* > 0$. Let p_n, $n \in \mathbb{N}_0$ be given by Eqn. (7).

Note that $\mu_n > 0$ for all $n \in \mathbb{N}$. Therefore, we can rewrite Eqn. (7), for all $n \in \mathbb{N}$, by dividing by μ_n. This gives

$$p_n = \frac{\frac{\lambda_n}{\mu_n}}{\frac{\lambda_n}{\mu_n} + \frac{\tau}{\mu_n} + (1 - p_{n-1})} \geq \frac{\frac{\lambda_n}{\mu_n}}{\frac{\lambda_n}{\mu_n} + \frac{\tau}{\mu_n} + 1}. \tag{9}$$

Hence,

$$p^- = \liminf_{n\to\infty} p_n \geq \liminf_{n\to\infty} \frac{\frac{\lambda_n}{\mu_n}}{\frac{\lambda_n}{\mu_n} + \frac{\tau}{\mu_n} + 1} = \frac{\alpha^*}{\alpha^* + \beta^* + 1} = 0, \tag{10}$$

since $\alpha^* = 0$ and $\beta^* > 0$.

Furthermore, it holds that

$$p_n = \frac{\frac{\lambda_n}{\mu_n}}{\frac{\lambda_n}{\mu_n} + \frac{\tau}{\mu_n} + (1 - p_{n-1})} \leq \frac{\frac{\lambda_n}{\mu_n}}{\frac{\lambda_n}{\mu_n} + \frac{\tau}{\mu_n}}.$$

Hence,

$$p^+ = \limsup_{n\to\infty} p_n \leq \limsup_{n\to\infty} \frac{\frac{\lambda_n}{\mu_n}}{\frac{\lambda_n}{\mu_n} + \frac{\tau}{\mu_n}} = \frac{\alpha^*}{\alpha^* + \beta^*} = 0. \tag{11}$$

Combining Eqns. (10) and (11) yields $\lim_{n\to\infty} p_n = 0$, which completes the proof of part 1 of Theorem 4.1. □

4.1 Application of Theorem 4.1 part 3

As an illustration of part 3 of Theorem 4.1, we consider the $M|M|\infty$-model. Recall that $\tau > 0$, $\lambda_n = \lambda > 0$ for all $n \in \mathbb{N}_0$, $\mu_0 = 0$ and $\mu_n = n\mu > 0$ for all $n \in \mathbb{N}$. Then, $\alpha^* = \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = 0$ and $\beta^* = \lim_{n\to\infty} \frac{\tau}{\mu_n} = 0$. An example of the graph of p_n is given in Figure 5.

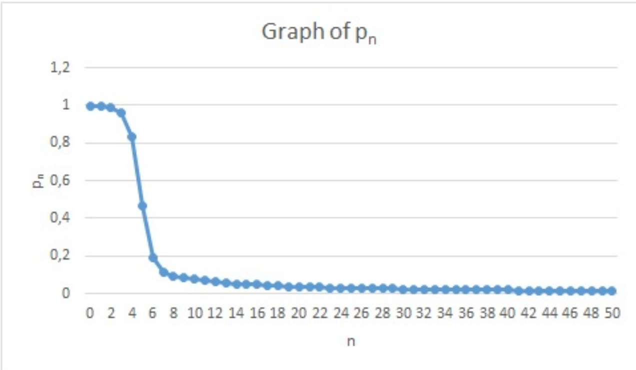

Figure 5: The graph of p_n for $n \in \{0, 1, \ldots, 50\}$ for $M|M|\infty$, with $\mu = 1$, $\lambda_n = \frac{3}{4}$ for all $n \in \mathbb{N}_0$, and $\tau = \frac{1}{1000}$.

Since the sequence $(\mu_n)_{n\in\mathbb{N}_0}$ has the property that it is monotonically increasing to ∞, we can show via a direct proof that $\lim_{n\to\infty} p_n = 0$. The proof of Lemma 4.2 was initially based on the proof of Lemma 3 in [2], but later we showed Lemma 3.5 part 3 and the proof below uses this lemma.

Lemma 4.2. *For the $M|M|\infty$ model it holds that $\lim_{n\to\infty} p_n = 0$.*

Proof. By Lemma 3.5 we know that the sequence $(p_n)_{n\in\mathbb{N}_0}$ is a decreasing sequence. Hence, for all $n \in \mathbb{N}$ it holds that $p_n \le p_0$. Therefore,

$$\begin{aligned} p_{n+1} &= \frac{\lambda}{\lambda + \tau + (n+1)\mu(1-p_n)} \\ &\le \frac{\lambda}{\lambda + \tau + (n+1)\mu(1-p_0)}, \end{aligned}$$

since $p_n \le p_0$.

Now we can take the limit of n to ∞ of the right-hand side. This gives

$$\lim_{n\to\infty} \frac{\lambda}{\lambda + \tau + (n+1)\mu(1-p_0)} = 0.$$

Thus,

$$p^* = \lim_{n\to\infty} p_n = 0.$$

□

4.2 Proof of parts 2 and 3 of Theorem 4.1

The proof of the second part of Theorem 4.1 consists of multiple steps. These steps are formulated as lemmas that will be proven in this subsection. The final proof of Theorem 4.1 part 2 can then be found at the end of this subsection and this proof will use all lemmas in this subsection. Throughout this section, we assume that $\alpha^* > 0$.

Recall Definition 2.6.1, i.e., $p^+ := \limsup_{n\to\infty} p_n$ and $p^- := \liminf_{n\to\infty} p_n$.

Lemma 4.3. *Let $\alpha^* > 0$ and $\beta^* \geq 0$. It holds that $p^+, p^- > 0$.*

Proof. Let $\alpha^* > 0$ and $\beta^* \geq 0$. Let p_n, $n \in \mathbb{N}_0$ be given by Eqn. (7). Analogously to the proof of the first part of Theorem 4.1 we get again Eqn. (9). Hence, also Eqn. (10) holds, which gives that

$$p^- \geq \frac{\alpha^*}{\alpha^* + \beta^* + 1} > 0,$$

since $\alpha^* > 0$.

Because $p^- = \liminf_{n\to\infty} p_n > 0$ it follows immediately that $p_n > 0$ for all $n \in \mathbb{N}_0$. Moreover, since $p^+ \geq p^-$ we also have that $p^+ > 0$, which concludes the proof. □

Lemma 4.4. *Let $\alpha^* > 0$. It holds that p^+ and p^- are both solutions of Eqn. (8). in other words,*

$$p^+, p^- \in \left\{ \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2}, \frac{(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} \right\}.$$

Proof. By virtue of Lemma 4.3 we have that $p^+, p^- > 0$. The proof of that lemma even yields that $p_n > 0$ for all $n \in \mathbb{N}_0$. Hence, dividing by p^+, p^- and p_n, $n \in \mathbb{N}_0$, is allowed.

Because $(p_n)_{n\in\mathbb{N}_0}$ is a sequence of probabilities, it holds that p^+ and p^- both exist with $p^+, p^- \in [0, 1]$. Hence, there exist a subsequence $(p_{n_k})_k \subseteq (p_n)_{n\in\mathbb{N}_0}$ such that $p_{n_k} \to p^+$ as $k \to \infty$ and there exists a subsequence $(p_{m_l})_l \subseteq (p_n)_{n\in\mathbb{N}_0}$ such that $p_{m_l} \to p^-$ as $l \to \infty$.

Recall Eqn. (7), so for all $n \in \mathbb{N}_0$ that $p_{n+1} = \frac{\lambda_{n+1}}{\lambda_{n+1}+\tau+\mu_{n+1}(1-p_n)}$. Therefore,

$$p_{n_k+1} = \frac{\lambda_{n_k+1}}{\lambda_{n_k+1} + \tau + \mu_{n_k+1}(1 - p_{n_k})}.$$

Note that $\lim_{k\to\infty} p_{n_k+1}$ exists, since $\lim_{k\to\infty} p_{n_k}$ exists.

Furthermore, it holds that

$$\lim_{k\to\infty} p_{n_k+1} \leq \limsup_{k\to\infty} \sup_{j\geq n_k+1} p_j \leq \limsup_{n\to\infty} p_n = p^+.$$

Thus, $\lim_{k\to\infty} p_{n_k+1} \leq p^+$, and so

$$\begin{aligned}
p^+ &\geq \lim_{k\to\infty} \frac{\lambda_{n_k+1}}{\lambda_{n_k+1} + \tau + \mu_{n_k+1}(1-p_{n_k})} \\
&= \lim_{k\to\infty} \frac{\frac{\lambda_{n_k+1}}{\mu_{n_k+1}}}{\frac{\lambda_{n_k+1}+\tau+\mu_{n_k+1}(1-p_{n_k})}{\mu_{n_k+1}}} \\
&= \frac{\alpha^*}{\alpha^* + \beta^* + 1 - \lim_{k\to\infty} p_{n_k}} \\
&= \frac{\alpha^*}{\alpha^* + \beta^* + 1 - p^+}.
\end{aligned}$$

So, we have that $p^+ \geq \frac{\alpha^*}{\alpha^*+\beta^*+1-p^+}$. This can be rewritten as

$$\begin{aligned}
&p^+ \left(\alpha^* + \beta^* + 1 - p^+\right) \geq \alpha^*, \\
\iff\ & -(p^+)^2 + (\alpha^* + \beta^* + 1)\, p^+ - \alpha^* \geq 0, \\
\iff\ & (p^+)^2 - (\alpha^* + \beta^* + 1)\, p^+ + \alpha^* \leq 0. \qquad (12)
\end{aligned}$$

Rewriting Eqn. (7) yields

$$p_{n-1} = \frac{p_n(\lambda_n + \tau + \mu_n) - \lambda_n}{\mu_n p_n}.$$

Therefore,

$$p_{n_k-1} = \frac{p_{n_k}(\lambda_{n_k} + \tau + \mu_{n_k}) - \lambda_{n_k}}{\mu_{n_k} p_{n_k}} = \frac{p_{n_k}\left(\frac{\lambda_{n_k}}{\mu_{n_k}} + \frac{\tau}{\mu_{n_k}} + 1\right) - \frac{\lambda_{n_k}}{\mu_{n_k}}}{p_{n_k}}. \qquad (13)$$

Because $\lim_{k\to\infty} p_{n_k} = p^+$ exists, it follows that $\lim_{k\to\infty} p_{n_k-1}$ also exists. It holds that

$$\lim_{k\to\infty} p_{n_k-1} \leq \limsup_{k\to\infty} \sup_{j\geq n_k-1} p_j \leq \limsup_{n\to\infty} p_n = p^+.$$

Since $\lim_{k\to\infty} p_{n_k-1} = \frac{p^+(\alpha^*+\beta^*+1)-\alpha^*}{p^+}$, we have that

$$\frac{p^+\left(\alpha^* + \beta^* + 1\right) - \alpha^*}{p^+} \leq p^+.$$

Therefore,

$$(p^+)^2 \geq p^+\left(\alpha^* + \beta^* + 1\right) - \alpha^*,$$

which yields

$$(p^+)^2 - p^+\left(\alpha^* + \beta^* + 1\right) + \alpha^* \geq 0. \qquad (14)$$

Combining Eqns. (12) and (14) yields that p^+ is a solution of Eqn. (8). Hence,

$$p^+ \in \left\{ \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2}, \frac{(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} \right\}.$$

Analogously (only the sign of the inequalities changes, because the limit inferior is considered instead of the limit superior), it follows that p^- is also a solution of Eqn. (8). This concludes the proof of the lemma. □

Remark 4.5. Since it holds that p^+ and p^- both exist, it necessarily must hold that both solutions of the quadratic expression exist.

Since $\beta^* \geq 0$ we only need to consider two situations, namely $\beta^* > 0$ and $\beta^* = 0$ in order to be able to prove the theorem.

First consider the case where $\beta^* > 0$. If

$$\frac{(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} > 1 \geq \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} \geq 0, \tag{15}$$

then it holds that

$$p^+ = p^- = \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2},$$

since $p^+, p^- \in [0, 1]$, which is the smallest solution of Eqn. (8).

Lemma 4.6. *Let $\alpha^*, \beta^* > 0$. Then Eqn. (15) holds.*

Proof. First we will prove that

$$\frac{(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} > 1 \tag{16}$$

holds. To show this, it is sufficient to prove that $(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} > 2$. Suppose that is not true. Then,

$$\sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \leq 2 - (\alpha^* + \beta^* + 1) = 1 - \alpha^* - \beta^*.$$

This implies that

$$((\alpha^* + \beta^*) + 1)^2 - 4\alpha^* \leq (1 - (\alpha^* + \beta^*))^2,$$

in other words,

$$(\alpha^* + \beta^*)^2 + 2(\alpha^* + \beta^*) + 1 - 4\alpha^* \leq 1 - 2(\alpha^* + \beta^*) + (\alpha^* + \beta^*)^2.$$

Rewriting yields

$$4(\alpha^* + \beta^*) - 4\alpha^* \leq 0,$$

so that $\beta^* \leq 0$. This contradicts the fact that $\beta^* > 0$. Hence, Eqn. (16) holds.

Next, we will prove that

$$0 \leq \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} \leq 1. \tag{17}$$

To show this, it is sufficient to prove that $0 \leq (\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \leq 2$.

The fact that $\alpha^*, \beta^* > 0$, implies that

$$(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \geq 0.$$

To prove that $(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \leq 2$, we will assume the contrary. In other words,

$$-\sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} > 2 - (\alpha^* + \beta^* + 1)\,,$$

or, equivalently,

$$\sqrt{((\alpha^* + \beta^*) + 1)^2 - 4\alpha^*} < \alpha^* + \beta^* - 1.$$

Similarly as before, this implies that $\beta^* < 0$. Thus, we have arrived at a contradiction, so that Eqn. (17) holds.

So, if $\beta^* > 0$, then it holds that

$$\frac{(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} > 1 \geq \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} \geq 0,$$

which concludes the proof of the claim. □

Corollary 4.7. *Let $\alpha^* > 0$. If $\beta^* > 0$, then*

$$p^+ = \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} = p^-,$$

which is the smallest solution of Eqn. (8).

Proof. By virtue of Lemma 4.4 and Lemma 4.6, and since $p^+, p^- \in [0, 1]$, we can conclude that for $\beta^* > 0$ it holds that

$$p^+ = \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2} = p^-,$$

which is the smallest solution of Eqn. (8). □

Corollary 4.7 concludes the proof of Theorem 4.1 in case $\beta^* > 0$.

Now it remains to consider the case $\beta^* = 0$.

Lemma 4.4 yields that p^+ and p^- are both solutions to Eqn. (8), which reduces to

$$x^2 - (\alpha^* + 1)\,x + \alpha^* = 0, \tag{18}$$

so that

$$\begin{aligned} p^+,\ p^- &\in \left\{ \frac{(\alpha^* + 1) - \sqrt{(\alpha^* + 1)^2 - 4\alpha^*}}{2}, \frac{(\alpha^* + 1) + \sqrt{(\alpha^* + 1)^2 - 4\alpha^*}}{2} \right\} \\ &= \left\{ \frac{(\alpha^* + 1) - \sqrt{(\alpha^* - 1)^2}}{2}, \frac{(\alpha^* + 1) + \sqrt{(\alpha^* - 1)^2}}{2} \right\}. \end{aligned} \tag{19}$$

Consider three situations, namely $\alpha^* = 1$, $\alpha^* > 1$ and $0 \leq \alpha^* < 1$. These three situations cover all possible situations, since $\alpha^* \geq 0$. These situations will be considered in Lemma 4.8 and Lemma 4.9, respectively.

Lemma 4.8. *Let $\beta^* = 0$. Let $\alpha^* \geq 1$. Then it holds that $p^+ = p^- = 1$, which is the smallest solution of Eqn. (18).*

Proof. Let $\alpha^* = 1$. Then 1 is the unique root of Eqn. (18), and so $p^+ = p^- = 1$. So, then it follows that $p^+ = 1 = p^-$, and hence p^* exists and is given by $p^* = 1$.

Let $\alpha^* > 1$. It holds that 1 and α^* are the roots of the quadratic Eqn. (18). Since $p^+, p^- \in [0, 1]$, necessarily $p^+ = p^- = 1$. □

Lemma 4.9. *Let $\beta^* = 0$. Let $0 < \alpha^* < 1$. Then $p^+,\ p^- \in \{\alpha^*, 1\}$, which are the solutions of the quadratic Eqn. (18).*

Proof. Since $0 < \alpha^* < 1$, Eqn. (19) reduces to

$$\begin{aligned} p^+,\ p^- &\in \left\{ \frac{\alpha^* + 1 - \sqrt{(1-\alpha^*)^2}}{2}, \frac{\alpha^* + 1 + \sqrt{(1-\alpha^*)^2}}{2} \right\} \\ &= \left\{ \frac{\alpha^* + 1 - (1-\alpha^*)}{2}, \frac{\alpha^* + 1 + (1-\alpha^*)}{2} \right\} \\ &= \left\{ \frac{2\alpha^*}{2}, \frac{2}{2} \right\} \\ &= \{\alpha^*, 1\}. \end{aligned}$$

However, because $0 < \alpha^* < 1$ this does not give a unique solution. Hence $p^+,\ p^- \in \{\alpha^*, 1\}$. □

However, we want to prove that for $\beta^* = 0$ and $0 < \alpha^* < 1$ we have that $p^+ = p^- = \alpha^* < 1$. Therefore we will consider the following lemma.

Lemma 4.10. *Let $\beta^* = 0$ and let $0 \leq \alpha^* < 1$. It holds that $p^+ = \limsup_{n\to\infty} p_n < 1$.*

Proof. It holds that $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = \alpha^* < 1$. Let $\epsilon > 0$ be such that $\alpha^* + \epsilon < 1$. Then there exists $N \in \mathbb{N}_0$, such that for all $n \geq N$ it holds that $\frac{\lambda_n}{\mu_n} \leq \alpha^* + \epsilon < 1$.

Define, for all $n \in \mathbb{N}_0$,

$$\lambda'_n := \begin{cases} \lambda_n & \text{if } n < N, \\ (\alpha^* + \epsilon)\,\mu_n & \text{if } n \geq N, \end{cases}$$

and

$$p'_n := \frac{\lambda'_n}{\lambda'_n + \tau + \mu_n(1 - p'_{n-1})}$$

For all $0 \leq n < N$ it holds that $\lambda'_n = \lambda_n$ and thus also that $p'_n = p_n$. For all $n \geq N$, it holds that that $\lambda'_n > \lambda_n$.

By virtue of Eqn. (7),

$$p_n = \frac{1}{1 + \frac{\tau}{\lambda_n} + \frac{\mu_n}{\lambda_n}(1 - p_{n-1})}$$

and

$$p'_n = \frac{1}{1 + \frac{\tau}{\lambda'_n} + \frac{\mu_n}{\lambda'_n}(1 - p'_{n-1})}.$$

Let $n = N$. Since $p'_{N-1} = p_{N-1}$ and $\lambda'_N > \lambda_N$, it holds that

$$\begin{aligned} 1 + \frac{\tau}{\lambda'_N} + \frac{\mu_N}{\lambda'_N}(1 - p'_{N-1}) &= 1 + \frac{\tau}{\lambda'_N} + \frac{\mu_N}{\lambda'_N}(1 - p_{N-1}) \\ &< 1 + \frac{\tau}{\lambda_N} + \frac{\mu_N}{\lambda_N}(1 - p_{N-1}). \end{aligned}$$

This yields

$$\begin{aligned} p'_N &= \frac{1}{1 + \frac{\tau}{\lambda'_N} + \frac{\mu_N}{\lambda'_N}(1 - p'_{N-1})} \\ &> \frac{1}{1 + \frac{\tau}{\lambda_N} + \frac{\mu_N}{\lambda_N}(1 - p_{N-1})} \\ &= p_N. \end{aligned}$$

Let $n > N$ and assume that $p'_{n-1} > p_{n-1}$. Then $1 - p'_{n-1} < 1 - p_{n-1}$. By assumption it holds that $\lambda'_n > \lambda_n$, hence $1/\lambda'_n < 1/\lambda_n$. Therefore,

$$\begin{aligned} 1 + \frac{\tau}{\lambda'_n} + \frac{\mu_n}{\lambda'_n}(1 - p'_{n-1}) &< 1 + \frac{\tau}{\lambda'_n} + \frac{\mu_n}{\lambda'_n}(1 - p_{n-1}) \\ &< 1 + \frac{\tau}{\lambda_n} + \frac{\mu_n}{\lambda_n}(1 - p_{n-1}), \end{aligned}$$

so that

$$\begin{aligned} p'_n &= \frac{1}{1 + \frac{\tau}{\lambda'_n} + \frac{\mu_n}{\lambda'_n}(1 - p'_{n-1})} \\ &> \frac{1}{1 + \frac{\tau}{\lambda_n} + \frac{\mu_n}{\lambda_n}(1 - p_{n-1})} \\ &= p_n. \end{aligned}$$

As a conclusion, $p_n \leq p'_n$ for all $n \in \mathbb{N}_0$, so that $\limsup_{n\to\infty} p_n \leq \limsup_{n\to\infty} p'_n$. Thus, it is sufficient to show that $\limsup_{n\to\infty} p'_n < 1$.

We will now prove by induction to n that $p'_n < 1$ for all $n \in \mathbb{N}_0$.

Notice that $p'_0 = \frac{\lambda'_0}{\lambda'_0 + \tau}$ and $p'_n = \frac{\lambda'_n}{\lambda'_n + \tau + \mu_n(1 - p'_{n-1})}$, $n \in \mathbb{N}$. Since we have for all $n \in \mathbb{N}$ that

$$\frac{\lambda'_n}{\mu_n} = \frac{(\alpha^* + \epsilon)\,\mu_n}{\mu_n} = \alpha^* + \epsilon =: c,$$

we can write, for all $k \in \mathbb{N}$,

$$p'_k = \frac{c}{c + \frac{\tau}{\mu_k} + 1 - p'_{k-1}}. \tag{20}$$

It holds that $\tau > 0$, hence $p'_0 < 1$. Then there exists $\sigma \in (0, 1)$ such that $p'_0 \leq \sigma < 1$. Without loss of generality we may assume that $c < \sigma$, otherwise we could have chosen σ larger.

Then,

$$p'_1 = \frac{c}{c + \frac{\tau}{\mu_1} + 1 - p'_0} \leq \frac{c}{c + 1 - p'_0} \leq \frac{c}{c + 1 - \sigma} < 1, \tag{21}$$

where the first inequality holds because $\frac{\tau}{\mu_1} \geq 0$.

Using induction to k, we will show that

$$p'_k \leq \frac{c}{c+1-\sigma} < 1. \tag{22}$$

Eqn. (21) gives

$$p'_1 \leq \frac{c}{c+1-\sigma},$$

which is Eqn. (22) for $k = 1$. Therefore,

$$1 - p'_1 \geq 1 - \frac{c}{c+1-\sigma} = \frac{c+1-\sigma-c}{c+1-\sigma} = \frac{1-\sigma}{c+1-\sigma}.$$

Using this for Eqn. (20) with $k = 2$, we get

$$p'_2 = \frac{c}{c + \frac{\tau}{\mu_2} + 1 - p'_1} \leq \frac{c}{c+1-p'_1} \leq \frac{c}{c + \frac{1-\sigma}{c+1-\sigma}}.$$

Since $c < \sigma$, we get $c + 1 - \sigma \leq 1$, hence $\frac{1}{c+1-\sigma} \geq 1$. Multiplying by $1 - \sigma$ then gives that $\frac{1-\sigma}{c+1-\sigma} \geq 1 - \sigma$. Therefore,

$$p'_2 \leq \frac{c}{c + \frac{1-\sigma}{c+1-\sigma}} \leq \frac{c}{c+1-\sigma},$$

which is Eqn. (22) for $k = 2$.

Assume that Eqn. (22) holds for $k = n - 1$. We will prove that Eqn. (22) also holds for $k = n$. Since $\frac{\tau}{\mu_n} \geq 0$, we get that

$$p'_n = \frac{c}{c + \frac{\tau}{\mu_n} + 1 - p'_{n-1}} \leq \frac{c}{c+1-p'_{n-1}} \leq \frac{c}{c+1-\frac{c}{c+1-\sigma}} = \frac{c}{c + \frac{1-\sigma}{c+1-\sigma}}.$$

Similarly to the case $k = 2$ we get

$$\frac{1-\sigma}{c+1-\sigma} \geq 1 - \sigma.$$

Therefore,

$$p'_n \leq \frac{c}{c + \frac{1-\sigma}{c+1-\sigma}} \leq \frac{c}{c+1-\sigma} < 1.$$

This proves Eqn. (22) for $k = n$.

Eqn. (22) implies that

$$\limsup_{n\to\infty} p'_n \leq \frac{c}{c+1-\sigma} < 1.$$

As a consequence, we get that

$$\limsup_{n\to\infty} p_n \leq \frac{c}{c+1-\sigma} < 1,$$

which is what we needed to show. □

Corollary 4.11. *For $\beta^* = 0$ and $0 < \alpha^* < 1$ it holds that $p^+ = p^- = \alpha^*$, and thus $p^* = \alpha^*$.*

Proof. From Lemma 4.10 it follows that $p^+ = \limsup_{n\to\infty} p_n < 1$. Since $p^+ \in \{\alpha^*, 1\}$ it follows that $p^+ = \alpha^* < 1$. Since it always holds that $p^- \le p^+$, we also have that $p^- = \alpha^* < 1$. Thus, $p^+ = p^- = \alpha^* < 1$. Therefore, p^* exists and is equal to $\alpha^* < 1$, which is the smallest solution of Eqn. (8). □

Now we can give the proof the second part of Theorem 4.1.

Proof of Theorem 4.1 part 2. Let $X = (X_t)_{t\ge 0}$ be a birth-death process with birth parameters $\lambda_n > 0$, $n \in \mathbb{N}_0$, and death parameters $\mu_n > 0$, $n \in \mathbb{N}$ and $\mu_0 = 0$. Let $\tau > 0$ be given. Assume that $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = \alpha^* > 0$ and $\lim_{n\to\infty} \frac{\tau}{\mu_n} = \beta^* \ge 0$ both exist. Let p_n, $n \in \mathbb{N}$, be given by Eqn. (7).

To prove that $\lim_{n\to\infty} p_n$ exists, it is sufficient to prove that $p^+ = p^-$, i.e., $\limsup_{n\to\infty} p_n = \liminf_{n\to\infty} p_n$.

Lemma 4.4 gives that p^+ and p^- are both solutions of Eqn. (8).

For $\beta^* > 0$, Lemma 4.6 gives that Eqn. (15) holds. Then Corollary 4.7 gives that

$$p^* = \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2},$$

which is the smallest solution of Eqn. (8).

Now, consider $\beta^* = 0$. Lemma 4.8 gives for $\alpha^* \ge 1$ that $p^* = 1$, which is the smallest solution of Eqn. (8). Moreover, Lemma 4.9 gives, for $0 < \alpha^* < 1$, that $p^+, p^- \in \{\alpha^*, 1\}$, which is not a unique solution yet. However, from Lemma 4.10 it follows that $p^+ \le \frac{c}{c+1-\sigma} < 1$. Then with Corollary 4.11 it follows that $p^* = \alpha^*$, which is the smallest solution of Eqn. (8).

Thus, for all $\beta^* \ge 0$ and for all $\alpha^* > 0$, it holds that p^* exists and it is the smallest solution of Eqn. (8), which is what we needed to show. □

To finish the proof of Theorem 4.1, it remains to prove part 3 of this theorem. This proof will now be given.

Proof of Theorem 4.1 part 3. Let $X = (X_t)_{t\ge 0}$ be a birth-death process with birth parameters $\lambda_n > 0$, $n \in \mathbb{N}_0$, and death parameters $\mu_n > 0$, $n \in \mathbb{N}$ and $\mu_0 = 0$. Let $\tau > 0$ be given. Assume that $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = \alpha^* = 0$ and $\lim_{n\to\infty} \frac{\tau}{\mu_n} = \beta^* = 0$. Let p_n, $n \in \mathbb{N}$, be given by Eqn. (7).

To prove that $p^* = \lim_{n\to\infty} p_n = 0$ exists, it is sufficient to prove that $p^+ = 0$, i.e., $\limsup_{n\to\infty} p_n = 0$.

Lemma 4.10 gives that $p^+ < 1$. So, to prove that $p^+ = 0$, we will assume the contrary and prove that $p^+ = 1$.

Therefore, assume that $p^+ > 0$. Hence, there exists a subsequence $(p_{n_k})_k \subseteq (p_n)_{n\in\mathbb{N}_0}$ such that

$p_{n_k} \to p^+$ as $k \to \infty$. Then, Eqn. (13) holds. Then, since $\alpha^* = \beta^* = 0$ we get

$$\begin{aligned}
\lim_{k\to\infty} p_{n_k-1} &= \lim_{k\to\infty} \frac{p_{n_k}\left(\frac{\lambda_{n_k}}{\mu_{n_k}} + \frac{\tau}{\mu_{n_k}} + 1\right) - \frac{\lambda_{n_k}}{\mu_{n_k}}}{p_{n_k}} \\
&= \lim_{k\to\infty} \frac{\lambda_{n_k}}{\mu_{n_k}} + \frac{\tau}{\mu_{n_k}} + 1 - \frac{\lambda_{n_k}}{\mu_{n_k} p_{n_k}} \\
&= 0 + 0 + 1 - 0.
\end{aligned}$$

Therefore, it holds that $p^+ = 1$. However, this is a contradiction to the fact that $p^+ < 1$. Therefore, $p^+ = 0$.

Since, $p^- \leq p^+$ and $p^+, p^- \in [0, 1]$, we immediately get that $p^* = 0$. □

4.3 More monotonicity and convergence results for the $M|M|c$-model (with $c \in \mathbb{N}$)

Let $c \in \mathbb{N}$ and consider the $M|M|c$-model. From [4] we know that we have to assume that $\lambda_n = \lambda < c\mu$, for all $n \in \mathbb{N}_0$, and

$$\mu_n = \begin{cases} n\mu & \text{for all } n < c, \\ c\mu & \text{for all } n \geq c, \end{cases}$$

where we assume that $\mu > 0$.

Recall Notation 3.1. This subsection uses the superscript index again.

From Theorem 4.1 we know that the sequence $(p_n^c)_{n\in\mathbb{N}_0}$ has a limit, therefore we introduce some notation for this limit.

Notation 4.12. Denote by $p^{*,c}$ the limiting value of the sequence $(p_n^c)_{n\in\mathbb{N}_0}$.

In this chapter we will prove that $p^{*,c}$ is monotonic in $c \in \mathbb{N}$ and compute the limit $\lim_{c\to\infty} p^{*,c}$.

For the $M|M|c$-model, $c \in \mathbb{N}$, it holds that

$$\alpha^* = \frac{\lambda}{c\mu} > 0 \quad \text{and} \quad \beta^* = \frac{\tau}{c\mu} > 0.$$

Lemma 4.13. *Let $c \in \mathbb{N}$. Then $p^{*,c}$ is non-increasing as a function of c, i.e., $p^{*,c} - p^{*,c+1} \geq 0$.*

Proof. By Theorem 4.1 we get that

$$p^{*,c} = \frac{\frac{\lambda}{c\mu} + \frac{\tau}{c\mu} + 1 - \sqrt{\left(\frac{\lambda}{c\mu} + \frac{\tau}{c\mu} + 1\right)^2 - 4\frac{\lambda}{c\mu}}}{2}.$$

Multiplication of both the denominator and the numerator by $c\mu$ yields

$$\begin{aligned}
p^{*,c} &= \frac{\lambda+\tau+c\mu-\sqrt{(c\mu)^2\cdot\left(\frac{\lambda}{c\mu}+\frac{\tau}{c\mu}+1\right)^2-4\lambda c\mu}}{2c\mu} \\
&= \frac{\lambda+\tau+c\mu-\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}}{2c\mu}.
\end{aligned}$$

To prove that $p^{*,c}$ is non-increasing in c, we will prove that the derivative to c is non-positive. The derivative of $p^{*,c}$ to c is given by

$$\begin{aligned}
\frac{d}{dc}p^{*,c} &= -\frac{\lambda+\tau}{2c^2\mu} - \left[-\frac{1}{2c^2\mu}\cdot\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu} + \frac{1}{2c\mu}\cdot\frac{d}{dc}\left[\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}\right]\right] \\
&= -\frac{\lambda+\tau}{2c^2\mu} + \frac{1}{2c^2\mu}\cdot\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu} \\
&\quad - \frac{1}{2c\mu}\cdot\left[\frac{1}{2\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}}\cdot\left(2\mu(\lambda+\tau)+2c\mu^2-4\mu\lambda\right)\right] \\
&= \frac{-(\lambda+\tau)+\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}}{2c^2\mu} - \frac{2\mu\left(-\lambda+\tau+c\mu\right)}{4c\mu\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}} \\
&= \frac{-(\lambda+\tau)\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}+(\lambda+\tau+c\mu)^2-4\lambda c\mu-c\mu\left(-\lambda+\tau+c\mu\right)}{2c^2\mu\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}}.
\end{aligned}$$

To prove that $\frac{d}{dc}p^{*,c} \leq 0$, we will assume the contrary.

Hence, we assume that

$$-(\lambda+\tau)\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}+(\lambda+\tau+c\mu)^2-4\lambda c\mu-c\mu\left(-\lambda+\tau+c\mu\right) > 0.$$

Then,

$$(\lambda+\tau+c\mu)^2-4\lambda c\mu > (\lambda+\tau)\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}+c\mu\left(-\lambda+\tau+c\mu\right),$$

in other words,

$$(\lambda+\tau)^2+2c\mu(\lambda+\tau)+c^2\mu^2-4c\mu\lambda > (\lambda+\tau)\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}+c^2\mu^2+c\mu\left(\tau-\lambda\right).$$

If $(\lambda+\tau)^2+c\mu(\tau-\lambda) < 0$, then we get a contradiction and hence the proof is complete. So, assume that $(\lambda+\tau)^2+c\mu(\tau-\lambda) \geq 0$. Then,

$$(\lambda+\tau)^2+c\mu(\tau-\lambda) > (\lambda+\tau)\sqrt{(\lambda+\tau+c\mu)^2-4\lambda c\mu}.$$

Taking the square on both sides yields,

$$(\lambda+\tau)^4+2c\mu(\tau-\lambda)(\lambda+\tau)^2+c^2\mu^2(\tau-\lambda)^2 > (\lambda+\tau)^2\cdot\left((\lambda+\tau+c\mu)^2-4\lambda c\mu\right).$$

Rewriting this gives

$$\begin{aligned}
&(\lambda+\tau)^4+2c\mu(\tau-\lambda)(\lambda+\tau)^2+c^2\mu^2(\tau-\lambda)^2 > (\lambda+\tau)^2\cdot\left((\lambda+\tau)^2+2c\mu(\lambda+\tau)+c^2\mu^2-4c\mu\lambda\right) \\
&\quad\Longleftrightarrow\; 2c\mu(\tau-\lambda)(\lambda+\tau)^2+c^2\mu^2(\tau-\lambda)^2 > 2c\mu(\lambda+\tau)^3+c^2\mu^2(\lambda+\tau)^2-4c\mu\lambda(\lambda+\tau)^2 \\
&\quad\Longleftrightarrow\; 2(\lambda+\tau)^2(\tau-\lambda-\lambda-\tau)+c\mu(\tau-\lambda)^2 > c\mu(\lambda+\tau)^2-4\lambda(\lambda+\tau)^2 \\
&\quad\Longleftrightarrow\; -4\lambda(\lambda+\tau)^2+c\mu(\tau-\lambda)^2 > c\mu(\lambda+\tau)^2-4\lambda(\lambda+\tau)^2 \\
&\quad\Longleftrightarrow\; (\tau-\lambda)^2 > (\lambda+\tau)^2.
\end{aligned}$$

However, since $\lambda, \tau > 0$, $(\tau - \lambda)^2 > (\lambda + \tau)^2$ gives a contradiction. Hence, the assumption that

$$-(\lambda+\tau)\sqrt{(\lambda+\tau+c\mu)^2 - 4\lambda c\mu} + (\lambda+\tau+c\mu)^2 - 4\lambda c\mu - c\mu\,(-\lambda+\tau+c\mu) > 0$$

is not correct. Thus, $\frac{d}{dc}p^{*,c} \leq 0$. □

Corollary 4.14. *Let $c_1, c_2 \in \mathbb{N}$. Then it holds that $p^{*,c_1} \geq p^{*,c_2}$.*

Proof. Apply Lemma 4.13 repeatedly. □

Lemma 4.15. *It holds that $p^{*,c} \leq \frac{\lambda}{c\mu}$, hence $\lim_{c\to\infty} p^{*,c} = 0$.*

Proof. Before proving the upper bound for $p^{*,c}$, we will first give a lower bound for $(\lambda+\tau+c\mu)^2 - 4c\mu\lambda$, which is the term under the square root in p^*.

For all $c \in \mathbb{N}$ we have that

$$\begin{aligned}
(\lambda+\tau+c\mu)^2 - 4c\mu\lambda &= (\lambda+\tau)^2 + 2(\lambda+\tau)c\mu + c^2\mu^2 - 4c\mu\lambda \\
&= \lambda^2 + 2\lambda\tau + \tau^2 + 2c\mu\lambda + 2c\mu\tau + c^2\mu^2 - 4c\mu\lambda \\
&= \lambda^2 + 2\lambda\tau + \tau^2 - 2c\mu\lambda + 2c\mu\tau + c^2\mu^2 \\
&= \lambda^2 - 2\lambda\tau + 4\lambda\tau + \tau^2 - 2c\mu\lambda + 2c\mu\tau + c^2\mu^2 \\
&= (\tau-\lambda)^2 + 2(\tau-\lambda)c\mu + c^2\mu^2 + 4\lambda\tau \\
&= (\tau-\lambda+c\mu)^2 + 4\lambda\tau \\
&\geq (\tau-\lambda+c\mu)^2 \geq 0,
\end{aligned}$$

where the last inequality holds since $\lambda, \tau \geq 0$ and $\lambda < c\mu$.

It holds that

$$\begin{aligned}
p^{*,c} &= \frac{\lambda+\tau+c\mu - \sqrt{(\lambda+\tau+c\mu)^2 - 4c\mu\lambda}}{2c\mu} \\
&= \frac{\lambda+\tau+c\mu - \sqrt{(\tau-\lambda+c\mu)^2 + 4\lambda\tau}}{2c\mu} \\
&\leq \frac{\lambda+\tau+c\mu - \sqrt{(\tau-\lambda+c\mu)^2}}{2c\mu} \\
&\overset{(*)}{=} \frac{\lambda+\tau+c\mu - (\tau-\lambda+c\mu)}{2c\mu} \\
&= \frac{2\lambda}{2c\mu} \\
&= \frac{\lambda}{c\mu},
\end{aligned}$$

where (*) holds because $\lambda < c\mu$.

So, when we take the limit of $c \to \infty$ we get

$$0 \leq \lim_{c\to\infty} p^{*,c} \leq \lim_{c\to\infty} \frac{\lambda}{c\mu} = 0.$$

□

Notation 4.16.

1. Let $c \in \mathbb{N}$. For all $n \in \mathbb{N}$ and for $\tau > 0$ denote by p_n^τ the probability p_n^c in the $M|M|c$-model.
2. Denote by $p^{*,\tau}$ the limiting value of the sequence $(p_n^\tau)_{n\in\mathbb{N}_0}$.

Lemma 4.17. *Let $c \in \mathbb{N}$. It holds that $\lim_{\tau\downarrow 0} p^{*,\tau} = \frac{\lambda}{c\mu}$.*

Proof. It holds that

$$\lim_{\tau\downarrow 0} p^{*,\tau} = \lim_{\tau\downarrow 0} \frac{\lambda+\tau+c\mu-\sqrt{(\lambda+\tau+c\mu)^2-4c\mu\lambda}}{2c\mu} = \frac{\lambda}{2c\mu}.$$

□

Proposition 4.18. *Let $c \in \mathbb{N}$ and let $\tau_1, \tau_2 > 0$ such that $\tau_1 < \tau_2$. Then, it holds for all $n \in \mathbb{N}_0$, that $p_n^{\tau_1} - p_n^{\tau_2} > 0$. Moreover, $p^{*,\tau_1} - p^{*,\tau_2} > 0$.*

Proof. We will give a proof by induction on n.

Let $n = 0$. Then,

$$p_0^{\tau_1} - p_0^{\tau_2} = \frac{\lambda_0}{\lambda_0+\tau_1} - \frac{\lambda_0}{\lambda_0+\tau_2} \overset{(*)}{>} \frac{\lambda_0}{\lambda_0+\tau_2} - \frac{\lambda_0}{\lambda_0+\tau_2} = 0,$$

where (*) holds because $\tau_1 < \tau_2$. So, for $n = 0$ it holds that $p_0^{\tau_1} - p_0^{\tau_2} > 0$.

Now consider $n = 1$. Then it holds that

$$\begin{aligned} p_1^{\tau_1} - p_1^{\tau_2} &= \frac{\lambda_1}{\lambda_1+\tau_1+\mu_1(1-p_0^{\tau_1})} - \frac{\lambda_1}{\lambda_1+\tau_2+\mu_1(1-p_0^{\tau_2})} \\ &\overset{(**)}{>} \frac{\lambda_1}{\lambda_1+\tau_1+\mu_1(1-p_0^{\tau_2})} - \frac{\lambda_1}{\lambda_1+\tau_2+\mu_1(1-p_0^{\tau_2})} \\ &\overset{(***)}{>} \frac{\lambda_1}{\lambda_1+\tau_2+\mu_1(1-p_0^{\tau_2})} - \frac{\lambda_1}{\lambda_1+\tau_2+\mu_1(1-p_0^{\tau_2})} \\ &= 0, \end{aligned}$$

where (**) holds because $p_0^{\tau_1} > p_0^{\tau_2}$, and where (***) holds because $\tau_1 < \tau_2$. So, for $n = 1$ it holds that $p_1^{\tau_1} - p_1^{\tau_2} > 0$.

Suppose that for $n-1$ it holds that $p_{n-1}^{\tau_1} - p_{n-1}^{\tau_2} > 0$. Then consider n. Analogously to the case $n = 1$ it follows that $p_n^{\tau_1} - p_n^{\tau_2} > 0$.

Thus, for all $n \in \mathbb{N}_0$, it holds that $p_n^{\tau_1} - p_n^{\tau_2} \geq 0$.

It holds that $p^{*,\tau} = \lim_{n\to\infty} p_n^{\tau}$. Theorem 4.1 states that this limit exists. Then for $\tau_1 < \tau_2$ we have that

$$\begin{aligned} p^{*,\tau_1} - p^{*,\tau_2} &= \lim_{n\to\infty} p_n^{\tau_1} - \lim_{n\to\infty} p_n^{\tau_2} \\ &= \lim_{n\to\infty} (p_n^{\tau_1} - p_n^{\tau_2}) \\ &\geq \lim_{n\to\infty} (0) \\ &= 0. \end{aligned}$$

So, it holds for $\tau_1 < \tau_2$ that $p^{*,\tau_1} - p^{*,\tau_2} \geq 0$. □

5 Convergence of q_n for a general birth-death process

Consider a birth-death process $X = (X_t)_{t\geq 0}$ with birth parameters $\lambda_n > 0$, $n \in \mathbb{N}_0$, and death parameters $\mu_n > 0$, $n \in \mathbb{N}$ and $\mu_0 = 0$. Let $\tau > 0$ be given. Assume that $\alpha^*, \beta^* \geq 0$ both exist.

Recall from Eqn. (3) that

$$q_n = \frac{\mu_n}{\mu_n + \tau + \lambda_n(1 - q_{n+1})}. \tag{23}$$

Lemma 5.1. *It holds, that $q_n > 0$ for all $n \in \mathbb{N}$. Moreover, it holds that*

$$q_{n+1} = \frac{\mu_n + \tau + \lambda_n}{\lambda_n} - \frac{\mu_n}{\lambda_n q_n}, \tag{24}$$

for all $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. Then,

$$\begin{aligned} q_n &= \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}(1 - q_{n+1})} \\ &\geq \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}}. \end{aligned}$$

Hence,

$$\liminf_{n\to\infty} q_n \geq \lim_{n\to\infty} \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}} = \frac{1}{1 + \beta^* + \alpha^*} > 0,$$

since $\beta^* \geq 0$ and $\alpha^* \geq 0$ both exist. Hence, $q_n > 0$ for all $n \in \mathbb{N}$.

It holds that $q_0 = 0$, therefore we consider only $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary. Recall that $\lambda_n > 0$ for all $n \in \mathbb{N}_0$. Lemma 5.1 states that $q_n > 0$ for all $n \in \mathbb{N}$, therefore dividing by q_n is allowed. Rewriting Eqn. (23) gives

$$q_n\left(\mu_n + \tau + \lambda_n(1 - q_{n+1})\right) = \mu_n.$$

Then,

$$q_{n+1} = \frac{\mu_n + \tau + \lambda_n}{\lambda_n} - \frac{\mu_n}{\lambda_n q_n},$$

which is equal to Eqn. (24), and which is what we needed to show. □

Note that since $q_0 = 0$, we do not get such a formula for q_1. Therefore, to prove that the sequence $(q_n)_{n\in\mathbb{N}_0}$ has a limit under mild conditions is harder than for the sequence $(p_n)_{n\in\mathbb{N}_0}$. However, in the remainder of this chapter we will prove that for a birth-death process the sequence $(q_n)_{n\in\mathbb{N}_0}$ has a limit under mild conditions.

Theorem 5.2. *Suppose that* $\alpha^* = \lim_{n\to\infty} \frac{\lambda_n}{\mu_n} \geq 0$ *and* $\beta^* = \lim_{n\to\infty} \frac{\tau}{\mu_n} \geq 0$ *both exist. Let* $q_0 = 0$ *and let* q_n, $n \in \mathbb{N}$, *be given by Eqn. (23).*

1. *Suppose* $\alpha^* = 0$. *Then* $\lim_{n\to\infty} q_n = q^*$ *exists with* $q^* = \frac{1}{\beta^*+1}$.
2. *Suppose* $\alpha^* > 0$. *Then* $\lim_{n\to\infty} q_n = q^*$ *exists with* q^* *the smallest solution of the equation*

$$\alpha^* x^2 - (\alpha^* + \beta^* + 1)\, x + 1 = 0, \tag{25}$$

in other words

$$q^* = \frac{\alpha^* + \beta^* + 1 - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*}.$$

The proof for $\alpha^* > 0$ requires more work than the proof for $\alpha^* = 0$. Therefore we will first prove part 1 of the theorem.

Proof of Theorem 5.2 part 1. Let $\alpha^* = 0$. Let $\beta^* = \lim_{n\to\infty} \frac{\tau}{\mu_n} \geq 0$ exist.

Note that $\mu_n > 0$, $n \in \mathbb{N}$. Hence, we can rewrite Eqn. (23) by dividing by μ_n. This gives

$$\begin{aligned} q_n &= \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}(1 - q_{n+1})} \\ &\geq \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}}. \end{aligned}$$

Hence,

$$\liminf_{n\to\infty} q_n \geq \lim_{n\to\infty} \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}} = \frac{1}{1 + \beta^* + \alpha^*} = \frac{1}{1 + \beta^*},$$

since $\beta^* \geq 0$ and $\alpha^* = 0$.

Furthermore, it holds that

$$\begin{aligned} q_n &= \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}(1 - q_{n+1})} \\ &\leq \frac{1}{1 + \frac{\tau}{\mu_n}}. \end{aligned}$$

Hence,

$$\limsup_{n\to\infty} q_n \leq \lim_{n\to\infty} \frac{1}{1 + \frac{\tau}{\mu_n}} = \frac{1}{1 + \beta^*}.$$

Hence, $\lim_{n\to\infty} q_n = \frac{1}{1+\beta^*}$, which completes the proof of part 1 of Theorem 5.2. □

5.1 Part 2 of Theorem 5.2

The proof of the second part of Theorem 5.2 consists of multiple steps that will be formulated and proven in this subsection. Similarly to the Theorem 4.1, the part of Theorem 5.2 concerning $\beta^* > 0$, can be proven directly. For $\beta^* = 0$ and $0 < \alpha^* \leq 1$, we can also give a direct proof. However, for the case that $\beta^* = 0$ and $\alpha^* > 1$, we will need to introduce the concept of stochastic monotonicity, discussed in Section 5.1.1. Stochastic monotonicity will be used to compare X with a birth-death process with slightly different birth- and death-rates in order to prove that $\limsup_{n\to\infty} p_n < 1$. Subsection 5.1.2 combines all results given in this subsection to give the proof of the second part of the theorem.

The conditions of part 2 of Theorem 5.2 are assumed to hold throughout this section, i.e., $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} = \alpha^* > 0$ and $\lim_{n\to\infty} \frac{\tau}{\mu_n} = \beta^* \geq 0$ both exist.

Recall Definition 2.6.2, i.e., $q^+ := \limsup_{n\to\infty} q_n$ and $q^- := \liminf_{n\to\infty} q_n$.

Because $(q_n)_{n\in\mathbb{N}}$ is a sequence of probabilities, it holds that q^+ and q^- both exist with $q^+, q^- \in [0,1]$.

Lemma 5.3. *It holds that $q^+, q^- > 0$.*

Proof. For the birth-death process X it holds that $\lambda_n > 0$, $n \in \mathbb{N}_0$ and $\mu_n > 0$, $n \in \mathbb{N}$. We assumed that $\alpha^* > 0$ and $\beta^* \geq 0$ both exist.

Analogously to the proof of Lemma 5.1 we get, for all $n \in \mathbb{N}$,

$$\begin{aligned} q_n &= \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}(1 - q_{n+1})} \\ &\geq \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}} > 0. \end{aligned}$$

Further,

$$q^+ \geq q^- \liminf_{n\to\infty} q_n \geq \lim_{n\to\infty} \frac{1}{1 + \frac{\tau}{\mu_n} + \frac{\lambda_n}{\mu_n}} = \frac{1}{1 + \beta^* + \alpha^*} > 0.$$

This completes the proof. □

Lemma 5.4. *Let $\alpha^* > 0$ and $\beta^* \geq 0$. It holds that both q^+ and q^- are solutions to the quadratic equation*

$$\alpha^* x^2 - (\alpha^* + \beta^* + 1)\, x + 1 = 0, \tag{26}$$

in other words,

$$q^+, q^- \in \left\{ \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*}, \frac{(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*} \right\},$$

Proof. Because $(q_n)_{n\in\mathbb{N}_0}$ is a sequence of probabilities, it holds that q^+ and q^- both exist with $q^+, q^- \in [0,1]$. Hence, there exist a subsequence $(q_{n_k})_k \subseteq (q_n)_{n\in\mathbb{N}_0}$ such that $q_{n_k} \to q^+$ as $k \to \infty$ and there exists a subsequence $(q_{m_l})_l \subseteq (q_n)_{n\in\mathbb{N}_0}$ such that $q_{m_l} \to q^-$ as $l \to \infty$.

By Lemma 5.3 $q^+, q^- > 0$. The proof of this lemma even gives that $q_n > 0$ for all $n \in \mathbb{N}$. Hence, dividing by q^+, q^- and q_n, $n \in \mathbb{N}$, is allowed.

Recall Eqn. (24). Therefore,

$$q_{n_k+1} = \frac{\mu_{n_k} + \tau + \lambda_{n_k}}{\lambda_{n_k}} - \frac{\mu_{n_k}}{\lambda_{n_k} q_{n_k}}. \tag{27}$$

By Eqn. (27) $\lim_{k\to\infty} q_{n_k+1}$ exists, since $\lim_{k\to\infty} q_{n_k} = q^+$ exists.

Furthermore, it holds that

$$\lim_{k\to\infty} q_{k+1} \leq \limsup_{k\to\infty} \sup_{j\geq n_k+1} q_j \leq \limsup_{n\to\infty} q_n = q^+.$$

So,

$$\begin{aligned}
q^+ &\geq \lim_{k\to\infty} \left[\frac{\mu_{n_k} + \tau + \lambda_{n_k}}{\lambda_{n_k}} - \frac{\mu_{n_k}}{\lambda_{n_k} q_{n_k}} \right] \\
&= \lim_{k\to\infty} \frac{\mu_{n_k}}{\lambda_{n_k}} + \lim_{k\to\infty} \frac{\tau}{\lambda_{n_k}} + 1 - \lim_{k\to\infty} \frac{\mu_{n_k}}{\lambda_{n_k}} \cdot \frac{1}{q_{n_k}} \\
&= \frac{1}{\alpha^*} + \lim_{k\to\infty} \frac{\tau}{\mu_{n_k}} \cdot \frac{\mu_{n_k}}{\lambda_{n_k}} + 1 - \frac{1}{\alpha^*} \cdot \lim_{k\to\infty} \frac{1}{q_{n_k}} \\
&= \frac{1}{\alpha^*} + \frac{\beta^*}{\alpha^*} + 1 - \frac{1}{\alpha^*} \cdot \frac{1}{q^+}.
\end{aligned}$$

Multiplying the above equation by α^* and q^+, we get

$$\alpha^* \left(q^+\right)^2 - \left(\alpha^* + \beta^* + 1\right) q^+ + 1 \geq 0. \tag{28}$$

By virtue of Eqn. (23) for $n = n_k - 1$ and dividing by μ_{n_k-1} we get for all $n_k \in \mathbb{N}_{\geq 2}$ that

$$q_{n_k-1} = \frac{1}{1 + \frac{\tau}{\mu_{n_k-1}} + \frac{\lambda_{n_k-1}}{\mu_{n_k-1}}(1 - q_{n_k})}. \tag{29}$$

Note that $\lim_{k\to\infty} q_{n_k-1}$ exists, since $\lim_{k\to\infty} q_{n_k} = q^+$ exists. It holds that

$$\lim_{k\to\infty} q_{n_k-1} \leq \limsup_{k\to\infty} \sup_{j\geq n_k-1} q_j \leq \limsup_{n\to\infty} q_n = q^+.$$

Since $\lim_{k\to\infty} q_{n_k-1}$ exists, Eqn. (29) yields $\lim_{k\to\infty} q_{n_k-1} = \frac{1}{1+\beta^*+\alpha^*(1-q^+)}$, and so

$$q^+ \geq \frac{1}{1 + \beta^* + \alpha^*(1 - q^+)}.$$

This yields,

$$\alpha^* \left(q^+\right)^2 - \left(\alpha^* + \beta^* + 1\right) q^+ + 1 \leq 0. \tag{30}$$

Combining Eqn. (28) and Eqn. (30) yields that $q+$ is a solution of the quadratic equation

$$\alpha^* x^2 - \left(\alpha^* + \beta^* + 1\right) x + 1 = 0. \tag{31}$$

Solving the quadratic expression gives

$$q^+ \in \left\{ \frac{\left(\alpha^* + \beta^* + 1\right) - \sqrt{\left(\alpha^* + \beta^* + 1\right)^2 - 4\alpha^*}}{2\alpha^*}, \frac{\left(\alpha^* + \beta^* + 1\right) + \sqrt{\left(\alpha^* + \beta^* + 1\right)^2 - 4\alpha^*}}{2\alpha^*} \right\}.$$

Analogously (only the sign of the inequalities change because the limit inferior is considered instead of the limit superior), it yields that q^- is also a solution of Eqn. (31). This concludes the proof of the lemma. □

Remark 5.5. Since q^+ and q^- both exist, it necessarily holds that both solutions of the quadratic expression are real. This can also be easily verified by noting that $(\alpha^* + \beta^* + 1)^2 - 4\alpha^* = (\beta^* + 1 - \alpha^*)^2 + 4\alpha^*\beta^* \geq 0$.

Since $\beta^* \geq 0$, we only need to consider two situations, namely $\beta^* = 0$ and $\beta^* > 0$.

First consider the case where $\beta^* > 0$. If it holds that

$$\frac{(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*} > 1 \geq \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*} \geq 0, \tag{32}$$

then, necessarily

$$q^+ = q^- = \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*},$$

since $q^+, q^- \in [0, 1]$.

Lemma 5.6. *Let $\beta^* > 0$. Then, Eqn. (32) holds. Moreover,*

$$q^+ = q^- = \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*},$$

which is the smallest solution of Eqn. (25).

Proof. The proof of this lemma consists of two parts.

- We will first prove that $\frac{(\alpha^*+\beta^*+1)+\sqrt{(\alpha^*+\beta^*+1)^2-4\alpha^*}}{2\alpha^*} > 1$. To show this, it is sufficient to prove that $(\alpha^* + \beta^* + 1) + \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} > 2\alpha^*$. Suppose that this is not true. Then,

  $$\sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \leq 2\alpha^* - (\alpha^* + \beta^* + 1) = \alpha^* - (\beta^* + 1).$$

  Taking squares yields

  $$(\alpha^* + (\beta^* + 1))^2 - 4\alpha^* \leq (\alpha^* - (\beta^* + 1))^2.$$

  This implies that

  $$(\alpha^*)^2 + 2\alpha^* (\beta^* + 1) + (\beta^* + 1)^2 - 4\alpha^* \leq (\alpha^*)^2 - 2\alpha^* (\beta^* + 1) + (\beta^* + 1)^2,$$

  in other words,

  $$0 \geq 4\alpha^* (\beta^* + 1) - 4\alpha^* = 4\alpha^*\beta^*.$$

  Since $\alpha^* > 0$, it follows that

  $$\beta^* \leq 0.$$

  This contradicts the assumption that $\beta^* > 0$. Therefore, the first part of Eqn. (32) holds.

- Now we will prove that

  $$0 \leq \frac{(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*}}{2\alpha^*} \leq 1.$$

To show this, it is sufficient to prove that

$$0 \leq (\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \leq 2\alpha^*.$$

Since $\alpha^*, \beta^* > 0$, clearly $(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \geq 0$.

It holds that

$$(\alpha^* + \beta^* + 1)^2 - 4\alpha^* = (\beta^* + 1 - \alpha^*)^2 + 4\alpha^*\beta^* \geq 0.$$

Thus, to prove that $(\alpha^* + \beta^* + 1) - \sqrt{(\alpha^* + \beta^* + 1)^2 - 4\alpha^*} \leq 2\alpha^*$, it is sufficient to prove that

$$(\alpha^* + \beta^* + 1) - \sqrt{(\beta^* + 1 - \alpha^*)^2 + 4\alpha^*\beta^*} \leq 2\alpha^*,$$

or

$$\beta^* + 1 - \alpha^* \leq \sqrt{(\beta^* + 1 - \alpha^*)^2 + 4\alpha^*\beta^*},$$

which is evidently true.

Hence, the remainder of Eqn. (32) also holds true.

Combination of the two parts yields the result. □

Now it remains to consider $\beta^* = 0$. Then, Eqn. (26) reduces to

$$\alpha^* x^2 - (\alpha^* + 1)\, x + 1 = 0. \tag{33}$$

Lemma 5.4 then gives,

$$q^+,\ q^- \in \left\{ \frac{(\alpha^* + 1) - \sqrt{(\alpha^* - 1)^2}}{2\alpha^*}, \frac{(\alpha^* + 1) + \sqrt{(\alpha^* - 1)^2}}{2\alpha^*} \right\}. \tag{34}$$

We consider three situations, namely $\alpha^* = 1$, $\alpha^* > 1$ and $0 < \alpha^* < 1$. These three situations cover all possible situations, because we assumed that $\alpha^* > 0$. These situations will be considered in the next lemmas.

Lemma 5.7. *Let $\beta^* = 0$ and let $0 < \alpha^* \leq 1$. Then $q^+ = q^- = 1$, which is the smallest solution of Eqn. (33).*

Proof. Suppose that $0 < \alpha \leq 1$. Then Eqn. (34) reduces to

$$\begin{aligned} q^+,\ q^- &\in \left\{ \frac{\alpha^* + 1 - \sqrt{(\alpha^* - 1)^2}}{2\alpha^*}, \frac{\alpha^* + 1 + \sqrt{(\alpha^* - 1)^2}}{2\alpha^*} \right\} \\ &= \left\{ \frac{\alpha^* + 1 - (1 - \alpha^*)}{2\alpha^*}, \frac{\alpha^* + 1 + (1 - \alpha^*)}{2\alpha^*} \right\} \\ &= \left\{ \frac{2\alpha^*}{2\alpha^*}, \frac{2}{2\alpha^*} \right\} \\ &= \left\{ 1, \frac{1}{\alpha^*} \right\}. \end{aligned}$$

Since $0 < \alpha^* \leq 1$, it holds that $\frac{1}{\alpha^*} \geq 1$. Since $q^+,\ q^- \in [0,1]$ it follows that $q^+ = q^- = 1$ and hence q^* exists and is given by $q^* = 1$, which is the smallest solution of Eqn. (33). ☐

Lemma 5.8. *Let $\beta^* = 0$ and let $\alpha^* > 1$. Then $q^+,\ q^- \in \left\{\frac{1}{\alpha^*}, 1\right\}$.*

Proof. Suppose that $\alpha^* > 1$. Then, analogously to the proof of Lemma 5.7, it follows that

$$q^+,\ q^- \in \left\{\frac{1}{\alpha^*}, 1\right\}.$$

However, because $\alpha^* > 1$ this does not give a unique solution. ☐

We would like to prove that $q^+ = q^- = \frac{1}{\alpha^*}$. The method we will use is to define a stochastically smaller system which will be a birth-death process Y with birth-rates λ'_n and death-rates μ'_n, and no extinction rate. We will derive the corresponding $\limsup_{n\to\infty} q'_n$ and prove that the limit superior is strictly smaller that 1. As a consequence, the $q^+ = \limsup_{n\to\infty} q_n < 1$ for the process X, and so $q^+ = q^- = \frac{1}{\alpha^*}$.

5.1.1 Stochastic monotonicity

This subsection will introduce the concept of stochastic monotonicity that will be needed to prove Theorem 5.2 for the case that $\beta^* > 0$ and $\alpha^* > 1$. This subsection is based on [5].

Let $S := \mathbb{N}_0 \cup \{\infty\}$.

Notation 5.9. Let $S_i := \{i, i+1, \ldots\} \cup \{\infty\}$, for all $n \in \mathbb{N}_0$.

Definition 5.10. Let p and q be two probability distributions on the state space S with the following property:

$$\sum_{i \geq I} p_i \leq \sum_{i \geq I} q_i \text{ for all } I \in S.$$

Then p is said to be stochastically not larger than q and we denote this by $p \preceq q$.

Lemma 5.11. *Let $f\colon S \to \mathbb{R}$ be non-decreasing. Let $p \preceq q$. Then it holds that*

$$pf = \sum_{i \in S} p_i f(i) \leq \sum_{i \in S} q_i f(i) = qf.$$

Proof. For notational purposes, define $f(-1) = 0$. For all $i \in S$ it holds that

$$f(i) = \sum_{j \in S} f(j) \mathbb{1}_{\{j\}}(i).$$

Since f is a non-decreasing function, it holds that

$$f(\infty) \geq \lim_{n\to\infty} f(n) =: f^*.$$

Recall $S_i := \{i, i+1, \ldots\} \cup \{\infty\}$ for all $i \in \mathbb{N}_0$ and $S_\infty := \{\infty\}$. Note that $S_0 = S$.

For all $i \in \mathbb{N}_0$ it holds that $\mathbb{1}_{S_\infty}(i) = 0$ and

$$\begin{aligned} f(i) &= f(0) + \sum_{j=1}^{i} [f(j) - f(j-1)] \\ &= f(0) - f(-1) + \sum_{j=1}^{i} [f(j) - f(j-1)] \\ &= \sum_{j=0}^{i} [f(j) - f(j-1)] \\ &= \sum_{j \in \mathbb{N}_0} [f(j) - f(j-1)] \, \mathbb{1}_{S_j}(i) + (f(\infty) - f^*) \, \mathbb{1}_{S_\infty}(i). \end{aligned}$$

For $i = \infty$ it holds that

$$\begin{aligned} f(\infty) &= f^* + f(\infty) - f^* \\ &= \sum_{j \in \mathbb{N}_0} [f(j) - f(j-1)] + f(\infty) - f^*, \end{aligned}$$

since it holds that $f^* = \lim_{n\to\infty} f(n) = \lim_{n\to\infty} \left(\sum_{j=0}^{n} [f(j) - f(j-1)] \right) = \sum_{j \in \mathbb{N}_0} [f(j) - f(j-1)]$. Therefore, we can write for all $i \in S$,

$$f(i) = \sum_{j \in \mathbb{N}_0} [f(j) - f(j-1)] \, \mathbb{1}_{S_j}(i) + (f(\infty) - f^*) \, \mathbb{1}_{S_\infty}(i).$$

Then,

$$\begin{aligned} pf &= \sum_{j \in S} p_j f(j) \\ &= \sum_{j \in S} \left[p_j \left(\sum_{k \in \mathbb{N}_0} [f(k) - f(k-1)] \, \mathbb{1}_{S_k}(j) + (f(\infty) - f^*) \, \mathbb{1}_{S_\infty}(j) \right) \right] \\ &= \sum_{j \in S} \left[p_j \left(\sum_{k \in \mathbb{N}_0} [f(k) - f(k-1)] \, \mathbb{1}_{S_k}(j) \right) \right] + \sum_{j \in S} [p_j \, (f(\infty) - f^*) \, \mathbb{1}_{S_\infty}(j)] \\ &= \sum_{k \in \mathbb{N}_0} \left[(f(k) - f(k-1)) \cdot \sum_{j \in S_k} p_j \right] + p_\infty \cdot (f(\infty) - f^*) \\ &\overset{(*)}{\leq} \sum_{k \in \mathbb{N}_0} \left[(f(k) - f(k-1)) \cdot \sum_{j \in S_k} q_j \right] + q_\infty \cdot (f(\infty) - f^*) \\ &= \sum_{j \in S} q_j f(j) \\ &= qf, \end{aligned}$$

where (*) holds since $\sum_{j \in S_k} p_j \leq \sum_{j \in S_k} q_j$ and $p_\infty \leq q_\infty$. Hence, this concludes the proof of the lemma. $\square$

Consider a Markov chain on the state space S and transition matrix P. Write $p_{i,\cdot}$ for the probability distribution $\{p_{i,0},\ p_{i,1}, \dots,\ p_{i,N}\}$.

Definition 5.12. We call the transition matrix P stochastically monotonic if

$$p_{i,\cdot} \preceq p_{i+1,\cdot} \text{ for all } i \in S.$$

Definition 5.13. Define $Pf\colon S \to \mathbb{R}$ as the function given by $Pf(i) = p_{i,\cdot}f$ for $i \in S$.

The following corollary of Lemma 5.11 holds.

Corollary 5.14. *Let $f\colon S \to \mathbb{R}$ be non-decreasing and let P be stochastically monotonic. Then it holds that $p_{i,\cdot}f \le p_{i+1,\cdot}f$ for all $i \in S$, hence Pf is a non-decreasing function.*

Let Q be a transition matrix with the property that $P \preceq Q$, i.e., $p_{i,\cdot} \preceq q_{i,\cdot}$ for all $i \in S$.

Theorem 5.15. *If $P \preceq Q$ and P or Q is stochastically monotonic, then*

$$P^n \preceq Q^n \text{ for all } n \in \mathbb{N}.$$

Moreover, if P (Q) is stochastically monotonic, then P^n (Q^n) is stochastically monotonic, $n \in \mathbb{N}$.

Proof. The proof of the theorem will be split into two parts. First we will prove that Q^n, $n \in \mathbb{N}$, is stochastically monotonic, if Q is stochastically monotonic. Then we will prove that $P^n \preceq Q^n$ for all $n \in \mathbb{N}$, if $P \preceq Q$ and P or Q is stochastically monotonic.

- Let Q be stochastically monotonic. We will prove by induction on n that Q^n, $n \in \mathbb{N}$ is stochastically monotonic. For $n = 1$ the statement holds. Therefore, assume that Q^n, $n \in \mathbb{N}$, is stochastically monotonic and consider $n+1$. To prove that Q^{n+1} is stochastically monotonic, it is sufficient to prove for all $i \in S$ that $q^{n+1}_{i,\cdot} \preceq q^{n+1}_{i+1,\cdot}$.

  Let $i \in S$. It holds, that

$$\begin{aligned}\sum_{j \ge I} q^{n+1}_{i,j} &= \sum_{j \ge I} \left(\sum_{k \in S} q^n_{i,k} \cdot q_{k,j} \right) \\ &= \sum_{k \in S} \left(q^n_{i,k} \sum_{j \ge I} q_{k,j} \right),\end{aligned}$$

  for all $I \in S$.

  Since, $\sum_{j \ge I} q_{k,j}$ is a non-decreasing function in k by monotonicity of Q and since Q^n is stochastically monotonic by assumption, we can apply Corollary 5.14. Then, for all $I \in S$,

$$\begin{aligned}\sum_{j \ge I} q^{n+1}_{i,j} &\le \sum_{k \in S} \left(q^n_{i+1,k} \sum_{j \ge I} q_{k,j} \right) \\ &= \sum_{j \ge I} \left(\sum_{k \in S} q^n_{i+1,k} \cdot q_{k,j} \right) \\ &= \sum_{j \ge I} q^{n+1}_{i+1,j}.\end{aligned}$$

  This proves that $q^{n+1}_{i,\cdot} \preceq q^{n+1}_{i+1,\cdot}$ for all $i \in S$. Thus, Q^{n+1} is stochastically monotonic.

- Now, assume that $P \preceq Q$, Q is stochastically monotonic. We will prove by induction on n that $P^n \preceq Q^n$. For $n = 1$ the statement holds, since we assume that $P \preceq Q$. Therefore, assume that $P^n \preceq Q^n$, $n \in \mathbb{N}$, and consider $n+1$. To prove that $P^{n+1} \preceq Q^{n+1}$, we need to prove for all $i \in S$ that $p^{n+1}_{i,\cdot} \preceq q^{n+1}_{i,\cdot}$.

  Let $i \in S$. For all $I \in S$ it holds that

$$\begin{aligned}\sum_{j\geq I} p^{n+1}_{i,j} &= \sum_{j \geq I}\left(\sum_{k\in S} p^n_{i,k}\cdot p_{k,j}\right)\\ &= \sum_{k\in S}\left(p^n_{i,k}\sum_{j\geq I} p_{k,j}\right)\\ &\overset{(*)}{\leq} \sum_{k\in S}\left(p^n_{i,k}\sum_{j\geq I} q_{k,j}\right)\\ &\overset{(**)}{\leq} \sum_{k\in S}\left(q^n_{i,k}\sum_{j\geq I} q_{k,j}\right)\\ &= \sum_{j\geq I} q^{n+1}_{i,j},\end{aligned}$$

  where (*) holds since $P \preceq Q$, and where (**) holds since $\sum_{j\geq I} q_{k,j}$ is a non-decreasing function in k by monotonicity of Q, $P^n \preceq Q^n$ by the induction hypothesis. Thus, $P^{n+1} \preceq Q^{n+1}$, which concludes the proof of the theorem.

□

5.1.2 Application of stochastic monotonicity

Recall that $\alpha^* > 1$. So, there exists an $\epsilon > 0$ such that $\alpha^* - \epsilon > 1$. Take such an ϵ. Since $\lim_{n\to\infty}\frac{\lambda_n}{\mu_n} = \alpha^*$ exists, there exists an $N \in \mathbb{N}_0$ such that for all $n \geq N$ it holds that $\frac{\lambda_n}{\mu_n} \geq \alpha^* - \epsilon$. Take such an $N \in \mathbb{N}_0$.

Now we want to construct a stochastic process Y and compare it with the process X using stochastic monotonicity. Therefore, let $p := \frac{\alpha^*-\epsilon}{1+\alpha^*-\epsilon}$. Then, $p > \frac{1}{2}$. The goal is to show that $q_n \leq \frac{1-p}{p} < 1$ for all $n \geq N$. The construction is based on stochastic monotonicity.

Consider the following birth-death process $Y = (Y_t)_{t\geq 0}$ on state space $S = \mathbb{N}_0 \cup \{\infty\}$. Let the birth-rates be given by $\lambda_n = p > \frac{1}{2}$, $n \in \mathbb{N}_0$, and the death-rates be given by $\mu_n = 1-p$, $n \in \mathbb{N}$. Assume that the states 0 and ∞ are absorbing states, i.e., $\mathbb{P}(Y_t = 0 \mid Y_0 = 0) = 1$ for all $t \geq 0$. Notice that from state $n < \infty$, the state ∞ is never reached by the process Y. Furthermore, consider the birth-death process X (as defined before). The process X jumps after an exponentially distributed time T to the exit state ∞.

Recall Notation 5.9.

Now, restrict both X and Y to the state space S_{N-1}.

The goal is to show that $q_N \leq \frac{1-p}{p}$. For computing this probability, it suffices to consider the corresponding jump chain. P and Q are the transition matrices of the jump chains associated with X and Y, respectively, restricted to S'.

The transition matrix $Q = (Q_{i,j})_{i,j \in S'}$ is given as follows:

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1-p & 0 & p & 0 & 0 & \dots & 0 \\ 0 & 1-p & 0 & p & 0 & \dots & 0 \\ 0 & 0 & 1-p & 0 & p & \dots & 0 \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \tag{35}$$

and let the transition matrix $P = (P_{i,j})_{i,j \in S'}$ be given as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \frac{\mu_N}{\mu_N+\tau+\lambda_N} & 0 & \frac{\lambda_N}{\mu_N+\tau+\lambda_N} & 0 & 0 & \dots & \frac{\tau}{\mu_N+\tau+\lambda_N} \\ 0 & \frac{\mu_{N+1}}{\mu_{N+1}+\tau+\lambda_{N+1}} & 0 & \frac{\lambda_{N+1}}{\mu_{N+1}+\tau+\lambda_{N+1}} & 0 & \dots & \frac{\tau}{\mu_{N+1}+\tau+\lambda_{N+1}} \\ 0 & 0 & \frac{\mu_{N+2}}{\mu_{N+2}+\tau+\lambda_{N+2}} & 0 & \frac{\lambda_{N+2}}{\mu_{N+2}+\tau+\lambda_{N+2}} & \dots & \frac{\tau}{\mu_{N+2}+\tau+\lambda_{N+2}} \\ \vdots & & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{36}$$

Lemma 5.16. *Q is stochastically monotonic.*

Proof. Consider $i = N-1$.

- Let $I = N-1$. Then it holds that $\sum_{j \geq I} Q_{i,j} = 1 = \sum_{j \geq I} Q_{i+1,j}$.
- Let $I \in \{N, N+1\}$. Then it holds that $\sum_{j \geq I} Q_{i,j} = 0 < p = \sum_{j \geq I} Q_{i+1,j}$.
- Let $I \geq N+2$. Then it holds that $\sum_{j \geq I} Q_{i,j} = 0 \leq \sum_{j \geq I} Q_{i+1,j}$.

So, for all $I \in S$ it holds that $Q_{N-1,\cdot} \preceq Q_{N,\cdot}$.

Consider $i \geq N$.

- Let $I \in \{N-1, \dots, i-1\}$. Then it holds that $\sum_{j \geq I} Q_{i,j} = 1 = \sum_{j \geq I} Q_{i+1,j}$.
- Let $I = i$. Then it holds that $\sum_{j \geq I} Q_{i,j} = p \leq 1 = \sum_{j \geq I} Q_{i+1,j}$.
- Let $I = i+1$. Then it holds that $\sum_{j \geq I} Q_{i,j} = p = \sum_{j \geq I} Q_{i+1,j}$.
- Let $I \geq i+2$. Then it holds that $\sum_{j \geq I} Q_{i,j} = 0 \leq \sum_{j \geq I} Q_{i+1,j}$.

So, for all $I \in S$ it holds that $Q_{i,\cdot} \preceq Q_{i+1,\cdot}$.

Thus, for all $i \geq N$ it holds that $Q_{i,\cdot} \preceq Q_{i+1,\cdot}$. Hence, Q is stochastically monotone. □

Lemma 5.17. *$Q \preceq P$.*

Proof. It holds that $Q_{N-1,\cdot} = P_{N-1,\cdot}$ and that $Q_{\infty,\cdot} = P_{\infty,\cdot}$. Hence, we have immediately that $Q_{N-1,\cdot} \preceq P_{N-1,\cdot}$ and that $Q_{\infty,\cdot} \preceq P_{\infty,\cdot}$.

Now, let $i \in S \backslash \{N-1, \infty\}$.

- Let $I \in \{N-1, \ldots, i-1\}$. Then, $\sum_{j \geq I} Q_{i,j} = 1 = \sum_{j \geq I} P_{i,j}$.
- Let $I \in \{i, i+1\}$. Then it holds that $\sum_{j \geq I} Q_{i,j} = p$ and that $\sum_{j \geq I} P_{i,j} = \frac{\lambda_i + \tau}{\mu_i + \tau + \lambda_i}$. Since $i \geq N-1$ we have that $\lambda_i \geq \mu_i (\alpha^* - \epsilon)$. Then, since $\tau > 0$, it holds that

  $$\lambda_i + \tau > \mu_i (\alpha^* - \epsilon).$$

  Adding $(\lambda_i + \tau)(\alpha^* - \epsilon)$ on both sides of this equation yields

  $$(\lambda_i + \tau)(1 + \alpha^* - \epsilon) > (\mu_i + \lambda_i + \tau)(\alpha^* - \epsilon).$$

  Since $\tau > 0$ it holds that $\mu_i + \tau + \lambda_i > 0$, therefore dividing both sides of the equation by $\mu_i + \tau + \lambda_i$ is allowed. Moreover, it holds that $\alpha^* - \epsilon > 0$, hence $1 + \alpha^* - \epsilon > 0$ and thus dividing both sides of the equation by $1 + \alpha^* - \epsilon$ is also allowed. This yields

  $$\frac{\lambda_i + \tau}{\mu_i + \tau + \lambda_i} > \frac{\alpha^* - \epsilon}{1 + \alpha^* - \epsilon} = p.$$

  Hence, it holds that $\sum_{j \geq I} Q_{i,j} \leq \sum_{j \geq I} P_{i,j}$.
- Let $I \in S \backslash \{N-1, \ldots, i, i+1\}$. Then, $\sum_{j \geq I} Q_{i,j} = 0 < \frac{\lambda_i + \tau}{\mu_i + \tau + \lambda_i} = \sum_{j \geq I} P_{i,j}$.

So, for all $i \in S$ it holds that $Q_{i,\cdot} \preceq P_{i,\cdot}$. Hence, Q is stochastically not larger than P. □

Definition 5.18. Let $(X_n)_{n \in \mathbb{N}_0}$ be a Markov chain on $\mathbb{N}_0$.

1. Let $f_{i,j}^{(=n)}$ be the probability that the Markov chain X, starting in state i, reaches state j for the first time at time $n \in \mathbb{N}$, i.e.,

   $$f_{i,j}^{(=n)} := \mathbb{P}\left(X_n = j, X_s \neq j \text{ for all } s = 1, 2, \ldots, n-1 \mid X_0 = i\right).$$

2. Let $f_{i,j}^{(n)}$ be the probability that the Markov chain X, starting in state i, has reached state j before or at time $n \in \mathbb{N}$, i.e.,

   $$f_{i,j}^{(n)} := \mathbb{P}\left(\exists t \in \{1, 2, \ldots, n\} \colon X_t = j \mid X_0 = i\right).$$

3. Let $f_{i,j}$ be the probability that the Markov chain X, starting in state i, reaches state j, i.e.,

   $$f_{i,j} := \mathbb{P}\left(\cup_{t=1}^{\infty} \{X_t = j\} \mid X_0 = i\right).$$

   We will call $f_{i,j}$ the first entrance probability if j, starting in i.

Remark 5.19. It holds that

$$f_{i,j} = \sum_{n=1}^{\infty} f_{i,j}^{(=n)},$$

$$f_{i,j} = \lim_{n\to\infty} f_{i,j}^{(n)},$$

and that

$$f_{i,j}^{(n)} = \sum_{k=1}^{n} f_{i,j}^{(=k)}.$$

Hence, both Definition 5.18.1 and Definition 5.18.2 can be used to describe the absorption probability $f_{i,j}$ as defined in Definition 5.18.3. We will use Definition 5.18.2 later, however the proof of Theorem 5.20 uses Definition 5.18.1.

Recall that q_n is that probability that the Markov chain, starting in state n, reaches state $n-1$ before time T. For the extended jump Markov chain with killed state ∞, it holds that $q_n = f_{n,n-1}$ for all $n \in \mathbb{N}_0$.

We recall that S_N is an infinite state space. Therefore the following theorem holds true for the models we consider.

Theorem 5.20. *Let $j \in S$ be given. Let $S' = \{i \in S \mid f_{i,j} > 0\}$. Then it holds that $\{f_{i,j}\}_{i\in S'}$ is the minimal, non-negative solution of the equations*

$$g_i = p_{i,j} + \sum_{k\neq j} p_{i,k} g_k, \text{ for all } i \in S'. \tag{37}$$

This theorem is proven as Theorem 1.13 in [3].

For the next lemma, we rename the states of the state space $S_N = \{N-1, N, \ldots\} \cup \{\infty\}$ by $S_0 = \{0, 1, \ldots\} \cup \{\infty\}$. Therefore, we are now interested in the probability $f_{1,0}$.

Lemma 5.21. *Consider the Markov chain with transition matrix Q as given in Eqn. (35). Then,*

$$f_{1,0} = \frac{1-p}{p}.$$

Proof. For the Markov chain with transition matrix Q, as given in Eqn. (35), with $j = 0$ it holds that

$$S' := \{i \in S \mid f_{i,j} > 0\} = \mathbb{N}_0.$$

It holds that $f_{\infty,0} = 0$ and $f_{i,0} = pf_{i+1,0} + (1-p)f_{i-1,0}$ for $i \in S'\backslash\{0\}$. Note that $f_{0,0} \neq 0$ necessarily. However, for convenience of notation, we write $f_{0,0} = 1$.

For all $i \in S'$ we can write $f_{i,0} = pf_{i,0} + (1-p)f_{i,0}$. Then,

$$p\left(f_{i,0} - f_{i+1,0}\right) = (1-p)\left(f_{i-1,0} - f_{i,0}\right).$$

Define $g(i) := f_{i,0} - f_{i+1,0}$ for all $i \in S'$. Then,

$$g(i) = \frac{1-p}{p} g(i-1),$$

by iteration,

$$g(i) = \left(\frac{1-p}{p}\right)^i g(0) = \left(\frac{1-p}{p}\right)^i \cdot (f_{0,0} - f_{1,0}) = \left(\frac{1-p}{p}\right)^i \cdot (1 - f_{1,0}),$$

since $f_{0,0} = 1$.

Assume that $\lim_{M\to\infty} f_{M,0} = 0$. Then, for all $i \in S'$,

$$\begin{aligned}
f_{i,0} &= \lim_{M\to\infty} [(f_{i,0} - f_{i+1,0}) + (f_{i+1,0} - f_{i+2,0}) + \ldots (f_{M-1,0} - f_{M,0}) + f_{M,0}] \qquad (38)\\
&= \lim_{M\to\infty} \left[\sum_{k=i}^{M-1} [g(k)] + f_{M,0}\right]\\
&= (1 - f_{1,0}) \cdot \left(\lim_{M\to\infty} \left[\sum_{k=i}^{M-1} \left(\frac{1-p}{p}\right)^k\right] + \lim_{M\to\infty} f_{M,0}\right)\\
&= (1 - f_{1,0}) \cdot \left(\lim_{M\to\infty} \left[\sum_{k=i}^{M-1} \left(\frac{1-p}{p}\right)^k\right] + 0\right)\\
&= (1 - f_{1,0}) \cdot \left[\lim_{M\to\infty} \left[\sum_{k=0}^{M-1} \left(\frac{1-p}{p}\right)^k\right] - \sum_{k=0}^{i-1} \left(\frac{1-p}{p}\right)^k\right]\\
&= (1 - f_{1,0}) \cdot \left[\lim_{M\to\infty} \frac{1 - \left(\frac{1-p}{p}\right)^M}{1 - \frac{1-p}{p}} - \frac{1 - \left(\frac{1-p}{p}\right)^i}{1 - \frac{1-p}{p}}\right]\\
&= (1 - f_{1,0}) \cdot \left[\lim_{M\to\infty} \frac{\left(\frac{1-p}{p}\right)^i - \left(\frac{1-p}{p}\right)^M}{1 - \frac{1-p}{p}}\right]\\
&= (1 - f_{1,0}) \cdot \left[\frac{\left(\frac{1-p}{p}\right)^i - 0}{1 - \frac{1-p}{p}}\right],
\end{aligned}$$

where we use that $\frac{1-p}{p} < 1$. Hence, for all $i \in S'$, we have that

$$f_{i,0} = (1 - f_{1,0}) \cdot \frac{\left(\frac{1-p}{p}\right)^i}{1 - \frac{1-p}{p}}, \qquad (39)$$

when we assume that $\lim_{M\to\infty} f_{M,0} = 0$. We can check that $f_{i,0}$, $i \in S'$, in Eqn. (39) are a solution to Eqn. (37). It remains to show that Eqn. (39) is the minimal solution.

However, we do not yet know that $\lim_{M\to\infty} f_{M,0} = 0$, but we do know that $f_{M,0} \geq 0$, for all $M \in S_0$. Then, the first equality of Eqn. (38) becomes an $\geq$, hence

$$f_{i,0} \geq (1 - f_{1,0}) \cdot \frac{\left(\frac{1-p}{p}\right)^i}{1 - \frac{1-p}{p}}. \qquad (40)$$

Substitution of $i = 2$ in Eqn. (40) gives that

$$f_{2,0} \geq (1 - f_{1,0}) \cdot \frac{\left(\frac{1-p}{p}\right)^2}{1 - \frac{1-p}{p}}.$$

Then, using that $f_{1,0} = 1 - p + pf_{2,0}$, we get

$$\begin{aligned}
f_{1,0} &\geq 1 - p + p \cdot (1 - f_{1,0}) \cdot \frac{\left(\frac{1-p}{p}\right)^2}{1 - \frac{1-p}{p}} \\
&\iff f_{1,0} \geq 1 - p + p^2 \cdot (1 - f_{1,0}) \cdot \frac{\left(\frac{1-p}{p}\right)^2}{2p - 1} \\
&\iff f_{1,0} \geq 1 - p + (1 - f_{1,0}) \cdot \frac{(1-p)^2}{2p - 1} \\
&\iff f_{1,0} \cdot \left(1 + \frac{(1-p)^2}{2p-1}\right) \geq 1 - p + \frac{(1-p)^2}{2p-1} \\
&\iff f_{1,0} \cdot \left(2p - 1 + (1-p)^2\right) \geq (1-p)(2p-1) + (1-p)^2 \\
&\iff f_{1,0} \cdot \left(2p - 1 + 1 - 2p + p^2\right) \geq 2p - 1 - 2p^2 + p + 1 - 2p + p^2 \\
&\iff f_{1,0} \cdot p^2 \geq p - p^2 \\
&\iff f_{1,0} \geq \frac{p - p^2}{p^2}.
\end{aligned}$$

Thus,

$$f_{1,0} \geq \frac{1-p}{p} \tag{41}$$

for any solution of Eqn. (37).

Rewriting Eqn. (39), with $i = 1$, gives the solution

$$f_{1,0} = \frac{1-p}{p}. \tag{42}$$

Hence, combining Eqn. (41) and Eqn. (42) gives that $f_{1,0} = \frac{1-p}{p}$ is the minimal solution of $f_{1,0}$. □

Recall that we only adjusted the numbering of the states for the previous lemma. So, we now we consider again the state space S_{N-1}.

Notice that $p > \frac{1}{2}$ implies for the Y-model that $f_{N,N-1} = \frac{1-p}{p} < 1$.

Lemma 5.22. *It holds that $q^+ \leq \frac{1-p}{p}$.*

Proof. Since $\alpha^* > 1$ there exists a $\epsilon > 0$ such that $\alpha^* - \epsilon > 1$. Take such an ϵ. Since α^* exists, there exists $N \in \mathbb{N}$ such that for all $n \geq N$ it holds that $\frac{\lambda_n}{\mu_n} \geq \alpha^* - \epsilon$. Take such a N.

Recall the processes X and Y. The corresponding transition matrices P and Q of the processes X and Y, respectively, with state space S_{N-1} are given by the matrices in Eqn. (36) and Eqn. (35).

In Lemma 5.16 it is proven that the transition matrix Q of the process Y is stochastically monotone. In Lemma 5.17 it is proven that $Q \preceq P$.

Let the function $f\colon S_{N-1} \to \mathbb{R}$ be given by $f(i) = \mathbb{1}_{S_N}(i)$. For notational purposes, define $f(N-2) = 0$. Then, f is a non-decreasing function.

Since f is a non-decreasing function, $Q \preceq P$ and Q is stochastically monotonic, Theorem 5.15 gives that $Q^n \preceq P^n$. Then, using Lemma 5.11, it follows that $Q^n f \leq P^n f$. Therefore,

$$\sum_{j \in S_{N-1}} Q^n_{i,j} f(j) \leq \sum_{j \in S_{N-1}} P^n_{i,j} f(j). \tag{43}$$

The entries of the matrices P and Q are given by $p_{i,j}$ and $q_{i,j}$, respectively. Furthermore, it holds that $P^n_{i,j} = \mathbb{P}(X_n = j \mid X_0 = i)$, i.e., the probability that the process is in state j after n jumps, given that it started in state i. For the remainder of this proof denote by $f^{(n)}_{i,j}(P)$ and $f^{(n)}_{i,j}(Q)$ the probability $f^{(n)}_{i,j}$ for the process X and Y, respectively. Moreover, denote by $f_{i,j}(P)$ and $f_{i,j}(Q)$ the first entrance probability $f_{i,j}$ for the process X and Y, respectively.

Let $n \in \mathbb{N}$ be arbitrary. For all $i \in S'$ it holds that

$$\begin{aligned}
\sum_{j \in S_{N-1}} P^n_{i,j} f(j) &= \sum_{j \in S_{N-1}} P^n_{i,j} \mathbb{1}_{S_N}(j) \\
&= \sum_{j \in S_N} P^n_{i,j} \\
&= \sum_{j \in S_N} \mathbb{P}\left(X_n = j \mid X_0 = i\right) \\
&\overset{(*)}{=} \mathbb{P}\left(X_t \neq N-1 \ \forall t \in \{1, 2, \ldots, n\} \mid X_0 = i\right) \\
&= 1 - \mathbb{P}\left(X_t = N-1 \text{ for some } t \in \{1, 2, \ldots, n\} \mid X_0 = i\right) \\
&= 1 - f^{(n)}_{i,N-1}(P),
\end{aligned} \tag{44}$$

where (*) holds since $N-1$ is an absorbing state, i.e., $\mathbb{P}\left(X_n = N-1 \mid X_0 = N-1\right) = 1$ for all $n \in \mathbb{N}$.

Analogously, it follows for all $i \in S_{N-1}$, that

$$\sum_{j \in S_{N-1}} Q^n_{i,j} f(j) = 1 - f^{(n)}_{i,j}(Q). \tag{45}$$

Using Eqn. (44) and Eqn. (45) together with Eqn. (43) gives for all $n \in \mathbb{N}$ that

$$1 - f^{(n)}_{i,N-1}(Q) \leq 1 - f^{(n)}_{i,N-1}(P),$$

hence

$$f^{(n)}_{i,N-1}(Q) \geq f^{(n)}_{i,N-1}(P).$$

For all $i \in S_{N-1}$ it holds that

$$f_{i,N-1}(P) = \lim_{n\to\infty} f^{(n)}_{i,N-1}(P) \leq \lim_{n\to\infty} f^{(n)}_{i,N-1}(Q) = f_{i,N-1}(Q). \tag{46}$$

Recall $f_{N,N-1}(P) = q_N$. Combining this with Eqn. (46) yields for all $N \in \mathbb{N}$, with $\frac{\lambda_n}{\mu_n} \geq \alpha^* - \epsilon$ for all $n \geq N$, that

$$q_N = f_{N,N-1}(P) \leq f_{N,N-1}(Q) = \frac{1-p}{p} < 1.$$

Consequently,

$$\limsup_{N\to\infty} q_N \leq \frac{1-p}{p} < 1.$$

□

Corollary 5.23. *It holds that* $q^+ = \frac{1}{\alpha^*}$.

Proof. To prove the statement, we will first prove that $\frac{1-p}{p} \geq \frac{1}{\alpha^*}$. It holds that

$$\begin{aligned}\frac{1-p}{p} &= \frac{1 - \frac{\alpha^*-\epsilon}{1+\alpha^*+\epsilon}}{\frac{\alpha^*-\epsilon}{1+\alpha^*+\epsilon}} \\ &= \frac{\frac{1}{1+\alpha^*-\epsilon}}{\frac{\alpha^*-\epsilon}{1+\alpha^*+\epsilon}} \\ &= \frac{1}{\alpha^*-\epsilon} \\ &\geq \frac{1}{\alpha^*}.\end{aligned}$$

In Lemma 5.8 it is shown that $q^+ \in \{\frac{1}{\alpha^*}, 1\}$. Hence, it immediately follows that $q^+ = \frac{1}{\alpha^*}$. □

5.2 Proof of part 2 of Theorem 5.2

Now we can prove the second part of Theorem 5.2.

Proof of part 2 of Theorem 5.2. Recall that $\alpha^* > 0$ and $\beta^* \geq 0$. Let $q_0 = 0$ and let q_n, $n \in \mathbb{N}$ be given by Eqn. (24), i.e.,

$$q_{n+1} = \frac{\mu_n + \tau + \lambda_n}{\mu_n} - \frac{\mu_n}{\lambda_n q_n}$$

for all $n \in \mathbb{N}$.

Combination of Lemma 5.4 and Lemma 5.6 proves the statement if the theorem for $\alpha^*, \beta^* > 0$. Lemma 5.7 proves the statement for $\beta^* = 0$ and $0 < \alpha^* \leq 1$.

Combination of Lemma 5.8, Lemma 5.22 and Corollary 5.23, proves the statement for $\alpha^* > 1$ and $\beta^* = 0$. □

6 Summary of the theoretical results in [1]

Ellens et al. describe in Section 2 of [1] a method to determine the probability that the maximum of a birth-death process over an interval with initial state and end state given exceeds a certain level. To determine this probability over an interval of deterministic length, they first determine this probability on an interval of exponential length.

After considering an interval of exponential length, Ellens et al. use the property that an Erlang distributed time converges to a deterministic time to calculate the probability that the maximum of a birth-death process over a deterministic interval with initial state and end state given exceeds a certain level.

Since the exponential interval is used to calculate the probability on a deterministic interval, in this chapter we will describe the steps Ellens et al. took in Section 2 in [1]. That way, we also give a motivation as to why we considered the killed birth-death process before.

In Subsection 6.1 we will summarize the results of Section 2.2 of [1], and in Subsection 6.2 we will summarize the results of Section 2.3 of [1]. Note that Section 2.1 of [1] is already discussed in this thesis in Chapter 2.

Since, we consider the stochastic process on an interval $[0, T]$ (of random duration) before we consider the process on an interval of deterministic length, we used the killed birth-death process in the previous chapters to model the process on an interval of exponential length. Since we use the exponential interval to later consider the deterministic interval, the deterministic interval is the motivation to consider the killed birth-death process.

6.1 Maximum over an exponential interval; initial and terminal state given

Section 2.2 in [1] derives an expression for the probability that the maximum of the process X is equal to m on a stochastic interval T given that it starts in state i and finishes in state j. In this subsection we will explain how Ellens et al. derived this expression and we will also give the expression for the probability that the maximum of the process X is at most m on a stochastic interval T given that it starts in state i and finishes in state j.

Recall Definition 2.7.

Definition 6.1.

1. Let $r_{m,i,j} := \mathbb{P}(\bar{X}_T = m \mid X_0 = i, X_T = j)$ be the probability that the maximum of the process X equals the level m, given that the process started in state i and finished at time T in state j.
2. Let $\bar{r}_{m,i,j} := \mathbb{P}(\bar{X}_T \leq m \mid X_0 = i, X_T = j)$ be the probability that the maximum of the process X does not exceed the level m, given that the process started in state i and finished at time T in state j.

Proposition 6.2.

1. *For* $i, j < m$ *it holds that* $r_{m,i,j} \leq \bar{r}_{m,i,j}$.
2. *For* $i = m$ *it holds that* $r_{m,i,j} = \bar{r}_{m,i,j}$.
3. *For* $j = m$ *it holds that* $r_{m,i,j} = \bar{r}_{m,i,j}$.
4. *For* $i > m$ *it holds that* $r_{m,i,j} = 0 = \bar{r}_{m,i,j}$.
5. *For* $j > m$ *it holds that* $r_{m,i,j} = 0 = \bar{r}_{m,i,j}$.

Proof.

1. Let $i, j < m$. Since

$$\mathbb{P}(\bar{X}_T \leq m \mid X_0 = i, X_T = j) = \sum_{k=1}^{m} \mathbb{P}(\bar{X}_T = k \mid X_0 = i, X_T = j) \geq \mathbb{P}(\bar{X}_T = m \mid X_0 = i, X_T = j),$$

it holds that $r_{m,i,j} \leq \bar{r}_{m,i,j}$.

2. Let $i = m$. Then,

$$r_{m,i,j} = r_{m,m,j} = \mathbb{P}(\bar{X}_T = m \mid X_0 = m, X_T = j)$$

and

$$\bar{r}_{m,i,j} = \bar{r}_{m,m,j} = \mathbb{P}(\bar{X}_T \leq m \mid X_0 = m, X_T = j) = \mathbb{P}(\bar{X}_T = m \mid X_0 = m, X_T = j).$$

Thus, $r_{m,i,j} = \bar{r}_{m,i,j}$ for $i = m$.

3. Similar as for the case where $i = m$, we get for $j = m$ that $r_{m,i,j} = \bar{r}_{m,i,j}$.
4. Let $i > m$. Then $\bar{X}_T \geq i > m$, hence $r_{m,i,j} = 0 = \bar{r}_{m,i,j}$.
5. Let $j > m$. Then, analogously to the case $i > m$, it holds that $\bar{X}_T \geq j > m$, and thus $r_{m,i,j} = 0 = \bar{r}_{m,i,j}$.

□

Now that we have proven some properties of the probabilities $r_{m,i,j}$ and $\bar{r}_{m,i,j}$, we can introduce the following concept.

Definition 6.3.

1. Let $s_{m,i,j} := \mathbb{P}(\bar{X}_T = m, X_T = j \mid X_0 = i)$ be the probability that the maximum of the process X equals the level m and finishes at time T in state j, given that the process started in state i.
2. Let $\bar{s}_{m,i,j} := \mathbb{P}(\bar{X}_T \leq m, X_T = j \mid X_0 = i)$ be the probability that the maximum of the process X does not exceed the level m and finishes at time T in state j, given that the process started in state i.

The above definition can be used to give an expression for the probabilities $r_{m,i,j}$ and $\bar{r}_{m,i,j}$ in the following way.

Proposition 6.4. *It holds that* $r_{m,i,j} = \frac{s_{m,i,j}}{\lim_{n\to\infty} \bar{s}_{n,i,j}}$ *and that* $\bar{r}_{m,i,j} = \frac{\bar{s}_{m,i,j}}{\lim_{n\to\infty} \bar{s}_{n,i,j}}$.

Proof. It holds that

$$\begin{aligned} r_{m,i,j} &= \mathbb{P}(\bar{X}_T = m \mid X_0 = i, X_T = j) \\ &= \frac{\mathbb{P}(\bar{X}_T = m, X_T = j \mid X_0 = i)}{\mathbb{P}(X_T = j \mid X_0 = i)} \\ &= \frac{\mathbb{P}(\bar{X}_T = m, X_T = j \mid X_0 = i)}{\lim_{n\to\infty} \mathbb{P}(\bar{X}_T \le n, X_T = j \mid X_0 = i)} \\ &= \frac{s_{m,i,j}}{\lim_{n\to\infty} \bar{s}_{n,i,j}}. \end{aligned}$$

Analogously, it holds that

$$\begin{aligned} \bar{r}_{m,i,j} &= \mathbb{P}(\bar{X}_T \le m \mid X_0 = i, X_T = j) \\ &= \frac{\mathbb{P}(\bar{X}_T \le m, X_T = j \mid X_0 = i)}{\mathbb{P}(X_T = j \mid X_0 = i)} \\ &= \frac{\mathbb{P}(\bar{X}_T \le m, X_T = j \mid X_0 = i)}{\lim_{n\to\infty} \mathbb{P}(\bar{X}_T \le n, X_T = j \mid X_0 = i)} \\ &= \frac{\bar{s}_{m,i,j}}{\lim_{n\to\infty} \bar{s}_{n,i,j}}. \end{aligned}$$

□

In Section 2.2 of [1], Ellens et al. state that

$$s_{m,i,j} = \begin{cases} (p_i p_{i+1} \cdot \ldots \cdot p_{m-1})\bar{p}_{m,j}, & i, j \le m, \\ 0, & i > m \text{ or } j > m, \end{cases} \tag{47}$$

and

$$\bar{p}_{m,j} = \begin{cases} \frac{\mu_m}{\lambda_m+\mu_m+\tau}(p_{m-1}\bar{p}_{m,j} + \bar{p}_{m-1,j}), & j < m, \\ \frac{\tau}{\lambda_m+\mu_m+\tau} + \frac{\mu_m}{\lambda_m+\mu_m+\tau} p_{m-1}\bar{p}_{m,m}, & j = m, \\ 0, & j > m. \end{cases} \tag{48}$$

Eqns. (47) and (48) are then used to prove

$$\bar{p}_{m,j} = \left(\frac{\mu_m p_m}{\lambda_m} \frac{\mu_{m-1} p_{m-1}}{\lambda_{m-1}} \cdots \frac{\mu_{j+1} p_{j+1}}{\lambda_{j+1}}\right) \cdot \frac{\tau p_j}{\lambda_j}$$

and

$$s_{m,i,j} = (p_i p_{i+1} \cdots p_{m-1}) \cdot (p_j p_{j+1} \cdots p_m) \cdot \left(\frac{\mu_{j+1}}{\lambda_{j+1}} \cdots \frac{\mu_m}{\lambda_m}\right) \cdot \frac{\tau}{\lambda_j}.$$

Proposition 6.5. $\bar{s}_{m,i,j} = \sum_{\max\{i,j\}}^{m} s_{k,i,j}$.

Proof. It holds that

$$\begin{aligned}
\bar{s}_{m,i,j} = \mathbb{P}(\bar{X}_T \leq m, X_T = j \mid X_0 = i) &= \sum_{k=0}^{m} s_{k,i,j} \\
&= \sum_{k=0}^{\max\{i,j\}-1} s_{k,i,j} + \sum_{\max\{i,j\}}^{m} s_{k,i,j} \\
&= \sum_{k=0}^{\max\{i,j\}-1} 0 + \sum_{\max\{i,j\}}^{m} s_{k,i,j} \\
&= \sum_{\max\{i,j\}}^{m} s_{k,i,j}.
\end{aligned}$$

□

Combining Proposition 6.5 with Proposition 6.4 gives $r_{m,i,j}$ and $\bar{r}_{m,i,j}$ in terms of the probabilities p_n, for which we have a recursive way of calculating them. This concludes the calculation of the probabilities $r_{m,i,j}$ and $\bar{r}_{m,i,j}$.

Before we consider the deterministic interval, we first will introduce two matrices.

Definition 6.6.

1. Let $\bar{S}_{m,T}$ be the matrix with (i,j)th entry given by $\bar{s}_{m,i,j}$.
2. Let $s_{m,T}$ be the matrix with (i,j)th entry given by $s_{m,i,j}$.

6.2 Maximum over a deterministic interval

Notation 6.7. Let T_k denote the sum of $k \in \mathbb{N}$ independent exponential distributed random variables, with mean $\frac{t}{k}$. So,

$$T_k = \sum_{n=1}^{k} T_n^k,$$

where T_n^k, $n = 1, \ldots, k$, are independent exponentially distributed random variables, all with mean $\frac{t}{k}$.

Then, T_k is Erlang distributed with mean t. With the Strong Law of Large Numbers it follows that $T_k \to t$ as $k \to \infty$, with probability 1. Since T_k converges to a deterministic time t as $k \to \infty$, we will use this to approximate $\bar{s}^*_{m,i,j,t}$ and $s^*_{m,i,j,t}$, where $\bar{s}^*_{m,i,j,t}$ and $s^*_{m,i,j,t}$ are defined as follows.

Definition 6.8.

1. Let $\bar{s}^*_{m,i,j,t} := \mathbb{P}(\bar{X}_t \leq m, X_t = j \mid X_0 = i)$.
2. Let $s^*_{m,i,j,t} := \mathbb{P}(\bar{X}_t = m, X_t = j \mid X_0 = i)$.

In the previous section we considered the maximum over an exponential time interval with the initial and terminal states of the process given. Section 2.3 of [1] studied the maximum over a deterministic interval with the initial and terminal states of the process given. So, the deterministic-time counterpart of $\bar{r}_{m,i,j}$, denoted by $\bar{q}_{m,i,j,t}$, is considered. Therefore, this subsection will evaluate not only $\bar{q}_{m,i,j,t}$, but also the deterministic-time counterpart of $r_{m,i,j}$, denoted by $q_{m,i,j,t}$. This will give an analogous relation as for $\bar{r}_{m,i,j}$ and $r_{m,i,j}$, when replacing the exponentially distributed time T by the deterministic time t. However, now the relation is less amenable.

Definition 6.9. Let $t \geq 0$ be given.

1. Let $\bar{q}_{m,i,j,t} := \mathbb{P}(\bar{X}_t \leq m \mid X_0 = i, X_t = j)$.
2. Let $q_{m,i,j,t} := \mathbb{P}(\bar{X}_t = m \mid X_0 = i, X_t = j)$ be the deterministic counterpart of $r_{m,i,j}$.

Let M be a sufficiently large truncation level.

Notation 6.10.

1. Let $\bar{S}^M_{m,T}$ be the $(M+1) \times (M+1)$-matrix with (i,j)th entry given by $\bar{s}_{m,i,j}$.
2. Let $S^M_{m,T}$ be the $(M+1) \times (M+1)$-matrix with (i,j)th entry given by $s_{m,i,j}$.

It holds that $\left(S^M_{m,T^k_1} \cdot \ldots \cdot S^M_{m,T^k_k}\right)_{i,j}$ is the probability that the truncated process never reaches a state above m during time T_k. Recall that T^k_1, ..., T^k_k are all independent and identically distributed. Therefore,

$$S^M_{m,T^k_1} \cdot \ldots \cdot S^M_{m,T^k_k} = \left(S^M_{m,T^k_1}\right)^k.$$

Analogously,

$$\bar{S}^M_{m,T^k_1} \cdot \ldots \cdot \bar{S}^M_{m,T^k_k} = \left(\bar{S}^M_{m,T^k_1}\right)^k.$$

Section 2.3 in [1] derives

$$\left(\lim_{k\to\infty} \left(\bar{S}^M_{m,T^k_1}\right)^k\right)_{i,j} = \left(\bar{S}^*_{m,t}\right)_{i,j} := \mathbb{P}(\bar{X}_t \leq m, X_t = j \mid X_0 = i) = \bar{s}^*_{m,i,j,t},$$

$$\left(\lim_{k\to\infty} \left(S^M_{m,T^k_1}\right)^k\right)_{i,j} = \left(S^*_{m,t}\right)_{i,j} := \mathbb{P}(\bar{X}_t = m, X_t = j \mid X_0 = i) = s^*_{m,i,j,t}.$$

To calculate the probabilities $\bar{q}_{m,i,j,t}$ and $q_{m,i,j,t}$ of interest we can use $\bar{s}^*_{m,i,j,t}$ and $s^*_{m,i,j,t}$, respectively. It holds that

$$\begin{aligned}
\bar{q}_{m,i,j,t} &= \mathbb{P}(\bar{X}_t \leq m \mid X_0 = i, X_t = j) \\
&= \frac{\mathbb{P}(\bar{X}_t \leq m, X_t = j \mid X_0 = i)}{\mathbb{P}(X_t = j \mid X_0 = i)} \\
&= \frac{\bar{s}^*_{m,i,j,t}}{\lim_{n\to\infty} \bar{s}^*_{m,i,j,t}}
\end{aligned}$$

and it holds that

$$\begin{aligned} q_{m,i,j,t} &= \mathbb{P}(\bar{X}_t = m \mid X_0 = i, X_t = j) \\ &= \frac{\mathbb{P}(\bar{X}_t = m, X_t = j \mid X_0 = i)}{\mathbb{P}(X_t = j \mid X_0 = i)} \\ &= \frac{s^*_{m,i,j,t}}{\lim_{n\to\infty} \bar{s}^*_{m,i,j,t}}. \end{aligned}$$

7 Discussion

We started this research by reading the first two chapters of the article of Ellens et al., [1]. While reading these chapters, we became interested in the behaviour of the probabilities p_n and q_n. Since the probabilities p_n have a forward recursive formula, these probabilities could be numerically analyzed in Excel for the $M|M|c$-model, $c \in \mathbb{N} \cup \{\infty\}$. That way we hoped to get more feeling with these probabilities before moving on to the next part of the article. However, during this numerical analysis, it turned out that we could give some interesting conjectures about the behaviour of the probabilities p_n. Therefore, we decided to prove some of these conjectures.

The original idea was that after studying the first two chapters we would study the rest of the article to see if we could extend this research. However, proving properties about the behaviour of the probabilities p_n turned out to be harder than anticipated. Therefore, we decided that we would spend more time on researching the properties of the probabilities p_n. We proved that under some mild restrictions the limit $p^* = \lim_{n\to\infty} p_n$ exists and we gave an expression for p^*. In the proof of this theorem, one case turned out to be harder than the other cases. However, using a upper bound for the limit superior, also this case could be proven.

Since the recursive formulas of p_n and q_n have a similar look, we decided that it would be interesting to research the probability q_n and see if these probabilities also converge. As is turns out, for most cases, the proof of this theorem goes in a similar way to the proof of the limiting value of p_n. However, one case turned out to be harder than the other cases. For this case we could not use the same trick as for the theorem about p^*. For this proof we needed to introduce the concept of stochastic monotonicity. Using this concept, we were able to also prove that under some mild conditions the limit $q^* = \lim_{n\to\infty} q_n$ exists and give an expression for q^*. Therefore, the main results of this thesis are the two theorems that give a statement about the limiting values p^* and q^*, i.e., Theorem 4.1 and Theorem 5.2. Some further research could be done to find if (and how) stochastic monotonicity can also be used to proof part of Theorem 4.1.

Appendix

Example	λ	μ	τ
1	0.75	1	2
2	0.75	1	1.75
3	0.75	1	1.5
4	0.75	1	1.25
5	0.75	1	1
6	0.75	1	0.75
7	0.75	1	0.5
8	0.75	1	0.4
9	0.75	1	0.3
10	0.75	1	0.25
11	0.75	1	0.2
12	0.75	1	0.15
13	0.75	1	0.1
14	0.75	1	0.05
15	0.75	1	0.01
16	0.75	1	0.001
17	0.75	1	0.0001

Table 1: Fixed λ and μ, with various τ.

Example	λ	μ	τ
1	0.9999	1	1
2	0.999	1	1
3	0.99	1	1
4	0.9	1	1
5	0.8	1	1
6	0.75	1	1
7	0.7	1	1
8	0.6	1	1
9	0.5	1	1
10	0.1	1	1

Table 2: Fixed μ and τ, with various λ.

Example	λ	μ	τ
1	0.99	1	0.5
2	0.75	1	0.5
3	0.5	1	0.5
4	0.1	1	0.5

Table 3: Fixed μ and τ, with various λ.

Example	λ	μ	τ
1	0.99	1	0.1
2	0.75	1	0.1
3	0.5	1	0.1
4	0.1	1	0.1

Table 4: Fixed μ and τ, with various λ.

Example	λ	μ	τ
1	0.99	1	0.01
2	0.75	1	0.01
3	0.5	1	0.01
4	0.1	1	0.01

Table 5: Fixed μ and τ, with various λ.

Example	λ	μ	τ
1	0.01	1	0.5
2	0.1	1	0.5
3	0.5	1	0.5
4	1	1	0.5
5	5	1	0.5
6	1	0.1	0.5
7	1	0.5	0.5
$8 = 4$	1	1	0.5
9	1	5	0.5
10	1	10	0.5
11	5	0.1	0.5
12	5	0.5	0.5
$13 = 5$	5	1	0.5
14	5	5	0.5
15	5	10	0.5
16	0.01	0.1	0.5
17	0.01	0.5	0.5
$18 = 1$	0.01	1	0.5
19	0.01	5	0.5
20	0.01	10	0.5
21	1	5	0.01
22	1	5	0.1
$23 = 9$	1	5	0.5
24	1	5	1
25	1	5	5
26	1	0.5	0.01
27	1	0.5	0.1
$28 = 7$	1	0.5	0.5
29	1	0.5	1
30	1	0.5	5
31	0.75	1	2
32	0.75	1	0.75
33	0.75	1	0.2
34	0.75	1	0.001
35	0.75	1	0.002
36	0.75	0.5	0.01

Table 6: 36 situations where λ, μ and τ can vary.

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