Processes with independent increments.

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Master Thesis

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Date master exam: 29 January 2016

Mathematisch Instituut, Universiteit Leiden

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1 Introduction

> ”Quien olvida su historia está condenado a repetirla”.
>
> *Jorge Agustín Nicolás Ruiz de Santayana y Borrás*

The aim of this thesis is to understand the sample path structure of processes with independent increments. The study of such a process goes back to [14, ChapitreVII, p-158]:

> ”Ce problème constitue une extension naturelle de celui des sommes ou sŕies à termes aléatoires indépendants”

A process with independent increments is the continuous time extension of the random walk $S_n = \sum_{i=1}^{n} X_i$ of independent random variables.

The French mathematician Paul Lévy studied processes with independent increments. Nowadays Lévy processes are defined to be processes with *stationary, independent* and some additional assumptions, see [20, Definition 1.6]. However Lévy determined and gave the ideas for investigating the path-wise behavior of processes with independent increments without assuming stationarity and the additional assumptions. We will consider processes with independent increments under minimal conditions.

The theory of processes with independent increments is connected with a limit theorem, see Theorem 2.1, for sums of independent random variables. This result is obtained from [6]. This limit theorem deserves in our opinion most of the attention for understanding the sample path structure of processes with independent increments. In Section 2.2 we will prove this theorem for sums of real-valued independent random variables. Furthermore with [12] we will extend the result for sums of independent, Banach space valued random variables, see Theorem 2.4. In Section 2.3, 2.4 we will use Theorem 2.4 to find the first regularity properties of sample paths. Also with Theorem 2.4 we are able to subtract jumps at fixed times. Then we are left with a process with independent increments that is continuous in probability, which we will call *additive processes*. In Section 2.5 we will show that for such a process there exists a càdlàg modification.

In 1942 Kiyosi Ito, in his first paper [10], succeeded in realizing an idea of Paul Lévy to describe the structure of additive processes. The fundamental theorem describing the path-wise structure of real-valued additive processes is the so called *Lévy-Ito decomposition*. For a complete proof and analysis we refer to [11]. For additive processes with *stationary* increments, nowadays martingale arguments are added to the analysis. We refer to [1],[4]. We will follow the path-wise approach to understand the Lévy-Ito decomposition for additive processes with values in separable Banach spaces. We will follow closely the analysis as in [10],[9] for the one dimensional case. In section 3.2,3.3 we analyze the jumps of additive processes. Using Theorem 2.4 and Theorem 3.9 we are able to prove Theorem

3.10. With the aid of Theorem 3.10 we are able decompose a general additive process with values in a separable Banach space E in a continuous part and a jump part.

Theorem 3.10 is a similar result as Theorem 2.1. Theorem 2.1 is used to subtract. jumps at fixed times. Theorem 3.10 is used to subtract jumps at random times. The use of Theorem 3.10 makes our approach different from the literature. At the same time Theorem 3.10 is inspired by Theorem 2.1, due to Paul Lévy, and proven with the aid of a recent (2013) result from [3].

2 Processes with independent increments

2.1 Introduction

We will consider processes with independent increments. We always let E be a separable Banach space unless otherwise stated. An E-valued stochastic process $\{X_t\}_{t\in\mathbb{R}_+}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a process with *independent increments* if for all $t_1 < t_2 < \ldots < t_n$ in $\mathbb{R}_+$ the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent. Let $\mathcal{F}_t := \sigma\{X_u : u \leq t\}$ be the σ-algebra generated by all random variables X_u with $u \in [0,t]$. A Stochastic process has independent increments if for all $s, t \in \mathbb{R}_+$ with $s < t$, the random variable $X_t - X_s$ is independent of $\mathcal{F}_s$. In most general form we define processes with independent increments as follows.

Definition 2.1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, \mathbb{P})$ be a filtered probability space and X be an adapted stochastic process with state space E. We call X a process with independent increments if the following conditions hold:

1. for every $\omega \in \Omega$, $X_0(\omega) = 0$;

2. for every $s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$.

Example 2.1. *Let $\{\tau_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{R}_+$ with $\lim_n \tau_n = \infty$. Let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of independent random variables. Let $S_t^- := \sum_{\tau_n \leq t} Z_n$ and $S_t^+ := \sum_{\tau_n < t} Z_n$, then $\{S_t^\pm\}_{t\in\mathbb{R}_+}$ are processes with independent increments. We call them pure jump processes.*

A sample path of a process with independent increments has no reason for being regular. The first natural question then immediatly arises: do paths have regularity properties? If in addition it is assumed that $\{X_t\}_{t\in\mathbb{R}_+}$ has the continuity in probability property, then there exists a modification with all paths càdlàg. This will be the content of Section 2.5. If stochastic continuity is not assumed, then there is a night and day difference.

A primary tool for analyzing processes with independent increments are characteristic functions. For the moment we take $E = \mathbb{R}^d$. For $0 = t_1 < t_2 < \ldots < t_n$ in $\mathbb{R}_+$ let $\mu_{t_1,t_2,\ldots,t_n}$ denote the distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$. For $s < t$ let $\varphi(s,t)(u)$ denote the characteristic function of $X_t - X_s$, i.e. for every $u \in \mathbb{R}^d$

$$\varphi(s,t)(u) := \mathbb{E}\left[e^{i\langle u, X_t - X_s\rangle}\right]. \tag{1}$$

For a definition and general properties of characteristic functions, see section 4.4. By independence of increments we find by Theorem 4.6 for $s < h < t$,

$$\varphi(s,h)(u) = \varphi(s,t)(u)\varphi(t,h)(u). \tag{2}$$

The distribution of $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ is uniquely determined by the characteristic function $\Phi_{(X_{t_1},X_{t_2},\ldots,X_{t_n})}(u),\ u \in \mathbb{R}^{nd}$ and is fully determined by the increments of $\{X_t\}_{t\in\mathbb{R}_+}$

$$\Phi_{(X_{t_1},\ldots,X_{t_n})}(u) = \prod_{i=1}^{n-1} \Phi_{(X_{t_{i+1}} - X_{t_i})}\left(\sum_{l=i}^{n} u_l\right), \quad \forall u \in \mathbb{R}^{nd}, \tag{3}$$

and $\Phi_{X_{t_1}}(u) = 1$.

2.2 Sums of independent random variables

We note that the theory of processes with independent increments is connected with limits of sums of independent random variables. Indeed for every choice $t_1 < t_2 < \ldots < t_n < t$ of time points we can represent X_t by

$$X_t = X_{t_1} + \sum_{i=1}^{n-1} \left(X_{t_{i+1}} - X_{t_i} \right) + \left(X_t - X_{t_n} \right), \tag{4}$$

which is a sum of independent random variables. If we approximate t b $\{t_n\}_{n\in\mathbb{N}}$, then $\lim_n X_{t_n}$ is a limit of sums of independent random variables. We state an important result for sums of independent random variables with values in a Banach space $(E, \|\cdot\|)$, see [11, chapter 1.3, Lemma 2.].

Remark 2.1. The sum of two random variables X, Y with values in a general Banach space $(E, \|\cdot\|)$ is not trivially a random variable. If we however assume E to be separable, then the collection of random variables is closed under summation. The following lemma holds for Banach spaces $(E, \|\cdot\|)$ where the collection of random variables is closed under summation.

Lemma 2.1. *Let $X_1, X_2 \ldots, X_N$ be independent random variables and $S_n = \sum_{i=1}^n X_i$, for $n = 1, \ldots, N$. Suppose that for some $a > 0$, $\mathbb{P}(||S_n|| > a) \leq \delta < \frac{1}{2}$ for all $n = 1, \ldots, N$. Then it holds that*

$$\mathbb{P}(\max_{0\leq p,q\leq N} ||S_p - S_q|| > 4a) \leq 4\mathbb{P}(||S_N|| > a).$$

Proof. By the triangle inequality $||S_k - S_l|| \leq ||S_N - S_k|| + ||S_N - S_l||$ it holds that

$$\mathbb{P}(\max_{1\leq k,l\leq N} ||S_k - S_l|| > 4a) \leq 2\mathbb{P}\left\{\max_{0\leq k\leq N} ||S_N - S_k|| > 2a\right\}.$$

Consider the events $A_k = \{||S_k|| \leq a\}$, $B_k = \left\{\max_{k<i\leq N} ||S_N - S_i|| \leq 2a, ||S_N - S_k|| > 2a\right\}$. The events B_k are disjoint and A_k, B_k are independent. Since $||S_N - S_k|| > 2a$ and $||S_k|| \leq a$ imply $||S_N|| > a$ it holds that $\{||S_N|| > a\} \supset \bigcup_k A_k \cap B_k$. Then it holds that

$$\mathbb{P}\left(||S_N|| > a\right) \geq \sum_k \mathbb{P}(A_k \cap B_k) = \sum_k \mathbb{P}(A_k)\mathbb{P}(B_k)$$

$$\geq (1-\delta)\sum_k \mathbb{P}(B_k) = (1-\delta)\mathbb{P}\left(\bigcup_k B_k\right) \geq \frac{1}{2}\mathbb{P}\left(\sup_{0\leq k\leq N} ||S_N - S_k|| > 2a\right)$$

which completes the proof.

□

We will often use a symmetrisation method in order to gain insights in sample path properties of processes with independent increments.

Definition 2.2. Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\in T}, \mathbb{P}\right)$ be a probability space and $\{X_t\}_{t\in T}$ be an adapted stochastic process. Let $\left(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_s\}_{s\in T}, \bar{\mathbb{P}}\right)$ and $\{\bar{X}_t\}_{t\in T}$ be independent copies. We define the product space

$$(\Omega^*, \mathcal{F}^*, \{\mathcal{F}^*_s\}_{s\in T}, \mathbb{P}^*) = \left(\Omega \times \bar{\Omega}, \mathcal{F} \otimes \bar{\mathcal{F}}, \{\mathcal{F}_s \otimes \bar{\mathcal{F}}_s\}_{s\in T}, \mathbb{P} \otimes \bar{\mathbb{P}}\right),$$

and the symmetrization of X_t as $X^s_t(\omega^*) = X_t(\omega) - \bar{X}_t(\bar{\omega}), \quad \forall \omega^* = (\omega, \bar{\omega}) \in \Omega^*$.

Remark 2.2. One important property of the symmetrization is $\Phi_{X^s}(u) = |\Phi_X|^2(u), \ \forall u \in \mathbb{R}$.

The importance of the following Theorem is clear from Eg. (4). The result is from [6]. The proof is partly taken from Doob.

Theorem 2.1. *Let $X_1, \dots, X_n, \dots$ be a sequence of independent random variables in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Suppose there is a random variables X so that for every $k = 1, 2, \dots$ the random variable Δ_k given by*

$$\Delta_k = X - \sum_{i=1}^{k} X_i \quad a.s,$$

and δ_k independent of $X_1, \dots, X_k$. Then there are constants m_k for $k = 1, 2, \dots$ such that

$$\lim_{N\to\infty} \sum_{k=1}^{N} (X_i - m_i),$$

exists with probability 1.

Proof. Let $\bar{X}_n$ be an exact copy of X_n (as in Definition 2.2) and let $X^s_n = X_n - \bar{X}_n$ be the symmetrisation of X_n. Define the sum $S^s_N = \sum_{n=1}^{N} X^s_n$. For every $n \in \mathbb{N}$,

$$\Phi_{S^s_n} = \prod_{i=1}^{n} |\Phi_{X_i}|^2 \geq \prod_{i=1}^{n} |\Phi_{X_i}|^2 |\Phi_{\Delta_n}|^2 = |\Phi_X|^2.$$

By properties of characteristic functions, see Theorem 4.5, it holds that $\Phi_X(0) = 1$ and that Φ_X is continuous on $\mathbb{R}$. For every $0 < \epsilon < 1$ there exists $\delta(\epsilon) > 0$ such that that $|\Phi_X(t)|^2 \geq 1 - \epsilon$, for all $t \in (-\delta(\epsilon), \delta(\epsilon))$. From this, for $t \in (-\delta(\epsilon), \delta(\epsilon))$,

$$\Phi_{S^s_n}(t) \geq |\Phi_X(t)|^2 \geq 1 - \epsilon.$$

For every $u \in \mathbb{R}$, $|\Phi_{S^s_n}(u)| = \prod_{i=1}^{n} |\Phi_{X_i}(u)|^2$ is non-decreasing in $u \in \mathbb{R}$ as $|\Phi_{X_i}| \leq 1$. A non-decreasing, bounded sequence in $\mathbb{R}_+$ converges, hence $|\Phi_{S^s_n}(u)|$ convergence pointwise to a limit, which we denote by $\varphi(u)$. We claim that the function φ is continuous. Let $\epsilon > 0$ be given, then for $u, v \in \mathbb{R}$,

$$\begin{aligned} |\varphi(u) - \varphi(v)| &= |\varphi(u) - \Phi_{S^s_N}(u) + \Phi_{S^s_N}(u) - \Phi_{S^s_N}(v) + \Phi_{S^s_N}(v) - \varphi(s)| \\ &\leq |\varphi(u) - \Phi_{S^s_N}(v)| + |\Phi_{S^s_N}(u) - \Phi_{S^s_N}(v)| + |\Phi_{S^s_N}(v) - \varphi(v)| \end{aligned} \tag{5}$$

Take $u, v \in \mathbb{R}$ such that $u - v \in (-\delta(\epsilon^2/18), \delta(\epsilon^2/18))$. By Lemma 4.7 it follows,

$$|\Phi_{S_n^s}(u) - \Phi_{S_n^s}(v)| \leq \sqrt{2|1 - \Phi_{S_n^s}(u-v)|} \leq \sqrt{2\frac{\epsilon^2}{18}} \leq \frac{\epsilon}{3}.$$

By taking N sufficiently large we can make sure that

$$|\varphi(u) - \Phi_{S_N^s}(u)| < \frac{\epsilon}{3}, \quad |\varphi(v) - \Phi_{S_N^s}(v)| < \frac{\epsilon}{3}.$$

From this it follows $|\varphi(u) - \varphi(v)| \leq \epsilon$ for $u, v \in \mathbb{R}$ such that $u - v \in (-\delta(\epsilon^2/18), \delta(\epsilon^2/18))$. By Dini's theorem, $\Phi_{S_n^s}$ converges uniformly on compact intervals. Note that by independence

$$\Phi_{S_n^s} = \Phi_{S_n^s - S_m^s} \Phi_{S_m^s}.$$

It holds that $\varphi(u) > 0$ for some interval $u \in (-\delta, \delta)$ around 0. From this and Lemma 4.7, for every compact interval $[-K, K]$ with $K > 0$,

$$\lim_{N\to\infty} \inf_{n,m\geq N} \Phi_{S_n^s - S_m^s}(u) = 1$$

uniformly on $[-K, K]$. By Lemma 4.6 it follows for n, m

$$\begin{aligned} \mathbb{P}(|S_n^s - S_m^s| \geq \epsilon) &\leq 7\epsilon \int_0^{1/\epsilon} \left[1 - \Re\left\{\Phi_{S_n^s - S_m^s}(v)\right\}\right] dv. \\ &= 7\epsilon \int_0^{1/\epsilon} \left[1 - |\Phi_{S_n - S_m}(v)|^2\right] dv. \end{aligned} \tag{6}$$

From this and uniform convergence of $\Phi_{S_n^s - S_m^s}$ for every $\epsilon > 0$,

$$\lim_{N\to\infty} \sup_{n,m\geq N} \mathbb{P}\left\{|S_n^s - S_m^s| > \epsilon\right\} = 0.$$

From Lemma 2.1 it follows $\lim_{N\to\infty} \mathbb{P}\left\{\sup_{n,m\geq N} |S_n^s - S_m^s| > 4\epsilon\right\} = 0$, from which we conclude a.s. convergence of S_n^s. Thus there exists a probability one set $\Omega^* \in \mathcal{F} \otimes \bar{\mathcal{F}}$ such that

$$S_n^s(\omega, \bar{\omega}) \to S(\omega, \bar{\omega}), \ \forall (\omega, \bar{\omega}) \in \Omega^*.$$

Let $\Omega_1^* = \left\{\bar{\omega} \in \bar{\Omega} : \exists \omega \in \Omega, \ (\omega, \bar{\omega}) \in \Omega^*\right\}$ and define for every $\bar{\omega} \in \Omega_1^*$, the set $\Omega_{\bar{\omega}}^* = \{\omega \in \Omega : (\omega, \bar{\omega}) \in \Omega^*\}$. Now it holds that

$$\mathbb{P} \otimes \bar{\mathbb{P}}(\Omega^*) = \int_{\bar{\omega}\in\Omega_1^*} \int_{\omega\in\Omega_{\bar{\omega}}^*} \mathbb{I}_{\Omega_{\bar{\omega}}^*}(\omega, \bar{\omega}) \ \ d\mathbb{P}(\omega) d\bar{\mathbb{P}}(\bar{\omega}) = 1$$

From this we find that there exists at least $\bar{\omega} \in \Omega_1^*$ such that $\mathbb{P}(\Omega_{\bar{\omega}}^*) = 1$. From this we find now that $S_n(\omega) - \bar{S}_n(\bar{\omega})$ converges $\forall \omega \in \Omega_{\bar{\omega}}^*$. This means that we can choose the centering constants $c_n = \bar{S}_n(\bar{\omega})$.

□

We will prove Theorem 2.1 for Banach space valued random variables. We take $(E, \|\cdot\|)$ a real separable Banach space and let E^* be its dual space, the set of all continuos linear functions, $x^* : E \to \mathbb{R}$.

Lemma 2.2. *There exists a sequence* $\{x_n^*\}_{n=1}^{\infty} \subset E^*$ *such that*

$$||x|| = \sup_n |\langle x_n^*, x\rangle|, \quad \forall x \in E. \tag{7}$$

Proof. The existence follows from separability of E, see [16, Lemma 1.1]. □

We denote by $\mathcal{B}(E)$ the Borel σ-algebra, the σ-algebra generated by the open sets of E. It holds that $\mathcal{B}(E)$ is the $\sigma-$algebra generated by the Cylinder sets, $\mathcal{C}$,

$$\{x \in E : \langle x, x_1^*\rangle \in B_1, \dots, \langle x, x_n^*\rangle \in B_n\},$$

where $B_1, \dots, B_n \in \mathcal{B}(\mathbb{R})$ and $x_1^*, \dots, x_n^* \in E^*$.

Lemma 2.3. *For a separable Banach space* E*,* $\sigma\{\mathcal{C}\} = \mathcal{B}(E)$*.*

Proof. Follows from the proof of [17, Theorem 2.8]. □

This means that for a function $X : \Omega \to E$ measurability is equivalent to measurability of $\langle x^*, X\rangle$, for every $x^* \in E^*$. From (7) we find that $||X - Y|| = \sup_n |\langle x_n^*, X - Y\rangle|$, which is measurable, hence we can define convergence in probability in the natural way.

Definition 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For a random variable $X : \Omega \to E$ we define the characteristic function $\Phi_X : E^* \to \mathbb{C}$ by,

$$\Phi_X(x^*) = \mathbb{E}e^{i\langle x^*, X\rangle}.$$

We denote $\mu_X(B) = \mathbb{P}\{X \in B\}, \forall B \in \mathcal{B}(E)$.

With Lemma 2.3 a similar result as to Theorem 4.4 holds for E-valued random variables.

Theorem 2.2. *Let* $X, Y : \Omega \to E$ *be two random variables with* $\Phi_X(x^*) = \Phi_Y(x^*)$*,* $\forall x^* \in E^*$*. Then we have* $\mu_X = \mu_Y$*.*

Proof. See [17, Theorem 2.8]. □

For real random variables $X_1, \dots, X_n$ it holds that $X_1, \dots, X_n$ are independent if and only if $\Phi_{(X_1,\dots,X_n)}(u) = \prod \Phi_{X_i}(u_i)), u \in \mathbb{R}^n$. The same result holds for random variables with values in a separable Banach space E. We recall that $\Phi_{(X_1,\dots,X_n)}(x_1^*, \dots, x_n^*) = \mathbb{E}e^{i\sum_j \langle x_j^*, X_j\rangle}$ for $x_1^*, \dots, x_n^* \in E^*$. The random variables $X_1, \dots, X_n$ are independent if and only if

$$\Phi_{(X_1,\dots,X_n)}(x_1^*, \dots, x_n^*) = \prod_{i=1}^{n} \Phi_{X_i}(x_i^*). \tag{8}$$

An important property of random variables with values in a separable Banach space E is that they are tight.

Lemma 2.4. *Let* X *be a random variable with values in a separable Banach space* E*, then* X *is tight, i.e. for every* ϵ *there is a compact set* $K_\epsilon \subset E$ *such that*

$$\mathbb{P}(K_\epsilon) \geq 1 - \epsilon.$$

Proof. See [17, Proposition 2.3]. □

Theorem 2.3 (Ito-Nisio)**.** *Let E be a separable Banach space. Suppose that X_i, $i = 1, 2\ldots$ are independent, symmetric*[1] *and E-valued random variables. For the sum $S_N = \sum_{i=1}^N X_i$ the following are equivalent,*

1. *S_N converges in distribution to a random variable S.*

2. *S_N converges in probability to a random variable S.*

3. *S_N converges a.s. to a random variable S.*

4. *The probability laws μ_N of S_N are uniformly tight.*

5. *There exists a random variable S such that $\langle x^*, S_N\rangle \xrightarrow{\mathbb{P}} \langle x^*, S\rangle$, for every $x^* \in E^*$.*

6. *$\mathbb{E}\left(e^{i\langle x^*, S_n\rangle}\right) \to \mathbb{E}\left(e^{i\langle x^*, S\rangle}\right)$, for every $x^* \in E^*$, for some random variable S.*

Proof. See [12]. □

Next we will use Theorem 2.3 to prove an extension of Theorem 2.1 for random variables with values in a separable Banach space.

Theorem 2.4. *Let $X_1, \ldots, X_n, \ldots$ be a sequence of symmetric, independent random variables in $(E, \mathcal{B}(E))$. Suppose there is a random variable X so that for every $k = 1, 2, \ldots$ there is a random variable Δ_k such that*

$$\Delta_k = X - \sum_{i=1}^{k} X_i \quad a.s,$$

and Δ_k is independent of $X_1, \ldots, X_k$. Then $S_N := \sum_{k=1}^N X_k$ converges with probability 1.

Proof. First we note that S_N and $X - S_N$ are independent random variables with values in E. We will show that this implies that S_N is uniformly tight. Let $K \subset E$ be a compact set. Now by the use of Fubini we find

$$\mathbb{P}(X \in K) = \int_E \mathbb{P}(S_N + x \in K)\mu_{X-S_N}(dx).$$

With this we can find an $x' \in E$ such that $\mathbb{P}(S_N + x' \in K) \geq \mathbb{P}(X \in K)$. Now set $K' = \left\{\frac{x-y}{2} : x, y \in K\right\}$. From the fact that $K \times K$ is also compact, the function $E \times E \to E, (x, y) \mapsto \frac{x-y}{2}$ is continuous and the image of a compact set under a continuous function is also compact, we conclude that K' is compact. Note that

$$\{S_N + x' \in K, -S_N + x' \in K\} \subset \{S_N \in K'\}$$

[1] A random variable X is called symmetric when $\mathbb{P}(X \in B) = \mathbb{P}(-X \in B)$, for every $B \in \mathcal{B}(E)$.

and by symmetry of S_N that

$$\begin{aligned}\mathbb{P}\{S_N \in K'\} &\geq \mathbb{P}\{S_N + x' \in K, -S_N + x' \in K\} \\ &\geq 1 - \mathbb{P}\{S_N + x' \notin K\} - \mathbb{P}\{-S_N + x' \notin K\} \\ &= 1 - 2\mathbb{P}\{S_N + x' \notin K\} \\ &\geq 1 - 2\mathbb{P}\{X \notin K\}\,.\end{aligned} \tag{9}$$

Next we will use Lemma 2.4 that every random variable with values in a separable Banach space is tight, i.e. for every $\epsilon > 0$ there is a compact set K_ϵ such that $\mathbb{P}(X \notin K_\epsilon) < \epsilon$. From this we find then that we can always find $K'_{\epsilon/2}$ such that

$$\mathbb{P}\left\{S_N \in K'_{\epsilon/2}\right\} \geq \epsilon.$$

Hence the collection of measures μ_{S_N} are uniformly tight. The statement now follows from Theorem 2.3. □

2.3 Symmetric processes with independent increments

We first consider symmetric processes with independent increments with values in a separable Banach space $(E, \mathcal{B}(E))$. We will show that for such processes we are led to study processes that are continuous in probability. We will show that every symmetric process with independent increments can be decomposed into independent parts: a part that is continuous in probability and a part that by approximation is a pure jump process, recall Example 2.1. The precise formulation is given in Theorem 2.5.

Definition 2.4. Let $\{X\}_{t\in\mathbb{R}_+}$ be a process with independent increments. We call X an *Additive process* if the following conditions hold,

1. For every $\omega \in \Omega$, $X_0(\omega) = 0$.

2. For every $s < t$, $X_t - X_s$ is independent of $\mathcal{F}_s$.

3. For every $t > 0$ and $\epsilon > 0$, $\lim_{s\to t} \mathbb{P}\{\|X_t - X_s\| > \epsilon\} = 0$, i.e. the process X is continuous in probability.

From Lemma 4.4 we know there is a metric $d_\mathbb{P}$ on $L^0_\mathbb{P}(\Omega; E)$ defined by

$$d_\mathbb{P}(X, Y) = \inf\{\epsilon \geq 0 : \mathbb{P}(||X - Y|| > \epsilon) \leq \epsilon\}\,,\ X, Y \in L^0_\mathbb{P}(\Omega; E),$$

that metrizes convergence in probability. We will consider processes as maps from $\mathbb{R}_+$ to the metric space $(L^0_\mathbb{P}(\Omega; E), d_\mathbb{P})$, see section 4.1. We can define regularity of this map in the sense of Definition 4.7.

Lemma 2.5. *Let $\{X\}_{t\in\mathbb{R}_+}$ be a symmetric stochastic process with independent increments and with values in $(E, \mathcal{B}(E))$, then $\{X\}_{t\in\mathbb{R}_+}$ is regular in probability, i.e. the function $X : \mathbb{R}_+ \to (L^0_\mathbb{P}(\Omega, \mathcal{F}; E, \mathcal{E}), d_\mathbb{P})$, $\ t \mapsto X_t$, is regular.*

Proof. Let $t > 0$ and $t_n \uparrow t$. We consider the sequence $(X_{t_n})_{n\in\mathbb{N}}$. It is possible to write X_t as a sum of independent variables, $X_t = X_{t_1} + \sum_{i=1}^{n-1}\left(X_{t_{i+1}} - X_{t_i}\right) + (X_t - X_{t_n})$. Note that the random variables $X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are symmetric random variables. By Theorem 2.4 the sequence X_{t_n} converges a.s. to a random variable X_{t-}, hence it converges in probability to X_{t-}. Let $s_n \uparrow t$ be another sequence. By the same arguments X_{s_n} converges in probability to a random variable X'_{t-}. We show that the limits are the same. First merge the two sequences together in one sequence $t'_n \uparrow t$. The sequence $(X_{t'_n})$ converges by the same arguments in probability to a random variable. This forces the limits X'_{t-} and X_{t-} to be the same. We can do the same for $t_n \downarrow t$. We conclude that for every $t > 0$ the limits $\lim_{h\uparrow t} X_h, \lim_{h\downarrow t} X_h$ exist in probability. □

Now we want to describe a procedure to define jumps of X. We recall once more that we have made no assumption yet about regularity of sample paths. We will use Lemma 2.5 to define jumps in probability.

Definition 2.5. Let X be a process with independent increments. Then for $s < t$ we define

$$\mathcal{F}^X_{s,t} = \sigma\left\{X_u - X_v : s \le u < v \le t\right\}, \tag{10}$$

to be the σ-algebra generated by all increments of X on $[s, t]$.

Lemma 2.6. *Let X be a process with independent increments. Then $\mathcal{F}_s$ and $\mathcal{F}^X_{s,t}$ are independent σ-algebra's.*

Let Y_n, $n = 1, 2, \ldots$ be a sequence of random variables, then the event that $\lim_n Y_n$ converges satisfies $\{\lim_n Y_n \text{ exists}\} \subset \bigcup_k \bigcap_{m\ge k} \{||Y_{m+1} - Y_m|| \le \epsilon_m\}$, for every sequence $\epsilon_n > 0$ with $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

Lemma 2.7. *Let Y_n, $n = 1, 2, \ldots$ be a sequence of random variables. If there exists a sequence $\epsilon_n > 0$ such that $\sum_{n=1}^{\infty} \mathbb{P}\{||Y_{n+1} - Y_n|| > \epsilon_n\} < \infty$ and $\sum_{n=1}^{\infty} \epsilon_n < \infty$, then $\mathbb{P}\{\lim_n Y_n \text{ exists}\} = 1$.*

Proof. First we note that $\{\lim_n X_n \text{ exists}\}^c \subset \bigcap_k \bigcup_{m\ge k} \{||Y_{m+1} - Y_m|| > \epsilon_m\}$. Also

$$\mathbb{P}\left\{\bigcup_{m\ge k} \{||Y_{m+1} - Y_m|| > \epsilon_m\}\right\} \le \sum_{n=k}^{\infty} \mathbb{P}\{||Y_{n+1} - Y_n|| > \epsilon_n\}.$$

From this it is clear that

$$\mathbb{P}\left(\left\{\lim_n Y_n \text{ exists}\right\}^c\right) \le \lim_{k\to\infty} \sum_{n=k}^{\infty} \mathbb{P}\{||Y_{n+1} - Y_n|| > \epsilon_n\} = 0.$$

□

Consequence of Lemma 2.7 is that we can define

$$Y(\omega) := \begin{cases} Y_1(\omega) + \sum_{n=1}^{\infty} (Y_{n+1}(\omega) Y_n(\omega)) & \text{if } \omega \in \{\lim_n Y_n \text{ exists}\} \\ 0 & \text{if } \omega \notin \{\lim_n Y_n \text{ exists}\} \end{cases}. \tag{11}$$

By Lemma 2.5, the a symmetric process with independent increments $X : \mathbb{R}_+ \mapsto L^0_{\mathbb{P}}(\Omega; E)$ is regular and thus has at most a countable number of jumps. We enumerate and denote the setof jumps with $J = \{t_n : n \in \mathbb{N}\}$. Let $t \in J$ and take some increasing sequence $s_n \uparrow t$. The sequence $Y_n := X_t - X_{s_n}$ is a Cauchy sequence in probability. We can take a subsequence s_{n_k} such that $\mathbb{P}\left\{||Y_{n_{k+1}} - Y_{n_k}|| > \frac{1}{2^k}\right\} \leq \frac{1}{2^k}$. By Lemma 2.7 we can define a random variable with (11),

$$\Delta X_{t-}(\omega) := \begin{cases} Y_{n_1}(\omega) + \sum_{k=1}^{\infty} \left(Y_{n_{k+1}}(\omega) - Y_{n_k}(\omega)\right) & \text{if } \omega \in \{\lim_k Y_{n_k} \text{ exists}\} \\ 0 & \text{if } \omega \notin \{\lim_k Y_{n_k} \text{ exists}\} \end{cases} .$$

It follows that ΔX_{t-} is $\bigcap_{k=1} \sigma\left\{Y_{n_k}, Y_{n_{k+1}}, \ldots\right\}$-measurable and $\lim_{s\uparrow t} X_t - X_s = \Delta X_{t-}$ in $(L^0_{\mathbb{P}}(\Omega; E), d_{\mathbb{P}})$. From this we find that for every $s < t$ it holds that ΔX_{t-} is $\mathcal{F}^X_{s,t}$-measurable. In the same way we define ΔX_{t+} for a sequence $s_n \downarrow t$ such that for every $s > t$ it holds that ΔX_{t+} is $\mathcal{F}^X_{t,s}$-measurable. Now we define the following processes.

Definition 2.6. Let X be a symmetric process with independent increments. We define the processes

$$S^-_N(t) = \sum_{n \leq N, t_n \leq t} \Delta X_{t_n-}, \quad S^+_N(t) = \sum_{n \leq N, t_n < t} \Delta X_{t_n+}, \tag{12}$$

where $\{t_n : n \in \mathbb{N}\}$ the set of jump points of X viewed as map from $\mathbb{R}_+$ to $L^0_{\mathbb{P}}(\Omega; E)$.

Remark 2.3. The increments $S^{\pm}_N(t) - S^{\pm}_N(s)$ are $\mathcal{F}^X_{s,t}$-measurable.

Definition 2.7. Let E be a separable Banach space and $T > 0$. We define $\mathbb{D}_E(T)$ to be the space of all càdlàg functions $f : [0, T] \to E$. We equip the space $\mathbb{D}_E(T)$ with the σ-algebra $\mathcal{D}_E(T)$ generated by the sets of the form,

$$\{f \in \mathcal{D}_E(T) | f(t_1) \in B_1, \ldots, f(t_n) \in B_n, 0 \leq t_1 < \ldots < t_n \leq T, B_i \in \mathcal{B}(E)\}.$$

On $\mathbb{D}_E(T)$ we define the supremum norm, $\|f\|_T := \sup_{t\in[0,T]} \|f(t)\|$.

Remark 2.4. For a stochastic process X the map $X : (\Omega, \mathcal{F}) \to (\mathbb{D}_E(T), \mathcal{D}_E(T))$ is measurable. For separable Banach spaces there exists a norming sequence $x^*_n \in E^*$ such that for $x \in E$, $\|x\| = \sup_n |\langle x^*_n, x\rangle|$, see Lemma 2.2. From the càdlàg property it follows that

$$\|f\|_T = \sup_{q\in[0,T]\cap(\mathbb{Q}\cup\{T\})} \|f(q)\| = \sup_{q\in[0,T]\cap(\mathbb{Q}\cup\{T\})} \sup_n |\langle x^*_n, f(q)\rangle| .$$

This implies that for the process X, the map $\omega \mapsto \|X(\omega)\|_T$ is measurable. The space $\mathbb{D}_E(T)$ equipped with the supremum norm $||\cdot||_T$ is a Banach space. The same we can say for the space $\mathbb{L}_E(T)$ of all càglàd functions $f : [0, T] \to E$.

The space $(\mathbb{D}_E(T), ||\cdot||_T)$ is Banach space, but not a separable Banach space.

Remark 2.5. For a fixed time horizon $T > 0$ it holds that $\left\{S^-_N(t)(\omega)\right\}_{t\in[0,T]} \in \mathbb{D}_E(T)$ and $\left\{S^+_N(t)(\omega)\right\}_{t\in[0,T]} \in \mathbb{L}_E(T)$, with S^-_N and S^+_N as in Definition 2.6.

If X is a symmetric process with independent increments, we can subtract jumps at fixed time points, $X_t - S_N^-(t) - S_N^+(t)$. If we take $N \to \infty$, then intuitively we expect $X - S_N^- - S_N^+$ to converge to a process that is continuous in probability. The main difficulty is that a priori it is not clear how $S_N^-(t), S_N^+(t)$ converges as $N \to \infty$. In order to understand the convergence of these processes we need the following lemmas.

Definition 2.8. Let $X, Y \in (\mathbb{D}_E(T), \mathcal{D}_E(T))$ be processes. We define

$$d_{\mathbf{ucp}}(X,Y) := \inf\left\{\epsilon > 0 : \mathbb{P}\left\{\sup_{0\le s\le T} ||X_s - Y_s|| > \epsilon\right\} \le \epsilon\right\}. \tag{13}$$

Definition 2.9. Let $X, X_1, X_2, \ldots, X_n, \ldots$ be random variables in $\mathbb{D}_E(T)$ or $(\mathbb{L}_E(T))$, then X_n converge uniform in probability to X if for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\{||X_n - X||_T > \epsilon\} = 0.$$

We denote this convergence by $X_n \overset{ucp}{\to} X$, $n \to \infty$

Lemma 2.8. *On $(\mathbb{D}_E(T), \mathcal{D}_E(T))$ $d_{\boldsymbol{ucp}}$ is a metric and $d_{ucp}(X_n, X) \to 0$ if and only if $X_n \overset{ucp}{\to} X$.*

Proof. The function d_{ucp} is non-negative, symmetric and $d_{ucp}(X,Y) = 0$ if and only if $\|X - Y\|_T = 0$ a.s. Next we will show the triangle inequality. Let $X, Y, Z \in \mathbb{D}_E(T)$, then by the triangle inequality $||X - Z||_T \le ||X - Y||_T + ||Y - Z||_T$ it yields

$$\begin{aligned}
&\mathbb{P}\{\|X - Z\|_T > d_{ucp}(X,Y) + d_{ucp}(Y,Z)\}\\
&\quad \le \mathbb{P}\{\|X - Y\|_T + \|Y - Z\|_T > d_{ucp}(X,Y) + d_{ucp}(Y,Z)\}\\
&\quad \le \mathbb{P}\{\|X - Y\|_T > d_{ucp}(X,Y)\} + \mathbb{P}\{\|Y - Z\| > d_{ucp}(Y,Z)\}\\
&\quad \le d_{ucp}(X,Y) + d_{ucp}(Y,Z).
\end{aligned} \tag{14}$$

By Definition 2.8 it follows that $d_{ucp}(X,Z) \le d_{ucp}(X,Y) + d_{ucp}(Y,Z)$.

Next, suppose that $X_n \overset{ucp}{\to} X$. Then for every ϵ there is K_ϵ such that

$$\sup_{N\ge K_\epsilon} \mathbb{P}\{\|X - X_N\|_T > \epsilon\} \le \epsilon.$$

From this it holds $\sup_{N\ge K_\epsilon} d_{ucp}(X, X_N) \le \epsilon$. we conclude that $d_{ucp}(X, X_n) \to 0$. Conversely suppose $\lim_{n\to\infty} d_{ucp}(X, X_n) = 0$. Then for every ϵ, there is a constant K_ϵ such that

$$\mathbb{P}\{\|X - X_N\|_T > \epsilon\} \le \epsilon,$$

for all $N \ge K_\epsilon$. From this it follows that $X_n \overset{ucp}{\to} X$, $n \to \infty$. □

Lemma 2.9. *Let X_n, $n = 1, 2\ldots$ be independent stochastic processes in $\mathbb{D}_E(T)$ (or $\mathbb{L}_E(T)$) such that for every $\epsilon > 0$,*

$$\lim_{N\to\infty} \mathbb{P}\left\{\sup_{n,m\ge N} ||X_n - X_m||_T > \epsilon\right\} = 0,$$

Then there exist $X \in \mathbb{D}_E(T)$ (or $\mathbb{L}_E(T)$), sych that $X_n \overset{ucp}{\to} X$, $n \to \infty$.

Proof. The event of convergence is given by,

$$\left\{\lim_n \ X_n \text{ exists}\right\} = \bigcap_m \bigcup_n \bigcap_{k,l\geq n} \left\{\|X_k - X_l\|_T < \frac{1}{m}\right\}.$$

By hypothesis, it follows that $\mathbb{P}\{\lim_n \ X_n \text{ exists}\} = 1$. Now define the random variable

$$X(\omega) := \begin{cases} \lim_{n\to\infty} X_n(\omega) & \text{if } \omega \in \{\lim_n X_n \text{ exists}\} \\ 0 & \text{if } \omega \notin \{\lim_n X_n \text{ exists}\} \end{cases}. \tag{15}$$

It holds that $X_n \overset{ucp}{\to} X$. From the fact that the space $(\mathbb{D}_E(T), \|\cdot\|)$ is a Banach space it follows that X has values in $\mathbb{D}_E(T)$. □

Lemma 2.10. *Let X_n, $n = 1, 2\dots$ be independent stochastic processes in $\mathbb{D}_E(T)$ (or $\mathbb{L}_E(T)$). Let $S_n = \sum_{i=1}^n X_i$ and suppose that*

$$\lim_{N\to\infty} \sup_{n,m\geq N} \mathbb{P}\{||S_n - S_m||_T > \epsilon\} = 0, \tag{16}$$

then there is a stochastic process S with values in $\mathbb{D}_E(T)$ (or $\mathbb{L}_E(T)$) such that

$$\lim_{n\to\infty} ||S - S_n||_T = 0 \quad a.s.$$

Proof. By hypothesis and Lemma 2.1 it holds for every $\epsilon > 0$ that

$$\lim_{N\to\infty} \mathbb{P}\left\{\sup_{n,m\geq N} ||S_n - S_m||_T > \epsilon\right\} = 0.$$

By Lemma 2.9 the statement follows. □

Lemma 2.11. *For $j = 1, 2, \dots, m$ let $X^{(j)}$ and $X_n^{(j)}$, $n \in \mathbb{N}$, be random variables in a separable Banach space E such that*

1. *$X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(m)}$ are independent random variables.*
2. *$X_n^{(j)} \overset{\mathbb{P}}{\to} X^{(j)}$, as $n \to \infty$, for $j = 1, \dots, m$.*

Then $X^{(1)}, \dots, X^{(m)}$ are independent.

Proof. The random variables $X^{(1)}, \dots, X^{(m)}$ are independent if and only if

$$\Phi_{(X^{(1)},\dots,X^{(m)})}(x_1^*, \dots, x_m^*) = \prod_{i=1}^m \Phi_{X^{(i)}}(x_i^*), \quad \forall x_1^*, \dots, x_m^* \in E^*.$$

Let $x_1^*, \dots, x_m^* \in E^*$. Now we consider $\left\langle x_j^*, X_n^{(j)} \right\rangle$ and $\langle x_j^*, X^{(j)}\rangle$, then it is clear that $\left\langle x_j^*, X_n^{(j)} \right\rangle$ are independent and $\left\langle x_j^*, X_n^{(j)} \right\rangle \overset{\mathbb{P}}{\to} \langle x_j^*, X^{(j)}\rangle$. Convergence in probability implies convergence in distribution. From this it follows $e^{\sum_{j=1}^m \left\langle x_j^*, X_n^{(j)} \right\rangle} \to e^{\sum_{j=1}^m \langle x_j^*, X^{(j)}\rangle}$ for every $x_1^*, \ldots, x_m^* \in E^*$. It also follows by independence and convergence in distribution that,

$$\Phi_{(X_n^{(1)},\ldots,X_n^{(m)})}(x_1^*, \ldots, x_m^*) = \prod_{j=1}^{m} \Phi_{\left\langle x_j^*, X_n^{(j)} \right\rangle}(1) \to \prod_{j=1}^{m} \Phi_{\left\langle x_j^*, X^{(j)} \right\rangle}(1) = \prod_{j=1}^{m} \Phi_{X^{(j)}}(x_j^*).$$

We conclude that

$$\Phi_{(X^{(1)},\ldots,X^{(m)})}(x_1^*, \ldots, x_m^*) = \prod_{i=1}^{m} \Phi_{X^{(i)}}(x_i^*), \quad x_1^*, \ldots, x_m^* \in E^*.$$

□

Lemma 2.12 (Lèvy's inequality)**.** *Let $X_1, \ldots, X_n$ be independent, symmetric and E-valued random variables. Let $S_k = \sum_{i=1}^{k} X_i$ be the sum for $k = 1, \ldots, n$. For every $r > 0$ we have*

$$\mathbb{P}\left\{\max_{1 \le k \le n} ||S_k|| > r\right\} \le 2\mathbb{P}\left\{||S_n|| > r\right\}. \tag{17}$$

Proof. See [17, Lemma 2.18]. □

The formulation and prove of the following theorem is inspired by [6], [11] and [21]. In the proof we use Theorem 2.4. Furthermore we use In. 17 for uniform convergence.

Theorem 2.5. *Let $\{X_t\}_{t \in \mathbb{R}_+}$ be a symmetric stochastic process with independent increments. Then the process can be written as*

$$X_t = X_t^c + S_t^- + S_t^+, \tag{18}$$

such that $X_t^c = X_t - S_t^- - S_t^+$ is an Additive process, S^-, S^+ are processes with independent increments and with values in $\mathbb{D}_E$ resp. $\mathbb{L}_E$ such for every $T > 0$

$$\lim_{N \to \infty} ||S^- - S_N^-||_T = 0, \quad \lim_{N \to \infty} ||S^+ - S_N^+||_T = 0, \quad \forall \omega \in \Omega',$$

where $\mathbb{P}\{\Omega'\} = 1$. Furthermore X^c, S^- and S^+ are independent processes.

Proof. Fix a time horizon $T > 0$. Order the jump points $t_n \in J$ with n up to N smaller than T,

$$\{\sigma_1 < \sigma_2 < \ldots < \sigma_k\} = \{t_n \le T : n \le N\}.$$

Consider the partitions,

$$\mathcal{P}_m = \left\{0 = \sigma_0 < \underline{s}_{1,m} < \sigma_1 < \overline{s}_{1,m} < \underline{s}_{2,m} < \sigma_2 \ldots < \underline{s}_{k,m} < \sigma_k < \overline{s}_{k,m}\right\},$$

such that $\underline{s}_{i,m} \uparrow \sigma_i$ and $\overline{s}_{i,m} \downarrow \sigma_i$. We write X_T as a random walk of increments,

$$X_T = \sum_{i=1}^{k} \Big((X_{\sigma_i} - X_{\underline{s}_{i,m}}) + (X_{\overline{s}_{i,m}} - X_{\sigma_i}) \Big) + \Delta_{N,m}.$$

where $\Delta_{N,m} = X_{\underline{s}_{1,m}} + \sum_{i=1}^{k-1}\left(-X_{\overline{s}_{i,m}} + X_{\underline{s}_{i+1,m}}\right) + X_T - X_{\overline{s}_{k,m}}$. This is a sum of independent increments. Let $m \to \infty$ and find

$$X_T = \underbrace{\sum_{i=1}^{k}(\Delta X_{\sigma_i -})}_{S_N^-(T)} + \underbrace{\sum_{i=1}^{k}(\Delta X_{\sigma_i +})}_{S_N^+(T)} + \underbrace{\Delta_N}_{\lim_k \Delta_{N,k}} \quad a.s..$$

Note that $S_N^-(T), S_N^+(T)$ and $\lim_k \Delta_{N,k}$ are independent by Lemma 2.11. We can do this for every $N \in \mathbb{N}$. By Theorem 2.4 it follows that $S_N^-(T)$ converges to a random variable S a.s. By (17) it holds for every $r > 0$,

$$\mathbb{P}\left\{\sup_{t\in[0,T]} ||S_N^-(t) - S_M^-(t)|| > r\right\} \leq 2\mathbb{P}\left\{||S_N^-(T) - S_M^-(T)|| > r\right\}.$$

By a.s. convergence of $S_N^-(T)$, for every $\epsilon > 0$,

$$\lim_{N,M\to\infty} \mathbb{P}\left\{\sup_{t\in[0,T]} ||S_N^-(t) - S_M^-(t)|| > \epsilon\right\} = 0.$$

By Lemma 2.10 $S_N^-(t)$ converges uniformly to a càdlàg stochastic process $\{S^-(t)\}_{t\in[0,T]}$. Note that $S_N^-(t)$ has independent increments. By uniform convergence it follows that S^- has independent increments. We can apply the same arguments to $S_N^+(t)$. There exists càglàd stochastic process $\{S^+(t)\}_{t\in[0,T]}$ such that S_N^+ converges uniformly to S^+. on $[0,T]$. Because $S_N^-(T), S_N^+(T)$ and Δ_N are independent, by Lemma 2.11 $S^-(T), S^+(T)$ and $X_T - S^-(T) - S^+(T)$ are independent for every $T > 0$.

Thus for every T there exists a probability one set Ω_T such that S_N^-, S_N^+ converge a.s. uniformly on $[0,T]$ to stochastic processes $S_T^-(t), S_T^+(t)$ in $\mathbb{D}_E(T)$ resp. $\mathbb{L}_E(T)$. Now by taking $\Omega^* = \bigcap_n \Omega_n$, we can find processes

$$S^-(t) = \sum_{n=1}^{\infty} S_n^-(t)\mathbb{I}_{[n-1,n)}(t), \quad S^+(t) = \sum_{n=1}^{\infty} S_n^+(t)\mathbb{I}_{(n-1,n]}(t),$$

in $\mathbb{D}_E$ resp. $\mathbb{L}_E$ such that for every $T > 0$ it follows that

$$\lim_{N\to\infty} ||S^-(\omega) - S_N^-(\omega)||_T = 0, \quad \lim_{N\to\infty} ||S^+(\omega) - S_N^+(\omega)||_T, \quad \forall \omega \in \Omega^*.$$

It follows by uniform convergence that S^- and S^+ are processes with independent increments. We want to show that $X^c = X - S^- - S^+$ is continuous in probability. Let $0 < s < t$ and $\omega \in \Omega$. Then it holds that

$$\begin{aligned} &||\Delta X_{t-} - \left(S^-(t) - S^-(s)\right)|| \\ &\quad = ||S_N^-(t) - S^-(t) + \left(\Delta X_{t-} - (S_N^-(t) - S_N^-(s))\right) + S^-(s) - S_N^-(s)|| \\ &\quad \leq ||S_N^-(t) - S^-(t)|| + ||\left(\Delta X_{t-} - (S_N^-(t) - S_N^-(s))\right)|| + ||S^-(s) - S_N^-(s)||. \end{aligned} \tag{19}$$

First we can take N such that $||S^- - S_N^-||_t < \epsilon$. Choose s so close to t such that

$$s > \max_{n \leq N, t_n < t} t_n.$$

In that case $\big(\Delta X_{t-} - (S_N^-(t) - S_N^-(s))\big) = 0, \quad a.s.$. Because ϵ was arbitrary we find that

$$\lim_{s \uparrow t} ||\Delta X_{t-} - \big(S^-(t) - S^-(s)\big)|| = 0 \quad a.s.$$

From right continuity of S^- it follows that $\lim_{s \downarrow t} S^-(s) - S^-(t) = 0 \ \ a.s.$ In the same way we can prove that

$$\lim_{s \downarrow t} ||\Delta X_{t+} - \big(S^+(s) - S^+(t)\big)|| = 0 \quad a.s.$$

By left continuity $\lim_{s \uparrow t} (S^+(t) - S^+(s)) = 0 \ \ a.s.$ We conclude that

$$\lim_{s \uparrow t} (X_t^c - X_s^c) = \lim_{s \uparrow t} (X_t - X_s) - \lim_{s \uparrow t} \big(S^-(t) - S^-(s)\big) = 0 \quad a.s.$$

and

$$\lim_{s \downarrow t} (X_s^c - X_t^c) = \lim_{s \downarrow t} (X_t - X_s) - \lim_{s \downarrow t} \big(S^+(s) - S^+(t)\big) = 0 \quad a.s.$$

Hence X^c is continuous in probability. For every $t > 0$ it holds that X_t^c, S_t^- and S_t^+ are independent for every $t \in \mathbb{R}_+$. From Lemma 3.10 and Remark 3.2 it follows that the processes X^c, S^- and S^+ are independent. □

2.4 Decomposition of processes with independent increments with values in $\mathbb{R}$

Now we will consider general processes X with independent increments and with state space $E = \mathbb{R}$. As in Section 2.3 we will consider processes as maps from $\mathbb{R}_+$ to the metric space $(L^0_{\mathbb{P}}(\Omega; E), d_{\mathbb{P}})$. We will show that a process with independent increments can be decomposed into independent parts: as a non-random function, a process continuous in probability and a process that by approximation is a pure jump process.

Paul Lèvy showed that a center $c[X]$ of random variables X can be defined such that $X_t - c[X_t]$ is regular in probability. The following center $c[\cdot]$ value defined by J.L.Doob, will do the job. See [6, Eq. (3.8)].

Definition 2.10. Let X be a random variable. The center $c[X]$ of X is defined by

$$\mathbb{E} \arctan [X - c[X]] = 0. \tag{20}$$

Existence and uniqueness of $c[X]$ follows from the proof of the next Lemma.

Lemma 2.13 (Lévy)**.** *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of random variables for which there exist a sequence of constants $(c_n)_{n \in \mathbb{N}}$ and a random variable X such that $X_n - c_n$ converges a.s. to X. Then $c_n - c[X_n]$ converges to a finite number c and $X_n - c[X_n]$ converges a.s. to $X - c$.*

Proof. It holds that $f(x) = \arctan(x)$ is bounded and measurable with $\lim_{x\to\pm\infty} \arctan(x) = \pm\frac{\pi}{2}$. By the dominated convergence theorem it holds that $\mathbb{E}f(X_n - x) \to \pm\frac{\pi}{2}$ as $x \to \pm\infty$. From this and the intermediate value theorem the constant $c[X_n]$ exists and are unique for all n. Then there are two posibilities for $c_n - c[X_n]$, namely the constants $c_n - c[X_n]$ are bounded or there exist an unbounded subsequence $c_{n(k)} - c[X_{n(k)}]$. In case the constants are bounded there exist a convergent subsequence $c_{n(k)} - c[X_{n(k)}]$ and by

$$0 = \mathbb{E}f(X_{n(k)} - c[X_{n(k)})]) = \mathbb{E}f(X_{n(k)} - c_{n(k)} + c_{n(k)} - c[X_{n(k)}])$$

there is only one posibility for this sequence to converge to. This subsequence converges to $c[X]$, hence $c_n - c[X_n]$ converge to $-c$. We conclude that $X_n - d_n$ converges to $X - c$. We conclude the proof by showing that sequence $c_n - c[X_n]$ cannot be unbounded. If it is unbounded we can take a subsequence $c_{h(n)} - c[X_{h(n)}] \uparrow \infty$ or $c_{h(n)} - c[X_{h(n)}] \downarrow -\infty$. In that case it holds that

$$X_{h(n)} - c[X_{h(n)}] = X_{h(n)} - c_{h(n)} + c_{h(n)} - c[X_{h(n)}] \to \infty \quad a.s.,$$

and by the dominated convergence theorem we find $0 = \lim_n \mathbb{E}\arctan(X_{h(n)} - c[X_{h(n)}]) = \frac{\pi}{2}$, which is a contradiction. □

Theorem 2.6. *Let X be a stochastic process with independent increments, then $X_t - c[X_t]$ is regular in probability.*

Proof. Let $t > 0$ and $t_n \uparrow t$. We consider the sequence $(X_{t_n})_{n\in\mathbb{N}}$. By Theorem 2.1 there are constants c_n such that $X_{t_n} - c_n$ converges in probability. By Lévy's lemma 2.13, $X_{t_n} - c[X_{t_n}]$ also converges in probability, the limit we denote by $X_{t-} \in L^0_{\mathbb{P}}(\Omega;\mathbb{R})$. Let $s_n \uparrow t$ be another sequence. By the same arguments $X_{s_n} - c[X_{s_n}]$ converges in probability to a limit $X'_{t-} \in L^0_{\mathbb{P}}(\Omega;\mathbb{R})$. Now merge $(s_n), (t_n)$ into one sequence (t'_n), which by the same arguments converges in $L^0_{\mathbb{P}}(\Omega;\mathbb{R})$. From this $X_{t-} = X'_{t-}$ in $L^0_{\mathbb{P}}(\Omega;\mathbb{R})$, hence $\lim_{s\uparrow t} X_s$ exists. The same can be concluded for $\lim_{s\downarrow t} X_s$. □

As we did for symmetric processes we can define the following jump processes. Let X be a process with independent increments. We denote the regularization by $Z = X - c[X]$. Let $J = \{t_n : n \in \mathbb{N}\}$ be the fixed jump points. We denote for every $N \in \mathbb{N}$,

$$S^-_N(t) = \sum_{n\leq N, t_n\leq t} \Delta Z_{t_n-}, \quad S^+_N(t) = \sum_{n\leq N, t_n< t} \Delta Z_{t_n+}. \tag{21}$$

Now we consider the symmetrization $Z^s = (X - c[X]) - \big(\bar{X} - c[\bar{X}]\big)$ defined on the probability space $(\Omega^*, \mathcal{F}^*, \{\mathcal{F}^*_s\}_{s\in T}, \mathbb{P}^*)$. Now we define the symmetric jump processes,

$$S^{s,-}_N(t) = \sum_{n\leq N, t_n\leq t} \left(\Delta Z_{t_n-} - \Delta\bar{Z}_{t_n-}\right), \quad S^{s,+}_N(t) = \sum_{n\leq N, t_n< t} \left(\Delta Z_{t_n+} - \Delta\bar{Z}_{t_n+}\right). \tag{22}$$

By Theorem 2.5 there is a probability one set Ω' and processes S^-, S^+ in $\mathbb{D}_\mathbb{R}$ resp. $\mathbb{L}_\mathbb{R}$ such that for every $T > 0$ it holds that

$$\lim_{N\to\infty} ||S^-(\omega.\bar{\omega}) - S^{s,-}_N(\omega.\bar{\omega})||_T = 0, \quad \lim_{N\to\infty} ||S^+(\omega.\bar{\omega}) - S^{s,+}_N(\omega.\bar{\omega})||_T = 0, \quad \forall(\omega,\bar{\omega}) \in \Omega'.$$

By Fubini's Theorem there is $\bar{\omega}$ and $\Omega_{\bar{\omega}} = \{\omega \in \Omega : (\omega, \bar{\omega}) \in \Omega'\}$ such that $\mathbb{P}\{\Omega_{\bar{\omega}}\} = 1$ and such that for every $T > 0$,

$$\lim_{N\to\infty} ||S^-(\omega,\bar{\omega}) - S_N^{s,-}(\omega,\bar{\omega})||_T = 0, \quad \lim_{N\to\infty} ||S^+(\omega,\bar{\omega}) - S_N^{s,+}(\omega,\bar{\omega})||_T, \forall \omega \in \Omega_{\bar{\omega}}.$$

We define $\Sigma_t(\omega) := S_t^-(\omega,\bar{\omega})$ and $\Pi_t := S_t^+(\omega,\bar{\omega})$.

Theorem 2.7. *Let X be a stochastic process with independent increments, then there is a deterministic function $f : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$X_t = f_t + X_t^c + \Sigma_t + \Pi_t, \tag{23}$$

where $X^c = X_t - \Sigma_t - \Pi_t$ is an Additive process. Furthermore X^c, Σ and Π are independent processes.

Proof. We first define $Y = Z - S^-(\cdot,\bar{\omega}) - S^+(\cdot,\bar{\omega})$. Now for every $t > 0$ we have by uniform convergence of $S_N^{s,-}, S_N^{s,+}$ that

$$\lim_{s\uparrow t} (Y_t - Y_s) = \Delta Z_{t-} - (\Delta Z_{t-} - \Delta Z_{t-}(\bar{\omega})) = \Delta Z_{t-}(\bar{\omega}) \;\; a.s.$$

and

$$\lim_{s\downarrow t} (Y_s - Y_t) = \Delta Z_{t+} - (\Delta Z_{t+} - \Delta Z_{t+}(\bar{\omega})) = \Delta Z_{t+}(\bar{\omega}) \;\; a.s.$$

Define $X^c = Y - c[Y]$ where $c[Y]_t = c[Y_t], t \geq 0$. We will prove that X^c is continuous in probability. First note that Y is regular in probability and thus $c[Y]$ is regular by lemma 2.13 . For $s > 0$ it holds

$$0 = \mathbb{E}\arctan(Y_s - c[Y_s]) = \mathbb{E}\arctan(Y_s - Y_t + Y_t + c[Y_t] - c[Y_t] - c[Y_s]).$$

We take $s \uparrow t$ and by the dominated convergence theorem (and considering arbitrary sequences $s_n \uparrow t$) it follows

$$\begin{aligned} 0 &= \lim_{s\uparrow t} \mathbb{E}\arctan(Y_s - Y_t + Y_t + c[Y_t] - c[Y_t] - c[Y_s]) \\ &= \mathbb{E}\arctan(-\Delta Y_{t-} + \lim_{s\uparrow t}(c[Y_t] - c[Y_s]) + Y_t - c[Y_t]) \end{aligned}$$

From this we find that $\lim_{s\uparrow t}(c[Y_t] - c[Y_s]) = -\Delta Y_{t-}$ a.s. The same argument can be used for $s \downarrow t$. Hence X^c is continuous in probability. We can write

$$X_t = X_t^c + S^-(\cdot,\bar{\omega}) + S^+(\cdot,\bar{\omega}) + c[X_t] + c[Y_t].$$

We define $f_t = c[X_t] + c[Y_t]$ and find the stated representation. □

Remark 2.6. This decomposition can be extended to a process with independent increments and values in $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Indeed, that

$$X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(n)} \end{pmatrix}$$

is a process with independent increments. For $i = 1, \dots, n$, $X^{(i)}$ has independent increments. From the previous result we can write,

$$X = \begin{pmatrix} f^{(1)} \\ f^{(2)} \\ \vdots \\ f^{(n)} \end{pmatrix} + \begin{pmatrix} X^{c,(1)} \\ X^{c,(2)} \\ \vdots \\ X^{c,(n)} \end{pmatrix} + \begin{pmatrix} \Sigma^{(1)} \\ \Sigma^{(2)} \\ \vdots \\ \Sigma^{(n)} \end{pmatrix} + \begin{pmatrix} \Pi^{(1)} \\ \Pi^{(2)} \\ \vdots \\ \Pi^{(n)} \end{pmatrix}.$$

What remains to show is that X^c is continuous in probability. We already know that $X^{c,(i)}$ are continuous in probability for each i. For $\epsilon > 0$ and $s, t \in \mathbb{R}_+$ it holds

$$\begin{aligned} \mathbb{P}(||X_t^c - X_s^c|| > \epsilon) &= \mathbb{P}(||X_t^c - X_s^c||^2 > \epsilon^2) \\ &\leq \sum_{i=1}^n \mathbb{P}(|X_t^{c,(i)} - X_s^{c,(i)}|^2 > \frac{\epsilon^2}{n}). \end{aligned} \tag{24}$$

From the continuity of $X^{c,(i)}$ we find that X^c is continuous in probability.

2.5 Càdlàg modification

We now consider Additive processes X with values in a separable Banach space E. For such a process we can construct a càdlàg modification. For processes with values in $\mathbb{R}$ this fact is proven by J.L. Doob [6]. For a martingale argument, see [1]. For a proof with Dynkin-Kinney Theorem, see [20, Chapter 2, Theorem 11.5].

We use Lemma 2.1 and [21] to construct the modification for Additive processes with values in separable Banach spaces.

Definition 2.11. Let X, Y be two stochastic processes defined on the same underlying probability space. We call Y a modification of X if for every $t > 0$

$$\mathbb{P}\{X_t = Y_t\} = 1.$$

Definition 2.12. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ be a filtration, then we define

$$\mathcal{F}_{s+} := \bigcap_{u > s} \mathcal{F}_u,$$

to be the right-continuous extension of $\mathcal{F}_t$.

The following lemma states that it costs nothing to replace the filtration $\mathcal{F}_s$ by its right-continuous version $\mathcal{F}_{s+}$.

Lemma 2.14. *Let X be an Additive process w.r.t. the filtration $\mathcal{F}_t$, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ for every $\geq t \geq s \geq 0$. Then X is also an Additve process w.r.t. $\mathcal{F}_{s+}$.*

Proof. First suppose that Z is a random variable independent of $\mathcal{F}_{s+}$. If Y is another random variable such that $\mathbb{P}\{Y = Z\} = 1$, then for every $F \in \mathcal{F}_{s+}$ and $B \in \mathcal{B}(E)$ it holds

$$\begin{aligned}\mathbb{P}\{\{Y \in B\} \cap F\} &= \mathbb{P}\{\{Y = Z\} \cap \{Y \in B\} \cap F\} \\ &= \mathbb{P}\{\{Z \in B\} \cap F\} \\ &= \mathbb{P}\{Z \in B\}\mathbb{P}\{F\} \\ &= \mathbb{P}\{Y \in B\}\mathbb{P}\{F\}.\end{aligned} \tag{25}$$

Notice that that $X_t - X_{s_n} \xrightarrow{\mathbb{P}} X_t - X_s$ for some sequence $s_n \uparrow t$. Then there is a subsequence such that $X_t - X_{s_{n_k}} \xrightarrow{a.s.} X_t - X_s$. Now we can define a random variable

$$Z(\omega) = \begin{cases} \lim\limits_{k\to\infty} X_t(\omega) - X_{s_{n_k}}(\omega) & \text{if } \omega \in \left\{\lim_{k\to\infty}\left(X_t - X_{s_{n_k}}\right) \text{exists}\right\} \\ 0 & \text{if } \omega \notin \left\{\lim_{k\to\infty}\left(X_t - X_{s_{n_k}}\right) \text{exists}\right\} \end{cases}$$

It holds that Z is $\bigcap_{s<u\le t}\mathcal{F}^X_{u,t}$-measurable. From this it holds that Z is independent of $\mathcal{F}_{s+}$. It holds that $Z = X_t - X_s$ a.s. We conclude that $X_t - X_s$ independent of $\mathcal{F}_{s+}$.

□

We will now consider oscillations of our sample paths of X. For this we need the following definitions.

Definition 2.13. Let X be a stochastic process. Let $T = \{t_1, \dots, t_n\} \subset \mathbb{R}_+$ be a finite set of time points. We define the number of oscillations of length $\delta > 0$ on T by

$$\mathbf{U}_X(T,\delta) := \sup\left\{k : \exists\, \tau_1 < \tau_2 < \ldots < \tau_{k+1} \text{ in } T, ||X_{\tau_{i+1}} - X_{\tau_i}|| > \delta, i = 1,\dots,k\right\}. \tag{26}$$

Definition 2.14. Let X be a stochastic process. Let $\mathbb{T} \subset \mathbb{R}_+$ be a countable subset. The number of oscillations of length $\delta > 0$ on $\mathbb{T}$ is defined by

$$\mathbf{U}_X(\mathbb{T},\delta) := \sup\{\mathbf{U}_X(T,\delta) : T \subset \mathbb{T},\ T \text{ finite}\}. \tag{27}$$

We will need an estimation for the number of oscillation of length $\delta > 0$. This will be the content of the following lemmas. Lemmas 2.15, 2.16 are proved in [21] for the real case. We easily can extend these for processes with values in separable Banach spaces.

Lemma 2.15. *Let $\{X\}$ be an Additive process. Let $T = \{t_1 < t_2 < \ldots < t_n\}$ be time points such that $\mathbb{P}\{||X_{t_k} - X_{t_1}|| > \delta\} \le \epsilon < \frac{1}{4}$, for $k = 1,\dots,n$. Then we have the following estimate,*

$$\mathbb{E}\,\boldsymbol{U}_X(T,4\delta) \le \frac{4\epsilon}{1-4\epsilon}.$$

Proof. Let $m \in \{1,\dots,n-1\}$ and define the sets $T_k = \{t_k, t_{k+1}, \dots, t_n\}$ and

$$\begin{aligned} A_k &= \left\{||X_{t_2} - X_{t_1}|| \le 4\delta, \dots, ||X_{t_{k-1}} - X_{t_{k-2}}|| \le 4\delta, ||X_{t_k} - X_{t_{k-1}}|| > 4\delta\right\} \\ B_k &= A_k \cap \{\mathbf{U}_X(T_k,4\delta) \ge m-1\}. \end{aligned}$$

Now it holds that A_k for $k = 2, \ldots n-1$ are disjoints sets and it holds that

$$\begin{aligned}
\mathbb{P}\left\{\mathbf{U}_X(T, 4\delta) \geq m\right\} &= \sum_{k=2}^{n-1} \mathbb{P}\left\{B_k\right\} \\
&= \sum_{k=2}^{n-1} \mathbb{P}\left\{\mathbf{U}_X(T_k, 4\delta) \geq m-1\right\} \mathbb{P}\left\{A_k\right\} \\
&\leq \sum_{k=2}^{n-1} \mathbb{P}\left\{\mathbf{U}_X(T, 4\delta) \geq m-1\right\} \mathbb{P}\left\{A_k\right\} \\
&\leq \mathbb{P}\left\{\mathbf{U}_X(T, 4\delta) \geq m-1\right\} \mathbb{P}\left\{\max_{1\leq k,l\leq n} ||X_{t_k} - X_{t_l}|| > 4\delta\right\} \\
&\leq 4\mathbb{P}\left\{\mathbf{U}_X(T, 4\delta) \geq m-1\right\} \mathbb{P}\left\{||X_{t_n} - X_{t_1}|| > \delta\right\} \\
&\leq \mathbb{P}\left\{\mathbf{U}_X(T, 4\delta) \geq m-1\right\} 4\epsilon
\end{aligned} \tag{28}$$

By iteration, $\mathbb{P}\left\{\mathbf{U}_X(T, 4\delta) \geq m\right\} = (4\epsilon)^m$. This yields

$$\mathbb{E}\mathbf{U}_X(T, 4\delta) = \sum_{m\geq 1} \mathbb{P}\left\{\mathbf{U}_X(T, 4\delta) \geq m\right\} \leq \frac{4\epsilon}{1-4\epsilon}.$$

□

Lemma 2.16. *Let X be an Additive process. Let $\mathbb{T} \subset [s_1, s_2]$ be a countably dense set of time points, $s_1, s_2 \in \mathbb{T}$. Suppose that $\mathbb{P}\left\{||X_t - X_{s_1}|| > \delta\right\} \leq \epsilon < \frac{1}{4}$, for all $t \in \mathbb{T}$. Then we have the following estimate,*

$$\mathbb{E}\,\boldsymbol{U}_X(\mathbb{T}, 4\delta) \leq \frac{4\epsilon}{1-4\epsilon}.$$

Proof. For every finite set $T \subset \mathbb{T}$ with $s_1 \in T$, we find by Lemma 2.16 it holds

$$\mathbb{E}\mathbf{U}_X(\mathbb{T}, 4\delta) \leq \frac{4\epsilon}{1-4\epsilon}.$$

Now by the monotone convergence theorem the statement follows. □

Lemma 2.17. *Let X be an Additive process and $\mathcal{D} \subset \mathbb{R}_+$ be countable dense subset. Then for every $T > 0$,*

$$\mathbb{P}\left\{\sup_{t\in\mathcal{D}\cap[0,T]} ||X_t|| < \infty\right\} = 1. \tag{29}$$

Proof. Fix $T > 0$ and let $0 = d_1 < d_2 < \ldots < d_n = T$, with $d_2, \ldots, d_{n-1} \in \mathcal{D}$. By continuity of the map

$$X : [0, T] \to L^0_{\mathbb{P}}(\Omega; E), \quad t \mapsto X_t,$$

it follows that the collection of random variables $(X_t)_{t\in[0,T]}$ is a compact subset of $L^0_{\mathbb{P}}(\Omega; E)$. We can take by Prohkorov's theorem 4.3 a real value $c > 0$ sufficiently large so that $\max_{1\leq i\leq n} \mathbb{P}(\{||X_{d_i}|| > c\}) \leq \delta < \frac{1}{2}$. Then from

$$\left\{\max_{1\leq i\leq n} ||X_{d_i}|| > 4c\right\} \subset \left\{\max_{1\leq i,j\leq n} ||X_{d_i} - X_{d_j}|| > 4c\right\},$$

and Lemma 2.1 we find

$$\begin{aligned}\mathbb{P}\left\{\max_{1\leq i\leq n}||X_{d_i}||>4c\right\} &\leq \mathbb{P}\left\{\max_{1\leq i,j\leq n}||X_{d_i}-X_{d_j}||>4c\right\} \\ &\leq 4\mathbb{P}\left\{||X_T||>c\right\}.\end{aligned} \tag{30}$$

From this we conclude that $\mathbb{P}\left(\left\{\sup_{d\in\mathcal{D}\cap[0,T]}||X_d||>4c\right\}\right)\leq 4\mathbb{P}\left((\{||X_T||>c\}\right)$. By taking $c\to\infty$ we find from tightness of X_T that $\sup_{d\in\mathcal{D}\cap[0,T]}||X_d||<\infty$ a.s. □

Theorem 2.8. *Let X be an Additive Process. Let $\mathcal{D}\subset\mathbb{R}_+$ be a countable, dense subset. Then for every $T>0$ there exists a set Ω_T of probability 1, such that*

$$\lim_{\substack{d\uparrow s\\ d\in\mathcal{D}}} X_d \quad\text{and}\quad \lim_{\substack{d\downarrow s,\\ d\in\mathcal{D}}} X_d,$$

exist and are finite for $s\in(0,T]$ and $s\in[0,T)$, respectively.

Proof. Let $t\in[0,T]$, $\delta>0$ and $0<\epsilon<\frac{1}{4}$. By Lemma 4.5 there exists a $\theta>0$ such that for all $u,v\in[0,T]$ with $|u-v|<\theta$, it holds that

$$\mathbb{P}\left\{||X_u-X_v||>\delta\right\}<\epsilon.$$

Take $s_1,s_2\in[0,T]\cap\mathcal{D}$ such that $s_1<t<s_2$ and $|s_1-s_2|<\theta$. Then by Lemma 2.15 it follows that

$$\mathbb{E}\mathbf{U}_X(\mathcal{D}\cap[s_1,s_2],4\delta)\leq\frac{4\epsilon}{1-4\epsilon}.$$

We can cover $[0,T]$ with a finite number of compact intervals $[s_1,s_2]$ and from this

$$\mathbb{E}\mathbf{U}_X(\mathcal{D}\cap[0,T],4\delta)\leq\frac{4\epsilon}{1-4\epsilon}.$$

We define now for $T>0$ the following set,

$$\Omega_T:=\left\{\omega\in\Omega:U_X(\mathcal{D}\cap[0,T],\frac{1}{n})<\infty,\forall n\text{ and }\sup_{d\in\mathcal{D}\cap[0,T]}||X_d||<\infty\right\}\in\mathcal{F}_T. \tag{31}$$

Note that by Lemma 2.17 and $\mathbf{U}_X(\mathcal{D}\cap[0,T],4\delta)<\infty$ *a.s.*, for every choice $\delta>0$, $\mathbb{P}\left\{\Omega_T\right\}=1$. Let $s\in(0,T]$ and suppose that for $\omega\in\Omega$, $\lim_{d\downarrow s}X_d$ does not exists. There are two possibilities:

1. There is a sequence $d_n\downarrow s$, $d_n\in\mathcal{D}$, such that $\lim_n||X_{d_n}||=\infty$.

2. There is a sequence $d_n\downarrow s$, $d_n\in\mathcal{D}$, and $\delta>0$ such that $\mathbf{U}_X(\{d_n\}_{n=1}^{\infty},\delta)=\infty$.

In both cases $\omega\notin\Omega_T$. For $s\in[0,T)$ and $\lim_{d\uparrow s}X_d$ we can apply the same arguments. We conclude that on Ω_T, the limits

$$\lim_{\substack{d\uparrow s\\ d\in\mathcal{D}}} X_d \quad\text{and}\quad \lim_{\substack{d\downarrow s,\\ d\in\mathcal{D}}} X_d$$

exist and are finite for $s\in(0,T]$ and $s\in[0,T)$, respectively. □

Theorem 2.9. *Let $\{X\}_{t\in\mathbb{R}_+}$ be an Additive process and $\mathcal{D}$ a countably dense in $\mathbb{R}_+$. Then there exists a modifiction $\bar{X}$ with independent increments adapted to $\{\mathcal{F}_{t+}\}_{t\in\mathbb{R}_+}$, with càdlàg paths for all $\omega \in \Omega$.*

Proof. Let Ω_T be defined as (31). It holds that $\Omega_T \subset \Omega_S$ for $T \geq S$. Define $\Omega_{T+} = \bigcup_{S>T} \Omega_S$. Put

$$\bar{X}_T(\omega) = \begin{cases} \lim\limits_{d\in\mathcal{D}, d\downarrow T} X_d(\omega) & \text{if } \omega \in \Omega_{T+} \\ 0 & \text{if } \omega \notin \Omega_{T+} \end{cases}$$

Let $T \in \mathbb{R}_+$, and suppose that $\omega \notin \Omega_{T+}$. Then for every $S > T$ it holds that $\omega \notin \Omega_{S+}$, Hence $\bar{X}_T = \bar{X}_S = 0$ and thus $\lim_{S\downarrow T} \bar{X}_S = \bar{X}_T$. Suppose $\omega \in \Omega_{T+}$, then right-continuity at T follows from existence of limits $\lim_{\substack{d\downarrow T, \\ d\in\mathcal{D}}} X_d$. Furthermore the existence of left-limits follows from right-continuity. We conclude that every path of $\bar{X}$ is càdlàg. For every $t > 0$, $\mathbb{P}\left\{X_t = \bar{X}_t\right\} = 1$ because X is continuous in probability. Let $S < T$, then

$$\bar{X}_T - \bar{X}_S = \lim_{n\to\infty} X_{t_n} - X_{s_n} \quad a.s.$$

for sequences $t_n \downarrow T$ and $s_n \downarrow S$. From this it holds that $\bar{X}_T - \bar{X}_S$ is independent of $\mathcal{F}_{S+}$. □

3 Lévy-Ito decomposition

3.1 Introduction

In this chapter we consider the structure of Additive processes, i.e. stochastic processes with independent increments that are continuous in probability. By Theorem 2.9 there exists a càdlàg modification. In this chapter an Additive process $\{X_t\}_{t\in\mathbb{R}_+}$ will therefore always be assumed to be càdlàg.

By the càdlàg property it is possible to define left-jumps, $\Delta X_{t-} := \lim_{s\uparrow t} X_t - X_s$. If now the condition of continuity in probability is dropped we can have fixed jumps at time $t > 0$. Suppose at $t > 0$ the process has a jump $\Delta X_{t-}(\omega) \neq 0$, we say that X has a fixed jump at time $t > 0$, if $\mathbb{P}\{\omega : \Delta X_{t-}(\omega) \neq 0\} \neq 0$. If $\{X_t\}_{t\in\mathbb{R}_+}$ is a process with independent increments and the càdlàg property, then continuity in probability has a natural interpretation. The condition of continuity in probability excludes fixed jumps. This makes Additive processes a suitable model for phenomena with jumps at unexpected times.

Suppose X has a jump at $t > 0$, i.e. $\Delta X_{t-} = X_t - X_{t-} = x \neq 0$. We represent such a jump at t with amplitude x as a point (t, x) in the (t, E)-plane $\mathbb{R}_+ \times (E \setminus 0)$. Note in $t = 0$ there is no jump. Let us fix some notation. We denote $S = \mathbb{R}_+ \times (E \setminus 0)$ and $\mathcal{S} = \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(E \setminus 0)$. We will count for every $A \in \mathcal{S}$ the number of points $(t, \Delta X_{t-}) \in A$. For $A \in \mathcal{S}$ define $A(t) = \{(s, x) \in A : 0 \leq s \leq t\}$ and $A^n = \left\{(s, x) \in A : ||x|| > \frac{1}{n}\right\}$. We say $A \in \mathcal{S}(u)$ for $u > 0$ if

$$A \subset \mathbb{R}_+ \times \{x \in E : ||x|| > u\} .$$

Let $\mathcal{S}^0 := \bigcup_{u>0} \mathcal{S}(u)$ be the collection of all Borel sets with positive distance from the origin. Note that $\mathcal{S}^0$ is a ring of subsets of S.

Definition 3.1. Let X be an Additive process.

1. For $A \in \mathcal{S}$, denote the number of jumps in A by $\mathcal{J}_X(\omega, A) := \#\{(t, \Delta X_t) \in A\}$.

2. For all $t > 0$ and $A \in \mathcal{S}^0$ the number of jumps that are in the Borel set A in the interval $[0, t]$ is denoted as $N_t(A) := \mathcal{J}_X(\cdot, A(t))$.

3. Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda}$, then the sum of all jumps that took place on $[0, t]$ and are in Λ is denoted as

$$\Delta X_t^\Lambda := \sum_{\substack{h\leq t,\\ \Delta X_{h-}\in\Lambda}} \Delta X_{h-}.$$

In section 3.2.1 we show that $\{\mathcal{J}_X(\cdot, A)\}_{A\in\mathcal{S}}$ is a random measure. In section 3.2.2 we consider $\{N_t(A)\}_{t\in\mathbb{R}_+}$ for every $A \in \mathcal{S}^0$. We will show $\{N_t(A)\}_{t\in\mathbb{R}_+}$ is Additive and Poisson. In section 3.2.3 it is shown that for $\Lambda \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda}$ the process ΔX_t^Λ is

Additive. This process can naturally be represented as an integral with respect to the random measure $\mathcal{J}_X$ counting the jumps of X. Next we define the sets

$$\Lambda_1 = \{x \in E : ||x|| > 1\}, \quad \Lambda_n = \left\{x \in E : \frac{1}{n} \leq ||x|| < \frac{1}{n-1}\right\}. \tag{32}$$

In section 3.3 we will show that $\Delta X^{\Lambda_1}, \dots, \Delta X^{\Lambda_k}, X - \Delta X^{\bigcup_i \Lambda_i}$ are independent processes that never jump together. The process $\{X_t\}_{t\in\mathbb{R}_+}$ can be represented as

$$X_t = \underbrace{\sum_{i=1}^{N} \Delta X_t^{\Lambda_i}}_{'Jump\ part'} + \underbrace{X_t - \Delta X_t^{\bigcup_i^N \Lambda_i}}_{'Continuous\ part'}.$$

By taking $N \to \infty$, we are exhausting the jumps of $\{X_t\}_{t\in\mathbb{R}_+}$ in the continuous part. We expect that $\{X_t\}_{t\in\mathbb{R}_+}$ consists of a continuous part and a jump part. This representation is the so called Lévy-Ito decomposition. The main difficulty is to show in what sense $\sum_{i=1}^{N} \Delta X^{\Lambda_i}$ converges.

"Mathematics consists of proving the most obvious thing in the least obvious way"

In section 3.5 we define the map

$$\mu_X : \mathcal{S} \to [0, \infty], \quad A \mapsto \mathbb{E}[\mathcal{J}_X(\cdot, A)],$$

and show that $\{\mathcal{J}_X(\cdot, A)\}_{A\in\mathcal{S}}$ is a random Poisson measure with Poisson intensity μ_X. We collect all observations and state the structure of Additive process in its most abstract form. We represent a general Additive process as an Additive process with a.s. continuous paths and an integral w.r.t. the random Poisson measure with Poisson intensity.

There are two ways of approaching the structure of Additive processes. One approach is using the correspondence of infinitely divisible distributions and the collection of Additive processes. A random variable X is infinitely divisible if for every $n \in \mathbb{N}$ it can be written as

$$X \overset{d}{=} X_{n,1} + \cdots X_{n,n}, \tag{33}$$

where $X_{n,1}, \dots, X_{n,n}$ are i.i.d.

Theorem 3.1. *If $\{X\}_{t\in\mathbb{R}_+}$ is Additive, then for every $t \in \mathbb{R}_+$, X_t is infinitely divisible.*

Proof. See [20, Theorem 9.1]. □

Then with the aid of the Lévy-Khintchine representation of characteristic functions of infinitely divisible distributions, see [20, Theorem 8.1], the Levy-Ito decomposition can be proved. For this approach we refer to [20].

Another approach is a direct analysis of jumps of sample paths. This approach goes back to Ito [10]. The ideas describing the structure of Additive processes comes from

Lévy and were realized by Ito. With every infinitely divisible distribution μ it is possible to construct a Lévy process $\{X_t\}_{t\in\mathbb{R}_+}$ such that X_1 has distribution μ, see [20, Theorem 7.10]. This direct analysis gives as by product the Lévy-Khintchine representation, [10]. We will follow this approach. In section 3.5 we give some concluding remarks.

3.2 Analysis of jumps

We consider the objects defined in Definition 3.1. It will be shown that for $\Lambda \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda}$ the process $\Delta \left\{X_t^\Lambda\right\}_{t\in\mathbb{R}_+}$ is additive. Let $\Lambda_1 = \{x \in E : ||x|| > 1\}$, then $X_t^1 := X_t - \Delta X_t^{\Lambda_1}, t \geq 0$ is additive with bounded jumps. If an Additive processes has bounded jumps, then the k-th moments are all finite, $k = 1, 2, \ldots$

Definition 3.2. Let $\{X\}_{t\in\mathbb{R}_+}$ be an Additive process, then we say that X has *bounded jumps* if for some $K > 0$,

$$\mathbb{P}\left\{\sup_{t\in\mathbb{R}_+} \|\Delta X_{t-}\| \leq K\right\} = 1.$$

Theorem 3.2. *Let X be an Additive process with values in a separable Banach space with bounded jumps. Then for every $s, t \in \mathbb{R}_+, s < t$,*

$$\mathbb{E}\left[\sup_{s\leq u\leq t} \|X_u - X_s\|^k\right] < \infty, \quad \text{for all } k \in \mathbb{N}.$$

Proof. See [9, Chapter IV, §1, Lemma 2 (13), p.267]. □

Lemma 3.1. *For every $\Lambda \in \mathcal{B}(E)$ with $||x|| \leq K$ for all $x \in \Lambda$ and some $K > 0$ it holds $\mathbb{E}[\sup_{t\in[0,T]} \left\|\Delta X_t^\Lambda\right\|^2] < \infty$ and $t \mapsto \mathbb{E}[\Delta X_t^\Lambda]$ is continuous.*

Proof. It holds $\left\{\Delta X_t^\Lambda\right\}_{t\in\mathbb{R}_+}$ has bounded jumps. The first statement follows from Theorem 3.2. The second statement follows from continuity in probability and the dominated convergence Theorem [16, Proposition 1.8]. □

With Additive process with bounded jumps we enter in the realm of martingales. Let $\{X_t\}_{t\in\mathbb{R}_+}$ be a real-valued Additive process. The process $X_t - \mathbb{E}X_t$ is a martingale. Now the whole martingale machinery can be used to investigate the structure of these real-valued processes, see [4].

3.2.1 Random jump measure

The goal is to prove that $\{\mathcal{J}_X(\cdot, A)\}_{A\in\mathcal{S}}$ is a random measure. We use [11].

Definition 3.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and (X, Σ) be a measure spaces. A random measure is a map $\mathcal{M} : \Omega \times \Sigma \to \mathbb{R}_+ \cup \{\infty\}$ such that

1. For every $A \in \Sigma$, $\omega \mapsto \mathcal{M}(\omega, A)$ is $\mathcal{F}$-measurable.

2. For every $\omega \in \Omega$, $A \mapsto \mathcal{M}(\omega, A)$ is a measure for (X, Σ).

If $\mathcal{M}$ takes values in $\mathbb{N} \cup \{\infty\}$, then we call $\mathcal{M}$ a random counting measure.

Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \bar{\Lambda}$, then by Lemma 4.2 the number of jumps with amplitude in Λ is finite for every $\omega \in \Omega$. It is clear that $\mathcal{J}_X(\cdot, (s, t] \times \Lambda)$ is determined by the increments on $[s, t]$, hence it is measurable w.r.t.

$$\sigma\left\{X_u - X_v : s \leq u < v \leq t\right\}.$$

This is the content of the following lemma.

Lemma 3.2. *Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \bar{\Lambda}$, then for $s < t$, $\mathcal{J}_X(\cdot, (s,t] \times \Lambda)$ is $\mathcal{F}^X_{s,t}$-measurable.*

Proof. First we define $\Lambda_n = \left\{x \in E : d(x, \Lambda) < \frac{1}{m}\right\}$. Suppose that $\mathcal{J}_X \geq n$, then there exist $t_1, \ldots, t_n \in (s,t]$ so that $\Delta X_{t_i} \in \Lambda$. For every $m, k \in \mathbb{N}$ we can find $p_i, q_i \in (s,t] \cap \mathbb{Q} \cup \{t\}$ and $q_n = t$ possibly, $i = 1, \ldots, n$ such that

$$s < p_1 < t_1 < q_1 < p_2 < t_2 < \ldots p_n < t_n < q_n \leq t$$

with $|p_i - q_i| < \frac{1}{k}$ and $X_{q_i} - X_{p_i} \in \Lambda_m$. From this we find

$$\{\mathrm{J}_X \geq n\} \subset \bigcap_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{\substack{p_1 < q_1 < \ldots < p_n < q_n \\ p_i, q_i \in (s,t] \cap \mathbb{Q}, \text{ or } q_n = t \\ |p_i - q_i| < \frac{1}{k}}} \bigcap_i \{X_{q_i} - X_{p_i} \in \Lambda_m\} = (*).$$

Suppose now the converse, $\omega \in (*)$. For every $m, k \in \mathbb{N}$ we can find $p_i^{(m,k)}, q_i^{(m,k)}$ with $|q_i^{(m,k)} - p_i^{(m,k)}| < \frac{1}{k}$ and $X_{q_i^{(m,k)}} - X_{p_i^{(m,k)}} \in \Lambda_m$ for $i = 1, 2, \ldots, n$. We take now $l = k = m$ and consider for every $l \in \mathbb{N}$,

$$s < p_1^{(l,l)} < q_1^{(l,l)} < p_2^{(l,l)} < \ldots p_n^{(l,l)} < q_n^{(l,l)} \leq t.$$

We can take a subsequence $(l(r))_{r \in \mathbb{N}}$ such that the sequences $p_i^{(l(r),l(r))}, q_i^{(l(r),l(r))}$ converge for every $i = 1, \ldots, n$. We have that $|q_i^{(l(r),l(r))} - p_i^{(l(r),l(r))}| < \frac{1}{l(r)}$ and $X_{q_i^{(l(r),l(r))}} - X_{p_i^{(l(r),l(r))}} \in \Lambda_{l(r)}$, we conclude that we can find $t_1, \ldots, t_n \in (s,t]$ such that $\Delta X_{t_i} \in \Lambda$, hence $\mathrm{J}_X \geq n$. It follows that

$$\{\mathrm{J}_X \geq n\} = \bigcap_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} \bigcup_{\substack{p_1 < q_1 < \ldots < p_n < q_n \\ p_i, q_i \in (s,t] \cap \mathbb{Q}, \text{ or } q_n = t \\ |p_i - q_i| < \frac{1}{k}}} \bigcap_i \{X_{q_i} - X_{p_i} \in \Lambda_m\} \in \mathcal{F}^X_{s,t}.$$

□

The next lemma shows that for every $A \in \mathcal{S}$ with $A \subset (s,t] \times \Lambda$, $\mathcal{J}_X(\cdot, A)$ is $\mathcal{F}^X_{s,t}$-measurable.

Lemma 3.3. *Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \bar{\Lambda}$ and define for $s < t$,*

$$\mathcal{D}^{\Lambda}_{s,t} = \left\{A \in \mathcal{S} : \mathcal{J}_X(\cdot, ((s,t] \times \Lambda) \cap A) \text{ is } \mathcal{F}^X_{s,t}\text{−measurable}\right\}.$$

Then it holds that $\mathcal{D}^{\Lambda}_{s,t} = \mathcal{S}$.

Proof. First we show that $\mathcal{D}^{\Lambda}_{s,t}$ is a D-system. By lemma 3.2 it holds that $S \in \mathcal{D}^{\Lambda}_{s,t}$. Let $A, B \in \mathcal{D}^{\Lambda}_{s,t}$ with $A \subset B$, then for $\omega \in \Omega$,

$$\mathcal{J}_X(\omega, ((s,t] \times \Lambda) \cap B \setminus A) = \mathcal{J}_X(\omega, ((s,t] \times \Lambda) \cap B) - \mathcal{J}_X(\omega, ((s,t] \times \Lambda) \cap A).$$

From this it follows $B \setminus A \in \mathcal{D}^{\Lambda}_{s,t}$. Let $A_n \in \mathcal{D}^{\Lambda}_{s,t}$ such that $A_n \subset A_{n+1}$ and $A = \bigcup_n A_n$, then

$$\mathcal{J}_X(\omega, ((s,t] \times \Lambda) \cap A) = \sup_n \mathcal{J}_X(\omega, ((s,t] \times \Lambda) \cap A_n).$$

From this we conclude $A \in \mathcal{D}^{\Lambda}_{s,t}$. The class of subsets $\mathcal{D}^{\Lambda}_{s,t}$ is a D-system. Now define the following class of subsets

$$\mathcal{C} = \left\{ (s,t] \times \Lambda : s < t, \Lambda \in \mathcal{B}(E \setminus 0), \text{ with } 0 \notin \bar{\Lambda} \right\}.$$

Note $\mathcal{C}$ is a π-system. Furthermore the σ-algebra generated by $\mathcal{C}$ is $\mathcal{S}$. It follows that $\mathcal{C} \subset \mathcal{D}^{\Lambda}_{s,t}$ and thus by Theorem 4.1, $\sigma\{\mathcal{C}\} \subset \mathcal{D}^{\Lambda}_{s,t}$. From this we conclude that $\mathcal{D}^{\Lambda}_{s,t} = \mathcal{S}$. □

Lemma 3.4. *Let X be an Additive process. Then $\{\mathcal{J}_X(\cdot, A)\}_{A \in \mathcal{S}}$ is a random measure, i.e. the map $\mathcal{J}_X : \Omega \times \mathcal{S} \to \mathbb{N} \cup \{\infty\}$ is a random counting measure.*

Proof. Let $A \in \mathcal{S}$ and consider $C_n = \left\{ (h,x) \in S : h \in (\frac{1}{n}, n], \frac{1}{n} < \|x\| \leq n \right\}$. It holds that $\mathcal{J}_X(\cdot, A) = \sup_n \mathcal{J}_X(\cdot, A \cap C_n)$. For all $\omega \in \Omega$ by Lemma 3.3 $\mathcal{J}_X(\omega, A \cap C_n)$ is measurable for each n and thus by the monotone convergence theorem it follows that $\mathcal{J}_X(\omega, A)$ is measurable. The map $A \mapsto \mathcal{J}_X(\omega, A)$ takes values in $\mathbb{N}$, possibly ∞. It is clear by definition that for every $\omega \in \Omega$, $A \mapsto \mathcal{J}_X(\omega, A)$ is a counting measure. □

3.2.2 Poisson processes

In this section we consider for every $A \in \mathcal{S}^0$ the process $N_t(A)$ defined as the number of jumps of $\{X_t\}_{t \in \mathbb{R}_+}$ with amplitudes in the Borel set A during the interval $[0,t]$. Note that A is a Borel set with positive distance from the origin. By Lemma 4.2, $X_t(\omega)$ has a finite number of jumps with amplitude bigger than $u > 0$. From this, $N_t(A)$ is finite for every realization. The first result we will obtain is that $\{N_t(A)\}_{t \in \mathbb{R}_+}$ is a Poisson process.

Definition 3.4. A stochastic process N_t is called a Poisson process with intensity $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ if,

1. For all $\omega \in \Omega$, $N_0(\omega) = 0$.
2. The intensity $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function and $\Lambda(0) = 0$.
3. For every set $t_1 < \ldots < t_n$ of timepoints, $N_{t_n} - N_{t_{n-1}}, N_{t_{n-1}} - N_{t_{n-2}}, \ldots, N_{t_1}$ are independent random variables.
4. For every $s < t, s, t \in \mathbb{R}_+$
$$N_t - N_s \sim \text{Poiss}(\Lambda(t) - \Lambda(s)).$$

The characteristic function of a Poisson process N_t with intensity $\Lambda(t)$ is

$$\Phi_{N_t}(u) = e^{\Lambda(t)\left(e^{i \cdot u} - 1\right)}.$$

Lemma 3.5. *For every $A \in \mathcal{S}^0$, the process $\{N_t(A)\}_{t \in \mathbb{R}_+}$ is Additive and Poisson.*

Proof. The process $N_t(A)$ is an increasing process with independent increments by lemma 3.2. The process grows with jumps of amplitude 1. This process has a finite number of jumps on $[0,t]$ and is càdlàg. That $N_t(A)$ is continuous in probability follows from continuity in probability of X. To see this let $t > 0$ and suppose that $\Delta N_{t-}(A) \neq 0$. By definition of $N_t(A)$, $\mathbb{P}(\Delta N_{t-}(A^n) \neq 0) \leq \mathbb{P}(\Delta X_{t-} \neq 0) = 0$. This shows left-continuity in probability. Right-continuity follows by definition. By lemma 4.9 it holds that $N_t(A^n)$ is a Poisson process. □

Lemma 3.6. *For every $A \in \mathcal{S}^0$, the random variable $\mathcal{J}_X(\omega, A)$ is Poisson distributed with*

$$\lim_{t\to\infty} \mathbb{E}[\mathcal{J}_X(\omega, A(t))] = \mathbb{E}[\mathcal{J}_X(\omega, A)] < \infty.$$

Proof. Let $A \in \mathcal{S}^0$, then $N_t(A)$ is Additive and Poisson. Let Λ be the intensity of $N_t(A)$. By Monotone convergence Theorem it holds that

$$\mathbb{E}[\mathcal{J}_X(\omega, A)] = \lim_{t\to\infty} \mathbb{E}[\mathcal{J}_X(\omega, A(t))].$$

From $\{N_t(A) > N\} \subset \{\mathcal{J}_X(A) > N\}$ it follows

$$\mathbb{P}\{\mathcal{J}_X(A) > N\} \geq \mathbb{P}\{N_t(A) > N\} = 1 - \sum_{i=0}^{N} \frac{(\Lambda^n(t))^i}{i!} e^{-\Lambda^n(t)}. \tag{34}$$

Suppose that $\lim_{t\to\infty} \mathbb{E}[\mathcal{J}_X(\omega, A(t))] = \lim_{t\to\infty} \Lambda(t) = \infty$, then with (34) we find that $\mathbb{P}\{\mathcal{J}_X(A) > N\} = 1$ for every $N \in \mathbb{N}$. Hence $\mathbb{P}\{\mathcal{J}_X(A) = \infty\} = 1$, but $\mathcal{J}_X(A)$ is finite for $A \in \mathcal{A}^0$. This is a contradiction. We conclude $\lim_{t\to\infty} \mathbb{E}[\mathcal{J}_X(\omega, A(t))] < \infty$. □

Lemma 3.7. *Let X be an Additive process. The map defined by,*

$$\mu_X : \mathcal{S}^0 \to [0, \infty], \ A \mapsto \mathbb{E}\left[\mathcal{J}_X(\cdot, A)\right], \tag{35}$$

is a σ-finite premeasure on $(S, \mathcal{S}^0)$.

Proof. By definition of $\mathcal{J}_X$, $\mathcal{J}_X(\omega, \emptyset) = 0$ and thus $\mu_X(\emptyset) = 0$. Let $A, A_n \in \mathcal{S}^0$ with $A_n \cap A_m = \emptyset$ and $A = \bigcup_n A_N$, then $\mathcal{J}_X(\omega, A) = \sum_n \mathcal{J}_X(\omega, A_n)$. By the Monotone convergence Theorem $\mathbb{E}\left[\mathcal{J}_X(\omega, A)\right] = \lim_{n\to\infty} \sum_{k\leq n} \mathbb{E}\left[\mathcal{J}_X(\omega, A_k)\right]$. We conclude that the map $A \mapsto \mathbb{E}\left[\mathcal{J}_X(\omega, A)\right]$ is a measure on $(S, \mathcal{S}^0)$. From Lemma 3.7 it holds that $\mu_X(A) < \infty$ for every $A \in \mathcal{S}^0$. From this it is clear that μ_X is σ-finite. □

3.2.3 The Jump processes

Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda}$, the goal is to show that ΔX_t^Λ is an Additive process. We will obtain the representation

$$\Delta X_t^\Lambda(\omega) = \int_0^t \int_\Lambda x \mathcal{J}_X(\omega, dsdx),$$

i.e. for every $\omega \in \Omega$, the process $\Delta X_t^\Lambda(\omega)$ is represented as a Bochner-Integral with respect to the random counting measure $\mathcal{J}_X$ that counts the jumps of the process. We will first introduce this E-valued integral. We follow [18, Chp. 1.1].

First suppose that X is simple,

$$X = \sum_i x_i \mathbb{I}_{A_i}, \quad A_i \in \mathcal{F}, \quad x_i \in E.$$

We set $\int_B X(\omega)\mathbb{P}(d\omega) := \sum_i x_i \mathbb{P}(A_i \cap B)$. The value does not depend on the representation of X as simple function. By the triangle inequality it holds that

$$\left\| \int_B X(\omega)\mathbb{P}(d\omega) \right\| \leq \int_B ||X(\omega)||\mathbb{P}(d\omega).$$

For a general random variable X there is a sequence of simple random variables X_m such that $||X(\omega) - X_m(\omega)||$ decreases pointwise for every $\omega \in \Omega$, monotonically to 0. To see this, let $\{e_1, \ldots, e_n, \ldots\}$ be a dense subset of E. Define $p_m = \min\{||X - e_k|| : k \leq m\}$, $k_m = \min\{k \leq m : p_m = ||X - e_k||\}$ and $X_m = e_{k_m}$. It is clear that $X_m \in \{e_1, \ldots, e_m\}$. It holds that $||X - X_m||$ monotonically decreases to 0, for every $\omega \in \Omega$.

Next we suppose that $\int_\Omega ||X(\omega)||\mathbb{P}(d\omega) < \infty$. We can show that $\int_\Omega X_m \mathbb{P}(d\omega)$ is a Cauchy sequence,

$$\left\| \int_\Omega X_n(\omega)\mathbb{P}(d\omega) - \int_\Omega X_m(\omega)\mathbb{P}(d\omega) \right\|$$
$$\leq \int_\Omega \|X(\omega) - X_n(\omega)||\mathbb{P}(d\omega) + \int_\Omega \|X(\omega) - X_m(\omega)||\mathbb{P}(d\omega).$$

The right term will go to 0 as $n, m \to \infty$. We define the integral of X now as

$$\int_\Omega X(\omega)\mathbb{P}(d\omega) := \lim_{m\to\infty} \int_\Omega X_m(\omega)\mathbb{P}(d\omega).$$

We will also use the notation $\mathbb{E}X$ for $\int_\Omega X\mathbb{P}(d\omega)$. If we take another sequence of simple functions X'_m such that $\int_\Omega ||X - X'_m||\mathbb{P}(d\omega) \to 0$ as $m \to \infty$, then we will get the same integral.

We also need a Bochner-integral on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. First note that for such a measure space there exist $(A_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ such that $\mu(A_n) < \infty$, $A_n \cap A_m = \emptyset$, for $n \neq m$ and $\bigcup_n A_n = \Omega$. Let $X : \Omega \to E$ be a measurable map and define $X^n = X \cdot \mathbb{I}_{A_n}$. As above we can find a sequence of simple random variables X_m^n such that $||X^n - X_m^n|| \to 0$ as $m \to \infty$. Define $X_m = \sum_{n=1}^m X_n^m$, then $||X(\omega) - X_m(\omega)|| \to 0$ for every $\omega \in \Omega$. Define

$$X'_m = \mathbb{I}_{\{||X_m|| \leq 2||X||\}} X_m, \tag{36}$$

then it holds that

$$||X'_m(\omega) - X(\omega)|| \to 0, \text{ as } m \to \infty \text{ and } ||X'_m(\omega) - X(\omega)|| \leq 3||X(\omega)||.$$

If $\int_\Omega ||X||\mu(d\omega) < \infty$, then by the Lebesque's dominated convergence theorem it holds that

$$\int_\Omega ||X(\omega) - X'_m(\omega)||\mu(d\omega) \to 0, \text{ as } m \to \infty.$$

We can define for a measurable function $X : \Omega \to E$ with $\int_\Omega ||X(\omega)||\mu(d\omega) < \infty$ the E-valued integral by

$$\int_\Omega X(\omega)\mu(d\omega) := \lim_{m\to\infty} \int_\Omega X'_m(\omega)\mu(d\omega).$$

Definition 3.5. Let $(\Omega, \mathcal{A}, \mu)$ a measure space and E a separable Banach space with Borel-σ-algebra $\mathcal{B}(E)$. A measurable function $f : \Omega \to E$ is Bochner-integrable if $\int ||f(\omega)||\mu(d\omega) < \infty$.

Remark 3.1. For a Bochner -integrable map $f : \Omega \to E$ it holds

$$\left\| \int_\Omega X\mu(d\omega) \right\| \leq \int_\Omega ||X||\mu(d\omega). \tag{37}$$

We would like to point out that we can define the conditional expectation in a general setting.

Theorem 3.3. *Let $X : \Omega \to E$ be a random variable in a separable Banach space E with $\mathbb{E}||X|| < \infty$ and $\mathcal{G}$ a sub-σ-algebra of $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exist a unique random E-valued, $\mathcal{G}$-measurable random variable Z , up to a probability 1 set, such that*

$$\int_A X(\omega)\mathbb{P}(d\omega) = \int_A Z(\omega)\mathbb{P}(d\omega), \quad \forall A \in \mathcal{G}. \tag{38}$$

The random variable Z will be denoted by $\mathbb{E}[X|\mathcal{G}]$.

Proof. See [18, Proposition 1.10] □

It also follows from the proof of this theorem that

$$\|\mathbb{E}[X|\mathcal{G}]\| \leq \mathbb{E}[||X|||\mathcal{G}]. \tag{39}$$

Definition 3.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Let X be an adapted E-valued stochastic process with E a separable Banach space. The process X is called a martingale if

$$\mathbb{E}[X_t|\mathcal{F}_s] = X_s, \ a.s. \tag{40}$$

Lemma 3.8. *Let X be a process with independent increments with $\mathbb{E}||X_t|| < \infty$. Then the process $X_t - \mathbb{E}X_t$ is a martingale with independent increments.*

Proof. Let $s < t$, then by the independent increment property it holds

$$\mathbb{E}[X_t - \mathbb{E}X_t|\mathcal{F}_s] = -\mathbb{E}X_t + \mathbb{E}[X_t - X_s + X_s|\mathcal{F}_s] = -\mathbb{E}X_t + \mathbb{E}[X_t - X_s] + X_s = X_s - \mathbb{E}X_s.$$

□

Theorem 3.4. *Let M be an E-valued martingale with càdlàg paths. Then for all $p \geq 1$ and $\lambda > 0$ we have*

$$\mathbb{P}\left\{ \sup_{t\in[0,T]} ||M_t|| \geq \lambda \right\} \leq \frac{1}{\lambda^p}\mathbb{E}\left[||M_T||^p\right] \tag{41}$$

Proof. Let $s < t$, then it holds by (39)

$$||M_s|| = ||\mathbb{E}[M_t|\mathcal{F}_s]|| \leq \mathbb{E}[||M_t|||\mathcal{F}_s].$$

From this we find that $||M_t||$ is a real-valued sub-martingale. For real-valued sub-martingales the equation (41) holds. □

Definition 3.7. Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda}$ and F a separable Banach space. Let $f : (S, \mathcal{S}) \to (F, \mathcal{B}(F))$ a measurable function. Then, we define

$$f(\Delta X^\Lambda)_t := \sum_{s \leq t} f(s, \Delta X_{s-}) \mathbb{I}_\Lambda(\Delta X_{s-}). \tag{42}$$

Lemma 3.9. *Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda}$ and F a separable Banach space. Suppose $f : (S, \mathcal{S}) \to (F, \mathcal{B}(F))$ a measurable function, then for every $\omega \in \Omega$ it holds $\int_0^T \int_\Lambda ||f(s,x)|| \mathcal{J}_X(\omega, dsdx) < \infty$ and*

$$f(\Delta X^\Lambda)_T = \int_0^T \int_\Lambda f(s,x) \mathcal{J}_X(\omega, ds\, dx). \tag{43}$$

Furthermore $\left\{ f(\Delta X^\Lambda)_t \right\}_{t \in \mathbb{R}_+}$ is an Additive process and for every $0 \leq u < v, u, v \in \mathbb{R}_+$,

$$f(\Delta X^\Lambda)_v - f(\Delta X^\Lambda)_u,$$

is $\mathcal{F}^X_{u,v}$-measurable.

Proof. Let $f_n = \sum_{j=1}^{m(n)} \alpha_j^{(n)} \mathbb{I}_{A_j^{(n)}}$ with $A_i^{(n)} \cap A_j^{(n)} = \emptyset$ defined as in (36) such that $f_n(s,x) \to f(s,x)$ for every $(s,x) \in \mathbb{R}_+ \times (E \setminus \{0\})$. Let $0 \leq u < v, u, v \in \mathbb{R}_+$. For every ω there are only a finite number of jumppoints $(s_n, \Delta X_{s_n-}) \in [u,v] \times \Lambda$, $n = 1, \ldots, \mathcal{J}_X(\omega, [u,v] \times \Lambda)$, see Lemma 4.2. For $\omega \in \Omega$ let $M(\omega) = \max_{s \leq T} ||f(s, \Delta X_{s-})||$.By the monotone convergence theorem it follows that

$$\int_u^v \int_\Lambda ||f(s,x)|| \mathcal{J}_X(\omega, dsdx)$$

$$\begin{aligned} &= \lim_{N \to \infty} \sum_{k=0}^{N2^N - 1} \frac{k}{2^N} \mathcal{J}_X \left(\omega, \left\{ \frac{k}{2^N} \leq ||f|| < \frac{k+1}{2^N} \right\} \cap ([u,v] \times \Lambda) \right) \\ &\leq M(\omega) \cdot \mathcal{J}_X(\omega, [u,v] \times \Lambda) < \infty. \end{aligned}$$

Hence $\int_u^v \int_\Lambda ||f(s,x)|| \mathcal{J}_X(\omega, dsdx) < \infty$. Now it holds that $f(s,x)$ is Bochner integrable and

$$\int_u^v \int_\Lambda f_n(s,x) \mathcal{J}_X(\omega, ds\, dx) \to \int_u^v \int_\Lambda f(s,x) \mathcal{J}_X(\omega, ds\, dx).$$

From this it follows

$$\int_u^v \int_\Lambda f(s,x) \mathcal{J}_X(\omega, ds\, dx) = \lim_{N \to \infty} \sum_{j=1}^{m(n)} \alpha_j^{(n)} \mathcal{J}_X \left(\omega, ([u,v] \times \Lambda) \cap A_j^{(n)} \right). \tag{44}$$

Note that $\mathcal{J}_X \left(\omega, ([u,v] \times \Lambda) \cap A_j^{(n)} \right)$ is $\mathcal{F}^X_{u,v}$-measurable and thus $\int_u^v \int_\Lambda f(s,x) \mathcal{J}_X(\omega, ds\, dx)$ is $\mathcal{F}^X_{u,v}$-measurable. There are only a finite number of jump points on compact intervals and thus for every $T > 0$ it holds

$$\max_{m=1,\ldots,\mathcal{J}_X(\omega,[0,T] \times \Lambda)} ||f_n(s_m, \Delta X_{s_m-}) - f(s_m, \Delta X_{s_m-})|| \to 0, \quad n \to \infty.$$

Note that $\int_0^T \int_\Lambda f_n(s,x)\mathcal{J}_X(\omega, ds\, dx) = \sum_{s\leq t} f_n(s,\Delta X_{s-})\mathbb{I}_\Lambda(\Delta X_{s-})$. From this it follows that

$$\int_0^T \int_\Lambda f(s,x)\mathcal{J}_X(\omega, ds\, dx) = \sum_{s\leq t} f(s,\Delta X_{s-})\mathbb{I}_\Lambda(\Delta X_{s-}).$$

Next we note that $f(\Delta X^\Lambda)_v - f(\Delta X^\Lambda)_u = \int_u^v \int_\Lambda f(s,x)\mathcal{J}_X(\omega, ds\, dx)$ is $\mathcal{F}^X_{u,v}$-measurable, hence $\left\{f(\Delta X^\Lambda)_t\right\}_{t\in\mathbb{R}_+}$ is an Additive process. □

3.3 Independence of processes with independent increments

Two processes defined on the same underlying probability space are called independent if the σ-algebra's $\mathcal{F}^X$ and $\mathcal{F}^Y$ are independent, see Definition 4.6. The main goal of this section is to prove the following theorem.

Theorem 3.5. *Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_k \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda_1}, \overline{\Lambda_2}, \ldots, \overline{\Lambda_k}$ and $\Lambda_j \cap \Lambda_j = \emptyset$, for $i \neq j$. Then the processes $\Delta X^{\Lambda_1}, \ldots, \Delta X^{\Lambda_k}, X - \Delta X^{\bigcup_i \Lambda_i}$ are independent.*

All the lemmas and theorems are inspired and based on results from [1],[9].

Note that the processes $\Delta X^{\Lambda_1}, \ldots, \Delta X^{\Lambda_k}, X - \Delta X^{\bigcup_i \Lambda_i}$ never jump together. This fact will be used to show independence. Two stochastic processes on the same underlying probability space are independent if for every $t_1, \ldots, t_n \in \mathbb{R}_+$, $(X_{t_1}, \ldots, X_{t_n})$ and $(Y_{t_1}, \ldots, Y_{t_n})$ are independent by Lemma 4.1. For the proof of Theorem 3.5 we need the following lemmas.

Lemma 3.10. *Let $\{X_t\}_{t\in\mathbb{R}_+}, \{Y_t\}_{t\in\mathbb{R}_+}$ be two real-valued stochastic processes on the same underlying filtered probability space such that $(X_t, Y_t)_{t\in\mathbb{R}_+}$ is a process with independent increments. Suppose*

1. *for all t, $t \geq 0$, X_t and Y_t are independent;*
2. *for all $s < t$, $s,t \in \mathbb{R}_+$, $X_t - X_s$ and $Y_t - Y_s$ are independent.*

Then $\{X_t\}_{t\in\mathbb{R}_+}, \{Y_t\}_{t\in\mathbb{R}_+}$ are independent processes.

Proof. Let $t_1, \ldots, t_n \in \mathbb{R}_+$, $0 = t_1 < t_2 < \ldots < t_n$. Recall that two random vectors $X = (X_1, \ldots, X_d)$ and $Y = (Y_1, \ldots, Y_d)$ are independent if and only if $\Phi_{(X,Y)}(u,v) = \Phi_X(u)\Phi_Y(v)$ for $u, v \in \mathbb{R}^d$. Let $u, v \in \mathbb{R}^n$,

$$\begin{aligned}
\Phi_{(X_{t_1},\ldots,X_{t_n},Y_{t_1},\ldots,Y_{t_n})}(u,v) &= \prod_{i=1}^n \Phi_{(X_{t_i}-X_{t_{i-1}},Y_{t_i}-Y_{t_{i-1}})}\left(\sum_{l=i}^n u_l, \sum_{l=i}^n v_l\right) \\
&= \prod_{i=1}^n \Phi_{X_{t_i}-X_{t_{i-1}}}\left(\sum_{l=i}^n u_l\right)\Phi_{Y_{t_i}-Y_{t_{i-1}}}\left(\sum_{l=i}^n v_l\right) \\
&= \Phi_{(X_{t_1},\ldots,X_{t_n})}(u)\Phi_{(Y_{t_1},\ldots,Y_{t_n})}(v)
\end{aligned} \tag{45}$$

Hence $(X_{t_1}, \ldots, X_{t_n})$ and $(Y_{t_1}, \ldots, Y_{t_n})$ are independent for all $n = 1, 2, \ldots, 0 \leq t_1 < t_2 \ldots < t_n$ and thus $\{X_t\}_{t\in\mathbb{R}_+}, \{Y_t\}_{t\in\mathbb{R}_+}$ are independent processes. □

Remark 3.2. Let X, Y be E-valued stochastic processes with independent increments and with E a separable Banach space. Suppose the same conditions as in lemma 3.10. We can prove the same statement.

$$\begin{aligned}
\Phi_{(X_{t_1},\ldots,X_{t_n},Y_{t_1},\ldots,Y_{t_n})}&(x_1^*,\ldots,x_n^*,y_1^*,\ldots,y_n^*) \\
&= \prod_{i=1}^n \Phi_{(X_{t_i}-X_{t_{i-1}},Y_{t_i}-Y_{t_{i-1}})}\left(\sum_{l=i}^n x_l^*, \sum_{l=i}^n y_l^*\right) \\
&= \prod_{i=1}^n \Phi_{X_{t_i}-X_{t_{i-1}}}\left(\sum_{l=i}^n x_l^*\right)\Phi_{Y_{t_i}-Y_{t_{i-1}}}\left(\sum_{l=i}^n y_l^*\right) \\
&= \Phi_{(X_{t_1},\ldots,X_{t_n})}(x_1^*,\ldots,x_n^*)\Phi_{(Y_{t_1},\ldots,Y_{t_n})}(y_1^*,\ldots,y_n^*),
\end{aligned} \tag{46}$$

from which we conclude that $(X_{t_1},\ldots,X_{t_n})$ and $(Y_{t_1},\ldots,Y_{t_n})$ are independent for all $n = 1, 2, \ldots, 0 \leq t_1 < t_2 \ldots < t_n$ and thus $\{X_t\}_{t\in\mathbb{R}_+}, \{Y_t\}_{t\in\mathbb{R}_+}$ are independent processes.

Let X be a real-valued process with independent increments. For $u \in \mathbb{R}^n$, recall (1) $\varphi(s,t)(u) = \Phi_{X_t-X_s}(u)$. By continuity in probability it follows from Theorem 4.7 that $t \mapsto \varphi(s,t)(u)$ is continuous, $t \geq s$. It holds that $\varphi(s,s)(u) = 1$. Define $T = \inf\{t \geq s : \varphi(s,t)(u) = 0\}$. Suppose that $T < \infty$ and let $s < h < T$, then

$$\varphi(s,T)(u) = \varphi(s,h)(u)\varphi(h,T)(u).$$

By definition $\varphi(s,T)(u) = 0$ and thus it holds $\varphi(h,T)(u) = 0$. Take $h \uparrow T$ and by continuity we it holds $\Phi(u,T,T) = 0$. Hence we found a contradiction and thus we conclude that $T = \infty$. For every $u \in \mathbb{R}^n$ $t \mapsto \varphi(0,t)(u)$ is continuous and for $t > 0$ it holds

$$\varphi(0,t)(u) \neq 0. \tag{47}$$

Lemma 3.11. *Let X be a stochastic process with independent increments, then by* (47) *for every $u \in \mathbb{R}^n$ we can define*

$$M_t^u := \frac{e^{i\langle u,X_t\rangle}}{\mathbb{E}[e^{i\langle u,X_t\rangle}]}, \tag{48}$$

and M^u is a complex martingale w.r.t. $\{\mathcal{F}_t\}_{t\in\mathbb{R}_{\geq 0}}$.

Proof. By independent increments we find that

$$\begin{aligned}
\mathbb{E}[M_t^u|\mathcal{F}_s] &= \frac{\mathbb{E}\left[e^{i\langle u,X_t-X_s+X_s\rangle}|\mathcal{F}_s\right]}{\mathbb{E}[e^{i\langle u,X_t-X_s+X_s\rangle}]} = \frac{e^{i\langle u,X_s\rangle}\mathbb{E}\left[e^{i\langle u,X_t-X_s\rangle}|\mathcal{F}_s\right]}{\mathbb{E}e^{i\langle u,X_t-X_s\rangle}\mathbb{E}[e^{i\langle u,X_s\rangle}]} \\
&=^{2} \frac{e^{i\langle u,X_s\rangle}\mathbb{E}e^{i\langle u,X_t-X_s\rangle}}{\mathbb{E}e^{i\langle u,X_t-X_s\rangle}\mathbb{E}e^{i\langle u,X_s\rangle}} = \frac{e^{i\langle u,X_s\rangle}}{\mathbb{E}e^{i\langle u,X_s\rangle}} = M_s^u.
\end{aligned}$$

and thus we have completed the proof. □

[2] Let $\mathcal{C}$ a sub-σ-algebra of $\mathcal{F}$. Suppose that X, Y are random variables in $\mathbb{R}^d$ such that X is $\mathcal{C}$-measurable and Y is independent of $\mathcal{C}$, then for every $\mathcal{B}(\mathbb{R}^{2d})$-measurable function f,

$$\mathbb{E}[f(X,Y)|\mathcal{C}] = g(X),$$

where $g(x) = \mathbb{E}(f(x,Y))$

Lemma 3.12. *Let N be a Poisson process with Poisson intensity $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ and $M_t^u = \frac{e^{iuN_t}}{\mathbb{E}[e^{iuN_t}]} - 1$, then for $s < t$,*

$$|M_t^u - M_s^u| \le 2\left(\lambda(t) - \lambda(s)\right) e^{4\lambda(t)} + e^{2\lambda(t)}|u||N_t - N_s|.$$

From this it follows that the variation of M^u is integrable.

Proof. The characteristic function of N is given by $\mathbb{E}[e^{iuN_t}] = e^{\lambda(t)(e^{iu}-1)}$. Let $\alpha = \mathbb{E}[e^{iuN_t}]$ and $\beta = \mathbb{E}[e^{iuN_s}]$. From $|e^{ix} - e^{iy}| \le |x-y|, x, y \in \mathbb{R}$ it holds

$$\begin{aligned} \left|\frac{e^{iuN_t}}{\alpha} - \frac{e^{iuN_s}}{\beta}\right| &= \frac{1}{|\alpha|}\left|e^{iuN_t} - \frac{\alpha}{\beta}e^{iuN_s}\right| \\ &= \frac{1}{|\alpha|}\left|e^{iuN_t} - e^{iuN_t}\frac{\alpha}{\beta} + e^{iuN_t}\frac{\alpha}{\beta} - \frac{\alpha}{\beta}e^{iuN_s}\right| \\ &\le \frac{1}{|\alpha|}\left|1 - \frac{\alpha}{\beta}\right| + \frac{1}{|\alpha|}\left|\frac{\alpha}{\beta}\right||u||N_t - N_s|. \end{aligned} \tag{49}$$

From $|e^z - 1| \le |z|e^{|z|}$ it follows that $\left|1 - \frac{\alpha}{\beta}\right| \le 2(\lambda(t) - \lambda(s))e^{2(\lambda(t)-\lambda(s))}$. It also holds, that $\left|\frac{\alpha}{\beta}\right| = e^{(\lambda(t)-\lambda(s))(\cos u - 1)}$ and $\frac{1}{|\alpha|} = \frac{1}{e^{\lambda(t)t(\cos u-1)}}$. Collecting all expressions in (49),

$$\begin{aligned} \left|\frac{e^{iuN_t}}{\alpha} - \frac{e^{iuN_s}}{\beta}\right| &\le \frac{2(\lambda(t) - \lambda(s))e^{2(\lambda(t)-\lambda(s))}}{e^{\lambda(t)t(\cos u-1)}} + \frac{e^{(\lambda(t)-\lambda(s))(\cos u-1)}}{e^{\lambda(t)(\cos u-1)}}|u||N_t - N_s| \\ &\le 2\left(\lambda(t) - \lambda(s)\right)e^{4\lambda(t)} + e^{2\lambda(t)}|u||N_t - N_s|. \end{aligned} \tag{50}$$

Let $T > 0$ then the variation $\mathbb{V}_T$ of M_t^u on [0,T] is bounded by $\mathbb{V}_T \le 2\lambda(T)e^{4\lambda(T)} + e^{2\lambda(T)}|u|N_T$ and thus the expectation $\mathbb{E}[\mathbb{V}_T] \le 2\lambda(T)e^{4\lambda(T)} + |u|e^{2\lambda(T)}\lambda(T)$. □

The following theorem is similar to [1, Proposition 2.4.1].

Theorem 3.6. *Let M, N be two square integrable martingales with all paths càdlàg. Suppose that $\sup_{s\in[0,t]}|M_s| < B_t < \infty$ for all paths, $M_0 = N_0 = 0$, N of bounded variation, $\mathbb{E}\left[\mathbb{V}_t^N\right] < \infty$ and M, N do never jump together. Then*

$$\mathbb{E}\left[M_t \cdot N_t\right] = 0.$$

Proof. First take the partitions $\mathcal{P}_n = \left\{0 = t_0 < t_1 < \ldots < t_{p(n)} = t\right\}$ such that the maximal width of the interval goes to zero, i.e. $\delta_n = \max_{i=0}^{p(n)-1}|t_{i+1} - t_i| \to 0$ as $n \to \infty$. We want to stress that for every n we take a partition of $[0, t]$. By lemma 4.3, for $\epsilon > 0$ we can construct a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_{k(\epsilon)} = t$ such that

$$\sup\left\{|M_u - M_v| : u, v \in [\tau_i, \tau_{i+1}),\ i = 0, \ldots, k(\epsilon) - 1\right\} \le \epsilon,$$

$$\sup\left\{|N_u - N_v| : u, v \in [\tau_i, \tau_{i+1}),\ i = 0, \ldots, k(\epsilon) - 1\right\} \le \epsilon.$$

Let $\delta > \epsilon$ and by taking n large enough we can bound,

$$\sum_{i=0}^{p(n)-1}|M_{t_{i+1}} - M_{t_i}||N_{t_{i+1}} - N_{t_i}| \le \sum_{i=1}^{k(\epsilon)-1}\left(|\Delta M_{\tau_i-}| + \delta\right)\left(|\Delta N_{\tau_i-}| + \delta\right) + \epsilon\mathbb{V}_t^N.$$

Let $U = \max_{i=1,\ldots,k(\epsilon)-1} |\Delta M_{\tau_i -}|$, note that for every $\omega \in \Omega$ it holds that $U(\omega) < \infty$. Under the assumption that M, N have for all paths no common jumptime it follows

$$\sum_{i=0}^{p(n)-1} |M_{t_{i+1}} - M_{t_i}||N_{t_{i+1}} - N_{t_i}| \leq \sum_{i=1}^{k(\epsilon)-1} (U+\delta)\,\delta + 2\epsilon \mathbb{V}_t^N.$$

Because δ was arbitrary, it holds $\sum_{i=0}^{p(n)-1} |M_{t_{i+1}} - M_{t_i}||N_{t_{i+1}} - N_{t_i}| \leq 2\epsilon \mathbb{V}_t^N$. For every $\epsilon > 0$ by taking n large enough we find $\sum_{i=0}^{p(n)-1} |M_{t_{i+1}} - M_{t_i}||N_{t_{i+1}} - N_{t_i}| \leq 2\epsilon \mathbb{V}_t^N$, we conclude that

$$\lim_{n\to\infty} \sum_{i=0}^{p(n)-1} |M_{t_{i+1}} - M_{t_i}||N_{t_{i+1}} - N_{t_i}| = 0 \quad a.s.$$

We know that $\sum_{i=0}^{p(n)-1} |M_{t_{i+1}} - M_{t_i}||N_{t_{i+1}} - N_{t_i}| \leq 2B_t \cdot \mathbb{V}_t^N$ and $\mathbb{E}\left[\mathbb{V}_t^N\right] < \infty$. Now we use the increments of the partitions to calculate the expectation. For square integrable martingales we have orthogonality of increments.

$$\begin{aligned}
\mathbb{E}\left[M_t \cdot N_t\right] &= \mathbb{E}\left(\sum_{i=0}^{p(n)-1} \left(M_{t_{i+1}} - M_{t_i}\right)\right)\left(\sum_{i=0}^{p(n)-1} \left(N_{t_{i+1}} - N_{t_i}\right)\right) \\
&= \mathbb{E} \sum_{i=0}^{p(n)-1} (M_{t_{i+1}} - M_{t_i})(N_{t_{i+1}} - N_{t_i}).
\end{aligned}$$

By using the lebesque dominated convergence theorem,

$$|\mathbb{E}\left[M_t \cdot N_t\right]| \leq \mathbb{E} \sum_{i=0}^{p(n)-1} |M_{t_{i+1}} - M_{t_i}||N_{t_{i+1}} - N_{t_i}| \to 0, \quad n \to \infty \tag{51}$$

We conclude that $\mathbb{E}\left[M_t \cdot N_t\right] = 0$. □

Theorem 3.7. *Let $\{X\}_{t\in\mathbb{R}_+}, \{Y\}_{t\in\mathbb{R}_+}$ be two real-valued processes on the same underlying probability space such that $(X_t, Y_t)_{t\in\mathbb{R}_+}$ is a process with independent increments. Suppose that X is a Poisson process and that X, Y have no common jump point for every $\omega \in \Omega$, then X, Y are independent processes.*

Proof. Define for every $u, v \in \mathbb{R}^n$ the processes

$$N^u = \frac{e^{i\langle u, X\rangle}}{\mathbb{E}[e^{i\langle u, X\rangle}]} - 1, \quad M^v = \frac{e^{i\langle v, Y\rangle}}{\mathbb{E}[e^{i\langle v, Y\rangle}]} - 1.$$

Both processes are square integrable martingales and both processes start at zero, $M_0^v = N_0^u = 0$. The variation $\mathbb{V}_T^N$ of N^u has finite expectation $\mathbb{E}[\mathbb{V}_t^N] < \infty$ by Lemma 3.12. Note that $s \mapsto \mathbb{E}[e^{i\langle u, Y_s\rangle}]$ is a continuous function. Also for every $s \geq 0$ it holds that $|\mathbb{E}[e^{i\langle v, Y_s\rangle}]| \neq 0$ by Eq. (47). Every continuous function has a maximum and it holds

$$|M_t^v| \leq \left|\frac{e^{i\langle v, Y_t\rangle}}{\mathbb{E}[e^{i\langle v, Y_t\rangle}]}\right| + 1.$$

Furthermore by assumption they have no common jump points for every $\omega \in \Omega$. By Theorem 3.6 it holds for every $t > 0$,

$$\mathbb{E}\left[N_t^u M_t^v\right] = 0.$$

From this it follows that $\mathbb{E}\left[e^{i\langle u, X_t\rangle} e^{i\langle v, Y_t\rangle}\right] = \mathbb{E}\left[e^{i\langle u, X_t\rangle}\right] \mathbb{E}\left[e^{i\langle v, Y_t\rangle}\right]$. The elements $u, v \in \mathbb{R}^n$ are arbitrary and thus for every $t > 0$, X_t and Y_t are independent. We can do exactly the same for $X_t - X_s$ and $Y_t - Y_s$, $s < t, s, t \in \mathbb{R}_+$. We conclude by Lemma 3.10 that X and Y are independent processes.

□

Remark 3.3. Let X, Y be two E-valued processes with independent increments on the same underlying probability space. Let N be a Poisson Process such that $Y = x \cdot N$, $x \in E$. Furthermore suppose that X, Y never jump together, then for every $x^*, y^* \in E^*$ it follows by 3.7 that $\langle x^*, X\rangle$ and $\langle y^*, Y\rangle$ are independent processes. This implies that X, Y are independent processes.

Theorem 3.8. *Let X be process with independent increments. Let $\Lambda \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda}$, then $X - \Delta X^\Lambda$ and ΔX^Λ are independent processes.*

Proof. Denote $f(s, x) = x$. Let $f_n = \sum_{j=1}^{m(n)} \alpha_j^{(n)} \mathbb{I}_{A_j^{(n)}}$ with $A_i^{(n)} \cap A_j^{(n)} = \emptyset$ defined as in (36) such that $f_n(s, x) \to f(s, x)$ for every $(s, x) \in \mathbb{R}_+ \times (E \setminus \{0\})$. It holds that $N_t := \mathcal{J}_X(\omega, ([0, t] \times \Lambda))$ is a Poisson process. Now note that

$$\mathcal{J}_N(\omega, A_j^{(n)}(t)) = \mathcal{J}_X\left(\omega, ([0, t] \times \Lambda) \cap A_j^{(n)}\right).$$

Now with Lemma 3.2 it follows that $\mathcal{J}_X\left(\omega, ([0, t] \times \Lambda) \cap A_j^{(n)}\right)$ is $\sigma\{N_s : s \leq t\}$-measurable. Now note N and $X - \Delta X^\Lambda$ never jump together, hence are independent processes. Recall Eq. (44). It follows that $X_t - \Delta X_t^\Lambda$ and $\sum_{j=1}^{m(n)} \alpha_j^{(n)} \mathcal{J}_X\left(\omega, ([0, t] \times \Lambda) \cap A_j^{(n)}\right)$ are independent. By using Lemma 2.11 it follows that $X_t - \Delta X_t^\Lambda$ and ΔX_t^Λ are independent. Property 2 of Lemma 3.10 is shown in the same way. Conclude by Lemma 3.10 that $X - \Delta X^\Lambda$ and ΔX^Λ are independent processes. □

Remark 3.4. Let $\Lambda_1, \Lambda_2 \in \mathcal{B}(E)$ with $0 \notin \overline{\Lambda_1}, \overline{\Lambda_2}$ and $\Lambda_1 \cap \Lambda_2 = \emptyset$, then ΔX^{Λ_1} and ΔX^{Λ_2} are independent processes. The proof goes in the same way as in the previous theorem.

Finally we will prove Theorem 3.5.

Proof. Let $A_i \in \sigma\left\{\Delta X^{\Lambda_i}\right\}$ for $i = 1, \ldots, k$ and $A_{k+1} \in \sigma\left\{X - \Delta X^{\bigcup_i \Lambda_i}\right\}$. By Theorem 3.8 $X - \Delta X^{\bigcup_i \Lambda_i}$ and $\Delta X^{\bigcup_i \Lambda_i}$ are independent processes. Then note that for every $i = 1, \ldots, k$,

$$\Delta X_t^{\Lambda_i} = \Delta(\Delta X_t^{\bigcup_j \Lambda_j})^{\Lambda_i}.$$

From Lemma 3.9 it follows that ΔX^{Λ_i} is $\sigma\left\{\Delta X^{\bigcup_j \Lambda_j}\right\}$-measurable. From this we find that

$$\mathbb{P}\left\{\bigcap_{i=1}^{k+1} A_i\right\} = \mathbb{P}\left\{\bigcap_{i=1}^{k} A_i\right\} \mathbb{P}\left\{A_{k+1}\right\}.$$

By Remark3.4 it holds that ΔX^{Λ_k} is independent of $\Delta X^{\bigcup_{j=1}^{k-1}\Lambda_j}$. Also for very $i = 1, \dots, k$, ΔX^{Λ_i} is determined by $\Delta X^{\bigcup_{j=1}^{i}\Lambda_j}$. Now it holds

$$\mathbb{P}\left\{\bigcap_{i=1}^{k} A_i\right\} = \mathbb{P}\left\{\bigcap_{i=1}^{k-1} A_i\right\}\mathbb{P}\left\{A_k\right\}.$$

By using repeatedly the same arguments we find,

$$\mathbb{P}\left\{\bigcap_{i=1}^{k+1} A_i\right\} = \prod_{i=1}^{k+1}\mathbb{P}\left\{A_i\right\}.$$

With this we conclude the Theorem.

□

3.4 The structure of processes with independent increments

In this section we will prove the Levy-Ito decomposition for Additive processes with values in a separable Banach space. Define the sets

$$\Lambda_1 = \{x \in E : ||x|| > 1\}, \quad \Lambda_n = \left\{x \in E : \frac{1}{n} \leq ||x|| < \frac{1}{n-1}\right\}. \tag{52}$$

The Levy-Ito decomposition expresses that sample paths of Additive processes can be decomposed as a sum of independent parts: a continuous part and a jump part. Recall the decomposition of the process $\{X\}_{t\in\mathbb{R}_+}$

$$X_t = \underbrace{\sum_{i=1}^{N}\Delta X^{\Lambda_i}}_{\text{"Jump part"}} + \underbrace{X_t - \Delta X^{\bigcup_i^N \Lambda_i}}_{\text{"Continuous part"}}.$$

As we take N larger, all the jumps of the process are getting subtracted. A process without jumps has continuous paths. The structure of a real-valued process with independent increments and a.s. continuous paths is known. The increments of such a process are Gaussian, see Theorem 4.9. It seems to be that the structure of Additive processes is a trivial matter. The opposite is true. In general we cannot express the jumps of the process as $\sum_{i=1}^{\infty}\Delta X^{\Lambda_i}$. This sum may be divergent. The convergence of this sum is a very delicate point. The content of this section is devoted to this matter.

If means are subtracted from the jump processes, then the sum in the jump part will converge. Note that we try to be as general as possible. We consider the structure of Additive process with values in a seperable Banach space. In that case means are Bochner-integrals. We will show X can be represented as

$$X_t = W_t + \lim_{n\to\infty}\sum_{i=2}^{n}\left(\Delta X_t^{\Lambda_i} - \mathbb{E}[\Delta X_t^{\Lambda_i}]\right) + \Delta X_t^{\Lambda_1}, \tag{53}$$

where W_t has independent increments and is a.s. continuous, the convergence of the centered sum $\lim_{n\to\infty}\sum_{i=2}^{n}\left(\Delta X_t^{\Lambda_i} - \mathbb{E}[\Delta X_t^{\Lambda_i}]\right)$ is uniformly a.s. on every bounded interval $[0,t]$.

We will use a symmetrisation argument, see Definition 2.2. We denote the independent copy of X by $\bar{X}$. Let us fix some notation. For every $t > 0$ and $N \in \mathbb{N}$ we define

$$\begin{aligned} X_t^0 &:= X_t - \Delta X^{\Lambda_1} \\ J_t^N &:= \sum_{n=2}^{N} \Delta X^{\Lambda_n} \\ J_t^{N,c} &:= \sum_{n=2}^{N} \Delta X^{\Lambda_n} - \mathbb{E}[\Delta X^{\Lambda_n}] \\ J_t^{*N}(\omega,\bar{\omega}) &:= J_t^N(\omega) - \bar{J}_t^N(\bar{\omega}). \end{aligned}$$

By Theorem 3.5 J_N is a sum of independent processes $\Delta X^{\Lambda_k}, k = 2,\dots,n$. The process X can be represented as a sum of independent processes

$$X_t = \sum_{n=1}^{N} \Delta X_t^{\Lambda_n} + X_t^0 - J_t^N. \tag{54}$$

The first step is to show that J_t^{*N} converges a.s. uniformly. Using Fubini's Theorem we can conclude, that there exists a centering sequence of functions $c_n \in \mathbb{D}_E(T)$ such that $\sum_{n=2}^{N}\left(\Delta X^{\Lambda_n} - c_n\right)$ converges uniformly a.s., similarly to the proof of Theorem 2.1. The next step is to show that as a centering function we can take $c_n(t) = \mathbb{E}[\Delta X^{\Lambda_n}]$.

The space $\mathbb{D}_E(T)$ equipped with the supremum norm $||\cdot||_T$ is a Banach space. Unfortunately it is not a separable Banach space, otherwise we could use Theorem 2.4 to show convergence of J^{*N} in $||\cdot||_T$. We will use instead the following Theorem.

Theorem 3.9 (Rosinsky-Basse-O'Conner). *Let X_j, $j \in \mathbb{N}$, be a sequence of independent random variables with values in $(\mathbb{D}_E(T), \mathcal{D}_E(T))$ and $S_n = \sum_{j=1}^{n} X_j$ be the sum. Suppose there exists a random variable $Y \in (\mathbb{D}_E(T), \mathcal{D}_E(T))$ and a dense subset $\mathcal{D} \subset [0,T]$ such that $T \in \mathcal{D}$ and for any $t_1,\dots,t_k \in \mathcal{D}$*

$$(S_n(t_1),\dots,S_n(t_k)) \xrightarrow{d} (Y(t_1),\dots,Y(t_k)). \tag{55}$$

Then there exists a random variable S with values in $(\mathbb{D}_E(T), \mathcal{D}_E(T))$ with the same distribution as Y such that

1. *If X_j are symmetric , then $\sup_{t\in[0,T]} ||S(t) - S_n(t)|| \xrightarrow{a.s.} 0$.*

2. *If X_j are not symmetric , then there exists $y_n \in \mathbb{D}_E(T)$ with $\lim_{n\to\infty} y_n(t) = 0$ for every $t \in \mathcal{D}$ such that $\sup_{t\in[0,T]} ||S(t) - S_n(t) - y_n(t)|| \xrightarrow{a.s.} 0$. Moreover if the family*

$\{||S(t)|| : t \in \mathcal{D}\}$ is uniformly integrable and the function $t \mapsto \mathbb{E}(X_n(t))$ belong to $\mathbb{D}_E(T)$, then

$$\sup_{t\in[0,T]} || \left(S(t) - \mathbb{E}(S(t)) - (S_n(t) - \mathbb{E}(S_n)(t)\right) || \overset{a.s.}{\to} 0.$$

Proof. See [3, Theorem 2.1]. □

Note that the following theorem is a similar statement to Theorem 2.4, but for random variables, with some additional conditions, with values in a general Banach space. We will prove this theorem with the aid of Theorem 3.9.

Theorem 3.10. *Let X_j, $j \in \mathbb{N}$, be independent Additive processes. Suppose there is a random variable X in $\mathbb{D}_E(T)$ such that for every $N \in \mathbb{N}$*

$$\Delta_n = X - \sum_{i=1}^{N} X_i.$$

independent of $X_1, \dots, X_n$. Then the following statements hold:

1. *Let $X_j^s = X_j - \bar{X}_j$ be the symmetrization of X_j, and $S_n^s = \sum_{i=1}^{n} X_n^s$, then there exists a random variable S such that $\|S - S_n^s\|_T \overset{a.s.}{\to} 0$.*
2. *There exists $c_n \in \mathbb{D}_E(T)$ such that $\sum_{n=1}^{N} (X_n - c_n)$ converges uniformly on $[0,T]$ a.s. where $c_n = \bar{X}_n(\bar{\omega})$ for some $\bar{\omega} \in \bar{\Omega}$.*

Proof. Let $t \in [0,T]$, by Theorem 2.4 there exists a random variable $S'(t)$ such that $S_n^s(t) \overset{a.s.}{\to} S'(t)$. This implies existence of a sequence $c_n \in E$ such that $S_n(t) - c_n \overset{a.s.}{\to} S_1$. From this it follows

$$S_n^s(t) = S_n(t) - \bar{S}_n(t) = S_n(t) - c_n + c_n - \bar{S}_n(t) \overset{a.s.}{\to} S_1 - \bar{S}_1,$$

recall the definition 2.2. It follows $S'(t)$ is symmetric. We will show that $S'(t)$ is continuous in probability.

Suppose for the moment that this is the case, then it is possible by Theorem 2.9 to construct a càdlàg modification. By the use Theorem 3.9 it is possible to find a process S with values in $\mathbb{D}_E(T)$ such that $||S - S_n^s||_T \to 0$ as $n \to \infty$.

We will use Ito-Nisio's theorem 2.3 to prove that S' is continuous in probability. Let $x^* \in E^*$ and define the function

$$\Phi_{n,t} = \mathbb{E}\left(e^{i\langle x^*, S_n^s(t)\rangle}\right) = \left|\Phi_{\langle x^*, S_n(t)\rangle}(1)\right|^2.$$

By the independent increment property it holds

$$\left|\Phi_{\langle x^*, S_n(t)\rangle}(1)\right|^2 = \left|\Phi_{\langle x^*, S_n(t)-S_n(s)\rangle}(1)\right|^2 \left|\Phi_{\langle x^*, S_n(t)\rangle}(1)\right|^2.$$

By independence of $X_j, j = 1, 2, \dots$ it follows

$$\left|\Phi_{\langle x^*, S_n(t)\rangle}(1)\right|^2 = \left|\Phi_{\langle x^*, S_{n-1}(t)\rangle}(1)\right|^2 \left|\Phi_{\langle x^*, X_n(t)\rangle}(1)\right|^2.$$

From this it follows $\Phi_{n,t}$ is non-increasing in both n, t. By Ito-Nisio's Theorem 2.3 it holds for every $t > 0$,

$$\Phi_{n,t} \to \mathbb{E}\left(e^{i\langle x^*, S'(t)\rangle}\right), \quad n \to \infty.$$

We denote $\Phi_t = \mathbb{E}\left(e^{i\langle x^*, S'(t)\rangle}\right)$ and note that Φ_t is also non-increasing because S' is symmetric and has independent increments. Suppose that $s_n \uparrow s$, then by Theorem 2.4 $S'(s_n)$ converges a.s. to a random variable S'' and by the Ito-Nisio's Theorem 2.3 it holds $\Phi_{s_n} \to \mathbb{E}\left(e^{i\langle x^*, S''\rangle}\right)$.We denote $\Phi'_s = \mathbb{E}\left(e^{i\langle x^*, S''\rangle}\right)$. Let $\epsilon > 0$ be given. Now choose N such that

$$|\Phi'_s - \Phi_{s_n}| < \epsilon, \quad \forall n \geq N.$$

Now for the moment take $n \geq N$. Choose M such that

$$|\Phi_{s_n} - \Phi_{m,s_n}| < \epsilon \ \text{ and } |\Phi_s - \Phi_{m,s}| < \epsilon, \quad \forall m \geq M.$$

Note that for $l \geq n$ it holds $|\Phi'_s - \Phi_{m,s_l}| < 2\epsilon$. Take $m \geq M$ and $n \geq N$ large enough so that $\Phi_{m,s} - \Phi_{m,s_n} < \epsilon$. Then it holds

$$|\Phi'_s - \Phi_s| \leq |\Phi'_s - \Phi_{m,s_n}| + |\Phi_{m,s_n} - \Phi_{m,s}| + |\Phi_{m,s} - \Phi_s| < 4\epsilon.$$

Because ϵ was arbitrary we find that $\Phi'_s = \Phi_s$. Because $x^* \in E^*$ was arbitrary, it holds by the Ito-Nisio Theorem 2.3 that $S'(s_n), \overset{a.s.}{\to} S'(s)$. We can do the same for $s_n \downarrow s$. We conclude that S' is process with independent increments and continuous in probability.

It is possible to construct a càdlàg modification S of S'. Conclude that there exists a random variable S with values in $(\mathbb{D}_E(T), \mathcal{D}_E(T))$, such that $S_n^s(t) \overset{a.s.}{\to} S(t)$, for every $t \in [0, T]$. Condition (55) in theorem 3.9 holds. We conclude by Theorem 3.9 that, $\sup_{t \in [0,T]} ||S(t) - S_n^s(t)|| \overset{a.s.}{\to} 0$. The second statement follows from Fubini's Theorem. □

Note that the previous Theorem is designed for J^{*N}. It was our first goal to show a.s. uniform convergence of J^{*N}.

Lemma 3.13. *For a fixed a time horizon $T > 0$, J^{*N} converge uniformly on $[0, T]$ a.s.*

Proof. Using Theorem 3.10 to 54 the statement follows. □

For every $n \in \mathbb{N}$, there is a probability one set $\Omega_n^* \subset \Omega \times \bar{\Omega}$ such that J^{*N} converge uniformly on $[0, n]$, $\forall(\omega, \bar{\omega}) \in \Omega_n^*$. We take $\Omega' = \bigcap_n \Omega_n^*$ and note that by Fubini's Theorem there is a $\bar{\omega}$ such that

$$\Omega_{\bar{\omega}} = \{\omega \in \Omega : (\omega, \bar{\omega}) \in \Omega'\},$$

has probability 1. We define the following process $\{S_t\}_{t \in \mathbb{R}_+}$ by

$$S_t(\omega) := \begin{cases} \lim\limits_{N \to \infty} J_t^N(\omega) - \bar{J}_t^N(\bar{\omega}) & \text{if } \omega \in \Omega_{\bar{\omega}} \\ 0 & \text{if } \omega \notin \Omega_{\bar{\omega}} \end{cases},$$

We have found centering function $c_n(t)$ such that the sum $\sum_{n=2}^{N} \Delta \left(X^{\Lambda_n} - c_n \right)$ converges uniformly a.s. on every bounded interval, where $c_n = \Delta \bar{X}^{\Lambda_n}(\bar{\omega})$. If we can show that $\{\|S_t\| : t \in [0, T]\}$ is uniformly integrable and $\mathbb{E}[\Delta X^{\Lambda_n} - c_n]$ is càdlàg, then by Theorem 3.9,2 we can show that

$$J^{N,c} = \sum_{n=2}^{N} \Delta X^{\Lambda_n} - \mathbb{E}[\Delta X^{\Lambda_n}],$$

converges uniformly a.s. to $S - \mathbb{E}[S]$.

Define the non-random function $F_t := \bar{X}_t(\bar{\omega}) - \Delta \bar{X}_t^{\Lambda_1}(\bar{\omega})$, then $S_t + F_t$ is an Additive process. Furthermore $S_t + F_t$ has bounded jumps.

Lemma 3.14. *The collection of random variables $\{\|S_t\| : t \in [0, T]\}$ is uniformly integrable.*

Proof. First note that S_t can be written as $S_t + F_t - F_t$ with F_t a càdlàg function and $S_t + F_t$ an Additive process with bounded jumps. By Theorem 3.2 for some $M_1 > 0$, $\sup_{t \in [0,T]} \mathbb{E}[\|S_t + F_t\|^2] \leq M_1 < \infty$. The non-random function F_t is càdlàg , hence $\sup_{t \in [0,T]} ||F_t|| \leq M_2$. From this it holds

$$\sup_{t \in [0,T]} \mathbb{E}[\|S_t\|^2] \leq \sup_{t \in [0,T]} \mathbb{E}[\|S_t + F_t\|^2] + \sup_{t \in [0,T]} ||F_t|| \leq M_1 + M_2.$$

From this it holds that the family of real-valued random variables $\{\|S_t\| : t \in [0, T]\}$ are uniformly integrable.

□

Define for every $t > 0$ and $n \in \mathbb{N}$, $S_t^c := S - \mathbb{E}[S_t]$.

Theorem 3.11. *For every fixed time horizon $T > 0$, $\lim_{N \to \infty} \left\| S^c - J^{N,c} \right\|_T = 0$, a.s.*

Proof. By Lemma 3.13 $J^N - \bar{J}^N(\bar{\omega})$ converge a.s. uniformly on $[0, T]$ to S. It follows $\mathbb{E}[\Delta X^{\Lambda_i}] - \Delta \bar{X}^{\Lambda_i}$ is a càdlàg function by Lemma 3.1. Furthermore the family $\{\|S_t\| : t \in [0, T]\}$ is uniformly integrable. By Theorem 3.9,2 the statement follows. □

Theorem 3.12 (Levy-Ito)**.** *Let X be an Additive process with values in a separable Banach space. Then X can be represented as*

$$X_t = W_t + \lim_{n \to \infty} \sum_{i=2}^{n} \left(\Delta X_t^{\Lambda_i} - \mathbb{E}[\Delta X_t^{\Lambda_i}] \right) + \Delta X_t^{\Lambda_1}, \tag{56}$$

where $\{W_t\}_{t \in \mathbb{R}_+}$ has independent increments and is a.s.continuous, the convergence of the centered sum $\lim_{n \to \infty} \sum_{i=2}^{n} \left(\Delta X^{\Lambda_i} - \mathbb{E}[\Delta X^{\Lambda_i}] \right)$ is uniformly on every bounded interval $[0, T]$. Furthermore the three terms are independent processes.

Proof. By Theorem 3.11 for every $T > 0$ there is a set Ω_T, $\mathbb{P}\{\Omega_T\} = 1$,

$$\left\| S^c(\omega) - J^{N,c}(\omega) \right\|_T \to 0, \; \forall \omega \in \Omega_T.$$

Now write $W_t = X_t - S_t^c - \Delta X_t^{\Lambda_1}$. We will show, that W has a.s. continuous paths.

Let $s, t \in [0, T]$ and define $W_t^N := X - J^{N,c} - \Delta X^{\Lambda_1}$. Then by taking N large enough we find on Ω_T, $||W - W^N||_T \leq \epsilon$. Now we find on Ω_T,

$$\begin{aligned} \|W_t - W_s\| &\leq \left\|W_t - W_t^N\right\| + \left\|W_t^N - W_s^N\right\| + \left\|W_s^N - W_s\right\| \\ &\leq 2\epsilon + \left\|W_t^N - W_s^N\right\| \\ &\leq 2\epsilon + \left\|\left(X_t - \sum_{i=1}^{N} \Delta X_t^{\Lambda_i}\right) - \left(X_s - \sum_{i=1}^{N} \Delta X_s^{\Lambda_i}\right)\right\| \\ &+ \left\|\left(\sum_{i=2}^{N} \mathbb{E}[\Delta X_t^{\Lambda_i}]\right) - \left(\sum_{i=2}^{N} \mathbb{E}[\Delta X_s^{\Lambda_i}]\right)\right\|. \end{aligned}$$

From the continuity of the map $t \mapsto \mathbb{E}[\Delta X_t^{\Lambda_i}]$, see Lemma 3.13, it follows that

$$\lim_{s \to t} \|W_t - W_s\| \leq 2\epsilon + \frac{1}{N+1}.$$

We conclude that on W is a.s. continuous. Note that W^N, $J^{N,c}$ and ΔX^{Λ_1} are independent process. By uniform convergence we find that W, S^c and ΔX^{Λ_1} are independent. □

3.5 Lévy measure and the Lévy-Ito representation

In this section we will collect all observations made and represent the structure of an Additive process in its most abstract form. For every Additive process $\{X_t\}_{t \in \mathbb{R}_+}$ there is a corresponding random counting measure $\mathcal{J}_X$. In Lemma 3.7 we proved

$$\mu_X : \mathcal{S}^0 \to [0, \infty], \ A \mapsto \mathbb{E}\left[\mathcal{J}_X(\cdot, A)\right], \tag{57}$$

is a σ-finite measure on the space $(S, \mathcal{S}^0)$. In Lemma 3.6 we showed that for every $A \in \mathcal{S}^0$, $\mathcal{J}_X(\cdot, A)$ has a Poisson distribution with $\mathbb{E}[\mathcal{J}_X(\cdot, A)] < \infty$. Note furthermore that every σ-finite μ measure on a ring $\mathcal{R}$, can be extended uniquely to a measure μ on $\sigma\{\mathcal{R}\}$. See [2, Theorem 5.1]

Definition 3.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(S, \mathcal{S}, \mu)$ be a σ-finite measure space. A random measure $\mathcal{M} : \Omega \times \mathcal{S} \to \overline{\mathbb{R}}_+$ with values in $\overline{\mathbb{N}}$ is called a random Poisson measure with Poisson intensity μ, if

1. For every $A \in \mathcal{S}$, the random variable $\mathcal{M}(\cdot, A)$ has a Poisson distribution with parameter $\mu(A)$:
   a) $\mathbb{P}\{\mathcal{M}(A) = k\} = \frac{\mu(A)^k e^{-\mu(A)}}{k!}$, if $\mu(A) < \infty$.
   b) $\mathbb{P}\{\mathcal{M}(A) = \infty\} = 1$, if $\mu(A) = \infty$.
2. If $A_1, \ldots, A_n \in \mathcal{S}$ and disjunct, then $\mathcal{M}(\cdot, A_1), \ldots, \mathcal{M}(\cdot, A_n)$ are independent.

Theorem 3.13. *Let X be an Additive process with corresponding random counting measure $\mathcal{J}_X$, then $\mathcal{J}_X$ is a random Poisson measure with Poisson intensity μ_X.*

Proof. For every $A \in \mathcal{S}^0$, Lemma 3.6 $\mathcal{J}_X(\cdot, A)$ has Poisson distributed with parameter $\mu_X(A) = \mathbb{E}[\mathcal{J}_X(\cdot, A)] < \infty$. Take $A \in \mathcal{S}$, then $A^n \in \mathcal{S}^0$. Now $\mathcal{J}_X(\cdot, A^n)$ has a Poisson distribution with parameter Λ^n. Now there are two possibilities:

1. $\Lambda := \lim_{n\to\infty} \Lambda^n < \infty$
2. $\Lambda := \lim_{n\to\infty} \Lambda^n = \infty$.

Let $\Phi_A(u)$ be the characteristic function of $\mathcal{J}_X(\omega, A)$ and $\Phi_{A^n}(u)$ of $\mathcal{J}_X(\omega, A^n)$. Note that

$$\Phi_{A^n(t)}(u) = e^{\Lambda^n(t)\left(e^{iu}-1\right)}.$$

In case (1) by Lebesque's dominated convergence theorem it follows that $\Phi_{A(t)}(u) = e^{\Lambda(t)\left(e^{iu}-1\right)}$. This implies that $\mathcal{J}_X(\omega, A(t))$ has Poisson distribution with parameter Λ. For case (2) we can use (34) to conclude that $\mathbb{P}\left\{\mathcal{J}_X(\omega, A(t)) \le N\right\} = 0$, for every N, hence

$$\mathbb{P}\left\{N_t(A) = \infty\right\} = 1.$$

Let $B_1, \ldots, B_n \in \mathcal{S}$ and disjunct, and define

$$B_j^m(t) = \left\{(s, x) \in B_j : s \le t, ||x|| > \frac{1}{m}\right\}.$$

Now it holds for every $t > 0$ and $m \in \mathbb{N}$,

$$\mathcal{J}_X(\omega, B_1^m(t)), \ldots, \mathcal{J}_X(\omega, B_n^m(t)).$$

are independent by Theorem 3.7. Let $t = m$ and take $m \to \infty$, then by Lemma 2.11 we find that the random variables $\mathcal{J}_X(\omega, B_1), \ldots, \mathcal{J}_X(\omega, B_n)$ are independent. □

Lemma 3.15. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E}, \mu)$ be a σ-finite measure space. Suppose $f : S \to F$ is a measurable map with F a separable Banach space and $\mathcal{M}$ a random Poisson measure with Poisson intensity μ. If (1) $\int ||f(s)||\mathcal{M}(\omega, ds) < \infty$ for every $\omega \in \Omega$ and (2) $\int ||f(s)||\mu(ds) < \infty$ holds, then*

$$\mathbb{E}\int f(s)M(\omega, ds) = \int f(s)\mu(ds). \tag{58}$$

Proof. Let f_n a sequence of simple functions $f_n = \sum_i x_i^n \mathbb{I}_{A_i^n}$ as in (36). Then it holds that

$$\mathbb{E}\int f_n(s)\mathcal{M}(ds) = \mathbb{E}\sum_i x_i\mathcal{M}(A_i) = \sum_i x_i\mu(A_i) = \int f(s)\mu(ds).$$

By (1), $\int f_n(s)\mathcal{M}(\omega, ds)$ converges to the Bochner integral $\int f(s)\mathcal{M}(\omega, ds)$. From the monotone convergence theorem it holds,

$$\int \|f(s)\|\,\mathcal{M}(ds) = \lim_{N\to\infty}\sum_{k=0}^{\infty}\frac{k}{2^N}\mathcal{M}\left(\left\{s \in S : \frac{k}{2^N} \le ||f(s)|| < \frac{k+1}{2^N}\right\}\right).$$

By using the monotone convergence theorem we get

$$\begin{aligned}\mathbb{E}\int \|f(s)\|\,\mathcal{M}(ds) &= \lim_{N\to\infty}\sum_{k=0}^{\infty}\frac{k}{2^N}\mu\left(\left\{s\in S:\frac{k}{2^N}\leq \|f(s)\|<\frac{k+1}{2^N}\right\}\right)\\ &= \int \|f(s)\|\,\mu(ds).\end{aligned}$$

It holds that $\int ||f(s)-f_n(s)||\mathcal{M}(ds)\leq 3\int ||f(s)||\mathcal{M}(ds)<\infty$. From (2) we find that $\mathbb{E}\int ||f(s)-f_n(s)||\mathcal{M}(ds)\leq 3\int ||f||\mu(ds)<\infty$, hence we have now the following estimate,

$$\begin{aligned}\left\|\mathbb{E}\left[\int f_n(s)-f(s)\mathcal{M}(ds)\right]\right\| &\leq \mathbb{E}\left[\left\|\int f_n(s)-f(s)\mathcal{M}(ds)\right\|\right]\\ &\leq \left[\int ||f_n(s)-f(s)||\mathcal{M}(ds)\right]\\ &= \int ||f(s)-f_n(s)||\mu(ds)\end{aligned}$$

By Lebesque's dominated convergence theorem we find that $\int ||f(s)-f_n(s)||\mu(ds)\to 0$, hence $\mathbb{E}\int f_n\mathcal{M}(ds)\to\mathbb{E}\int f(s)\mathcal{M}(ds)$. Above we showed that for simple functions it holds that $\mathbb{E}\int f_n\mathcal{M}(ds)=\int f_n\mu(ds)$ and from (2) we conclude that

$$\int f_n\mu(ds)\to\int f\mu(ds).$$

We conclude that $\mathbb{E}\int f\mathcal{M}(\omega,ds)=\int f\mu(ds)$.

□

Remark 3.5. Let X be an Additive process. For $\Lambda\in\mathcal{B}(E)$ with $0\notin\overline{\Lambda}$ it holds by Lemma 3.9 that $\Delta X_t^{\Lambda}=\int_0^t\int_\Lambda x\mathcal{J}_x(ds\,dx)$. If $\int_{[0,t]\times\Lambda}||x||\mu_X(ds\,dx)<\infty$ then by lemma 3.15

$$\mathbb{E}\left[\Delta X_t^{\Lambda}\right]=\int_0^t\int_\Lambda x\mu_X(ds\,dx). \tag{59}$$

Definition 3.9. Let X be an Additive process and for every $0<\delta<1$, we define the process with finite jumps $S_t^{\delta}:=\Delta X_t^{\Lambda_{\delta,1}}$ where $\Lambda_{\delta,1}=\{x\in E:\delta\leq||x||\leq 1\}$. We define the centered process $\tilde{S}_t^{\delta}=S_t^{\delta}-\mathbb{E}S_t^{\delta}$.

For $E=\mathbb{R}^n$, the following theorem can be found in [20]. For E a separable Banach space, the representation can be found in [5, Theorem 2.1].

Theorem 3.14. *Let X be an Additive process. Then X can be represented as*

$$X_t=W_t+\underbrace{\lim_{\delta\downarrow 0}\int_0^t\int_{\delta\leq\|x\|\leq 1}x\left[\mathcal{J}_X(\omega,dsdx)-\mu_X(dsdx)\right]}_{J_t^1}+\underbrace{\int_0^t\int_{\|x\|>1}x\mathcal{J}_X(\omega,dsdx)}_{J_t^2}, \tag{60}$$

where W, J^1 and J^2 are independent processes, W_t is a.s. continuous, J^1 describes all the jumps with amplitude $\delta\leq 1$ and J^2 are the jumps with amplitude strictly larger than 1.

Proof. First we note that S^δ has bounded jumps. For every realization ω it follows that $S^\delta(\omega)$ has only a finite number of jumps on $[0,T]$. For $0<\delta<\delta'$, it holds that

$$\sup_{t\in[0,T]}\left\|\mathbb{E}[S_t^\delta(\omega)-S_t^{\delta'}(\omega)]\right\|\leq\int_0^T\int_{\{\delta\leq||x||<\delta'\}}||x||\mu(dsdx).$$

From this it follows that

$$\tilde{S}:(0,1]\to\mathbb{D}_E(T),\quad \delta\mapsto\left\{\tilde{S}_t^\delta\right\}_{t\in[0,T]},$$

is continuous. Furthermore by Theorem 3.12 it follows that $\tilde{S}^{\frac{1}{n}}$ converges. From this it follows that $\lim_{\delta\downarrow 0}\tilde{S}^\delta$ exists. By the use of Theorem 3.12 we find the representation of the process $\{X_t\}_{t\in\mathbb{R}_+}$

$$X_t=W_t+\lim_{\delta\downarrow 0}\int_0^t\int_{\delta\leq\|x\|\leq 1}x\left[\mathcal{J}_X(\omega,dsdx)-\mu_X(dsdx)\right]+\int_0^t\int_{\|x\|>1}x\mathcal{J}_X(\omega,dsdx).$$

□

Now we want to give some results on integrability properties of the Poisson intensity measure μ_X. For the case $E=\mathbb{R}^n$, it holds for every $T>0$,

$$\int_0^T\int_{E\setminus 0}\min(1,||x||^2)\mu_X(dsdx)<\infty. \tag{61}$$

We will show this result in case the state spaces is a Hilbert space $E=\mathcal{H}$. In the general case when E is a separable Banach space this is not necessarily the case, see [5].

Lemma 3.16. *Let X,Y be independent random variables, with values in a separable Banach space, with $\mathbb{E}[Y]=0$ and $c:\mathbb{R}_+\to\mathbb{R}_+$ a convex function, then*

$$\mathbb{E}[c\,(||X+Y||)]\geq\mathbb{E}[c\,(||X||)].$$

Proof. Let μ_X and μ_Y be the distributions of X and Y. By Fubini's Theorem and Jensen inequality we find

$$\begin{aligned}
\mathbb{E}[c\,(||X+Y||)] &= \int\int c\,(||x+y||)\,\mu_X(dy)\mu_Y(dx)\\
&\geq \int c\left(\left\|\int(x+y)\ \mu_Y(dy)\right\|\right)\mu_X(dx)\\
&= \int c\,(\|x\|)\,\mu_X(dx)\\
&= \mathbb{E}[c\,(||X||)].
\end{aligned}$$

□

Definition 3.10. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E}, \mu)$ be a finite measure space. Let $\mathcal{M} : \Omega \times \mathcal{E} \to \overline{\mathbb{N}}$ be a random Poisson measure with intensity μ. The compensated random Poisson measure $\mathcal{M}_c : \Omega \times \mathcal{S} \to \overline{\mathbb{R}}_+$ is defined as

$$\mathcal{M}_c(\omega, A) := \mathcal{M}(\omega, A) - \mu(A). \tag{62}$$

Remark 3.6. If $(E, \mathcal{E}, \mu)$ is σ-finite measure space, then we note that $\mathcal{M}_c$ is well defined for $A \in \mathcal{E}$ with $\mu(A) < \infty$.

Definition 3.11. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E}, \mu)$ be a finite measure space. Let $\mathcal{M} : \Omega \times \mathcal{E} \to \overline{\mathbb{N}}$ be a random Poisson measure with intensity μ. Let $f : E \to F$ be a measurable map with F and suppose that (1) $\int ||f(s)||\mathcal{M}(\omega, ds) < \infty$ for every $\omega \in \Omega$ and (2) $\int ||f(s)||\mu(ds) < \infty$ holds. Then we define

$$\int f(s)\mathcal{M}_c(ds) := \int f(s)\mathcal{M}(ds) - \int f(s)\mu(ds) \tag{63}$$

Lemma 3.17. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E}, \mu)$ be a σ-finite measure space. Let $\mathcal{M} : \Omega \times \mathcal{E} \to \overline{\mathbb{N}}_+$ be a random Poisson measure with Poisson intensity μ. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be its Borel-σ-algebra and $f : (E, \mathcal{E}) \to (\mathcal{H}, \mathbb{B}(\mathcal{H}))$ be a measurable map such that (1) $\int ||f(s)||\mathcal{M}(ds) < \infty$ (2) $\int ||f(s)||\mu(ds)$ and (3) $\int ||f(s)||^2\mu(ds) < \infty$, then*

$$\mathbb{E}\left\|\int f(s)\mathcal{M}_c(\omega, ds)\right\|^2 = \int ||f(s)||^2\mu(ds). \tag{64}$$

Proof. Let $f = \sum_{i=1}^n x_i \mathbb{I}_{A_i}$ be a simple function with $A_n \cap A_m = \emptyset$, $n \neq m$. It holds [3]

$$\begin{aligned}\mathbb{E}\left\|\int f(s)\mathcal{M}_c(\omega, ds)\right\|^2 &= \mathbb{E}\sum_{i=1}^n\sum_{j=1}^n \langle x_i|x_j\rangle \left(\mathcal{M}(A_i) - \mu(A_i)\right)\left(\mathcal{M}(A_j) - \mu(A_j)\right) \\ &= \sum_{i=1}^n \langle x_i|x_i\rangle\, \mathbb{E}\left(\mathcal{M}(A_i) - \mu(A_i)\right)^2 \\ &= \sum_{i=1}^n ||x_i||^2\mu(A_i) = \int ||f(s)||^2\mu(ds).\end{aligned} \tag{65}$$

Let $f : S \to \mathcal{H}$ be a general measurable map with $\int ||f(s)||^2\mu(ds) < \infty$. Let f_n be a sequence of simple functions as in (36) $f_n = \sum_i x_i^n \mathbb{I}_{A_i^n}$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, such that $f_n \to f$ for all $s \in E$. Properties (1) and (2) imply that $\int f_n\mathcal{M}(ds) \to \int f\mathcal{M}(ds)$ and $\int f_n\mu(ds) \to \int f\mu(ds)$. From this we conclude that $\int f_n\mathcal{M}_c \to \int f\mathcal{M}_c$. It follows that

[3] The variance of a Poisson random variable X_λ with parameter λ is given by

$$\mathbb{E}\left[(X_\lambda - \mathbb{E}[X_\lambda])^2\right] = \lambda.$$

$$\lim_{n\to\infty}\left\|\int f_n\mathcal{M}_c\right\| = \left\|\int f\mathcal{M}_c\right\|.$$

It holds that $\mathbb{E}\left\|\int f_n\mathcal{M}_c\right\|^2 = \int ||f_n||^2\mu(ds) \leq 2\int ||f||^2\mu(ds) < \infty$. This means that the random variables $\left\|\int f_n\mathcal{M}_c\right\|$ is bounded in L^2. Thus, this class of random variables is uniformly integrable. Hence,

$$\lim_{n\to\infty}\mathbb{E}\left\|\int f_n\mathcal{M}_c\right\|^2 = \mathbb{E}\left\|\int f\mathcal{M}_c\right\|^2.$$

By using once more the Lebesgue's dominated convergence theorem it follows

$$\begin{aligned}
\mathbb{E}\left\|\int f(s)\mathcal{M}_c\right\|^2 &= \lim_{n\to\infty}\mathbb{E}\left\|\int f_n(s)\mathcal{M}_c\right\|^2 \\
&= \lim_{n\to\infty}\int ||f_n(s)||^2\mu(ds) \\
&= \int ||f(s)||^2\mu(ds).
\end{aligned}$$

□

It holds that

$$\tilde{S}_t^\delta = \int_0^t\int_{\delta\leq\|x\|\leq 1} x\left[\mathcal{J}_X(\omega, ds\,dx) - \mu_X(ds\,dx)\right].$$

Let $Y_t := X_t - \Delta X^{\Lambda_1} - \mathbb{E}[X_t - \Delta X^{\Lambda_1}]$, Now we can write

$$Y_t = Y_t - \tilde{S}_t^\delta + \tilde{S}_t^\delta.$$

Now it holds by Lemma 3.16 that

$$\mathbb{E}||Y_t||^2 \geq \mathbb{E}||\tilde{S}_t^\delta||^2. \quad (66)$$

Note Y_t has bounded jumps and is an Additive process. By Theorem 3.2 it follows that for every $T > 0$

$$\mathbb{E}||Y_T||^2 < \infty.$$

If we assume $E = \mathcal{H}$ to be a Hilbert space, then we find with Lemma 3.17 and (66)

$$\int_0^T\int_{\delta\leq||x||\leq 1} ||x||^2\mu(dsdx) \leq \mathbb{E}||Y_T||^2 < \infty,$$

for every $0 < \delta \leq 1$. From this we conclude that for every $T > 0$,

$$\int_0^T\int_{E\backslash 0} \min(1, ||x||^2)\mu_X(dsdx) < \infty, \quad (67)$$

in case the state space $E = \mathcal{H}$ is a Hilbert space.

We would like to end with some observations without proofs:

1. For every $\Lambda \in \mathcal{B}(E)$ with $0 \notin \bar{\Lambda}$ and every $x \in E^*$ it holds
$$\mathbb{E}\left[e^{i\left\langle x^*, \Delta X_t^\Lambda \right\rangle}\right] = \exp\left\{\int_0^t \int_\Lambda \left(e^{i\langle x^*, x\rangle} - 1\right) \mu_X(dsdx)\right\}.$$
In case $E = \mathbb{R}^n$ it holds for every $u \in \mathbb{R}^n$
$$\mathbb{E}\left[e^{i\left\langle u, \Delta X_t^\Lambda \right\rangle}\right] = \exp\left\{\int_0^t \int_\Lambda \left(e^{i\langle u, x\rangle} - 1\right) \mu_X(dsdx)\right\}.$$

2. Let T_δ be the first time that the process X_t jumps with amplitude bigger than $\delta > 0$, then
$$\mathbb{P}\left\{T_\delta \leq t\right\} = 1 - \exp\left\{-\mu_X\left([0,t] \times \{||x|| > \delta\}\right)\right\}.$$

3. If we take as state space $E = \mathbb{R}$, then W_t is a Gaussian process with independent increments, see Theorem 4.9. If we take as state space $E = \mathbb{R}^n$ and denote $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$,then for every vector $(u_1, \ldots, u_n) \in \mathbb{R}^n$
$$G_t = \sum_{i=1}^n u_i W_t^{(1)},$$
is a real-valued Additive process with a.s. continuous paths. It holds for every vector $(u_1, \ldots, u_n) \in \mathbb{R}^n$ that $\sum_{i=1}^n u_i W_t^{(i)}$ is a Gaussian random variable. From this, $W_t = (W_t^{(1)}, \ldots, W_t^{(n)})$ is a multivariate normal distribution. A vector $X = (X^{(1)}, \ldots, X^{(n)})$ is a multivariate normal distribution if and only if for every vector $(u_1, \ldots, u_n) \in \mathbb{R}^n$ it holds that $\sum_{i=1}^n u_i X^{(i)}$ is normally distributed if and only if there is a vector $\mu \in \mathbb{R}^n$, and a nonnegative-definite $n \times n$-matrix Σ such that the characteristic function of X is given by
$$\Phi_X(u) = \exp\left\{i \cdot u^T \mu - u^T \Sigma u\right\}.$$
From this the characteristic functions of W_t and $W_t - W_s$ are given by
$$\Phi_{W_t}(u) = \exp\left\{i \cdot u^T \mu_t - u^T \Sigma_t u\right\}, \quad \Phi_{W_t - W_s}(u) = \exp\left\{i \cdot u^T \mu_{s,t} - u^T \Sigma_{s,t} u\right\}$$
Because it is necessary for $\Phi_{W_t}(u)$ to be continuous as function of $t \in \mathbb{R}_+$ it must hold that μ_t and $u^T \Sigma_t u$ are continuous.

4. For $E = \mathbb{R}^n$, the characteristic function of $\{X_t\}_{t \in \mathbb{R}_+}$ is
$$\phi_{X_t}(u) = \mathbb{E}\left[e^{i\langle u, W_t\rangle}\right] \mathbb{E}\left[e^{i\left\langle u, J_t^1\right\rangle}\right] \mathbb{E}\left[e^{i\left\langle u, J_t^2\right\rangle}\right].$$
$$\begin{aligned}
\mathbb{E}\left[e^{i\langle u, W_t\rangle}\right] &= \exp\left\{i \cdot u^T \mu_t - u^T \Sigma_t u\right\} \\
\mathbb{E}\left[e^{i\left\langle u, J_t^1\right\rangle}\right] &= \exp\left\{\int_0^t \int_{\{||x|| \leq 1\}} \left(e^{i\langle u, x\rangle} - 1 - \langle u, x\rangle\right) \mu_X(dsdx)\right\}. \\
\mathbb{E}\left[e^{i\left\langle u, J_t^2\right\rangle}\right] &= \exp\left\{\int_0^t \int_{\{||x|| > 1\}} \left(e^{i\langle u, x\rangle} - 1\right) \mu_X(dsdx)\right\}.
\end{aligned}$$

Note that the distribution is determined by $\mu_t \in \mathbb{R}_+$, a non-negative $n \times n$-matrix Σ_t and a measure μ_X satisfying for every $T > 0$,

$$\int_0^T \int_{E \setminus 0} \min(1, ||x||^2) \mu_X(dsdx) < \infty. \tag{68}$$

4 Appendix

4.1 Stochastic processes

Definition 4.1. Let $(E, \mathcal{E})$ be a measurable space, T a set and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A stochastic process with time parameter set T, state-space $(E, \mathcal{E})$ and underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $(X_t : t \in T)$ of $(E, \mathcal{E})$- valued random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.Generally we take $T = \mathbb{N}, \mathbb{R}_+$

We will regard stochastic processes in different ways:

1. For any measurable space $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$, let $\mathcal{L}^0(\Omega, \mathcal{F}; E, \mathcal{E})$ be the set of all measurable function from Ω to E. We regard a stochastic process as

$$X : T \to \mathcal{L}^0, \quad t \mapsto X_t.$$

   i.e. a function from the time index T to the set of all measurable functions from Ω to E.

2. Let $\mathcal{P}_E$ be the collection of all probability measures μ on $(E, \mathcal{E})$. Define for every $t \in T$, the probability measure

$$\mu_{X_t} = \mathbb{P} \circ X_t^{-1}.$$

   With this we can regard our stochastic process as

$$X : T \to \mathcal{P}_E, \quad t \mapsto \mu_{X_t} \in \mathcal{P}_E.$$

   i.e. a function from the time index T to thecollection of all probability measures on $(E, \mathcal{E})$.

3. One can also view a stochastic process as a sample path realization. For every $\omega \in \Omega$, the map

$$X(\omega) : T \to E, \quad t \mapsto X_t(\omega),$$

   is called a sample path realization. This leads to an alternative view as a map

$$X : \Omega \to E^T, \quad \omega \mapsto (t \mapsto X_t(\omega)),$$

   i.e. the stochastic process is viewed as an $(E^T, \mathcal{E}^T)$-valued map.

Definition 4.2. A **finite-dimensional rectangle** in E^T is a set of the form

$$\left\{x \in E^T | x_{t_1} \in B_1, \dots x_{t_n} \in B_n\right\},$$

for $\{t_1, \dots, t_n\} \subset T$ and $E_i \in \mathcal{E}$. The set of all **finite-dimensional rectangles** is denoted by $\mathcal{C}^T$.

It is clear that $\mathcal{C}^T$ is $\pi - system$, i.e. for all $A, B \in \mathcal{C}^T$ i holds that $A \cap B \in \mathcal{C}^T$. The σ-algebra $\mathcal{E}^T$ is the σ-algebra generated by this π-system,

$$\mathcal{E}^T = \sigma\left\{\mathcal{C}^T\right\}.$$

It is the smallest σ-algebra such that the map $X : \Omega \to E^T$ is measurable.

Definition 4.3. Let $(X_t)_{t\in T}$ be a stochastic process with state space $(E, \mathcal{E})$. We denote $\mathcal{F}^X$ as the smallest σ-algebra on Ω such that for every $t \in T$, the map $X : \Omega \to E$ is measurable.

It is clear that $\mathcal{F}^X = \sigma\left\{X^{-1}(\mathcal{C}^T)\right\}$, i.e. $\mathcal{F}^X$ is generated by sets of the form,

$$\{X_{t_1} \in B_1,\ X_{t_2} \in B_2 \dots,\ X_{t_n} \in B_n\},$$

where $t_1 < t_2 < \dots < t_n$ and $B_i \in \mathcal{E}$, $i = 1, \dots, n$.

Definition 4.4. A collection Σ of subset in Ω is called a π-system if it is closed under finite intersections. A collection $\mathcal{D}$ is called a D-set if it contains Ω , for every $A, B \in D$ with $A \subset B$ implies $B \setminus A \in \mathcal{D}$ and for every $D_n \in \mathcal{D}$ with $D_n \subset D_{n+1}$ it holds $\bigcup_n D_n \in \mathcal{D}$.

Theorem 4.1 (Dynkin)**.** *If a π-system Σ is contained in a D-set $\mathcal{D}$, then $\sigma\{\Sigma\} \subset \mathcal{D}$.*

Proof. See [13, Theorem 1.1] □

Definition 4.5. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{C}_i \subset \mathcal{F}$, $i \in \mathcal{I}$, be classes events such that for every $i_1, \dots, i_n \in \mathcal{I}$ and $A_{i_k} \in \mathcal{C}_{i_k}$,

$$\mathbb{P}\left\{\bigcap_{k=1}^{n} A_{i_k}\right\} = \prod_{k=1}^{n} \mathbb{P}\left\{A_{i_k}\right\}.$$

Then the classes of events $\mathcal{C}_i$, $i \in \mathcal{I}$, are called independent.

Definition 4.6. Let $\left\{X_t^{(n)}\right\}_{t\in T}$, $n = 1, 2, \dots$ be stochastic processes on the same underlying probability space. We call them independent if $\mathcal{F}^{X^{(n)}} = \sigma\left\{X_t^{(n)} : t \in T\right\}$,$n = 1, 2, \dots,$ are independent.

Lemma 4.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{C}_i \subset \mathcal{F}$, $i \in \mathcal{I}$ be independent classes of events such that $\mathcal{C}_i$ is a π-system for every $i \in \mathcal{I}$, then $\sigma\{C_i\}$, $i \in \mathcal{I}$, are also independent.*

Proof. See [13, Lemma 2.6] □

4.2 Stochastic processes viewed as random path realization

A stochastic process X can be viewed as a path realization $t \mapsto X_t$ for every $\omega \in \Omega$, i.e. as a map from $(\Omega, \mathcal{F})$ to $(E^T, \mathcal{E}^T)$. We will assume conditions on the path-realizations of the stochastic processes.

Definition 4.7. Let $(E, ||\cdots||)$ be a normed space. A function $f : \mathbb{R}_+ \to E$ is called regular if for every $t \in \mathbb{R}_+$

1. the left limit f(t-) $= \lim_{s \uparrow t} f(s)$ exists;
2. the right limit f(t+) $= \lim_{s \downarrow t} f(q)$ exits.

If in addition for every $t \in \mathbb{R}_+$, $f(t+) = f(t)$, then f is called càdlàg.

Definition 4.8. Let X be stochastic process, then we call the process Càdlàg when for all $\omega \in \Omega$, the paths $t \mapsto X_t$ are càdlàg.

Definition 4.9. Let $(E, ||\cdot||)$ be a normed space. For a regular function $f : \mathbb{R}_+ \to E$, the left- and right-jump are defined as

$$\Delta f(t+) = f(t+) - f(t), \quad \Delta f(t-) = f(t) - f(t-).$$

Definition 4.10. Let $(E, ||\cdot||)$ be a normed space. For a regular function $f : \mathbb{R}_+ \to E$ define the following sets

$$J^{\pm}(f) := \{t \in \mathbb{R} : \Delta f(t\pm) \neq 0\}.$$

and for $k > 0$

$$J_k^{\pm}(f) := \{t \in \mathbb{R} : ||\Delta f(t\pm)|| > k\}.$$

Finally define $J(f) := J^+(f) \cup J^-(f)$ and $J_k := J_k^+(f) \cup J_k^-(f)$.

Lemma 4.2. *Let $(E, ||\cdot||)$ be a normed space and $f : \mathbb{R} \to M$ a regular function, then for $k > 0$, J_k is finite and the set $J(f)$ is at most countable.*

Proof. Fix an interval $[a, b]$. Suppose that $t \in J(f)$, then there is $n \in \mathbb{N}$ such that for every $\delta > 0$ there is $s \in B_\delta(t) = \{s \in [a, b] : |t - s| \leq \delta\}$ such that

$$||f(t) - f(s)|| > \frac{1}{n}.$$

Let $t^* \in [a, b]$. Then there exists $\epsilon_{t^*} > 0$, such that $(B_{\epsilon_{t^*}} \setminus t^*) \cap J_{\frac{1}{n}} = \emptyset$. Otherwise there exists a sequence $t_m \uparrow t^*$ with $t_m \in J_{\frac{1}{n}}$(or $t_m \downarrow t^*$). In that case, a sequence τ_m, $t_m < \tau_m < t_{m+1}$, can be chosen such that $||f(t_m), f(\tau_m)|| > \frac{1}{n}$, hence

$$\lim_m f(\tau_m) \neq \lim_m f(t_m).$$

This contradicts regularity of f. Now $\bigcup_{t^* \in [a,b]} B_{\epsilon_{t^*}}$ is a open subcovering of $[a,b]$ and by compactness there are $s_1, s_2, \ldots, s_N \in [a,b]$ with

$$[a,b] \subset \bigcup_{i}^{N} B_{\epsilon_{s_i}}(s_i).$$

Hence it holds that $J_{\frac{1}{n}} \subset \{s_1, \ldots, s_N\}$ and thus J_n is finite. Conclude that $J \cap [a,b]$ is countable and thus J is countable.

□

Lemma 4.3. *Let f be a càdlàg function. For every $\epsilon > 0$, there exists a partition $s = t_0 < t_1 < \ldots < t_{p(n)} = t$ such that*

$$\sup\{|f(v) - f(w)| : v, w \in [t_i, t_{i+1})\} \leq \epsilon, \quad i = 0, \ldots, p(n) - 1$$

Proof. Let ϵ be given. For every $s \in [0,t]$ there exists a $\delta(s) > 0$ such that for $k, h \in (s - \delta(s), s + \delta(s))$ with $h, k \neq 0$ or $h, k \geq 0$, it holds that $|f(h) - f(k)| \leq \epsilon$. Now it holds that $\{(s - \delta(s), s + \delta(s)) : s \in [0,t]\}$ is an open subcover of $[0,t]$. By compactness there exist $s_1, s_2, \ldots, s_N$ such that $[0,t] \subset \bigcup_i (s_i - \delta(s_i), s + \delta(s_i))$. Now with these s_i we can construct our partition. □

Definition 4.11. Let f be càdlàg function. Let $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a partition and $\mathbb{V}_t^f(\Pi) = \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$. Then f is of finite variation if for all $t \geq 0$,

$$\mathbb{V}_t^f = \sup_{\Pi} \mathbb{V}_t^f(\Pi) < \infty,$$

where the supremum is taken over all partitions of [0,t].

Remark 4.1. Let f be càdlàg function and $\Pi = \{0 = t_0 < t_1 < \ldots < t_n = t\}$ be a partition. Then define for every t_i a decreasing seguence $(t_i^k)_k$ such that $t_i < t_i^k < t_{i+1}$, $t_i^k \in \mathbb{Q}$ and $t_i^k \downarrow t_i$. Let $\Pi_k = \left\{0 = t_0^k < t_1^k < \ldots < t_n^k = t\right\}$, then it holds that

$$\begin{aligned}
|\mathbb{V}_t^f(\Pi) - \mathbb{V}_t^f(\Pi_k)| &\leq \sum_{i=0}^{n-1} \left| |f(t_{i+1}) - f(t_i)| - |f(t_{i+1}^k) - f(t_i^k)| \right| \\
&\leq \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_{i+1}^k)| + |f(t_i) - f(t_i^k)|.
\end{aligned}$$

By taking k large enough, we see by right-continuity that $\mathbb{V}_t^f(\Pi_k) \to \mathbb{V}_t^f(\Pi)$. We see thus that the supremum can be taken over all partitions in $\mathbb{Q}$, which are countable. If we consider a càdlàg stochastic process $\{X\}_{t \in T}$, then it holds that $\mathbb{V}_t^X$ is the supremum of a countable set of random variables and thus measurable.

4.3 Convergence in probability and distribution.

Most of the results in this section can be found in [7]. Let $(\Omega,\mathcal{F})$,$(E,\mathcal{E})$ be measurable spaces and $\mathcal{L}^0(\Omega,\mathcal{F};E,\mathcal{E})$ be the set of all measurable maps from Ω to E. If E is a metric space then we take $\mathcal{E}$ always to be $\mathcal{B}(E)$, the Borel σ-algebra. For a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$ we let $L^0_{\mathbb{P}}(\Omega,\mathcal{F};E,\mathcal{E})$ be the set of all equivalence classes of $\mathcal{L}^0(\Omega,\mathcal{F};E,\mathcal{E})$ with equivalence relation $\mathbb{P}$-a.s. equality. For $X\in\mathcal{L}^0(\Omega,\mathcal{F};E,\mathcal{E})$ we denote the equivalence class of X in $L^0_{\mathbb{P}}(\Omega,\mathcal{F};E,\mathcal{E})$ by $[X]_{\mathbb{P}}$.

Definition 4.12. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and (E,d) a **separable metric space**. For $X_n,X\in\mathcal{L}^0(\Omega,\mathcal{F};E,\mathcal{E})$ convergence in probability of X_n to X denoted by $X_n\xrightarrow{\mathbb{P}}X$ holds if for every $\epsilon>0$

$$\mathbb{P}(d(X_n,X)>\epsilon)\to 0,\quad \text{as } n\to\infty.$$

Remark 4.2. For separable metric spaces $S_1,S_2,\dots,S_n$ it holds

$$\mathcal{B}(S_1\times S_2\times\ldots\times S_n)=\mathcal{B}(S_1)\times\mathcal{B}(S_2)\times\ldots\times\mathcal{B}(S_n),$$

see [13, Lemma1.2]. For a metric space (S,d) it holds that $d:S\times S\to\mathbb{R}_+$ is a continuous map. Now it follows that $d(X,Y)$ is measurable because $\mathcal{B}(S\times S)=\mathcal{B}(S)\times\mathcal{B}(S)$.

Definition 4.13. Let $(X_t)_{t\in Y}$ be a stochastic process, then (X_t) is called continuous in probability if for every $t\in\mathbb{R}_+$ and $\epsilon>0$, $\lim_{s\to t}\mathbb{P}(d(X_s,X_t)>\epsilon)=0$.

Remark 4.3. X_n tends to X in probability if and only if there exist a subsequence $k(n)$ such that $X_{k(n)}$ converges to X a.s.

For $X,Y\in\mathcal{L}^0(\Omega,\mathcal{F};E,\mathcal{E})$ with metric space (E,d) we define

$$d_{\mathbb{P}}:=\inf\left\{\epsilon\geq 0\,|\,\mathbb{P}(d(X,Y)>\epsilon)\leq\epsilon\right\}. \tag{69}$$

Let $X'\in[X]_{\mathbb{P}}$ and $Y'\in[Y]_{\mathbb{P}}$, then $\mathbb{P}(d(X,Y)>\epsilon)=\mathbb{P}(d(X',Y')>\epsilon)$ and thus

$$d_{\mathbb{P}}(X,Y)=d_{\mathbb{P}}(X',Y').$$

We can define $d_{\mathbb{P}}([X]_{\mathbb{P}},[Y]_{\mathbb{P}})=d_{\mathbb{P}}(X,Y)$.

Lemma 4.4 (Ky Fan Metric)**.** *Let $(\Omega,\mathbb{F},\mathbb{P})$ be a probability space and (E,d) be a separable metric space, then on $L^0_{\mathbb{P}}(\Omega,\mathcal{F};E,\mathcal{E})$, $d_{\mathbb{P}}$ is metric, which metrizes convergence in probability so that $d_{\mathbb{P}}([X_n]_{\mathbb{P}},[X]_{\mathbb{P}})\to 0$ if and only if $X_n\xrightarrow{\mathbb{P}}X$.*

Proof. See[7, Theorem 9.2.2.]. □

Lemma 4.5. *Let X be a stochastic process. Suppose that X is continuous in probability. Fix $T>0$. Then for every $\epsilon>0$ and $\gamma>0$ there exists a $\theta>0$ such that, for $s,t\in[0,T]$ with $|t-s|<\theta$, it holds that*

$$\mathbb{P}(d(X_t,X_s)>\epsilon)<\gamma.$$

Proof. Let $t \in \cap[0,T]$, then for every $c > 0$, there exists a $\delta_t > 0$ such that $d_{\mathbb{P}}(X_t, X_s) < c$ for all $s \in B_{\delta_t}(t)$. By compactness of $[0,T]$ we can find $t_1, \dots t_n \in \cap[0,T]$ such that $[0,T] \subset \bigcup_{i=1}^{n} B_{\frac{\delta_{t_i}}{2}}(t_i)$. Let ϵ and γ be given. Take $c = \frac{1}{2}\min\{\epsilon, \gamma\}$ and $\delta = \min\left\{\frac{\delta_{t_1}}{2}, \dots, \frac{\delta_{t_n}}{2}\right\}$. Now it holds for $s, t \in \cap[0,T]$ with $|t-s| < \delta$ that $s, t \in B_{\delta_{t_i}}(t_i)$ for some i. From this we find that

$$d_{\mathbb{P}}(X_t, X_s) \leq d_{\mathbb{P}}(X_{t_i}, X_t) + d_{\mathbb{P}}(X_{t_i}, X_s) < 2c.$$

Now it holds that $\mathbb{P}(d(X_t, X_s) > 2c) < 2c$, from which we find that $\mathbb{P}(d(X_t, X_s) > \epsilon) < \gamma$. □

Let (E,d) a metric space and denote

$$C_b(E) = \{f : E \to \mathbb{R} : \text{f is continuous and bounded}\}.$$

Definition 4.14. Let $\mu, \mu_1, \mu_2, \dots$ be finite Borel measures on E. We say that $(\mu_n)_{n\in\mathbb{N}}$ converges in distribution to μ if for every $f \in C_b(E)$

$$\int f d\mu_n \to \int f d\mu.$$

Notation : $\mu_n \rightsquigarrow \mu$

Remark 4.4. $X_n \xrightarrow{\mathbb{P}} X \Rightarrow \mu_{X_n} \rightsquigarrow \mu_X$.

Remark 4.5. We recal that the bounded Lipschitz Metric d_{BL} is defined as, for μ_1, μ_2

$$d_{BL}(\mu_1, \mu_2) := \sup\left\{\left|\int_E f d(\mu_1 - \mu_2)\right| : ||f||_{BL} \leq 1\right\},$$

where $||f||_{BL} := \sup_{x\in E}|f(x)| + \sup_{x\neq y}\frac{|f(x)-f(y)|}{d(x,y)}$. This metric metrizes convergence in distribution on $\mathcal{P}_E$ under the condition that (E,d) is a separable metric space.

Theorem 4.2. *If (E,d) is a separable metric space , then for any $\mu, \mu_1, \dots \in \mathcal{P}_E$ one has*

$$\mu_n \rightsquigarrow \mu \ \text{ if and only if } d_{BL}(\mu_n, \mu) \to 0.$$

Definition 4.15. A set Γ of Borel probability measures on E are uniformly tight if for every $\epsilon > 0$ there exists a compact subset K of E such that

$$\mu(K) \geq 1 - \epsilon, \quad \text{for all } \mu \in \Gamma.$$

Theorem 4.3 (Prokhorov Criterion)**.** *Let (E,d) be a complete separable metric space and let Γ be a subset of $\mathcal{P}_E$. Then the following statements are equivalent*

1. *$\overline{\Gamma}$ is compact in $\mathcal{P}_E$.*
2. *Γ is uniformly tight.*

4.4 Characteristics

We list some basic properties of characteristic functions. We refer to [19] for a complete treatment of characteristic functions.

Definition 4.16. Let X be a random vector in $\mathbb{R}^d$ with distribution μ_X. The characteristic function Φ_X is defined as

$$\Phi_X(u) = \mathbb{E}[e^{i(u,X)}], \quad u \in \mathbb{R}^d. \tag{70}$$

Theorem 4.4. *Let X, Y be two random vectors, then X, Y are identically distributed if and only if $\Phi_X(u) = \Phi_Y(u)$ for all $u \in \mathbb{R}^d$.*

Theorem 4.5. *Every characteristic function Φ_X has the following properties:*

1. $\Phi_X(0) = 1$
2. $|\Phi_X(u)| \leq 1, \quad \forall u \in \mathbb{R}^d$.
3. $\Phi_X(-u) = \bar{\Phi}_X(u), , \quad \forall u \in \mathbb{R}^d$.
4. Φ_X *is uniformly continuous on* $\mathbb{R}^d$.

Theorem 4.6. *Random variables $X_1, \ldots, X_d$ with characteristic functions $\Phi_{X_i}(u_i)$ are independent if and only if*

$$\Phi_X(u) = \prod_{i=1}^{d} \Phi_{X_i}(u_i), \tag{71}$$

where $X = (X_1, \ldots, X_d)$ and $u = (u_1, \ldots, u_n)$.

Theorem 4.7. *Let $X, X_1, X_2, \ldots, X_n, \ldots$ be random vectors with distributions μ_X, μ_{X_i}. $\mu_{X_n} \rightsquigarrow \mu_X$ if and only if Φ_{X_n} converges uniformly on every compact set to Φ_X.*

Lemma 4.6. *Let X be a random variable in $(\mathbb{R}, \mathcal{B}(\mathbb{R})$ and Φ_X the corresponding characteristic function , then for $u > 0$*

$$\mathbb{E}X^2\mathbb{I}_{|X|<\frac{1}{u}} \leq \frac{3}{u^2}(1 - \Re\{\Phi_X(u)\}) \tag{72}$$

and

$$\mathbb{P}(|X| \geq \frac{1}{u}) \leq \frac{7}{u}\int_0^u 1 - \Re\{\Phi_X(v)\}dv. \tag{73}$$

Proof. See [15, p-209, B' Inequality]. □

Lemma 4.7. *Let X be a random variable in $(\mathbb{R}, \mathcal{B}(\mathbb{R})$ and Φ_X the corresponding characteristic function, then*

$$|\Phi_X(t) - \Phi_X(s)| \leq \sqrt{2|1 - \Phi_X(t-s)|}. \tag{74}$$

Proof. See [15, P-208,B. Inequality]. □

4.5 Buildings blocks of Additive processes

In this section we consider the building blocks of Additive processes:

(1) Additive processes with values in $\mathbb{Z}_+$ and jumps of amplitude 1.

(2) Additive processes with a.s. continuous paths

Processes with property (1) have Poisson distributed increments. Processes with property (2) have Gaussian distributed increments. A random variable X is Poisson distributed with intensity $\lambda > 0$ if

$$\mathbb{P}\{X = k\} = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k \in \mathbb{N}.$$

The characteristic function is $\Phi_X(u) = e^{\lambda(e^{iu}-1)}, \quad \forall u \in \mathbb{R}$. A random variable Y is Gaussian distributed with mean m and variance σ^2 if

$$\mathbb{P}\{Y \in A\} = \frac{1}{\sigma\sqrt{2\pi}} \int_A e^{\frac{-(x-m)^2}{2\sigma^2}} dx, \quad A \in \mathcal{B}(\mathbb{R}).$$

The characteristic function is given by $\Phi_X(u) = e^{imu - \frac{\sigma^2 u^2}{2}}, \quad \forall u \in \mathbb{R}$.

The content of this section is to show that Additive processes with property (1) have Poisson distributed increments and Additive processes with property (2) have Gaussian increments. These results and proofs are taken from [11] and [9]. In order to prove the statements we need an elementary relation

$$-\log(1-x) = x + o(x), \text{as } x \to 0. \tag{75}$$

Suppose that for every $n \in \mathbb{N}$ there are positive values $x_1^n, \dots, x_{r_n}^n$ such that the maximum $\lim_{n\to\infty} \max_{1\le i \le r_n} x_i^n = 0$, then from (75),

$$\lim_{n\to\infty} \sum_{i=1}^{r_n} x_i^n \to c \iff \lim_{n\to\infty} \prod_{i=1}^{r_n} (1 - x_i^n) \to e^{-c}. \tag{76}$$

Suppose that we are given N independent Bernoulli random variables $X_1, \dots, X_N$. If $X_1, \dots, X_N$ are identically distributed, with N very large and $\lambda = pN$, where $\mathbb{P}\{X_i = 1\} = p$, then the Poisson distribution with intensity λ appears to be a good approximation of $\sum_{i=1}^N X_i$. The following theorem gives a bound. The importance of the Poisson Approximation becomes clear in Theorem 4.9.

Theorem 4.8 (Poisson Approximation)**.** *Let $X_1, \dots, X_n$ be Bernoulli distributed with $\mathbb{P}(X_i = 1) = p_i$. Let $\lambda = \sum_i p_i$, $M = \max_i p_i$ and $\mathbb{P}_1, \mathbb{P}_2$ be two probability measures on $\mathbb{N}$, such that $\mathbb{P}_1$ is the probability distribution of $\sum_i X_i$ and $\mathbb{P}_2$ a Poisson distribution with parameter λ. Then it holds that*

$$\sup_{A \subset \mathbb{N}} |\mathbb{P}_1(A) - \mathbb{P}_2(A)| \le M\lambda. \tag{77}$$

Proof. See [22, page 12-14]. □

Theorem 4.9. *Let $\{X\}_{t\in\mathbb{R}_+}$ be an Additive process with property (1). Then there exists a non-decreasing continuous function $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\mathbb{P}\{X_t - X_s = k\} = \frac{(\Lambda(t) - \Lambda(s))^k}{k!} e^{-(\Lambda(t)-\Lambda(s))},$$

i.e. the increment $X_t - X_s$ has a Poisson distribution with parameter $\Lambda(t) - \Lambda(s)$.

Proof. For $s < t$ and every $n \in \mathbb{N}$ let $t_{nk} = \frac{(t-s)k}{n}$ with $k = 0, 1, \ldots, n-1$. Define $Z_{nk} = X_{t_{n(k+1)}} - X_{t_{nk}}$. By continuity in probability by Lemma 4.5, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \max_{1\leq k\leq n-1} \mathbb{P}\{Z_{nk} \geq \epsilon\} = 0.$$

Now we use the properties that the process has independent increments, takes values in $\mathbb{Z}_+$, all paths are increasing and jumps of amplitude 1. Because for every realization, the path is non-decreasing, $\{X_t - X_s < \epsilon\} \subset \cap_{k=0}^{n-1} \{Z_{nk} < \epsilon\}$ and thus,

$$\mathbb{P}\{X_t - X_s < \epsilon\} \leq \prod_{k=0}^{n-1} \mathbb{P}\{Z_{nk} < \epsilon\}.$$

The process takes values in $\mathbb{Z}_+$ from which it follows that

$$\mathbb{P}\{X_t - X_s = 0\} \leq \prod_{k=0}^{n-1} \mathbb{P}\{Z_{nk} = 0\}.$$

For the moment, assume $|t-s|$ sufficiently small, such that $\mathbb{P}(X_t - X_s = 0) > 0$. Then we find from (76) that $\sum_{i=0}^{n-1} \mathbb{P}(Z_{ni} \geq 1)$ converges to a finite value. Now we define the following process,

$$Z'_{nk} = \begin{cases} 0 & \text{if } Z_{nk} = 0 \\ 1 & \text{if } Z_{nk} \geq 1. \end{cases}$$

Let $Z'_n = \sum_{i=0}^{n-1} Z'_{ni}$, then it holds that $Z'_n \overset{a.s.}{\to} X_t - X_s$ and by Theorem 4.8 we find the bound

$$\sup_{A\subset\mathbb{N}} |\mathbb{P}(Z'_n \in A) - \mathbb{P}_{\lambda_n}(A)| \leq M_n\lambda_n.$$

where $M_n = \max_i \mathbb{P}(Z_{ni} \geq 1)$, $\lambda_n = \sum_i \mathbb{P}(Z_{ni} \geq 1)$ and $\mathbb{P}_{\lambda_n}$ is a Poisson measure with parameter λ_n. Now we find with Lemma 4.5

$$\sup_{A\subset\mathbb{N}} |\mathbb{P}(Z'_n \in A) - \mathbb{P}_{\lambda_n}(A)| \to 0, \quad \text{as } n \to \infty.$$

From this it follows that $X_t - X_s$ is Poisson distributed with parameter

$$\lambda = \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbb{P}(Z_{ni} \geq 1).$$

We know that for independent random variables Y_i, with a Poisson distribution and with parameter λ_i, the sum $\sum_i Y_i$ is again Poisson distributed with parameter $\lambda = \sum_i \lambda_i$.

From this it follows that for every $s < t$, $X_t - X_s$ is Poisson distributed. Now denote the parameter of X_t by Λ_t. Because the paths of X_t are always non-decreasing, it is clear that $t \mapsto \Lambda_t$ is non-decreasing. Now the characteristic function of X_t is given by $\Phi_{X_t}(u) = e^{(e^{iu}-1)\lambda_t}$. By continuity in probability we must have for every $u \in \mathbb{R}$, $t \mapsto \Phi_{X_t}(u)$ is continuous. We conclude that $t \mapsto \Lambda_t$ is continuous. □

Let $\{X_t\}_{t\in\mathbb{R}_{\geq 0}}$ be an Additive process in $\mathbb{R}$ with property 2. We will consider the process on $[t_0, t_1]$. We follow [11, Section 1.4] For given δ, ϵ define the set

$$D_{\epsilon,\delta} := \left\{ \sup_{\substack{|t-s|<\delta \\ t,s\in[t_0,t_1]}} |X_t - X_s| < \epsilon \right\}. \tag{78}$$

Lemma 4.8. *Let $(X_t)_{t\in\mathbb{R}_{\geq 0}}$ be a stochastic process with continuous paths. Fix an interval $[t_0, t_1]$, then $D_{\epsilon,\delta}$ is measurable and for every $\epsilon > 0$ there exist $\delta(\epsilon)$ such that $\mathbb{P}\{D_{\epsilon,\delta(\epsilon)}\} > 1 - \epsilon$*

Proof. Because of continuity of the paths it holds that

$$D_{\epsilon,\delta} = \bigcup_{m\in\mathbb{N}} \bigcap_{\substack{q_1,q_2\in\mathbb{Q}\cap[t_0,t_1] \\ |q_1-q_2|<\delta}} \left\{ |X_{q_1} - X_{q_2}| < \epsilon - \frac{1}{m} \right\}.$$

The case "$\subset$" is clear. However the case "$\supset$" need some extra argument. Suppose that ω is in the right part. Then there is a m such that for all $q_1, q_2 \in \mathbb{Q}\cap[t_0, t_1]$ with $|q_1 - q_2| < \delta$ it holds

$$|X_{q_1}(\omega) - X_{q_2}(\omega)| < \epsilon - \frac{1}{m}.$$

Let $s, t \in [t_0, t_1]$ with $|t - s| < \delta$. Without loss of generality assume $s < t$ and take $h_n \downarrow s$ and $k_n \uparrow t$ with $h_n, k_n \in [s, t] \cap \mathbb{Q}$. Now it holds

$$\begin{aligned} |X_t(\omega) - X_s(\omega)| &\leq |X_t(\omega) - X_{k_n}(\omega) + X_{k_n}(\omega) - X_{h_n}(\omega) + X_{h_n}(\omega) - X_s(\omega)| \\ &\leq |X_t(\omega) - X_{k_n}(\omega)| + |X_{k_n}(\omega) - X_{h_n}(\omega)| + |X_{h_n}(\omega) - X_s(\omega)| \end{aligned}$$

Now by letting $n \to \infty$

$$|X_t(\omega) - X_s(\omega)| \leq \epsilon - \frac{1}{m}$$

and thus

$$\sup_{\substack{|t-s|<\delta \\ t,s\in[t_0,t_1]}} |X_t - X_s| \leq \epsilon - \frac{1}{m} < \epsilon.$$

Now, $D_{\epsilon,\delta}$ is a countable union of measurable sets, hence it is measurable. Consider

$$(D_{\epsilon,\delta})^c = \{\omega \in \Omega : \exists s, t \in [t_0, t_1],\ |t - s| < \delta,\ |X_t - X_s| \geq \epsilon\},$$

and define $\delta_n = \frac{1}{2^n}$, then $(D_{\epsilon,\delta_{n+1}})^c \subset (D_{\epsilon,\delta_n})^c$ and by uniform continuity of continuous functions on compact sets it holds that $\bigcap(D_{\epsilon,\delta_n})^c$ is the set of all ω for which $t \to X_t(\omega)$ is not continuous. From this it is clear that $\lim_{n\to\infty} \mathbb{P}\{(D_{\epsilon,\delta_n})^c\} = 0$, hence for $\epsilon > 0$ there exist $\delta(\epsilon) > 0$ such that $\mathbb{P}\{D_{\epsilon,\delta(\epsilon)}\} > 1 - \epsilon$. □

Before we prove the next lemma a little remark for symmetric random variables is needed. If a random variable is symmetric, i.e. $\mathbb{P}(X \in A) = \mathbb{P}(X \in -A)$ for all $A \in \mathcal{B}(\mathbb{R}^d)$. Then the characteristic function is given by

$$\Phi_X(u) = \mathbb{E}(\cos(uX)).$$

Lemma 4.9. *Let $\{X_t\}_{t\in\mathbb{R}_{\geq 0}}$ be an Additive process with property (2). Fix an interval $[t_0, t_1]$, then $X_{t_1} - X_{t_0}$ is Gaussian distributed.*

Proof. Let $X_t^s = X_t - \bar{X}_t$ be the symmetrization, recall (cf. 2.2). For $\epsilon, \delta > 0$ define

$$D^s_{\epsilon,\delta} := \left\{ \sup_{\substack{|t-h|<\delta \\ t,h\in[t_0,t_1]}} |X_t^s - X_h^s| < \epsilon \right\}.$$

For every $\epsilon > 0$ by Lemma 4.8 there is an $\delta(\epsilon) > 0$ such that $\mathbb{P}\{D^s_{\epsilon,\delta(\epsilon)}\} > 1 - \epsilon$. Take a sequence $\epsilon_n \downarrow 0$ and a sequence of partitions

$$t_0 = t_{n,0} < t_{n,1} < \ldots < t_{n,k_n} = t_1, \quad t_{n,i+1} - t_{n,i} < \delta(\epsilon_n)$$

Define the truncations $Y_{n,k} := (X^s_{t_{n,k}} - X^s_{t_{n,k-1}})\mathbb{I}_{\{|X^s_{t_{n,k}} - X^s_{t_{n,k-1}}| \leq \epsilon_n\}}$ and $S_n := \sum_{i=1}^{k_n} Y_{n,i}$. From

$$D^s_{\epsilon,\delta(\epsilon)} \subset \bigcap_{k=1}^{k_n} \{|X^s_{t_{n,k}} - X^s_{t_{n,k-1}}| \leq \epsilon_n\} \subset \{S_n = X^s_{t_1} - X^s_{t_0}\},$$

it follows that S_n converges in probability to $X^s_{t_1} - X^s_{t_0}$. Convergence in probability implies convergence in distribution. This implies that the characteristic functions Φ_{S_n} converge uniformly on compact intervals to the characteristic function of $X^s_{t_1} - X^s_{t_0}$,

$$\begin{aligned} |\Phi_{X_{t_1}-X_{t_0}}(u)|^2 &= \Phi_{X^s_{t_1}-X^s_{t_0}}(u) \\ &= \lim_{n\to\infty} \Phi_{S_n}(u) \\ &= \lim_{n\to\infty} \prod_{i=j}^{k_n} \mathbb{E}\cos(uY_{n,j}). \end{aligned} \tag{79}$$

The last equality holds because the random variables $Y_{n,i}$ are symmetric. We use the series representation of $\cos(x)$

$$\begin{aligned} &= \lim_{n\to\infty} \prod_{i=1}^{k_n} \mathbb{E}\left(1 - \frac{Y_{n,i}^2 u^2}{2!} + \frac{Y_{n,i}^4 u^4}{4!} - \ldots\right) \\ &= \lim_{n\to\infty} \prod_{i=1}^{k_n} \mathbb{E}\left(1 - \frac{Y_{n,i}^2 u^2}{2!} + \frac{Y_{n,i}^2 u^2}{2!} O(\epsilon_n)\right) \\ &= \lim_{n\to\infty} \prod_{i=1}^{k_n} \left(1 - \frac{\mathbb{E}(Y_{n,i}^2) u^2}{2!} (1 + O(\epsilon_n))\right). \end{aligned} \tag{80}$$

Note that as $n \to \infty$, $\lim_{n\to\infty} \max_{i \le k_n} \frac{\mathbb{E}(Y_{n,i}^2)u^2}{2!}(1 + O(\epsilon_n)) = 0$. For u sufficiently close to zero, $|\Phi_{X_{t_1}-X_{t_0}}(u)|^2 > 0$. We conclude by (76) that

$$c = \lim_{n\to\infty} \sum_{i=1}^{k_n} \frac{\mathbb{E}(Y_{n,i}^2)}{2} < \infty.$$

Furthermore by (76) we find for every u,

$$\lim_{n\to\infty} \prod_{i=1}^{k_n} \left(1 - \frac{\mathbb{E}(Y_{n,i}^2)u^2}{2!}(1 + O(\epsilon_n))\right) = e^{-cu^2}.$$

The characteristic function of $X_{t_1} - X_{t_0}$ is of the form

$$\Phi_{X_{t_1}-X_{t_0}}(u) = e^{i\xi(u) - cu^2\frac{1}{2}}.$$

For $\epsilon, \delta > 0$ define

$$D_{\epsilon,\delta} := \left\{ \sup_{\substack{|t-h|<\delta \\ t,h\in[t_0,t_1]}} |X_t - X_h| < \epsilon \right\}.$$

For every $\epsilon > 0$ by Lemma 4.8 there is an $\delta(\epsilon) > 0$ such that $\mathbb{P}\{D_{\epsilon,\delta(\epsilon)}\} > 1 - \epsilon$. Take a sequence $\epsilon_n \downarrow 0$ and a sequence of partitions

$$t_0 = t_{n,0} < t_{n,1} < \ldots < t_{n,k_n} = t_1, \quad t_{n,i+1} - t_{n,i} < \delta(\epsilon_n).$$

Define the truncations $Z_{n,k} := (X_{t_{n,k}} - X_{t_{n,k-1}})\mathbb{I}_{\{|X_{t_{n,k}} - X_{t_{n,k-1}}| \le \epsilon_n\}}$ and $Z_n := \sum_{i=1}^{k_n} Z_{n,i}$. From

$$D_{\epsilon,\delta(\epsilon)} \subset \bigcap_{k=1}^{n} \{|X_{t_{n,k}} - X_{t_{n,k-1}}| \le \epsilon_n\} \subset \{Z_n = X_{t_1} - X_{t_0}\},$$

it follows that Z_n converges in probability to $X_{t_1} - X_{t_0}$. Now write $\Phi_{Z_n}(u) = e^{im_n u}\Re(\Phi_{Z_n}(u))$. Now for every $u \in \mathbb{R}$

$$e^{im_n u} \to \Phi_{X_{t_1}-X_{t_0}}(u)e^{cu^2}, \text{ as } n \to \infty.$$

The m_n are bounded. Otherwise, there is a unbounded subsequence $\{m_{n_k}\}_k$ from which

$$\begin{aligned} \left|\int_0^\epsilon \Phi_{X_{t_1}-X_{t_0}}(u)e^{cu^2}du\right| &= \lim_k \left|\int_0^\epsilon e^{im_{n_k}u}du\right| \\ &= \lim_k \left|\frac{e^{im_{n_k}\epsilon} - 1}{im_{n_k}}\right| = 0. \end{aligned}$$

This is a contradiction, for ϵ sufficiently close to 0, $\left|\int_0^\epsilon \Phi_{X_{t_1}-X_{t_0}}(u)e^{cu^2}du\right| > 0$. We conclude that $\{m_n\}$ are bounded. We can take a converging subsequence with limit m. We conclude that $\xi(u) = mu$ and thus the characteristic function is given by

$$\Phi_{X_{t_1}-X_{t_0}}(u) = e^{imu - cu^2\frac{1}{2}}.$$

Hence $X_{t_1} - X_{t_0}$ is Gaussian distributed. □

References

[1] D. Applebaum. *Lèvy processes and stochastic calculus. Cambridge: Cambridge University Press*. 2009.

[2] H. Bauer. *Measure and Integration theory*. De Gruyter Studies in Mathematics 26. 2001.

[3] A. Basse. O. Connor, J. Rosiǹsky. *On the uniform convergence of random series in Skohorod space and representations of càdlàg infinitely divisible processes*. The Annals of Probability. Vol.41. No. 6..4317-4241, 2013.

[4] J.L. Bretagnolle, P. Ouwehand. *The Levy-Ito Decomposition theorem*. arXiv:1506.06624 [math.PR]. 2015.

[5] E. Dettweiler. *Banach valued processes with independent increments and stochastic integration*. Probability in Banach spaces IV. Proceedings. Oberwolfach. Springer, 1982.

[6] J.L. Doob. *stochastic processes depending on continuous time parameter*. Trans.Amer. Math. Soc., **42**, 107-140 1937.

[7] R.M. Dudley. *Real analysis and Probability*. Cambridge: Cambridge University press, 2004.

[8] O. van Gaans. *Probability measures on metric spaces*, 2006.

[9] I.I. Gihman, A.V. Skohorod. *The Theory of Stochastic Processes II*. Springer, 1975.

[10] K. Ito. *On stochastic processes* (infinitely divisible laws of probability) (Doctoral thesis). Japan. Jouro. Math. XVIII. 261-301, 1942.

[11] K. Ito. *Sochastic processes : Lectures given at Aarhus Univeristy*. Springer, 2004.

[12] K. Ito and M. Nisio. *On the convergence of independent sums in Banach space*. Osaka Journal of Mathematics. 5(1) P.35-P.48, 1968.

[13] O. Kallenberg. Foundations of Modern probability. Probability and its Applications. Springer, 1997.

[14] P. Lévy, *Theories de L'addition Aleatoires*. Paris: Gauthier-Villars, 1937.

[15] M. Loève. *Probability Theory I*. Springer , 1977.

[16] J.M.A.M van Neerven. *I Integration in Banach spaces.*

[17] J.M.A.M van Neerven. *II Random variables in Banach spaces.*

[18] G. Da Prato, J. Zabczyck. *Stochastic Equations in Infinite Dimensions. Cambridge: Cambridge University press*, 1992.

[19] Z. Sasvari, *Multivariate characteristic and correlation functions.* De Gruyter Studies in Mathematics 50, 2013.

[20] K. Sato. *Levy processes and Infinitely Divisible Distributions.* Cambridge: Cambridge University Press, 1999.

[21] A.V. Skohorod. *Random processes with independent increments.* Kluwer Academic Publishers, 1991

[22] H. Thorisson. *Coupling, stationarity and regeneration.* Probability and its applications. Springer, 2000.