The varieties of e-th powers

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Contents

Notation

A family $(A_i)_{i\in I}$ is a rule that assigns to each object i of an index class I an object A_i. Let S_1, S_2 be sets. Then we denote the set of elements of S_1 which are not elements of S_2 by $S_1 - S_2$.

Let X be a set. A permutation of X is bijective map $\sigma\colon X \to X$. Denote the set of all permutations of X by $S(X)$. The set $S(X)$ has the structure of a group with the identity map $\mathrm{id}\colon X \to X$ as identity element and the composition $\sigma \circ \tau$ as the product $\sigma\tau$ for all $\sigma, \tau \in S(X)$. For all $n \in \mathbb{Z}_{\geq 0}$, denote $S(\{1, \ldots, n\})$ by S_n and denote the sign map $S_n \to \{\pm 1\}$ by sgn.

Denote the ring of natural numbers by $\mathbb{Z}$ and denote the algebraically closed field of complex numbers by $\mathbb{C}$. Let R be a ring. Then we denote the group of units of R by R^*. An R-module is an abelian group M together with a homomorphism of rings $R \to \mathrm{End}(M)$. Let M be an R-module and let $\eta\colon R \to \mathrm{End}(M)$ be the associated homomorphism of rings. Then we denote $\eta(r)(m)$ by $r \cdot m$ for all $r \in R$ and $m \in M$. Let M, N be R-modules. Then a map $\ell\colon M \to N$ is called R-linear if $\ell(r \cdot m) = r \cdot \ell(m)$ for all $r \in R$ and $m \in M$.

Let K be a field. Then we denote the characteristic of K by $\mathrm{char}(K)$. Let V be a vector space over K. Then we denote the dual of V by $V^\times$. Let $\ell\colon V \to W$ be a K-linear map. Then we call the K-linear map

$$\begin{aligned} \ell^\times\colon W^\times &\to V^\times \\ \varphi &\mapsto \varphi \circ \ell \end{aligned}$$

the dual of ℓ. We denote the group of invertible K-linear maps $V \to V$ by $\mathrm{GL}(V)$ and we denote the subgroup of $\mathrm{GL}(V)$ consisting of all maps with determinant 1 by $\mathrm{SL}(V)$.

A K-algebra A is a (not necessarily commutative) ring A that comes with a homomorphism of rings $\iota\colon K \to A$ such that each element of the image of ι commutes with all elements of A. Let A_1, A_2 be K-algebras. Then a homomorphism of K-algebras $A_1 \to A_2$ is a homomorphism of rings $A_1 \to A_2$ that is K-linear.

Introduction

Let K be an algebraically closed field. If the characteristic of K does not equal 2, then it is well known that for all elements $a, b, c \in K$, the polynomial

$$az^2 + bz + c \in K[z]$$

is a square if and only if its discriminant $b^2 - 4ac$ is zero. The previous sentence has a homogeneous analogue: if the characteristic of K does not equal 2, then it is well known that for all elements $a, b, c \in K$, the polynomial

$$ax^2 + bxy + cy^2 \in K[x, y]$$

is a square if and only if $b^2 - 4ac$ is zero. We see that, under some assumptions about the characteristic of the field K, we can determine whether a homogeneous polynomial of degree two in two variables is a square by checking whether a certain polynomial in its coefficients is zero.

Suppose that the characteristic of K does not divide 6 and let a, b, c, d be elements of K. Then one can check that the polynomial

$$ax^3 + bx^2y + cxy^2 + dy^3 \in K[x, y]$$

is a cube if and only if we have $bc - 9ad = b^2 - 3ac = c^2 - 3bd = 0$.

Suppose that the characteristic of K equals zero. Then it is possible to prove that there exist seven polynomials $q_1, \ldots, q_7$ in $K[x_0, \ldots, x_4]$ such that for all elements $a, b, c, d, e \in K$ the polynomial

$$ax^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4 \in K[x, y]$$

is a square if and only if $q_i(a, b, c, d, e) = 0$ for each $i \in \{1, \ldots, 7\}$.

Seeing the previous statements, the obvious question to ask is whether these statements generalize is some way.

For each integer $n \in \mathbb{Z}_{\geq 0}$, denote the subspace of $K[x, y]$ consisting of all homogeneous polynomials degree n and zero by V_n. Let $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$ be integers and let CT_d be the subset of V_{de} consisting of all e-th powers of polynomials in V_d. The main question of this thesis is: how can we tell whether a polynomial $g \in V_{de}$ is an element of CT_d or not?

This question can be asked for all algebraically closed fields K, all integers $d \in \mathbb{Z}_{\geq 0}$ and all integers $e \in \mathbb{Z}_{\geq 1}$. We will mostly assume the field K and the integer e to be fixed, which is why we do not include these symbols in the notation for the set CT_d and many of the objects that we will define later.

One possible way to answer the question is: the set CT_d turns out to be an affine variety inside the affine space $\mathbb{A}(V_{de})$. This means that there exists a prime ideal I_d of $K[x_0, \ldots, x_{de}]$ such that for all elements $c_0, \ldots, c_{de} \in K$, the polynomial

$$g = c_0 y^{de} + c_1 x y^{de-1} + \cdots + c_{de-1} x^{de-1} y + c_{de} x^{de} \in V_{de}$$

is an element of CT_d if and only if $f(c_0, \ldots, c_{de}) = 0$ for each $f \in I_d$. Since the ring $K[x_0, \ldots, x_{de}]$ is Noetherian, the prime ideal I_d is generated by finitely many polynomials. So as suggested in the beginning, for all algebraically closed fields K and integers $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$, there exists a finite list of polynomial giving us a membership test for CT_d which only requires a finite number of computations per polynomial $g \in V_{de}$. Moreover, this list of polynomials can be chosen such that it generates a prime ideal. The problem we will work on in this thesis is to find such a list of generators explicitly.

Related to the set CT_d is the homogeneous polynomial map

$$\begin{array}{rcl} \mathrm{pow}_d \colon V_d & \to & V_{de} \\ f & \mapsto & f^e \end{array}$$

of degree e whose image equals CT_d. To the polynomial map pow_d, we will associate a homomorphism of K-algebras pow_d^* from the K-algebra of polynomials on V_{de} to the K-algebra of polynomials in V_d. We will show that the kernel of the map pow_d^* is equal to the ideal I_d. Since the polynomial map pow_d is homogeneous of degree e, the homomorphism of K-algebras pow_d^* restricts to a K-linear map

$$\mathrm{pow}_{d,(i)}^* \colon \mathrm{Sym}^i(V_{de}^\times) \to \mathrm{Sym}^{ie}(V_d^\times)$$

for each integer $i \in \mathbb{Z}_{\geq 0}$. We will study the ideal I_d by studying the maps $\mathrm{pow}_{d,(i)}^*$ and the function

$$\begin{array}{rcl} \mathbb{Z}_{\geq 0} & \to & \mathbb{Z}_{\geq 0} \\ i & \mapsto & \dim_K \left(\ker \mathrm{pow}_{d,(i)}^* \right), \end{array}$$

which is called the Hilbert function of the ideal I_d.

In this thesis we will state two conjectures. The first conjecture states that if the characteristic of K is not divisible by $(de)!$, then the ideal I_d is equal to an ideal J_d of which we have an explicit list of generators which are all homogeneous of degree $d+1$. The second conjecture states that the K-linear map $\mathrm{pow}^*_{d,(d)}$ is injective, which is implied by the first conjecture in the case where the characteristic of K is not divisible by $(de)!$. We will show that if the second conjecture is true, then the map $\mathrm{pow}^*_{d,(i)}$ is surjective for all $i \geq d$ and

$$\begin{array}{rcl} Z_{\geq 0} & \to & \mathbb{Z}_{\geq 0} \\ i & \mapsto & \left\{ \begin{array}{cl} 0 & \text{if } i \leq d \\ \binom{de+i}{i} - \binom{ie+d}{d} & \text{if } i > d \end{array} \right. \end{array}$$

is the Hilbert function of I_d. We will also prove that the first and second conjectures are equivalent for $d = 1$ and that the second conjecture holds for $d = 1$ and $d = 2$.

Relation to other work

I found out in the late stages of writing this thesis that this problem has been worked on before me by Abdelmalek Abdesselam and Jaydeep Chipalkatti. See [AC1] and [AC2].

The affine variety CT_d is the cone over an projective variety T_d inside the projective space $\mathbb{P}(V_{de})$. This projective variety T_d is also defined in the beginning of section 3 of [AC2]. In Proposition 3.1 of [AC2], Abdelmalek Abdesselam and Jaydeep Chipalkatti give an alternate characterisation of this subset T_d of $\mathbb{P}(V_{de})$ and use this characterisation to give an explicit list of homogeneous generators of degree $d+1$ for a homogeneous ideal whose zero set equals T_d. Conjecture 5.1 of [AC2] then states that this homogeneous ideal is in fact equal to the ideal I_d. In Chapter 4 of this thesis, we similarly give an alternate characterisation of the subset T_d of $\mathbb{P}(V_{de})$ and use this characterisation to give an explicit list of homogeneous generators of degree $d+1$ for a homogeneous ideal J_d whose zero set equals T_d. Our first conjecture then states that this homogeneous ideal J_d is in fact equal to the ideal I_d.

We will prove that the map $\mathrm{pow}^*_{i,(d)}$ is injective for all $i \leq 2$ when the field K equals $\mathbb{C}$, which implies the second conjecture for $K = \mathbb{C}$ and either $d = 1$ or $d = 2$ by taking $i = d$. One of the main steps in this proof is to relate the map $\mathrm{pow}^*_{d,(i)}$ to a homomorphism $\Psi_{i,d}$ of representations of $\mathrm{GL}_2(K)$ and to prove that these map $\Psi_{i,d}$ are injective if we have $i \leq 2$. For all integers $i, d \in \mathbb{Z}_{\geq 0}$, the dual of the map $\Psi_{i,d}$ can be identified with the map $\Psi_{d,i}$. Proving that the map $\Psi_{i,d}$ is injective is equivalent to proving that its dual map is surjective. So we can reformulate one of the previous statements as: the map $\Psi_{d,i}$ is surjective if $i \leq 2$. This reformulated statement has already been proved by Abdelmalek Abdesselam and Jaydeep Chipalkatti in the case $i = 2$. See Theorem 1.1 of [AC1].

Chapter 1

Category theory

In this chapter, let K be any field.

In this thesis, we will see various correspondences which are best stated in the language of category theory. Many of the categories we will come across are abelian and many of the functors are additive and either invertible or an equivalence of categories. The goal of this chapter is to define these terms.

1.1 Categories

Definition 1.1. A category $\mathbf{C}$ consists of the following data:

(i) a class $|\,\mathbf{C}\,|$ of objects of $\mathbf{C}$,

(ii) a set $\mathrm{Hom}_{\mathbf{C}}(A, B)$ of morphisms $A \to B$ for every pair of objects (A, B) of $\mathbf{C}$,

(iii) a composition map

$$\mathrm{Hom}_{\mathbf{C}}(B, C) \times \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$$

for all objects $A, B, C \in |\,\mathbf{C}\,|$ and

(iv) an identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathbf{C}}(A, A)$ for each object $A \in |C|$.

Let $A, B, C \in |\,\mathbf{C}\,|$ be objects. Then we write $f\colon A \to B$ to indicate that f is an element of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ and we write $g \circ f$ for the composition of two morphisms $f\colon A \to B$ and $g\colon B \to C$.

To be a category, these data $\mathbf{C}$ must satisfy the following conditions:

(a) for all objects $A, B, C, D \in |\,\mathbf{C}\,|$ and all morphisms $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$, we have $h \circ (g \circ f) = (h \circ g) \circ f$;

(b) for all objects $A, B \in |\,\mathbf{C}\,|$ and each morphism $f\colon A \to B$, we have

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A;$$

(c) for all objects $A, A', B, B' \in |\,\mathbf{C}\,|$ such that $(A, B) \neq (A', B')$, the sets $\mathrm{Hom}_{\mathbf{C}}(A, B)$ and $\mathrm{Hom}_{\mathbf{C}}(A', B')$ are disjoint.

Examples 1.2.

(i) The sets form the class of objects of the category $\underline{\mathrm{Set}}$ whose morphisms are maps.

(ii) The vector spaces over K form the class of objects of the category $\underline{\mathrm{Vect}}_K$ whose morphisms are K-linear maps.

(iii) Let R be a ring. Then the R-modules form the class of objects of the category R-$\underline{\mathrm{Mod}}$ whose morphisms are R-linear maps.

Let $\mathbf{C}$ be a category

Definition 1.3. Let $A, B \in |\,\mathbf{C}\,|$ be objects and let $f\colon A \to B$ be a morphism.

(i) We call f an isomorphism if there exists a morphism $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

(ii) We call f a monomorphism when we have $g = h$ for all morphisms $g, h\colon C \to A$ such that $f \circ g = f \circ h$.

(iii) We call f an epimorphism when we have $g = h$ for all morphisms $g, h\colon B \to C$ such that $g \circ f = h \circ f$.

Definition 1.4. A subcategory of $\mathbf{C}$ is a category $\mathbf{D}$ such that the following conditions hold:

(i) we have $|\,\mathbf{D}\,| \subseteq |\,\mathbf{C}\,|$;

(ii) we have $\mathrm{Hom}_{\mathbf{D}}(A, B) \subseteq \mathrm{Hom}_{\mathbf{C}}(A, B)$ for all objects $A, B \in |\,\mathbf{D}\,|$;

(iii) the composition map

$$\mathrm{Hom}_{\mathbf{D}}(B, C) \times \mathrm{Hom}_{\mathbf{D}}(A, B) \to \mathrm{Hom}_{\mathbf{D}}(A, C)$$

is the restriction of the composition map

$$\mathrm{Hom}_{\mathbf{C}}(B, C) \times \mathrm{Hom}_{\mathbf{C}}(A, B) \to \mathrm{Hom}_{\mathbf{C}}(A, C)$$

for all objects $A, B, C \in |\,\mathbf{D}\,|$;

(iv) for each object $A \in |\,\mathbf{D}\,|$, the identity morphism of A is the same in the categories $\mathbf{C}$ and $\mathbf{D}$.

Definition 1.5. Let $\mathbf{C}$ be a category and let $\mathbf{D}$ be a subcategory of $\mathbf{C}$. Then $\mathbf{D}$ is called a full subcategory of $\mathbf{C}$ if $\mathrm{Hom}_{\mathbf{D}}(A, B) = \mathrm{Hom}_{\mathbf{C}}(A, B)$ for all objects $A, B \in |\,\mathbf{D}\,|$.

Let $\mathbf{C}$ be a category and let $\mathcal{P}$ be a property that an object of $\mathbf{C}$ might or might not have. Then there exists a unique full subcategory $\mathbf{D}$ of $\mathbf{C}$ such that $|\,\mathbf{D}\,|$ is the class of objects of $\mathbf{C}$ that have the property $\mathcal{P}$. We call this category $\mathbf{D}$ the full subcategory of $\mathbf{C}$ consisting of all objects of $\mathbf{C}$ that have the property $\mathcal{P}$.

Example 1.6. The finite-dimensional vector spaces over K form the class of objects of the category $\underline{\text{fVect}}_K$ whose morphisms are K-linear maps. The category $\underline{\text{fVect}}_K$ is the full subcategory of $\underline{\text{Vect}}_K$ consisting of all vector spaces over K that are finite dimensional.

1.2 Functors

Let $\mathbf{C}, \mathbf{D}, \mathbf{E}$ be categories.

Definition 1.7. A covariant functor $\mathbf{F}\colon\ \mathbf{C} \to \mathbf{D}$ is a rule, which assigns to each object $A \in |\,\mathbf{C}\,|$ an object $\mathbf{F}(A) \in |\,\mathbf{D}\,|$ and to each morphism $f\colon A \to B$ a morphism $\mathbf{F}(f)\colon\ \mathbf{F}(A) \to \mathbf{F}(B)$, such that the following conditions hold:

(a) for all objects $A, B, C \in |\,\mathbf{C}\,|$ and all morphisms $f\colon A \to B$ and $g\colon B \to C$, we have $\mathbf{F}(g \circ f) = \mathbf{F}(g) \circ \mathbf{F}(f)$;

(b) for each object $A \in |\,\mathbf{C}\,|$, we have $\mathbf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{F}(A)}$.

Definition 1.8. Let $\mathbf{F}\colon\ \mathbf{C} \to \mathbf{D}$ be a covariant functor. Then we call $\mathbf{F}$ invertible if the following conditions hold:

(a) for every object $B \in |\,\mathbf{D}\,|$, there exists precisely one object $A \in |\,\mathbf{C}\,|$ such that $\mathbf{F}(A) = B$;

(b) for all objects $A, B \in |\,\mathbf{C}\,|$, the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{C}}(A, B) & \to & \mathrm{Hom}_{\mathbf{D}}(\mathbf{F}(A), \mathbf{F}(B)) \\ f & \mapsto & \mathbf{F}(f) \end{array}$$

is bijective.

Example 1.9. Let $\mathrm{id}_{\mathbf{C}}\colon\ \mathbf{C} \to \mathbf{C}$ be the rule that assigns to each object $A \in |\,\mathbf{C}\,|$ the object A itself and to each morphism $f\colon A \to B$ the morphism f itself. Then $\mathrm{id}_{\mathbf{C}}$ is an invertible covariant functor. We call $\mathrm{id}_{\mathbf{C}}$ the identity functor on $\mathbf{C}$.

Definition 1.10. A contravariant functor $\mathbf{F}\colon\ \mathbf{C} \to \mathbf{D}$ is a rule, which assigns to each object $A \in |\,\mathbf{C}\,|$ an object $\mathbf{F}(A) \in |\,\mathbf{D}\,|$ and to each morphism $f\colon A \to B$ a morphism $\mathbf{F}(f)\colon\ \mathbf{F}(B) \to \mathbf{F}(A)$, such that the following conditions hold:

(a) for all objects $A, B, C \in |\mathbf{C}|$ and all morphisms $f\colon A \to B$ and $g\colon B \to C$, we have $\mathbf{F}(g \circ f) = \mathbf{F}(f) \circ \mathbf{F}(g)$;

(b) for each object $A \in |\mathbf{C}|$, we have $\mathbf{F}(\mathrm{id}_A) = \mathrm{id}_{\mathbf{F}(A)}$.

Example 1.11. Let $(-)^\times\colon \underline{\mathrm{Vect}}_K \to \underline{\mathrm{Vect}}_K$ be the rule that assigns to each vector space V over K its dual $V^\times$ and to each K-linear map ℓ its dual $\ell^\times$. Then $(-)^\times$ is a contravariant functor. The dual of a finite-dimensional vector space over K is finite dimensional over K. So $(-)^\times$ restricts to a contravariant functor $(-)^\times_{\mathrm{f}}\colon \underline{\mathrm{fVect}}_K \to \underline{\mathrm{fVect}}_K$.

Definition 1.12. Let $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ be a contravariant functor. Then we call $\mathbf{F}$ invertible if the following conditions hold:

(a) For every object $B \in |\mathbf{D}|$, there exists precisely one object $A \in |\mathbf{C}|$ such that $\mathbf{F}(A) = B$.

(b) For all objects $A, B \in |\mathbf{C}|$, the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{C}}(A, B) & \to & \mathrm{Hom}_{\mathbf{D}}(\mathbf{F}(B), \mathbf{F}(A)) \\ f & \mapsto & \mathbf{F}(f) \end{array}$$

is bijective.

1.13. By a functor, we mean a covariant functor or a contravariant functor. Let $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ and $\mathbf{G}\colon \mathbf{D} \to \mathbf{E}$ be functors. Then we get a functor $\mathbf{G} \circ \mathbf{F}\colon \mathbf{C} \to \mathbf{E}$ by taking the composition of the rules $\mathbf{F}$ and $\mathbf{G}$. If $\mathbf{F}$ and $\mathbf{G}$ are both covariant or both contravariant, then $\mathbf{G} \circ \mathbf{F}$ is covariant. If one of $\mathbf{F}$ and $\mathbf{G}$ is covariant and the other is contravariant, then $\mathbf{G} \circ \mathbf{F}$ is contravariant.

1.14. Let $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ be an invertible covariant functor. Then the rule $\mathbf{G}\colon \mathbf{D} \to \mathbf{C}$ which assigns to each object $B \in |\mathbf{D}|$ the unique object $A \in |\mathbf{C}|$ such that $\mathbf{F}(A) = B$ and assigns to each morphism g the unique morphism f such that $\mathbf{F}(f) = g$, is also an invertible covariant functor. By construction, we have $\mathbf{G} \circ \mathbf{F} = \mathrm{id}_{\mathbf{C}}$ and $\mathbf{F} \circ \mathbf{G} = \mathrm{id}_{\mathbf{D}}$.

Definition 1.15. Let $\mathbf{F}, \mathbf{G}\colon \mathbf{C} \to \mathbf{D}$ be covariant functors. A natural transformation $\mu\colon \mathbf{F} \Rightarrow \mathbf{G}$ is a family of morphisms $(\mu_A\colon \mathbf{F}(A) \to \mathbf{G}(A))_{A \in |\mathbf{C}|}$ such that for all objects $A, B \in |\mathbf{C}|$ and each morphism $f\colon A \to B$ the diagram

$$\begin{array}{ccc} \mathbf{F}(A) & \xrightarrow{\mathbf{F}(f)} & \mathbf{F}(B) \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ \mathbf{G}(A) & \xrightarrow{\mathbf{G}(f)} & \mathbf{G}(B) \end{array}$$

commutes.

Example 1.16. Let $(-)^\times \colon \underline{\mathrm{Vect}}_K \to \underline{\mathrm{Vect}}_K$ be the contravariant functor from Example 1.11. By taking the composition of $(-)^\times$ with itself, we get the covariant functor $(-)^{\times\times} \colon \underline{\mathrm{Vect}}_K \to \underline{\mathrm{Vect}}_K$.

For a vector space V over K, let $\varepsilon_V \colon V \to V^{\times\times}$ be the K-linear map sending v to the K-linear map $(\varphi \mapsto \varphi(v))$. Let V, W be vector spaces over K, let $\ell \colon V \to W$ be a K-linear map and let v be an element of V. Then we have

$$\ell^{\times\times}(\varepsilon_V(v)) = \ell^{\times\times}(\varphi \mapsto \varphi(v)) = (\varphi \mapsto \varphi(v)) \circ \ell^\times$$
$$= \big(\phi \mapsto \ell^\times(\phi)(v)\big) = (\phi \mapsto (\phi \circ \ell)(v)) = (\phi \mapsto \phi(\ell(v))) = \varepsilon_W(\ell(v)).$$

Therefore the diagram

$$\begin{array}{ccc} V & \xrightarrow{\ \ell\ } & W \\ {\scriptstyle \varepsilon_V}\downarrow & & \downarrow{\scriptstyle \varepsilon_W} \\ V^{\times\times} & \xrightarrow{\ \ell^{\times\times}\ } & W^{\times\times} \end{array}$$

commutes. So we see that $\{\varepsilon_V \colon V \to V^{\times\times}\}_{V \in |\underline{\mathrm{Vect}}_K|}$ is a natural transformation $\mathrm{id}_{\underline{\mathrm{Vect}}_K} \Rightarrow (-)^{\times\times}$.

Let the contravariant functor $(-)^\times_{\mathrm{f}} \colon \underline{\mathrm{fVect}}_K \to \underline{\mathrm{fVect}}_K$ be the restriction of $(-)^\times$ and let $(-)^{\times\times}_{\mathrm{f}} \colon \underline{\mathrm{fVect}}_K \to \underline{\mathrm{fVect}}_K$ be the composition of $(-)^\times_{\mathrm{f}}$ with itself. Then $\{\varepsilon_V \colon V \to V^{\times\times}\}_{V \in |\underline{\mathrm{fVect}}_K|}$ is a natural transformation $\mathrm{id}_{\underline{\mathrm{fVect}}_K} \Rightarrow (-)^{\times\times}_{\mathrm{f}}$.

Definition 1.17. Let $\mathbf{F}, \mathbf{G} \colon \mathbf{C} \to \mathbf{D}$ be covariant functors. Let $\mu \colon \mathbf{F} \Rightarrow \mathbf{G}$ be a natural transformation. Then we call μ a natural isomorphism if μ_A is an isomorphism for all objects $A \in |A|$.

Proposition 1.18. Let V be a vector space over K and let $\varepsilon_V \colon V \to V^{\times\times}$ be the K-linear map sending v to the K-linear map $(\varphi \mapsto \varphi(v))$. Then ε_V is injective. In particular, if V is finite dimensional over K, then ε_V is an isomorphism.

Proof. Let $v \in V$ be a non-zero element. Then there exists a basis $(v_i)_{i \in I}$ of V containing v. Let $\phi \colon V \to K$ be the K-linear map sending v_i to 1 for all $i \in I$. Then we see that $\phi(v) = 1$. Hence the K-linear map $\varepsilon_V(v) = (\varphi \mapsto \varphi(v))$ is non-zero. Hence ε_V is injective.

Suppose that V is finite dimensional over K. Then V, $V^\times$ and $V^{\times\times}$ all have the same dimension over K. So since ε_V is injective, we see that ε_V is an isomorphism. $\square$

Example 1.19. Consider the covariant functor $(-)^{\times\times}_{\mathrm{f}} \colon \underline{\mathrm{fVect}}_K \to \underline{\mathrm{fVect}}_K$ from Example 1.16. By the Proposition 1.18, we see that

$$\{\varepsilon_V \colon V \to V^{\times\times}\}_{V \in |\underline{\mathrm{fVect}}_K|}$$

is a natural isomorphism $\mathrm{id}_{\underline{\mathrm{fVect}}_K} \Rightarrow (-)^{\times\times}$.

Definition 1.20. Let $\mathbf{F}\colon \mathbf{C} \to \mathbf{D}$ and $\mathbf{G}\colon \mathbf{D} \to \mathbf{C}$ either both be covariant functors or both be contravariant functors. If there exist natural isomorphisms $\mu\colon \mathrm{id}_{\mathbf{C}} \Rightarrow \mathbf{G} \circ \mathbf{F}$ and $\nu\colon \mathrm{id}_{\mathbf{D}} \Rightarrow \mathbf{F} \circ \mathbf{G}$, then we call $\mathbf{F}$ and $\mathbf{G}$ equivalences of categories. We call the categories $\mathbf{C}$ and $\mathbf{D}$ equivalent if there exists an equivalence of categories $\mathbf{C} \to \mathbf{D}$.

Examples 1.21.

(i) Any invertible covariant functor is an equivalence of categories.

(ii) The contravariant functor $(-)^{\times}_{\mathrm{f}}\colon \underline{\mathrm{fVect}}_K \to \underline{\mathrm{fVect}}_K$ from Example 1.11 is an equivalence of categories by Example 1.19.

1.3 Abelian categories

Definition 1.22. A category $\mathbf{L}$ is called linear if for all objects $A, B \in |\,\mathbf{L}\,|$ the set of morphism $\mathrm{Hom}_{\mathbf{L}}(A, B)$ is an abelian group and for all objects $A, B, C \in |\,\mathbf{L}\,|$ the composition map

$$\mathrm{Hom}_{\mathbf{L}}(B, C) \times \mathrm{Hom}_{\mathbf{L}}(A, B) \to \mathrm{Hom}_{\mathbf{L}}(A, C)$$

is bilinear.

Let $\mathbf{L}$ be a linear category.

Definition 1.23. A direct sum of a pair (A, B) of objects of $\mathbf{L}$ is an object $S \in |\,\mathbf{L}\,|$ together with morphisms $i_A\colon A \to S$ and $i_B\colon B \to S$ such that for each object $C \in |\,\mathbf{L}\,|$ and all morphisms $f\colon A \to C$ and $g\colon B \to C$, there exists a unique morphism $h\colon S \to C$ such that $f = h \circ i_A$ and $g = h \circ i_B$.

When a direct sum of a pair (A, B) of objects of $\mathbf{L}$ exists, it is unique up to a unique isomorphism and we denote it by $A \oplus B$.

Definition 1.24. An object $Z \in |\,\mathbf{L}\,|$ is called a zero object if for each object $A \in |\,\mathbf{L}\,|$ there exists a unique morphism $A \to Z$ and a unique morphism $Z \to A$.

When a zero object exists, it is unique up to a unique isomorphism and we denote it by 0.

Definition 1.25. A linear category $\mathbf{A}$ is called additive if it has a zero object and it has a direct sum $A \oplus B$ for all pairs (A, B) of objects of $\mathbf{A}$.

Examples 1.26.

(i) The categories $\underline{\mathrm{Vect}}_K$ and $\underline{\mathrm{fVect}}_K$ are additive.

(ii) Let R be a ring. Then the category R-$\underline{\mathrm{Mod}}$ is additive.

Definition 1.27. Let $\mathbf{F}\colon \mathbf{A} \to \mathbf{B}$ be a covariant functor between additive categories. Then $\mathbf{F}$ is called additive if the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{A}}(A,B) & \to & \mathrm{Hom}_{\mathbf{B}}(\mathbf{F}(A),\mathbf{F}(B)) \\ f & \mapsto & \mathbf{F}(f) \end{array}$$

is a homomorphism of groups for all objects $A, B \in |\,\mathbf{A}\,|$.

Definition 1.28. Let $\mathbf{F}\colon \mathbf{A} \to \mathbf{B}$ be a contravariant functor between additive categories. Then $\mathbf{F}$ is called additive if the map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{A}}(A,B) & \to & \mathrm{Hom}_{\mathbf{B}}(\mathbf{F}(B),\mathbf{F}(A)) \\ f & \mapsto & \mathbf{F}(f) \end{array}$$

is a homomorphism of groups for all objects $A, B \in |\,\mathbf{A}\,|$.

Remark 1.29. One can check that additive functors between additive categories preserve zero objects and direct sums.

Example 1.30. The contravariant functor $(-)^{\times}\colon \underline{\mathrm{Vect}}_K \to \underline{\mathrm{Vect}}_K$ from Example 1.11 is additive.

Definition 1.31. Let $\mathbf{L}$ be a linear category, let $A, B \in |\,\mathbf{L}\,|$ be objects and let $f\colon A \to B$ be a morphism.

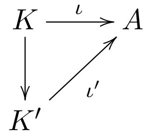
(i) A kernel of f is a morphism $\iota\colon K \to A$ such that the following conditions hold:

- we have $f \circ \iota = 0$;
- for each morphism $\iota^\dagger\colon K^\dagger \to A$ such that $f \circ \iota^\dagger = 0$, there exists a unique morphism $e\colon K^\dagger \to K$ such that $\iota^\dagger = \iota \circ e$.

(ii) A cokernel of f is a morphism $\pi\colon B \to Q$ such that the following conditions hold:

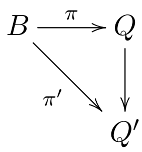
- we have $\pi \circ f = 0$;
- for each morphism $\pi^\dagger\colon B \to Q^\dagger$ such that $\pi^\dagger \circ f = 0$, there exists a unique morphism $e\colon Q \to Q^\dagger$ such that $\pi^\dagger = e \circ \pi$.

Let $f\colon A \to B$ be a morphism. If $\iota\colon K \to A$ and $\iota'\colon K' \to A$ are kernels of f, then there exists a unique isomorphism $K \to K'$ such that the diagram



commutes. So if f has a kernel, we denote it by $\iota\colon \ker(f) \to A$.

If $\pi\colon B \to Q$ and $\pi'\colon B \to Q'$ are cokernels of f, then there exists a unique isomorphism $Q \to Q'$ such that the diagram



commutes. So if f has a cokernel, then we denote it by $\pi\colon B \to \operatorname{coker}(f)$.

Definition 1.32. An additive category $\mathbf{A}$ is called abelian if the following conditions hold:

(i) every morphism of $\mathbf{A}$ has a kernel and a cokernel;

(ii) every monomorphism of $\mathbf{A}$ is the kernel of its cokernel;

(iii) every epimorphism of $\mathbf{A}$ is the cokernel of its kernel.

Examples 1.33.

(i) The categories $\underline{\text{Vect}}_K$ and $\underline{\text{fVect}}_K$ are abelian categories.

(ii) Let R be a ring. Then the category R-$\underline{\text{Mod}}$ is an abelian category.

Chapter 2

Basic algebraic geometry

In this chapter, let K be an algebraically closed field.

Algebraic geometry starts with the statement that a polynomial induces a function: every polynomial $f \in K[x_1, \ldots, x_n]$ gives rise to a polynomial function

$$\begin{array}{rcl} K^n & \to & K \\ (x_1, \ldots, x_n) & \mapsto & f(x_1, \ldots, x_n) \end{array}$$

which we identify with f. Algebraic geometry is the study of zeros of polynomial functions.

The vector space K^n comes with the standard basis $(e_1, \ldots, e_n)$. Note that the basis dual to this standard basis is $(x_1, \ldots, x_n)$, i.e., for each $i \in \{1, \ldots, n\}$ the function x_i sends $(a_1, \ldots, a_n)$ to a_i and we have

$$v = x_1(v)e_1 + \cdots + x_n(v)e_n$$

for all $v \in K^n$. The ring $K[x_1, \ldots, x_n]$ is the algebra of polynomials on K^n. So algebraic geometry typically actually starts with the choice of a standard basis of a finite-dimensional vector space over K. This choice is not necessary however.

In this chapter, we define what a polynomial on a finite-dimensional vector space over K is without choosing a standard basis. We then use this definition to give an introduction to algebraic geometry. In particular, we define symmetric powers of a vector space, polynomial maps, affine and projective varieties and morphisms between such varieties.

The content of this chapter is mostly based on [Mo], but written in a way that does not require the choice of a basis. The propositions in this chapter that are stated without proof can be translated to propositions from [Mo] by picking a basis of each vector space.

2.1 Tensor products, symmetric powers and alternating powers

Let U, V, W be vector spaces over K and let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer.

2.1. We denote the tensor product of V and W over K by $V \otimes W$. Recall that for each bilinear map $\omega \colon V \times W \to U$, there exists a unique K-linear map $\ell \colon V \otimes W \to U$ such that $\ell(v \otimes w) = \omega(v, w)$ for all $v \in V$ and $w \in W$. We call this the universal property of the tensor product of V and W. We see that

$$\begin{array}{rcl} \ell \colon V \otimes W & \to & U \\ v \otimes w & \mapsto & \omega(v, w) \end{array}$$

is a valid way to define a K-linear map $\ell \colon V \otimes W \to U$ whenever ω is a bilinear map $V \times W \to U$ and we will frequently define maps this way.

2.2. We call the tensor product of n copies of V the n-th tensor power of V and denote it by $V^{\otimes n}$. Note that for all multilinear maps $\omega \colon V^n \to U$, there exists a unique K-linear map $\ell \colon V^{\otimes n} \to U$ such that

$$\ell(v_1 \otimes \cdots \otimes v_n) = \omega(v_1, \ldots, v_n)$$

for all $v_1, \ldots, v_n \in V$. We use this universal property frequently to define K-linear maps from $V^{\otimes n}$.

For example, for each K-linear map $\ell \colon V \to W$ the map

$$\begin{array}{rcl} \omega \colon V^n & \to & W^{\otimes n} \\ (v_1, \ldots, v_n) & \mapsto & \ell(v_1) \otimes \cdots \otimes \ell(v_n) \end{array}$$

is multilinear and hence corresponds to the K-linear map

$$\begin{array}{rcl} V^{\otimes n} & \to & W^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n & \mapsto & \ell(v_1) \otimes \cdots \otimes \ell(v_n) \end{array}$$

which we will denote by $\ell^{\otimes n}$.

Let $\ell_1 \colon U \to V$ and $\ell_2 \colon V \to W$ be K-linear maps. Then we have $\ell_2^{\otimes n} \circ \ell_1^{\otimes n} = (\ell_2 \circ \ell_1)^{\otimes n}$. So we see that we get a functor

$$(-)^{\otimes n} \colon \underline{\mathrm{Vect}}_K \to \underline{\mathrm{Vect}}_K$$

So if ℓ_1 is an isomorphism, then $\ell_1^{\otimes n}$ is also an isomorphism. Also note that if ℓ_1 is injective, then $\ell_1^{\otimes n}$ is also injective and that if ℓ_1 is surjective, then $\ell_1^{\otimes n}$ is also surjective.

Definition 2.3. Define the n-th symmetric power $\mathrm{Sym}^n(V)$ of V to be the quotient of $V^{\otimes n}$ by its subspace generated by

$$v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for all $v_1, \ldots, v_n \in V$ and $\sigma \in S_n$.

By definition, the n-th symmetric power $\mathrm{Sym}^n(V)$ of V comes with a projection map $\pi_V^n \colon V^{\otimes n} \to \mathrm{Sym}^n(V)$. For elements $v_1, \ldots, v_n \in V$, we denote the element $\pi_V^n(v_1 \otimes \cdots \otimes v_n)$ of $\mathrm{Sym}^n(V)$ by $v_1 \odot \cdots \odot v_n$.

2.4. Let $(v_i)_{i \in I}$ be a totally ordered basis of V. Then

$$(v_{i_1} \otimes \cdots \otimes v_{i_n} | i_1, \ldots, i_n \in I)$$

is a basis of $V^{\otimes n}$. So we see that

$$\{v_{i_1} \odot \cdots \odot v_{i_n} | i_1, \ldots, i_n \in I\}$$

spans $\mathrm{Sym}^n(V)$. By reordering the v_{i_k} of an element $v_{i_1} \odot \cdots \odot v_{i_n}$, we get the same element of $\mathrm{Sym}^n(V)$ and these relations span all relations between the elements of this spanning set. So we see that

$$(v_{i_1} \odot \cdots \odot v_{i_n} | i_1, \ldots, i_n \in I, i_1 \leq \cdots \leq i_n)$$

is a basis of $\mathrm{Sym}^n(V)$.

Suppose that V has dimension m over K and let I be the set $\{1, \ldots, m\}$ with the obvious ordering. Note that

$$\binom{n+m-1}{m-1}$$

is the number of ways we can order n symbols $\bullet$ and $m-1$ symbols $\#$. An element $v_{i_1} \odot \cdots \odot v_{i_n}$ of the basis of $\mathrm{Sym}^n(V)$ corresponds to the ordering of these symbols such that for all $j \in \{1, \ldots, m\}$ the number of $\bullet$ symbols between the $(j-1)$-th and j-th symbols $\#$ equals $\#\{k | i_k = j\}$. For example, the ordering $\bullet \bullet \# \bullet \#$ corresponds to the element $v_1 \odot v_1 \odot v_2$ when the dimension of V over K equals 3. We see that this correspondence is one to one. So if the dimension of V over K equals m, then the dimension of $\mathrm{Sym}^n(V)$ over K equals

$$\binom{n+m-1}{m-1}.$$

2.5. Let $\omega \colon V^n \to U$ be a symmetric multilinear map. Then there exists a unique K-linear map $\ell \colon \mathrm{Sym}^n(V) \to U$ such that

$$\ell(v_1 \odot \cdots \odot v_n) = \omega(v_1, \ldots, v_n)$$

for all $v_1, \ldots, v_n \in V$. We call this the universal property of the n-th symmetric power of V. This shows that

$$\begin{array}{rcl} \ell\colon \operatorname{Sym}^n(V) & \to & U \\ v_1 \odot \cdots \odot v_n & \mapsto & \omega(v_1, \ldots, v_n) \end{array}$$

is a valid way to define a K-linear map $\ell\colon \operatorname{Sym}^n(V) \to U$ whenever we have a symmetric multilinear map $\omega\colon V^n \to U$.

Let $\ell\colon V \to W$ be a K-linear map. Then the map

$$\begin{array}{rcl} \omega\colon V^n & \to & \operatorname{Sym}^n(W) \\ (v_1, \ldots, v_n) & \mapsto & \ell(v_1) \odot \cdots \odot \ell(v_n) \end{array}$$

is multilinear and symmetric. We denote the corresponding K-linear map $\operatorname{Sym}^n(V) \to \operatorname{Sym}^n(W)$ by $\operatorname{Sym}^n(\ell)$. We get a functor

$$\operatorname{Sym}^n(-)\colon \underline{\mathrm{Vect}}_K \to \underline{\mathrm{Vect}}_K$$

Note that similar to the map $\ell^{\otimes n}$ from 2.2, the map $\operatorname{Sym}^n(\ell)$ is injective whenever ℓ is injective and surjective whenever ℓ is surjective.

2.6. Let $\ell\colon V \to W$ be a K-linear map. Then the diagram

$$\begin{array}{ccc} V^{\otimes n} & \xrightarrow{\ell^{\otimes n}} & W^{\otimes n} \\ {\scriptstyle \pi_V^n}\downarrow & & \downarrow{\scriptstyle \pi_W^n} \\ \operatorname{Sym}^n(V) & \xrightarrow{\operatorname{Sym}^n(\ell)} & \operatorname{Sym}^n(W) \end{array}$$

commutes. So we see that the family π^n of K-linear maps π_V^n over all vector spaces V over K is a natural transformation $(-)^{\otimes n} \Rightarrow \operatorname{Sym}^n(-)$.

2.7. Suppose that $\operatorname{char}(K) \nmid n!$. Then the K-linear map

$$\begin{array}{rcl} \iota_V^n\colon \operatorname{Sym}^n(V) & \to & V^{\otimes n} \\ v_1 \odot \cdots \odot v_n & \mapsto & \dfrac{1}{n!} \displaystyle\sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \end{array}$$

is a section of π_V^n. Note that the family ι^n of K-linear maps ι_V^n over all vector spaces V over K is a natural transformation $\operatorname{Sym}^n(-) \Rightarrow (-)^{\otimes n}$.

2.8. The group S_n acts on $V^{\otimes n}$ by the homomorphism

$$\begin{array}{rcl} S_n & \to & \operatorname{GL}\left(V^{\otimes n}\right) \\ \sigma & \mapsto & \left(v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}\right). \end{array}$$

Let $(V^{\otimes n})^{S_n}$ be the subspace of $V^{\otimes n}$ that is fixed by S_n. Then we see that $\iota_V^n \circ \pi_V^n$ is an idempotent endomorphism of $V^{\otimes n}$ with image $(V^{\otimes n})^{S_n}$. So we see that ι_V^n is an isomorphism onto $(V^{\otimes n})^{S_n}$ with the restriction of π_V^n as inverse.

Definition 2.9. Define the symmetric algebra $\mathrm{Sym}(V)$ of V to be the commutative graded K-algebra

$$\bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V)$$

where the product map $\mathrm{Sym}(V) \times \mathrm{Sym}(V) \to \mathrm{Sym}(V)$ is the unique bilinear map which sends $(v_1 \odot \cdots \odot v_n, w_1 \odot \cdots \odot w_m)$ to $v_1 \odot \cdots \odot v_n \odot w_1 \odot \cdots \odot w_m$ for all $v_1, \ldots, v_n, w_1, \ldots, w_m \in V$.

2.10. Note that $V = \mathrm{Sym}^1(V)$ is a subspace of $\mathrm{Sym}(V)$. Let A be a commutative K-algebra and let $\ell \colon V \to A$ be a K-linear map. Then there exists a unique homomorphism of K-algebras

$$\eta \colon \mathrm{Sym}(V) \to A$$

such that $\eta|_V = \ell$. We call this the universal property of the symmetric algebra of V. This unique homomorphism of K-algebras η is the unique K-linear map $\mathrm{Sym}(V) \to A$ which sends $v_1 \odot \cdots \odot v_n$ to $\ell(v_1) \cdots \ell(v_n)$ for all elements $v_1, \ldots, v_n \in V$.

We see that K-linear maps $\ell \colon V \to A$ correspond one to one with homomorphisms of K-algebras $\eta \colon \mathrm{Sym}(V) \to A$. We call η the extension of ℓ to $\mathrm{Sym}(V)$ and we call ℓ the restriction of η to V.

Example 2.11. Let $(v_1, \ldots, v_n)$ be a basis of V over K. Then the unique homomorphism of K-algebras

$$\eta \colon \mathrm{Sym}(V) \to K[x_1, \ldots, x_n]$$

such that $\eta(v_i) = x_i$ for all $i \in \{1, \ldots, n\}$ is an isomorphism.

Definition 2.12. Define the n-th alternating power $\Lambda^n V$ of V to be the quotient of $V^{\otimes n}$ by its subspace generated by

$$\{v_1 \otimes \cdots \otimes v_n | v_1, \ldots, v_n \in V, v_i = v_j \text{ for some } i \neq j\}.$$

By definition, the n-th alternating power $\Lambda^n V$ of V comes with a projection map $\pi \colon V^{\otimes n} \to \Lambda^n V$. For elements $v_1, \ldots, v_n \in V$, we denote the element $\pi(v_1 \otimes \cdots \otimes v_n)$ of $\Lambda^n V$ by $v_1 \wedge \cdots \wedge v_n$.

2.13. Let $v_1, \ldots, v_n$ be elements of V. Then for all $1 \leq i < j \leq n$, the element

$$v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n$$

of $V^{\otimes n}$ is equal to the difference between

$$v_1 \otimes \cdots \otimes (v_i + v_j) \otimes \cdots \otimes (v_i + v_j) \otimes \cdots \otimes v_n$$

and

$$(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n + v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n).$$

So we see that

$$v_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge v_j \wedge \cdots \wedge v_n = -v_1 \wedge \cdots \wedge v_j \wedge \cdots \wedge v_i \wedge \cdots \wedge v_n$$

for all $1 \leq i < j \leq n$ and therefore we have

$$v_1 \wedge \cdots \wedge v_n = \operatorname{sgn}(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)}$$

for all $\sigma \in S_n$.

Let $(v_i)_{i \in I}$ be a totally ordered basis of V. Then

$$(v_{i_1} \otimes \cdots \otimes v_{i_n} | i_1, \ldots, i_n \in I)$$

is a basis of $V^{\otimes n}$. So we see that

$$\{v_{i_1} \wedge \cdots \wedge v_{i_n} | i_1, \ldots, i_n \in I, i_1 < \cdots < i_n\}$$

spans $\Lambda^n V$.

2.14. Let $\omega \colon V^n \to U$ be a multilinear map with the property that

$$\omega(v_1, \ldots, v_n) = 0$$

for all $v_1, \ldots, v_n \in V$ such that $v_i = v_j$ for some $i \neq j$. Then there exists a unique K-linear map $\ell \colon \Lambda^n V \to U$ such that $\ell(v_1 \wedge \cdots \wedge v_n) = \omega(v_1, \ldots, v_n)$ for all $v_1, \ldots, v_n \in V$. We call this the universal property of the n-th alternating power of V. It shows that

$$\begin{aligned} \ell \colon \Lambda^n V &\to U \\ v_1 \wedge \cdots \wedge v_n &\mapsto \omega(v_1, \ldots, v_n) \end{aligned}$$

is a valid way to define a K-linear map ℓ whenever we have a multilinear map $\omega \colon V^n \to U$ with the property that $\omega(v_1, \ldots, v_n) = 0$ for all $v_1, \ldots, v_n \in V$ such that $v_i = v_j$ for some $i \neq j$.

Let $\ell \colon V \to W$ be a K-linear map. Then the map

$$\begin{aligned} \omega \colon V^n &\to \Lambda^n W \\ (v_1, \ldots, v_n) &\mapsto \ell(v_1) \wedge \cdots \wedge \ell(v_n) \end{aligned}$$

is a multilinear map with the property that $\omega(v_1, \ldots, v_n) = 0$ for all elements $v_1, \ldots, v_n \in V$ such that $v_i = v_j$ for some $i \neq j$. We denote the corresponding K-linear map $\Lambda^n V \to \Lambda^n W$ by $\Lambda^n \ell$. This gives us the functor $\Lambda^n(-) \colon \underline{\text{Vect}}_K \to \underline{\text{Vect}}_K$.

2.15. Consider the multilinear map

$$\begin{array}{rcl} \omega\colon (K^n)^n & \to & K \\ (v_1, \dots, v_n) & \mapsto & \det(v_1 \ \cdots \ v_n) \end{array}$$

where $(v_1 \ \cdots \ v_n)$ is the $n \times n$ matrix whose i-th column equals v_i for all $i \in \{1, \dots, n\}$. The map ω is multilinear and we have $\det(v_1 \ \cdots \ v_n) = 0$ for all $v_1, \dots, v_n \in K^n$ such that $v_i = v_j$ for some $i \neq j$. Let

$$\begin{array}{rcl} \det\colon \Lambda^n(K^n) & \to & K \\ v_1 \wedge \cdots \wedge v_n & \mapsto & \det(v_1, \dots, v_n) \end{array}$$

be the K-linear map corresponding to ω and let $(e_1, \dots, e_n)$ be the standard basis of K^n. Then $e_1 \wedge \cdots \wedge e_n$ spans $\Lambda^n(K^n)$ and we have

$$\det(e_1 \wedge \cdots \wedge e_n) = 1.$$

So $e_1 \wedge \cdots \wedge e_n$ is a non-zero element of $\Lambda^n(K^n)$ and hence a basis of $\Lambda^n(K^n)$.

2.16. Let $(v_i)_{i \in I}$ be a totally ordered basis of V. Then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_n} | i_1, \dots, i_n \in I, i_1 < \cdots < i_n\}$$

spans $\Lambda^n V$. Let $i_1, \dots, i_n \in I$ be such that $i_1 < \cdots < i_n$. Let

$$\varphi_{i_1 \dots i_n}\colon \Lambda^n V \to K$$

be the K-linear map $\det \circ \Lambda^n \ell$ where $\ell\colon V \to K^n$ is the K-linear map sending v_{i_k} to e_k for all $k \in \{1, \dots, n\}$ and sending v_i to 0 for all $i \in I - \{i_1, \dots, i_n\}$. One can check that

$$\varphi_{i_1 \dots i_n}(v_{j_1} \otimes \cdots \otimes v_{j_n}) = \begin{cases} 1 & \text{if } i_k = j_k \text{ for all } k \\ 0 & \text{otherwise} \end{cases}$$

for all $j_1, \dots, j_n \in I$ such that $j_1 < \cdots < j_n$. Hence

$$(v_{i_1} \wedge \cdots \wedge v_{i_n} | i_1, \dots, i_n \in I, i_1 < \cdots < i_n)$$

is a basis of $\Lambda^n V$ with dual basis $(\varphi_{i_1 \dots i_n} | i_1, \dots, i_n \in I, i_1 < \cdots < i_n)$. In particular, we see that if V has dimension m over K, then $\Lambda^n V$ has dimension $\binom{m}{n}$ over K. We also see that for elements $w_1, \dots, w_n \in V$, the element $w_1 \wedge \cdots \wedge w_n$ of $\Lambda^n V$ is non-zero if and only if $w_1, \dots, w_n$ are linearly independent over K.

Let $w_1, \dots, w_n$ be elements of V and write

$$w_k = \sum_{i \in I} a_{ik} v_i$$

for each $k \in \{1, \dots, n\}$. Let $i_1, \dots, i_n \in I$ be such that $i_1 < \cdots < i_n$. Then we see that

$$\varphi_{i_1 \dots i_n}(w_1 \wedge \cdots \wedge w_n) = \det((a_{1i_1}, \dots a_{1i_n}) \ \cdots \ (a_{ni_1}, \dots a_{ni_n}))$$

is the determinant of the $n \times n$ matrix $(a_{ji_k})_{j,k=1}^n$.

2.2 Polynomial functions

Let V be a finite-dimensional vector space over K.

2.17. An element of $V^\times$ is a K-linear map $V \to K$. So $V^\times$ is a subset of the K-algebra $\mathrm{Map}(V, K)$ consisting of all maps $V \to K$. The K-linear map

$$\begin{array}{rcl} \mathrm{Sym}\left(V^\times\right) & \to & \mathrm{Map}(V,K) \\ \varphi_1 \odot \cdots \odot \varphi_n & \mapsto & (v \mapsto \varphi_1(v) \dots \varphi_n(v)) \end{array}$$

is the extension of the inclusion map $V^\times \to \mathrm{Map}(V, K)$ to $\mathrm{Sym}(V^\times)$.

Since the field K is infinite, the following proposition holds.

Proposition 2.18. The homomorphism of K-algebras

$$\begin{array}{rcl} \mathrm{Sym}\left(V^\times\right) & \to & \mathrm{Map}(V,K) \\ \varphi_1 \odot \cdots \odot \varphi_n & \mapsto & (v \mapsto \varphi_1(v) \cdots \varphi_n(v)) \end{array}$$

is injective.

Definition 2.19. Define the algebra $P(V)$ of polynomials on V to be the commutative graded K-algebra $\mathrm{Sym}(V^\times)$. We call an element $f \in P(V)$ a polynomial on V and we call the image of f in $\mathrm{Map}(V, K)$ a polynomial function on V.

Proposition 2.18 tells us that we can identify polynomials on V with polynomial functions on V. Let $f \in P(V)$ be a polynomial on V and let v be an element of V. Then we denote the value of the polynomial function f on V at v by $f(v)$.

Definition 2.20. Let U be a vector space over K and let $\varepsilon_U : U \to U^{\times\times}$ be the K-linear map sending u to $(\varphi \mapsto \varphi(u))$. For an element u of U, let $\mathrm{eval}_u : P(U) \to K$ be the extension of the K-linear map $\varepsilon_U(u): U^\times \to K$ to $P(U)$. Define $\mathrm{eval}_{(-)} : U \to P(U)^\times$ to be the map sending u to eval_u.

Note that $\mathrm{eval}_v(f) = f(v)$ for all $v \in V$ and $f \in P(V)$.

2.3 Polynomial maps

Let U, V, W be finite-dimensional vector spaces over K.

Definition 2.21. Let $\alpha : V \to W$ be a map. Then we say that α is a polynomial map if for all $\varphi \in W^\times$ the composition $\varphi \circ \alpha$ is a polynomial function on V.

By the kernel of a polynomial map $\alpha : V \to W$, we mean the set $\ker \alpha$ consisting of all elements $v \in V$ such that $\alpha(v) = 0$.

2.22. Let $\alpha\colon V \to W$ be a map and let $(\varphi_1, \ldots, \varphi_n)$ be a basis of $V^\times$. Then α is a polynomial map if and only if the composition $\varphi_i \circ \alpha$ is a polynomial function on V for each $i \in \{1, \ldots, n\}$, because a linear combination of polynomial functions on V is again a polynomial function on V. In particular, a map $V \to K$ is a polynomial map if and only if it is a polynomial function on V, because the identity map id_K is a basis of $K^\times$.

Definition 2.23. Let $\alpha\colon V \to W$ be a polynomial map. Then α gives us the K-linear map $\ell\colon W^\times \to P(V)$ sending a K-linear map $\varphi\colon W \to K$ to the polynomial on V corresponding to the polynomial function $\varphi \circ \alpha$ on V. Define the homomorphism of K-algebras $\alpha^*\colon P(W) \to P(V)$ to be the extension of ℓ to $P(W)$.

Let $\eta\colon P(W) \to P(V)$ be a homomorphism of K-algebras and let

$$\ell\colon W^\times \to P(V)$$

be the restriction of η to $W^\times$. Since W is finite dimensional over K, the K-linear map $\varepsilon_W\colon W \to W^{\times\times}$ sending w to $(\varphi \mapsto \varphi(w))$ is an isomorphism by Proposition 1.18. So there exists a unique map $\alpha\colon V \to W$ making the diagram

$$\begin{array}{ccc} P(V)^\times & \xrightarrow{\ \ell^\times\ } & W^{\times\times} \\ \Big\uparrow{\scriptstyle \mathrm{eval}_{(-)}} & & \Big\uparrow{\scriptstyle \varepsilon_W} \\ V & \xrightarrow[\ \alpha\ ]{} & W \end{array}$$

commute.

Lemma 2.24. The map α is a polynomial map and we have $\alpha^* = \eta$.

Proof. Let φ be an element of $W^\times$. To prove that α is a polynomial map such that $\alpha^* = \eta$, it suffices to prove that $\varphi \circ \alpha$ is the polynomial function on V associated to the polynomial $\ell(\varphi)$ on V, because η is the unique extension of ℓ to $P(W)$.

Let v be an element of V. Note that the diagram

$$\begin{array}{ccccc} P(V)^\times & \xrightarrow{\ \ell^\times\ } & W^{\times\times} & \xrightarrow{\ \varepsilon_{W^\times}(\varphi)\ } & K \\ \Big\uparrow{\scriptstyle \mathrm{eval}_{(-)}} & & \Big\uparrow{\scriptstyle \varepsilon_W} & & \Big\| \\ V & \xrightarrow[\ \alpha\ ]{} & W & \xrightarrow[\ \varphi\ ]{} & K \end{array}$$

commutes. So we have $(\varphi \circ \alpha)(v) = \varepsilon_{W^\times}(\varphi)(\ell^\times(\mathrm{eval}_v))$. Recall that the map $\ell^\times\colon P(V)^\times \to W^{\times\times}$ sends ϕ to the K-linear map $\phi \circ \ell$. So we have

$$\varepsilon_{W^\times}(\varphi)(\ell^\times(\mathrm{eval}_v)) = (\mathrm{eval}_v \circ \ell)(\varphi) = \mathrm{eval}_v(\ell(\varphi)) = \ell(\varphi)(v).$$

We see that $\varphi \circ \alpha$ is indeed the polynomial function on V associated to the polynomial $\ell(\varphi)$ on V. Hence α is a polynomial map and since α^* and η are both the extension of ℓ to $P(W)$, we see that $\alpha^* = \eta$. □

We see that polynomial maps $V \to W$ and homomorphisms of K-algebras $P(W) \to P(V)$ correspond one to one.

Proposition 2.25. Let $\alpha\colon V \to W$ be a polynomial map and let $f\colon W \to K$ be a polynomial function on W. Then $f \circ \alpha$ is the polynomial function on V associated to polynomial $\alpha^*(f)$ on V.

Proof. Recall that $\alpha^*\colon P(W) \to P(V)$ is the extension of the K-linear map $\ell\colon W^\times \to P(V)$ to $P(W)$ where ℓ sends φ to the polynomial on V corresponding to the polynomial function $\varphi \circ \alpha$ on V. So we know that α^* satisfies

$$\begin{aligned}
\alpha^*(\varphi_1 \odot \cdots \odot \varphi_n) &= \ell(\varphi_1)\cdots\ell(\varphi_n)\\
&= (\varphi_1 \circ \alpha)\cdots(\varphi_n \circ \alpha)\\
&= (v \mapsto \varphi_1(\alpha(v))\cdots\varphi_n(\alpha(v)))\\
&= (w \mapsto \varphi_1(w)\cdots\varphi_n(w)) \circ \alpha
\end{aligned}$$

for all $\varphi_1, \ldots, \varphi_n \in W^\times$. Note that $(v \mapsto \varphi_1(v)\cdots\varphi_n(v))$ is the polynomial function on W corresponding to the polynomial $\varphi_1 \odot \cdots \odot \varphi_n$ on W for all $\varphi_1, \ldots, \varphi_n \in W^\times$. So since such polynomials on W span $P(W)$ as a vector space over K, we see that $f \circ \alpha$ is the polynomial function on V corresponding to the polynomial $\alpha^*(f)$ on V for all $f \in P(W)$. □

Corollary 2.26. Let $\alpha\colon U \to V$ and $\beta\colon V \to W$ be polynomial maps. Then $\beta \circ \alpha$ is a polynomial map and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.

Proof. Let $f\colon W \to K$ be a polynomial function on W. Then $f \circ \beta$ is the polynomial function on V corresponding to the polynomial $\beta^*(f)$ on V. Therefore $(f \circ \beta) \circ \alpha$ is the polynomial function on U corresponding to the polynomial $\alpha^*(\beta^*(f))$ on U. In particular, we see that $\varphi \circ (\beta \circ \alpha)$ is the polynomial function on U corresponding to the polynomial $(\alpha^* \circ \beta^*)(\varphi)$ on U for all $\varphi \in W^\times$. Hence $\beta \circ \alpha$ is a polynomial map and $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$. □

We get a contravariant functor from the category whose objects are finite-dimensional vector spaces over K and whose morphisms are polynomial maps to the category of K-algebras.

Definition 2.27. Let $\alpha\colon V \to W$ be a polynomial map and let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. We say that α is homogeneous of degree n if the restriction of α^* to $W^\times$ is a non-zero K-linear map $\ell\colon W^\times \to \mathrm{Sym}^n(V^\times)$.

Remark 2.28. Any K-linear map $\ell\colon W^\times \to P(V)$ can be written uniquely as the sum of K-linear maps $\ell_i\colon W^\times \to \mathrm{Sym}^i(V^\times)$ for $i \in \mathbb{Z}_{\geq 0}$. Since W is finite dimensional over K, only finitely many of these maps ℓ_i can be non-zero. As a consequence, any polynomial map $V \to W$ can be uniquely written as a finite sum of homogeneous polynomial maps $V \to W$ of distinct degrees.

2.29. Let $\alpha\colon V \to W$ be a map. Then α is a homogeneous polynomial map of degree zero if and only if α is constant and non-zero and α is a homogeneous polynomial map of degree one if and only if α is K-linear and non-zero.

A K-linear combination of two homogeneous polynomial maps $V \to W$ of degree $n \in \mathbb{Z}_{\geq 0}$ is either zero or a homogeneous polynomial map $V \to W$ of degree n.

Let $\alpha\colon U \to V$ and $\beta\colon V \to W$ be homogeneous polynomial maps of degree n and m. Then $\beta \circ \alpha$ is either zero or a homogeneous polynomial map of degree nm.

The next proposition will give us a useful way to construct homogeneous polynomial maps. To prove the proposition, we use a following lemma.

Lemma 2.30. Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer.

(a) The K-linear map

$$\begin{array}{rcl} V^{\times} \otimes W & \to & \mathrm{Hom}_K(V, W) \\ \varphi \otimes w & \mapsto & (v \mapsto \varphi(v)w) \end{array}$$

is an isomorphism.

(b) The K-linear map

$$\begin{array}{rcl} \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)) & \to & \mathrm{Hom}_K(U \otimes V, W) \\ g & \mapsto & (u \otimes v \mapsto g(u)(v)) \end{array}$$

is an isomorphism.

(c) The K-linear map

$$\begin{array}{rcl} V^{\times} \otimes W^{\times} & \to & (V \otimes W)^{\times} \\ \varphi \otimes \phi & \mapsto & (v \otimes w \mapsto \varphi(v)\phi(w)) \end{array}$$

is an isomorphism.

(d) The K-linear map

$$\begin{array}{rcl} \mu\colon \left(V^{\times}\right)^{\otimes n} & \to & \left(V^{\otimes n}\right)^{\times} \\ \varphi_n \otimes \cdots \otimes \varphi_n & \mapsto & (v_1 \otimes \cdots \otimes v_n \mapsto \varphi_1(v_1) \ldots \varphi_n(v_n)) \end{array}$$

is an isomorphism.

(e) Let $\pi_V^n \colon V^{\otimes n} \to \mathrm{Sym}^n(V)$ and $\pi_{V^\times}^n \colon (V^\times)^{\otimes n} \to \mathrm{Sym}^n(V^\times)$ be the projection maps and let ν be the K-linear map making the diagram

$$\begin{array}{ccc} (V^{\otimes n})^\times & \xrightarrow{\mu^{-1}} & (V^\times)^{\otimes n} \\ {\scriptstyle \pi_V^{n\times}}\uparrow & & \downarrow{\scriptstyle \pi_{V^\times}^n} \\ \mathrm{Sym}^n(V)^\times & \xrightarrow{\nu} & \mathrm{Sym}^n(V^\times) \end{array}$$

commute. If $\mathrm{char}(K) \nmid n!$, then ν is an isomorphism.

Proof.

(a) Let $(v_1, \dots, v_n)$ be a basis of V and let $(\varphi_1, \dots, \varphi_n)$ be its dual basis. Then the K-linear map

$$\begin{array}{rcl} \mathrm{Hom}_K(V, W) & \to & V^\times \otimes W \\ f & \mapsto & \displaystyle\sum_{i=1}^n \varphi_i \otimes f(v_i) \end{array}$$

is the inverse.

(b) The K-linear map

$$\begin{array}{rcl} \mathrm{Hom}_K(U \otimes V, W) & \to & \mathrm{Hom}_K(U, \mathrm{Hom}_K(V, W)) \\ f & \mapsto & (u \mapsto (v \mapsto f(u \otimes v))) \end{array}$$

is the inverse.

(c) Using part (a) and (b), we have

$$\begin{array}{rcl} V^\times \otimes W^\times & \cong & \mathrm{Hom}_K(V, W^\times) = \mathrm{Hom}_K(V, \mathrm{Hom}_K(W, K)) \\ & \cong & \mathrm{Hom}_K(V \otimes W, K) = (V \otimes W)^\times. \end{array}$$

This isomorphism sends $\varphi \otimes \phi$ to $(v \mapsto \varphi(v)\phi)$ to $(v \otimes w \mapsto \varphi(v)\phi(w))$ for all $\varphi \in V^\times$ and $\phi \in W^\times$.

(d) We will prove part (d) using induction on n. Part (d) holds for $n = 0, 1$. Suppose that part (d) holds for $n \in \mathbb{Z}_{\geq 1}$. Then we see using part (c) that

$$\begin{array}{rcl} \left(V^\times\right)^{\otimes n+1} = \left(V^\times\right)^{\otimes n} \otimes V^\times & \cong & \left(V^{\otimes n}\right)^\times \otimes V^\times \\ & \cong & \left(V^{\otimes n} \otimes V\right)^\times = \left(V^{\otimes n+1}\right)^\times. \end{array}$$

For all $\varphi_1, \dots, \varphi_{n+1} \in V^\times$, this isomorphism sends $\varphi_n \otimes \cdots \otimes \varphi_{n+1}$ to

$$(v_1 \otimes \cdots \otimes v_n \mapsto \varphi_1(v_1) \dots \varphi_n(v_n)) \otimes \varphi_{n+1}$$

to $(v_1 \otimes \cdots \otimes v_{n+1} \mapsto \varphi_1(v_1) \dots \varphi_{n+1}(v_{n+1}))$. Therefore part (d) holds for $n + 1$. So by induction, part (d) holds for all $n \in \mathbb{Z}_{\geq 0}$.

(e) Suppose that $\mathrm{char}(K) \nmid n!$. Let

$$\begin{array}{rcl} \iota_V^n\colon \mathrm{Sym}^n(V) & \to & V^{\otimes n} \\ v_1 \odot \cdots \odot v_n & \mapsto & \dfrac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \end{array}$$

be the section of π_V^n from 2.7 and similarly let $\iota_{V^\times}^n$ be the section of $\pi_{V^\times}^n$. Let ν' be the K-linear map making the diagram

$$\begin{array}{ccc} (V^{\otimes n})^\times & \xleftarrow{\ \mu\ } & (V^\times)^{\otimes n} \\ \downarrow{\scriptstyle \iota_V^{n\times}} & & \uparrow{\scriptstyle \iota_{V^\times}^n} \\ \mathrm{Sym}^n(V)^\times & \xleftarrow[\ \nu'\ ]{} & \mathrm{Sym}^n(V^\times) \end{array}$$

commute. Then we have

$$\begin{array}{rcl} \nu \circ \nu' & = & \pi_{V^\times}^n \circ \mu^{-1} \circ \pi_V^{n\times} \circ \iota_V^{n\times} \circ \mu \circ \iota_{V^\times}^n \\ & = & \pi_{V^\times}^n \circ \mu^{-1} \circ (\iota_V^n \circ \pi_V^n)^\times \circ \mu \circ \iota_{V^\times}^n \\ & = & \pi_{V^\times}^n \circ \mu^{-1} \circ \mathrm{id}_{\mathrm{Sym}^n(V)}^\times \circ \mu \circ \iota_{V^\times}^n \\ & = & \pi_{V^\times}^n \circ \mu^{-1} \circ \mathrm{id}_{\mathrm{Sym}^n(V)^\times} \circ \mu \circ \iota_{V^\times}^n \\ & = & \pi_{V^\times}^n \circ \iota_{V^\times}^n \\ & = & \mathrm{id}_{\mathrm{Sym}^n(V^\times)} \end{array}$$

and we similarly have $\nu' \circ \nu = \mathrm{id}_{\mathrm{Sym}^n(V)^\times}$. So ν' is the inverse of ν. □

Remark 2.31. In the language of category theory, part (b) of the proposition states that the functor $- \otimes V\colon \underline{\mathrm{fVect}}_K \to \underline{\mathrm{fVect}}_K$ is left adjoint to the functor $\mathrm{Hom}_K(V, -)\colon \underline{\mathrm{fVect}}_K \to \underline{\mathrm{fVect}}_K$.

Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer and let $\delta\colon V \to \mathrm{Sym}^n(V)$ be the map sending v to $v^{\odot n}$.

Proposition 2.32. For each K-linear map $\ell\colon \mathrm{Sym}^n(V) \to W$, the map $\ell \circ \delta$ is either zero or a homogeneous polynomial map of degree n. In particular, the map δ is a homogeneous polynomial map of degree n if V is not zero.

Proof. Let $\nu\colon \mathrm{Sym}^n(V)^\times \to \mathrm{Sym}^n(V^\times)$ be the K-linear map from part (e) of Lemma 2.30. Let $\ell\colon \mathrm{Sym}^n(V) \to W$ be a K-linear map and let

$$\eta\colon P(W) \to P(V)$$

be the extension of the K-linear map $\nu \circ \ell^\times\colon W^\times \to \mathrm{Sym}(V^\times)$ to $P(W)$. Let $\alpha\colon V \to W$ be the polynomial map corresponding to η. By the construction of α, the diagram

$$\begin{array}{ccc} P(V)^\times & \xrightarrow{(\nu \circ \ell^\times)^\times} & W^{\times\times} \\ \uparrow{\scriptstyle \mathrm{eval}_{(-)}} & & \uparrow{\scriptstyle \varepsilon_W} \\ V & \xrightarrow[\ \alpha\ ]{} & W \end{array}$$

commutes. Recall that ε_W is an isomorphism. It suffices to prove that $\alpha = \ell \circ \delta$ or equivalently that the diagram

$$\begin{array}{ccccc} P(V)^\times & & \xrightarrow{(\nu\circ\ell^\times)^\times} & & W^{\times\times} \\ \uparrow{\scriptstyle \mathrm{eval}_{(-)}} & & & & \uparrow{\scriptstyle \varepsilon_W} \\ V & \xrightarrow{\delta} & \mathrm{Sym}^n(V) & \xrightarrow{\ell} & W \end{array}$$

commutes.

Let v be an element of V and let φ be an element of $W^\times$. Then we have

$$(\varepsilon_W \circ \ell \circ \delta)(v)(\varphi) = \varepsilon_W(\ell(v^{\odot n}))(\varphi) = \varphi(\ell(v^{\odot n})) = (\varphi \circ \ell)(v^{\odot n})$$

and

$$\left((\nu \circ \ell^\times)^\times \circ \mathrm{eval}_{(-)}\right)(v)(\varphi) = (\nu \circ \ell^\times)^\times(\mathrm{eval}_v)(\varphi) = (\mathrm{eval}_v \circ \nu)(\varphi \circ \ell).$$

So it is suffices to check that $\mathrm{eval}_v \circ \nu \colon \mathrm{Sym}^n(V)^\times \to K$ sends ϕ to $\phi(v^{\odot n})$.

Note that the K-linear map

$$\begin{array}{rcl} \varepsilon\colon (V^\times)^{\otimes n} & \to & K \\ \varphi_1 \otimes \cdots \otimes \varphi_n & \mapsto & \varphi_1(v) \cdots \varphi_n(v) \end{array}$$

makes the diagram

$$\begin{array}{ccc} (V^{\otimes n})^\times & \xrightarrow{\varepsilon_{V^{\otimes n}}(v^{\otimes n})} & K \\ \downarrow{\scriptstyle \mu^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} \\ (V^\times)^{\otimes n} & \xrightarrow{\varepsilon} & K \\ \downarrow{\scriptstyle \pi_{V^\times}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Sym}^n(V^\times) & \xrightarrow{\mathrm{eval}_v} & K \end{array}$$

commute. So the diagram

$$\begin{array}{ccccc} & & (V^{\otimes n})^\times & \xrightarrow{\varepsilon_{V^{\otimes n}}(v^{\otimes n})} & K \\ & \nearrow^{\mathrm{id}} & \downarrow{\scriptstyle \mu^{-1}} & & \downarrow{\scriptstyle \mathrm{id}} \\ (V^{\otimes n})^\times & \xrightarrow{\mu^{-1}} & (V^\times)^{\otimes n} & \xrightarrow{\varepsilon} & K \\ \uparrow{\scriptstyle \pi_V^\times} & & \downarrow{\scriptstyle \pi_{V^\times}} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{Sym}^n(V)^\times & \xrightarrow{\nu} & \mathrm{Sym}^n(V^\times) & \xrightarrow{\mathrm{eval}_v} & K \end{array}$$

also commutes. Let ϕ be an element of $\mathrm{Sym}^n(V)^\times$. Then we see that

$$(\mathrm{eval}_v \circ \nu)(\phi) = \varepsilon_{V^{\otimes n}}(v^{\otimes n})(\pi_V^\times(\phi)) = (\phi \circ \pi_V)(v^{\otimes n}) = \phi(v^{\odot n}).$$

So $\ell \circ \delta = \alpha$ is a polynomial map which is either zero or homogeneous of degree n. Taking ℓ to be the identity on $\mathrm{Sym}^n(V)$, we see that δ is a homogeneous polynomial of degree n if V is not zero. $\square$

Let V, W be finite dimensional vector spaces over K and let $r \in \mathbb{Z}_{\geq 0}$. Let $(v_1, \ldots, v_n)$ be a basis of V. Let $(w_1, \ldots, w_m)$ be a basis of W and let $(\phi_1, \ldots, \phi_m)$ be its dual basis. For $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, let $\varphi_{ij} \in \mathrm{Hom}_K(V, W)^\times$ be the map

$$\begin{array}{rcl} \varphi_{ij} \colon \mathrm{Hom}_K(V, W) & \to & K \\ \ell & \mapsto & \phi_j(\ell(v_i)) \end{array}$$

Then $\varphi_{11}, \ldots, \varphi_{nm}$ form a basis of $\mathrm{Hom}_K(V, W)^\times$. Since we have chosen bases for V and W, we can identify a K-linear map $\ell \colon V \to W$ with the matrix $(a_{ij})_{ij}$ such that $\ell(v_i) = \sum_{j=1}^m a_{ij} w_j$ for all $i \in \{1, \ldots, n\}$. Note that this matrix is precisely $(\varphi_{ij}(\ell))_{ij}$.

Since $v_1, \ldots, v_n$ form a basis of V, we know that

$$(v_{i_1} \wedge \cdots \wedge v_{i_r} | 1 \leq i_1 < \cdots < i_r \leq n)$$

is a basis of $\Lambda^r V$. Since $w_1, \ldots, w_m$ form a basis of W, we similarly know that

$$(w_{j_1} \wedge \cdots \wedge w_{j_r} | 1 \leq j_1 < \cdots < j_r \leq m)$$

is a basis of $\Lambda^r W$. Let $(\phi_{j_1 \ldots j_r} | 1 \leq j_1 < \cdots < j_r \leq m)$ be its dual basis. For $1 \leq i_1 < \cdots < i_r \leq n$ and $1 \leq j_1 < \cdots < j_r \leq m$, let $\phi_{i_1 \ldots i_r j_1 \ldots j_r}$ be the K-linear map

$$\begin{array}{rcl} \phi_{i_1 \ldots i_r j_1 \ldots j_r} \colon \mathrm{Hom}_K(\Lambda^r V, \Lambda^r W) & \to & K \\ \ell & \mapsto & \phi_{j_1 \ldots j_r}(\ell(v_{i_1} \wedge \cdots \wedge v_{i_r})) \end{array}$$

Then the $\phi_{i_1 \ldots i_r j_1 \ldots j_r}$ form for a basis of $\mathrm{Hom}_K(\Lambda^r V, \Lambda^r W)^\times$.

Proposition 2.33.

(i) The map

$$\begin{array}{rcl} \alpha \colon \mathrm{Hom}_K(V, W) & \to & \mathrm{Hom}_K(\Lambda^r V, \Lambda^r W) \\ \ell & \mapsto & \Lambda^r \ell \end{array}$$

is a homogeneous polynomial map of degree r.

(ii) The homomorphism of K-algebras α^* sends $\phi_{i_1 \ldots i_r j_1 \ldots j_r}$ to the determinant of the $r \times r$ submatrix of the matrix

$$M = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1m} \\ \vdots & & \vdots \\ \varphi_{n1} & \cdots & \varphi_{nm}. \end{pmatrix}$$

over $P(\mathrm{Hom}_K(V, W))$ consisting of the rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$ for all $1 \leq i_1 < \cdots < i_r \leq n$ and $1 \leq j_1 < \cdots < j_r \leq m$.

Proof. Let $\ell\colon V \to W$ be a K-linear map. Let $1 \le i_1 < \cdots < i_r \le n$ and $1 \le j_1 < \cdots < j_r \le m$ be integers. By 2.16, we see that

$$(\phi_{i_1 \ldots i_r j_1 \ldots j_r} \circ \alpha)(\ell) = \phi_{j_1 \ldots j_r}(\ell(v_{i_1}) \wedge \cdots \wedge \ell(v_{i_r}))$$

is the determinant of the $r \times r$ submatrix of M consisting of the rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$, which is a polynomial function on $\mathrm{Hom}_K(V, W)$. So since the $\phi_{i_1 \ldots i_r j_1 \ldots j_r}$ form a basis of $\mathrm{Hom}_K(\Lambda^r V, \Lambda^r W)$ we see that α is polynomial. We also see that (b) holds. Since the determinant of a matrix is homogeneous in the entries of the matrix, we see that α is homogeneous of degree r. □

2.4 Affine varieties

Let V be a finite-dimensional vector space over K.

Definition 2.34. Define the affine space $\mathbb{A}(V)$ to be the set V.

Definition 2.35. Let S be a subset of $P(V)$. Define the zero set of S to be the subset

$$Z_{\mathbb{A}(V)}(S) := \{P \in \mathbb{A}(V) | f(P) = 0 \text{ for all } f \in S\}$$

of $\mathbb{A}(V)$.

Proposition 2.36.

(i) Let S be a subset of $P(V)$ and let I be the ideal of $P(V)$ generated by S. Then we have $Z_{\mathbb{A}(V)}(S) = Z_{\mathbb{A}(V)}(I)$.

(ii) We have $Z_{\mathbb{A}(V)}(\emptyset) = \mathbb{A}(V)$ and $Z_{\mathbb{A}(V)}(P(V)) = \emptyset$.

(iii) Let I be a set and let $(S_i)_{i \in I}$ be a family of subsets of $P(V)$. Then we have

$$Z_{\mathbb{A}(V)}\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} Z_{\mathbb{A}(V)}(S_i).$$

(iv) Let I, J be ideals of $P(V)$. Then we have

$$Z_{\mathbb{A}(V)}(I \cap J) = Z_{\mathbb{A}(V)}(IJ) = Z_{\mathbb{A}(V)}(I) \cup Z_{\mathbb{A}(V)}(J).$$

By Proposition 2.36, the subsets of $\mathbb{A}(V)$ of the form $Z_{\mathbb{A}(V)}(S)$ for some subset $S \subseteq P(V)$ form the closed subsets of a topology on $\mathbb{A}(V)$.

Definition 2.37. Define the Zariski topology on $\mathbb{A}(V)$ to be the topology on $\mathbb{A}(V)$ whose closed subsets are the subsets of the form $Z_{\mathbb{A}(V)}(S)$ for some subset S of $P(V)$.

Remark 2.38. Let $(x_1, \dots, x_n)$ be a basis of $V^\times$. Then $P(V)$ is isomorphic to $K[x_1, \dots, x_n]$ by Example 2.11. So the ring $P(V)$ is Noetherian. Therefore all closed subsets of $\mathbb{A}(V)$ are of the form $Z_{\mathbb{A}(V)}(S)$ for some finite subset S of $P(V)$.

Definition 2.39. Let X be a subset of $\mathbb{A}(V)$. Define the ideal of X to be the ideal

$$I_{\mathbb{A}(V)}(X) := \{f \in P(V) | f(P) = 0 \text{ for all } P \in X\}$$

of $P(V)$.

Proposition 2.40. Let $\alpha\colon V \to W$ be a polynomial map. Then we have $I_{\mathbb{A}(V)}(\operatorname{im}\alpha) = \ker \alpha^*$ and $Z_{\mathbb{A}(V)}(\alpha^*(W^\times)) = \ker \alpha$.

Proof. The ideal of $\operatorname{im}\alpha$ is the ideal

$$\{f \in P(W) | f(P) = 0 \text{ for all } P \in \operatorname{im}\alpha\}$$

of $P(W)$. Let $f \in P(W)$ be a polynomial on W. Then we see that $f(P) = 0$ for all $P \in \operatorname{im}\alpha$ if and only if $(f \circ \alpha)(P) = 0$ for all $P \in \mathbb{A}(V)$. Recall from Proposition 2.25 that $\alpha^*\colon P(W) \to P(V)$ sends f to the polynomial associated to the polynomial function $f \circ \alpha$. So we see that $(f \circ \alpha)(P) = 0$ for all $P \in \mathbb{A}(V)$ if and only $\alpha^*(f)$ is the polynomial on V associated to the zero function $V \to K$. The polynomial associated to the zero function is the zero polynomial. Hence $I_{\mathbb{A}(V)}(\operatorname{im}\alpha) = \ker \alpha^*$.

The zero set of $\alpha^*(W^\times)$ is the subset

$$\{P \in \mathbb{A}(V) | f(P) = 0 \text{ for all } f \in \alpha^*(W^\times)\}$$

of $\mathbb{A}(V)$. Let $P \in \mathbb{A}(V)$ be a point. Then we have $f(P) = 0$ for all $f \in \alpha^*(W^\times)$ if and only if we have $\varphi(\alpha(P)) = (\varphi \circ \alpha)(P) = 0$ for all $\varphi \in W^\times$. So we see that P is an element of the zero set of $\alpha^*(W^\times)$ if and only if $\alpha(P)$ is contained in the zero set of $W^\times$, which is $\{0\}$. Hence $Z_{\mathbb{A}(V)}(\alpha^*(W^\times)) = \ker \alpha$. □

Proposition 2.41.

(i) The map $S \mapsto Z_{\mathbb{A}(V)}(S)$ is inclusion reversing.

(ii) The map $X \mapsto I_{\mathbb{A}(V)}(X)$ is inclusion reversing.

(iii) Let X be a subset of $\mathbb{A}(V)$. Then $Z_{\mathbb{A}(V)}(I_{\mathbb{A}(V)}(X))$ is the closure of X in $\mathbb{A}(V)$.

Theorem 2.42 (Hilbert's Nullstellensatz)**.** Let I be an ideal of $P(V)$. Then $I_{\mathbb{A}(V)}(Z_{\mathbb{A}(V)}(I))$ is the radical ideal of I.

Proof. See 1.10 from [Mo]. □

Corollary 2.43. The map $S \mapsto Z_{\mathbb{A}(V)}(S)$ induces a bijection between the set of closed subsets of $\mathbb{A}(V)$ and the set of radical ideals of $P(V)$.

Definition 2.44. A topological space X is called irreducible when for all closed subsets X_1, X_2 of X with $X_1 \cup X_2 = X$, we have $X_1 = X$ or $X_2 = X$.

Proposition 2.45. Let X be a closed subset of $\mathbb{A}(V)$. Then X is irreducible if and only if the radical ideal $I_{\mathbb{A}(V)}(X)$ of $P(V)$ is prime.

Definition 2.46. An affine variety inside $\mathbb{A}(V)$ is an irreducible closed subset of $\mathbb{A}(V)$. An affine variety is an affine variety inside some affine space.

Example 2.47. The zero ideal of $P(V)$ is prime. Hence $\mathbb{A}(V)$ is an affine variety inside $\mathbb{A}(V)$.

Proposition 2.48. Let V, W be finite dimensional vector spaces over K and let $r \in \mathbb{Z}_{\geq 0}$. Then the kernel of the homogeneous polynomial map

$$\begin{array}{rcl} \alpha\colon \operatorname{Hom}_K(V,W) & \to & \operatorname{Hom}_K(\Lambda^r V, \Lambda^r W) \\ \varphi & \mapsto & \Lambda^r \varphi \end{array}$$

of degree r from Proposition 2.33 is the affine variety inside $\mathbb{A}(\operatorname{Hom}_K(V,W))$ consisting of the linear maps $V \to W$ with rank lower than r. Its corresponding prime ideal is generated by $\alpha^*(\operatorname{Hom}_K(\Lambda^r V, \Lambda^r W)^\times)$.

Proof. Recall that for $w_1, \dots, w_r \in W$, the element $w_1 \wedge \cdots \wedge w_n$ of $\Lambda^r W$ is non-zero if and only if $w_1, \dots, w_r$ are linearly independent over K. Combining this with the fact that a K-linear map $\ell\colon V \to W$ has rank r if and only if there exist $v_1, \dots, v_r \in V$ such that $\ell(v_1), \dots, \ell(v_r)$ are linearly independent over K, we see that the kernel of α is the zero of $\operatorname{im} \alpha^*$. By Theorem 2.10 from [BV] applied with $B = K$ and $m = r$, we see that the ideal generated by $\alpha^*(\operatorname{Hom}_K(\Lambda^r V, \Lambda^r W)^\times)$ is prime. Hence the kernel of α is the affine variety inside $\mathbb{A}(\operatorname{Hom}_K(V,W))$ corresponding to the prime ideal generated by $\alpha^*(\operatorname{Hom}_K(\Lambda^r V, \Lambda^r W)^\times)$. □

Let X be an affine variety inside $\mathbb{A}(V)$.

Definition 2.49. Define the Zariski topology on X to be the induced topology on X from the Zariski topology on $\mathbb{A}(V)$.

2.50. Let $f \in P(V)$ be a polynomial on V. Then the polynomial function $f\colon V \to K$ restricts to a function $f|_X\colon X \to K$. This gives us a homomorphism of K-algebras $\eta\colon P(V) \to \operatorname{Map}(X,K)$ sending $f \in P(V)$ to $f|_X$. By definition, the ideal $I_{\mathbb{A}(V)}(X)$ of $P(V)$ is the kernel of η. So we see that an element of $P(V)/I_{\mathbb{A}(V)}(X)$ corresponds to a map $X \to K$.

Definition 2.51. Define the coordinate ring of X to be the K-algebra $K[X] := P(V)/I_{\mathbb{A}(V)}(X)$. For an element $f \in K[X]$ and a point $P \in X$, we denote the value of the map corresponding to f at P by $f(P)$.

(i) Let S be a subset of $K[X]$. Define the zero set of S to be the subset

$$Z_X(S) := \{P \in X | f(P) = 0 \text{ for all } f \in S\}$$

of X.

(ii) Let Y be a subset of X. Define the ideal of Y to be the ideal

$$I_X(Y) := \{f \in K[X] | f(P) = 0 \text{ for all } P \in Y\}$$

of $K[X]$.

Proposition 2.52. Let X be an affine variety.

(i) The map $S \mapsto Z_X(S)$ is inclusion reversing.

(ii) The map $Y \mapsto I_X(Y)$ is inclusion reversing.

(iii) Let Y be a subset of X. Then $Z_X(I_X(Y))$ is the closure of Y in X.

(iv) Let I be an ideal of $K[X]$. Then $I_X(Z_X(I))$ is the radical ideal of I.

(v) The map $S \mapsto Z_X(S)$ induces a bijection between the closed subsets of X and the radical ideals of $K[X]$.

(vi) Let Y be a closed subset of X. Then Y is irreducible if and only if the radical ideal $I_X(Y)$ of $K[X]$ is prime.

2.5 Morphisms of affine varieties

Let V, W be finite dimensional vector spaces over K. Let X be an affine variety inside $\mathbb{A}(V)$ and let Y be an affine variety inside $\mathbb{A}(W)$.

Definition 2.53. Let $\Phi\colon X \to Y$ be a map. We say that Φ is a morphism of affine varieties if there exists a polynomial map $\alpha\colon V \to W$ such that $\Phi(P) = \alpha(P)$ for all $P \in X$.

2.54. Let $\Phi\colon X \to Y$ be a morphism and let $\alpha\colon V \to W$ be a polynomial map such that $\Phi(P) = \alpha(P)$ for all $P \in X$. Consider the homomorphism of K-algebras

$$\alpha^*\colon P(W) \to P(V)$$

which sends a polynomial f on W to the polynomial on V associated to the polynomial function $f \circ \alpha$ on V by Proposition 2.25. Let $\pi_X\colon P(V) \to K[X]$ and $\pi_Y\colon P(W) \to K[Y]$ be the projection maps. Let $f \in I_{\mathbb{A}(W)}(Y)$ be a polynomial on W that is zero on Y. Then we see that $f \circ \alpha$ is zero on X.

So $\alpha^*(I_{\mathbb{A}(W)}(Y)) \subseteq I_{\mathbb{A}(V)}(X)$. Therefore α^* induces a homomorphism of K-algebras $\Phi^*\colon K[Y] \to K[X]$ making the diagram

$$\begin{array}{ccc} P(W) & \xrightarrow{\alpha^*} & P(V) \\ \big\downarrow{\scriptstyle \pi_Y} & & \big\downarrow{\scriptstyle \pi_X} \\ K[Y] & \xrightarrow[\Phi^*]{} & K[X] \end{array}$$

commute.

Let $\beta\colon V \to W$ also be a polynomial map such that $\Phi(P) = \beta(P)$ for all $P \in X$. Then we see that $(\alpha - \beta)(x) = 0$ for all $x \in X$. So the polynomial function $f \circ (\alpha - \beta)$ on V is zero for all $f \in P(W)$. Hence the image of the homomorphism of K-algebras $(\alpha - \beta)^*\colon P(W) \to P(V)$ is contained in $I_{\mathbb{A}(V)}(X)$. So we see that α and β induce the same homomorphism of K-algebras $K[Y] \to K[X]$. Hence Φ^* is well-defined.

2.55. Let $\theta\colon K[Y] \to K[X]$ be a homomorphism of K-algebras. Consider the restriction $\bar{\ell}$ of the composition $\theta \circ \pi_Y\colon P(W) \to K[X]$ to $W^\times$. Let $\ell\colon W^\times \to P(V)$ be any K-linear map making the diagram

$$\begin{array}{ccc} W^\times & \xrightarrow{\ell} & P(V) \\ & \searrow^{\bar{\ell}} & \big\downarrow{\scriptstyle \pi_X} \\ & & K[X] \end{array}$$

commute and let $\eta\colon P(W) \to P(V)$ be the extension of ℓ to $P(W)$. Then the diagram

$$\begin{array}{ccc} W^\times & \xrightarrow{\ell} & P(V) \\ \big\downarrow{\scriptstyle \pi_Y|_{W^\times}} & \searrow^{\bar{\ell}} & \big\downarrow{\scriptstyle \pi_X} \\ K[Y] & \xrightarrow{\theta} & K[X] \end{array}$$

commutes and therefore the diagram

$$\begin{array}{ccc} P(W) & \xrightarrow{\eta} & P(V) \\ \big\downarrow{\scriptstyle \pi_Y} & & \big\downarrow{\scriptstyle \pi_X} \\ K[Y] & \xrightarrow{\theta} & K[X] \end{array}$$

also commutes. Let $\alpha\colon V \to W$ be the polynomial map associated to η. Then $\alpha^*(I_{\mathbb{A}(W)}(Y)) \subseteq I_{\mathbb{A}(V)}(X)$. So we have $\alpha(X) \subseteq Z_{\mathbb{A}(W)}(I_{\mathbb{A}(W}(Y)) = Y$.

Let $\Phi\colon X \to Y$ be the morphism of affine varieties we get by restricting

α to X. Then we have $\Phi^* = \theta$, because the diagram

$$\begin{array}{ccc} P(W) & \xrightarrow{\alpha^*} & P(V) \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ K[Y] & \xrightarrow{\theta} & K[X] \end{array}$$

commutes. So we see that the morphism Φ is uniquely determined by θ.

Theorem 2.56. The morphisms of affine varieties $X \to Y$ correspond one to one with the homomorphisms of K-algebras $K[Y] \to K[X]$.

Let U be an open subset of the affine variety X inside $\mathbb{A}(V)$ and let U have the induced topology of X.

Definition 2.57. Let $P \in U$ be a point and let $f\colon U \to K$ be a function.

(i) The function f is called regular at P if there exists an open subset U' of U containing P together with polynomial functions g, h on V such that $h(x) \neq 0$ and $f(x) = g(x)/h(x)$ for all $x \in U'$.

(ii) The function f is called regular if it is regular at all points of U.

(iii) Define $\mathcal{O}_X(U)$ to be the K-algebra of regular functions on U.

Proposition 2.58.

- Let $f\colon U \to K$ be a regular function and consider K as the topological space $\mathbb{A}(K)$ with the Zariski topology. Then f is continuous.
- Let $f, g\colon U \to K$ be regular functions such that $f|_{U'} = g|_{U'}$ for some non-empty open subset U' of U. Then $f = g$.
- The natural map $K[X] \to \mathcal{O}_X(X)$ is an isomorphism.

Proposition 2.59. Let X, Y be affine varieties and let $\Phi\colon X \to Y$ be a map. Then the following are equivalent:

- the map Φ is a morphism of affine varieties;
- the map Φ is continuous and for every open subset U of Y and every regular function $f \in \mathcal{O}_Y(U)$, the function $f \circ \Phi\colon \Phi^{-1}(U) \to K$ is regular on $\Phi^{-1}(U)$.

2.6 Projective varieties

Let V be a finite-dimensional vector space over K.

Definition 2.60. Define the projective space $\mathbb{P}(V)$ to be the set of one-dimensional subspaces of V. For a non-zero element $v \in V$, denote the one-dimensional subspace of V spanned by v by $[v]$.

Let v, w be non-zero element of V. Then we see that $[v] = [w]$ if and only if $v = \lambda w$ for some $\lambda \in K^*$.

2.61. We call an ideal I of the graded K-algebra $P(V)$ homogeneous if it is generated by homogeneous polynomials F on V. Let v be an element of V, let $F \in P(V)$ be a homogeneous polynomial on V of degree d and let λ be a non-zero element of K. Then we see that $F(\lambda v) = \lambda^d F(v)$. So whether $F(v) = 0$ holds or not depends on only $[v]$. We write $F([v]) = 0$ when $F(v) = 0$.

Definition 2.62. Let S be a subset of $P(V)$ consisting of homogeneous polynomials on V. Define the zero set of S to be the subset

$$Z_{\mathbb{P}(V)}(S) := \{P \in \mathbb{P}(V) | F(P) = 0 \text{ for all } F \in S\}.$$

of $\mathbb{P}(V)$. Let I be a homogeneous ideal of $P(V)$. Then we define the zero set $Z_{\mathbb{P}(V)}(I)$ of I to be the zero set of the set of all homogeneous polynomial on V contained in I.

Proposition 2.63.

(i) Let S be a subset of $P(V)$ consisting of homogeneous polynomials on V and let I be the homogeneous ideal generated by S. Then we have $Z_{\mathbb{P}(V)}(I) = Z_{\mathbb{P}(V)}(S)$.

(ii) We have $Z_{\mathbb{P}(V)}(\emptyset) = \mathbb{P}(V)$ and $Z_{\mathbb{P}(V)}(P(V)) = \emptyset$.

(iii) Let I be a set and let $(S_i)_{i \in I}$ be a family of subsets of $P(V)$ whose members consists of homogeneous polynomials on V. Then we have

$$Z_{\mathbb{P}(V)}\left(\bigcup_{i \in I} S_i\right) = \bigcap_{i \in I} Z_{\mathbb{P}(V)}(S_i).$$

(iv) Let I, J be homogeneous ideals of $P(V)$. Then we have

$$Z_{\mathbb{P}(V)}(I \cap J) = Z_{\mathbb{P}(V)}(IJ) = Z_{\mathbb{P}(V)}(I) \cup Z_{\mathbb{P}(V)}(J).$$

Proposition 2.63 tell us that the subsets of $\mathbb{P}(V)$ of the form $Z_{\mathbb{P}(V)}(S)$ for some subset S of $P(V)$ which consists of homogeneous polynomials on V, form the closed subsets of a topology on $\mathbb{P}(V)$.

Definition 2.64. Define the Zariski topology on $\mathbb{P}(V)$ to be the topology on $\mathbb{P}(V)$ whose closed subsets are the subsets of the form $Z_{\mathbb{P}(V)}(S)$ for some subset S of $P(V)$ which consists of homogeneous polynomials on V.

Let $(v_1, \ldots, v_n)$ be a basis of V and let $(x_1, \ldots, x_n)$ be its corresponding dual basis.

2.65. Let $i \in \{1, \ldots, n\}$ be an integer. Consider the open subset

$$U_i := \{[v] \in \mathbb{P}(V) | x_i(v) \neq 0\}$$

of $\mathbb{P}(V)$. Every point $P \in U_i$ is of the form $[v]$ for some unique $v \in V$ such that $x_i(v) = 1$. So the map

$$\begin{array}{rcl} \psi_i \colon Z_{\mathbb{A}(V)}(x_i - 1) & \to & U_i \\ v & \mapsto & [v] \end{array}$$

is a bijection. The inverse of ψ_i is the map $U_i \to Z_{\mathbb{A}(V)}(x_i - 1)$ sending $[v]$ to $x_i(v)^{-1}v$.

Since the ideal $(x_i - 1)$ is a prime ideal of $P(V) \cong K[x_1, \ldots, x_n]$, we see that $I_{\mathbb{A}(V)}(Z_{\mathbb{A}(V)}(x_i - 1)) = (x_i - 1)$. So the coordinate ring of $Z_{\mathbb{A}(V)}(x_i - 1)$ is the K-algebra $P(V)/(x_i - 1)$. Note that the homomorphism of K-algebras

$$\begin{array}{rcl} P(V)/(x_i - 1) & \to & P(\ker x_i) \\ x_j + (x_i - 1) & \mapsto & \begin{cases} x_j & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases} \end{array}$$

is an isomorphism, allowing us to identify these two K-algebra with each other. We call the homomorphism of K-algebras

$$\mathrm{dehom}_i \colon P(V) \to P(\ker x_i)$$

we get by taking the composition of this isomorphism with the projection map $P(V) \to P(V)/(x_i - 1)$ the dehomogenisation map with respect to x_i.

Let $f \in P(\ker x_i)$ be a polynomial of degree d. Then there exists a unique homogeneous polynomial $F \in P(V)$ of degree d such that $\mathrm{dehom}_i(F) = f$. We call the map $\mathrm{hom}_i \colon P(\ker x_i) \to P(V)$ sending f to F the homogenisation map.

Proposition 2.66. Write $X = Z_{\mathbb{A}(V)}(x_i - 1)$.

- Let S be a subset of $P(V)$ consisting of homogeneous polynomials. Then we have

$$\psi_i^{-1}\left(Z_{\mathbb{P}(V)}(S) \cap U_i\right) = Z_X(\mathrm{dehom}_i(S)).$$

- Let I be an ideal of $P(\ker x_i)$ and take $Y = Z_X(I)$. Then the closure of $\psi_i(Y)$ in $\mathbb{P}(V)$ is the zero set of $\{\mathrm{hom}_i(f) | f \in I\}$.

- The bijection $\psi_i\colon X \to U_i$ is a homeomorphism.

We call the maps $\psi_1, \dots, \psi_n$ the affine charts of $\mathbb{P}(V)$ corresponding to the basis $(v_1, \dots, v_n)$ of V. We call the map $\psi_i\colon Z_{\mathbb{A}(V)}(x_i - 1) \to U_i$ the affine chart of $\mathbb{P}(V)$ corresponding to $x_i = 1$.

Definition 2.67. Let X be a subset of $\mathbb{P}(V)$. Define the ideal of X to be the homogeneous ideal $I_{\mathbb{P}(V)}(X)$ of $P(V)$ generated by all homogeneous polynomials $f \in P(V)$ such that $f(P) = 0$ for all $P \in X$.

Definition 2.68. A projective variety inside $\mathbb{P}(V)$ is a closed irreducible subset of $\mathbb{P}(V)$. A projective variety is a projective variety inside some projective space.

Definition 2.69. Let X be a projective variety inside $\mathbb{P}(V)$. Define the Zariski topology on X to be the induced topology from $\mathbb{P}(V)$.

Let X be a projective variety inside $\mathbb{P}(V)$ and let U be an open subset of X.

Proposition 2.70. Let $(v_1, \dots, v_n)$ be a basis of V and let $(x_1, \dots, x_n)$ be its dual basis. Let $\psi_i\colon Z_{\mathbb{A}(V)}(x_i - 1) \to U_i$ be the associated affine charts. Let $f\colon U \to K$ be a map, let $P \in U$ be a point and let $i \in \{1, \dots, n\}$ be an integer such that $P \in U_i$. Then the following are equivalent:

- the function $f \circ \psi_i\colon \psi_i^{-1}(U \cap U_i) \to K$ is regular at $\psi_i^{-1}(P)$;
- there exists an open subset U' of U containing P together with homogeneous polynomial maps $G, H \in P(V)$ of the same degree such that $H(v) \neq 0$ and $f([v]) = G(v)/H(v)$ for all $[v] \in U'$.

Definition 2.71.

- Let $P \in U$ be a point. Then a function $f\colon U \to K$ is called regular at P if the equivalent conditions of the previous proposition hold.
- A function $f\colon U \to K$ is called regular if it is regular at all points of U.
- Define $\mathcal{O}_X(U)$ to be the K-algebra of regular functions on U.

Definition 2.72. Let X, Y be affine or projective varieties. Let $\Phi\colon X \to Y$ be a map. Then we call Φ a morphism of varieties if Φ is continuous and for all open subsets U of Y and all regular functions $f \in \mathcal{O}_Y(U)$, the function $f \circ \Phi\colon \Phi^{-1}(U) \to K$ is regular on $\Phi^{-1}(U)$.

Proposition 2.73. Let $\alpha\colon V \to W$ be a homogeneous polynomial map such that $\alpha^{-1}(0) = \{0\}$. Then the map

$$\begin{array}{rcl} \Phi\colon \mathbb{P}(V) & \to & \mathbb{P}(W) \\ {[v]} & \mapsto & [\alpha(v)] \end{array}$$

is a morphism of projective varieties.

Proof. Let S be a subset of $P(W)$ consisting of homogeneous polynomials. Then the subset

$$\begin{aligned}\Phi^{-1}(Z_{\mathbb{P}(W)}(S)) &= \Phi^{-1}(\{P \in \mathbb{P}(W) | F(P) = 0 \text{ for all } F \in S\}) \\ &= Z_{\mathbb{P}(V)}(\{F \circ \alpha | F \in S\})\end{aligned}$$

is closed in $\mathbb{P}(V)$. Hence Φ is continuous.

Let U be an open subset of $\mathbb{P}(W)$, let $f \in \mathcal{O}_{\mathbb{P}(W)}(U)$ be a regular function and let $[v_0] \in \Phi^{-1}(U)$ be a point. Since the function f is regular on U, there exists an open subset U' of U containing $\Phi([v_0])$ together with homogeneous polynomial maps $G, H \in P(W)$ of the same degree such that $H(w) \neq 0$ and $f([w]) = G(w)/H(w)$ for all $[w] \in U'$. So we have the open subset $\Phi^{-1}(U')$ of $\Phi^{-1}(U)$ containing $[v_0]$ together with the homogeneous polynomial maps $G \circ \alpha, H \circ \alpha \in P(V)$ of the same degree such that $(H \circ \alpha)(v) \neq 0$ and $(f \circ \Phi)([v]) = (G \circ \alpha)(v)/(H \circ \alpha)(v)$ for all $[v] \in \Phi^{-1}(U')$. Hence $f \circ \Phi$ is regular at $[v_0]$. So $f \circ \Phi$ is a regular function on $\Phi^{-1}(U)$. Hence Φ is a morphism. □

Definition 2.74. Let X be a projective variety inside $\mathbb{P}(V)$. Define the cone of X to be the affine variety

$$\text{cone}(X) := \{v \in \mathbb{A}(V) | [v] \in X\} \cup \{0\}$$

inside $\mathbb{A}(V)$ which corresponds to the prime ideal $I_{\mathbb{P}(V)}(X)$ of $P(V)$.

Chapter 3

The varieties of e-th powers

In this chapter, let K be an algebraically closed field and let $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$ be integers.

For an integer $n \in \mathbb{Z}_{\geq 0}$, recall that the vector subspace of $K[x, y]$ spanned by the homogeneous polynomials degree n is denoted by V_n. Let CT_d be the subset of V_{de} consisting of all e-powers of polynomials in V_d. Consider the map

$$\begin{array}{rcl} \mathrm{pow}_d \colon V_d & \to & V_{de} \\ f & \mapsto & f^e \end{array}$$

whose image equals CT_d. By the universal property of the e-th symmetric power of V_d, the symmetric multilinear map $\omega \colon V_d^e \to V_{de}$ sending $(f_1, \ldots, f_e)$ to $f_1 \cdots f_e$ corresponds to the K-linear map

$$\begin{array}{rcl} \ell \colon \mathrm{Sym}^e(V_d) & \to & V_{de} \\ f_1 \odot \cdots \odot f_e & \mapsto & f_1 \cdots f_e. \end{array}$$

Let $\delta \colon V_d \to \mathrm{Sym}^e(V_d)$ be the map sending a polynomial f to $f^{\odot e}$. Then the diagram

$$\begin{array}{ccc} V_d & \xrightarrow{\mathrm{pow}_d} & V_{de} \\ {\scriptstyle \delta}\downarrow & \nearrow_{\ell} & \\ \mathrm{Sym}^e(V_d) & & \end{array}$$

commutes. So by Proposition 2.32, the map $\mathrm{pow}_d \colon V_d \to V_{de}$ is a homogeneous polynomial map of degree e. Corresponding to pow_d, we have the homomorphism of K-algebras

$$\begin{array}{rcl} \mathrm{pow}_d^* \colon P(V_{de}) & \to & P(V_d) \\ f & \mapsto & f \circ \mathrm{pow}_d \end{array}$$

where we identify polynomial functions on V_n with their corresponding polynomial maps on V_n for $n = d$ and $n = de$. By Proposition 2.40 we know that the ideal $I_d := I_{\mathbb{A}(V_{de})}(CT_d)$ corresponding to the subset CT_d of $\mathbb{A}(V_{de})$ is equal to the kernel of pow_d^*.

Since the field K is algebraically closed, each element $\lambda \in K$ is the e-th power of some element $\lambda^\dagger \in K$. So for all polynomials $g \in CT_d$ and elements $\lambda \in K$, we have $\lambda g \in CT_d$. Consider the subset $T_d := \{[g] \in \mathbb{P}(V_{de}) | g \in CT_d\}$ of $\mathbb{P}(V_{de})$. The polynomial map $\text{pow}_d \colon V_d \to V_{de}$ is homogeneous and satisfies $\text{pow}_d^{-1}(0) = \{0\}$. So by Proposition 2.73, we see that the map

$$\begin{array}{rrcl} \Pi_d \colon & \mathbb{P}(V_d) & \to & \mathbb{P}(V_{de}) \\ & [f] & \mapsto & [f^e] \end{array}$$

is a morphism of projective varieties. Since the topological space $\mathbb{P}(V_d)$ is irreducible and the map Π_d is continuous, we know that the image T_d of Π_d is irreducible. By Theorem 7.8 from [Mo], the projective space $\mathbb{P}(V_d)$ is complete. So by Corollary 7.6 of [Mo], we also know that the image T_d of Π_d is closed. Hence T_d is a projective variety inside $\mathbb{P}(V_{de})$. Note that CT_d is the cone of T_d and therefore an affine variety inside $\mathbb{A}(V_{de})$ as we claimed before. We see that I_d is equal to $I_{\mathbb{P}(V_{de})}(T_d)$ and hence I_d is a homogeneous ideal of $P(V_{de})$.

In this chapter, we will explain why we may restrict ourselves to the case where $\text{char}(K) \nmid e$. Assume that $\text{char}(K) \nmid e$. The e-th power of a monic polynomial is monic. We will use this fact to show that Π_d is an isomorphism onto its image when restricted to the affine chart M_d of $\mathbb{P}(V_d)$ consisting of all d-monic polynomials in $\mathbb{A}(V_d)$. By shifting this affine chart M_d, we can cover the whole of $\mathbb{P}(V_d)$. We will use this shifting and the fact that we get an isomorphism from M_d to its image, to show that Π_d is an isomorphism onto its image. Lastly, we will give two methods using Gröbner bases to compute the ideal I_d for instances of $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$.

3.1 Reducing to the case where $\text{char}(K) \nmid e$

In the section, denote the morphism

$$\begin{array}{rcl} \mathbb{P}(V_{d^\dagger}) & \to & \mathbb{P}(V_{d^\dagger e^\dagger}) \\ [f] & \mapsto & [f^{e^\dagger}] \end{array}$$

by $\Pi_{d^\dagger}^{e^\dagger}$ for all $d^\dagger \in \mathbb{Z}_{\geq 0}$ and $e^\dagger \in \mathbb{Z}_{\geq 1}$ and denote its image by $T_{d^\dagger}^{e^\dagger}$.

Let $e_1, e_2 \in \mathbb{Z}_{\geq 1}$ be integers. Then we have $\Pi_d^{e_1 e_2} = \Pi_{de_1}^{e_2} \circ \Pi_d^{e_1}$. So a factorisation of e allows us to write $\Pi_d = \Pi_d^e$ as a composition of morphisms of the form $\Pi_{d^\dagger}^{e^\dagger}$ with $d^\dagger \in \mathbb{Z}_{\geq 0}$ and $e^\dagger \in \mathbb{Z}_{\geq 1}$.

Lemma 3.1. Suppose that $e = \mathrm{char}(K) > 0$. Then T_d^e consists of the classes of homogeneous polynomials in $K[x^e, y^e]$ of degree de.

Proof. Let $[f] \in \mathbb{P}(V_d)$ be the class of a homogeneous polynomial $f \in V_d$ of degree d. Then we have

$$f = b_0 y^d + b_1 x y^{d-1} + \cdots + b_d x^d$$

for some $b_0, \ldots, b_d \in K$ not all zero. Since $e = \mathrm{char}(K)$, the e-th power map is additive. Hence

$$f^e = b_0^e y^{de} + b_1^e x^e y^{(d-1)e} + \cdots + b_d^e x^{de}$$

is a homogeneous polynomial in $K[x^e, y^e]$ of degree de. So we see that T_d^e is contained in the set of classes of homogeneous polynomials in $K[x^e, y^e]$ of degree de.

Let

$$g = c_0 y^{de} + c_e x^e y^{(d-1)e} + \cdots + c_{de} x^{de}$$

be a homogeneous polynomial in $K[x^e, y^e]$ of degree de. Since K is algebraically closed, there exist $b_0, \ldots, b_d \in K$ such that $b_i^e = c_{ie}$ for all $i \in \{0, \ldots, d\}$. We have

$$g = b_0^e y^{de} + b_1^e x^e y^{(d-1)e} + \cdots + b_d^e x^{de} = (b_0 y^d + b_1 x y^{d-1} + \cdots + b_d x^d)^e.$$

So we see that T_d^e is the set of classes of homogeneous polynomials in $K[x^e, y^e]$ of degree de. $\square$

3.2. Note that the map $V_d \to V_{de}$ which sends f to $f(x^e, y^e)$ is K-linear and injective. So by Proposition 2.73, we see that the map

$$\begin{array}{rccc} \Gamma_d^e \colon & \mathbb{P}(V_d) & \to & \mathbb{P}(V_{de}) \\ & [f] & \mapsto & [f(x^e, y^e)] \end{array}$$

is a morphism of projective varieties. Let $n \in \mathbb{Z}_{\geq 1}$ be an integer. Then we have

$$(f^e)(x^n, y^n) = (f(x^n, y^n))^e$$

for each polynomial $f \in V_d$. So we see that the diagram

$$\begin{array}{ccc} \mathbb{P}(V_d) & \xrightarrow{\Pi_d^e} & \mathbb{P}(V_{de}) \\ \downarrow{\scriptstyle \Gamma_d^n} & & \downarrow{\scriptstyle \Gamma_{de}^n} \\ \mathbb{P}(V_{dn}) & \xrightarrow{\Pi_{dn}^e} & \mathbb{P}(V_{den}) \end{array}$$

commutes.

Proposition 3.3. Suppose that $\mathrm{char}(K) = p$ for some prime number $p > 0$. Then $T_d^{ep^n}$ is the image of the morphism

$$\Gamma^p_{dep^{n-1}} \circ \cdots \circ \Gamma^p_{de} \circ \Pi_d : \mathbb{P}(V_d) \to \mathbb{P}(V_{de})$$

for all $n \in \mathbb{Z}_{\geq 0}$.

Proof. We will prove the proposition using induction. We have $\Pi_d^e = \Pi_d$. So the proposition holds for $n = 0$. Suppose that we have

$$T_d^{ep^n} = \mathrm{im}\left(\Gamma^p_{dep^{n-1}} \circ \cdots \circ \Gamma^p_{de} \circ \Pi_d\right).$$

We have $\Pi_d^{ep^{n+1}} = \Pi_{dp}^{ep^n} \circ \Pi_d^p$. By Lemma 3.1, the morphisms Π_d^p and Γ_d^p have the same image. So we see that

$$T_d^{ep^{n+1}} = \mathrm{im}\left(\Pi_d^{ep^{n+1}}\right) = \mathrm{im}\left(\Pi_{dp}^{ep^n} \circ \Gamma_d^p\right).$$

So since the diagram

$$\begin{array}{ccc} \mathbb{P}(V_d) & \xrightarrow{\Pi_d^{ep^n}} & \mathbb{P}(V_{dep^n}) \\ \downarrow{\scriptstyle \Gamma_d^p} & & \downarrow{\scriptstyle \Gamma^p_{dep^n}} \\ \mathbb{P}(V_{dp}) & \xrightarrow{\Pi_{dp}^{ep^n}} & \mathbb{P}(V_{dep^{n+1}}) \end{array}$$

commutes, we see that

$$\begin{aligned} T_d^{ep^{n+1}} &= \mathrm{im}\left(\Gamma^p_{dep^n} \circ \Pi_d^{ep^n}\right) \\ &= \Gamma^p_{dep^n}\left(T_d^{ep^n}\right) \\ &= \mathrm{im}\left(\Gamma^p_{dep^n} \circ \cdots \circ \Gamma^p_{de} \circ \Pi_d\right). \end{aligned}$$

□

The morphism $\Gamma_d^e \colon \mathbb{P}(V_d) \to \mathbb{P}(V_{de})$ is easy to understand. Therefore it suffices to consider the case where $\mathrm{char}(K) \nmid e$.

3.2 The affine variety of de-monic e-th powers

In this section, assume that $\mathrm{char}(K) \nmid e$.

Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer and consider the K-linear map

$$\begin{aligned} \mathrm{dehom} \colon K[x,y] &\to K[z] \\ f &\mapsto f(z,1). \end{aligned}$$

This map induces a one-to-one correspondence between the elements of V_n and the polynomials in $K[z]$ of degree at most n.

Definition 3.4. Let $f \in V_n$ be a homogeneous polynomial of degree n and let $m \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. Then we say that f is m-monic if $\operatorname{dehom}(f)$ is monic of degree m.

We see that every element of $\mathbb{P}(V_n)$ can be written uniquely as $[f]$ for some polynomial $f \in V_n$ that is m-monic for some $m \in \mathbb{Z}_{\geq 0}$.

3.5. Consider the basis $(y^n, xy^{n-1}, \ldots, x^n)$ of V_n. Let $(c_0, \ldots, c_n)$ be its dual basis. So we have

$$f = c_0(f)y^n + c_1(f)xy^{n-1} + \cdots + c_n(f)x^n$$

for all $f \in V_n$. Denote the affine variety $Z_{\mathbb{A}(V_n)}(c_n - 1)$ inside $\mathbb{A}(V_n)$ by M_n and let

$$\begin{array}{rcl} \psi_n \colon M_n & \to & U_n \\ f & \mapsto & [f] \end{array}$$

be the affine chart corresponding to $c_n = 1$ as in 2.65. Recall that M_n has coordinate ring $P(V_n)/(c_n - 1) \cong P(\ker c_n)$.

3.6. Let $(b_0, \ldots, b_d)$ be the basis of $V_d^\times$ dual to $(y^d, xy^{d-1}, \ldots, x^d)$ and let $(c_0, \ldots, c_{de})$ be the basis of $V_{de}^\times$ dual to $(y^{de}, xy^{de-1}, \ldots, x^{de})$. Then we have $\Pi_d^{-1}(\psi_{de}(M_{de})) = \psi_d(M_d)$, because an e-th power of a polynomial $f \in V_d$ is de-monic if and only if f is d-monic. The map

$$\begin{array}{rcl} \Upsilon_d \colon M_d & \to & M_{de} \\ f & \mapsto & f^e \end{array}$$

which sends a d-monic polynomial $f \in V_n$ to its e-th power is a morphism of affine varieties, because it is the restriction of the polynomial map $\operatorname{pow}_d \colon V_d \to V_{de}$. Note that the diagram

$$\begin{array}{ccc} \mathbb{P}(V_d) & \xrightarrow{\Pi_d} & \mathbb{P}(V_{de}) \\ {\scriptstyle \psi_d}\uparrow & & \uparrow{\scriptstyle \psi_{de}} \\ M_d & \xrightarrow{\Upsilon_d} & M_{de} \end{array}$$

commutes. Let S_d be the image of Υ_d.

Studying n-monic homogeneous polynomials in V_n is the same as studying monic polynomials in $K[z]$ of degree n. So we start by determining when the latter is an e-th power.

Lemma 3.7. Let R be a commutative ring, let $f \in R[z]$ be a polynomial of degree d and let $b_0, \ldots, b_d$ be elements of K such that

$$f = b_0 + b_1 z + \cdots + b_{d-1}z^{d-1} + b_d z^d.$$

Then we have

$$f^e = \sum_{j=0}^{de} \left(\sum_{\substack{0 \le i_1,\dots,i_e \le d \\ i_1+\dots+i_e=j}} b_{i_1} \dots b_{i_e} \right) z^j.$$

Proof. Trivial. □

Lemma 3.8. Let R be a commutative ring such that $e \in R^*$, let $g \in R[z]$ be a monic polynomial of degree de and let $c_0, \dots, c_{de} \in R$ be such that

$$g = c_0 + c_1 z + \dots + c_{de-1} z^{de-1} + c_{de} z^{de}.$$

(a) There is a unique monic polynomial $f \in R[z]$ of degree d such that the degree of $g - f^e$ is lower than $de - d$.

(b) Define the elements $b_0, \dots, b_d \in R$ recursively and in opposite order by the equations

$$b_i = \frac{1}{e} \left(c_{de-d+i} - \sum_{\substack{i < i_1,\dots,i_e \le d \\ i_1+\dots+i_e=de-d+i}} b_{i_1} \dots b_{i_e} \right)$$

for $i \in \{0, \dots, d-1\}$ and $b_d = 1$. Then the unique polynomial f from part (a) is equal to $\sum_{i=0}^{d} b_i z^i$.

Proof. Let $f \in R[z]$ be a monic polynomial of degree d and let $b_0, \dots, b_d$ be elements of R such that $f = b_0 + b_1 z + \dots + b_{d-1} z^{d-1} + b_d z^d$. Then we have

$$f^e = \sum_{j=0}^{de} \left(\sum_{\substack{0 \le i_1,\dots,i_e \le d \\ i_1+\dots+i_e=j}} b_{i_1} \dots b_{i_e} \right) z^j.$$

by Lemma 3.7. So we see that

$$g - f^e = \sum_{j=0}^{de} \left(c_j - \sum_{\substack{0 \le i_1,\dots,i_e \le d \\ i_1+\dots+i_e=j}} b_{i_1} \dots b_{i_e} \right) z^j.$$

Note that $c_{de} = b_d = 1$. So there exists a unique monic polynomial $f \in K[z]$ of degree d such that the degree of $g - f^e$ is lower than $de - d$ if and only if we can solve the system of equations

$$\left\{ c_j = \sum_{\substack{0 \le i_1,\dots,i_e \le d \\ i_1+\dots+i_e=j}} b_{i_1} \dots b_{i_e} \;\middle|\; j \in \{de-d, \dots, de-1\} \right\}$$

uniquely for $b_0, \dots, b_{d-1}$. Substituting $j = de - d + i$, we get the equations

$$\left\{ c_{de-d+i} = \sum_{\substack{0 \le i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = de-d+i}} b_{i_1} \dots b_{i_e} \;\middle|\; i \in \{0, \dots, d-1\} \right\}.$$

Note that if $0 \le i_1, \dots, i_e \le d$ and $i_1 + \dots + i_e = de - d + i$, then either we have $i_1, \dots, i_e > i$ or we have $\#\{k | i_k = i\} = 1$ and $\#\{k | i_k = d\} = e - 1$. Since $b_d = 1$, we see that

$$\sum_{\substack{0 \le i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = de-d+i}} b_{i_1} \dots b_{i_e} = e b_i + \sum_{\substack{i < i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = de-d+i}} b_{i_1} \dots b_{i_e}.$$

for all $i \in \{1, \dots, d-1\}$. So since $e \in R^*$, we can rewrite the equations to

$$\left\{ b_i = \frac{1}{e} \left(c_{de-d+i} - \sum_{\substack{i < i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = de-d+i}} b_{i_1} \dots b_{i_e} \right) \;\middle|\; i \in \{0, \dots, d-1\} \right\}.$$

We see that if we know $b_{i+1}, \dots, b_d$, then we can solve the i-th equation uniquely for b_i. Therefore we can solve uniquely for $b_0, \dots, b_{d-1}$, because we know that b_d is equal to 1. We see from the last set of equations that part (b) holds. □

Applying the lemma with $R = K$, we see that for each monic polynomial in $g \in K[z]$ of degree de there is a unique monic polynomial $f \in K[z]$ of degree d such that the degree of $g - f^e$ is lower than $de - d$.

3.9. Let R be the ring $K[c_0, \dots, c_{de}]/(c_{de} - 1)$ and let $g \in R[z]$ be the polynomial $g = c_0 + c_1 z + \dots + c_{de-1} z^{de-1} + c_{de} z^{de}$. Let $f = \sum_{i=0}^{d} p_i z^i \in R[z]$ be the polynomial defined recursively and in opposite order by the equations

$$p_i = \frac{1}{e} \left(c_{de-d+i} - \sum_{\substack{i < i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = de-d+i}} p_{i_1} \dots p_{i_e} \right)$$

for $i \in \{0, \dots, d-1\}$ and $p_d = 1$. Then Lemma 3.8 tells us that f is the unique monic polynomial $R[z]$ of degree d such that the degree of $g - f^e$ is lower than $de - d$. For each integer $j \in \{0, \dots, de\}$, take

$$q_j = c_j - \sum_{\substack{0 \le i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = j}} p_{i_1} \dots p_{i_e}.$$

Then we have $g - f^j = \sum_{j=0}^{de} q_j z^j$ by Lemma 3.7. We see that $q_j \in R$ is zero for all $j \in \{de-d, \dots, de\}$. Note that p_i is a polynomial which contains only the variables $c_{de-d+i}, \dots, c_{de-1}, c_{de} = 1$ for each $i \in \{0, \dots, d\}$. Therefore the polynomial $q_j - c_j$ contains only the variables $c_{de-d}, \dots, c_{de-1}, c_{de} = 1$ for each $j \in \{0, \dots, de-d-1\}$.

3.10. Let $g \in K[z]$ be a monic polynomial of degree de and let $a_0, \dots, a_{de}$ be elements of K such that $g = a_0 + a_1 z + \cdots + a_{de-1} z^{de-1} + a_{de} z^{de}$. Then

$$f = \sum_{i=0}^{d} p_i(a_0, \dots, a_{de}) z^i$$

is a monic polynomial in $K[z]$ of degree d such that

$$g - f^e = \sum_{j=0}^{de} q_j(a_0, \dots, a_{de}) z^j = \sum_{j=0}^{de-d-1} q_j(a_0, \dots, a_{de}) z^j.$$

So we see that this polynomial f is the unique monic polynomial $f \in K[z]$ of degree d such that the degree of $g - f^e$ is lower than $de - d$. Hence the monic polynomial $g \in K[z]$ of degree de is an e-th power if and only if $q_j(a_0, \dots, a_{de}) = 0$ for all $j \in \{0, \dots, de-d-1\}$.

Proposition 3.11. The image S_d of the morphism $\Upsilon_d \colon M_d \to M_{de}$ is the subvariety of M_{de} corresponding to the prime ideal $(q_0, \dots, q_{de-d-1})$ of $K[M_{de}]$.

Proof. By 3.10 and the correspondence between elements of V_{de} and polynomial in $K[z]$ of degree at most de, we see that $S_d = I_{M_{de}}(q_0, \dots, q_{de-d-1})$. So it suffices to show that this ideal $(q_0, \dots, q_{de-d-1})$ of $K[M_{de}]$ is prime.

Recall that $K[M_{de}] = K[c_0, \dots, c_{de}]/(c_{de} - 1)$. Also recall from 3.9 that for each $j \in \{0, \dots, de-d-1\}$, the polynomial $q_j - c_j$ contains only the variables $c_{de-d}, \dots, c_{de-1}, c_{de} = 1$. So the homomorphism of K-algebras

$$\begin{aligned} K[M_{de}]/(q_0, \dots, q_{de-d-1}) &\to K[c_{de-d}, \dots, c_{de-1}] \\ c_j &\mapsto \begin{cases} c_j - q_j & \text{if } j \in \{0, \dots, de-d-1\} \\ c_j & \text{if } j \in \{de-d, \dots, de-1\} \\ 1 & \text{if } j = de \end{cases} \end{aligned}$$

is an isomorphism from the quotient of $K[M_{de}]$ by $(q_0, \dots, q_{de-d-1})$ to a domain. Hence the ideal $(q_0, \dots, q_{de-d-1})$ is prime corresponds to the subvariety S_d of M_{de}. $\square$

Theorem 3.12. The morphism $\Upsilon_d \colon M_d \to S_d$ of affine varieties is an isomorphism.

Proof. The map

$$\begin{aligned} \alpha\colon V_{de} &\to V_d \\ g &\mapsto \sum_{i=0}^{d} p_i(c_0(g), \ldots, c_{de-1}(g), 1)x^i y^{d-i} \end{aligned}$$

is a polynomial map. So the restriction

$$\begin{aligned} \Phi\colon M_{de} &\to M_d \\ g &\mapsto \sum_{i=0}^{d} p_i(c_0(g), \ldots, c_{de}(g))x^i y^{d-i} \end{aligned}$$

of α to M_{de} is a morphism of affine varietes. By 3.10 and the correspondence between elements of V_{de} and polynomial in $K[z]$ of degree at most de, we see that Φ sends a de-monic polynomial $g \in M_{de}$ to the unique d-monic polynomial $f \in M_d$ such that the degree of $g(z, 1)-f(z, 1)^e$ is lower than $de-d$. So we see that $\Phi|_{S_d}$ is an inverse of Υ_d. Hence Υ_d is an isomorphism. □

Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. We want to use Theorem 3.12 to prove that the morphisms of projective varieties $\Pi_d\colon \mathbb{P}(V_d) \to \mathbb{P}(V_{de})$ is an isomorphism. To do that, we first give an alternate description of the open subset $\psi_n(M_n)$ of $\mathbb{P}(V_n)$.

Proposition 3.13. The open subset

$$\psi_n(M_n) = \{[f] \in \mathbb{P}(V_n) | c_n(f) \neq 0\}$$

of $\mathbb{P}(V_n)$ consists of the classes $[f]$ of all non-zero polynomials $f \in V_n$ such that $f(1, 0) \neq 0$.

Proof. We have $c_n(f) = f(1, 0)$ for all $f \in V_n$. □

Let $[f] \in \mathbb{P}(V_n)$ be a point. Then the polynomial $f \in V_n$ is non-zero. Therefore there exists a pair (x_0, y_0) of elements of K such that $f(x_0, y_0) \neq 0$. We can use this fact to give an open cover of $\mathbb{P}(V_n)$ which consists of open subset which are similar to $\psi_n(M_n)$.

3.14. Let $(x_0, y_0) \in K^2$ be a non-zero vector. Consider the K-linear map

$$\begin{aligned} \mathrm{eval}_{(x_0,y_0)}\colon K[x, y] &\to K \\ f &\mapsto f(x_0, y_0) \end{aligned}$$

The restriction of $\mathrm{eval}_{(x_0,y_0)}$ to V_n is an element of $V_n^\times$, which is a subset of $P(V_n)$. So we see that the set of classes of non-zero polynomials $f \in V_n$ such that $f(x_0, y_0) = 0$ is closed in $\mathbb{P}(V_n)$. So the subset

$$U_{n,(x_0,y_0)} := \{[f] \in \mathbb{P}(V_n) | f(x_0, y_0) \neq 0\}$$

of $\mathbb{P}(V_n)$ is open. The polynomial on K^2 corresponding to the zero function on K^2 is zero. So we see that

$$\mathbb{P}(V_n) = \bigcup_{(x_0,y_0)\in K^2-\{0\}} U_{n,(x_0,y_0)}.$$

3.15. Let

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be an element of $\mathrm{GL}_2(K)$. Then we have

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Consider the map

$$\begin{array}{rcl} \ell_A \colon V_n & \to & V_n \\ f & \mapsto & f(ax+by, cx+dy). \end{array}$$

One can check that this map is K-linear and invertible and that the map

$$\begin{array}{rcl} \rho\colon\ \mathrm{GL}_2(K) & \to & \mathrm{GL}(V_n) \\ A & \mapsto & \ell_A \end{array}$$

is a homomorphism.

Since ℓ_A is a K-linear automorphism, we see that the map

$$\begin{array}{rcl} \Phi_{n,A}\colon\ \mathbb{P}(V_n) & \to & \mathbb{P}(V_n) \\ [f] & \mapsto & [\ell_A(f)] \end{array}$$

is an automorphism of $\mathbb{P}(V_n)$. Let $(x_0, y_0) \in K^2$ be a non-zero vector. Then we see that

$$\Phi_{n,A}^{-1}(U_{n,(x_0,y_0)}) = \{f \in V_n | f(ax_0 + by_0, cx_0 + dy_0) = 0\} = U_{n,(x_0,y_0)A^T}.$$

Every non-zero element of K^2 is part of some basis of K^2. So for all $(x_0, y_0), (x_1, y_1) \in K^2$ non-zero, there exists a matrix $A \in \mathrm{GL}_2(K)$ such that $(x_0, y_0)A^T = (x_1, y_1)$. So we see that the action of $\mathrm{GL}_2(K)$ on

$$\{U_{n,(x_0,y_0)} | (x_0, y_0) \in K^2 - \{0\}\}$$

is transitive.

3.16. Let $(x_0, y_0) \in K^2$ be a non-zero vector and let

$$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$$

be a matrix such that $(1,0)A^T = (x_0, y_0)$. Then the automorphism

$$\begin{array}{rccl} \Phi_{n,A}\colon & \mathbb{P}(V_n) & \to & \mathbb{P}(V_n) \\ & [f] & \mapsto & [f(ax+by, cx+dy)] \end{array}$$

sends $\psi_n(M_n) = U_{n,(1,0)}$ to $U_{n,(x_0,y_0)}$ for all $n \in \mathbb{Z}_{\geq 0}$. Consider the diagram

$$\begin{array}{ccc} M_d & \xrightarrow{\Upsilon_d} & M_{de} \\ \downarrow{\scriptstyle \Phi_{d,A}\circ\psi_d} & & \downarrow{\scriptstyle \Phi_{de,A}\circ\psi_{de}} \\ U_{d,(x_0,y_0)} & \xrightarrow{\Pi_d} & U_{de,(x_0,y_0)} \end{array}$$

We have

$$(f^e)(ax+by, cx+dy) = (f(ax+by, cx+dy))^e$$

for all $f \in V_d$. So we see that the diagram commutes. Since $\Phi_{d,A}$, $\Phi_{de,A}$, ψ_d and ψ_{de} are isomorphisms and Υ_d is an isomorphism onto its image, we see that Π_d restricts to an isomorphism from $U_{d,(x_0,y_0)}$ onto its image.

Theorem 3.17. The morphism of projective varieties $\Pi_d\colon \mathbb{P}(V_d) \to T_d$ is an isomorphism.

Proof. First note that the map $\Pi_d\colon \mathbb{P}(V_d) \to T_d$ is bijective. Since we have an open covers of $\mathbb{P}(V_{de})$ such that Π_d restricts to an isomorphism on each member of the open cover, the inverse map is an morphism of projective varieties. □

3.3 Calculating the ideal corresponding to T_d

3.18. Let $m, n \in \mathbb{Z}_{\geq 0}$ be non-negative integers such that $m \leq n$. Consider the injective K-linear map $V_m \to V_n$ sending f to fy^{n-m}. This map is polynomial, so by Proposition 2.73 the map

$$\begin{array}{rccl} \Phi_{m,n}\colon & \mathbb{P}(V_m) & \to & \mathbb{P}(V_n) \\ & [f] & \mapsto & [fy^{n-m}] \end{array}$$

is a morphism of projective varieties. Let $f \in V_m$ be a k-monic polynomial for some $k \in \mathbb{Z}_{\geq 0}$. Then we see that $\Phi_{m,n}([f])$ is also k-monic. The map $\Phi_{m,n}$ is a bijection between the set $\mathbb{P}(V_m)$ and the closed subset

$$\{[f] \in \mathbb{P}(V_n) | f \in V_n \ k\text{-monic for some } k \leq m\}$$

of $\mathbb{P}(V_n)$. We see that $\mathbb{P}(V_n)$ is the disjoint union of the sets $\Phi_{0,n}(M_0)$, $\Phi_{1,n}(M_1)$,…,$\Phi_{n,n}(M_n)$. We also see that T_d is the disjoint union of the sets $\Phi_{0,de}(S_0)$, $\Phi_{e,de}(S_1)$,…,$\Phi_{de,de}(S_d)$. This allows us the prove the following proposition.

Proposition 3.19. The closure of S_d in $\mathbb{P}(V_{de})$ is T_d.

Proof. The closed set T_d is the disjoint union of $\Phi_{0,de}(S_0)$, $\Phi_{e,de}(S_1)$, $\ldots$, $\Phi_{de,de}(S_d)$. Therefore the closed set T_d is the disjoint union of the closures of $\Phi_{0,de}(S_0), \Phi_{e,de}(S_1), \ldots, \Phi_{de,de}(S_d)$ in $\mathbb{P}(V_{de})$. Since T_d is irreducible, this means that T_d is the closure of $\Phi_{ie,de}(S_i)$ in $\mathbb{P}(V_{de})$ for some $i \in \{0, \ldots, d\}$. Note that $\Phi_{ie,de}(S_i)$ is contained in the closed subset $\Phi_{ie,de}(\mathbb{P}(V_{ie}))$ of $\mathbb{P}(V_{de})$. So the closure of $\Phi_{ie,de}(S_i)$ is also contained in $\Phi_{ie,de}(\mathbb{P}(V_{ie}))$. The element $[x^{de}] \in \mathbb{P}(V_{de})$ is an element of T_d which is not contained in $\Phi_{j,de}(\mathbb{P}(V_j))$ for any $j \in \{0, \ldots, de-1\}$. So we see that T_d is the closure of $S_d = \Phi_{de,de}(S_d)$ in $\mathbb{P}(V_{de})$. □

We can use Gröbner bases to compute generators for $I_d = I_{\mathbb{P}(V_{de})}(T_d)$ using that we know $I_{M_{de}}(S_d)$. We will use the notation and definitions from chapter 9 of [Ke]. Let $\preceq$ be a monomial ordering on a polynomial ring with variables $x_1, \ldots, x_n$.

Proposition 3.20. Let X be an affine variety inside $\mathbb{A}(K^n)$ for some integer $n \in \mathbb{Z}_{\geq 0}$ and let $I = I_{\mathbb{A}(K^n)}(X)$ be its corresponding prime ideal of $K[x_1, \ldots, x_n]$. Suppose that $\preceq$ is a total degree ordering and let G be a Gröbner basis of I relative to $\preceq$. Then the homogeneous prime ideal of $K[x_0, \ldots, x_n]$ corresponding to the closure of X in $\mathbb{P}(K^{n+1})$ is generated by $\{\hom(g) | g \in G\}$.

Proof. See Lemma 2.50 from [DP]. □

To make Gröbner bases useful when working over the algebraically closed field K, we have the following proposition.

Proposition 3.21. Let κ be the prime field of K and let $f_1, \ldots, f_m$ be elements of $\kappa[x_1, \ldots, x_n]$. Then a Gröbner basis of the ideal of $\kappa[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_m$ relative to $\preceq$ is a Gröbner basis of the ideal of $K[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_m$ relative to $\preceq$.

Proof. Let G be a Gröbner basis of the ideal of $\kappa[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_m$ relative to $\preceq$. Then the ideal of $K[x_1, \ldots, x_n]$ generated by $f_1, \ldots, f_m$ is also generated by G. Since G satisfies Buchberger's criterion, it is a Gröbner basis for the ideal of $K[x_1, \ldots, x_n]$ it generates. □

By Proposition 3.11, we know the ideal of $K[c_0, \ldots, c_{de}]$ corresponding to S_d. So using these two propositions, we can compute the homogeneous ideal I_d of $K[c_0, \ldots, c_{de}]$ corresponding to T_d for (small) instances of $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$.

3.22. Again consider the polynomial map

$$\begin{array}{rccc} \mathrm{pow}_d\colon & V_d & \to & V_{de} \\ & f & \mapsto & f^e \end{array}$$

whose image is CT_d. We can also view pow_d as a morphism of affine varieties $\mathbb{A}(V_d) \to \mathbb{A}(V_{de})$. This gives us another way to compute $I_d = I_{\mathbb{A}(V_{de})}(CT_d)$.

Proposition 3.23. Let V, W be finite-dimensional vector space over K. Let $\Phi\colon \mathbb{A}(V) \to \mathbb{A}(W)$ be a morphism of affine varieties. Let $(v_1, \ldots, v_n)$ be a basis of V and let $(x_1, \ldots, x_n)$ be its dual basis. Let $(w_1, \ldots, w_m)$ be a basis of W and let $(y_1, \ldots, y_m)$ be its dual basis. Then Φ^* is a homomorphism of K-algebras $K[y_1, \ldots, y_m] \to K[x_1, \ldots, x_n]$. Let I be the ideal of $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by $y_1 - \Phi^*(y_1), \ldots, y_m - \Phi^*(y_m)$. Then $I_{\mathbb{A}(W)}(\mathrm{im}\,\Phi) = I \cap K[y_1, \ldots, y_m]$.

Proof. Let $(\mathrm{Id}, \Phi)\colon \mathbb{A}(V) \to \mathbb{A}(V) \times \mathbb{A}(W)$ be the morphism sending $P \in \mathbb{A}(V)$ to $(P, \Phi(P))$. Then we have $\mathrm{im}(\mathrm{Id}, \Phi) = \{(P, \Phi(P)) | P \in \mathbb{A}(V)\}$. We see that $\mathrm{im}(\mathrm{Id}, \Phi)$ is the affine variety corresponding to the ideal I of $K[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by $y_1 - \Phi^*(y_1), \ldots, y_m - \Phi^*(y_m)$.

Let $f \in I_{\mathbb{A}(W)}(\mathrm{im}\,\Phi)$ and $Q \in \mathrm{im}(\mathrm{Id}, \Phi)$. Then there exists a $P \in \mathbb{A}(V)$ such that $Q = (P, \Phi(P))$. We have $f(Q) = f(\Phi(P)) = 0$. Therefore $f \in I$ and hence $f \in I \cap K[y_1, \ldots, y_m]$. Let $f \in I \cap K[y_1, \ldots, y_m]$ and let $P \in \mathbb{A}(V)$. Take $Q = (P, \Phi(P))$. Then we see that $f(\Phi(P)) = f(Q) = 0$. Hence $f \in I_{\mathbb{A}(W)}(\mathrm{im}\,\Phi)$. So $I_{\mathbb{A}(W)}(\mathrm{im}\,\Phi) = I \cap K[y_1, \ldots, y_m]$. □

Since CT_d is the image of pow_d, the previous proposition allows us to write I_d as the intersection of a known ideal I with a polynomial ring in fewer variables. This intersection is called an elimination ideal. See [Ke] for how to calculate elimination ideals using Gröbner bases.

Chapter 4

Conjectures

In this chapter, let K be an algebraically closed field and let $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$ be integers such that $\mathrm{char}(K) \nmid e$. Let

$$\begin{aligned} \mathrm{pow}_d \colon V_d &\to V_{de} \\ f &\mapsto f^e \end{aligned}$$

be the e-th power homogeneous polynomial map with image CT_d. Let $\Pi_d \colon \mathbb{P}(V_d) \to \mathbb{P}(V_{de})$ be the e-th power morphism of projective varieties with image T_d. Let I_d be the homogeneous ideal

$$I_{\mathbb{A}(V_{de})}(CT_d) = I_{\mathbb{P}(V_{de})}(T_d) = \ker \mathrm{pow}_d^*$$

of $P(V_{de})$.

In this chapter, we state two conjectures; the second being a (possibly not strictly) weakened version of the first. The first conjecture states that the ideal I_d of $P(V_{de})$ is generated by its degree $d+1$ part. The second conjecture states that I_d contains no homogeneous polynomials of degree d.

We will motivate our first conjecture by proving that T_d is the zero set of an ideal generated by homogeneous polynomials on V_{de} of degree $d+1$ and that this ideal and the ideal I_d become equal when we dehomogenize with respect to c_{de} if $\mathrm{char}(K) \nmid (de)!$. We will also show that if the second conjecture holds, then we know the Hilbert function of I_d. Lastly we will show that, in the case $d = 1$, the second conjecture implies the first conjecture making these conjectures equivalent.

4.1 Another description of the projective variety of e-th powers

Definition 4.1. Let

$$f = \sum_{i=0}^{n} c_i z^i$$

be a polynomial in $K[z]$. Define the derivative of f to be the polynomial

$$f' := \sum_{i=1}^{n} i c_i z^{i-1}$$

in $K[z]$.

4.2. Let $f_1, f_2 \in \mathbb{C}[z]$ be polynomials. Then we have $(f_1 f_2)' = f_1' f_2 + f_1 f_2'$. Now suppose that f_1 and f_2 are non-zero. Then we have

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

in $\mathbb{C}(z)$. We see that the map

$$\begin{array}{ccc} \mathbb{C}[z] - \{0\} & \to & \mathbb{C}(z) \\ f & \mapsto & \frac{f'}{f} \end{array}$$

which sends a polynomial to its logarithmic derivative, sends products to sums. Now consider the map

$$\begin{array}{rcl} \ln(-)' \colon \mathbb{C}(z)^* & \to & \mathbb{C}(z) \\ \frac{f}{g} & \mapsto & \frac{f'g - fg'}{fg} \end{array}$$

which sends a rational function to its logarithmic derivative. One can check that $\ln(-)'$ is a homomorphism of groups whose kernel equals $\mathbb{C}^*$.

Let $g \in K[z]$ be a non-zero polynomial of degree de. If $g = f^e$ for some non-zero polynomial $f \in K[z]$ of degree d, then we have

$$\ln(g)' = \frac{g'}{g} = \frac{(f^e)'}{f^e} = \frac{ef'f^{e-1}}{f^e} = e\frac{f'}{f} = e\ln(f)'.$$

If we have $\ln(g)' = e\ln(f)'$ for some non-zero polynomial $f \in K[z]$, then we have $\ln(g)' = \ln(f^e)'$ and therefore $f = \lambda f^e$ for some $\lambda \in \mathbb{C}^*$. So since K is algebraically closed, we see that g is an e-th power of a polynomial in $f \in K[z]$ of degree d if and only if we have $fg' - ef'g$.

This proof can be generalized in the following way.

Theorem 4.3. Suppose that $\mathrm{char}(K) \nmid (de)!$. Let $g \in K[z]$ be a non-zero polynomial of degree at most de and let $f \in K[z]$ be a non-zero polynomial of degree at most d. Then the following are equivalent:

(a) we have $g = \lambda f^e$ for some $\lambda \in K^*$;

(b) the polynomial $g'f - ef'g$ is zero;

(c) the polynomial $g'f - ef'g$ has degree lower than $\max(0, d-1)$.

We will prove this theorem another way, which gives us a bit more information about how to check whether a polynomial f satisfying these equivalent conditions exists. We start with a lemma that should be compared to Lemma 3.8.

4.4. Before we state the lemma, note that if $f, g \in R[z]$ are polynomials of degree n and m where R is a commutative ring, then we have

$$g'f - ef'g = \sum_{k=0}^{n+m-1} \left(\sum_{\substack{0 \le \alpha \le n \\ 0 \le \beta \le m \\ \alpha+\beta=k+1}} (\beta - e\alpha) b_\alpha c_\beta \right) z^k.$$

In particular, note that that if $m = en$, then the degree of $g'f - ef'g$ is at most $n + en - 2$, because

$$\sum_{\substack{0 \le \alpha \le n \\ 0 \le \beta \le en \\ \alpha+\beta=n+en}} (\beta - e\alpha) b_\alpha c_\beta = (en - en) b_n c_{en} = 0.$$

Lemma 4.5. Let R be a commutative ring such that $e, 1, \ldots, d \in R^*$, let $g \in R[z]$ be a monic polynomial of degree de and let $c_0, \ldots, c_{de} \in R$ be such that

$$g = c_0 + c_1 z + \cdots + c_{de-1} z^{de-1} + c_{de} z^{de}.$$

(a) There is a unique monic polynomial $f \in R[z]$ of degree d such that the degree of $g'f - ef'g$ is lower than $de - 1$.

(b) Define the elements $b'_0, \ldots, b'_d \in R$ recursively and in opposite order by

$$b'_i = -\frac{1}{(d-i)e} \sum_{j=1}^{d-i} ((d-i-j)e - j) b'_{i+j} c_{de-j}$$

for $i \in \{0, \ldots, d-1\}$ and $b'_d = 1$. Then the unique polynomial f from part (a) is equal to $\sum_{i=0}^{d} b'_i z^i$.

Proof. Let $f \in R[z]$ be a monic polynomial of degree d and let $b_0, \ldots, b_d$ be elements of R such that $f = b_0 + b_1 z + \cdots + b_{d-1} z^{d-1} + b_d z^d$. Then we have

$$g'f - ef'g = \sum_{k=0}^{de+d-2} \left(\sum_{\substack{0 \le \alpha \le d \\ 0 \le \beta \le de \\ \alpha+\beta=k+1}} (\beta - e\alpha) b_\alpha c_\beta \right) z^k.$$

So we see that there exists a unique monic polynomial $f \in R[z]$ of degree d such that the degree of $g'f - ef'g$ is lower than $de - 1$ if and only if we can solve the system of equations

$$\left\{ \sum_{\substack{0 \le \alpha \le d \\ 0 \le \beta \le de \\ \alpha+\beta=k+1}} (\beta - e\alpha) b_\alpha c_\beta = 0 \;\middle|\; k \in \{de-1, \dots, de+d-2\} \right\}$$

uniquely for $b_0, \dots, b_{d-1}$. Substituting $k + 1 = de + i$, we get the equations

$$\left\{ \sum_{\substack{0 \le \alpha \le d \\ 0 \le \beta \le de \\ \alpha+\beta=de+i}} (\beta - e\alpha) b_\alpha c_\beta = 0 \;\middle|\; i \in \{0, \dots, d-1\} \right\}.$$

Note that if $0 \le \alpha \le d$, $0 \le \beta \le de$ and $\alpha + \beta = de + i$, then we have $\alpha \in \{i, \dots, d\}$ and $\beta = de + i - \alpha$. So we have

$$\sum_{\substack{0 \le \alpha \le d \\ 0 \le \beta \le de \\ \alpha+\beta=de+i}} (\beta - e\alpha) b_\alpha c_\beta = \sum_{\alpha=i}^{d} ((d-\alpha)e + i - \alpha) b_\alpha c_{de+i-\alpha}$$

for all $i \in \{0, \dots, d-1\}$. Substituting $\alpha = i + j$, we get the equations

$$\left\{ \sum_{j=0}^{d-i} ((d-i-j)e - j) b_{i+j} c_{de-j} = 0 \;\middle|\; i \in \{0, \dots, d-1\} \right\}.$$

Since $e, d - i \in R^*$, we can take the summand $j = 0$ to the other side and divide by $-(d-i)e$ for each equation. Since $c_{de} = 1$, this gives us

$$\left\{ b_i = -\frac{1}{(d-i)e} \sum_{j=1}^{d-i} ((d-i-j)e - j) b_{i+j} c_{de-j} \;\middle|\; i \in \{0, \dots, d-1\} \right\}.$$

So if we know $b_{i+1}, \dots, b_d$, then we can solve the i-th equation uniquely for b_i. Therefore we can solve uniquely for $b_0, \dots, b_{d-1}$, because we know that $b_d = 1$. We see from the last set of equations that f satisfies the equation from part (b). □

Recall that $\mathrm{char}(K) \nmid e$. If we in addition assume that $\mathrm{char}(K) \nmid d!$, then we can apply the lemma with $R = K$ to see that for each monic polynomial $g \in K[z]$ of degree de, there is a unique monic polynomial $f \in K[z]$ of degree d such that the degree of $g'f - ef'g$ is lower than $de - 1$. By applying the lemma with $R = K[c_0, \dots, c_{de}]$ and $g = \sum_{j=0}^{de} c_j z^j$, we see that the coefficients of f are polynomials in the coefficients of g.

Lemma 4.6. Let R be a commutative ring such that $e \in R^*$, let $g \in R[z]$ be a monic polynomial of degree de and let $f \in K[z]$ be the unique monic polynomial of degree d from Lemma 3.8 such that the degree of $g - f^e$ is lower than $de - d$. Then the degree of $g'f - ef'g$ is lower than $de - 1$.

Proof. Take $h = g - f^e$. Then we have

$$g'f - ef'g = (f^e + h)'f - ef'(f^e + h) = h'f - ef'h.$$

Therefore

$$\deg(g'f - ef'g) = \deg(h'f - ef'h) \leq \deg(f) + \deg(h) - 1 < de - 1$$

since $\deg(h) < de - d$ and $\deg(f) = d$. □

By the lemma, we see that the b_i from Lemma 3.8 and the b_i' from Lemma 4.5 are in fact equal. As a consequence, the polynomials in the coefficients of g that give the coefficients of f are the same in both lemmas.

Lemma 4.7. Suppose that $\mathrm{char}(K) \nmid (de)!$. Let $g \in K[z]$ be a monic polynomial of degree de. Let $f \in K[z]$ be the unique monic polynomial of degree d such that the degree of $h = g - f^e$ is lower than $de - d$. Then $h = 0$ if and only if $g'f - ef'g = 0$. If $h \neq 0$, then we have $d > 0$ and $\deg(g'f - ef'g) = \deg(h) + d - 1$.

Proof. We have

$$g'f - ef'g = (f^e + h)'f - ef'(f^e + h) = h'f - ef'h.$$

So if $h = 0$, then $g'f - ef'g = 0$.

Suppose that $h \neq 0$. Then we have

$$0 \leq \deg(h) < de - d$$

and hence $d > 0$. We have

$$-de \leq \deg(h) - de < -d < 0$$

and $\mathrm{char}(K) \nmid (de)!$. So we see that $\deg(h) - de \neq 0$ in K. We know that $\deg(h'f - ef'h) \leq \deg(h) + d - 1$. Let $c \in K^*$ be the leading coefficient of h. Then the coefficient of $h'f - ef'h$ at $z^{\deg(h)+d-1}$ equals

$$\lambda \deg(h) - de\lambda = \lambda(\deg(h) - de) \neq 0.$$

So we see that $\deg(g'f - ef'g) = \deg(h'f - ef'h) = \deg(h) + d - 1$. In particular, we see that if $h \neq 0$, then $g'f - ef'g \neq 0$. □

Proof of Theorem 4.3. Note that whether $f^e - \lambda g = 0$ holds for some $\lambda \in K^*$ and whether some coefficient of $g'f - ef'g$ is zero does not change if we scale f or g by some element of K^*. So we may assume that both f and g are monic. We see that there exist a $\lambda \in K^*$ such that $f^e = \lambda g$ if and only if $f^e = g$, because both f and g are monic.

(a)⇒(b) Suppose that $f^e = g$, then we have $g'f - ef'g = 0$.

(b)⇒(c) Suppose that $g'f - ef'g = 0$. Then the degree of $g'f - ef'g$ is lower than $\max(0, d-1)$.

(c)⇒(a) Suppose that $g'f - ef'g$ has degree lower than $\max(0, d-1)$. Note that the coefficient of $g'f - ef'g$ at $z^{de+\deg(f)-1}$ equals $de - e\deg(f)$. We have $de + \deg(f) - 1 \geq d - 1$, so we see that $de \equiv e\deg(f) \mod \operatorname{char}(K)$. So since $\operatorname{char}(K) \nmid e$, we have $\deg(f) \equiv d \mod \operatorname{char}(K)$. Hence the degree of f equals d, because $0 \leq \deg(f) \leq d$ and $\operatorname{char}(K) \nmid (de)!$. So by Lemmas 4.5 and 4.7, we see that $f^e = g$. □

4.8. Let $n \in \mathbb{Z}_{\geq 1}$ and $i \in \{1, \ldots, n\}$ be integers. Recall that the map

$$\frac{\partial}{\partial x_i} \colon K[x_1, \ldots, x_n] \to K[x_1, \ldots, x_n]$$

sending a polynomial in $K[x_1, \ldots, x_n]$ to its derivative with respect to x_i is the unique K-linear map sending 1 to 0, sending x_i to 1, sending x_j to 0 for all $j \neq i$ and sending fg to $f\frac{\partial g}{\partial x_i} + g\frac{\partial f}{\partial x_i}$ for all $f, g \in K[x_1, \ldots, x_n]$.

4.9. Let $g \in V_{de}$ be a homogeneous polynomial of degree de and let $f \in V_d$ be a homogeneous polynomial of degree d. Then

$$f\frac{\partial g}{\partial x} - eg\frac{\partial f}{\partial x}$$

is either zero or a homogeneous polynomial of degree $de + d - 1$. Let b_d be the coefficient of f at x^d and let c_{de} be the coefficient of g at x^{de}. Then we see that the coefficient of $f\frac{\partial g}{\partial x}$ and $eg\frac{\partial f}{\partial x}$ at x^{de+d-1} both equal deb_dc_{de}. Hence the polynomial

$$f\frac{\partial g}{\partial x} - eg\frac{\partial f}{\partial x}$$

is divisible by y.

Using the correspondence between elements of V_n and polynomials in $K[z]$ of degree at most n for integers $n \in \mathbb{Z}_{\geq 0}$, Theorem 4.3 has the following corollary.

Corollary 4.10. Suppose that $\operatorname{char}(K) \nmid (de)!$. Let $g \in V_{de}$ be a non-zero polynomial. Then the following are equivalent:

(a) we have $[g] \in T_d$;

(b) there exists a non-zero $f \in V_d$ such that $\frac{1}{y}\left(\frac{\partial g}{\partial x} f - e\frac{\partial f}{\partial x} g\right) = 0$;

(c) there exists a non-zero $f \in V_d$ such that the coefficients of

$$\frac{1}{y}\left(\frac{\partial g}{\partial x} f - e\frac{\partial f}{\partial x} g\right)$$

at $x^{d-1}y^{de-2}, \ldots, x^{de+d-2}$ are all zero.

Proof. This corollary is an analogue of Theorem 4.3. □

Remark 4.11. Proposition 3.1 of [AC2] also gives a similar alternate characterisation of the subset T_d of $\mathbb{P}(V_{de})$ which is similar to 4.10

4.2 The projective variety of e-th powers as the zero set of an ideal generated by determinants

4.12. Consider the bilinear map

$$\begin{aligned} \omega \colon V_d \times V_{de} &\to V_{de+d-2} \\ (f, g) &\mapsto \frac{1}{y}\left(\frac{\partial g}{\partial x} f - e\frac{\partial f}{\partial x} g\right). \end{aligned}$$

For every $g \in V_{de}$, the bilinear map ω gives us the K-linear map

$$\begin{aligned} V_d &\to V_{de+d-2} \\ f &\mapsto \frac{1}{y}\left(\frac{\partial g}{\partial x} f - e\frac{\partial f}{\partial x} g\right). \end{aligned}$$

This gives us the K-linear map

$$\begin{aligned} \ell_d \colon V_{de} &\to \mathrm{Hom}_K(V_d, V_{de+d-2}) \\ g &\mapsto \left(f \mapsto \frac{1}{y}\left(\frac{\partial g}{\partial x} f - e\frac{\partial f}{\partial x} g\right)\right). \end{aligned}$$

Corollary 4.10 tells us that that a non-zero polynomial $g \in V_{de}$ is contained in $CT_d = \mathrm{cone}(T_d)$ if and only if $\ell_d(g)$ has a non-trivial kernel.

4.13. The map ℓ_d is K-linear and hence polynomial. The image of CT_d is contained in the subset of $\mathrm{Hom}_K(V_d, V_{de+d-2})$ consisting of all K-linear maps $V_d \to V_{de+d-2}$ which have rank at most d. Denote this subset of $\mathrm{Hom}_K(V_d, V_{de+d-2})$ by L_d. Recall from Proposition 2.48 that L_d is the affine variety inside $\mathbb{A}(\mathrm{Hom}_K(V_d, V_{de+d-2}))$ corresponding to the prime ideal generated by

$$\alpha_d^*\left(\mathrm{Hom}_K(\Lambda^{d+1}V_d, \Lambda^{d+1}V_{de+d-2})^\times\right)$$

where

$$\alpha_d\colon \operatorname{Hom}_K(V_d, V_{de+d-2}) \to \operatorname{Hom}_K(\Lambda^{d+1}V_d, \Lambda^{d+1}V_{de+d-2})$$

is the homogeneous polynomial map of degree $d+1$ that sends a K-linear map $\ell\colon V_d \to V_{de+d-2}$ to $\Lambda^{d+1}\ell$.

4.14. The map

$$\begin{array}{rcl} \Phi_d\colon CT_d & \to & L_d \\ g & \mapsto & \left(f \mapsto \dfrac{1}{y}\left(\dfrac{\partial g}{\partial x}f - e\dfrac{\partial f}{\partial x}g\right)\right) \end{array}$$

is a restriction of the polynomial map ℓ_d and therefore Φ_d is a morphism of affine varieties. So we get a commutative diagram

$$\begin{array}{ccc} P(\operatorname{Hom}_K(V_d, V_{de+d-2})) & \xrightarrow{\ell_d^*} & P(V_{de}) \\ \downarrow & & \downarrow \\ K[L_d] & \xrightarrow[\Phi_d^*]{} & K[CT_d] \end{array}$$

where the vertical maps are the projection maps. The kernel of the projection map on the left is the ideal generated by

$$\alpha_d^*\left(\operatorname{Hom}_K(\Lambda^{d+1}V_d, \Lambda^{d+1}V_{de+d-2})^\times\right).$$

Since the diagram commutes, we see that the image of this ideal under ℓ_d^* is contained in the kernel I_d of the projection map on the right. Since ℓ_d is injective, we know that $\ell_d^\times$ is surjective and hence ℓ_d^* is also surjective. Therefore the image of the ideal generated by

$$\alpha_d^*\left(\operatorname{Hom}_K(\Lambda^{d+1}V_d, \Lambda^{d+1}V_{de+d-2})^\times\right)$$

under ℓ_d^* is an ideal of $P(V_{de})$. Denote this ideal by J_d.

Conjecture 1. The ideal $J_d \subseteq I_d$ of $P(V_{de})$ is equal to I_d.

Remark 4.15. If Conjecture 1 holds, then the ideal I_d is generated by its degree $d+1$ part. Conjecture 5.1 of [AC2] also states that the ideal I_d is generated by its degree $d+1$ part together with a statement similar to Conjecture 1.

4.16. For each integer $n \in \mathbb{Z}_{\geq 0}$, choose $(x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n)$ as a basis of V_n. Let $n, m \in \mathbb{Z}_{\geq 0}$ be integers. Then a K-linear map $\ell\colon V_n \to V_m$ corresponds with the $m \times n$ matrix $(a_{ki})_{ki}$ such that

$$\ell(x^{n-i}y^i) = \sum_{k=0}^{m} a_{ki}x^{m-k}y^k$$

for all $i \in \{0, \ldots, n\}$.

4.17. Let $g \in V_{de}$ be a polynomial and consider the K-linear map

$$\begin{aligned} \ell_d(g)\colon V_d &\rightarrow V_{de+d-2} \\ f &\mapsto \frac{1}{y}\left(\frac{\partial g}{\partial x}f - e\frac{\partial f}{\partial x}g\right). \end{aligned}$$

Let $i \in \{0, \ldots, d\}$ and $j \in \{0, \ldots, de\}$ be integers. Take $f = x^{d-i}y^i$ and $g = x^{de-j}y^j$. Then we have

$$\frac{1}{y}\left(\frac{\partial g}{\partial x}f - e\frac{\partial f}{\partial x}g\right) = (ie-j)x^{de+d-i-j-1}y^{i+j-1}.$$

So we see that if

$$g = c_0y^{de} + c_1xy^{de-1} + \cdots + c_{de-1}x^{de-1}y + c_{de}x^{de},$$

then we have

$$\ell_d(g)(x^{d-i}y^i) = \sum_{j=0}^{de} c_{de-j}(ie-j)x^{de+d-i-j-1}y^{i+j-1}$$

for all $i \in \{0, \ldots, d\}$. So in this case, the matrix

$$\begin{pmatrix}
-c_{de-1} & ec_{de} & 0 & \dots & \dots & \dots & 0 \\
-2c_{de-2} & (e-1)c_{de-1} & 2ec_{de} & 0 & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & & \ddots & 0 & \vdots \\
\vdots & \vdots & \vdots & & & (d-1)ec_{de} & 0 \\
\vdots & \vdots & \vdots & & & \vdots & dec_{de} \\
\vdots & \vdots & \vdots & & & \vdots & \vdots \\
-dec_0 & \vdots & \vdots & & & \vdots & \vdots \\
0 & -(d-1)ec_0 & \vdots & & & \vdots & \vdots \\
\vdots & 0 & -(d-2)ec_0 & & & \vdots & \vdots \\
\vdots & \vdots & 0 & \ddots & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \dots & 0 & -ec_0 & c_1
\end{pmatrix}$$

is the matrix corresponding to the K-linear map $\ell_d(g)$.

Let $(c_0, \ldots, c_{de})$ be the dual basis of $(y^{de}, xy^{de-1}, \ldots, x^{de-1}y, x^{de})$ and let M be $(de+d-1) \times (d+1)$ the matrix over $P(V_{de})$ written above. Then we see that by Corollary 4.10 a polynomial $g \in \mathbb{A}(V_{de})$ is an element of CT_d if and only if g is contained in the zero set of the ideal J_d of $P(V_{de})$ generated the determinants of the $(d+1) \times (d+1)$ submatrices of M.

4.18. Note that if a polynomial $g \in V_{de}$ is contained in the zero set of the ideal $J_d^\dagger$ of $P(V_{de})$ generated the determinants of the $(d+1) \times (d+1)$ submatrices of M that do not contain any of the bottom $d-1$ rows, then g is an element of CT_d by condition (c) of Corollary 4.10.

Example 4.19. For $d = 0$ or $e = 1$, Conjecture 1 is trivially true. Using the matrix above and one of the techniques from section 3.3, we can check Conjecture 1 for instances of $d > 0$ and $e > 1$. We will assume in this example that $K = \mathbb{C}$. For $d = 1$ and $e \in \{2, \ldots, 10\}$, for $d = 2$ and $e \in \{1, \ldots, 9\}$, for $d = 3$ and $e \in \{2, 3, 4\}$ and for $(d, e) = (4, 2)$, we find that the conjecture holds. In the cases where $d > 1$, we also find that $J_d^\dagger$ and $J_d = I_d$ are not equal.

We have the ideals I_d, J_d and $J_d^\dagger$ of $P(V_{de})$ which have the same zero set in $\mathbb{A}(V_{de})$. So we have $J_d^\dagger \subseteq J_d \subseteq I_d$ and radical ideals of J_d and $J_d^\dagger$ are both equal to the prime ideal I_d. From the example, we see that the equality $J_d^\dagger = I_d$ does not hold in general. It is however true that the dehomogenisations of I_d, J_d and $J_d^\dagger$ with respect to c_{de} are equal if $\mathrm{char}(K) \nmid (de)!$. This is what we are going to prove next.

4.20. Let A be a K-algebra and let N be an $n \times m$ matrix over A where $n, m \in \mathbb{Z}_{\geq 0}$ are integers. Assume that $n \geq m$. Let V be the vector subspace of A spanned by the determinants of all $m \times m$ submatrices of N. Now suppose that we multiply a row or column of N by a non-zero constant $\lambda \in K^*$. Then all determinants of submatrices of N that contain that row or column get multiplied with λ. So we see that the vector space V does not change. One can check that if we add a multiple of a row or column of N to another row or column of N, then the vector space V also does not change.

4.21. Assume that $\mathrm{char}(K) \nmid (de)!$. Recall that the ideal $J_d^\dagger$ is generated by the determinants of the submatrices of maximal size of M that do not contain any of the bottom $d-1$ rows. We now want to consider the dehomogenisation of $J_d^\dagger$ with respect to c_{de}. This dehomogenisation is equal to the ideal of $K[M_{de}] = P(\ker c_{de})$ generated by the determinants of the $(d+1) \times (d+1)$ submatrices of the matrix

$$\begin{pmatrix}
-c_{de-1} & ec_{de} & 0 & \cdots & \cdots & \cdots & 0 \\
-2c_{de-2} & (e-1)c_{de-1} & 2ec_{de} & \ddots & & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & & \ddots & 0 \\
\vdots & \vdots & \vdots & & & & dec_{de} \\
\vdots & \vdots & \vdots & & & & \vdots \\
-dec_0 & (e+1-de)c_1 & (2e+2-de)c_2 & \cdots & \cdots & \cdots & dc_d
\end{pmatrix}$$

where we replace c_{de} by 1. If we replace c_{de} by 1, we get a matrix of the form

$$\begin{pmatrix}
-c_{de-1} & e & 0 & \dots & \dots & \dots & 0 \\
-2c_{de-2} & \bullet c_{de-1} & 2e & \ddots & & & \vdots \\
\vdots & \vdots & \bullet c_{de-1} & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & & & \ddots & de \\
\vdots & \vdots & \vdots & & & & \bullet c_{de-1} \\
\vdots & \vdots & \vdots & & & & \vdots \\
-dec_0 & \bullet c_1 & \bullet c_2 & \dots & \dots & \dots & \bullet c_d
\end{pmatrix}$$

where we denote all elements of K also by $\bullet$. Now we are going to apply row and column operations to this matrix. Recall that this does not change the vector subspace spanned by determinants of maximal submatrices of this matrix. So it also does not change the ideal generated by these determinants.

We start by multiplying the k-th row by $1/k$ for all $k \in \{1, \dots, de\}$, by multiplying the first column by -1 and by multiplying the other columns by $1/e$. This gives us a matrix of the form

$$\begin{pmatrix}
c_{de-1} & 1 & 0 & \dots & \dots & \dots & 0 \\
c_{de-2} & \bullet c_{de-1} & 1 & \ddots & & & \vdots \\
\vdots & \vdots & \bullet c_{de-1} & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & & & \ddots & 1 \\
\vdots & \vdots & \vdots & & & & \bullet c_{de-1} \\
\vdots & \vdots & \vdots & & & & \vdots \\
c_0 & \bullet c_1 & \bullet c_2 & \dots & \dots & \dots & \bullet c_d
\end{pmatrix}.$$

Next we add multiples of the first row to all rows below it in a way such that the second column becomes $(1, 0, \dots, 0)^T$. This gives us a matrix of the form

$$\begin{pmatrix}
c_{de-1} & 1 & 0 & \cdots & \cdots & \cdots & 0 \\
c_{de-2} - f_{de-1} & 0 & 1 & \ddots & & & \vdots \\
\vdots & \vdots & \bullet c_{de-1} & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & & & \ddots & 1 \\
\vdots & \vdots & \vdots & & & & \bullet c_{de-1} \\
\vdots & \vdots & \vdots & & & & \vdots \\
c_0 - f_0 & 0 & \bullet c_2 & \cdots & \cdots & \cdots & \bullet c_d
\end{pmatrix}.$$

where for each $i \in \{0, \ldots, de-2\}$ the polynomial $f_i \in P(\ker c_{de})$ is a polynomial in the variables $c_{i+1}, \ldots, c_{de-1}$. We repeat this for the second till d-th row to get a matrix of the form

$$\begin{pmatrix}
c_{de-1} & 1 & 0 & \cdots & \cdots & 0 \\
c_{de-2} - g_{de-2} & 0 & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & 0 \\
\vdots & \vdots & & & \ddots & 1 \\
\vdots & \vdots & & & & 0 \\
\vdots & \vdots & & & & \vdots \\
c_0 - g_0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}$$

where for each $i \in \{0, \ldots, de-2\}$ the polynomial g_i is a polynomial in the variables $c_{i+1}, \ldots, c_{de-1}$. Lastly, for the rows $d+1$ till $de+d-1$, we add a multiple of the row to the rows below it to get a matrix of the form

$$\begin{pmatrix}
q_{de-1} & 1 & 0 & \cdots & \cdots & 0 \\
c_{de-2} - h_{de-2} & 0 & \ddots & \ddots & & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & 0 \\
\vdots & \vdots & & & \ddots & 1 \\
\vdots & \vdots & & & & 0 \\
\vdots & \vdots & & & & \vdots \\
c_0 - h_0 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}$$

where for each $i \in \{0, \ldots, de-2\}$ the polynomial h_i is a polynomial in the variables $c_{de-d}, \ldots, c_{de-1}$.

Now consider the determinants of the $(d+1) \times (d+1)$ submatrices of this matrix. If such a submatrix does not contain the first d rows, then one of its columns is zero and hence its determinant is zero. So the non-zero determinants we get are up to a minus sign

$$\{c_i - h_i | i \in \{0, \dots, de-d\}\}.$$

Recall that this set of determinants is a subset of the prime ideal

$$(q_0, \dots, q_{de-d})$$

of $P(\ker c_{de})$ corresponding to the affine variety S_d of de-monic polynomials $g \in V_{de}$ that are the e-th power of some d-monic polynomial in V_d. For each $i \in \{0, \dots, de-d\}$, the polynomial q_i is is the difference of c_i and some polynomial $h_i^\dagger$ in the variables $c_{de-d}, \dots, c_{de-1}$. For each $i \in \{0, \dots, de-d\}$, we see that $h_i - h_i^\dagger$ is contained in $(q_0, \dots, q_{de-d}) \cap K[c_{de-d}, \dots, c_{de-1}] = 0$. Hence $c_i - h_i = q_i$ for all $i \in \{0, \dots, de-d\}$.

4.3 The Hilbert function of the ideals associated to the projective varieties of e-th powers

Since pow_d is a homogeneous polynomial map of degree e, we see that for each $i \in \mathbb{Z}_{\geq 0}$ we get a K-linear map

$$\mathrm{pow}^*_{d,(i)} \colon \mathrm{Sym}^i(V_{de}^\times) \to \mathrm{Sym}^{ie}(V_d^\times).$$

If Conjecture 1 holds, then I_d is generated by homogeneous polynomials of degree $d+1$ and so the degree d part of I_d must be zero. Recall that I_d is the kernel of the homomorphism of K-algebras $\mathrm{pow}^*_d \colon P(V_{de}) \to P(V_d)$. So we can weaken Conjecture 1 in the following way.

Conjecture 2. The map $\mathrm{pow}^*_{d,(d)} \colon \mathrm{Sym}^d(V_{de}^\times) \to \mathrm{Sym}^{de}(V_d^\times)$ is injective.

4.22. Since pow^*_d is a homomorphism of K-algebras, we see that from Conjecture 2 easily follows that

$$\mathrm{pow}^*_{d,(i)} \colon \mathrm{Sym}^i(V_{de}^\times) \to \mathrm{Sym}^{ie}(V_d^\times)$$

is injective for all $i \leq d$. Recall that for an integer $n \in \mathbb{Z}_{\geq 0}$ and a vector space V over K of dimension m, the vector space $\mathrm{Sym}^n(V)$ has dimension

$$\binom{n+m-1}{m-1}$$

over K. So $\mathrm{Sym}^i(V_{de}^\times)$ has dimension

$$\binom{i+de}{de}$$

over K and $\mathrm{Sym}^{ie}(V_d^\times)$ has dimension

$$\binom{ie+d}{d}$$

over K. Note that these two binomials are equal when $i = d$. So if the K-linear map $\mathrm{pow}_{d,(d)}^*\colon \mathrm{Sym}^d(V_{de}^\times) \to \mathrm{Sym}^{de}(V_d^\times)$ is injective as Conjecture 2 states, then it is also surjective.

4.23. Let $(c_0, \dots, c_{de})$ be the basis of $V_{de}^\times$ dual to $(y^{de}, xy^{de-1}, \dots, x^{de})$ and let $(b_0, \dots, b_d)$ be the basis of $V_d^\times$ dual to $(y^d, xy^{d-1}, \dots, x^d)$. Let $f \in V_d$ be a polynomial and take $g = f^e$. Then we have

$$f = b_0(f)y^d + b_1(f)xy^{d-1} + \cdots + b_d(f)x^d$$

and

$$g = \sum_{k=0}^{de} \left(\sum_{\substack{0 \le i_1, \dots, i_e \le d \\ i_1 + \cdots + i_e = k}} b_{i_1}(f) \dots b_{i_e}(f) \right) x^k y^{de-k}.$$

So we see for $k \in \{0, \dots, de\}$ that

$$\sum_{\substack{0 \le i_1, \dots, i_e \le d \\ i_1 + \cdots + i_e = k}} b_{i_1} \odot \cdots \odot b_{i_e}$$

is the polynomial on V_d associated to the polynomial function $c_k \circ \mathrm{pow}_d$. This determines the map $\mathrm{pow}_{d,(1)}^*\colon V_{de}^\times \to \mathrm{Sym}^e(V_d^\times)$ and therefore it determines the whole map pow_d^*. Recall from 2.10 that we have

$$\mathrm{pow}_{d,(k)}^*(\varphi_1 \odot \cdots \odot \varphi_k) = \mathrm{pow}_{d,(1)}^*(\varphi_1) \odot \cdots \odot \mathrm{pow}_{d,(1)}^*(\varphi_k)$$

for all $k \in \mathbb{Z}_{\ge 1}$ and $\varphi_1, \dots, \varphi_k \in V_{de}^\times$, because pow_d^* is the extension of $\mathrm{pow}_{d,(1)}^*$ to $P(V_{de})$.

Theorem 4.24. Let $k \in \mathbb{Z}_{\ge d}$ be an integer and suppose that the K-linear map

$$\mathrm{pow}_{d,(k)}^*\colon \mathrm{Sym}^k(V_{de}^\times) \to \mathrm{Sym}^{ke}(V_d^\times)$$

is surjective. Then the map

$$\mathrm{pow}_{d,(k+1)}^*\colon \mathrm{Sym}^{k+1}(V_{de}^\times) \to \mathrm{Sym}^{(k+1)e}(V_d^\times)$$

is also surjective.

Proof. Consider the commuting diagram

$$
\begin{array}{ccc}
\mathrm{Sym}^k(V_{de}^\times) \otimes V_{de}^\times & \xrightarrow{\mathrm{pow}_{d,(k)}^* \otimes \mathrm{pow}_{d,(1)}^*} & \mathrm{Sym}^{ke}(V_d^\times) \otimes \mathrm{Sym}^e(V_d^\times) \\
\downarrow & \overset{\ell}{\searrow} & \downarrow \\
\mathrm{Sym}^{k+1}(V_{de}^\times) & \xrightarrow[\mathrm{pow}_{d,(k+1)}^*]{} & \mathrm{Sym}^{(k+1)e}(V_d^\times)
\end{array}
$$

where the vertical maps come from the bilinear multiplication maps on $P(V_{de})$ and $P(V_d)$. Note that $\mathrm{pow}_{d,(k+1)}^*$ is surjective if ℓ is surjective.

We have the basis $(b_0, \dots, b_d)$ for $V_d^\times$. This gives us a basis

$$(b_{i_1} \odot \cdots \odot b_{i_n} | 0 \leq i_1 \leq \cdots \leq i_n \leq d)$$

for $\mathrm{Sym}^n(V_d^\times)$. We order this basis by saying that $b_{i_1} \odot \cdots \odot b_{i_n} \prec b_{j_1} \odot \cdots \odot b_{j_n}$ if $(i_1, \dots, i_n) \neq (j_1, \dots, j_n)$ and we have $i_k < j_k$ where k is the smallest integer such that $i_k \neq j_k$. We say that $b_{i_1} \odot \cdots \odot b_{i_n} \preceq b_{j_1} \odot \cdots \odot b_{j_n}$ if $b_{i_1} \odot \cdots \odot b_{i_n} \prec b_{j_1} \odot \cdots \odot b_{j_n}$ or $b_{i_1} \odot \cdots \odot b_{i_n} = b_{j_1} \odot \cdots \odot b_{j_n}$. This gives us a totally ordered basis for $\mathrm{Sym}^n(V_d^\times)$.

Let $i \in \{0, \dots, de\}$ and consider the element

$$\mathrm{pow}_d^*(c_i) = \mathrm{pow}_{d,(1)}^*(c_i) = \sum_{\substack{0 \leq i_1, \dots, i_e \leq d \\ i_1 + \cdots + i_e = i}} b_{i_1} \odot \cdots \odot b_{i_e}$$

of $\mathrm{Sym}^e(V_d^\times)$. Take $q \in \{0, \dots, d\}$ and $r \in \{0, \dots, e-1\}$ such that $i = qe + r$. Then the $\preceq$-maximal element of

$$\left\{ b_{i_1} \odot \cdots \odot b_{i_e} \;\middle|\; \begin{array}{c} 0 \leq i_1, \dots, i_e \leq d \\ i_1 + \cdots + i_e = i \end{array} \right\}$$

is $b_q^{\odot e-r} \odot b_{q+1}^{\odot r}$. In particular, for all $i \in \{0, \dots, d\}$ we see that

$$\mathrm{pow}_{d,(1)}^*(c_{ie}) - b_i^{\odot e}$$

is a linear combination of basis elements of $\mathrm{Sym}^e(V_d^\times)$ that are smaller than $b_i^{\odot e}$.

We want to prove that ℓ is surjective. So it suffices to prove that every element of the basis

$$(b_{i_1} \odot \cdots \odot b_{i_{(k+1)e}} | 0 \leq i_1 \leq \cdots \leq i_{(k+1)e} \leq d)$$

of $\mathrm{Sym}^{(k+1)e}(V_d^\times)$ is contained in the image of ℓ. We will prove this using induction on this totally ordered set.

Let $0 \leq i_1 \leq \cdots \leq i_{(k+1)e} \leq d$ and suppose that all basis elements smaller than $b_{i_1} \odot \cdots \odot b_{i_{(k+1)e}}$ are contained in the image of ℓ. We have

$(k+1)e \geq (d+1)e$. So we see that there must be some $i \in \{0, \ldots, d\}$ such that $\#\{j | i_j = i\} \geq e$. We have $b_{i_1} \odot \cdots \odot b_{i_{(k+1)e}} = b_i^{\odot e} \odot t$ for some

$$t \in \{b_{i_1} \odot \cdots \odot b_{i_{ke}} | 0 \leq i_1 \leq \cdots \leq i_{ke} \leq d\}.$$

Since $\mathrm{pow}^*_{d,(k)}$ is surjective, we have $t = \mathrm{pow}^*_{d,(k)}(s)$ for some $s \in \mathrm{Sym}^k(V_{de}^\times)$. So we have

$$b_{i_1} \odot \cdots \odot b_{i_{(k+1)e}} = \left(\mathrm{pow}^*_{d,(1)}(c_{ie}) - b_i^{\odot e}\right) \odot t + \mathrm{pow}^*_{d,(1)}(c_{ie}) \odot \mathrm{pow}^*_{d,(k)}(s).$$

We know that $\mathrm{pow}^*_{d,(1)}(c_{ie}) - b_i^{\odot e}$ is a linear combination of basis elements of $\mathrm{Sym}^e(V_d^\times)$ that are smaller than $b_i^{\odot e}$. Therefore

$$\left(\mathrm{pow}^*_{d,(1)}(c_{ie}) - b_i^{\odot e}\right) \odot t$$

is a linear combination of basis elements of $\mathrm{Sym}^{(k+1)e}(V_d^\times)$ that are smaller than $b_i^{\odot e} \odot t = b_{i_1} \odot \cdots \odot b_{i_{(k+1)e}}$. We also know that

$$\mathrm{pow}^*_{d,(1)}(c_{ie}) \odot \mathrm{pow}^*_{d,(k)}(s) = \ell(s \otimes c_{ie}).$$

So both are contained in the image of ℓ. Hence $b_{i_1} \odot \cdots \odot b_{i_{(k+1)e}}$ is contained in the image of ℓ. □

Remark 4.25. For $d = 2$ and $K = \mathbb{C}$, the surjectivity of the maps $\mathrm{pow}^*_{d,(k)}$ for $k \geq 2$ also follows from Theorem 1.1 of [AC1], because the maps from that theorem can be obtained as the composition of the maps $\mathrm{pow}^*_{d,(k)}$ with Howe's isomorphisms from Section 8.2.

Corollary 4.26. Assume that Conjecture 2 holds. Then the dimension of the degree k part of I_d equals zero for $k \leq d$ and equals

$$\binom{k+de}{de} - \binom{ke+d}{d}$$

for $k > d$.

Proof. If Conjecture 2 holds, then the map $\mathrm{pow}^*_{d,(i)}$ is injective for all $i \leq d$ and surjective for all $i \geq d$. □

4.4 The relation between conjectures 1 and 2

The second conjecture is a weakened version of the first conjecture, but is some cases it is not strictly weaker. If we prove Conjecture 2, we know the Hilbert function of the ideal I_d. So to prove or disprove the first conjecture, it then suffices to compute the Hilbert function of the ideal J_d. We will do just that for $d = 1$ in this section. Assume that the characteristic of K does not divide $e!$.

Lemma 4.27. The dimension of the degree k part of J_1 equals zero for $k \leq 1$ and is at least
$$\binom{k+e}{e} - (ke+1)$$
for $k > 1$.

Proof. The ideal J_1 is generated by the determinants of the 2×2 submatrices of the matrix
$$\begin{pmatrix} -c_{e-1} & -2c_{e-2} & \dots & (1-e)c_1 & -ec_0 \\ ec_e & (e-1)c_{e-1} & \dots & 2c_2 & c_1 \end{pmatrix}$$
and therefore the dimensions of the degree zero and one parts of the ideal J_1 are zero. Note that the degree k part of $P(V_e)$ equals
$$\binom{k+e}{e}.$$
So it suffices to prove that for each $k > 1$ the degree k part of the K-algebra $P(V_e)/J_1$ has dimension at most $ie+1$.

Let $k \in \mathbb{Z}_{>1}$ be an integer. Then the degree k part of $P(V_e)/J_1$ is spanned by $c_0^{\alpha_0} \cdots c_e^{\alpha_e}$ for all $(\alpha_0, \dots, \alpha_e) \in \mathbb{Z}_{\geq 0}^{e+1}$ such that $\alpha_0 + \dots + \alpha_e = 0$. Note that J_1 is generated by the elements
$$i(j-e-1)c_i c_{j-1} - j(i-e-1)c_{i-1}c_j$$
for all $1 \leq i < j \leq e$. Since $\mathrm{char}(K) \nmid e!$, we see that i, j, $(i-e-1)$ and $(j-e-1)$ are non-zero for all $1 \leq i < j \leq e$. So we see that for all $1 \leq i < j \leq e$, there exists a $\lambda \in K^*$ such that $c_i c_{j-1} = \lambda c_{i-1} c_j$ in $P(V_e)/J_1$. It follows that for all $(\alpha_0, \dots, \alpha_e), (\beta_0, \dots, \beta_e) \in \mathbb{Z}_{\geq 0}^{e+1}$ such that
$$\sum_{h=0}^{e} h\alpha_h = \sum_{h=0}^{e} h\beta_h,$$
there exists a $\lambda \in K^*$ such that $c_0^{\alpha_0} \cdots c_e^{\alpha_e} = \lambda c_0^{\beta_0} \cdots c_e^{\beta_e}$ in $P(V_e)/J_1$. So the dimension of the degree k part of $P(V_e)/J_1$ is bounded from above by the size of
$$\left\{ \sum_{h=0}^{e} h\alpha_h \,\middle|\, \begin{array}{c} \alpha_0, \dots, \alpha_e \in \mathbb{Z}_{\geq 0} \\ \alpha_0 + \dots + \alpha_e = k \end{array} \right\}.$$
Let $\alpha_0, \dots, \alpha_e \in \mathbb{Z}_{\geq 0}$ be integers such that $\alpha_0 + \dots + \alpha_e = k$. Then we have
$$0 \leq \sum_{h=0}^{e} h\alpha_h \leq e \sum_{h=0}^{e} \alpha_h = ke$$
Hence the degree k part of $P(V_e)/J_1$ has dimension at most $ke+1$. $\square$

Corollary 4.28. If Conjecture 2 holds for $d = 1$, then Conjecture 1 holds for $d = 1$.

Proof. The result follows by comparing the dimensions of the degree i parts of the ideals J_1 and I_1 for all integers $i \in \mathbb{Z}_{\geq 0}$. □

Chapter 5

Bimodules and commutants

In this chapter, let K be an algebraically closed field.

To use Schur-Weyl duality, we need the Double Commutant Theorem. So in this chapter, we prove this theorem.

5.1 Modules

We will assume knowledge about modules over a ring and semisimple modules comparable to the first three paragraphs of chapter III and the first two paragraphs of chapter XVII of [La]. Let A be a K-algebra.

5.1. Let V be an A-module. Then the composition of the homomorphism of rings $\iota\colon K \to A$ coming from the K-algebra and the homomorphism of rings $A \to \mathrm{End}_{\mathbb{Z}}(V)$ is a homomorphism of rings $K \to \mathrm{End}_{\mathbb{Z}}(V)$. This homomorphism gives V the structure of a vector space over K such that the map

$$\begin{array}{rcl} V & \to & V \\ v & \mapsto & a \cdot v \end{array}$$

is K-linear for all $a \in A$, because each element of the image of ι commutes with every element of A.

Let V, W be A-modules and consider V and W as vector spaces over K using the induced structure from A. Let $\ell\colon V \to W$ be an A-linear map. Then we have

$$\ell(\lambda \cdot v) = \ell(\iota(\lambda) \cdot v) = \iota(\lambda) \cdot \ell(v) = \lambda \cdot \ell(v)$$

for all $\lambda \in K$ and $v \in V$. Hence the map $\ell\colon V \to W$ is K-linear.

Lemma 5.2 (Schur's Lemma)**.** Let V, W be simple A-modules. Then any non-zero A-linear map $V \to W$ is an isomorphism. The map

$$\begin{array}{rcl} K & \to & \operatorname{End}_A(V) \\ \lambda & \mapsto & \lambda \operatorname{id}_V \end{array}$$

is an isomorphism of K-algebras.

Proof. Let $\ell\colon V \to W$ be a non-zero A-linear map. Then the image of ℓ is a non-zero A-invariant subspace of W. So since W is simple, we see that ℓ is surjective. The kernel of ℓ is an A-invariant subspace of V which is not equal to V. So since V is simple, we see that ℓ is injective. Hence ℓ is an isomorphism. The map

$$\begin{array}{rcl} K & \to & \operatorname{End}_A(V) \\ \lambda & \mapsto & \lambda \operatorname{id}_V \end{array}$$

is injective, because V is not zero.

Let $\ell\colon V \to V$ be an A-linear endomorphism. Then ℓ is also a K-linear map. Since K is algebraically closed, we know that ℓ has an eigenvalue $\lambda \in K$. Note that $\ell - \lambda \operatorname{id}_V \in \operatorname{End}_A(V)$ is not an isomorphism. Hence $\ell - \lambda \operatorname{id}_V$ is zero. So every element of $\operatorname{End}_A(V)$ is of the form $\lambda \operatorname{id}_V$ for some $\lambda \in K$. □

Proposition 5.3. Let

$$A = \bigoplus_{i \in I} M_i$$

be a decomposition of the A-module A into a sum of simple submodules. Then every A-module is isomorphic to a direct sum of a family of members of the family $(M_i)_{i \in I}$.

Proof. Let W be an A-module and let $(w_j)_{j \in J}$ be a basis of W over K. The the map

$$\begin{array}{rcl} \ell\colon \bigoplus_{j \in J} A & \to & W \\ (a_j)_j & \mapsto & \sum_{j \in J} a_j \cdot w_j \end{array}$$

is a surjective homomorphism of A-modules. For all $j \in J$ and $i \in I$, let $M_{j,i}$ be M_i. Then we have

$$\bigoplus_{j \in J} A = \bigoplus_{j \in J} \bigoplus_{i \in I} B_{j,i}.$$

So we see that

$$W = \bigoplus_{j \in J} \bigoplus_{i \in I} \ell(B_{j,i})$$

is a decomposition of W into submodules. Since the A-modules $B_{j,i}$ are simple, the A-modules $\ell(B_{j,i})$ are zero or simple. When $\ell(B_{j,i})$ is simple, the map $\ell|_{B_{j,i}}$ is an isomorphism of A-modules. So we see that W isomorphic to a direct sum of a family of members of the family $(M_i)_{i\in I}$. □

Corollary 5.4. If the ring A is semisimple, then each A-module is semisimple and each simple A-module is isomorphic to a submodule of A.

Proof. Suppose that the ring A is semisimple. Then the A-module A has a decomposition into simple submodules. So we see that every A-module is semisimple by the previous theorem. The theorem also tells us that each simple A-module is isomorphic to a submodule of A. □

For each positive integer $n \in \mathbb{Z}_{>0}$, denote the K-algebra consisting of all $n \times n$ matrices over K by $M_n(K)$.

Example 5.5. Let $n \in \mathbb{Z}_{>0}$ be a positive integer and let A be the K-algebra $M_n(K)$. Then the vector space K^n naturally has the structure of an A-module. Let V be a non-zero submodule of K^n and let $v \in V$ be a non-zero element. For each element $w \in K^n$, there exists a matrix $M \in A$ such that $Av = w$. Hence $V = K^n$. So we see that the A-module K^n is simple.

Let $j \in \{1, \ldots, n\}$ be an integer. Denote the subspace of $M_n(K)$ consisting of all $n \times n$ matrices whose entries outside the j-th column are zero by $M_n(K)_j$. Then $M_n(K)_j$ is a submodule of $M_n(K)$. Then map $M_n(K)_j \to K^n$ sending a matrix to its j-th column is an isomorphism of A-modules. Therefore

$$A = \bigoplus_{j=1}^{n} M_n(K)_j$$

is a decomposition of the A-module A into simple submodules. We see that the ring A is semisimple and that every simple A-modules is isomorphic to K^n.

Example 5.6. Let $n_1, \ldots, n_s \in \mathbb{Z}_{>0}$ be positive integers and let A be the K-algebra

$$M_{n_1}(K) \times \cdots \times M_{n_s}(K).$$

Let $i \in \{1, \ldots, s\}$ be an integer. Then the homomorphism of rings

$$\begin{array}{rcl} A & \to & \mathrm{End}_{\mathbb{Z}}\left(M_{n_i}(K)\right) \\ (M_1, \ldots, M_s) & \mapsto & (M \mapsto M_i M) \end{array}$$

gives $M_{n_i}(K)$ the structure of an A-module. Let $\iota_i\colon M_{n_i}(K) \to A$ be the homomorphism of A-modules sending a matrix M to the s-tuple

$$(0, \ldots, 0, M, 0, \ldots, 0)$$

where the matrix M is in the i-th place. Then we see that

$$A = \bigoplus_{i=1}^{s} \iota_i(M_{n_i}(K))$$

is a decomposition of the A-module A into submodules.

The homomorphism of rings

$$\begin{array}{rcl} A & \to & \mathrm{End}_{\mathbb{Z}}(K^{n_i}) \\ (M_1, \ldots, M_s) & \mapsto & (v \mapsto M_i v) \end{array}$$

gives the abelian group K^{n_i} the structure of a simple A-module. For each integer $j \in \{1, \ldots, n_i\}$, the subspace $M_{n_i}(K)_j$ of $M_{n_i}(K)$ is a submodule of $M_{n_i}(K)$. The map $M_{n_i}(K)_j \to K^{n_i}$ sending a matrix to its j-th column is an isomorphism of A-modules. So we see that

$$A \cong \bigoplus_{i=1}^{s} \bigoplus_{j=1}^{n_i} K^{n_i}$$

is a decomposition of A into a direct sum of simple A-modules. Hence the ring A is semisimple and every simple A-module W there is an integer $i \in \{1, \ldots, s\}$ such that W is isomorphic to the A-module K^{n_i} such that

$$(M_1, \ldots, M_s) \cdot v = M_i v$$

for all $(M_1, \ldots, M_s) \in A$ and $v \in K^{n_i}$.

Remark 5.7. Let $i, i' \in \{1, \ldots, s\}$ be distinct integers. Then the A-modules K^{n_i} and $K^{n_{i'}}$ constructed above are not isomorphic as A-modules, even when $n_i = n_{i'}$, because the actions of A on these abelian groups are different.

Theorem 5.8 (Artin-Wedderburn)**.** If the K-algebra A is finite dimensional over K, then the ring A is semisimple if and only if A is isomorphic to

$$M_{n_1}(K) \times \cdots \times M_{n_s}(K)$$

as K-algebra for some positive integers $n_1, \ldots, n_s \in \mathbb{Z}_{>0}$.

Proof. The previous example show that the ring

$$M_{n_1}(K) \times \cdots \times M_{n_s}(K)$$

is semisimple for all positive integers $n_1, \ldots, n_s \in \mathbb{Z}_{>0}$. For the other direction, see Proposition 5.2.6 of [Co]. □

Let I be an ideal of the K-algebra A and let $\pi \colon A \to A/I$ be the projection map.

5.9. Let V be an A/I-module. Then the composition $A \to \operatorname{End}_{\mathbb{Z}}(V)$ of the associated homomorphism of rings $A/I \to \operatorname{End}_{\mathbb{Z}}(V)$ with π gives V the structure of an A-module such that $a \cdot v = 0$ for all $a \in I$ and $v \in V$.

Let W be an A-module such that $a \cdot w = 0$ for all $a \in I$ and $w \in W$. Then the homomorphism of rings $A \to \operatorname{End}_{\mathbb{Z}}(W)$ factors through π.

We see that the A/I-modules correspond one to one with the A-modules such that I is contained in the kernel of the associated homomorphism of rings. Let V, W be A/I-modules and let $\ell \colon V \to W$ be a map. Then ℓ is A/I-linear if and only if ℓ is A-linear. We see that the correspondence is a functor.

Let $\mathbf{C}$ be the full subcategory of A-$\underline{\text{Mod}}$ consisting of all A-modules such that I is contained in the kernel of the associated homomorphism of rings.

Theorem 5.10. The additive covariant functor A/I-$\underline{\text{Mod}} \to \mathbf{C}$ sending an A/I-module V to V with its induced A-modules structure and sending an A/I-linear map to itself is invertible.

Proof. This theorem is a reformulation of 5.9. □

Let V be an A/I-module. Then a subset W of V is A/I-invariant if and only if it is A-invariant. So we see that V is a simple A/I-module if and only if V is a simple A-module. We also see that V is a semisimple A/I-module if and only if V is a semisimple A-module. The gives us the following corollary.

Corollary 5.11. Suppose that the ring A is semisimple. Then the ring A/I is also semisimple.

Proof. Suppose that the ring A is semisimple. Then every A/I-module has a decomposition as an A-module into a direct sum of a family of simple A-modules. This decomposition is also a decomposition as an A/I-module into a direct sum of a family of simple A/I-modules. □

5.2 Bimodules

Definition 5.12. Let M be an abelian group. Then we say that homomorphisms of rings $\eta \colon R \to \operatorname{End}_{\mathbb{Z}}(M)$ and $\theta \colon S \to \operatorname{End}_{\mathbb{Z}}(M)$ commute if $\eta(r) \circ \theta(s) = \theta(s) \circ \eta(r)$ for all $r \in R$ and $s \in S$.

Definition 5.13. Let R, S be rings. Define an (R, S)-bimodule to be an abelian group M that comes with with a pair of commuting homomorphisms of rings $R \to \operatorname{End}_{\mathbb{Z}}(M)$ and $S \to \operatorname{End}_{\mathbb{Z}}(M)$.

Let M, N be (R, S)-bimodules. Then we call a map $M \to N$ an (R, S)-linear map is it is both R-linear and S-linear.

Remark 5.14. Similar to the category $R\text{-}\underline{\mathrm{Mod}}_K$, one can check that the category $(R,S)\text{-}\underline{\mathrm{biMod}}$ of (R,S)-bimodules is abelian.

Let A_1, A_2 be K-algebras.

Definition 5.15. Define $(A_1,A_2)\text{-}\underline{\mathrm{biMod}}_K$ to be the full subcategory of the category $(A_1,A_2)\text{-}\underline{\mathrm{biMod}}$ consisting of all (A_1,A_2)-bimodules V such that the diagram

$$\begin{array}{ccc} K & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & \mathrm{End}_{\mathbb{Z}}(V) \end{array}$$

commutes.

5.16. Let V be an (A_1,A_2)-bimodule. Then V is both an A_1-module and an A_2-module. So V inherits the structure of a vector space over K from both A_1 and A_2. The (A_1,A_2)-bimodule V is an object of $(A_1,A_2)\text{-}\underline{\mathrm{biMod}}_K$ precisely when these induced vector spaces structure are the same.

Suppose that V is an object of $(A_1,A_2)\text{-}\underline{\mathrm{biMod}}_K$. Let $\eta\colon A_1 \to \mathrm{End}_{\mathbb{Z}}(V)$ and $\theta\colon A_2 \to \mathrm{End}_{\mathbb{Z}}(V)$ be the associated homomorphisms of rings. Then $\eta(a_1)$ and $\theta(a_2)$ are both K-linear maps $V \to V$ by 5.1 for all $a_1 \in A$ and $a_2 \in A_2$. The map

$$\begin{array}{rcl} A_1 \times A_2 & \to & \mathrm{End}_K(V) \\ (a_1,a_2) & \mapsto & \eta(a_1)\circ\theta(a_2) \end{array}$$

is K-bilinear. Note that the corresponding K-linear map

$$\begin{array}{rcl} A_1 \otimes_K A_2 & \to & \mathrm{End}_K(V) \\ a_1 \otimes a_2 & \mapsto & \eta(a_1)\circ\theta(a_2) \end{array}$$

is a homomorphism of K-algebras, because η and θ commute. The composition $A_1 \otimes_K A_2 \to \mathrm{End}_{\mathbb{Z}}(V)$ of this map with the inclusion map from $\mathrm{End}_K(V)$ to $\mathrm{End}_{\mathbb{Z}}(V)$ is a homomorphism of rings. This gives V the structure of an $(A_1 \otimes_K A_2)$-module.

Let W be an $(A_1 \otimes_K A_2)$-module. Then the homomorphisms of rings $A_1 \to A_1 \otimes_K A_2$ sending a_1 to $a_1 \otimes 1$ and $A_2 \to A_1 \otimes_K A_2$ sending a_2 to $1 \otimes a_2$ induce commuting homomorphism of rings $A_1 \to \mathrm{End}_{\mathbb{Z}}(W)$ and $A_2 \to \mathrm{End}_{\mathbb{Z}}(W)$. This gives W the structure of an (A_1,A_2)-bimodule.

We see that the objects of $(A_1,A_2)\text{-}\underline{\mathrm{biMod}}_K$ correspond one to one with the $(A_2 \otimes_K A_2)$-modules.

Let V, W be $(A_1 \otimes_K A_2)$-modules and let $\ell\colon V \to W$ be a map. Then ℓ is $(A_1 \otimes_K A_2)$-linear if and only if ℓ is (A_1,A_2)-linear. So the correspondence between the objects of $(A_1,A_2)\text{-}\underline{\mathrm{biMod}}_K$ and $(A_1 \otimes_K A_2)\text{-}\underline{\mathrm{Mod}}$ is a functor.

Theorem 5.17. The additive covariant functor

$$(A_1 \otimes_K A_2)\text{-}\underline{\text{Mod}} \to (A_1, A_2)\text{-}\underline{\text{biMod}}_K$$

sending an $A_1 \otimes A_2$-module V to V with its associated structure of an (A_1, A_2)-bimodule and sending an $(A_1 \otimes_K A_2)$-linear map to itself is invertible.

Proof. This theorem is a reformulation of 5.16. □

5.18. Let V be an A_1-module and let W be an A_2-module. View V and W as vector spaces over K. Then the map

$$\begin{array}{rcl} A_1 \times A_2 & \to & \text{End}_K(V \otimes_K W) \\ (a_1, a_2) & \mapsto & (v \otimes w \mapsto (a_1 \cdot v) \otimes (a_2 \cdot w)) \end{array}$$

is K-bilinear. The corresponding K-linear map $A_1 \otimes_K A_2 \to \text{End}_K(V \otimes_K W)$ is a homomorphism of K-algebras. So the composition of this map with the inclusion map

$$\text{End}_K(V \otimes_K W) \to \text{End}_{\mathbb{Z}}(V \otimes_K W)$$

is a homomorphism of rings. This homomorphism gives $V \otimes_K W$ the structure of an $(A_1 \otimes_K A_2)$-module.

5.19. Let V be an (A_1, A_2)-bimodule. The homomorphisms $A_1 \to \text{End}_{\mathbb{Z}}(V)$ and $A_2 \to \text{End}_{\mathbb{Z}}(V)$ associated to V commute. So the map

$$\begin{array}{rcl} V & \to & V \\ v & \mapsto & a_1 \cdot v \end{array}$$

is A_2-linear for all $a_1 \in A_1$ and the map

$$\begin{array}{rcl} V & \to & V \\ v & \mapsto & a_2 \cdot v \end{array}$$

is A_1-linear for all $a_2 \in A_2$. The maps

$$\begin{array}{rcl} \eta\colon A_1 & \to & \text{End}_{A_2}(V) \\ a_1 & \mapsto & (v \mapsto a_1 \cdot v) \end{array}$$

and

$$\begin{array}{rcl} \theta\colon A_2 & \to & \text{End}_{A_1}(V) \\ a_2 & \mapsto & (v \mapsto a_2 \cdot v) \end{array}$$

are homomorphisms of K-algebras.

Let W be an A_1-module. Then the map

$$\begin{array}{rcl} A_2 & \to & \mathrm{End}_{\mathbb{Z}}\left(\mathrm{Hom}_{A_1}(W,V)\right) \\ a_2 & \mapsto & (\ell \mapsto \theta(a_2)\circ \ell) \end{array}$$

is a homomorphism of rings. This gives $\mathrm{Hom}_{A_1}(W,V)$ the structure of an A_2-module. Let U be an A_2-module. Then $\mathrm{Hom}_{A_2}(U,V)$ similarly gets the structure of an A_1-module using the map η.

5.3 Commutants

Let V be a vector space over K.

Definition 5.20. Let S be a subset of $\mathrm{End}_K(V)$. Define the commutant S' of S to be the subset $\{\ell' \in \mathrm{End}_K(V) \mid \ell' \circ \ell = \ell \circ \ell' \text{ for all } \ell \in S\}$ of $\mathrm{End}_K(V)$.

Examples 5.21. Let V be the vector space K^3 and identify the K-algebra $\mathrm{End}_K(V)$ with $M_3(K)$.

(i) Let T be the subalgebra $M_3(K)$ consisting of all multiples of the identity matrix. Then the commutant T' of T equals $M_3(K)$. The commutant T'' of T' equals T, because the multiples of the identity matrix are the only matrices that commute with all matrices of $M_3(K)$.

(ii) Let T be the subalgebra of $M_3(K)$ consisting of the matrices

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

for all $\lambda, \mu \in K$. Then one can check that T' consists of the matrices

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}$$

for all $a, b, c, d, e \in K$. Since $T \subseteq T'$, we have $(T')' \subseteq T'$. One can check that $T'' = T$.

(iii) Let T be the subalgebra of $M_3(K)$ consisting of the matrices

$$\begin{pmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \mu \end{pmatrix}$$

for all $\lambda, \mu \in K$. Then one can check that T' consists of the matrices

$$\begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

for all $a, b, c \in K$. Since $T \subseteq T'$, we have $(T')' \subseteq T'$. One can check that $T'' = T'$.

Let T be a subalgebra of $\mathrm{End}_K(V)$. Then V naturally has the structure of a T-module. The abelian group V inherits the structure of a vector space over K from T. Note that this vector space structure is the same as the vector space structure that V already had. By 5.1, we know that any T-linear endomorphism of V is also K-linear. So we see that the commutant T' of T equals $\mathrm{End}_T(V)$. In particular, we see that T' is a subalgebra of $\mathrm{End}_K(V)$. So V also has the structure of a T'-module. By definition of the commutant, we see that the homomorphisms of rings $T \to \mathrm{End}_{\mathbb{Z}}(V)$ and $T' \to \mathrm{End}_{\mathbb{Z}}(V)$ commute. This gives V the structure of a $(T \otimes_K T')$-module.

Theorem 5.22 (Double Commutant Theorem)**.** Suppose that the vector space V is finite dimensional over K and the ring T is semisimple. Then the ring T' is semisimple and $T'' = T$. There exists a complete family $(W_i)_{i \in J}$ of simple T-modules. For each $i \in J$, the map

$$\begin{array}{rcl} T' & \to & \mathrm{End}_{\mathbb{Z}}\left(\mathrm{Hom}_T(W_i, V)\right) \\ \ell' & \mapsto & \left(\ell \mapsto \ell' \circ \ell\right) \end{array}$$

gives $\mathrm{Hom}_T(W_i, V)$ the structure of a T'-module. The family

$$(\mathrm{Hom}_T(W_i, V))_{i \in J}$$

is a complete family of simple T'-modules and

$$V \cong \bigoplus_{i \in J} W_i \otimes_K \mathrm{Hom}_T(W_i, V)$$

is a decomposition of V as a $(T \otimes_K T')$-module.

Proof. Since V is finite dimensional over K, we see that $\mathrm{End}_K(V)$ is finite dimensional over K and therefore T is finite dimensional over K. So by the Artin-Wedderburn Theorem, we know that

$$T \cong M_{n_1}(K) \times \cdots \times M_{n_s}(K)$$

as K-algebras for some positive integers $n_1, \ldots, n_s \in \mathbb{Z}_{>0}$. Let J be the set $\{1, \ldots, s\}$ and for each integer $i \in J$, let W_i be the simple T-module K^{n_i} from Example 5.6. Then $(W_i)_{i \in J}$ is a complete family of simple T-modules.

Since the ring T is semisimple, we know that the T-module V is semisimple. So we have

$$V \cong \bigoplus_{i \in J} W_i^{\oplus e_i}$$

for some integers $e_1, \ldots, e_s \in \mathbb{Z}_{\geq 0}$. If $e_i = 0$ for some $i \in J$, then we see that the ideal

$$0 \times \cdots \times 0 \times M_{n_i}(K) \times 0 \times \cdots \times 0$$

of T is contained in the kernel of the homomorphism of rings $T \to \operatorname{End}_{\mathbb{Z}}(V)$. So since this homomorphism is injective, we have $e_i > 0$ for all $i \in J$.

We have $T' = \operatorname{End}_T(V)$. So using Schur's Lemma, we see that

$$T' = \operatorname{End}_T\left(\bigoplus_{i \in J} W_i^{\oplus e_i}\right) \cong \prod_{i \in J} \operatorname{End}_T\left(W_i^{\oplus e_i}\right) \cong \prod_{i \in J} M_{e_i}(K).$$

So by the Artin-Wedderburn Theorem, we see that T' is semisimple. For each integer $i \in J$, let $\operatorname{Hom}_T(W_i, V)$ have the structure of a T'-module as in 5.19. Then we have $\ell' \cdot \ell = \ell' \circ \ell$ for all $\ell' \in T'$ and $\ell \in \operatorname{Hom}_T(W_i, V)$. Using Schur's Lemma, we see that

$$\operatorname{Hom}_T\left(W_i, \bigoplus_{j \in J} W_j^{\oplus e_j}\right) \cong \operatorname{Hom}_T\left(W_i, W_i^{\oplus e_i}\right) \cong K^{e_i}.$$

The identification of T' with $\prod_{i \in J} M_{e_i}(K)$ and $\operatorname{Hom}_T(W_i, V)$ with K^{e_i} shows that $(\operatorname{Hom}_T(W_i, V))_{i \in J}$ is a complete family of simple T'-modules by Example 5.6.

For each integer $i \in J$, the K-bilinear map

$$\begin{aligned} W_i \times \operatorname{Hom}_T(W_i, V) &\to V \\ (v, \ell) &\mapsto \ell(v) \end{aligned}$$

gives us a K-linear map $W_i \otimes_K \operatorname{Hom}_T(W_i, V) \to V$. Let this tensor product have the structure of and $(T \otimes_K T')$-module as in 5.18. Then one can check that this K-linear map $W_i \otimes_K \operatorname{Hom}_T(W_i, V) \to V$ is also $(T \otimes_K T')$-linear. Together these $(T \otimes_K T')$-linear maps form a $(T \otimes_K T')$-linear map

$$\bigoplus_{i \in J} W_i \otimes \operatorname{Hom}_T(W_i, V) \to V$$

Using Schur's Lemma, one can check that this map is an isomorphism of vector spaces and hence an isomorphism of $(T \otimes_K T')$-modules.

Note that

$$V \cong \bigoplus_{i \in J} \operatorname{Hom}_T(W_i, V)^{\oplus \dim_K(W_i)}$$

is a decomposition of V into simple T'-modules. So since $\dim_K(W_i) = n_i$ for all $i \in \{1, \dots, s\}$, we see that $T'' = \mathrm{End}_{T'}(V)$ is isomorphic to

$$T \cong M_{n_1}(K) \times \cdots \times M_{n_s}(K)$$

by Schur's Lemma. Since $T \subseteq T''$, we see that we must have $T = T''$. □

Corollary 5.23. Let A_1, A_2 be K-algebras and let V be an object of (A_1, A_2)-$\underline{\mathrm{biMod}}_K$. Suppose that V is finite dimensional over K, the K-algebra A_2 is semisimple and the homomorphisms of K-algebras η and θ from 5.19 are surjective. Let $(W_i)_{i\in I}$ be a complete family of simple A_2-modules. Then $(\mathrm{Hom}_{A_2}(W_i, V))_{i\in I}$ is a family of A_1-modules, each member of which is simple or zero, such that

$$V \cong \bigoplus_{i\in I} W_j \otimes \mathrm{Hom}_{A_2}(W_i, V)$$

is a decomposition of V as $(A_1 \otimes_K A_2)$-module.

Proof. Let T be the subalgebra $\mathrm{End}_{A_1}(V) = \mathrm{im}(\theta)$ of $\mathrm{End}_K(V)$. The K-linear endomorphisms $V \to V$ sending v to $a_2 \cdot v$ and sending v to $\theta(a_2) \cdot v$ are equal for all $a_2 \in A_2$. So we see that a map $V \to V$ is A_2-linear if and only if it is T-linear. Therefore the commutant $T' = \mathrm{End}_T(V)$ of T is equal to $\mathrm{End}_{A_2}(V)$. Note that T is a quotient of A_2, because θ is surjective. So T is semisimple since A_2 is semisimple and by Theorem 5.10 the simple T-modules correspond to a subset of the simple A_2-modules.

Let $(W_i)_{i\in I}$ be a complete family of simple A_2-modules. For each element $i \in I$, give $\mathrm{Hom}_{A_2}(W_i, V)$ the structure of an A_1-module as in 5.19. Then $(\mathrm{Hom}_{A_2}(W_i, V))_{i\in I}$ is a family of A_1-modules, each member of which is simple or zero, such that

$$V \cong \bigoplus_{i\in I} W_j \otimes \mathrm{Hom}_{A_2}(W_i, V)$$

is a decomposition of V as $(A_1 \otimes_K A_2)$-module by the theorem. □

Chapter 6

Representations and Schur-Weyl duality

In this chapter, let K be an algebraically closed field.

In this chapter, we define representations of a group, we show that the vector spaces V_n for $n \in \mathbb{Z}_{\geq 0}$ can be given the structure of a representation of $\mathrm{GL}_2(K)$ such that the homomorphism of K-algebras pow_d^* also is a homomorphism of representations of $\mathrm{GL}_2(K)$ for all integers $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$ and we introduce Schur-Weyl duality.

6.1 Representations

Let G be a group.

Definition 6.1. A representation of G is a vector space V over K that comes with a homomorphism $G \to \mathrm{GL}(V)$. Let V be a representation of G and let $\rho\colon G \to \mathrm{GL}(V)$ be the associated homomorphism. Then we denote $\rho(g)(v)$ by $g \cdot v$ for all $g \in G$ and $v \in V$. We call the representation V finite dimensional if the vector space V is finite dimensional over K.

Examples 6.2. Let V be a vector space over K.

(i) The homomorphism $G \to \mathrm{GL}(V)$ sending every element of G to id_V gives V the structure of a representation of G. We call this representation of G on V trivial.

(ii) The identity map $\mathrm{GL}(V) \to \mathrm{GL}(V)$ gives V the structure of a representation of $\mathrm{GL}(V)$. We call this representation the standard representation of $\mathrm{GL}(V)$.

(iii) Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. Then the homomorphism

$$\begin{array}{rcl} S_n & \to & \mathrm{GL}\left(V^{\otimes n}\right) \\ \sigma & \mapsto & \left(v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}\right) \end{array}$$

from 2.7 gives $V^{\otimes n}$ the structure of a representation of S_n.

Definition 6.3. Let V, W be representations of G. A homomorphism of representations of G is a K-linear map $\ell\colon V \to W$ such that

$$\ell(g \cdot v) = g \cdot \ell(v)$$

for all $g \in G$ and $v \in V$. We denote the vector space over K consisting of all homomorphisms $V \to W$ of representations by $\mathrm{Hom}_G(V, W)$. We denote the K-algebra of endomorphisms $V \to V$ of representation by $\mathrm{End}_G(V)$.

Example 6.4. Let V be a representation of G. Then $\lambda\, \mathrm{id}_V\colon V \to V$ is an endomorphism of representation for each $\lambda \in K$.

Definition 6.5. Define G-$\underline{\mathrm{Rep}}$ to be the category of representations of G.

6.6. Let H be a subgroup of G. For each representation V of G, denote the subspace

$$\{v \in V | g \cdot (v) = v \text{ for all } g \in H\}$$

of V by V^H.

Let $\ell\colon V \to W$ be a homomorphism of representations of G. Then we see that

$$g \cdot \ell(v) = \ell(g \cdot v) = \ell(v)$$

for all $v \in V^H$ and $g \in H$. So we see that ℓ restricts to a K-linear map

$$\ell^H\colon V^H \to W^H.$$

This gives us a functor $(-)^H\colon G\text{-}\underline{\mathrm{Rep}} \to \underline{\mathrm{Vect}}_K$.

Definition 6.7. Define the group ring of G to be the K-algebra $K[G]$ that has G as a basis over K and where the product of g_1 and g_2 in $K[G]$ is $g_1 g_2$ for all $g_1, g_2 \in G$.

Remark 6.8. The group ring $K[G]$ is a commutative K-algebra if and only if the group G is abelian.

6.9. Let V be a representation of G. Then the K-linear map

$$K[G] \to \mathrm{End}_{\mathbb{Z}}(V) \qquad g \mapsto \rho_V(g)$$

is a homomorphism of rings. This homomorphism gives V the structure of a $K[G]$-module.

Let W be a $K[G]$-module. Then W inherits the structure of a vector space over K from the K-algebra $K[G]$. Recall from 5.1 that for all $g \in G$ the map

$$\begin{aligned} W &\to W \\ w &\mapsto g \cdot W \end{aligned}$$

is K-linear. The map

$$\begin{array}{rcl} G & \to & \mathrm{GL}(W) \\ g & \mapsto & (w \mapsto g \cdot w) \end{array}$$

is a homomorphism. This homomorphism gives W the structure of a representation of G.

We see that the representations of G correspond one to one with the $K[G]$-modules. Let V, W be representations of G and let $\ell \colon V \to W$ be a map. Then ℓ is a homomorphism of representations if and only if ℓ is a $K[G]$-linear map. So the correspondence between representations of G and $K[G]$-modules is a functor.

Theorem 6.10. The covariant functor G-$\underline{\mathrm{Rep}} \to K[G]$-$\underline{\mathrm{Mod}}$, sending a representation V of G to V with its associated structure of an $K[G]$-module and sending a homomorphism ℓ to itself, is invertible.

Proof. This theorem is a reformulation of 6.9. □

Definition 6.11. Let V be a representation of G. We call a subspace W of V a G-invariant subspace if $g \cdot w \in W$ for all $g \in G$ and $w \in W$. The representation V is called irreducible if V has precisely two G-invariant subspaces. The representation V is called completely reducible if it is isomorphic to a direct sum of a family of irreducible representations of G.

6.12. By Theorem 6.10, the category G-$\underline{\mathrm{Rep}}$ is the same as the category $K[G]$-$\underline{\mathrm{Mod}}$. So all statements about the category $K[G]$-$\underline{\mathrm{Mod}}$ can be translated to statements about the category G-$\underline{\mathrm{Rep}}$.

(i) The category G-$\underline{\mathrm{Rep}}$ is abelian.

(ii) A homomorphism of representations of G which is both injective and surjective is an isomorphism of representations.

(iii) Let V be a representation of G. Then a subspace W of V is G-invariant if and only if W is a $K[G]$-invariant subspace of the $K[G]$-module V. So if V is irreducible, then the G-invariant subspaces of V are 0 and V itself.

(iv) Schur's Lemma: let V, W be irreducible representations of G. Then any non-zero homomorphism $\ell \colon V \to W$ of representation is an isomorphism. The map

$$\begin{array}{rcl} K & \to & \mathrm{End}_G(V) \\ \lambda & \mapsto & \lambda \, \mathrm{id}_V \end{array}$$

is an isomorphism of K-algebras.

(v) A representation of G is completely reducible if and only if its corresponding $K[G]$-module is semisimple.

Lemma 6.13. Let V be a representation of G. Then the following are equivalent:

(i) the representation V is a sum of simple submodules;

(ii) the representation V is completely reducible;

(iii) for each G-invariant subspace W of V, there exists a G-invariant subspace U of V such that $W \oplus U = V$.

Proof. See paragraph 2 of chapter XVII of [La]. □

Theorem 6.14 (Maschke's theorem)**.** Suppose that G is a finite group whose order is not divisible by the characteristic of K. Then any representation V of G is completely reducible.

Proof. Let V be a representation of G and let W be a G-invariant subspace of V. Let $\pi\colon V \to V$ be a K-linear map such that $\pi^2 = \pi$ and $\operatorname{im} \pi = W$. Then we have

$$\sum_{g \in G} g \cdot \pi(g^{-1} \cdot h \cdot v) = h \cdot \sum_{g \in G} (h^{-1}g) \cdot \pi\left((h^{-1}g)^{-1} \cdot v\right) = h \cdot \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v)$$

for all $h \in G$ and $v \in V$ and hence the K-linear map

$$\begin{aligned} \overline{\pi}\colon V &\to W \\ v &\mapsto \frac{1}{|G|} \sum_{g \in G} g \cdot \pi(g^{-1} \cdot v) \end{aligned}$$

is a homomorphism of representations of G. Since $\pi|_W = \mathrm{id}_W$ and W is G-invariant, we see that $\overline{\pi}|_W = \mathrm{id}_W$. So $\overline{\pi}$ is surjective. We get a short exact sequence

$$0 \longrightarrow \ker \overline{\pi} \longrightarrow V \xrightarrow{\overline{\pi}} W \longrightarrow 0$$

of representations of G. The inclusion map $W \to V$ is a section of this short exact sequence, so we see that $V = W \oplus \ker \overline{\pi}$. Now we see that the representation V is completely reducible by the previous lemma. □

Corollary 6.15. Suppose that G is a finite group whose order is not divisible by the characteristic of K. Then the group ring $K[G]$ is a semisimple ring.

Proof. Suppose that G is a finite group whose order is not divisible by the characteristic of K. Then the representation $K[G]$ of G is completely reducible. Therefore the $K[G]$-module $K[G]$ is semisimple. □

6.2 The dual functor of representations

Let G be a group.

6.16. Let V be a representation of G and let $\rho\colon G \to \mathrm{GL}(V)$ be the associated homomorphism. Then the map

$$\begin{array}{rcl} V^\times & \to & V^\times \\ \varphi & \mapsto & \varphi \circ \rho(g)^{-1} \end{array}$$

is a K-linear isomorphism for all $g \in G$. Consider the map

$$\begin{array}{rcl} \rho^\times : G & \to & \mathrm{GL}(V^\times) \\ g & \mapsto & \left(\varphi \mapsto \varphi \circ \rho(g)^{-1}\right). \end{array}$$

Let g, h be elements of G. Then we have

$$\begin{array}{rcl} \rho^\times(g)(\rho^\times(h)(\varphi)) & = & \rho^\times(g)(\varphi \circ \rho(h)^{-1}) = \varphi \circ \rho(h)^{-1} \circ \rho(g)^{-1} \\ & = & \varphi \circ (\rho(g) \circ \rho(h))^{-1} = \varphi \circ \rho(gh)^{-1} \\ & = & \rho^\times(gh)(\varphi) \end{array}$$

for all $\varphi \in V^\times$. So we see that $\rho^\times$ is a homomorphism. The homomorphism $\rho^\times$ gives $V^\times$ the structure of a representation of G. We call this representation the dual representation of V.

6.17. Let V, W be representations of G and let $\ell\colon V \to W$ be a homomorphism of representations. Then ℓ is also a K-linear map. One can check that the dual map $\ell^\times : W^\times \to V^\times$ sending an element $\varphi \in W^\times$ to $\varphi \circ \ell$ is a homomorphism of representations. We call $\ell^\times$ the dual homomorphism of ℓ.

Note that ℓ is injective if and only if $\ell^\times$ is surjective and ℓ is surjective if and only if $\ell^\times$ is injective, because ℓ is a K-linear map.

We get the additive contravariant functor $(-)^\times : G\text{-}\underline{\mathrm{Rep}} \to G\text{-}\underline{\mathrm{Rep}}$ sending a representation of G to its dual and a homomorphism to its dual.

Definition 6.18. Define $G\text{-}\underline{\mathrm{fRep}}$ to be the full subcategory of $G\text{-}\underline{\mathrm{Rep}}$ consisting of all finite-dimensional representations of G.

The dual of a finite-dimensional representation of G is finite dimensional. So we also get an additive contravariant functor $(-)^\times : G\text{-}\underline{\mathrm{fRep}} \to G\text{-}\underline{\mathrm{fRep}}$ sending a representation of G to its dual and a homomorphism to its dual.

Proposition 6.19. The additive contravariant functor

$$(-)^\times : G\text{-}\underline{\mathrm{fRep}} \to G\text{-}\underline{\mathrm{fRep}}$$

is an equivalence of categories.

Proof. Recall from Proposition 1.18 that for each finite-dimensional vector space V over K, we have the isomorphism $\varepsilon_V \colon V \to V^{\times\times}$ sending an element v to the K-linear map $(\varphi \mapsto \varphi(v))$. Let V be a finite-dimensional representation of G. Then one can check that ε_V is a homomorphism of representations and hence an isomorphism of representations. So

$$\{\varepsilon_V \colon V \to V^{\times\times}\}_{V \in |G\text{-}\underline{\mathrm{fRep}}|}$$

is a natural isomorphism $\mathrm{id}_{G\text{-}\underline{\mathrm{fRep}}} \Rightarrow (-)^\times \circ (-)^\times$. Hence the functor $(-)^\times \colon G\text{-}\underline{\mathrm{fRep}} \to G\text{-}\underline{\mathrm{fRep}}$ is an equivalence of categories. □

6.3 Examples from previous chapters

Let G be a group. Let V, W be representations of G and let $n \in \mathbb{Z}_{\geq 1}$ be an integer. In this section we will generalize some constructions for vector spaces over K to the setting of representations of G.

6.20. The map

$$\begin{aligned} G &\to \mathrm{GL}(V \otimes W) \\ g &\mapsto (v \otimes w \mapsto (g \cdot v) \otimes (g \cdot w)) \end{aligned}$$

is a homomorphism. This homomorphism gives $V \otimes W$ the structure of a representation of G. We call $V \otimes W$ the tensor product of the representations V and W.

6.21. The map

$$\begin{aligned} G &\to \mathrm{GL}(V^{\otimes n}) \\ g &\mapsto (v_1 \otimes \cdots \otimes v_n \mapsto (g \cdot v_1) \otimes \cdots \otimes (g \cdot v_n)) \end{aligned}$$

is a homomorphism. This homomorphism gives $V^{\otimes n}$ the structure of a representation of G. We call $V^{\otimes n}$ the n-th tensor power of the representation V.

Let $\ell \colon V \to W$ be a homomorphism of representations. Then the map

$$\begin{aligned} \ell^{\otimes n} \colon V^{\otimes n} &\to W^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n &\mapsto \ell(v_1) \otimes \cdots \otimes \ell(v_n) \end{aligned}$$

is also a homomorphism of representations. So we get a functor

$$(-)^{\otimes n} \colon G\text{-}\underline{\mathrm{Rep}} \to G\text{-}\underline{\mathrm{Rep}}.$$

6.22. The subspace of $V^{\otimes n}$ spanned by

$$v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$$

for all $v_1, \ldots, v_n \in V$ and $\sigma \in S_n$ is a G-invariant subspace of $V^{\otimes n}$. So the representation structure on $V^{\otimes n}$ induced by the representation V of G is inherited by $\mathrm{Sym}^n(V)$. We call the representation $\mathrm{Sym}^n(V)$ of G the n-th symmetric power of the representation V.

Let $\ell \colon V \to W$ be a homomorphism of representations. Then the map

$$\begin{array}{rcl} \mathrm{Sym}^n(\ell)\colon \mathrm{Sym}^n(V) & \to & \mathrm{Sym}^n(W) \\ v_1 \odot \cdots \odot v_n & \mapsto & \ell(v_1) \odot \cdots \odot \ell(v_n) \end{array}$$

is also a homomorphism of representations. So we get a functor

$$\mathrm{Sym}^n(-)\colon G\text{-}\underline{\mathrm{Rep}} \to G\text{-}\underline{\mathrm{Rep}}\,.$$

6.23. The projection map $\pi_V^n \colon V^{\otimes n} \to \mathrm{Sym}^n(V)$ is homomorphism of representations of G. The family π^n consisting of the projection maps π_V^n over all representations V of G is a natural transformation.

If the characteristic of K does not divide $n!$, then the section

$$\iota_V^n \colon \mathrm{Sym}^n(V) \to V^{\otimes n}$$

of π_V^n from 2.7 is also a homomorphism of representations of G. The family ι^n consisting of the sections ι_V^n over all representations V of G is also a natural transformation.

6.24. Recall that $\mathrm{Sym}(V)$ is the direct sum of $\mathrm{Sym}^i(V)$ over all integers $i \in \mathbb{Z}_{\geq 0}$. We define the representation $\mathrm{Sym}(V)$ of G to be the direct sum of the representations $\mathrm{Sym}^i(V)$ of G over all integers $i \in \mathbb{Z}_{\geq 0}$. We define the representation $P(V)$ of G to be $\mathrm{Sym}(V^\times)$.

6.25. The subspace of $V^{\otimes n}$ spanned by

$$\{v_1 \otimes \cdots \otimes v_n | v_1, \ldots, v_n \in V, v_i = v_j \text{ for some } i \neq j\}$$

is a G-invariant subspace of $V^{\otimes n}$. So the representation structure of $V^{\otimes n}$ induced by the representation V of G is inherited by its quotient $\Lambda^n V$ from Definition 2.12. We call the representation $\Lambda^n V$ the n-th alternating power of the representation V.

6.26. The map

$$\begin{array}{rcl} G & \to & \mathrm{GL}(\mathrm{Map}(V,K)) \\ g & \mapsto & \big(f \mapsto (v \mapsto f(g^{-1} \cdot v)\big) \end{array}$$

gives the K-algebra of maps $V \to K$ the structure of a representation of G. One can check that the map $P(V) \to \mathrm{Map}(V,K)$ from Proposition 2.18 sending a polynomial on V to its associated polynomial function on V is an injective homomorphism of representations.

Proposition 6.27. Let $\alpha\colon V \to W$ be a polynomial map such that $\alpha(g\cdot v) = g\cdot\alpha(v)$ for all $g \in G$ and $v \in V$. Then the homomorphism of K-algebras

$$\alpha^*\colon P(W) \to P(V)$$

is also a homomorphism of representations of G.

Proof. Let $f \in P(W)$ be a polynomial. Then we have $g^{-1}\cdot\alpha(v) = \alpha(g^{-1}\cdot v)$ for all $g \in G$ and $v \in V$. Therefore we have

$$\begin{aligned}\alpha^*(g\cdot f) &= (w \mapsto f(g^{-1}\cdot w))\circ\alpha \\ &= (v \mapsto f(g^{-1}\cdot\alpha(v))) \\ &= (v \mapsto f(\alpha(g^{-1}\cdot v))) \\ &= g\cdot\alpha^*(f)\end{aligned}$$

for all $g \in G$. Hence α^* is a homomorphism of representations of G. □

6.28. The map

$$\begin{aligned} G &\to \mathrm{GL}\left(\mathrm{Hom}_K(V,W)\right) \\ g &\mapsto \left(\ell \mapsto (g\cdot\ell(g^{-1}\cdot v))\right)\end{aligned}$$

is a homomorphism. This homomorphism gives $\mathrm{Hom}_K(V,W)$ the structure of a representation of G. One can check that the maps from Lemma 2.30 are all homomorphisms of representations. Let $\ell\colon V \to W$ be a map. Then we see that ℓ is a homomorphism of representations if and only if ℓ is an element of $\mathrm{Hom}_K(V,W)^G$.

6.29. For a polynomial $f \in K[x,y]$ and a vector $\binom{g}{h} \in K[x,y]^2$, denote the polynomial

$$f(g(x,y),h(x,y)) \in K[x,y]$$

by $f\binom{g}{h}$. Let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. Recall that V_n is the subspace of $K[x,y]$ consisting of all homogeneous polynomial of degree n together with the zero polynomial. View the elements of $\mathrm{GL}_2(K)$ as matrices over the ring $K[x,y]$. Then the map

$$\begin{aligned}\mathrm{GL}_2(K) &\to \mathrm{GL}(V_n) \\ A &\mapsto \left(f \mapsto f\left(A^{-1}\begin{pmatrix}x\\y\end{pmatrix}\right)\right)\end{aligned}$$

is a homomorphism. This homomorphism gives V_n the structure of a representation of $\mathrm{GL}_2(K)$.

Let $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$ be integers.

6.30. Note that the polynomial map

$$\begin{array}{rcl}\mathrm{pow}_d\colon V_d & \to & V_{de} \\ f & \mapsto & f^e\end{array}$$

sends $f\binom{g}{h}$ to $(f^e)\binom{g}{h}$ for all $g, h \in V_1$. So for all $A \in \mathrm{GL}_2(K)$ and $f \in V_d$ we see that $A \cdot \mathrm{pow}_d(f) = \mathrm{pow}_d(A \cdot f)$. So by Proposition 6.27, the homomorphism of K-algebras $\mathrm{pow}_d^*\colon P(V_{de}) \to P(V_d)$ is also a homomorphism of representations of $\mathrm{GL}_2(K)$.

6.31. Let

$$\begin{array}{rcl}\ell_d\colon V_{de} & \to & \mathrm{Hom}_K(V_d, V_{de+d-2}) \\ g & \mapsto & \left(f \mapsto \dfrac{1}{y}\left(\dfrac{\partial g}{\partial x} f - e\dfrac{\partial f}{\partial x} g\right)\right)\end{array}$$

be the K-linear map from 4.12. Then a tedious direct computation shows that for all $A \in \mathrm{GL}_2(K)$ and $g \in V_{de}$ the equality

$$\det(A) \cdot \ell_d(A \cdot g) = A \cdot \ell_d(g)$$

holds. There is also an alternative proof using theory will will not use here: every element of $\mathrm{GL}_2(K)$ is the product of a diagonal matrix and an element of $\mathrm{SL}_2(K)$ and therefore is suffices to prove the equality for all matrices A that are diagonal or an element of $\mathrm{SL}_2(K)$. Proving the equality in the case where A is a diagonal matrix is easy. Proving the equality in the case where A is an element of $\mathrm{SL}_2(K)$ can be done by proving that ℓ_d is a homomorphism of representations of the Lie algebra $\mathfrak{sl}_2(K)$. See section 8.1 for more information on Lie algebras.

6.32. Let $r \in \mathbb{Z}_{\geq 0}$ be an integer. Then one can easily check that the homogeneous polynomial map

$$\begin{array}{rcl}\alpha\colon \mathrm{Hom}_K(V, W) & \to & \mathrm{Hom}_K(\Lambda^r V, \Lambda^r W) \\ \ell & \mapsto & \Lambda^r \ell\end{array}$$

of degree r from Proposition 2.33 sends $g \cdot \ell$ to $g \cdot \alpha(\ell)$ for all $g \in G$ and $\ell \in \mathrm{Hom}_K(V, W)$. So by Proposition 6.27, the homomorphism of K-algebras α^* is also a homomorphism of representations of G.

6.4 Birepresentations

Let G_1, G_2 be groups.

Definition 6.33. Let V be a vector space over K and let $\rho_1\colon G_1 \to \mathrm{GL}(V)$ and $\rho_2\colon G_2 \to \mathrm{GL}(V)$ homomorphisms. We say that ρ_1 and ρ_2 commute if

$$\rho_1(g_1) \circ \rho_2(g_2) = \rho_2(g_2) \circ \rho_1(g_1)$$

for all $g_1 \in G_1$ and $g_2 \in G_2$.

Definition 6.34. Define a (G_1, G_2)-birepresentation to be a vector space V over K that comes with a pair of commuting homomorphisms $G_1 \to \mathrm{GL}(V)$ and $G_2 \to \mathrm{GL}(V)$.

Definition 6.35. Let V, W be (G_1, G_2)-birepresentations. A homomorphism of birepresentations is a K-linear map $\ell\colon V \to W$ that is both a homomorphism of representations of G_1 and a homomorphism of representations of G_2.

Definition 6.36. Let $(G_1, G_2)\text{-}\underline{\mathrm{biRep}}$ be the category of (G_1, G_2)-birepresentations.

6.37. Let V be a (G_1, G_2)-birepresentation. Then V also has the structures of a representation of G_1, a representation of G_2, a $K[G_1]$-module and a $K[G_2]$-module.

Since the homomorphisms $G_1 \to \mathrm{GL}(V)$ and $G_2 \to \mathrm{GL}(V)$ commute, the homomorphisms of rings $K[G_1] \to \mathrm{End}_{\mathbb{Z}}(V)$ and $K[G_2] \to \mathrm{End}_{\mathbb{Z}}(V)$ also commute. So the (G_1, G_2)-birepresentation structure on V induces the structure of a $(K[G_1], K[G_2])$-bimodule on the abelian group V. The birepresentation V is even an object of the full subcategory $(K[G_1], K[G_2])\text{-}\underline{\mathrm{biMod}}_K$ of $(K[G_1], K[G_2])\text{-}\underline{\mathrm{biMod}}$, because the vector space structures on V induces by $K[G_1]$ and $K[G_2]$ are both the same as the original vector space structure on V.

Recall that for K-algebras A_1, A_2, we have a correspondence between the objects of $(A_1, A_2)\text{-}\underline{\mathrm{biMod}}_K$ and the $(A_1 \otimes_K A_2)$-modules.

Lemma 6.38. The K-linear map

$$\begin{array}{rcl} K[G_1] \otimes_K K[G_2] & \to & K[G_1 \times G_2] \\ g_1 \otimes g_2 & \mapsto & (g_1, g_2) \end{array}$$

is an isomorphism of K-algebras.

By Lemma 6.38, we see that the corresponding statement for representations is a correspondence between (G_1, G_2)-birepresentation and representation of $(G_1 \times G_2)$. Let $i_1\colon G_1 \to G_1 \times G_2$ and $i_2\colon G_1 \to G_2 \times G_2$ be the inclusions maps.

6.39. Let V be a (G_1, G_2)-birepresentation. Then the map

$$\begin{array}{rcl} G_1 \times G_2 & \to & \mathrm{GL}(V) \\ (g_1, g_2) & \mapsto & (v \mapsto g_1 \cdot (g_2 \cdot v)) \end{array}$$

is a homomorphism. This homomorphism gives V the structure of a representation of $G_1 \times G_2$.

Let W be a representation of $G_1 \times G_2$. The the maps

$$\begin{array}{rcl} G_1 & \to & \mathrm{End}(W) \\ g & \mapsto & (w \mapsto (g,1) \cdot w) \end{array}$$

and

$$\begin{array}{rcl} G_2 & \to & \mathrm{End}(W) \\ g & \mapsto & (w \mapsto (1,g) \cdot w) \end{array}$$

are commuting homomorphisms. These homomorphisms give W the structure of a (G_1, G_2)-birepresentation.

Let V, W be representations of $G_1 \times G_2$ and let $\ell\colon V \to W$ be a map. Then ℓ is a homomorphism of representations of $G_1 \times G_2$ if and only if ℓ is a homomorphism of (G_1, G_2)-birepresentations.

Theorem 6.40. The covariant functor

$$G_1 \times G_2\text{-}\underline{\mathrm{Rep}} \to (G_1, G_2)\text{-}\underline{\mathrm{biRep}}$$

sending a representation V to V with its associated (G_1, G_2)-birepresentation structure and sending a homomorphism ℓ to itself is invertible.

Proof. This is a reformulation of 6.39. □

6.41. Let V be a representation of G_1 and let W be a representation of G_2. Then the map

$$\begin{array}{rcl} G_1 \times G_2 & \to & \mathrm{End}_K(V \otimes W) \\ (g_1, g_2) & \mapsto & (v \otimes w \mapsto (g_1 \cdot v) \otimes (g_2 \cdot w)) \end{array}$$

is a homomorphism. This homomorphism gives $V \otimes W$ the structure of a representation of $G_1 \times G_2$.

6.42. Let V be a (G_1, G_2)-birepresentation. Then the associated homomorphisms $G_1 \to \mathrm{GL}(V)$ and $G_2 \to \mathrm{GL}(V)$ commute. So the map

$$\begin{array}{rcl} V & \to & V \\ v & \mapsto & g_1 \cdot v \end{array}$$

is a homomorphism of representations of G_2 for all $g_1 \in G_1$ and the map

$$\begin{array}{rcl} V & \to & V \\ v & \mapsto & g_2 \cdot v \end{array}$$

is a homomorphism of representations of G_1 for all $g_2 \in G_2$. The maps

$$\begin{array}{rcl} \rho\colon G_1 & \to & \mathrm{End}_{G_2}(V) \\ g_1 & \mapsto & (v \mapsto g_1 \cdot v) \end{array}$$

and

$$\begin{array}{rcl} \varrho\colon G_2 & \to & \mathrm{End}_{G_1}(V) \\ g_2 & \mapsto & (v \mapsto g_2 \cdot v) \end{array}$$

are homomorphisms.

Let W be a representation of G_1. Then the map

$$\begin{array}{rcl} G_2 & \to & \mathrm{GL}\left(\mathrm{Hom}_{G_1}(W,V)\right) \\ g_2 & \mapsto & (\ell \mapsto \varrho(g_2)\circ \ell) \end{array}$$

is a homomorphism. This homomorphism gives $\mathrm{Hom}_{G_1}(W,V)$ the structure of a representation of G_2. Let $\ell\colon U \to W$ be a homomorphism of representations of G_1. Then we denote the homomorphism

$$\begin{array}{rcl} \mathrm{Hom}_{G_1}(W,V) & \to & \mathrm{Hom}_{G_1}(U,V) \\ \ell' & \mapsto & \ell'\circ \ell \end{array}$$

of representations of G_2 by $\mathrm{Hom}_{G_1}(\ell,V)$. This gives us a functor

$$\mathrm{Hom}_{G_1}(-,V)\colon G_1\text{-}\underline{\mathrm{Rep}} \to G_2\text{-}\underline{\mathrm{Rep}}\,.$$

By switching the role of G_1 and G_2, we similarly get a functor

$$\mathrm{Hom}_{G_2}(-,V)\colon G_2\text{-}\underline{\mathrm{Rep}} \to G_1\text{-}\underline{\mathrm{Rep}}\,.$$

Example 6.43. Let U,V be representations of G_1 and let W be a representation of G_2. Then the map

$$\begin{array}{rcl} \mathrm{Hom}_{G_1}(U,V)\otimes W & \to & \mathrm{Hom}_{G_1}(U,V\otimes W) \\ \ell\otimes w & \mapsto & (u \mapsto \ell(u)\otimes w) \end{array}$$

is an isomorphism of representations of G_2. Where $\mathrm{Hom}_{G_1}(U,V)\otimes W$ is the tensor product of the trivial representation $\mathrm{Hom}_{G_1}(U,V)$ of G_2 with W.

6.5 Schur-Weyl duality

Let V be a finite-dimensional vector space over K and let $n \in \mathbb{Z}_{\geq 0}$ be a non-negative integer. Recall from 6.2 that the identity map $\mathrm{GL}(V) \to \mathrm{GL}(V)$ gives V the structure of a representation of $\mathrm{GL}(V)$. The representation structure on V gives the vector space $V^{\otimes n}$ the structure of a representation of $\mathrm{GL}(V)$. Also recall that the homomorphism

$$\begin{array}{rcl} S_n & \to & \mathrm{GL}(V^{\otimes n}) \\ \sigma & \mapsto & \left(v_1\otimes\cdots\otimes v_n \mapsto v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}\right) \end{array}$$

gives the vector space $V^{\otimes n}$ the structure of a representation of S_n. Note that homomorphisms $\mathrm{GL}(V) \to \mathrm{GL}(V^{\otimes n})$ and $S_n \to \mathrm{GL}(V^{\otimes n})$ commute. So $V^{\otimes n}$ has the structure of a $(\mathrm{GL}(V), S_n)$-birepresentation.

6.44. Since $V^{\otimes n}$ is a representation of $\mathrm{GL}(V)$, it is also a $K[\mathrm{GL}(V)]$-module. Since $V^{\otimes n}$ is a representation of S_n, it is also a $K[S_n]$-module. The associated homomorphisms of rings commutes. Hence $V^{\otimes n}$ has the structure of a $(K[\mathrm{GL}(V)], K[S_n])$-bimodule. We get the homomorphisms of K-algebras

$$\begin{array}{rcl} \eta\colon K[\mathrm{GL}(V)] & \to & \mathrm{End}_{K[S_n]}(V^{\otimes n}) \\ a & \mapsto & (t \mapsto a \cdot t) \end{array}$$

and

$$\begin{array}{rcl} \theta\colon K[S_n] & \to & \mathrm{End}_{K[\mathrm{GL}(V)]}(V^{\otimes n}) \\ \chi & \mapsto & (t \mapsto \chi \cdot t) \end{array}$$

Theorem 6.45 (Schur-Weyl duality)**.** The homomorphisms η and θ are surjective.

Proof. See Theorem 1 of [Do], which is the main theorem of [Do]. □

From now on, suppose that the characteristic of K does not divide $n!$.

6.46. By Maschke's theorem, the K-algebra $K[S_n]$ is semisimple. Let $(W_i)_{i\in I}$ be a complete family of simple $K[S_n]$-modules. By Corollary 5.23

$$V^{\otimes n} \cong \bigoplus_{i\in I} \mathrm{Hom}_{K[S_n]}(W_i, V^{\otimes n}) \otimes W_i$$

is a decomposition of the $K[\mathrm{GL}(V)] \otimes K[S_n]$-module $V^{\otimes n}$.

The family $(W_i)_{i\in I}$ is also a complete family of irreducible representations of S_n and for each element $i \in I$ we have

$$\mathrm{Hom}_{K[S_n]}(W_i, V^{\otimes n}) = \mathrm{Hom}_{S_n}(W_i, V^{\otimes n}).$$

We see that

$$V^{\otimes n} \cong \bigoplus_{i\in I} \mathrm{Hom}_{S_n}(W_i, V^{\otimes n}) \otimes W_i$$

is a decomposition of the $(\mathrm{GL}(V), S_n)$-birepresentation $V^{\otimes n}$.

6.47. Let i be an element of I. Then the Schur-Weyl dual of the representation W_i of S_n is the representation $\mathrm{Hom}_{S_n}(W_i, V^{\otimes n})$ of $\mathrm{GL}(V)$. Note that if $\mathrm{Hom}_{S_n}(W_i, V^{\otimes n})$ is non-zero, then

$$\begin{array}{rcl} W_i & \cong & \mathrm{Hom}_{\mathrm{GL}(V)}(\mathrm{Hom}_{S_n}(W_i, V^{\otimes n}), \mathrm{Hom}_{S_n}(W_i, V^{\otimes n}) \otimes W_i) \\ & \cong & \mathrm{Hom}_{\mathrm{GL}(V)}(\mathrm{Hom}_{S_n}(W_i, V^{\otimes n}), V^{\otimes n}) \end{array}$$

by Schur's Lemma. So in this case, the Schur-Weyl dual of the representation $\mathrm{Hom}_{S_n}(W_i, V^{\otimes n})$ of $\mathrm{GL}(V)$ is isomorphic to the representation W_i of S_n.

Let J be the subset of I consisting of all i such that $\mathrm{Hom}_{S_n}(W_i, V^{\otimes n})$ is non-zero. Then we see that every simple subrepresentation of the representation $V^{\otimes n}$ of $\mathrm{GL}(V)$ is isomorphic to precisely one member of the family $(\mathrm{Hom}_{S_n}(W_i, V^{\otimes n}))_{i\in J}$ and that every simple subrepresentation of the representation $V^{\otimes n}$ of S_n is isomorphic to precisely one member of the family $(W_i)_{i\in J}$. The functors

$$\mathrm{Hom}_{S_n}(-, V^{\otimes n})\colon S_n\text{-}\underline{\mathrm{Rep}} \to \mathrm{GL}(V)\text{-}\underline{\mathrm{Rep}}$$

and

$$\mathrm{Hom}_{\mathrm{GL}(V)}(-, V^{\otimes n})\colon \mathrm{GL}(V)\text{-}\underline{\mathrm{Rep}} \to S_n\text{-}\underline{\mathrm{Rep}}$$

induce a one-to-one correspondence between the isomorphism classes of direct sums of members of $(W_i)_{i\in J}$ and isomorphism classes of direct sums of members of $(\mathrm{Hom}_{S_n}(W_i, V^{\otimes n}))_{i\in J}$. So we call these functors the Schur-Weyl duality functors.

Examples 6.48.

(1) Let the vector space K have the structure of a trivial representation of S_n. Then one can check that the map

$$\begin{array}{rcl} \mathrm{Hom}_{S_n}(K, V^{\otimes n}) & \to & \mathrm{Sym}^n(V) \\ \ell & \mapsto & \pi_V^n(\ell(1)) \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(V)$.

(2) The homomorphism

$$\begin{array}{rcl} S_n & \to & \mathrm{GL}(K) \\ \sigma & \mapsto & \mathrm{sgn}(\sigma)\cdot \mathrm{id}_K \end{array}$$

gives the vector space K the structure of a representation of S_n. One can check that the map

$$\begin{array}{rcl} \mathrm{Hom}_{S_n}(K, V^{\otimes n}) & \to & \Lambda^n V \\ \ell & \mapsto & \pi(\ell(1)) \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(V)$ where $\pi\colon V^{\otimes n} \to \Lambda^n V$ is the projection map.

(3) The homomorphism

$$\begin{array}{rcl} S_n & \to & \mathrm{GL}(K[S_n]) \\ \sigma & \mapsto & (\chi \mapsto \sigma\chi) \end{array}$$

gives the group ring $K[S_n]$ the structure of a representation of S_n. One can check that the map

$$\begin{array}{rcl} \mathrm{Hom}_{S_n}(K[S_n], V^{\otimes n}) & \to & V^{\otimes n} \\ \ell & \mapsto & \ell(1) \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(V)$.

Proposition 6.49. Let $\ell\colon U \to W$ be a homomorphism of representations of S_n. Then the following statements hold:

(i) if the map ℓ is surjective, then its Schur-Weyl dual $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$ is injective;

(ii) if the map ℓ is injective, then its Schur-Weyl dual $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$ is surjective.

If the double Schur-Weyl dual of U is isomorphic to U and the double Schur-Weyl dual of W is isomorphic to W. Then the following statements also hold:

(iii) if the map $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$ is surjective, then the map ℓ is injective;

(iv) if the map $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$ is injective, then the map ℓ is surjective.

Proof. Recall that the map

$$\begin{array}{rcl} \mathrm{Hom}_{S_n}(\ell, V^{\otimes n})\colon \mathrm{Hom}_{S_n}(W, V^{\otimes n}) & \to & \mathrm{Hom}_{S_n}(U, V^{\otimes n}) \\ \ell' & \mapsto & \ell' \circ \ell \end{array}$$

is the Schur-Weyl dual of ℓ.

(i) Suppose that the map ℓ is surjective. Then we see that for all maps $\ell_1, \ell_2\colon W \to V^{\otimes n}$ such that $\ell_1 \circ \ell = \ell_2 \circ \ell$ holds that $\ell_1 = \ell_2$. Hence the Schur-Weyl dual of ℓ is injective.

(ii) Suppose that ℓ is injective. Then we may use ℓ to identify U with a subrepresentation of W in such a way that ℓ is the inclusion map. By Lemma 6.13, we see that

$$W = U \oplus U'$$

for some subrepresentation U' of W. So any homomorphism of representations of S_n from U can be extended to a homomorphism from W using the zero map from U'. So we see that the Schur-Weyl dual of ℓ is surjective.

(iii) Assume that the double Schur-Weyl duals of U and W are isomorphic to U and W respectively. Suppose that the map $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$ is surjective. Since the representation U is completely reducible and the double Schur-Weyl dual of U is isomorphic to U, we see that by 6.47 the double Schur-Weyl dual of any subrepresentation U' of U is isomorphic to U'. So the Schur-Weyl dual of any non-zero subrepresentation of U is non-zero.

Suppose that $\ker \ell$ is non-zero. Then there exists a non-zero homomorphism of representations $\ker \ell \to V^{\otimes n}$. By Lemma 6.13, this non-zero homomorphism can be extended to a homomorphism of representations $\ell^\dagger \colon U \to V^{\otimes n}$. Since $\ker \ell$ is not contained in $\ker \ell^\dagger$, we see that $\ell^\dagger$ can not be an element of the image of the map $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$. Contradiction, so $\ker \ell$ is zero.

(iv) Assume that the double Schur-Weyl duals of U and W are isomorphic to U and W respectively. Suppose that the map $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$ is injective. Since the representation W is completely reducible, there exists a subrepresentation W' of W such that $W = \mathrm{im}\,\ell \oplus W'$. Suppose that W' is non-zero. Then as in the previous part, we see that the Schur-Weyl dual of W' is non-zero. Let $\overline{\ell}' \colon W' \to V^{\otimes n}$ be a non-zero homomorphism of representations and let $\ell' \colon W \to V^{\otimes n}$ be the homomorphism of representations extending $\overline{\ell}'$ by zero on $\mathrm{im}\,\ell$. Then ℓ' is a non-zero element of the kernel of $\mathrm{Hom}_{S_n}(\ell, V^{\otimes n})$. Contradiction, so $\mathrm{im}\,\ell = W$. □

Proposition 6.50. Let $\ell \colon U \to W$ be a homomorphism of representations of $\mathrm{GL}(V)$ and suppose that U and W are both completely reducible. Then the following statements hold:

(i) if the map ℓ is surjective, then its Schur-Weyl dual $\mathrm{Hom}_{\mathrm{GL}(V)}(\ell, V^{\otimes n})$ is injective;

(ii) if the map ℓ is injective, then its Schur-Weyl dual $\mathrm{Hom}_{\mathrm{GL}(V)}(\ell, V^{\otimes n})$ is surjective.

If in addition the double Schur-Weyl dual of U is isomorphic to U and the double Schur-Weyl dual of W is isomorphic to W. Then the following statements also hold:

(iii) if the map $\mathrm{Hom}_{\mathrm{GL}(V)}(\ell, V^{\otimes n})$ is surjective, then the map ℓ is injective;

(iv) if the map $\mathrm{Hom}_{\mathrm{GL}(V)}(\ell, V^{\otimes n})$ is injective, then the map ℓ is surjective.

Proof. This proposition is proven similarly to the previous proposition. □

Chapter 7

Ikenmeyer's method

In this chapter we will work over the the algebraically closed field $\mathbb{C}$ of complex numbers. Fix an integer $e \in \mathbb{Z}_{\geq 1}$ and a finite-dimensional vector space U over $\mathbb{C}$.

Ikenyemer proves in [Ik] that within a certain family of homomorphisms $\Psi_{a\times b}$ of representations of S_{ab} over all $a, b \in \mathbb{Z}_{\geq 0}$ the following statement holds: if $a, b \in \mathbb{Z}_{\geq 0}$ are integers such that $a < b$ and $\Psi_{a\times b-1}$ is injective, then $\Psi_{a\times b}$ is also injective. In this chapter, we will construct a similar family of homomorphisms $\Psi_{i,d}$ of representations of S_{die} over all $i, d \in \mathbb{Z}_{\geq 0}$ that is Ikenmeyers family if $e = 1$. We will then prove that if $i, d \in \mathbb{Z}_{\geq 0}$ are integers such that $i < d$ and $\Psi_{i,d-1}$ is injective, then $\Psi_{i,d}$ is also injective. We will also show that $\Psi_{1,1}$ and $\Psi_{2,2}$ are injective and that for all $i, d \in \mathbb{Z}_{\geq 0}$ such that $\Psi_{i,d}$ is injective, the homomorphism

$$\text{pow}^*_{d,(i)} \colon \text{Sym}^i(V^\times_{de}) \to \text{Sym}^{ie}(V^\times_d)$$

is also injective. Combining these results, we then conclude that the second conjecture holds for $d = 2$.

7.1 Preparations

In this section, we introduce some notation that we will use later in this chapter. Let $a, b, n \in \mathbb{Z}_{\geq 0}$ be integers.

7.1. The identity map $\text{GL}(U) \to \text{GL}(U)$ gives the vector space U the structure of a representation of $\text{GL}(U)$. This representation structure induces the structure of a representation of $\text{GL}(U)$ on each vector space constructed naturally from U.

For each vector space V over $\mathbb{C}$, the homomorphism

$$\begin{array}{rcl} S_n & \to & \text{GL}\left(V^{\otimes n}\right) \\ \sigma & \mapsto & \left(v_1 \otimes \cdots \otimes v_n \mapsto v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}\right) \end{array}$$

gives the vector space $V^{\otimes n}$ the structure of a representation of S_n and the corresponding structure of a $\mathbb{C}[S_n]$-module. The homomorphism

$$\begin{array}{rcl} S_n & \to & \mathrm{GL}(\mathbb{C}[S_n]) \\ \sigma & \mapsto & (\chi \mapsto \sigma\chi) \end{array}$$

gives the vector space $\mathbb{C}[S_n]$ the structure of a representation of S_n.

7.2. Let $u_{1,1}, \ldots, u_{a,b}$ be elements of U. Then, to avoid confusion in the notation, we denote the element $(u_{1,1} \otimes \cdots \otimes u_{1,b}) \otimes \cdots \otimes (u_{a,1} \otimes \cdots \otimes u_{a,b})$ of $(U^{\otimes b})^{\otimes a}$ by

$$(u_{1,1} \circledast \cdots \circledast u_{1,b}) \otimes \cdots \otimes (u_{a,1} \circledast \cdots \circledast u_{a,b})$$

and we denote the element $(u_{1,1} \odot \cdots \odot u_{1,b}) \odot \cdots \odot (u_{a,1} \odot \cdots \odot u_{a,b})$ of $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ by

$$(u_{1,1} \circledcirc \cdots \circledcirc u_{1,b}) \odot \cdots \odot (u_{a,1} \circledcirc \cdots \circledcirc u_{a,b}).$$

7.3. We will now construct some natural transformations that will be important later.

(i) For each representation V of $\mathrm{GL}(U)$, the map

$$\begin{array}{rcl} \mathrm{unord}_V^{a\times b}\colon \left(V^{\otimes b}\right)^{\otimes a} & \to & V^{\otimes ab} \\ \bigotimes_{i=1}^{a} \bigcircledast_{j=1}^{b} v_{i,j} & \mapsto & \bigotimes_{i=1}^{a} \bigotimes_{j=1}^{b} v_{i,j} \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(U)$. The family $\mathrm{unord}^{a\times b}$ consisting of the homomorphisms $\mathrm{unord}_V^{a\times b}$ over all representations V of $\mathrm{GL}(U)$ is a natural isomorphism $((-)^{\otimes b})^{\otimes a} \Rightarrow (-)^{\otimes ab}$.

(ii) For each representation V of $\mathrm{GL}(U)$, the map

$$\begin{array}{rcl} \mathrm{transp}_V^{a\times b}\colon \left(V^{\otimes b}\right)^{\otimes a} & \to & \left(V^{\otimes a}\right)^{\otimes b} \\ \bigotimes_{i=1}^{a} \bigcircledast_{j=1}^{b} v_{i,j} & \mapsto & \bigotimes_{j=1}^{b} \bigcircledast_{i=1}^{a} v_{i,j} \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(U)$. The family $\mathrm{transp}^{a\times b}$ consisting of the homomorphisms $\mathrm{transp}_V^{a\times b}$ over all representations V of $\mathrm{GL}(U)$ is a natural isomorphism $((-)^{\otimes b})^{\otimes a} \Rightarrow ((-)^{\otimes a})^{\otimes b}$.

(iii) For each representation V of $\mathrm{GL}(U)$, let $\Theta_V^{a\times b}$ be the homomorphism of representations of $\mathrm{GL}(U)$ making the diagram

$$\begin{array}{ccc}
\left(V^{\otimes b}\right)^{\otimes a} & \xrightarrow{\operatorname{transp}_V^{a\times b}} & \left(V^{\otimes a}\right)^{\otimes b} \\
\uparrow {\scriptstyle \iota^a_{V^{\otimes b}}} & & \downarrow {\scriptstyle \pi^b_{V^{\otimes a}}} \\
\operatorname{Sym}^a\left(V^{\otimes b}\right) & & \operatorname{Sym}^b\left(V^{\otimes a}\right) \\
\uparrow {\scriptstyle \operatorname{Sym}^a(\iota^b_V)} & & \downarrow {\scriptstyle \operatorname{Sym}^b(\pi^a_V)} \\
\operatorname{Sym}^a(\operatorname{Sym}^b(V)) & \xrightarrow{\Theta_V^{a\times b}} & \operatorname{Sym}^b(\operatorname{Sym}^a(V))
\end{array}$$

commute. Then the family $\Theta^{a\times b}$ consisting of the homomorphisms $\Theta_V^{a\times b}$ over all representations V of $\mathrm{GL}(U)$ is a natural transformation $\operatorname{Sym}^a(\operatorname{Sym}^b(-)) \Rightarrow \operatorname{Sym}^b(\operatorname{Sym}^a(-))$ since it is obtained as the composition of natural transformations.

(iv) For each representation V of $\mathrm{GL}(U)$, the map

$$\begin{array}{rcl}
\pi_V^{a\times b}\colon V^{\otimes ab} & \to & \operatorname{Sym}^a(\operatorname{Sym}^b(V)) \\
v_1 \otimes \cdots \otimes v_{ab} & \mapsto & (v_1 \circledcirc \cdots \circledcirc v_b) \odot \cdots \odot (v_{(a-1)b+1} \circledcirc \cdots \circledcirc v_{ab})
\end{array}$$

is equal to the the homomorphisms of representations

$$\operatorname{Sym}^a(\pi^b_V) \circ \pi^a_{V^{\otimes b}} \circ \left(\operatorname{unord}_V^{a\times b}\right)^{-1}.$$

The family $\pi^{a\times b}$ consisting of the homomorphisms $\pi_V^{a\times b}$ over all representations V of $\mathrm{GL}(U)$ is a natural transformation

$$(-)^{\otimes ab} \Rightarrow \operatorname{Sym}^a(\operatorname{Sym}^b(-))$$

since it is obtained as the composition of natural transformations.

(v) For each representation V of $\mathrm{GL}(U)$, let

$$\iota_V^{a\times b}\colon \operatorname{Sym}^a(\operatorname{Sym}^b(V)) \to V^{\otimes ab}$$

be the injective homomorphism of representations of $\mathrm{GL}(U)$

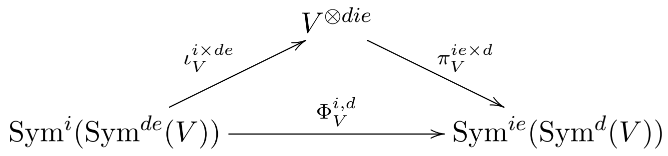
$$\operatorname{unord}_V^{a\times b} \circ \iota^a_{V^{\otimes b}} \circ \operatorname{Sym}^a(\iota^b_V).$$

Then the family $\iota^{a\times b}$ consisting of the homomorphisms $\iota_V^{a\times b}$ over all representations V of $\mathrm{GL}(U)$ is a natural transformation

$$\operatorname{Sym}^a(\operatorname{Sym}^b(-)) \Rightarrow (-)^{\otimes ab}$$

since it is obtained as the composition of natural transformations.

(vi) Let $d, i \in \mathbb{Z}_{\geq 0}$ be integers. For each representation V of $\mathrm{GL}(U)$, let $\Phi_V^{i,d}$ be the homomorphism of representations of $\mathrm{GL}(U)$ making the diagram



commute. Then the family $\Phi^{a\times b}$ consisting of the homomorphisms $\Phi_V^{a\times b}$ over all representations V of $\mathrm{GL}(U)$ is a natural transformation

$$\mathrm{Sym}^i(\mathrm{Sym}^{de}(-)) \Rightarrow \mathrm{Sym}^{ie}(\mathrm{Sym}^d(-))$$

since it is obtained as the composition of natural transformations.

7.4. Denote the set of all functions $r\colon \{1, \ldots, ab\} \to \{1, \ldots, a\}$ such that $\#r^{-1}(i) = b$ for all integers $i \in \{1, \ldots, a\}$ by $\Omega_{a\times b}$. Note that for all functions $r \in \Omega_{a\times b}$ and permutations $\tau \in S_a$, the function $\tau \circ r$ is an element of $\Omega_{a\times b}$. So we see that S_a acts on $\Omega_{a\times b}$ by $\tau \cdot r = \tau \circ r$ for all $\sigma \in S_a$ and $r \in \Omega_{a\times b}$. We call elements of $\Omega_{a\times b}$ that are in the same orbit of this action equivalent. Denote the set of equivalence classes under this equivalence relation by $\Omega_{a\times b}/\sim$. Denote the class of an element $r \in \Omega_{a\times b}$ by $[r]$.

Next note that for all functions $r \in \Omega_{a\times b}$ and permutations $\sigma \in S_{ab}$, the function $r \circ \sigma^{-1}$ is an element of $\Omega_{a\times b}$. Also note that for all functions $r, s \in \Omega_{a\times b}$, there exist a permutation $\sigma \in S_{ab}$ such that $r = s \circ \sigma^{-1}$. So we see that S_{ab} acts transitively on $\Omega_{a\times b}$ by $\sigma \cdot r = r \circ \sigma^{-1}$ for all $\sigma \in S_{ab}$ and $r \in \Omega_{a\times b}$. The actions of S_{ab} and S_a on $\Omega_{a\times b}$ commute. So we see that for all permutations $\sigma \in S_{ab}$ and equivalent functions $r, s \in \Omega_{a\times b}$, the functions $\sigma \cdot r$ and $\sigma \cdot s$ are equivalent. So we see that $\sigma \cdot [r] = [\sigma \cdot r]$ for all $\sigma \in S_{ab}$ and $r \in \Omega_{a\times b}$ defines an action of S_{ab} on $\Omega_{a\times b}/\sim$.

Let $r_{a\times b} \in \Omega_{a\times b}$ be the function sending $(i-1)b + j$ to i for all integers $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, b\}$. Denote the stabilizer of the element $[r_{a\times b}]$ of $\Omega_{a\times b}/\sim$ by $H_{a\times b}$. Then we see that $H_{a\times b}$ consists of all permutations $\sigma \in S_{ab}$ such that for each $i \in \{1, \ldots, a\}$ we have

$$\sigma\left(\{(i-1)b + 1, \ldots, ib\}\right) = \{(j-1)\,b + 1, \ldots, jb\}.$$

for some $j \in \{1, \ldots, a\}$.

Let V be a representation of $\mathrm{GL}(U)$ and suppose that $x_1, \ldots, x_{ab}$ are linearly independent elements of V. Then we see that the subset

$$\{v_1 \otimes \cdots \otimes v_{ab} | v_1, \ldots, v_{ab} \in \{x_1, \ldots, x_{ab}\}\}$$

of $V^{\otimes ab}$ is linearly independent. Note that $H_{a\times b}$ equals the subset of S_{ab} consisting of all permutations σ such that

$$\pi_V^{a\times b}(\sigma \cdot (x_1 \otimes \cdots \otimes x_{ab})) = (x_1 \circledcirc \cdots \circledcirc x_b) \odot \cdots \odot (x_{(a-1)b+1} \circledcirc \cdots \circledcirc x_{ab}).$$

Let W be a representation of $\mathrm{GL}(U)$, then the section $\iota_W^{a\times b}$ of $\pi_W^{a\times b}$ is the map

$$\begin{array}{rcl} \mathrm{Sym}^a(\mathrm{Sym}^b(W)) & \to & W^{\otimes ab} \\ \bigodot_{i=1}^{a} \bigodot_{j=1}^{b} w_{(i-1)b+j} & \mapsto & \frac{1}{\#(H_{a\times b})} \sum_{\sigma\in H_{a\times b}} \sigma \cdot (w_1 \otimes \cdots \otimes w_{ab}) \,. \end{array}$$

The image of $\iota_W^{a\times b}$ is exactly the subspace $(W^{\otimes ab})^{H_{a\times b}}$ of $W^{\otimes ab}$.

7.2 Relation to the previous chapters

In this section, let U be the vector space $\mathbb{C}^2$. Identify $\mathrm{GL}(U)$ with $\mathrm{GL}_2(\mathbb{C})$ using the standard basis (e_1, e_2) of U.

7.5. Denote the dual basis of the standard basis (e_1, e_2) of U by (x, y). Then we see that $U^\times$ is the vector space V_1. For each integer $n \in \mathbb{Z}_{\geq 0}$, let V_n be the representation of $\mathrm{GL}_2(\mathbb{C})$ from 6.29. Then one can check that V_1 is the representation $U^\times$ of $\mathrm{GL}(U)$ and that the map

$$\begin{array}{rcl} \mathrm{Sym}^n(V_1) & \to & V_n \\ f_1 \odot \cdots \odot f_n & \mapsto & f_1 \cdots f_n \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(U)$ for each integer $n \in \mathbb{Z}_{\geq 0}$.

Let $d, i \in \mathbb{Z}_{\geq 0}$ be integers.

7.6. The representation V_1 of $\mathrm{GL}(U)$ equals $U^\times$. So the representation $U^{\times\times}$ of $\mathrm{GL}(U)$ equals $V_1^\times$. Since the vector space U is finite dimensional over $\mathbb{C}$, the homomorphism of representations $\varepsilon_U \colon U \to V_1^\times$ of $\mathrm{GL}(U)$ from the proof of Proposition 6.19 is an isomorphism. By applying the natural transformation

$$\Phi^{i,d} \colon \mathrm{Sym}^i(\mathrm{Sym}^{de}(-)) \Rightarrow \mathrm{Sym}^{ie}(\mathrm{Sym}^d(-))$$

to ε_U, we see that the diagram

$$\begin{array}{ccc} \mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) & \xrightarrow{\Phi_U^{i,d}} & \mathrm{Sym}^{ie}(\mathrm{Sym}^d(U)) \\ \downarrow{\scriptstyle \mathrm{Sym}^i(\mathrm{Sym}^{de}(\varepsilon_U))} & & \downarrow{\scriptstyle \mathrm{Sym}^{ie}(\mathrm{Sym}^d(\varepsilon_U))} \\ \mathrm{Sym}^i(\mathrm{Sym}^{de}(V_1^\times)) & \xrightarrow{\Phi_{V_1^\times}^{i,d}} & \mathrm{Sym}^{ie}(\mathrm{Sym}^d(V_1^\times)) \end{array}$$

commutes. Since ε_U is an isomorphism, we know that $\mathrm{Sym}^i(\mathrm{Sym}^{de}(\varepsilon_U))$ and $\mathrm{Sym}^{ie}(\mathrm{Sym}^d(\varepsilon_U))$ are isomorphisms too. So we see that $\Phi_U^{i,d}$ is injective if and only if $\Phi_{V_1^\times}^{i,d}$ is injective.

7.7. Recall from 6.30 that the $\mathbb{C}$-linear map

$$\mathrm{pow}^*_{d,(i)} \colon \mathrm{Sym}^i(V_{de}^\times)) \to \mathrm{Sym}^{ie}(V_d^\times))$$

is a homomorphism of representations of $\mathrm{GL}(U)$. Let $(c_0, \dots, c_{de})$ be the dual basis of the basis $(y^{de}, xy^{de-1}, \dots, x^{de})$ of V_{de}, let $(b_0, \dots, b_d)$ be the dual basis of the basis $(y^d, xy^{d-1}, \dots, x^d)$ of V_d and let (a_0, a_1) be the dual basis of the basis (y, x) of V_1. By taking the composition of the dual of the isomorphism from 7.5 with the isomorphism from Lemma 2.30(e), we get the isomorphisms

$$\begin{array}{rcl} \ell_{de} \colon V_{de}^\times & \to & \mathrm{Sym}^{de}(V_1^\times) \\ c_k & \mapsto & \binom{de}{k} \cdot a_0^{\odot de-k} \odot a_1^{\odot k} \end{array}$$

and

$$\begin{array}{rcl} \ell_d \colon V_d^\times & \to & \mathrm{Sym}^{d}(V_1^\times) \\ b_k & \mapsto & \binom{d}{k} \cdot a_0^{\odot d-k} \odot a_1^{\odot k} \end{array}$$

of representations of $\mathrm{GL}(U)$. For each integer $k \in \{0, \dots, de\}$, we have

$$\begin{aligned} & \mathrm{Sym}^e(\ell_d)(\mathrm{pow}^*_{d,(1)}(c_k)) \\ = \ & \mathrm{Sym}^e(\ell_d) \left(\sum_{\substack{0 \le i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = k}} b_{i_1} \odot \dots \odot b_{i_e} \right) \\ = \ & \sum_{\substack{0 \le i_1, \dots, i_e \le d \\ i_1 + \dots + i_e = k}} \left(\binom{d}{i_1} \cdot a_0^{\circledcirc d-i_1} \circledcirc a_1^{\circledcirc i_1} \right) \odot \dots \odot \left(\binom{d}{i_e} \cdot a_0^{\circledcirc d-i_e} \circledcirc a_1^{\circledcirc i_e} \right) \\ = \ & \Phi^{1,d}_{V_1^\times} \left(\binom{de}{k} \cdot a_0^{\odot de-k} \odot a_1^{\odot k} \right) \\ = \ & \Phi^{1,d}_{V_1^\times} \left(\ell_{de}(c_k) \right). \end{aligned}$$

The second to last equality can be seen to be true as follows: we have

$$\Phi^{1,d}_{V_1^\times} \left(\binom{de}{k} \cdot a_0^{\odot de-k} \odot a_1^{\odot k} \right) = \pi^{e \times d}_{V_1^\times} \left(\binom{de}{k} \cdot \iota^{de}_{V_1^\times} \left(a_0^{\odot de-k} \odot a_1^{\odot k} \right) \right).$$

Recall that $\iota^{de}_{V_1^\times}(a_0^{\odot de-k} \odot a_1^{\odot k})$ is the average of of $\sigma \cdot (a_0^{\otimes de-k} \otimes a_1^{\otimes k})$ over all $\sigma \in S_{de}$. If we multiply this element of $(V_1^\times)^{\otimes de}$ by $\binom{de}{k}$, then we get the sum of $x_1 \otimes \dots \otimes x_{de}$ over all $x_1 \otimes \dots \otimes x_{de}$ such that we have $\#\{h | x_h = a_0\} = de-k$ and $\#\{h | x_k = a_1\} = k$. Let $x_1 \otimes \dots \otimes x_{de}$ be such that $\#\{h | x_h = a_0\} = de-k$

and $\#\{h|x_k = a_1\} = k$. For each integer $j \in \{1, \dots, e\}$, let i_j be the number of a_1 among $x_{(j-1)d+1}, \dots, x_{jd}$. Then we have

$$\pi^{e\times d}_{V_1^\times}(x_1 \otimes \cdots \otimes x_{de}) = \left(a_0^{\circledcirc d-i_1} \circledcirc a_1^{\circledcirc i_1}\right) \odot \cdots \odot \left(a_0^{\circledcirc d-i_e} \circledcirc a_1^{\circledcirc i_e}\right).$$

The equality now follows from a counting argument.

For all integers $k_1, \dots, k_i \in \{0, \dots, de\}$, we have

$$\begin{aligned}
& \mathrm{Sym}^{ie}(\ell_d)(\mathrm{pow}^*_{d,(i)}(c_{k_1} \odot \cdots \odot c_{k_i})) \\
= \; & \mathrm{Sym}^{ie}(\ell_d)(\mathrm{pow}^*_{d,(1)}(c_{k_1}) \odot \cdots \odot \mathrm{pow}^*_{d,(1)}(c_{k_i})) \\
= \; & \mathrm{Sym}^{e}(\ell_d)(\mathrm{pow}^*_{d,(1)}(c_{k_1})) \odot \cdots \odot \mathrm{Sym}^{e}(\ell_d)(\mathrm{pow}^*_{d,(1)}(c_{k_i})) \\
= \; & \Phi^{1,d}_{V_1^\times}(\ell_{de}(c_{k_1})) \odot \cdots \odot \Phi^{1,d}_{V_1^\times}(\ell_{de}(c_{k_i})) \\
= \; & \Phi^{i,d}_{V_1^\times}(\ell_{de}(c_{k_1}) \odot \cdots \odot \ell_{de}(c_{k_i})) \\
= \; & \Phi^{i,d}_{V_1^\times}(\mathrm{Sym}^{i}(\ell_{de})(c_{k_1} \odot \cdots \odot c_{k_i})).
\end{aligned}$$

Hence the diagram

$$\begin{array}{ccc}
\mathrm{Sym}^i(\mathrm{Sym}^{de}(V_1^\times)) & \xrightarrow{\Phi^{i,d}_{V_1^\times}} & \mathrm{Sym}^{ie}(\mathrm{Sym}^d(V_1^\times)) \\
\uparrow{\scriptstyle \mathrm{Sym}^i(\ell_{de})} & & \uparrow{\scriptstyle \mathrm{Sym}^{ie}(\ell_d)} \\
\mathrm{Sym}^i(V_{de}^\times) & \xrightarrow{\mathrm{pow}^*_{d,(i)}} & \mathrm{Sym}^{ie}(V_d^\times)
\end{array}$$

commutes.

Proposition 7.8. The following are equivalent:

- the homomorphism $\mathrm{pow}^*_{d,(i)}$ is injective;
- the homomorphism $\Phi^{i,d}_{V_1^\times}$ is injective;
- the homomorphism $\Phi^{i,d}_{U}$ is injective.

Proof. This proposition combines the results from 7.6 and 7.7. □

7.9. Note that the maps $\Phi^{i,d}_U$ depend on only the vector space U. Let $\ell\colon U \to V$ be an injective $\mathbb{C}$-linear map. Then we see that the diagram

$$\begin{array}{ccc}
\mathrm{Sym}^i(\mathrm{Sym}^{de}(V)) & \xrightarrow{\Phi^{i,d}_{V}} & \mathrm{Sym}^{ie}(\mathrm{Sym}^d(V)) \\
\uparrow{\scriptstyle \mathrm{Sym}^i(\mathrm{Sym}^{de}(\ell))} & & \uparrow{\scriptstyle \mathrm{Sym}^{ie}(\mathrm{Sym}^d(\ell))} \\
\mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) & \xrightarrow{\Phi^{i,d}_{U}} & \mathrm{Sym}^{ie}(\mathrm{Sym}^d(U))
\end{array}$$

commutes and that the vertical maps are injective. So we see that if the map $\Phi^{i,d}_V$ is injective, then so is the map $\Phi^{i,d}_U$.

7.3 Symmetric powers of symmetric powers

Let $a, b \in \mathbb{Z}_{\geq 0}$ be integers and let

$$\eta \colon \mathbb{C}[S_{ab}] \to \mathrm{End}_{\mathbb{Z}}(V^{\otimes ab})$$

be the homomorphism of rings associated to the $\mathbb{C}[S_{ab}]$-module $V^{\otimes ab}$.

7.10. Let χ be an element of the group ring $\mathbb{C}[S_{ab}]$. Then χ induces the homomorphism $\mathrm{Sym}^a(\mathrm{Sym}^b(U)) \to U^{\otimes ab}$ of representations of $\mathrm{GL}(U)$ making the diagram

$$\begin{array}{ccc} U^{\otimes ab} & \xrightarrow{\eta(\chi)} & U^{\otimes ab} \\ {\scriptstyle \iota_U^{a\times b}}\uparrow & \nearrow & \\ \mathrm{Sym}^a(\mathrm{Sym}^b(U) & & \end{array}$$

commute. Note that the $\mathbb{C}$-linear map

$$\begin{array}{rcl} q_{a\times b} \colon \mathbb{C}[S_{ab}] & \to & \mathrm{Hom}_{\mathrm{GL}(U)}\left(\mathrm{Sym}^a(\mathrm{Sym}^b(U)), U^{\otimes ab}\right) \\ \chi & \mapsto & \eta(\chi) \circ \iota_U^{a\times b} \end{array}$$

sends $\sigma \cdot \chi$ to $\sigma \cdot q_{a\times b}(\chi)$ for all $\sigma \in S_{ab}$ and $\chi \in \mathbb{C}[S_{ab}]$. So we see that $q_{a\times b}$ is a homomorphism of representations of S_{ab}.

Note that the codomain of the map $q_{a\times b}$ is the Schur-Weyl dual of the representation $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ of $\mathrm{GL}(U)$. Since $\iota_U^{a\times b}$ is injective and the representation $U^{\otimes ab}$ of $\mathrm{GL}(U)$ is completely decomposable, we see that

$$\begin{array}{rcl} \mathrm{End}_{\mathrm{GL}(U)}\left(U^{\otimes ab}\right) & \to & \mathrm{Hom}_{\mathrm{GL}(U)}\left(\mathrm{Sym}^a(\mathrm{Sym}^b(U)), U^{\otimes ab}\right) \\ \ell & \mapsto & \ell \circ \iota_U^{a\times b} \end{array}$$

is a surjective homomorphism of representations of S_{ab}. So using Theorem 6.45, we see that $q_{a\times b}$ is surjective. Therefore the Schur-Weyl dual of the representation $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ of $\mathrm{GL}(U)$ is the representation

$$\mathbb{C}[S_{ab}]/\ker(q_{a\times b})$$

of S_{ab}. Recall that the image of $\iota_U^{a\times b}$ is fixed by the subgroup $H_{a\times b}$ of S_{ab}. So we see that $q_{a\times b}(\sigma) = q_{a\times b}(\tau)$ for all $\sigma, \tau \in S_{ab}$ such that $\sigma H_{a\times b} = \tau H_{a\times b}$. Let $K_{a\times b}$ be the subrepresentation of $\mathbb{C}[S_{ab}]$ generated by $1 - \sigma$ for all $\sigma \in H_{a\times b}$. Then one can check that $K_{a\times b}$ is the S_{ab}-invariant subspace

$$\left\{ \sum_{\sigma \in S_{ab}} c_\sigma \cdot \sigma \;\middle|\; \sum_{\sigma \in \tau H_{a\times b}} c_\sigma = 0 \text{ for all } \tau \in S_{ab} \right\}$$

of $\mathbb{C}[S_{ab}]$ and is contained in $\ker(q_{a\times b})$.

In the remainder of this section, assume that $\dim_{\mathbb{C}}(U) \geq a$ and let $(x_1, \dots, x_m)$ be a basis of U over $\mathbb{C}$.

7.11. The family

$$(u_1 \otimes \cdots \otimes u_{ab} | u_1, \dots, u_{ab} \in \{x_1, \dots, x_m\})$$

is a basis of $U^{\otimes ab}$. For each $r \in \Omega_{a\times b}$, let u_r be the element

$$\frac{1}{a!} \sum_{\tau \in S_a} x_{\tau(r(1))} \otimes \cdots \otimes x_{\tau(r(ab))}$$

of $U^{\otimes ab}$. Note that for each $r \in \Omega_{a\times b}$, the element u_r is the average of the elements $x_{s(1)} \otimes \cdots \otimes x_{s(ab)}$ over all $a!$ elements $s \in \Omega_{a\times b}$ that are the equivalent to r. Also note that these elements $x_{s(1)} \otimes \cdots \otimes x_{s(ab)}$ are all elements of the basis

$$(u_1 \otimes \cdots \otimes u_{ab} | u_1, \dots, u_{ab} \in \{x_1, \dots, x_m\})$$

of $U^{\otimes ab}$. So in particular, we see that the family $(u_r | [r] \in \Omega_{a\times b}/\sim)$ is linearly independent.

7.12. Let $r \in \Omega_{a\times b}$ be a function. Then we have

$$u_{\sigma\cdot r} = \frac{1}{a!} \sum_{\tau \in S_a} x_{\tau(r(\sigma^{-1}(1)))} \otimes \cdots \otimes x_{\tau(r(\sigma^{-1}(ab)))} = \sigma \cdot u_r$$

for all $\sigma \in S_{ab}$. So we see that the action of S_{ab} on $\Omega_{a\times b}$ corresponds to the action of S_{ab} on $\{u_r | r \in \Omega_{a\times b}\}$ induced by the action on $U^{\otimes ab}$. Hence S_{ab} acts transitively on $\{u_r | r \in \Omega_{a\times b}\}$ and $H_{a\times b}$ is the stabilizer of $u_{r_{a\times b}}$ under the action of S_{ab}. Let $u_{a\times b}$ be the element

$$\pi_U^{a\times b}(u_{r_{a\times b}}) = (x_1 \circledcirc \cdots \circledcirc x_1) \odot (x_2 \circledcirc \cdots \circledcirc x_2) \odot \cdots \odot (x_a \circledcirc \cdots \circledcirc x_a)$$

of $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$.

Proposition 7.13. Let χ be an element of $\mathbb{C}[S_{ab}]$. Then the following are equivalent:

(i) the element χ of $\mathbb{C}[S_{ab}]$ is contained in the kernel of $q_{a\times b}$;

(ii) the element $u_{a\times b}$ of $U^{\otimes ab}$ is contained in the kernel of $q_{a\times b}(\chi)$;

(iii) the element χ of $\mathbb{C}[S_{ab}]$ is contained in the subspace $K_{a\times b}$ of $\mathbb{C}[S_{ab}]$.

Proof. It is clear that (i) implies (ii) and we have already seen that (iii) implies (i). For (ii) implies (iii), suppose that $u_{a\times b}$ is contained in the kernel of $q_{a\times b}(\chi)$. We have

$$q_{a\times b}(\chi)(u_{a\times b}) = \chi \cdot \iota_U^{a\times b}(u_{a\times b}) = \chi \cdot u_{r_{a\times b}}.$$

Write $\chi = \sum_{\sigma \in S_{ab}} c_\sigma \cdot \sigma$. Then we have

$$\chi \cdot u_{r_{a\times b}} = \sum_{\tau \in S_{ab}/H_{a\times b}} \left(\sum_{\sigma \in \tau H_{a\times b}} c_\sigma \right) \cdot u_{\tau \cdot r_{a\times b}}.$$

Since the set

$$\{u_{\tau \cdot r_{a\times b}} | \tau \in S_{ab}/H_{a\times b}\} = \{u_r | [r] \in \Omega_{a\times b}/\sim\}$$

is linearly independent, we see that

$$\sum_{\sigma \in \tau H_{a\times b}} c_\sigma = 0$$

for all $\tau \in S_{ab}$. Hence χ is contained in $K_{a\times b}$. □

Corollary 7.14.

(i) The Schur-Weyl dual of the representation $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ of $\mathrm{GL}(U)$ is the representation $\mathbb{C}[S_{ab}]/K_{a\times b}$ of S_{ab}.

(ii) The Schur-Weyl dual of the representation $\mathbb{C}[S_{ab}]/K_{a\times b}$ of S_{ab} is the representation $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ of $\mathrm{GL}(U)$.

Proof. Part (i) follows directly from the proposition. Part (ii) follows from the fact that any subrepresentation of the representation $U^{\otimes ab}$ of $\mathrm{GL}(U)$ is isomorphic to its double Schur-Weyl dual. □

Remark 7.15. We will see later that part (ii) of this corollary also holds without the restriction on the dimension of U over $\mathbb{C}$.

Corollary 7.16. Let V be a subrepresentation of the representation $U^{\otimes ab}$ of $\mathrm{GL}(U)$ and let $\ell_1, \ell_2 \colon \mathrm{Sym}^a(\mathrm{Sym}^b(U)) \to V$ be homomorphisms of representations. Then $\ell_1 = \ell_2$ if and only if $\ell_1(u_{a\times b}) = \ell_2(u_{a\times b})$.

Proof. Note that both ℓ_1 and ℓ_2 are also homomorphisms of representations of $\mathrm{GL}(U)$ from $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ to $U^{\otimes ab}$. The map $q_{a\times b}$ is surjective. Therefore there exists a $\chi \in \mathbb{C}[S_{ab}]$ such that $\ell_1 - \ell_2 = q_{a\times b}(\chi)$. So by the proposition, we have $\ell_1 = \ell_2$ if and only if $\ell_1(u_{a\times b}) = \ell_2(u_{a\times b})$. □

Corollary 7.17. The representation $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ of $\mathrm{GL}(U)$ is generated by $u_{a\times b}$.

Proof. Let V be the subrepresentation of the representation $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ of $\mathrm{GL}(U)$ generated by $u_{a\times b}$. Since the representation $U^{\otimes ab}$ of $\mathrm{GL}(U)$ is completely decomposable, the representation $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ is completely decomposable too. Therefore we have

$$\mathrm{Sym}^a(\mathrm{Sym}^b(U)) = V \oplus W$$

for some subrepresentation W of $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$. Note that the zero homomorphism and the projection map $V \oplus W \to W$ are equal on $u_{a\times b}$. Hence the map $\mathrm{id}_W : W \to W$ is the zero homomorphism by the previous corollary. □

Let $d, i \in \mathbb{Z}_{\geq 0}$ be integers.

7.18. Suppose that $\dim_{\mathbb{C}}(U) = m$ is at least i. We see that by Corollary 7.17 the homomorphism

$$\Phi_U^{i,d} \colon \mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) \to \mathrm{Sym}^{ie}(\mathrm{Sym}^{de}(U))$$

of representations of $\mathrm{GL}(U)$ is completely determined by its value at $u_{i\times de}$. Recall that the diagram

$$\begin{array}{ccc} & U^{\otimes die} & \\ {\scriptstyle \iota_U^{i\times de}}\nearrow & & \searrow{\scriptstyle \pi_U^{ie\times d}} \\ \mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) & \xrightarrow{\Phi_U^{i,d}} & \mathrm{Sym}^{ie}(\mathrm{Sym}^{d}(U)) \end{array}$$

commutes. We have $u_{i\times de} = \bigodot_{k=1}^{i} x_k^{\circledcirc de}$ and

$$\iota_U^{i\times de}(u_{i\times de}) = \frac{1}{i!}\sum_{\sigma\in S_i}\bigotimes_{k=1}^{i} x_{\sigma(k)}^{\otimes de}.$$

So we see that

$$\Phi_U^{i,d}(u_{i\times de}) = \pi_U^{ie\times d}\left(\frac{1}{i!}\sum_{\sigma\in S_i}\bigotimes_{k=1}^{i} x_{\sigma(k)}^{\otimes de}\right) = \frac{1}{i!}\sum_{\sigma\in S_i}\pi_U^{ie\times d}\left(\bigotimes_{k=1}^{i} x_{\sigma(k)}^{\otimes de}\right)$$

$$= \frac{1}{i!}\sum_{\sigma\in S_i}\bigodot_{k=1}^{i}\left(x_{\sigma(k)}^{\circledcirc d}\right)^{\odot e} = \bigodot_{k=1}^{i}\left(x_k^{\circledcirc d}\right)^{\odot e} = \left(x_1^{\circledcirc d}\odot\cdots\odot x_i^{\circledcirc d}\right)^{\odot e}.$$

7.19. Suppose that $\dim_{\mathbb{C}}(U) = m$ is at least a. Note that $\Theta_U^{a\times b}$ is the homomorphism of representations of $\mathrm{GL}(U)$ making the diagram

$$\begin{array}{ccc} \left(U^{\otimes b}\right)^{\otimes a} & \xrightarrow{\mathrm{transp}_U^{a\times b}} & \left(U^{\otimes a}\right)^{\otimes b} \\ \uparrow{\scriptstyle (\mathrm{unord}_U^{a\times b})^{-1}} & & \downarrow{\scriptstyle \mathrm{unord}_U^{b\times a}} \\ U^{\otimes ab} & & U^{\otimes ab} \\ \uparrow{\scriptstyle \iota_U^{a\times b}} & & \downarrow{\scriptstyle \pi_U^{b\times a}} \\ \mathrm{Sym}^a(\mathrm{Sym}^b(U)) & \xrightarrow{\Theta_U^{a\times b}} & \mathrm{Sym}^b(\mathrm{Sym}^a(U)) \end{array}$$

commute. The homomorphism $\Theta_U^{a\times b}$ is completely determined by its value at $u_{a\times b}$. If we follow the diagram clockwise from $\mathrm{Sym}^a(\mathrm{Sym}^b(U))$ to the representation $\mathrm{Sym}^b(\mathrm{Sym}^a(U))$, we see that

$$\begin{aligned}
u_{a\times b} &= \bigodot_{i=1}^{a} x_i^{\circledcirc b} \\
&\mapsto \frac{1}{a!}\sum_{\sigma\in S_a}\bigotimes_{i=1}^{a} x_{\sigma(i)}^{\otimes b} \\
&\mapsto \frac{1}{a!}\sum_{\sigma\in S_a}\bigotimes_{i=1}^{a} x_{\sigma(i)}^{\circledast b} \\
&\mapsto \frac{1}{a!}\sum_{\sigma\in S_a}\bigotimes_{j=1}^{b} x_{\sigma(1)}\circledast\cdots\circledast x_{\sigma(a)} \\
&\mapsto \frac{1}{a!}\sum_{\sigma\in S_a}\bigotimes_{j=1}^{b} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(a)} \\
&\mapsto \frac{1}{a!}\sum_{\sigma\in S_a}\bigodot_{j=1}^{b} x_{\sigma(1)}\circledcirc\cdots\circledcirc x_{\sigma(a)} \\
&= (x_1\circledcirc\cdots\circledcirc x_a)^{\odot b}.
\end{aligned}$$

So we have $\Theta_U^{a\times b}(u_{a\times b}) = (x_1\circledcirc\cdots\circledcirc x_a)^{\odot b}$. It follows that for all elements $u_{1,1},\ldots,u_{a,b}\in U$, we have

$$\Theta_U^{a\times b}\left(\bigodot_{i=1}^{a}\mathop{\circledcirc}_{j=1}^{b} u_{i,j}\right) = \left(\frac{1}{b!}\right)^{a}\sum_{\sigma_1,\ldots,\sigma_a\in S_b}\bigodot_{j=1}^{b}\mathop{\circledcirc}_{i=1}^{a} u_{i,\sigma_i(j)},$$

because the homomorphism

$$\begin{aligned}
\mathrm{Sym}^a(\mathrm{Sym}^b(U)) &\to \mathrm{Sym}^b(\mathrm{Sym}^a(U)) \\
\bigodot_{i=1}^{a}\mathop{\circledcirc}_{j=1}^{b} u_{i,j} &\mapsto \left(\frac{1}{b!}\right)^{a}\sum_{\sigma_1,\ldots,\sigma_a\in S_b}\bigodot_{j=1}^{b}\mathop{\circledcirc}_{i=1}^{a} u_{i,\sigma_i(j)}
\end{aligned}$$

of representations of $\mathrm{GL}(U)$ also sends $u_{a\times b}$ to $(x_1\circledcirc\cdots\circledcirc x_a)^{\odot b}$.

7.20. Denote the homomorphism

$$\Theta_U^{ie\times d}\circ\Phi_U^{i,d} : \mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) \to \mathrm{Sym}^d(\mathrm{Sym}^{ie}(U))$$

of representations of $\mathrm{GL}(U)$ by $\Psi_{i,d}$. Note that to prove the injectivity of $\Phi_U^{i,d}$, it suffices to prove the injectivity of $\Psi_{i,d}$. The homomorphism $\Psi_{i,d}$ is also completely determined by its value at $u_{i\times de}$. We have

$$\Phi_U^{i,d}(u_{i\times de}) = \left(x_1^{\circledcirc d}\odot\cdots\odot x_i^{\circledcirc d}\right)^{\odot e}.$$

So we see that

$$\Psi_{i,d}(u_{i\times de}) = \left(x_1^{\circledcirc e} \circledcirc \cdots \circledcirc x_i^{\circledcirc e}\right)^{\odot d}.$$

One can check now that

$$\Psi_{i,d}\left(\bigodot_{k=1}^{i} \bigcircledcirc_{j=1}^{de} u_{k,j}\right) = \left(\frac{1}{(de)!}\right)^{i} \sum_{\sigma_1,\ldots,\sigma_i \in S_{de}} \bigodot_{j=1}^{d} \bigcircledcirc_{k=1}^{i} \bigcircledcirc_{h=(j-1)e+1}^{je} u_{k,\sigma_k(h)}$$

for all $u_{1,1},\ldots,u_{i,de} \in U$.

7.4 Proving injectivity for $d = i = 2$

We are mainly interested in the endomorphism $\Psi_{d,d}$ for integers $d \in \mathbb{Z}_{\geq 0}$, because injectivity of $\Psi_{d,d}$ implies that the second conjecture holds for that d. Note that the endomorphism of representations of $\mathrm{GL}(U)$

$$\Psi_{1,1}\colon \mathrm{Sym}^e(U) \quad\to\quad \mathrm{Sym}^e(U)$$

is just the identity map. So we see that the second conjecture holds for $d = 1$. Recall that the first and second conjecture are equivalent for $d = 1$. So we see that the first conjecture also holds for $d = 1$. Next we consider the case $d = 2$. Let $(x_1, x_2, \ldots, x_m)$ be a basis of U with $m \geq 2$.

7.21. The endomorphism of representations of $\mathrm{GL}(U)$

$$\Psi_{2,2}\colon \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) \to \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U))$$

is completely determined by the fact that it sends the element $x_1^{\circledcirc 2e} \odot x_2^{\circledcirc 2e}$ to the element $(x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e})$. To prove that $\Psi_{2,2}$ is injective, it suffices to prove that there exists an endomorphism of representations of $\mathrm{GL}(U)$

$$\mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) \to \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U))$$

which sends $(x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e})$ to $x_1^{\circledcirc 2e} \odot x_2^{\circledcirc 2e}$. We know that for all endomorphisms of representations of $\mathrm{GL}(U)$

$$\ell\colon \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) \to \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U))$$

there exists an element $\chi \in \mathbb{C}[S_{4e}]$ such that the diagram

$$\begin{array}{ccc} U^{\otimes 4e} & \xrightarrow{\quad \eta(\chi) \quad} & U^{\otimes 4e} \\ \big\uparrow {\scriptstyle \iota_U^{2\times 2e}} & & \big\downarrow {\scriptstyle \pi_U^{2\times 2e}} \\ \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) & \xrightarrow{\quad \ell \quad} & \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) \end{array}$$

commutes where $\eta(\chi)$ is the map sending t to $\chi(t)$ for all $t \in U^{\otimes 4e}$.

7.22. For each integer $i \in \{0, \ldots, 2e\}$, let $\sigma_i \in S_{4e}$ be the permutation that sends k to $4e+1-k$ for all $k \in \{1, \ldots, i\} \cup \{4e+1-i, \ldots, 4e\}$ and that sends k to k for all $k \in \{i+1, \ldots, 4e-i\}$. Then we have

$$\sigma_i \cdot \left(u_1^{\otimes 2e} \otimes u_2^{\otimes 2e}\right) = u_2^{\otimes i} \otimes u_1^{\otimes 2e-i} \otimes u_2^{\otimes 2e-i} \otimes u_1^{\otimes i}$$

for all $i \in \{0, \ldots, 2e\}$ and $u_1, u_2 \in U$. For each integer $i \in \{0, \ldots, 2e\}$, let ℓ_i be the homomorphism of representations of $\mathrm{GL}(U)$ making the diagram

$$\begin{array}{ccc} U^{\otimes 4e} & \xrightarrow{\varrho(\sigma_i)} & U^{\otimes 4e} \\ {\scriptstyle \iota_U^{2\times 2e}} \uparrow & & \downarrow {\scriptstyle \pi_U^{2\times 2e}} \\ \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) & \xrightarrow{\ell_i} & \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) \end{array}$$

commute where $\varrho(\sigma_i)$ is the map sending t to $\sigma_i \cdot t$ for all $t \in U^{\otimes 4e}$.

7.23. Note that σ_0 is the trivial permutation. So we have

$$\ell_0\left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e})\right) = (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}).$$

Consider the homomorphism ℓ_1. Note that

$$\iota_U^{2\times 2e}\left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e})\right)$$

is the average over all $u_1 \otimes \cdots \otimes u_{4e}$ such that $u_k \in \{x_1, x_2\}$ for all integers $k \in \{1, \ldots, 4e\}$ and

$$\#\{k \in \{1, \ldots, 2e\} | u_k = x_1\} = \#\{k \in \{2e+1, \ldots, 4e\} | u_k = x_1\} = e,$$

$$\#\{k \in \{1, \ldots, 2e\} | u_k = x_2\} = \#\{k \in \{2e+1, \ldots, 4e\} | u_k = x_2\} = e.$$

For each such $u_1 \otimes \cdots \otimes u_{4e}$, the homomorphism $\varrho(\sigma_1)$ exchanges the positions of u_1 and u_{4e}. There are now two possibilities. The first is that u_1 and u_{4e} are equal. In this case, the homomorphism $\varrho(\sigma_1)$ does not change $u_1 \otimes \cdots \otimes u_{4e}$. Denote the set of such $u_1 \otimes \cdots \otimes u_{4e}$ by $\delta_{1,0}$. The second possibility is that u_1 and u_{4e} are different. In this case, the homomorphism $\varrho(\sigma_1)$ does change $u_1 \otimes \cdots \otimes u_{4e}$. Denote the set of such $u_1 \otimes \cdots \otimes u_{4e}$ by $\delta_{1,1}$. Note that

$$\ell_1\left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e})\right)$$

is equal to

$$\left(\frac{1}{2e!}\right)^2 \left(\sum_{t \in \delta_{1,0}} \pi_U^{2\times 2e}(\sigma_1 \cdot t) + \sum_{t \in \delta_{1,1}} \pi_U^{2\times 2e}(\sigma_1 \cdot t)\right).$$

Note that $\delta_{1,0}$ and $\delta_{1,1}$ both have size $((de)!)^2/2$. For all $t \in \delta_{1,0}$, we have

$$\pi_U^{2\times 2e}(\sigma_1 \cdot t) = \pi_U^{2\times 2e}(t) = (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}).$$

For all $t \in \delta_{1,1}$, we have

$$\pi_U^{2\times 2e}(\sigma_1 \cdot t) = (x_1^{\circledcirc e+1} \circledcirc x_2^{\circledcirc e-1}) \odot (x_1^{\circledcirc e-1} \circledcirc x_2^{\circledcirc e+1}),$$

because one pair x_1 and x_2 got switched. So we see that

$$\ell_1 \left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \right)$$

equals

$$\frac{1}{2} \left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) + (x_1^{\circledcirc e+1} \circledcirc x_2^{\circledcirc e-1}) \odot (x_1^{\circledcirc e-1} \circledcirc x_2^{\circledcirc e+1}) \right).$$

7.24. Let $i \in \{0, \dots, 2e\}$ be an integer. We know that

$$\iota_U^{2\times 2e} \left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \right)$$

is the average over all $u_1 \otimes \cdots \otimes u_{4e}$ such that $u_k \in \{x_1, x_2\}$ for all integers $k \in \{1, \dots, 4e\}$ and

$$\#\{k \in \{1, \dots, 2e\} | u_k = x_1\} = \#\{k \in \{2e+1, \dots, 4e\} | u_k = x_1\} = e,$$

$$\#\{k \in \{1, \dots, 2e\} | u_k = x_2\} = \#\{k \in \{2e+1, \dots, 4e\} | u_k = x_2\} = e.$$

For each such $u_1 \otimes \cdots \otimes u_{4e}$, the homomorphism $\varrho(\sigma_i)$ exchanges the positions of u_k and u_{4e+1-k} for each $k \in \{1, \dots, i\}$. For each $j \in \{0, \dots, i\}$, let $\delta_{i,j}$ be the set of $u_1 \otimes \cdots \otimes u_{4e}$ such that $\#\{k \in \{1, \dots, i\} | u_k \neq u_{4e+1-k}\} = j$. Then we see that each set $\delta_{i,j}$ has a strictly positive size and we have

$$\ell_i \left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \right) = \left(\frac{1}{2e!} \right)^2 \sum_{j=0}^{i} \sum_{t \in \delta_{i,j}} \pi_U^{2\times 2e}(\sigma_i \cdot t).$$

For all integers $j \in \{0, \dots, i\}$ and elements $t \in \delta_{i,j}$, we have

$$\pi_U^{2\times 2e}(\sigma_i \cdot t) = (x_1^{\circledcirc e+j} \circledcirc x_2^{\circledcirc e-j}) \odot (x_1^{\circledcirc e-j} \circledcirc x_2^{\circledcirc e+j})$$

because j pairs x_1 and x_2 got switched. So we see that

$$\ell_i \left((x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \right)$$

equals

$$\left(\frac{1}{2e!} \right)^2 \sum_{j=0}^{i} \#\delta_{i,j} \cdot (x_1^{\circledcirc e+j} \circledcirc x_2^{\circledcirc e-j}) \odot (x_1^{\circledcirc e-j} \circledcirc x_2^{\circledcirc e+j}).$$

7.25. Now back to our problem. We want to prove that there exists a $\chi \in \mathbb{C}[S_{4e}]$ such that the homomorphism ℓ of representations of $\mathrm{GL}(U)$ making the diagram

$$\begin{array}{ccc} U^{\otimes 4e} & \xrightarrow{\quad \eta(\chi) \quad} & U^{\otimes 4e} \\ \Big\uparrow {\scriptstyle \iota_U^{2\times 2e}} & & \Big\downarrow {\scriptstyle \pi_U^{2\times 2e}} \\ \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) & \xrightarrow{\quad \ell \quad} & \mathrm{Sym}^2(\mathrm{Sym}^{2e}(U)) \end{array}$$

commute, sends $(x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e}) \odot (x_1^{\circledcirc e} \circledcirc x_2^{\circledcirc e})$ to $x_1^{\circledcirc 2e} \odot x_2^{\circledcirc 2e}$. We claim that χ can be chosen to be a linear combination

$$\sum_{i=0}^{2e} \lambda_i \sigma_i$$

of $\sigma_0, \ldots, \sigma_{2e}$. By our previous computations, we see that this claim holds if and only if the linear system

$$\left(\frac{1}{2e!}\right)^2 \begin{pmatrix} \#\delta_{0,0} & \#\delta_{1,0} & \cdots & \cdots & \#\delta_{2e-1,0} & \#\delta_{2e,0} \\ 0 & \#\delta_{1,1} & & & \vdots & \vdots \\ \vdots & 0 & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \#\delta_{2e-1,2e-1} & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & \#\delta_{2e,2e} \end{pmatrix} \begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{2e} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

has a solution. This is indeed the case, because $\#\delta_{i,j} > 0$ for all integers $0 \leq j \leq i \leq 2e$. So we see that $\Psi_{2,2}$ is indeed injective. This also proves the second conjecture for $d = 2$.

7.5 Ikenmeyer's construction

For each integer $n \in \mathbb{Z}_{\geq 0}$, denote the set $\{1, \ldots, n\}$ by $[n]$. Let $d, i \in \mathbb{Z}_{\geq 0}$ be integers and let V be the vector space $\mathbb{C}^i \oplus \mathbb{C}^d$. Let $(e_1, \ldots, e_i)$ be the standard basis of $\mathbb{C}^i$ and let $(f_1, \ldots, f_d)$ be the standard basis of $\mathbb{C}^d$.

7.26. Let L_i be the group S_i. Then the homomorphism

$$\begin{array}{rcl} L_i & \to & \mathrm{GL}(V) \\ \sigma & \mapsto & \left(\begin{array}{ll} e_j & \mapsto e_{\sigma(j)} \\ f_j & \mapsto f_j \end{array} \right) \end{array}$$

gives V the structure of a representation of L_i. Let R_d be the group S_d. Then the homomorphism

$$\begin{array}{rcl} R_d & \to & \mathrm{GL}(V) \\ \sigma & \mapsto & \left(\begin{array}{ll} e_j & \mapsto e_j \\ f_j & \mapsto f_{\sigma(j)} \end{array} \right) \end{array}$$

gives V the structure of a representation of R_d. Note that the above homomorphisms commute. So V has the structure of a representation of $L_i \times R_d$.

Let $n \in \mathbb{Z}_{\geq 0}$ be an integer and let $\alpha \in \mathbb{Z}^i_{\geq 0}$ and $\beta \in \mathbb{Z}^d_{\geq 0}$ be vectors.

7.27. Denote the span of

$$\left\{ v_1 \otimes \cdots \otimes v_n \;\middle|\; \begin{array}{c} v_1, \dots, v_n \in \{e_1, \dots, e_i, f_1, \dots, f_d\} \\ \#\{k | v_k = e_j\} = \alpha_j \text{ for all } j \in [i] \\ \#\{k | v_k = f_j\} = \beta_j \text{ for all } j \in [d] \end{array} \right\}$$

inside $V^{\otimes n}$ by $V^{\otimes n}_{(\alpha,\beta)}$. Note that the above set is invariant under the action of S_n on $V^{\otimes n}$. So we see that $V^{\otimes n}_{(\alpha,\beta)}$ is an S_n-invariant subspace of $V^{\otimes n}$. Therefore $V^{\otimes n}_{(\alpha,\beta)}$ has the structure of a subrepresentation of the representation $V^{\otimes n}$ of S_n.

7.28. Let H be a subgroup of $L_i \times R_d$. Denote the subspace

$$\left\{ t \in V^{\otimes n}_{(\alpha,\beta)} \;\middle|\; g \cdot t = t \text{ for all } g \in H \right\}$$

by $V^{\otimes n,H}_{(\alpha,\beta)}$. Since the homomorphisms

$$L_i \times R_d \to \mathrm{GL}\left(V^{\otimes n}\right)$$

and

$$S_n \to \mathrm{GL}\left(V^{\otimes n}\right)$$

commute, we see that $V^{\otimes n,H}_{(\alpha,\beta)}$ is an S_n-invariant subspace of $V^{\otimes n}$. Hence $V^{\otimes n,H}_{(\alpha,\beta)}$ has the structure of a subrepresentation of the representation $V^{\otimes n}$ of S_n.

For integers $a, b \in \mathbb{Z}_{\geq 0}$, denote the vector $(b, \dots, b) \in \mathbb{Z}^a_{\geq 0}$ by $a \times b$.

Example 7.29. The vector space $V^{\otimes die}_{(i \times de, 0)}$ is the span of the set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{c} v_1, \dots, v_{die} \in \{e_1, \dots, e_i\} \\ \#\{k | v_k = e_j\} = de \text{ for all } j \in [i] \end{array} \right\}.$$

Recall that $\Omega_{i \times de}$ is the set of all functions $r\colon \{1, \dots, die\} \to \{1, \dots, i\}$ such that $\#r^{-1}(j) = de$ for all $j \in [i]$. Note that the set written above is equal to the set $\{e_{r(1)} \otimes \cdots \otimes e_{r(die)} | r \in \Omega_{i \times de}\}$. Next note that this set of invariant under the actions of L_i, R_d and S_{die}. For each $r \in \Omega_{i \times de}$, denote the element

$$\frac{1}{i!} \sum_{\sigma \in L_i} e_{\sigma(r(1))} \otimes \cdots \otimes e_{\sigma(r(die))}$$

of $V^{\otimes die}$ by v_r. Then we see that $V^{\otimes die, L_i}_{(i\times de,0)}$ is the span of the linearly independent elements $\{v_r | [r] \in \Omega_{i\times de}/\sim\}$ and that we have $\sigma \cdot v_r = v_{\sigma\cdot r}$ for all $\sigma \in S_{die}$ and $r \in \Omega_{i\times de}$. Let v_{L} be the element

$$v_{r_{i\times de}} = \frac{1}{i!} \sum_{\sigma\in L_i} \bigotimes_{k=1}^{i} e^{\otimes de}_{\sigma(k)}$$

of $V^{\otimes die}$. Then we see that the representation $V^{\otimes die, L_i}_{(i\times de,0)}$ of S_{die} is generated by the element v_{L}.

Proposition 7.30. The map

$$\begin{array}{rcl} \ell\colon \mathbb{C}[S_{die}]/K_{i\times de} & \to & V^{\otimes die, L_i}_{(i\times de,0)} \\ \chi & \mapsto & \chi \cdot v_{\mathrm{L}} \end{array}$$

is an isomorphism of representations of S_{die}.

Proof. We already know that ℓ is surjective. Let χ be an element of $\mathbb{C}[S_{die}]$ and write $\chi = \sum_{\sigma\in S_{ab}} c_\sigma \cdot \sigma$. Then we have

$$\chi \cdot v_{\mathrm{L}} = \sum_{\tau\in S_{die}/H_{i\times de}} \left(\sum_{\sigma\in\tau H_{i\times de}} c_\sigma \right) \cdot v_{\tau\cdot r_{i\times de}}.$$

Since $\{v_{\tau\cdot r_{i\times de}} | \tau \in S_{die}/H_{i\times de}\} = \{u_r | [r] \in \Omega_{i\times de}/\sim\}$ is linearly independent, we see that

$$\sum_{\sigma\in\tau H_{i\times de}} c_\sigma = 0$$

for all $\tau \in S_{die}$. Hence χ is contained in $K_{i\times de}$. □

Proposition 7.31. The map

$$\begin{array}{rcl} \delta_{\mathrm{L}}\colon \mathrm{Hom}_{S_{die}}\left(V^{\otimes die, L_i}_{(i\times de,0)}, U^{\otimes die}\right) & \to & \mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) \\ \ell & \mapsto & \pi^{i\times de}_U(\ell(v_{\mathrm{L}})) \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(U)$.

Proof. Let

$$\ell\colon V^{\otimes die, L_i}_{(i\times de,0)} \to U^{\otimes die}$$

be a homomorphism of representations of S_{die}. Then we see that

$$\begin{array}{rcl} \delta(g\cdot\ell) & = & \pi^{i\times de}_U(g\cdot\ell(v_{\mathrm{L}})) \\ & = & g\cdot\pi^{i\times de}_U(\ell(v_{\mathrm{L}})) \\ & = & g\cdot\delta(\ell). \end{array}$$

for all $g \in \mathrm{GL}(U)$, because $\pi_U^{i\times de}$ is a homomorphism of representations of $\mathrm{GL}(U)$. Therefore δ is a homomorphism of representation of $\mathrm{GL}(U)$.

Consider the element $\ell(v_{\mathrm{L}})$ of $U^{\otimes die}$. For each permutation $\sigma \in H_{i\times de}$, we have

$$\sigma \cdot \ell(v_{\mathrm{L}}) = \ell(\sigma \cdot v_{\mathrm{L}}) = \ell(v_{\mathrm{L}}),$$

because we have $\sigma \cdot r_{i\times de} = r_{i\times de}$ for all $\sigma \in H_{i\times de}$. So we see that

$$\ell(v_{\mathrm{L}}) = \iota_U^{i\times de}(\pi_U^{i\times de}(\ell(v_{\mathrm{L}}))) = \iota_U^{i\times de}(\delta(\ell)).$$

Since v_{L} generates the representation $V_{(i\times de,0)}^{\otimes die,L_i}$ of S_{die}, we see that the homomorphism ℓ is completely determined by $\ell(v_{\mathrm{L}})$. So $\iota_U^{i\times de} \circ \delta$ is injective and therefore δ is injective.

Let t be a element of $\mathrm{Sym}^i(\mathrm{Sym}^{de}(U))$. Then one can check that the $\mathbb{C}$-linear map

$$\begin{aligned} \ell\colon V_{(i\times de,0)}^{\otimes die,L_i} &\to U^{\otimes die} \\ v_{\sigma\cdot r_{i\times de}} &\mapsto \sigma \cdot \iota_U^{i\times de}(t) \end{aligned}$$

is a homomorphism of representations of S_{die} which sends v_{L} to $\iota_U^{i\times de}(t)$. So we see that δ is also surjective and hence an isomorphism. □

7.32. Denote the inclusion homomorphism of representations

$$V_{(i\times de,0)}^{\otimes die,L_i} \to V^{\otimes die}$$

by $\iota_{\mathrm{L}}^{i,d}$ and denote the surjective homomorphism of representations

$$V^{\otimes die} \to V_{(i\times de,0)}^{\otimes die,L_i}$$

that sends for all $v_1, \dots, v_{die} \in \{e_1, \dots, e_i, f_1, \dots, f_d\}$ the element

$$v_1 \otimes \cdots \otimes v_{die}$$

to

$$\frac{1}{i!} \sum_{\sigma \in L_i} \sigma \cdot (v_1 \otimes \cdots \otimes v_{die})$$

if $\#\{k | v_k = e_j\} = de$ for all $j \in \{1, \dots, i\}$ and to zero otherwise by $\pi_{\mathrm{L}}^{i,d}$.

Example 7.33. The vector space $V_{(0,d\times ie)}^{\otimes die}$ is the span of the set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \,\middle|\, \begin{array}{c} v_1, \dots, v_{die} \in \{f_1, \dots, f_d\} \\ \#\{k|v_k = f_j\} = ie \text{ for all } j \in [d] \end{array} \right\}.$$

Recall that $\Omega_{d\times ie}$ is the set of all functions $r\colon \{1, \dots, die\} \to \{1, \dots, d\}$ such that $\#r^{-1}(j) = ie$ for all $j \in [d]$. Note that the set written above is equal to the set $\{f_{r(1)} \otimes \cdots \otimes f_{r(die)} | r \in \Omega_{d\times ie}\}$. Next note that this set of invariant under the actions of L_i, R_d and S_{die}. For each $r \in \Omega_{d\times ie}$, denote the element

$$\frac{1}{d!} \sum_{\sigma \in R_d} f_{\sigma(r(1))} \otimes \cdots \otimes f_{\sigma(r(die))}$$

of $V^{\otimes die}$ by w_r. Then we see that $V^{\otimes die, R_d}_{(0,d\times ie)}$ is the span of the linearly independent elements $\{w_r | [r] \in \Omega_{d\times ie}/\sim\}$ and that we have $\sigma \cdot w_r = w_{\sigma\cdot r}$ for all $\sigma \in S_{die}$ and $r \in \Omega_{d\times ie}$. Let w_{R} be the element

$$w_{r_{d\times ie}} = \frac{1}{d!} \sum_{\sigma \in R_d} \bigotimes_{k=1}^{d} f_{\sigma(k)}^{\otimes ie}$$

of $V^{\otimes die}$. Then we see that the representation $V^{\otimes die, R_d}_{(0,d\times ie)}$ of S_{die} is generated by the element w_{R}.

Proposition 7.34. The map

$$\begin{array}{rcl} \mathbb{C}[S_{die}]/K_{d\times ie} & \to & V^{\otimes die, R_d}_{(0,d\times ie)} \\ \chi & \mapsto & \chi \cdot w_{\mathrm{R}} \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(U)$.

Proof. This proposition is proven similarly to Proposition 7.30. $\square$

Proposition 7.35. The map

$$\begin{array}{rcl} \delta_{\mathrm{R}} \colon \mathrm{Hom}_{S_{die}}\left(V^{\otimes die, R_d}_{(0,d\times ie)}, U^{\otimes die}\right) & \to & \mathrm{Sym}^d(\mathrm{Sym}^{ie}(U)) \\ \ell & \mapsto & \pi_U^{d\times ie}(\ell(w_{\mathrm{R}})) \end{array}$$

is an isomorphism of representations of $\mathrm{GL}(U)$.

Proof. This proposition is proven similarly to Proposition 7.31. $\square$

7.36. Denote the inclusion homomorphism of representations

$$V^{\otimes die, R_d}_{(0,d\times ie)} \to V^{\otimes die}$$

by $\iota_{\mathrm{R}}^{i,d}$ and denote the surjective homomorphism of representations

$$V^{\otimes die} \to V^{\otimes die, R_d}_{(0,d\times ie)}$$

that sends for all $v_1, \ldots, v_{die} \in \{e_1, \ldots, e_i, f_1, \ldots, f_d\}$ the element

$$v_1 \otimes \cdots \otimes v_{die}$$

to

$$\frac{1}{d!} \sum_{\sigma \in R_d} \sigma \cdot (v_1 \otimes \cdots \otimes v_{die})$$

if $\#\{k | v_k = f_j\} = ie$ for all $j \in \{1, \ldots, d\}$ and to zero otherwise by $\pi_{\mathrm{R}}^{i,d}$.

7.37. For all integers $n \in \mathbb{Z}_{\geq 0}$, $j \in \{1, \ldots, i\}$ and $k \in \{1, \ldots, d\}$, denote the $\mathbb{C}$-linear map $V \to V$ that sends e_h to f_k if $h = j$ and to zero otherwise and that sends f_h to zero for all h by $\phi_{j,k}$ and denote the homomorphism of representations of S_n

$$\begin{array}{rcl} V^{\otimes die} & \to & V^{\otimes die} \\ v_1 \otimes \cdots \otimes v_n & \mapsto & \displaystyle\sum_{h=1}^{n} v_1 \otimes \cdots \otimes \phi_{j,k}(v_h) \otimes \cdots \otimes v_n \end{array}$$

by $\varphi_{j,k}$.

Proposition 7.38. For all $j, j^\dagger \in \{1, \ldots, i\}$ and $k, k^\dagger \in \{1, \ldots, d\}$, the maps $\varphi_{j,k}$ and $\varphi_{j^\dagger,k^\dagger}$ commute.

Proof. We have

$$\varphi_{j,k}(\varphi_{j^\dagger,k^\dagger}(v_1 \otimes \cdots \otimes v_{die})) = \sum_{h \neq h^\dagger} v_1 \otimes \cdots \otimes \phi_{j,k}(v_h) \otimes \cdots \otimes \phi_{j^\dagger,k^\dagger}(v_{h^\dagger}) \otimes \cdots \otimes v_{die}$$

for all $v_1, \ldots, v_{die} \in V$. Since the right hand side is symmetric in the pairs (j, k) and $(j^\dagger, k^\dagger)$, we see that $\varphi_{j,k}$ and $\varphi_{j^\dagger,k^\dagger}$ commute. $\square$

7.39. For all integers $n \in \mathbb{Z}_{\geq 0}$, $j \in \{1, \ldots, i\}$ and $k \in \{1, \ldots, d\}$, denote the $\mathbb{C}$-linear map $V \to V$ that sends f_h to e_j if $h = k$ and to zero otherwise and that sends e_h to zero for all h by $\phi^*_{j,k}$ and denote the homomorphism of representations of S_n

$$\begin{array}{rcl} V^{\otimes die} & \to & V^{\otimes die} \\ v_1 \otimes \cdots \otimes v_n & \mapsto & \displaystyle\sum_{h=1}^{n} v_1 \otimes \cdots \otimes \phi^*_{j,k}(v_h) \otimes \cdots \otimes v_n \end{array}$$

by $\varphi^*_{j,k}$.

Proposition 7.40. For all $j, j^\dagger \in \{1, \ldots, i\}$ and $k, k^\dagger \in \{1, \ldots, d\}$, the maps $\varphi^*_{j,k}$ and $\varphi^*_{j^\dagger,k^\dagger}$ commute.

Proof. This proposition is proven similarly to Proposition 7.38. $\square$

7.41. Let $(e_1^*, \ldots, e_i^*, f_1^*, \ldots, f_d^*)$ be the dual basis of $(e_1, \ldots, e_i, f_1, \ldots, f_d)$. Then we see that for all integers $j \in \{1, \ldots, i\}$ and $k \in \{1, \ldots, d\}$, the $\mathbb{C}$-linear map $\phi^\times_{j,k} \colon V^\times \to V^\times$ sends e_h^* to zero for all h and sends f_h^* to e_j^* if $h = k$ and to zero otherwise. Let $\kappa_1 \colon V \to V^\times$ be the isomorphism of vector spaces that sends e_h to e_h^* and f_h to f_h^* for all h. Then we see that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi^*_{j,k}} & V \\ {\scriptstyle \kappa_1}\downarrow & & \downarrow{\scriptstyle \kappa_1} \\ V^\times & \xrightarrow{\phi^\times_{j,k}} & V^\times \end{array}$$

commutes for all $j \in \{1, \dots, i\}$ and $k \in \{1, \dots, d\}$. Let $n \in \mathbb{Z}_{\geq 0}$ be an integer. Note that the $\mathbb{C}$-linear map

$$\begin{aligned} \kappa_n \colon V^{\otimes n} &\to (V^{\otimes n})^\times \\ \bigotimes_{k=1}^{n} v_k &\mapsto \left(\bigotimes_{k=1}^{n} w_k \mapsto \prod_{k=1}^{n} \kappa_1(v_k)(w_k) \right) \end{aligned}$$

is the composition of the isomorphism $\kappa_1^{\otimes n}$ with the isomorphism for Lemma 2.30(d). So we see that κ_n is an isomorphism as well. One can check that the diagram

$$\begin{array}{ccc} V^{\otimes n} & \xrightarrow{\varphi_{j,k}^*} & V^{\otimes n} \\ \downarrow{\scriptstyle \kappa_n} & & \downarrow{\scriptstyle \kappa_n} \\ (V^{\otimes n})^\times & \xrightarrow{\varphi_{j,k}^\times} & (V^{\otimes n})^\times \end{array}$$

commutes for all $n \in \mathbb{Z}_{\geq 0}$, $j \in \{1, \dots, i\}$ and $k \in \{1, \dots, d\}$.

7.42. Consider the homomorphism of representations of S_{die}

$$\ell_{\mathrm{L}} := \varphi_{1,1}^e \circ \cdots \circ \varphi_{i,d}^e \colon V_{(i \times de, 0)}^{\otimes die} \to V_{(0, d \times ie)}^{\otimes die}.$$

Let v be the element $e_1^{\otimes de} \otimes \cdots \otimes e_i^{\otimes de}$ of $V_{(i \times de, 0)}^{\otimes die}$. Now consider the element $(\varphi_{1,1}^e \circ \cdots \circ \varphi_{1,d}^e)(v)$ of $V_{((0,de,\dots,de),d \times e)}^{\otimes die}$. Each map $\varphi_{1,k}$ replaces one of the de entries e_1 at the start of v and we sum over all the possibilities. So we see that

$$(\varphi_{1,1}^e \circ \cdots \circ \varphi_{1,d}^e)(v) = \sum_{\sigma \in S_{de}} \left(\sigma \cdot (f_1^{\otimes e} \otimes \cdots \otimes f_d^{\otimes e}) \right) \otimes e_2^{\otimes de} \otimes \cdots \otimes e_i^{\otimes de}.$$

By repeating this, we see that $(\varphi_{1,1}^e \circ \cdots \circ \varphi_{j,d}^e)(v)$ is equal to

$$\bigotimes_{k=1}^{j} \left(\sum_{\sigma \in S_{de}} \sigma \cdot (f_1^{\otimes e} \otimes \cdots \otimes f_d^{\otimes e}) \right) \otimes e_{j+1}^{\otimes de} \otimes \cdots \otimes e_i^{\otimes de}$$

for all $j \in \{1, \dots, i\}$. So in particular, we see that

$$(\varphi_{1,1}^e \circ \cdots \circ \varphi_{i,d}^e)(v) = \bigotimes_{k=1}^{i} \left(\sum_{\sigma \in S_{de}} \sigma \cdot (f_1^{\otimes e} \otimes \cdots \otimes f_d^{\otimes e}) \right)$$

is an element of $V_{(0,d \times ie)}^{\otimes die, R_d}$. So since the element v generates the representation $V_{(i \times de, 0)}^{\otimes die}$ of S_{die}, we see that the map $\varphi_{1,1}^e \circ \cdots \circ \varphi_{i,d}^e$ restricts to a homomorphism

$$\psi_{i,d} \colon V_{(i \times de, 0)}^{\otimes die, L_i} \to V_{(0, d \times ie)}^{\otimes die, R_d}$$

of representation of S_{die}. One can similarly check that the homomorphism

$$\ell_{\mathrm{R}} := (\varphi_{1,1}^*)^e \circ \cdots \circ (\varphi_{i,d}^*)^e \colon V^{\otimes die}_{(0,d\times ie)} \to V^{\otimes die}_{(i\times de,0)}$$

of representations of S_{die} restricts to a homomorphism

$$\psi_{i,d}^* \colon V^{\otimes die, R_d}_{(0,d\times ie)} \to V^{\otimes die, L_i}_{(i\times de,0)}$$

of representations of S_{die}. One can also check that the diagram

$$\begin{array}{ccc}
V^{\otimes die,R_d}_{(0,d\times ie)} & \longrightarrow & \left(V^{\otimes die,R_d}_{(0,d\times ie)}\right)^{\times} \\
\iota_{\mathrm{R}}^{i,d} \big\downarrow & & \big\downarrow (\pi_{\mathrm{L}}^{i,d})^{\times} \\
V^{\otimes die}_{(0,d\times ie)} & \longrightarrow & \left(V^{\otimes die}_{(0,d\times ie)}\right)^{\times} \\
\ell_{\mathrm{R}} \big\downarrow & & \big\downarrow \ell_{\mathrm{L}}^{\times} \\
V^{\otimes die}_{(i\times de,0)} & \longrightarrow & \left(V^{\otimes die}_{(i\times de,0)}\right)^{\times} \\
\pi_{\mathrm{R}}^{i,d} \big\downarrow & & \big\downarrow (\iota_{\mathrm{L}}^{i,d})^{\times} \\
V^{\otimes die,L_i}_{(i\times de,0)} & \longrightarrow & \left(V^{\otimes die,L_i}_{(i\times de,0)}\right)^{\times}
\end{array}$$

commutes where the horizontal maps sends an element $t \in V^{\otimes die}$ to the $\mathbb{C}$-linear map $\kappa_{die}(t)$ restricted to the appropriate domain. Each of the horizontal maps is an isomorphism and the vertical maps have compositions $\psi_{i,d}^*$ and $\psi_{i,d}^{\times}$. So we see that $\psi_{i,d}^*$ is surjective if and only if $\psi_{i,d}$ is injective.

Consider the Schur-Weyl dual of the homomorphism $\psi_{i,d}^*$ of representations of S_{die}. Using Propositions 7.31 and 7.35, we get a homomorphism

$$\mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) \to \mathrm{Sym}^d(\mathrm{Sym}^{ie}(U))$$

of representations of $\mathrm{GL}(U)$.

Proposition 7.43. The homomorphism of representations of $\mathrm{GL}(U)$

$$\mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) \to \mathrm{Sym}^d(\mathrm{Sym}^{ie}(U))$$

is a non-zero multiple of $\Psi_{i,d}$.

Proof. We may assume that the dimension of U is at least i. Our goal is the prove that the diagram

$$\begin{array}{ccc}
\mathrm{Sym}^i(\mathrm{Sym}^{de}(U)) & \xrightarrow{\lambda\Psi_{i,d}} & \mathrm{Sym}^d(\mathrm{Sym}^{ie}(U)) \\
\delta_{\mathrm{L}} \big\uparrow & & \big\uparrow \delta_{\mathrm{R}} \\
\mathrm{Hom}_{S_{die}}\left(V^{\otimes die,L_i}_{(i\times de,0)}, U^{\otimes die}\right) & \xrightarrow{\mathrm{Hom}_{S_{die}}\left(\psi_{i,d}^*, U^{\otimes die}\right)} & \mathrm{Hom}_{S_{die}}\left(V^{\otimes die,R_d}_{(0,d\times ie)}, U^{\otimes die}\right)
\end{array}$$

commutes for some $\lambda \in \mathbb{C}^*$. It suffices to show that the homomorphism

$$\ell\colon V^{\otimes die, L_i}_{(i\times de,0)} \to U^{\otimes die}$$

of representations of S_{die} such that $\ell(v_{\mathrm{L}}) = \iota_U^{i\times de}(u_{i\times de})$ also satisfies

$$\pi_U^{d\times ie}(\ell(\psi^*_{i,d}(w_{\mathrm{R}}))) = \lambda \Psi_{i,d}(u_{i\times de})$$

for some $\lambda \in \mathbb{C}^*$. One can check that

$$\psi^*_{i,d}(w_{\mathrm{R}}) = \bigotimes_{k=1}^{d} \left(\sum_{\sigma \in S_{ie}} \sigma \cdot (e_1^{\otimes e} \otimes \cdots \otimes e_i^{\otimes e}) \right).$$

Let $H^\dagger_{d\times ie}$ be the subgroup of $H_{d\times ie}$ consisting of all permutations $\sigma \in S_{die}$ such that

$$\sigma(\{(j-1)ie+1,\dots,jie\}) = \{(j-1)ie+1,\dots,jie\}$$

for all $j \in \{1,\dots,d\}$. Then we see that

$$\psi^*_{i,d}(w_{\mathrm{R}}) = \left(\sum_{\sigma \in H^\dagger_{d\times ie}} \sigma \right) \cdot \left(\bigotimes_{k=1}^{d} \left(e_1^{\otimes e} \otimes \cdots \otimes e_i^{\otimes e}\right) \right).$$

So we have

$$\pi_U^{d\times ie}(\ell(\psi^*_{i,d}(w_{\mathrm{R}}))) = \#H^\dagger_{d\times ie} \cdot \bigodot_{k=1}^{d} \left(e_1^{\circledcirc e} \circledcirc \cdots \circledcirc e_i^{\circledcirc e}\right) = \#H^\dagger_{d\times ie} \cdot \Psi_{i,d}(u_{i\times de}).$$

□

Corollary 7.44.

(i) If the homomorphism $\psi_{i,d}$ is injective, then the homomorphism $\Psi_{i,d}$ is also injective.

(i) If the homomorphism $\Psi_{i,d}$ is injective and the dimension of U is at least $\max(i,d)$, then the homomorphism $\psi_{i,d}$ is also injective.

Proof. This corollary combines the results from the previous proposition, 7.42, Corollary 7.14 and Proposition 6.49. □

7.6 Induction on d

Let $d, i \in \mathbb{Z}_{\geq 0}$ be integers such that $d \geq 1$. For each integer $n \in \mathbb{Z}_{\geq 0}$, denote the set $\{1, \dots, n\}$ by $[n]$. Our goal in this section is the prove the following theorem.

Theorem 7.45. If the homomorphism $\psi_{i,d-1}$ is injective and $i \leq d-1$, then the homomorphism

$$\psi_{i,d} \colon V^{\otimes die, L_i}_{(i \times de, 0)} \to V^{\otimes die, R_d}_{(0, d \times ie)}$$

is also injective.

To prove this theorem, we will imitate the proof of Ikenmeyer and split $\psi_{i,d}$ into two parts. Note that $\psi_{i,d}$ is the restriction of

$$(\varphi^e_{1,1} \circ \cdots \circ \varphi^e_{i,d-1}) \circ (\varphi^e_{1,d} \circ \cdots \circ \varphi^e_{i,d})$$

to $V^{\otimes die, L_i}_{(i \times de, 0)}$.

Lemma 7.46. Suppose that $\psi_{i,d}$ is injective. Then the map

$$\varphi^e_{1,1} \circ \cdots \circ \varphi^e_{i,d-1} \colon V^{\otimes die, L_i}_{(i \times (d-1)e, (0,\dots,0,ie))} \to V^{\otimes die}_{(0, d \times ie)}$$

is also injective.

Proof. Recall that $V^{\otimes die}_{(i \times (d-1)e, (0,\dots,0,ie))}$ is the span of the linearly independent set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \dots, v_{die} \in \{e_1, \dots, e_i, f_d\} \\ \#\{k | v_k = e_j\} = (d-1)e \text{ for all } j \in [i] \\ \#\{k | v_k = f_d\} = ie \end{array} \right\}$$

and $V^{\otimes die}_{(0, d \times ie)}$ is the span of the linearly independent set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \dots, v_{die} \in \{f_1, \dots, f_d\} \\ \#\{k | v_k = f_j\} = ie \text{ for all } j \in [d] \end{array} \right\}.$$

We will first decompose these spaces in to a direct sum of subspaces in such a way that the map $\varphi^e_{1,1} \circ \cdots \circ \varphi^e_{i,d-1}$ splits as well.

Let $\mathcal{S}$ be the set of subsets S of $[die]$ of size ie. For each $S \in \mathcal{S}$, let $V^{\otimes die}_{(i \times (d-1)e, (0,\dots,0,ie)), S}$ be the span of the linearly independent elements

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \dots, v_{die} \in \{e_1, \dots, e_i, f_d\} \\ \#\{k | v_k = e_j\} = (d-1)e \text{ for all } j \in [i] \\ v_k = f_d \text{ for all } k \in S \end{array} \right\},$$

take

$$V^{\otimes die, L_i}_{(i \times (d-1)e, (0,\dots,0,ie)), S} := V^{\otimes die, L_i}_{(i \times (d-1)e, (0,\dots,0,ie))} \cap V^{\otimes die}_{(i \times (d-1)e, (0,\dots,0,ie)), S}$$

and let $V^{\otimes die}_{(0,d\times ie),S}$ be the span of the linearly independent elements

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \ldots, v_{die} \in \{f_1, \ldots, f_d\} \\ \#\{k | v_k = f_j\} = ie \text{ for all } j \in [d-1] \\ v_k = f_d \text{ for all } k \in S \end{array} \right\}.$$

Then we see that the following statements hold:

- We have
$$V^{\otimes die}_{(i\times(d-1)e,(0,\ldots,0,ie))} = \bigoplus_{S\in\mathcal{S}} V^{\otimes die}_{(i\times(d-1)e,(0,\ldots,0,ie)),S}.$$

- We have
$$V^{\otimes die,L_i}_{(i\times(d-1)e,(0,\ldots,0,ie))} = \bigoplus_{S\in\mathcal{S}} V^{\otimes die,L_i}_{(i\times(d-1)e,(0,\ldots,0,ie)),S}.$$

- We have
$$V^{\otimes die}_{(0,d\times ie)} = \bigoplus_{S\in\mathcal{S}} V^{\otimes die}_{(0,d\times ie),S}$$

- We have
$$\left(\varphi^e_{1,1} \circ \cdots \circ \varphi^e_{i,d-1}\right)\left(V^{\otimes die}_{(i\times(d-1)e,(0,\ldots,0,ie)),S}\right) \subseteq V^{\otimes die}_{(0,d\times ie),S}$$

  for all $S \in \mathcal{S}$.

For each $S \in \mathcal{S}$, let

$$\ell_S \colon V^{\otimes die,L_i}_{(i\times(d-1)e,(0,\ldots,0,ie)),S} \to V^{\otimes die}_{(0,d\times ie),S}$$

be the restriction of $\varphi^e_{1,1} \circ \cdots \circ \varphi^e_{i,d-1}$ to $V^{\otimes die,L_i}_{(i\times(d-1)e,(0,\ldots,0,ie)),S}$. We will prove the lemma by proving that ℓ_S is injective for each $S \in \mathcal{S}$.

Let S be an element of $\mathcal{S}$. Let W be the subspace of V spanned by $e_1, \ldots, e_i, f_1, \ldots, f_{d-1}$, i.e., what V would be if we lower d by one. Then we see that the $\mathbb{C}$-linear maps

$$\begin{array}{rcl} V^{\otimes die}_{(i\times(d-1)e,(0,\ldots,0,ie)),S} & \to & W^{\otimes(d-1)ie}_{(i\times(d-1)e,0)} \\ v_1 \otimes \cdots \otimes v_{die} & \mapsto & \bigotimes_{k\in S} v_k \end{array}$$

and

$$\begin{array}{rcl} V^{\otimes die}_{(0,d\times ie),S} & \to & W^{\otimes(d-1)ie}_{(0,(d-1)\times ie)} \\ v_1 \otimes \cdots \otimes v_{die} & \mapsto & \bigotimes_{k\in S} v_k \end{array}$$

are isomorphisms. Visually, we just remove the tensor factor f_d, which we fixed in place before, from the notation. Now note that the diagram

$$\begin{array}{ccc} V^{\otimes die, L_i}_{(i\times(d-1)e,(0,\ldots,0,ie)),S} & \xrightarrow{\ell_S} & V^{\otimes die}_{(0,d\times ie),S} \\ \downarrow & & \downarrow \\ W^{\otimes(d-1)ie, L_i}_{(i\times(d-1)e,0)} & \xrightarrow{\psi_{i,d-1}} & W^{\otimes(d-1)ie}_{(0,(d-1)\times ie)} \end{array}$$

commutes. So since $\psi_{i,d-1}$ is injective, we see that ℓ_S is injective for each $S \in \mathcal{S}$. Hence

$$\varphi^e_{1,1} \circ \cdots \circ \varphi^e_{i,d-1} : V^{\otimes die, L_i}_{(i\times(d-1)e,(0,\ldots,0,ie))} \to V^{\otimes die}_{(0,d\times ie)}$$

is also injective. □

For the second part, we can use a part of Ikenmeyer's proof directly. Let $\mathbb{C}^2$ have basis (e, f). For each integer $n \in \mathbb{Z}_{\geq 0}$, denote the $\mathbb{C}$-linear map $\mathbb{C}^2 \to \mathbb{C}^2$ sending e to f and f to zero by ϕ and denote the $\mathbb{C}$-linear map

$$\begin{array}{rcl} (\mathbb{C}^2)^{\otimes n} & \to & (\mathbb{C}^2)^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n & \mapsto & \displaystyle\sum_{h=1}^{n} v_1 \otimes \cdots \otimes \phi(v_h) \otimes \cdots \otimes v_n \end{array}$$

by φ.

Lemma 7.47 (Ikenmeyer)**.** Let $a, b \in \mathbb{Z}_{\geq 0}$ be integers be such that $a \geq b$. Then the map

$$\varphi\colon (\mathbb{C}^2)^{\otimes a+b}_{(a+1,b-1)} \to (\mathbb{C}^2)^{\otimes a+b}_{(a,b)}$$

is injective.

Proof. See the map ζ in the proof of Claim 5.1 of [Ik]. □

Lemma 7.48. If we have $i \leq d-1$, then the map

$$\varphi^e_{1,d} \circ \cdots \circ \varphi^e_{i,d}\colon V^{\otimes die}_{(i\times de,0)} \to V^{\otimes die}_{(i\times(d-1)e,(0,\ldots,0,ie))}$$

is injective.

Proof. For all integers $k \in \{1, \ldots, i\}$ and $h \in \{0, \ldots, e\}$, denote the vector

$$(de, \ldots, de, (d-1)e + h, (d-1)e, \ldots, (d-1)e) \in \mathbb{Z}^i_{\geq 0}$$

by $\gamma_{k,h}$ where the value $(d-1)e + h$ is in the k-th entry. We will prove the lemme by proving that for all integers $k \in \{1, \ldots, i\}$ and $h \in \{1, \ldots, e\}$, the map

$$\varphi_{k,d}\colon V^{\otimes die}_{(\gamma_{k,h},(0,\ldots,0,(i-k+1)e-h))} \to V^{\otimes die}_{(\gamma_{k,h-1},(0,\ldots,0,(i-k+1)-h+1))}$$

is injective.

Let $k \in \{1, \ldots, i\}$ and $h \in \{1, \ldots, e\}$ be integers. Note that the vector space $V^{\otimes die}_{(\gamma_{k,h},(0,\ldots,0,(i-k+1)e-h))}$ is the span of the linearly independent set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \ldots, v_{die} \in \{e_1, \ldots, e_i, f_d\} \\ \#\{a | v_a = e_b\} = de \text{ for all } b \in \{1, \ldots, k-1\} \\ \#\{a | v_a = e_k\} = (d-1)e + h \\ \#\{a | v_a = e_b\} = (d-1)e \text{ for all } b \in \{k+1, \ldots, i\} \\ \#\{a | v_a = f_d\} = (i-k+1)e - h \end{array} \right\}$$

and $V^{\otimes die}_{(\gamma_{k,h-1},(0,\ldots,0,(i-k+1)-h+1))}$ is the span of the linearly independent set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \ldots, v_{die} \in \{e_1, \ldots, e_i, f_d\} \\ \#\{a | v_a = e_b\} = de \text{ for all } b \in \{1, \ldots, k-1\} \\ \#\{a | v_a = e_k\} = (d-1)e + h - 1 \\ \#\{a | v_a = e_b\} = (d-1)e \text{ for all } b \in \{k+1, \ldots, i\} \\ \#\{a | v_a = f_d\} = (i-k+1)e - h + 1 \end{array} \right\}.$$

Let $\mathcal{S}$ be the set of subsets S of $[die]$ of size $(d+i-k-1)e$. For each element $S \in \mathcal{S}$, let Ω_S be the set of functions $r \colon S \to \{e_1, \ldots, e_i\} - \{e_k\}$ such that $\#r^{-1}(e_j) = de$ if $j < k$ and $\#r^{-1}(e_j) = (d-1)e$ if $j > k$. Now for all $S \in \mathcal{S}$ and $r \in \Omega_S$, denote the span of the linearly independent set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \ldots, v_{die} \in \{e_1, \ldots, e_i, f_d\} \\ \#\{a | v_a = e_b\} = de \text{ for all } b \in \{1, \ldots, k-1\} \\ \#\{a | v_a = e_k\} = (d-1)e + h \\ \#\{a | v_a = e_b\} = (d-1)e \text{ for all } b \in \{k+1, \ldots, i\} \\ \#\{a | v_a = f_d\} = (i-k+1)e - h \\ v_a = r(a) \text{ for all } a \in S \end{array} \right\}$$

by $V^{\otimes die}_{(\gamma_{k,h},(0,\ldots,0,(i-k+1)e-h)),S,r}$ and denote the span of the linearly independent set

$$\left\{ v_1 \otimes \cdots \otimes v_{die} \;\middle|\; \begin{array}{l} v_1, \ldots, v_{die} \in \{e_1, \ldots, e_i, f_d\} \\ \#\{a | v_a = e_b\} = de \text{ for all } b \in \{1, \ldots, k-1\} \\ \#\{a | v_a = e_k\} = (d-1)e + h - 1 \\ \#\{a | v_a = e_b\} = (d-1)e \text{ for all } b \in \{k+1, \ldots, i\} \\ \#\{a | v_a = f_d\} = (i-k+1)e - h + 1 \\ v_a = r(a) \text{ for all } a \in S \end{array} \right\}$$

by $V^{\otimes die}_{(\gamma_{k,h-1},(0,\ldots,0,(i-k+1)-h+1)),S,r}$. Then we see that the following statements hold:

- We have

$$V^{\otimes die}_{(\gamma_{k,h},(0,\ldots,0,(i-k+1)e-h))} = \bigoplus_{S \in \mathcal{S}} \bigoplus_{r \in \Omega_S} V^{\otimes die}_{(\gamma_{k,h},(0,\ldots,0,(i-k+1)e-h)),S,r}.$$

- We have

$$V^{\otimes die}_{(\gamma_{k,h-1},(0,\dots,0,(i-k+1)-h+1))} = \bigoplus_{S\in\mathcal{S}} \bigoplus_{r\in\Omega_S} V^{\otimes die}_{(\gamma_{k,h-1},(0,\dots,0,(i-k+1)-h+1)),S,r}.$$

- For all $S \in \mathcal{S}$ and $r \in \Omega_S$, we have

$$\varphi_{k,d}\left(V^{\otimes die}_{(\gamma_{k,h},(0,\dots,0,(i-k+1)e-h)),S,r}\right) \subseteq V^{\otimes die}_{(\gamma_{k,h-1},(0,\dots,0,(i-k+1)-h+1)),S,r}.$$

For all $S \in \mathcal{S}$ and $r \in \Omega_S$, let

$$\ell_{S,r} \colon V^{\otimes die}_{(\gamma_{k,h},(0,\dots,0,(i-k+1)e-h)),S,r} \to V^{\otimes die}_{(\gamma_{k,h-1},(0,\dots,0,(i-k+1)-h+1)),S,r}$$

be the restriction of $\varphi_{k,d}$ to $V^{\otimes die}_{(\gamma_{k,h},(0,\dots,0,(i-k+1)e-h)),S,r}$. Then it suffices to prove that $\ell_{S,r}$ is injective for all $S \in \mathcal{S}$ and $r \in \Omega_S$.

Let $S \in \mathcal{S}$ and $r \in \Omega_S$ be elements. Let W be the subspace of V spanned by e_k and f_d. Then the $\mathbb{C}$-linear maps

$$\begin{array}{rcl} V^{\otimes die}_{(\gamma_{k,h},(0,\dots,0,(i-k+1)e-h)),S,r} & \to & W^{\otimes(d+i-k-1)e}_{((d-1)e+h,(i-k+1)e-h)} \\ v_1 \otimes \cdots \otimes v_{die} & \mapsto & \bigotimes_{a\in S} v_a \end{array}$$

and

$$\begin{array}{rcl} V^{\otimes die}_{(\gamma_{k,h-1},(0,\dots,0,(i-k+1)-h+1)),S,r} & \to & W^{\otimes(d+i-k-1)e}_{((d-1)e+h-1,(i-k+1)e-h+1)} \\ v_1 \otimes \cdots \otimes v_{die} & \mapsto & \bigotimes_{a\in S} v_a \end{array}$$

are isomorphisms and the diagram

$$\begin{array}{ccc} V^{\otimes die}_{(\gamma_{k,h},(0,\dots,0,(i-k+1)e-h)),S,r} & \xrightarrow{\ \ell_{S,r}\ } & V^{\otimes die}_{(\gamma_{k,h-1},(0,\dots,0,(i-k+1)-h+1)),S,r} \\ \downarrow & & \downarrow \\ W^{\otimes(d+i-k-1)e}_{((d-1)e+h,(i-k+1)e-h)} & \xrightarrow{\ \varphi\ } & W^{\otimes(d+i-k-1)e}_{((d-1)e+h-1,(i-k+1)e-h+1)} \end{array}$$

commutes. We have

$$(d-1)e + h - 1 \geq (i-k+1)e - h + 1,$$

because $1 \leq k$, $1 \leq h$ and $i \leq d-1$. So by the previous lemma, we see that φ and $\ell_{S,r}$ are injective. □

Proof of Theorem 7.45. We see that by combining Lemma's 7.46 and 7.48, we get Theorem 7.45. □

Corollary 7.49. For all integers $i, d \in \mathbb{Z}_{\geq 0}$ such that $i \leq 2$, the map $\Psi_{i,d}$ is injective.

Proof. We know from Section 7.4 that the maps $\Psi_{1,1}$ and $\Psi_{2,2}$ are injective for all vector spaces U of dimension al least two. Therefore the maps $\psi_{1,1}$ and $\psi_{2,2}$ are injective by Corollary 7.44. So by Theorem 7.45, the map $\psi_{i,d}$ is injective for all integers $i, d \in \mathbb{Z}_{\geq 0}$ such that $i \leq 2$. So the map $\Psi_{i,d}$ is injective for all integers $i, d \in \mathbb{Z}_{\geq 0}$ such that $i \leq 2$ by Corollary 7.44. $\square$

Remark 7.50. Let $i, d \in \mathbb{Z}_{\geq 0}$ be integers and consider the dual

$$\Psi_{i,d}^{\times}\colon \operatorname{Sym}^d(\operatorname{Sym}^{ie}(U))^{\times} \to \operatorname{Sym}^i(\operatorname{Sym}^{de}(U))^{\times}$$

of the $\mathbb{C}$-linear map $\Psi_{i,d}$. By applying the isomorphism from Lemma 2.30(e) repeatedly, we can identify the map $\Psi_{i,d}^{\times}$ with a map

$$\operatorname{Sym}^d(\operatorname{Sym}^{ie}(U^{\times})) \to \operatorname{Sym}^i(\operatorname{Sym}^{de}(U^{\times})).$$

Let $\kappa\colon U \to U^{\times}$ be any isomorphism of vector spaces. Then one can check that the diagram

$$\begin{array}{ccc}
\operatorname{Sym}^d(\operatorname{Sym}^{ie}(U^{\times})) & \longrightarrow & \operatorname{Sym}^i(\operatorname{Sym}^{de}(U^{\times})) \\
{\scriptstyle \operatorname{Sym}^d(\operatorname{Sym}^{ie}(\kappa))}\uparrow & & \uparrow{\scriptstyle \operatorname{Sym}^i(\operatorname{Sym}^{de}(\kappa))} \\
\operatorname{Sym}^d(\operatorname{Sym}^{ie}(U)) & \xrightarrow{\Psi_{d,i}} & \operatorname{Sym}^i(\operatorname{Sym}^{de}(U))
\end{array}$$

commutes. So we see that the map $\Psi_{i,d}$ is injective if and only if the map $\Psi_{d,i}$ is surjective. So the previous corollary is equivalent to the statement that for all integers $i, d \in \mathbb{Z}_{\geq 0}$ such that $i \leq 2$, the map $\Psi_{d,i}$ is surjective. For $i = 2$, this is the statement of Theorem 1.1 from [AC1].

Chapter 8

Other methods and things left to do

There are still a lot of things to be done. First of all is of course proving or disproving the conjectures. More concrete examples of things that we still have not explored in this thesis are the following questions.

(i) We defined for each integer $n \in \mathbb{Z}_{\geq 0}$ the vector space V_n as the homogeneous part of degree n of the graded K-algebra $K[x, y]$. What changes if we replace $K[x, y]$ by $K[x_1, \ldots, x_m]$ for some integer $m \in \mathbb{Z}_{\geq 3}$?

(ii) Does the previous chapter generalize from the field $\mathbb{C}$ to algebraically closed fields K with the property that $\mathrm{char}(K)$ does not divide $n!$ for some integer $n \in \mathbb{Z}_{\geq 0}$ chosen sufficiently high?

(iii) We know that the ideals J_d and I_d become equal after dehomogenisation with respect to c_{de}. So the ideal I_d is equal to the homogenisation of the dehomogenisation of J_d. Can we somehow use this fact to prove that the the ideals J_d and I_d are equal?

In this chapter, we list some more interesting ideas which might be of some use to people working on this problem.

8.1 Lie algebras

In this section, we work over the field $\mathbb{C}$. Each representation of $\mathrm{GL}_2(\mathbb{C})$ also naturally has the structure of a representation of $\mathrm{SL}_2(\mathbb{C})$ and each homomorphism of representations of $\mathrm{GL}_2(\mathbb{C})$ is also a homomorphism of representations of $\mathrm{SL}_2(\mathbb{C})$. This allows us to consider all homomorphisms of representations of $\mathrm{GL}_2(\mathbb{C})$ we have seen in this thesis as homomorphisms of representations of $\mathrm{SL}_2(\mathbb{C})$. Representations of $\mathrm{SL}_2(\mathbb{C})$ are closely related to representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. It is well known that all finite-dimensional representations of $\mathrm{SL}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{C})$ are completely reducible and that the representations from 7.5 give rise to a complete family of finite-dimensional irreducible representations. See [Hu] for more information.

Example 8.1. As an example of why using this approach might be useful, recall that in Corollary 7.49 we have proved that certain homomorphisms of the form

$$\mathrm{Sym}^2(\mathrm{Sym}^{de}(V_1)) \to \mathrm{Sym}^{2e}(\mathrm{Sym}^d(V_1))$$

are injective where $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$ are integers. It is well know that the representation $\mathrm{Sym}^2(\mathrm{Sym}^k(V_1))$ is isomorphic to

$$\bigoplus_{h=0}^{\lfloor k/2 \rfloor} \mathrm{Sym}^{2k-4h}(V_1)$$

as representations of $\mathfrak{sl}_2(\mathbb{C})$ for each $k \in \mathbb{Z}_{\geq 0}$. So using Schur's Lemma, we see that to check the injectivity of a homomorphism of the form

$$\mathrm{Sym}^2(\mathrm{Sym}^{de}(V_1)) \to \mathrm{Sym}^{2e}(\mathrm{Sym}^d(V_1))$$

it suffices to check that for each $h \in \{0, \dots, \lfloor k/2 \rfloor\}$ some non-zero element of the summand $\mathrm{Sym}^{2k-4h}(V_1)$ is not sent to zero.

For more results obtained using this approach on problems the same or similar to the ones we discussed in this thesis, see for example [AC1] and [AC2].

8.2 Howe's isomorphism

Let K be an algebraically closed field and let $d \in \mathbb{Z}_{\geq 0}$ and $e \in \mathbb{Z}_{\geq 1}$ be integers. In this section, we generalize a construction of Roger Howe from [Ho].

8.2. Consider the map

$$\begin{array}{rcl} \alpha\colon V_d^e & \to & V_{de} \\ (f_1, \dots, f_e) & \mapsto & f_1 \cdots f_e. \end{array}$$

Let $(c_0, \dots, c_{de})$ be the dual basis of $(y^{de}, xy^{de-1} \dots, x^{de})$. For each integer $i \in \{1, \dots, e\}$, let $\pi_i\colon V_d^e \to V_d$ be the projection map on the i-th factor. Let $(b_0, \dots, b_d)$ be the dual basis of $(y^d, xy^{d-1}, \dots, x^d)$. For all integers $i \in \{1, \dots, e\}$ and $j \in \{0, \dots, d\}$, take $b_{i,j} = b_j \circ \pi_i$. Then

$$(b_{i,j} | i \in \{1, \dots, e\}, j \in \{0, \dots, d\})$$

is a basis of $(V_d^e)^\times$. For each $(f_1, \dots, f_e) \in V_d^e$, we have

$$\alpha(f_1, \dots, f_e) = \sum_{k=0}^{de} \left(\sum_{\substack{0 \leq j_1, \dots, j_e \leq e \\ j_1 + \dots + j_e = k}} b_{j_1}(f_1) \cdots b_{j_e}(f_e) \right) x^k y^{de-k}.$$

So we see that for all $k \in \{0, \ldots, de\}$, the function $c_k \circ \alpha \colon V_d^e \to K$ is the polynomial function on V_d^e associated to the polynomial

$$\sum_{\substack{0 \leq j_1, \ldots, j_e \leq e \\ j_1 + \cdots + j_e = k}} b_{1,j_1} \odot \cdots \odot b_{e,j_e}$$

on V_d^e. So α is a homogeneous polynomial map of degree e.

By the fundamental theorem of algebra, we see that α is surjective. By Proposition 2.40, the kernel of the homomorphism of K-algebras

$$\begin{array}{rcl} \alpha^* \colon P(V_{de}) & \to & P(V_d^e) \\ c_k & \mapsto & \displaystyle\sum_{\substack{0 \leq j_1, \ldots, j_e \leq e \\ j_1 + \cdots + j_e = k}} b_{1,j_1} \odot \cdots \odot b_{e,j_e} \end{array}$$

equals the ideal of $P(V_{de})$ corresponding to the image of α. So since α is surjective, we see that α^* is injective, because $I_{\mathbb{A}(V_{de})}(\mathbb{A}(V_{de})) = 0$.

The vector space $P(V_d)^{\otimes e}$ is a K-algebras with the rule

$$(f_1 \otimes \cdots \otimes f_e) \cdot (g_1 \otimes \cdots \otimes g_e) = (f_1 g_1 \otimes \cdots \otimes f_e g_e)$$

for all $f_1, \ldots, f_e, g_1, \ldots, g_e \in P(V_d)$. For all integers $i \in \{1, \ldots, e\}$ and polynomials $f \in P(V_d)$, note that the function $f \circ \pi_i$ is a polynomial function on V_d^e. We identify this polynomial function with its corresponding polynomial. The group S_e acts on $P(V_d)^{\otimes e}$ by permuting the tensor factors with the homomorphism

$$\begin{array}{rcl} S_e & \to & \mathrm{GL}\left(P(V_d)^{\otimes e}\right) \\ \sigma & \mapsto & \left(f_1 \otimes \cdots \otimes f_e \mapsto f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(e)}\right). \end{array}$$

The homomorphism $S_e \to \mathrm{GL}\left(P(V_d^e)\right)$ sending σ to the endomorphism of K-algebras

$$\begin{array}{rcl} P(V_d^e) & \to & P(V_d^e) \\ b_{i,j} & \mapsto & b_{\sigma^{-1}(i),j} \end{array}$$

gives us an action of S_e on $P(V_d^e)$.

Lemma 8.3. The map

$$\begin{array}{rcl} \ell \colon P(V_d)^{\otimes e} & \to & P(V_d^e) \\ f_1 \otimes \cdots \otimes f_e & \mapsto & (f_1 \circ \pi_1) \odot \cdots \odot (f_e \circ \pi_e) \end{array}$$

is an isomorphism of K-algebras and an isomorphism of representations.

Proof. The homomorphism of K-algebras $P(V_d^e) \to P(V_d)^{\otimes e}$ extending the K-linear map

$$\begin{array}{rcl} (V_d^e)^\times & \to & P(V_d)^{\otimes e} \\ b_{i,j} & \mapsto & 1 \otimes \cdots \otimes 1 \otimes b_j \otimes 1 \otimes \cdots \otimes 1 \end{array}$$

to $P(V_d^e)$ is the inverse. $\square$

8.4. Let $h \in \mathbb{Z}_{\geq 0}$ be an integer. Then the map ℓ restricts to an injective map

$$\ell' \colon \left(\mathrm{Sym}^h(V_d^\times)^{\otimes e}\right)^{S_e} \to \mathrm{Sym}^{he}((V_d^e)^\times)^{S_e}.$$

The map α^* gives us an injective map

$$\alpha^*_{(h)} \colon \mathrm{Sym}^h(V_{de}^\times) \to \mathrm{Sym}^{he}((V_d^e)^\times).$$

One can check that the image of $\alpha^*_{(h)}$ is contained in the image if ℓ'. It follows that there exists a unique injective map

$$\mathrm{Sym}^h(V_{de}^\times) \to \left(\mathrm{Sym}^h(V_d^\times)^{\otimes e}\right)^{S_e}$$

making the diagram

$$\begin{array}{ccc} \mathrm{Sym}^h(V_{de}^\times) & \longrightarrow & \left(\mathrm{Sym}^h(V_d^\times)^{\otimes e}\right)^{S_e} \\ & \searrow^{\alpha^*_{(h)}} & \downarrow \ell' \\ & & \mathrm{Sym}^{he}((V_d^e)^\times)^{S_e} \end{array}$$

commute.

Suppose that the characteristic of K does not divide $e!$. Then we can identify $\left(\mathrm{Sym}^h(V_d^\times)^{\otimes e}\right)^{S_e}$ with $\mathrm{Sym}^e(\mathrm{Sym}^h(V_d^\times))$. This gives us an injective map

$$\mathrm{Sym}^h(V_{de}^\times) \to \mathrm{Sym}^e(\mathrm{Sym}^h(V_d^\times)).$$

The dimension of its domain is

$$\binom{h+de}{h}$$

and the dimension of it codomain is

$$\binom{e + \binom{h+d}{d} - 1}{e}.$$

Take $d = 1$. Then the map is an isomorphism. This is the statement of 5.4.1.1 of [Ho]. Therefore we call this isomorphism Howe's isomorphism.

8.3 More on polynomial maps

Let K be a field and let V, W be vector spaces over K.

8.5. Let $\alpha\colon V \to W$ be a polynomial map and let v be an element of V. Then the map $V \to V$ sending x to $x+v$ is polynomial. Therefore the composition

$$\begin{array}{rcl} V & \to & W \\ x & \mapsto & \alpha(x+v) \end{array}$$

is a polynomial map. The difference between polynomial maps is a polynomial map. Hence the function

$$\begin{array}{rcl} \Delta_v\alpha\colon V & \to & W \\ x & \mapsto & f(x+v)-f(x) \end{array}$$

is also a polynomial map.

Let $k \in \mathbb{Z}_{\geq 0}$ be an integer and let $\delta\colon V \to \mathrm{Sym}^k(V)$ be the map sending v to $v^{\odot k}$. Then for all K-linear maps $\ell\colon \mathrm{Sym}^k(V) \to W$, the map $\ell \circ \delta$ is zero or a homogeneous polynomial map of degree k by Proposition 2.32.

Theorem 8.6 (Cartan)**.** Let $\ell\colon \mathrm{Sym}^k(V) \to W$ be a K-linear map and take $\alpha = \ell \circ \delta$. Then for all $v_1, \ldots, v_k \in V$, the polynomial map $\Delta_{v_1} \cdots \Delta_{v_k}\alpha$ is constant with value $k! \cdot \ell(v_1 \odot \cdots \odot v_k)$.

Proof. See Theorem 6.3.1(ii) from [Ca]. $\square$

Corollary 8.7. Let $k \in \mathbb{Z}_{\geq 0}$ be an integer such that $\mathrm{char}(K) \nmid k!$. Then the set $\{v^{\odot k} | v \in V\}$ spans $\mathrm{Sym}^k(V)$.

Proof. We apply the theorem with ℓ the identity map. We see that for all $v_1, \ldots, v_k \in V$, we have

$$v_1 \odot \cdots \odot v_k = \frac{1}{k!}\left(\Delta_{v_1} \cdots \Delta_{v_k}\delta\right)(0).$$

Using induction on k, one can check that

$$\left(\Delta_{v_1} \cdots \Delta_{v_k}\delta\right)(x) = \sum_{I \subseteq \{1,\ldots,k\}} (-1)^{k-\#I}\delta\left(x + \sum_{i \in I} v_i\right)$$

for all $x \in V$. So we see that $v_1 \odot \cdots \odot v_k$ is a linear combination of the elements

$$\left\{ \left(\sum_{i \in I} v_i\right)^{\odot k} \middle| I \subseteq \{1, \ldots, k\} \right\}.$$

$\square$

8.8. Let $d, i \in \mathbb{Z}_{\geq 0}$ be integers, let V be a vector space over $\mathbb{C}$ and let $\Phi_V^{i,d}$ be the $\mathbb{C}$-linear map from 7.3 making the diagram

$$\begin{array}{ccc} & V^{\otimes die} & \\ {\scriptstyle \iota_V^{i\times de}} \nearrow & & \searrow {\scriptstyle \pi_V^{ie\times d}} \\ \mathrm{Sym}^i(\mathrm{Sym}^{de}(V)) & \xrightarrow{\Phi_V^{i,d}} & \mathrm{Sym}^{ie}(\mathrm{Sym}^d(V)) \end{array}$$

commute. We want to prove that $\Phi_V^{i,d}$ is surjective for $i \geq d$ if the dimension of V equals two. By the previous corollary, we know that $\mathrm{Sym}^{de}(V)$ is spanned by $\{v^{\circledcirc de} | v \in V\}$. If a subset S of a vector space W spans W, then one can easily check that the set

$$\{w_1^{\circledcirc de} \odot \cdots \odot w_i^{\circledcirc de} | w_1, \ldots, w_i \in S\}$$

spans $\mathrm{Sym}^i(W)$. So we see that $\mathrm{Sym}^i(\mathrm{Sym}^{de}(V))$ is spanned by

$$\{v_1^{\circledcirc de} \odot \cdots \odot v_i^{\circledcirc de} | v_1, \ldots, v_i \in V\}.$$

Let $v_1, \ldots, v_i$ be elements of V. Then we have

$$\iota_V^{i\times de}(v_1^{\circledcirc de} \odot \cdots \odot v_i^{\circledcirc de}) = \frac{1}{i!} \sum_{\sigma \in S_i} \bigotimes_{j=1}^{i} v_{\sigma(j)}^{\otimes de}$$

and hence

$$\Phi_V^{i,d}(v_1^{\circledcirc de} \odot \cdots \odot v_i^{\circledcirc de}) = (v_1^{\circledcirc d})^{\odot e} \odot \cdots \odot (v_i^{\circledcirc d})^{\odot e}.$$

Since $\mathrm{Sym}^d(V)$ is spanned by $\{v^{\circledcirc d} | v \in V\}$, we know that $\mathrm{Sym}^{ie}(\mathrm{Sym}^d(V))$ is spanned by

$$\{v_1^{\circledcirc d} \odot \cdots \odot v_{ie}^{\circledcirc d} | v_1, \ldots, v_{ie} \in V\}$$

and now we see that to prove that the map $\Phi_V^{i,d}$ is surjective, it suffices to prove that $\mathrm{Sym}^{ie}(\mathrm{Sym}^d(V))$ is spanned by

$$\{(v_1^{\circledcirc d})^{\odot e} \odot \cdots \odot (v_i^{\circledcirc d})^{\odot e} | v_1, \ldots, v_i \in V\}.$$

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