ALGANT MASTER THESIS

FINITE DIMENSIONAL MOTIVES

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Preface

The idea of motives, envisaged by *Alexander Grothendieck*, goes back to the Weil's conjectures about Zeta functions on varieties over finite fields and to the notion of *Weil cohomology theories*. The research of a cohomology theory in characteristic p, led to the construction of many Weil cohomology theories by Grothendieck himself which were linked by various comparison isomorphisms, suggesting the existence of a *universal cohomology theory*.

To provide such a universal cohomology, Grothendieck built different categories of motives starting from the category of smooth projective varieties and enlarging the set of morphisms between two varieties by considering *algebraic cycles* on their product, modulo some adequate equivalence relation.

An important feature among the numerous properties that Weil cohomology theories share is that the vector spaces in which they take values are finite dimensional. This suggests that a similar property should be true for motives and it gives rise to the task of finding a suitable definition of finite dimensionality in the context of motives.

Shun-Ichi Kimura and Peter O'Sullivan, in [Kim04; Sul05] addressed this question, giving a definition of finite dimensionality for motives. Recall that a characterization for the dimension of a finite dimensional vector space V is given by the fact that, if d is the dimension of V, the exterior power $\Lambda^{d+1}(V)$ vanishes. The definition of finite dimensionality for motives is based on this principle, involving however both symmetric and exterior powers.

Outline

In this thesis we will illustrate the aforementioned concept of finite dimensionality and we will collect the most important results concerning it.

In Chapter 1 we will develop the categorical formalisms needed in order to deal with the concept of Kimura-finiteness in an abstract context. In particular, in Section 1.1 we will recall the definition and properties of rigid tensor categories and we will introduce useful tools such as the trace of a morphism and the rank of an object, while in Section 1.5 we will present the concepts of even and odd objects which are crucial to define Kimura-finiteness. Moreover, we will collect the most important results about the behaviour of Kimura-finiteness under various operations.

In Chapter 2 we will illustrate Grothendieck's construction of pure motives and a few crucial examples will be given. In particular, Section 2.2 will be devoted to the properties of the motive of an abelian variety and in Section 2.6 we will show the decompositions of the motive of a projective bundle and of a blow-up. Furthermore, in Section 2.5 we will show how we can recover the classical theory of adequate equivalence relations through the language of tensor categories and tensor ideals and we will use it to present Janssen's fundamental theorem about semi-simplicity of numerical motives.

To conclude, in Chapter 3 we will show how to apply the results of Section 1.5 to the framework of motives. In section 3.2 we will state the Kimura-O'Sullivan conjecture, we will show that motives of abelian varieties are finite dimensional and we will show that Kimura-finiteness is a birational invariant under certain conditions. We will conclude the chapter with Section 3.3, with some results about cohomology of finite dimensional motives.

Notation and conventions

All categories we consider are *locally small*, *i.e.* the class of morphisms between every two objects is a *set*. If $\mathcal{C}$ is a category, X and Y are objects in $\mathcal{C}$ we denote the set of morphisms from X to Y either as $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ or as $\mathcal{C}(X, Y)$. Functors between additive categories are assumed to be additive.

For the categories Mod_R, Vec_K and $\mathsf{Rep}_K(G)$ of modules over a ring, vector spaces over a field or representations of affine group schemes, we denote as mod_R, vec_K and $\mathsf{rep}_K(G)$ their full subcategories of finite-dimensional objects.

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Chapter 1

Finite dimensionality in tensor categories

In this chapter we develop in an abstract setting the tools needed in the subsequent chapters, in which we will apply them to the study of motives. In particular, the results in Section 1.5, which is devoted to the study of finite dimensionality in abstract rigid pseudo-abelian tensor categories, will have important consequences in Chapter 3.

1.1 Tensor categories

We present definitions and results concerning symmetric monoidal categories and rigid tensor categories that will be useful later. For more details we refer to [Mac78, Chapter VII] and [Del90].

1.1.1. – Recall that a *symmetric monoidal category* or *tensor category* is a 5-tuple

$$(\mathcal{C}, \otimes, \varphi, \psi, (\mathbb{1}, e))$$

where $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is a bifunctor, subject to the following constraints:

(1) An *associativity constraint*, which is a natural isomorphism,

$$\varphi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

satisfying the so called *pentagon axiom* [Mac78, p. 162].

(2) A *commutativity constraint*, which is a natural isomorphism,

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X$$

such that $\psi_{Y,X} \circ \psi_{X,Y}$ is the identity on $X \otimes Y$, for every $X, Y \in \mathcal{C}$ and satisfying together with φ a compatibility axiom, called *hexagon axiom* [Mac78, p. 184].

(3) An *identity object* $\mathbb{1} \in \mathcal{C}$ for which the rule $X \mapsto \mathbb{1} \otimes X$ defines an autoequivalence of $\mathcal{C}$, which comes equipped with an isomorphism $e : \mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$.

In the following, a tensor category will be simply denoted as $(\mathcal{C}, \otimes, \mathbb{1})$ or even just $\mathcal{C}$ if no ambiguity will arise.

Remark 1.1.2. Let $(\mathcal{C}, \otimes, 1)$ be a symmetric monoidal category.

(1) It is possible to build up natural isomorphisms $l_X : X \to \mathbb{1} \otimes X$ and $r_X : X \to X \otimes \mathbb{1}$, in a compatible way with the constraints and in such a way $l_{\mathbb{1}}$ and $r_{\mathbb{1}}$ coincide with the given isomorphism $\mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$.

(2) If $\mathbb{1}$ and $\mathbb{1}'$ are identity objects equipped with isomorphisms $e : \mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$ and $e' : \mathbb{1}' \to \mathbb{1}' \otimes \mathbb{1}'$ respectively, there exists a unique isomorphism $a : \mathbb{1} \to \mathbb{1}'$ such that $(a \otimes a) \circ e = e' \circ a$.

(3) The compatibility axioms in a tensor category allow the tensor product of any finite family of objects to be well defined up to isomorphism.

Example 1.1.3. We present some examples of tensor categories.

(1) Every category $\mathcal{C}$ with finite products and a terminal object $*$ gives rise to a symmetric monoidal category $(\mathcal{C}, \prod, *)$.

(2) If $(\mathcal{C}, \otimes)$ is a symmetric monoidal category, then for any small category $\mathcal{I}$ the functor category $\mathrm{Fun}(\mathcal{I}, \mathcal{C})$ inherits a monoidal structure which is symmetric. The same is true for the opposite category $\mathcal{C}^{\mathrm{op}}$.

(3) If R is a commutative ring, the category Mod_R of modules over R has a tensor structure $(\mathsf{Mod}_R, \otimes_R, R)$ given by the usual tensor product and the obvious constraints. Every free R-module of rank 1 is an identity object with respect to the tensor structure.

Let $(\mathcal{C}, \otimes, \mathbb{1})$ and $(\mathcal{C}', \otimes', \mathbb{1}')$ be tensor categories. Recall that a *tensor functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ equipped with natural isomorphisms

$$F(X) \otimes' F(Y) \simeq F(X \otimes Y)$$

and an isomorphism

$$\mathbb{1}' \simeq F(\mathbb{1})$$

which are compatible with the commutativity and associativity constraints.

Morphisms of tensor functors are natural transformations compatible with the tensor structure. If $F : \mathcal{C} \longrightarrow \mathcal{C}'$ and $G : \mathcal{C} \longrightarrow \mathcal{C}'$ are tensor functors, we will denote the set of morphisms from F to G as $\mathrm{Hom}^{\otimes}(F, G)$.

A *tensor equivalence* of tensor categories is a tensor functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ which is an equivalence of the underlying categories. Indeed, it is possible to show that, if F is a tensor equivalence, there exists a tensor functor $G : \mathcal{C}' \longrightarrow \mathcal{C}$ such that $F \circ G$ and $G \circ F$ are tensor isomorphic to the identity functors.

1.1.4. Rigidity – We introduce the definition of rigid tensor category, which is necessary in order to define the trace of a morphism and the rank of an object.

Definition 1.1.5. A *rigid tensor category* is a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$ such that:

(1) There exists an auto-duality
$$(-)^\vee : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}.$$

(2) For every object $X \in \mathcal{C}$, the functor $- \otimes X^\vee$ is left adjoint to $- \otimes X$ and the functor $X^\vee \otimes -$ is right adjoint to $X \otimes -$.

Notation. If $\mathcal{C}$ is a rigid tensor category, given a morphism $f : X \to Y$ in $\mathcal{C}$ we will use the notation ${}^{\mathrm{t}}f$ in place of $f^\vee$.

Definition 1.1.6. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a tensor category. If $\mathcal{C}'$ is a strictly full subcategory (i.e.: full and stable under isomorphisms) of $\mathcal{C}$, it is said to be a *tensor subcategory* if it is stable under tensor products.

A tensor subcategory $\mathcal{C}'$ of a rigid tensor category $\mathcal{C}$ is said to be a *rigid tensor subcategory* if it is also stable under taking duals.

Remark 1.1.7. If $\mathcal{C}$ is a rigid tensor category, point 2. of Definition 1.1.5 implies the existence of unit and counit morphisms:
$$\begin{aligned} \mathrm{ev}_X &: X \otimes X^\vee \to \mathbb{1} \\ \mathrm{coev}_X &: \mathbb{1} \to X^\vee \otimes X \end{aligned} \tag{1.1}$$
which are called *evaluation* and *coevaluation* morphisms. They satisfy the equalities:
$$\begin{aligned} \mathrm{id}_X &= (\mathrm{ev}_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_X \otimes \mathrm{coev}_X), \\ \mathrm{id}_{X^\vee} &= (\mathrm{id}_{X^\vee} \otimes \mathrm{ev}_X) \circ (\mathrm{coev}_X \otimes \mathrm{id}_{X^\vee}). \end{aligned} \tag{1.2}$$
For any couple (X, Y) of objects in $\mathcal{C}$, by adjunction we have a natural isomorphism:
$$\iota_{X,Y} : \mathcal{C}(\mathbb{1}, X^\vee \otimes Y) \to \mathcal{C}(X, Y) \tag{1.3}$$
which sends $u : \mathbb{1} \to X^\vee \otimes Y$ to the morphism
$$X \xrightarrow{\mathrm{id}_X \otimes u} X \otimes X^\vee \otimes Y \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_Y} Y$$
Moreover, one can write the composition of two morphisms $f : X \to Y$ and $g : Y \to Z$ in terms of tensor products as follows:
$$g \circ f = \iota_{X,Z}\Big[(\mathrm{id}_{X^\vee} \otimes \mathrm{ev}_{Y^\vee} \otimes \mathrm{id}_Z) \circ \big(\iota^{-1}_{X,Y}(f) \otimes \iota^{-1}_{Y,Z}(g)\big)\Big] \tag{1.4}$$
Example 1.1.8. Let $\mathcal{C} = \mathsf{vec}_K$ be the category of finite dimensional vector spaces over a field K and let us consider morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{C}$. Let $\{x_i\}_i$ and $\{y_j\}_j$ be bases for X and Y respectively and let $\{\varepsilon^i\}_i$ and $\{\varphi^j\}_j$ be their dual bases. The morphism $\iota^{-1}_{X,Y}(f) : K \to X^\vee \otimes Y$ sends 1 to $\sum_i \varepsilon^i \otimes f(x_i)$ and, similarly, $\iota^{-1}_{Y,Z}(g) : K \to Y^\vee \otimes Z$ sends 1

to $\sum_j \varphi^j \otimes g(y_j)$ and it can be proved that they do not depend on the choice of the bases. We have:

$$(\mathrm{id}_{X^\vee} \otimes \mathrm{ev}_{Y^\vee} \otimes \mathrm{id}_Z)\big(\iota^{-1}_{X,Y}(f) \otimes \iota^{-1}_{Y,Z}(g)\big)(1) = (\mathrm{id}_{X^\vee} \otimes \mathrm{ev}_{Y^\vee} \otimes \mathrm{id}_Z)\left(\sum_{i,j} \varepsilon^i \otimes f(x_i) \otimes \varphi^j \otimes g(y_j)\right) =$$
$$= \sum_{i,j} \varphi^j\big(f(x_i)\big)\big(\varepsilon^i \otimes g(y_j)\big).$$

Hence, for finite dimensional vector spaces, formula (1.4) gives us, for any i, the relation:

$$(g \circ f)(x_i) = \sum_{i,j} \varphi^j\big(f(x_i)\big) \otimes \varepsilon^i(x_i) \otimes g(y_j) = \sum_j \varphi^j\big(f(x_i)\big)g(y_j).$$

1.1.9. Trace and rank – Let now K be a field, recall that functors between K-linear categories are required to be K-linear.

Remark 1.1.10. If $(\mathcal{C}, \otimes, \mathbb{1})$ is a tensor category, the set of endomorphisms of the identity object $R := \mathrm{End}(\mathbb{1})$, is indeed a commutative unitary ring and it induces an R-linear structure on $\mathcal{C}$.

Definition 1.1.11. A tensor category $(\mathcal{C}, \otimes, \mathbb{1})$ is said to be a *K-linear tensor category* if $\mathrm{End}(\mathbb{1})$ is isomorphic to K and $\otimes$ is a K-bilinear functor.

Definition 1.1.12. Let $(\mathcal{C}, \otimes, \varphi, \psi, \mathbb{1})$ be a rigid K-linear tensor category.

(1) If $f : X \to X$ is an endomorphism in $\mathcal{C}$, we define the *trace* of f as the element in $\mathrm{End}(\mathbb{1}) \simeq K$ given by the composition:

$$\mathrm{Tr}_X(f) : \mathbb{1} \xrightarrow{\mathrm{coev}_X} X^\vee \otimes X \xrightarrow{\mathrm{id}_{X^\vee} \otimes f} X^\vee \otimes X \xrightarrow{\psi_{X^\vee,X}} X \otimes X^\vee \xrightarrow{\mathrm{ev}_X} \mathbb{1}$$

where $\psi_{X^\vee,X} : X^\vee \otimes X \to X \otimes X^\vee$ is the commutativity constraint as defined in 1.1.1.

(2) In particular, if X is an object in $\mathcal{C}$, we define the *rank* or *dimension* of X as the trace of the identity and we put:

$$\mathrm{rank}(X) := \mathrm{Tr}_X(\mathrm{id}_X). \tag{1.5}$$

Lemma 1.1.13. *Let $F : \mathcal{C} \to \mathcal{C}'$ be a tensor functor between rigid K-linear tensor categories, let X be an object of $\mathcal{C}$ and $f : X \to X$ be an endomorphism of X. Then the following formulas hold:*

$$\mathrm{Tr}_{F(X)}(F(f)) = \mathrm{Tr}_X(f),$$
$$\mathrm{rank}(X) = \mathrm{rank}(F(X)),$$
$$F(X)^\vee = F(X^\vee).$$

1.2 Schur functors

Let K be a field of characteristic 0 and let $\mathcal{C}$ be a *pseudo-abelian* K-linear tensor category. Recall that, for $\mathcal{C}$ to be pseudo-abelian it means that every idempotent endomorphism in $\mathcal{C}$ (usually called *projector*) has a kernel and an image. For every object $X \in \mathcal{C}$ and every $n \geq 1$, the symmetric group $\mathfrak{S}_n$ acts canonically on $X^{\otimes n}$, which implies the existence of a morphism of $\mathbb{Q}$-algebras $\mathbb{Q}[\mathfrak{S}_n] \to \mathrm{End}(X^{\otimes n})$.

Recall that there exists a bijection between the classes of isomorphisms V_λ of irreducible $\mathbb{Q}$-representations of $\mathfrak{S}_n$ and the partitions λ of n.

We introduce some notation:

Notation. Let $\lambda := (\lambda_1, \ldots, \lambda_k)$, with $0 \leq \lambda_1 \leq \ldots \leq \lambda_k$, be a partition of n, we put $|\lambda| = n$. We denote the *diagram of lambda* as $[\lambda]$ which is by definition the set of couples (i, j) of integers $i, j \geq 1$ such that $j \leq \lambda_i$.

The following isomorphism holds

$$\mathbb{Q}[\mathfrak{S}_n] = \prod_{\lambda,\, |\lambda|=n} \mathrm{End}_{\mathbb{Q}} V_\lambda \tag{1.6}$$

and we denote by c_λ the unique idempotent element of $\mathbb{Q}[\mathfrak{S}_n]$ corresponding to the element which is the identity on V_λ and 0 elsewhere.

Definition 1.2.1. For any partition λ of n, we still denote by c_λ the corresponding endomorphism of $X^{\otimes n}$, then we define the λ-*Schur functor* on the objects of $\mathcal{C}$ as:

$$\begin{aligned} \mathrm{S}_\lambda : \mathcal{C} &\longrightarrow \mathcal{C} \\ X &\longmapsto c_\lambda(X^{\otimes n}) \end{aligned} \tag{1.7}$$

and analogously on morphisms.

Example 1.2.2. (1) If $\lambda = (n)$ then V_λ is isomorphic to K with the trivial action and in this case we put:

$$\mathsf{S}^n(X) = \mathrm{S}_\lambda(X), \tag{1.8}$$

for the n^{th}-*symmetric power* of X.

Explicitly, $\mathsf{S}^n(X)$ is defined as the image of the projector:

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \sigma : X^{\otimes n} \to X^{\otimes n}.$$

(2) Analogously, if we choose λ to be ${}^{\mathrm{t}}(n) = (1, \ldots, 1)$, the representation V_λ is isomorphic to K with the sign action and we put

$$\Lambda^n(X) = \mathrm{S}_\lambda(X), \tag{1.9}$$

for the n^{th}-*exterior power* of X, which is defined as the image of the following projector:

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma)\sigma : X^{\otimes n} \to X^{\otimes n}.$$

Proposition 1.2.3. *Let $\mathcal{C}$ be rigid, let X be an object of $\mathcal{C}$ with* $\mathrm{rank}(X) = d$. *For any $n \in \mathbb{N}$ we have:*

$$\mathrm{rank}\left(\Lambda^n(X)\right) = \binom{d}{n} := \frac{d(d-1)\ldots(d-n+1)}{n!} \tag{1.10}$$

and, analogously:

$$\operatorname{rank}\big(\mathsf{S}^n(X)\big) = \binom{d+n-1}{n} := \frac{d(d+1)\dots(d+n-1)}{n!}. \tag{1.11}$$

Notice that, in particular, for negative values of d the rank of $\Lambda^n(X)$ is non-zero. Analogously, if $d > 0$ then the rank of $S^n(X)$ is non-zero for all n.

Proof. See [AK02, Proposition 7.2.4.] □

Definition 1.2.4. (1) An object $X \in \mathcal{C}$ is said to be *Schur finite*, if there exists $n \in \mathbb{N}$ and a partition λ of n such that

$$\mathrm{S}_\lambda(X) = 0.$$

(2) The category $\mathcal{C}$ is said to be a *Schur-finite category* if every object $X \in \mathcal{C}$ is Schur-finite.

Notation. Let μ and ν be partitions of p and q and λ be a partition of $n = p+q$. In the sequel we will denote as $[V_\lambda : V_\mu \otimes V_\nu]$ the Littlewood-Richardson coefficient, *i.e.* the multiplicity of the irreducible representation $V_\mu \otimes V_\nu$ of $\mathfrak{S}_p \times \mathfrak{S}_q$ in the restriction of V_λ from $\mathfrak{S}_n$ to $\mathfrak{S}_p \times \mathfrak{S}_q$.

Similarly, if λ, μ and ν are partitions of n, we denote by $[V_\mu \otimes V_\nu : V_\lambda]$ the multiplicity of the irreducible representation V_λ into $V_\mu \otimes V_\nu$, *i.e.* the coefficient a_λ in the decomposition of representations $V_\mu \otimes V_\nu = \sum_\lambda a_\lambda V_\lambda$.

Lemma 1.2.5. *Let X and Y be objects in $\mathcal{C}$. The following formulas hold:*

(1) Let μ and ν be partitions of p and q:

$$\mathrm{S}_\mu(X) \otimes \mathrm{S}_\nu(X) \simeq \bigoplus_{|\lambda|=|\mu|+|\nu|} \mathrm{S}_\lambda(X)^{[V_\lambda : V_\mu \otimes V_\nu]}.$$

(2) If $[\mu] \subset [\lambda]$,

$$\mathrm{S}_\mu(X) = 0 \ \Rightarrow \mathrm{S}_\lambda(X) = 0.$$

(3) Let λ be a partition of n:

$$\mathrm{S}_\lambda(X \oplus Y) \simeq \bigoplus_{|\mu|+|\nu|=n} \big(\mathrm{S}_\mu(X) \oplus \mathrm{S}_\nu(Y)\big)^{[V_\lambda : V_\mu \otimes V_\nu]}.$$

(4) Let λ be a partition of n:

$$\mathrm{S}_\lambda(X \otimes Y) \simeq \bigoplus_{|\mu|,|\nu|=n} \big(\mathrm{S}_\mu(X) \otimes \mathrm{S}_\nu(Y)\big)^{[V_\mu \otimes V_\nu : V_\lambda]}.$$

(5) Let $\mathcal{C}$ be also rigid, then:

$$\mathrm{S}_\lambda(X)^\vee \simeq \mathrm{S}_\lambda(X^\vee).$$

Proof. See [Del02, Section 1].

□

As an immediate corollary of the previous lemma we get:

Corollary 1.2.6. *Let $\mathcal{C}$ be rigid. The full subcategory of Schur-finite objects of $\mathcal{C}$ is a rigid K-linear tensor subcategory,* i.e. *it is closed under direct sums, duals and tensor products.*

1.3 Tannakian categories

We briefly recall the definitions and the most important results about the theory of Tannakian categories, which historically provided the formalism needed in order to deal with a Galois theory for motives. We refer to [Saa72] and [Del90] for a more detailed exposition.

Fix a field K of characteristic 0 and let $\mathcal{A}$ be an abelian, K-linear, rigid, tensor category.

Definition 1.3.1. (1) An L-valued *fibre functor* is an exact faithful tensor functor:

$$\omega : \mathcal{A} \longrightarrow \mathsf{vec}_L,$$

where $K \subset L$ is a field extension.

(1) The category $\mathcal{A}$ is said to be *Tannakian* if there exists a fibre functor:

$$\omega : \mathcal{A} \longrightarrow \mathsf{vec}_L.$$

(2) If $\mathcal{A}$ is a Tannakian category, it is said to be *neutralized* if it comes equipped with a K-valued fibre functor:

$$\omega : \mathcal{A} \longrightarrow \mathsf{vec}_K.$$

If G is an affine group scheme over K, the forgetful functor $\mathsf{rep}_K(G) \longrightarrow \mathsf{vec}_K$ presents $\mathsf{rep}_K(G)$ as a neutralized Tannakian category. In fact also the converse holds, namely given a neutralized Tannakian category $(\mathcal{A}, \omega)$ we can associate to it an affine K-group scheme $G_{\mathcal{A},\omega}$ defined as follows. If R is a commutative K-algebra, the R-points of $G_{\mathcal{A},\omega}$ are given by:

$$G_{\mathcal{A},\omega}(R) := \mathrm{Aut}^{\otimes}(\varphi_R \circ \omega)$$

where $\varphi_R : \mathsf{vec}_K \to \mathsf{mod}_R$ is the functor "extension of scalars". Then we can recover $\mathcal{A}$ entirely from the associated affine K-group scheme as follows.

Theorem 1.3.2 (N. Saavedra [Saa72])**.** *Let $(\mathcal{A}, \omega)$ be a neutralized Tannakian category. The fibre functor ω lifts to an equivalence of tensor categories:*

$$\begin{array}{ccc} \mathcal{A} & \overset{\overline{\omega}}{\dashrightarrow} & \mathsf{rep}_K(G_{\mathcal{A},\omega}) \\ & \underset{\omega}{\searrow} & \downarrow \scriptstyle{\textit{Forgetful}} \\ & & \mathsf{vec}_K \end{array}$$

However, in [Del90], Deligne gave an internal characterization of Tannakian categories, namely a criterion for a category to be Tannakian that does not involve a fibre functor. This is indeed an important tool for proving that a category is Tannakian without exhibiting a specific fibre functor and we recall it in the following theorem.

Theorem 1.3.3 (P. Deligne [Del90])**.** *Let $\mathcal{A}$ be an abelian K-linear rigid tensor category. The following conditions are equivalent.*

- *The category $\mathcal{A}$ is Tannakian.*
- *For each object $M \in \mathcal{A}$ there exists a positive integer n such that*

$$\Lambda^n(M) = 0.$$

- *For each object $M \in \mathcal{A}$ the rank of M is a non-negative integer.*

1.4 Tensor ideals

1.4.1. – Let R be a fixed ring. An R-algebra can be seen as an R-linear category with just one object. Taking the opposite point of view, one can see R-linear categories as "R-algebras with more than one object". This point of view allows us to borrow from non-commutative algebra many useful concepts. We refer to [AK02] for a detailed exposition of the topic.

Definition 1.4.2. Let $\mathcal{C}$ be an R-linear category. An *ideal* $\mathcal{I}$ of R is the data, for each couple of objects (M, N), of an R-submodule $\mathcal{I}(M, N)$ of $\mathcal{C}(M, N)$ such that, for every couple of morphisms $\big(f \in \mathcal{C}(A', A), g \in \mathcal{C}(B, B')\big)$:

$$g \circ \mathcal{I}(A, B) \circ f \subset \mathcal{I}(A', B').$$

If $\mathcal{I}$ is an ideal of $\mathcal{C}$, we will denote the quotient category by $\mathcal{C}/\mathcal{I}$. It has the same objects of $\mathcal{C}$ and for any couple (X, Y) of objects in $\mathcal{C}/\mathcal{I}$,

$$\mathcal{C}/\mathcal{I}(X, Y) := \mathcal{C}(X, Y)/\mathcal{I}(X, Y).$$

As in the classical case, it is possible to define in a natural way *sums* and *intersections* of ideals of a given category $\mathcal{C}$. Furthermore, if $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathcal{C}$ we define their product $\mathcal{I} \cdot \mathcal{J}$ by taking as $(\mathcal{I} \cdot \mathcal{J})(X, Y)$ the set of finite sums of compositions $\sum_i (g_i \circ f_i)$ with $f_i \in \mathcal{I}(X, Z_i)$ and $g_i \in \mathcal{J}(Z_i, Y)$ for some objects $Z_i \in \mathcal{C}$.

Example 1.4.3. (1) For S a set of objects in $\mathcal{C}$ one can form the ideal of the maps in $\mathcal{C}$ that factor through an object of S, denoted as $\mathcal{I}_S$.

(2) If $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is an R-linear functor one can consider the ideal $\operatorname{Ker} F$ of the morphisms f in $\mathcal{C}$ with $F(f) = 0$. Then F induces an equivalence of categories between $\mathcal{C}/\operatorname{Ker} F$ and a subcategory of $\mathcal{C}'$.

(3) Let us define, for any $M, N \in \mathcal{C}$,

$$\mathcal{R}(M, N) := \big\{ f \in \mathcal{C}(M, N) \mid \mathrm{id}_M - g \circ f \text{ is invertible, for any } g \in \mathcal{C}(N, M) \big\}$$

then $\mathcal{R}$ is an ideal, called the *Kelly-radical* of $\mathcal{C}$.

Definition 1.4.4. Let now $(\mathcal{C}, \otimes, \mathbb{1})$ be an R-linear tensor category. An ideal $\mathcal{I} \subset \mathcal{C}$ is said to be a *tensor ideal* or *monoidal ideal* if, for every couple (M, N) of objects in $\mathcal{C}$, given $f \in \mathcal{I}(M, N)$ and for every morphism $f' : M' \to N'$ in $\mathcal{C}$, one has $f \otimes f' \in \mathcal{I}(M \otimes M', N \otimes N')$ and $f' \otimes f \in \mathcal{I}(M' \otimes M, N' \otimes N)$.

Remark 1.4.5. (1) It is possible to show that, for an ideal $\mathcal{I}$ of a tensor category $\mathcal{C}$ to be a tensor ideal, it suffices to be stable under $- \otimes \mathrm{id}_X$, for every object $X \in \mathcal{C}$.

(2) If $\mathcal{I}$ is a tensor ideal of a tensor category $\mathcal{C}$, the quotient category $\mathcal{C}/\mathcal{I}$ naturally inherits a tensor structure.

(3) The notion of tensor ideal is stable under sums, intersections and products.

Example 1.4.6. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an R-linear tensor category:

(1) It admits a unique *maximal monoidal ideal* denoted by $\mathcal{N}$, which is defined, for any $X, Y \in \mathcal{C}$, as:

$$\mathcal{N}(X,Y) = \{f \in \mathcal{C}(X,Y) \mid \operatorname{Tr}_X(gf) = 0, \text{ for every } g \in \mathcal{C}(X,Y)\}. \tag{1.12}$$

One can prove that $\mathcal{N}$ is also the smallest tensor ideal containing $\mathcal{R}$.

(2) The morphisms for which a tensor power is zero form an ideal, the $\otimes$-*nilradical* of $\mathcal{C}$ denoted as $\sqrt[\otimes]{0}$. The tensor functor $\mathcal{C} \to \mathcal{C}/\sqrt[\otimes]{0}$ is conservative, i.e. it reflects the isomorphisms.

(3) If $F : (\mathcal{C}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}', \otimes', \mathbb{1}')$ is a tensor functor, its kernel $\operatorname{Ker} F$ is a tensor ideal.

1.4.7. Jannsen's semi-simplicity theorem – We conclude this section with a purely categorical version of "Jannsen's semi-simplicity theorem" that will be applied in Section 2.5 in the framework of motives, in order to prove it in its original fashion.

Let $\mathcal{C}$ be an R-linear category, recall that a (left) $\mathcal{C}$-module is an R-linear functor $M : \mathcal{C} \longrightarrow R\text{-}\mathsf{Mod}$. The $\mathcal{C}$-modules, with natural transformations between them, form an R-linear abelian category, denoted as $\mathcal{C}\text{-}\mathsf{Mod}$.

Definition 1.4.8. Let $\mathcal{C}$ be an R-linear category and let $\mathcal{A}$ be an abelian, R-linear category.

(1) An object $X \in \mathcal{A}$ is said to be *simple* if it is non-zero and it has no non-trivial subobjects.

(2) An object $X \in \mathcal{A}$ is said to be *semi-simple* if there exist $X_1, \dots, X_n \in \mathcal{A}$ simple objects such that

$$X = \bigoplus_{i=1}^{n} X_i.$$

(3) The abelian category $\mathcal{A}$ is said *semi-simple* if every object $X \in \mathcal{A}$ is semi-simple.

(4) We call $\mathcal{C}$ a *semi-simple* category if the abelian category $\mathcal{C}\text{-}\mathsf{Mod}$ is semi-simple.

Lemma 1.4.9 ([AK02, p. 2.1.2.])**.** *Let K be a field, $\mathcal{C}$ be a K-linear tensor category. The following are equivalent:*

(1) $\mathcal{C}$ is semi-simple.

(2) For every $X \in \mathcal{C}$, the K-algebra $\operatorname{Hom}_{\mathcal{C}}(X,X)$ *is semi-simple.*

Moreover, under these conditions, $\mathcal{C}$ is pseudo-abelian if and only if it is abelian.

Proof. See [AK02, A.2.10] for the proof and more equivalent conditions. □

Recall that, for a K-linear tensor category $\mathcal{C}$, we denote by $\mathcal{N}$ the maximal tensor ideal of $\mathcal{C}$ as defined in Example 1.4.6, (1). The following result holds.

Proposition 1.4.10. *Let $\mathcal{C}$ be a K-linear rigid tensor category with* $\operatorname{End}(\mathbb{1})$ *isomorphic to K. Assume there exists a K-linear tensor functor $H : \mathcal{C} \longrightarrow \mathsf{sVec}_L$, where $K \subset L$ is a field extension. Then the pseudo-abelian envelope of $\mathcal{C}/\mathcal{N}$ is an abelian semi-simple category in which the* Hom*-sets have finite dimension over K.*

Proof. Since the kernel of H is a monoidal ideal, it is contained in $\mathcal{N}$ and we can substitute $\mathcal{T}$ with $\mathcal{T}/\operatorname{Ker} H$, so that H is a faithful functor.

Let us first prove the case $L = K$. Since for every $X, Y \in \mathcal{C}$, $\operatorname{Hom}_{\mathsf{sVec}_K}(H(X), H(Y))$ is finitely dimensional, the same is true for $\mathcal{C}(X, Y)$. In particular, the radical of the K-algebra $\mathcal{C}(X, X)$ is nilpotent and its quotient by the radical is semi-simple, so it is enough to prove that every nilpotent ideal of $\mathcal{C}(X, X)$ is contained in $\mathcal{N}(X, X)$. Now, let I be a nilpotent ideal of $\mathcal{N}(X, X)$ for any $g \in \mathcal{C}(X, X)$ the composition $g \circ f$ is nilpotent, so $H(g \circ f)$ is nilpotent as well and we have

$$H(\operatorname{Tr}_X(g \circ f)) = \operatorname{Tr}_{H(X)}(H(g \circ f)) = 0.$$

This implies that $\operatorname{Tr}(g \circ f) = 0$ by faithfulness of H, so that $f \in \mathcal{N}(X, X)$. The claim follows by the previous lemma.

The general case can be reduced to the previous one as shown in [And04a, Proposition 2.6.] □

1.5 Finite dimensionality

Let K be a field of characteristic zero and let us fix a rigid, pseudo-abelian, K-linear tensor category $(\mathcal{T}, \otimes, \mathbb{1})$ with $\operatorname{End}(\mathbb{1}) = K$. Moreover, we suppose that there exists a non-zero tensor functor $H : \mathcal{T} \to \mathsf{sVec}_L$ for some field extension $K \subset L$.

This last assumption will make the proof of Corollary 1.5.11 easier, even though all the results we are presenting below are still proved to be true without making it. In any case, the condition is fulfilled in the motivic framework, where we take $\mathcal{T}$ to be the category of Chow motives, by choosing as H any Weil cohomology theory, as we will see in Chapter 3.

1.5.1. Even and odd objects – It is well known that finite dimensionality of a vector space V is characterized by V having n^{th}-exterior power vanishing, for some integer n. This is not anymore the case if we deal with finite dimensional *super vector spaces*, where "odd" parts come into the picture. In fact, the same idea fits perfectly in our abstract framework.

Definition 1.5.2. Let $\mathcal{T}$ be as above.

(1) An object $M \in \mathcal{T}$ is said to be *even of finite dimension*, or just *even*, if there exists an integer $n \in \mathbb{N}$ such that:
$$\mathsf{\Lambda}^n(M) = 0.$$

(2) An object $M \in \mathcal{T}$ is said to be *odd of finite dimension*, or just *odd*, if there exists an integer $n \in \mathbb{N}$ such that:
$$\mathsf{S}^n(M) = 0.$$

(3) Let M be an even object of $\mathcal{T}$. We define the *Kimura dimension* of M as:
$$\operatorname{kim}(M) := \min\{n \in \mathbb{N} \mid \mathsf{\Lambda}^{n+1}(M) = 0\}.$$

(4) Let M be an odd object of $\mathcal{T}$. We define the *Kimura dimension* of M as:
$$\operatorname{kim}(M) := \min\{n \in \mathbb{N} \mid \mathsf{S}^{n+1}(M) = 0\}.$$

Remark 1.5.3. It follows from point ((2)) of Lemma 1.2.5 that, for an even (resp. odd) object M of dimension d, we have $\mathsf{\Lambda}^n(M) = 0$ (resp. $\mathsf{S}^n(M) = 0$) for every $n \geq d$. There seems to be an ambiguity on which definition one has to choose if a given object is both even and odd. This will disappear as we will point out, in Corollary 1.5.16, that the only object which is both even and odd is the zero object.

Proposition 1.5.4 ([AK02, Prop. 9.1.4.]). *Let $\mathcal{T}$ be as above, the following properties hold.*

(1) A direct sum of odd objects in $\mathcal{T}$ is odd. Every direct summand of an odd object is odd. The same is true for even objects.

(2) Let $M, N \in \mathcal{T}$ be evenly or oddly finite dimensional objects. Then, $M \otimes N$ is even if the kind of finite dimensionality of M and N is the same , odd if otherwise. Moreover

$$\mathrm{kim}(M \otimes N) \leq \mathrm{kim}(M)\mathrm{kim}(N).$$

(3) The dual of an odd (resp. even) object is odd (resp. even).

(4) If $M \in \mathcal{T}$ is even, $\mathsf{\Lambda}^n(M)$ is even for every n.

(5) If $M \in \mathcal{T}$ is odd, $\mathsf{S}^n(M)$ is odd if n is odd, it is even if n is even.

Proof. (1) It results from point (3) of Lemma 1.2.5, which in our cases yields the following isomorphisms:

$$\mathsf{\Lambda}^n(M \otimes N) \simeq \bigoplus_{p+q=n} \mathsf{\Lambda}^p(M) \otimes \mathsf{\Lambda}^q(N) \tag{1.13}$$

$$\mathsf{S}^n(M \oplus B) \simeq \bigoplus_{p+q=n} \mathsf{S}^p(M) \otimes \mathsf{S}^q(N) \tag{1.14}$$

(2) Cf. [Kim04, Proposition. 5.10].

(3) It follows from point (5) of Lemma 1.2.5.

(4) If M is even, so is $M^{\otimes n}$ for every n by (2). Since $\mathsf{\Lambda}^n(M)$ is a direct factor of $M^{\otimes n}$ the assertion follows by (1).

(5) If M is odd, $M^{\otimes n}$ is either even or odd depending on the parity of n, by point (2) and we get the claim as in the previous item, by (1).

□

Lemma 1.5.5. *Let M be an object in $\mathcal{T}$. Then the ideal $\sqrt[\otimes]{0}(M, M)$ is a nilideal,* i.e. *for every $f \in \sqrt[\otimes]{0}(M, M)$ there exists a positive integer $n \in \mathbb{N}$ such that $f^n = 0$. Moreover, the ideal generated by any element of $\sqrt[\otimes]{0}(M, M)$ is nilpotent.*

Proof. Let $f \in \sqrt[\otimes]{0}(M, M)$ and choose n such that $f^{\otimes n} = 0$. By the commutativity constraint we know that for any $n + 1$-tuple $(g_1, \ldots, g_{n+1})$ of endomorphisms in $\mathcal{T}(M, M)$:

$$g_{n+1} \otimes f \otimes g_n \otimes \ldots \otimes f \otimes g_1 = 0.$$

By induction, using formula (1.4) we get that:

$$g_{n+1} \circ f \circ g_n \circ \ldots \circ f \circ g_1 = 0$$

which proves the last claim. The first assertion follows choosing $g_i = \mathrm{id}_M$. □

1.5.6. A nilpotence theorem – If M is any object of $\mathcal{T}$, given a morphism $g \in \mathcal{T}(M^{\otimes n}, M^{\otimes n})$ we can produce for any n a morphism $(g)_n \in \mathcal{T}(M, M)$ defined as the image of g through the following map

$$\mathcal{T}(M^{\otimes n}, M^{\otimes n}) \xrightarrow{\iota^{-1}_{M^{\otimes n}}} \mathcal{T}(\mathbb{1}, M^{\otimes n \vee} \otimes M^{\otimes n}) \xrightarrow{\varepsilon_{M^{\otimes n-1}}} \mathcal{T}(\mathbb{1}, M^{\vee} \otimes M) \xrightarrow{\iota_M} \mathcal{T}(M, M)$$

where, for any M, we denote by ι_M the natural isomorphism $\iota_{M,M}$ defined in formula (1.3) and $\varepsilon_{M^{\otimes n-1}}$ stands for the morphism $\mathrm{id}_M^{\vee} \otimes \mathrm{ev}_{M^{\otimes n-1 \vee}} \otimes \mathrm{id}_M : M^{\otimes n \vee} \otimes M^{\otimes n} \to M^{\vee} \otimes M$ which is the evaluation on the central $2n-2$ factors and the identity on the extreme terms.

Example 1.5.7. We give a baby example to show how this construction works. Let $\mathcal{C}$ be the category vec_K of finite dimensional vector spaces over some field K, we put $n := 2$ and we consider $g := f^{\otimes 2} : V^{\otimes 2} \to V^{\otimes 2}$ for an endomorphism $f : V \to V$, where V is a finite dimensional vector space. If we fix a basis $\{e_i\}_i$ of V and we consider $\{\varepsilon^i\}_i$ its dual basis, then $(f^{\otimes 2})_2 : V \to V$ is equal to

$$\mathrm{Tr}(f) \cdot f : V \to V.$$

Indeed, $\iota^{-1}_{V^{\otimes 2}}(f) : K \to V^{\otimes 2 \vee} \otimes V^{\otimes 2}$ is the morphism sending 1 to the linear combination $\sum_{i,j} \varepsilon^j \otimes \varepsilon^i \otimes f(e_i) \otimes f(e_j)$ which, after composing with the evaluation on the central factors, yields the linear map

$$\begin{aligned} K &\to V^{\vee} \otimes V \\ 1 &\mapsto \sum_{i,j} \varepsilon^i(f(e_i)) \cdot \varepsilon^j \otimes f(e_j). \end{aligned}$$

Thus, the linear map $(f^{\otimes 2})_2 : V \to V$ sends an element of the basis e_k to

$$\sum_{i,j} \varepsilon^i(f(e_i)) \cdot \varepsilon^j(e_k) \cdot f(e_j) = \sum_i \varepsilon^i(f(e_i)) \cdot f(e_k) = \mathrm{Tr}(f) \cdot f(e_k).$$

This can be generalized for any n to the formula

$$(f^{\otimes n})_n = \mathrm{Tr}(f)^{n-1} \cdot f$$

as one can show with a similar argument, or which can be deduced by formula (1.15) in an abstract setting.

We will use this construction in order to prove the following proposition.

Proposition 1.5.8. *Let M be a non-zero, either odd or even object in $\mathcal{T}$ of Kimura dimension d. Then $\mathcal{N}(M, M)$ is a nil-ideal of $\mathcal{T}(M, M)$ with nilpotence degree at most $d+1$. This implies that $\mathcal{N}(M, M)$ is a nilpotent ideal of $\mathcal{T}(M, M)$, with nilpotence degree bounded by $2^{d+1}-1$. In particular, the image of M in $\mathcal{T}/\mathcal{N}$ is non-zero.*

Indeed, the proof of this fact relies on a computation via traces of the morphisms $(\sigma \circ f^{\otimes n})_n$ for a given permutation $\sigma \in \mathfrak{S}_n$ and for any endomorphism $f \in \mathcal{N}(M, M)$. We first fix some notation.

Notation. Given a permutation $\sigma \in \mathfrak{S}_n$, we denote by Σ_σ the set of of orbits of σ in $\{1, \dots, n\}$ and by $\Sigma_{\sigma,n}$ the subset of orbits which do not contain n. The orbit of n will be denoted as $\mathcal{O}_n$.

Lemma 1.5.9. *Let $\sigma \in \mathfrak{S}_n$ be a permutation and $f \in \mathcal{T}(M, M)$ an endomorphism. We have the following formula*

$$(\sigma \circ f^{\otimes n})_n = \left(\prod_{\mathcal{O} \in \Sigma_{\sigma,n}} \mathrm{Tr}\left(f^{|\mathcal{O}|}\right) \right) \cdot f^{|\mathcal{O}_n|} \tag{1.15}$$

Proof. See [Ivo06, Proposition 4.14.] □

Lemma 1.5.10 (Nagata-Higman)**.** *Let n be a positive integer and K a commutative unitary ring in which $n!$ is invertible. Let R be an associative, non-unitary K-algebra in which, every $x \in R$ satisfies $x^n = 0$. Then $R^{2^n - 1} = 0$.*

Proof. See [AK02, Lemma 7.2.8.]. □

Proof of Proposition 1.5.8. Let $M \neq 0$ be an even or odd object in $\mathcal{T}$ and let $f \in \mathcal{N}(M, M)$ be a nilpotent endomorphism of M. By definition of the ideal $\mathcal{N}$ the trace of f^k is 0 for all positive integers k. Therefore, for any permutation $\sigma \in \mathfrak{S}_n$ we see from Lemma 1.5.9 that

$$(\sigma \circ f^{\otimes n})_n = \begin{cases} f^n & \text{if } \sigma \text{ is a } n\text{-cycle} \\ 0 & \text{otherwise.} \end{cases} \tag{1.16}$$

which implies that

$$(\mathsf{S}^n(f))_n = \frac{1}{n!} f^n \quad \text{and} \quad (\mathsf{\Lambda}^n(f))_n = \frac{(-1)^{n-1}}{n!} f^n.$$

By definition of even and odd objects, the first part of the proposition follows. The second part is ensured to be true by Nagata-Higman Lemma stated above. □

Corollary 1.5.11 ([AK02, p. 9.1.6])**.** *If M is even or odd of finite dimension and* $\mathrm{rank}(M) = 0$ *then $M = 0$.*

Proof. By the assumption made at the beginning of this section, we have that $H(M) \in \mathsf{sVec}_L$ is again either even or odd and since $\mathrm{rank}(M) = 0$ then also $\mathrm{rank}(H(M)) = 0$ which implies that $H(M)$ is 0. So M belongs to the kernel of H which is contained in $\mathcal{N}$, being the largest tensor ideal in $\mathcal{T}$. Hence, the image of M in $\mathcal{T}/\mathcal{N}$ is 0 and by Proposition 1.5.8 it follows that $M = 0$. □

Clearly, the previous result fails to be true if we do not assume M to be even or odd.

1.5.12. Kimura dimension – We now prove some results about Kimura dimension of even and odd objects. Moreover we show, as already announced, that the only object in $\mathcal{T}$ which is both even and odd is the zero object.

Proposition 1.5.13 ([AK02, Theorem 9.1.7.])**.** *Let M be an object of $\mathcal{T}$ which is either even or odd. Then the rank of M is a non-negative integer, and*

$$\mathrm{kim}(M) = \begin{cases} \mathrm{rank}(M) & \textit{if } M \textit{ is even,} \\ -\mathrm{rank}(M) & \textit{if } M \textit{ is odd.} \end{cases} \tag{1.17}$$

Proof. Let $M \in \mathcal{T}$ be an even object. Put $d := \operatorname{rank}(M)$ and $k := \operatorname{kim}(M)$. By definition we know that $\Lambda^{k+1}(M) = 0$ and, by Proposition 1.2.3

$$\operatorname{rank}(\Lambda^{k+1}(M)) = \binom{d}{k+1}$$

which is 0 only if d is an integer with $0 \leq d \leq k$.

On the other hand

$$\operatorname{rank}(\Lambda^{d+1}(M)) = \binom{d}{d+1} = 0$$

implies that $\Lambda^{d+1}(M) = 0$ by Corollary 1.5.11, since $\Lambda^n(M)$ is an even object by point 4. of Proposition 1.5.4.

If M is odd the proof is similar. □

Corollary 1.5.14. *Let M, N be objects of $\mathcal{T}$.*

(1) If M and N are even, then $\operatorname{kim}(M \oplus N) = \operatorname{kim}(M) + \operatorname{kim}(N)$.

(2) If M and N are odd, then $\operatorname{kim}(M \oplus N) = \operatorname{kim}(M) + \operatorname{kim}(N)$.

Proposition 1.5.15 ([Kim04, Proposition 6.1])**.** *Let $M_+, M_- \in \mathcal{T}$ be respectively even and odd objects. For every $f \in \mathcal{T}(M_+, M_-)$ and for every $g \in (M_-, M_+)$ one has:*

$$f^{\otimes r} = 0 \text{ and } g^{\otimes r} = 0 \text{ for every } r > \operatorname{kim}(M_-) \cdot \operatorname{kim}(M_+).$$

Proof. Thanks to Proposition 1.5.4 (2) and by duality we can restrict to the case of f with $M_+ = \mathbb{1}$. The morphism $f^{\otimes r} : \mathbb{1} = \mathbb{1}^{\otimes r} \to M_-^{\otimes r}$ is $\mathfrak{S}_n$-equivariant, which implies that it factors through $\mathsf{S}^r(M_-) = 0$ for every $r > \operatorname{kim}(M_-)$. □

Corollary 1.5.16. *If an object M in $\mathcal{T}$ is both even and odd then $M = 0$.*

Proof. Immediate, combining Proposition 1.5.13 and Corollary 1.5.11. Alternative proof: by the previous proposition $\mathrm{id}_M \in \sqrt[\otimes]{0}(M, M)$ and by Lemma 1.5.5 it is nilpotent, which implies that M is 0. □

Proposition 1.5.17 ([Kim04, Proposition 6.3])**.** *Let M be an object in $\mathcal{T}$ and let*

$$M \simeq M_+ \oplus M_- \simeq M'_+ \oplus M'_-$$

be two decompositions with M_+, M'_+ even and M_-, M'_- odd. Then $M_+ \simeq M'_+$ and $M_- \simeq M'_-$.

Proof. Let $p : M \to M$ be the projector corresponding to the decomposition $M_+ \oplus M_-$ and, analogously, let p' be the one corresponding to the decomposition $M'_+ \oplus M'_-$. Then $\mathrm{id}_M - p'$ is an idempotent endomorphism of M which surjects to M' and the composition

$$p - p' \circ p = (\mathrm{id}_M - p') \circ p$$

is tensor nilpotent by Proposition 1.5.15 which implies that it is nilpotent by Lemma 1.5.5. Hence we have an expression

$$(p - p' \circ p)^n = 0 \tag{1.18}$$

By expanding the left hand side of (1.18) we get a relation $p = h \circ p' \circ p$ for some morphism $h \in \mathcal{T}(M, M)$. Seeing $p' \circ p$ as a morphism from M_+ to M'_+, formula (1.18) tells us that h is a section of $p' \circ p$ so that M_+ is a direct summand of M'_+.

Hence there exists an object N with $M'_+ \simeq M_+ \oplus N$. By Corollary 1.5.14 we have $\mathrm{kim}(M_+) \leq \mathrm{kim}(M'_+)$ and, since the other inequality is true by the same argument, we have in fact $\mathrm{kim}(M_+) = \mathrm{kim}(M'_+)$. Then again Corollary 1.5.14 ensures that Kimura dimension of N is zero and the result follows from Corollary 1.5.11. □

Remark 1.5.18. (1) As pointed out by André and Kahn in [AK02], Kimura's proof of Proposition 1.5.17 was not complete since he did not prove Corollary 1.5.14.

(2) If an object is decomposable into an even and an odd part its decomposition is unique up to isomorphism, which is not unique in general. We will show an example of a non-canonical decomposition in Remark 3.2.3.

1.5.19. Kimura finiteness – We end this chapter by stating the definition of finiteness we are interested in and by collecting some crucial results.

Definition 1.5.20. (1) An object $M \in \mathcal{T}$ is said to be *Kimura-finite*, if there exists a decomposition

$$M \simeq M_+ \oplus M_-$$

where M_+ is even and M_- is odd.

(2) If $M \in \mathcal{T}$ is a Kimura-finite object, $M = M_+ \oplus M_-$ we define the *Kimura dimension* of M as

$$\mathrm{kim}(M) := \mathrm{kim}(M_+) + \mathrm{kim}(M_-). \tag{1.19}$$

(3) We denote by $\mathcal{T}^{\mathrm{kim}}$ the full subcategory of Kimura-finite objects in $\mathcal{T}$. Moreover, if any object $M \in \mathcal{T}$ is Kimura-finite, $\mathcal{T}$ is said to be a *Kimura-O'Sullivan category.*

The following lemma yields a result analogous to Proposition 1.5.8 for Kimura-finite objects, that we will need in Section 3.3.

Lemma 1.5.21. *Let K be a field and let $\mathcal{T}$ be a rigid K-linear pseudo abelian tensor category. Let M be a Kimura-finite object in $\mathcal{T}$ then $\mathcal{N}(M, M)$ is a nilpotent ideal with nilpotence index bounded in function of* $\mathrm{kim}(M)$. *Moreover, if we denote by $\overline{\mathcal{T}}$ the quotient $\mathcal{T}/\mathcal{N}$ and by $\overline{M}$ the image of M in $\overline{\mathcal{T}}$, the K-algebra $\overline{\mathcal{T}}(\overline{M}, \overline{M})$ is semi-simple of finite dimension.*

Proof. See [And04a, Theorem 3.14]. □

We conclude the section collecting the main properties of Kimura-finite objects.

Theorem 1.5.22. *Let $\mathcal{T}$ be a rigid pseudo-abelian K-linear tensor category. Kimura-finiteness is stable under direct sums, tensor products, direct summands and duals. In particular, the category $\mathcal{T}^{\mathrm{kim}}$ is a rigid K-linear tensor subcategory of $\mathcal{T}$.*

Proof. By Proposition 1.5.4 it follows immediately that direct sums, duals and tensor products of Kimura-finite objects are Kimura-finite. The case of direct summands is a bit trickier, since given a decomposition $M \simeq M_+ \oplus M_-$ in even and odd objects, is not immediate that any direct summands N of M will inherit such a decomposition. See [Ivo06, Proposition 4.27] for the proof. The last assertion is simply a reformulation of the first part. □

Chapter 2

Pure motives

Let k be a field of any characteristic and let us denote by Sch/k the category of schemes over $\mathrm{Spec}(k)$. Throughout this thesis a *variety over k* (or *k-variety*) will be a separated reduced scheme of finite type over k (non-necessarily irreducible) and we denote by Var/k the category of varieties and morphisms between them. We are particularly interested in its full subcategory SmProj/k of *smooth projective k-varieties.*

Recall that, since the class of smooth projective morphisms is stable under base change, SmProj/k is in particular closed under finite products. By Example 1.1.3 this makes $(\mathsf{SmProj}/k, \times, \mathrm{Spec}(k))$ into a symmetric monoidal category.

2.1 Chow Rings

Definition 2.1.1. Let X be a smooth projective k-variety.

(1) Let $\mathcal{Z}^r(X)$ be the set of irreducible closed subvarieties of codimension r in X. We denote by $\mathrm{Z}^r(X)$ the free abelian group with basis $\mathcal{Z}^r(X)$ and by $\mathrm{Z}^*(X) = \oplus_i \mathrm{Z}^i(X)$ the resulting graded abelian group. An element $\alpha \in \mathrm{Z}^r(X)$ will be called *algebraic cycle* on X of *codimension r* or simply *cycle*. Analogously, we can consider the free abelian group of cycles graded by dimension, denoted as $\mathrm{Z}_*(X)$. Clearly the underlying groups (forgetting the grading) coincide and we will denote them simply as $\mathrm{Z}(X)$.

(2) Let X be a variety over k and $W \in \mathcal{Z}^{i-1}(X)$ a subvariety of codimension $i-1$. To any non-zero rational function $f \in k(W)$ we can associate its *divisor* $\mathrm{div}_W(f)$, see [Ful98, Section 1.3]. We define the group of *algebraic cycles rationally equivalent to* 0 of codimension i on X as:

$$\mathrm{Z}^i_{\mathrm{rat}}(X) = \langle \mathrm{div}_W(f) \mid W \in \mathcal{Z}^{i-1}(X), f \in k(W)^* \rangle. \tag{2.1}$$

(3) We will denote by $\mathrm{CH}^i(X)_{\mathbb{Z}} := \mathrm{Z}^i(X)/\,\mathrm{Z}^i_{\mathrm{rat}}(Z)$ the *Chow group* of cycles of codimension i on X. Analogously we define $\mathrm{CH}_i(X)_{\mathbb{Z}}$ as the *Chow group* of cycles of dimension i and we

put

$$\mathrm{CH}^*(X)_{\mathbb{Z}} = \bigoplus_i \mathrm{CH}^i(X), \qquad \mathrm{CH}_*(X)_{\mathbb{Z}} = \bigoplus_i \mathrm{CH}_i(X). \tag{2.2}$$

$$\mathrm{CH}^*(X) = \mathrm{CH}^*(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad \mathrm{CH}_*(X) = \mathrm{CH}_*(X)_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{2.3}$$

for the resulting graded abelian groups.

If Y is a subvariety of X of codimension r as a slight abuse of notation we will still denote by Y its image via the natural map (of sets) $\mathcal{Z}^r(X) \to \mathrm{CH}^r(X)$.

Example 2.1.2. (1) The terminal object Spec(k) of SmProj/k, is sent by both Z^* and Z_* to $\mathbb{Z}$ seen as a graded abelian group concentrated in degree 0.

(2) For any $X \in \mathsf{SmProj}/k$, the algebraic cycles on X of codimension 1 are usually called *(Weil) divisors* on X. The first Chow group $\mathrm{CH}^1(X)$ is isomorphic to $\mathrm{Pic}(X)$, the *Picard group* of X; see for instance [Har77, Ch. II, Corollary 6.16].

(3) If X is an equi-dimensional smooth projective variety of dimension d, for any $r \in \mathbb{Z}$:

$$\mathrm{CH}^r(X) = \mathrm{CH}_{d-r}(X). \tag{2.4}$$

(4) A *zero-cycle* on X is a finite formal sum $\sum_\alpha n_\alpha P_\alpha$ where P_α runs over the set of closed points of X.

(5) If Y is any subscheme of X and $Y_1, \dots, Y_t$ are the irreducible components of Y, then the local rings $\mathcal{O}_{Y,Y_i}$ are zero-dimensional Artinian rings and we put $n_i := \mathrm{length}_{\mathcal{O}_{Y,Y_i}}(\mathcal{O}_{Y,Y_i})$. We define the cycle associated to Y as,

$$\mathrm{cyc}(Y) := \sum_i n_i Y_i. \tag{2.5}$$

(6) For every $X \in \mathsf{SmProj}/k$ we denote the diagonal morphism as $\delta_X : X \to X \times X$ and for every morphism $f : X \to Y$ in SmProj/k we denote the graph morphism of f as $\gamma_f : X \to X \times Y$. Then we define:

$$\Delta_X := \mathrm{Im}(\delta_X) \in \mathrm{CH}(X \times X), \tag{2.6}$$

$$\Gamma_f := \mathrm{Im}(\gamma_f) \in \mathrm{CH}(X \times Y). \tag{2.7}$$

They will be called the *diagonal* cycle of X and the *graph* cycle of f, respectively.

2.1.3. – We recall some well-known properties of the Chow groups, they can be found with complete proofs in [Ful98, Ch. 1].

Cartesian Product Let X and Y be smooth projective varieties. For any $W \subset X$ and $Z \subset Y$ irreducible subvarieties, it is possible to consider their cartesian product $W \times Z \in \mathrm{CH}(X \times Y)$. This extends to a bilinear map:

$$\times : \mathrm{CH}^*(X) \times \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X \times Y).$$

Covariant Functoriality Let $f : X \to Y$ be a morphism in SmProj/k, if $Z \subset X$ is an irreducible closed subvariety, one sets:

$$\deg(Z/F(Z)) = \begin{cases} [k(Z) : k(f(Z))] & \text{if } \dim f(Z) = \dim Z \\ 0 & \text{if } \dim f(Z) < \dim Z \end{cases}$$

Then, putting:

$$f_*(Z) := \deg(Z/f(Z)) \cdot f(Z),$$

and extending by linearity, we get a well defined group homomorphism:

$$f_* : \mathrm{CH}_*(X) \to \mathrm{CH}_*(Y). \tag{2.8}$$

Example 2.1.4. In particular, applying CH_0 to the structural morphism of any smooth projective variety $\pi_X : X \to \mathrm{Spec}(k)$, we obtain $\deg_X := (\pi_X)_*$ the so called *degree map of* X, defined by:

$$\begin{aligned} \pi_{X*} : \mathrm{CH}_0(X) \to \mathbb{Z}. \\ \sum_\alpha n_\alpha P_\alpha \mapsto \sum_\alpha n_\alpha [k(P_\alpha) : k] \end{aligned} \tag{2.9}$$

Intersection Product Let X be a smooth projective variety, if V and W are two subvarieties of codimension i and j, they intersect each other in a union of subvarieties.

$$V \cap W = \bigcup_{\alpha \in \Lambda} Z_\alpha,$$

with $\mathrm{codim}_X Z_\alpha \geq i + j$ for every $\alpha \in \Lambda$, see [Har77, p. 48].

Definition 2.1.5. Let X be a smooth projective variety, V and W two subvarieties as above.

(1) We say that V an W *intersect properly* in X if, for every $\alpha \in \Lambda$,

$$\mathrm{codim}_X Z_\alpha = i + j.$$

(2) If V and W are properly intersecting subvarieties of X which intersect in $\bigcup_\alpha Z_\alpha$, we define their *Serre's intersection numbers* as:

$$i(V \cdot_X W; Z_\alpha) := \sum_r (-1)^r \mathrm{length}_{\mathcal{O}_{X,Z_\alpha}} \left(\mathrm{Tor}_i^{\mathcal{O}_{X,Z_\alpha}} (\mathcal{O}_{V,Z_\alpha}, \mathcal{O}_{W,Z_\alpha}) \right). \tag{2.10}$$

(3) If V and W intersect properly, we define their *intersection product* in X as:

$$V \cdot_X W := \sum_\alpha i(V \cdot_X W; Z_\alpha) Z_\alpha \in \mathrm{Z}(X). \tag{2.11}$$

We will often omit the subscript X, when the ambient variety is clear from the context.

The intersection product on X extends to a well defined multiplication on the Chow group of X which makes $\mathrm{CH}^*(X)$ into a graded commutative ring.

Contravariant Functoriality Let $f : X \to Y$ be a morphism in SmProj/k, if Z is a closed subvariety of Y, we define:

$$f^*(Z) := (\mathrm{pr}_X)_*\big(\Gamma_f \cdot_{X\times Y} (X \times Z)\big), \tag{2.12}$$

which extends to a homomorphism of graded commutative rings:

$$f^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X).$$

Push-Pull For every cartesian square in SmProj/k,

$$\begin{array}{ccc} W & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & \ulcorner & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[g]{} & X \end{array}$$

the following equality holds:

$$f'_* \circ g'^* = g^* \circ f_*. \tag{2.13}$$

Projection Formula Let $f : X \to Y$ be a morphism in SmProj/k. For any $a \in \mathrm{CH}^*(X)$ and $b \in \mathrm{CH}^*(Y)$ *projection formula* holds:

$$f_*(a \cdot f^*(b)) = f_*(a) \cdot b \in \mathrm{CH}^*(Y). \tag{2.14}$$

2.2 Correspondences

Definition 2.2.1. Let X and Y be smooth projective varieties over k.

(1) A *correspondence* from X to Y is an algebraic cycle α in $\mathrm{CH}^*(X \times Y)$, which will be denoted as $\alpha : X \vdash Y$.

(2) If X is an equi-dimensional variety, we define the *group of correspondences of degree* r as

$$\mathrm{Corr}^r(X, Y) := \mathrm{CH}^{r+\dim(X)}(X \times Y). \tag{2.15}$$

Dropping the assumption on $X \in \mathsf{SmProj}/k$, let $X = \coprod_i X_i$ be its decomposition in irreducible subvarieties, we put

$$\mathrm{Corr}^r(X, Y) := \bigoplus_i \mathrm{Corr}^r(X_i, Y) \subset \mathrm{CH}^*(X \times Y). \tag{2.16}$$

(3) Given a correspondence $\alpha : X \vdash Y$, let $\sigma : X \times Y \to Y \times X$ the natural isomorphism switching the two factors. We define the *transpose* of α as

$${}^{\mathrm{t}}\alpha = \sigma^*(\alpha) \in \mathrm{CH}^*(Y \times X).$$

(4) If $\alpha \in \mathrm{Corr}^r(X, Y)$ is a correspondence of degree r, it induces a homomorphism of graded abelian groups.

$$\begin{aligned} \alpha_* : \mathrm{CH}^*(X) &\to \mathrm{CH}^{*+r}(Y) \\ x &\mapsto (\mathrm{pr}_Y)_*(\alpha \cdot \mathrm{pr}_X^*(x)) \end{aligned} \tag{2.17}$$

Remark 2.2.2. Notice that, if $f : X \to Y$ is a morphism in SmProj/k, then

$$f^* = ({}^{\mathrm{t}}\Gamma_f)_*, \quad f_* = (\Gamma_f)_*.$$

Moreover f^* has degree 0 and, if X and Y are equi-dimensional varieties, f_* has degree $\dim(Y) - \dim(X)$.

Lemma 2.2.3. *Let X_1, X_2 and X_3 be smooth projective varieties over k. Let us denote by pr_{ij} the projection of $X_1 \times X_2 \times X_3$ into $X_i \times X_j$ for $i < j$. For any two correspondences $\alpha \in \mathrm{Corr}^r(X_1, X_2)$ and $\beta \in \mathrm{Corr}^s(X_2, X_3)$, we define their composition as*

$$\beta \circ \alpha := (\mathrm{pr}_{13})_*\big(\mathrm{pr}_{12}^*(\alpha) \cdot \mathrm{pr}_{23}^*(\beta)\big) \in \mathrm{Corr}^{r+s}(X_1, X_3). \tag{2.18}$$

With this definition, and for any such a triple, the map

$$\circ : \mathrm{Corr}^*(X_2, X_3) \times \mathrm{Corr}^*(X_1, X_2) \to \mathrm{Corr}^*(X_1, X_3) \tag{2.19}$$

is a well defined composition law, with identities $\Delta_{X_i} \in \mathrm{Corr}^0(X_i, X_i)$.

Definition 2.2.4. We define the category of (Chow) *Correspondences* of degree 0 over k, denoted as $\mathsf{Corr}^0(k)$, with the same objects as SmProj/k and, for any two smooth projective varieties X and Y, with set of morphisms given by

$$\mathsf{Corr}^0(k)(X, Y) := \mathrm{Corr}^0(X, Y).$$

Remark 2.2.5. The symmetric monoidal structure on SmProj/k induces a tensor structure on $\mathsf{Corr}^0(k)$, namely $\big(\mathsf{Corr}^0(k), \times, \mathrm{Spec}(k)\big)$ is a tensor category. On the other hand $\mathsf{Corr}^0(k)$ is $\mathbb{Q}$-linear, with biproduct given by $X \oplus Y := X \coprod Y$. Cartesian product is easily seen to be bilinear, which makes $\mathsf{Corr}^0(k)$ into a $\mathbb{Q}$-linear tensor category.

We can define a faithful functor $(\mathsf{SmProj}/k)^{\mathrm{op}} \longrightarrow \mathsf{Corr}^0(k)$ sending an object to itself and every morphism $f : X \to Y$ to the transpose of its graph ${}^{\mathrm{t}}\Gamma_f$, which respects the monoidal structure.

2.3 The category of Chow motives

In this section, we present the construction of effective pure motives.

Definition 2.3.1. We define the category of (Chow) *effective pure motives* $\mathsf{Mot}^{\mathrm{eff}}(k) := \mathsf{Mot}^{\mathrm{eff}}_{\mathrm{rat}}(k)$ as the pseudo-abelian envelope of the category of correspondences.

$$\mathsf{Mot}^{\mathrm{eff}}(k) = \mathsf{Corr}^0(k)^{\natural} \tag{2.20}$$

By composition we obtain a faithful functor

$$\mathfrak{h} : (\mathsf{SmProj}/k)^{\mathrm{op}} \longrightarrow \mathsf{Mot}^{\mathrm{eff}}(k). \tag{2.21}$$

which sends a variety X to the couple (X, Δ_X) and assigns to any morphism $f : X \to Y$ the correspondence ${}^{\mathrm{t}}\Gamma_f : (Y, \Delta_Y) \to (X, \Delta_X)$.

Remark 2.3.2. (1) Recall that, if $\mathcal{C}$ is an additive category, a *pseudo-abelian envelope* (or *Karoubi completion*) of $\mathcal{C}$ is a couple $(\mathcal{C}^\natural, i)$, where $\mathcal{C}^\natural$ is a pseudo-abelian category and $i : \mathcal{C} \longrightarrow \mathcal{C}^\natural$ is a fully faithful functor, which is universal among the couples $(\mathcal{D}, F : \mathcal{C} \longrightarrow \mathcal{D})$, with $\mathcal{D}$ pseudo-abelian.

(2) One can prove that any additive category $\mathcal{C}$ has a pseudo-abelian envelope $(C^\natural, i)$. Following the general construction, we can describe the objects of $\mathsf{Mot}^{\mathrm{eff}}(k)$ as couples $M = (X, e)$, where X is a smooth projective variety over k and e is a projector in $\mathrm{Corr}^0(X, X)$, and the morphisms from $M = (X, p)$ to $N = (Y, q)$ as correspondences $\alpha : X \vdash Y$ of the form $\alpha = q \circ \alpha' \circ p$ for some $\alpha' \in \mathrm{Corr}^0(X, Y)$.

The category $\mathsf{Mot}^{\mathrm{eff}}(k)$ is a pseudo-abelian $\mathbb{Q}$-linear tensor category. For any two motives $M = (X, p)$ and $N = (Y, q)$ the tensor structure is given by:

$$M \otimes N := (X \times Y, p \otimes q), \tag{2.22}$$

where $p \otimes q$ is the cycle $p \times q$ seen as an endo-correspondence of $X \times Y$ and the identity object is given by the *motive of the point* $\mathbb{1} := \mathfrak{h}\big(\mathrm{Spec}(k)\big)$.

While the additive structure is defined as:

$$M \oplus N = \left(X \coprod Y, p \coprod q\right). \tag{2.23}$$

2.3.3. Pure Motives – The category of effective motives is not rigid, for this reason we introduce a sort of "Tate twist" that will allow us to dualize every object.

Definition 2.3.4. We define the category $\mathsf{Mot}(k) := \mathsf{Mot}_{\mathrm{rat}}(k)$ of *(Chow) motives.*

The objects are triples (X, p, r) where X is a smooth projective k-variety, p is a projector in $\mathrm{Corr}^0(X, X)$ and r is an integer. The set of morphisms between two such triples $M = (X, p, r)$ and $M' = (X', p', r')$ is defined as

$$\mathrm{Hom}_{\mathsf{Mot}(k)}(M, M') := p' \circ \mathrm{Corr}^{r'-r}(X, X') \circ p$$

The category of effective Chow motives has a natural embedding in $\mathsf{Mot}(k)$:

$$\begin{aligned} \mathsf{Mot}^{\mathrm{eff}}(k) &\longrightarrow \mathsf{Mot}(k) \\ (X, e) &\longmapsto (X, e, 0). \end{aligned} \tag{2.24}$$

For a motive (X, e, r) we will also use the notation $e\mathfrak{h}(X)(r)$.

We will deliberately confuse $\mathsf{Mot}^{\mathrm{eff}}(k)$ with its essential image in $\mathsf{Mot}(k)$ and we will still call $\mathfrak{h}$ the composition

$$\mathfrak{h} : \mathsf{SmProj}/k \longrightarrow \mathrm{CH}^0(k) \longrightarrow \mathsf{Mot}^{\mathrm{eff}}(k) \longrightarrow \mathsf{Mot}(k).$$

The category of Chow motives $\mathsf{Mot}(k)$ inherits a tensor structure:

$$\Big(\mathsf{Mot}(k), \otimes, \mathbb{1} := \mathfrak{h}\big(\mathrm{Spec}(k)\big)\Big),$$

with tensor product given by

$$e\mathfrak{h}(X)(r) \otimes e'\mathfrak{h}(Y)(r') = (e \otimes e')\mathfrak{h}(X \times Y)(r + r').$$

In particular, for any integer $r \in \mathbb{Z}$ we can consider the autofunctor given, on objects, by

$$\begin{aligned} \mathsf{Mot}(k) &\longrightarrow \mathsf{Mot}(k) \\ M &\longmapsto M(r) = M \otimes \mathbb{1}(r) \end{aligned} \tag{2.25}$$

which will be called *Tate twist* of degree r.

We now present some basic examples of motives.

Example 2.3.5. Let X be a connected smooth projective k-variety of dimension d, with a rational point $x \in X(k)$.

(1) The cycles

$$p_0(X) = x \times X, \quad p_{2d}(X) = X \times x$$

are orthogonal projectors in $\mathrm{Corr}^0(X, X)$. They define motives:

$$\mathfrak{h}^0(X) = (X, p_0(X)), \quad \mathfrak{h}^{2d}(X) = (X, p_{2d}(X))$$

that are unique up to a (non-canonical) isomorphism, which depends on the rational equivalence class of the chosen point. In fact it can be proved that the structural morphism $X \to \mathrm{Spec}(k)$ and the point $x : \mathrm{Spec}(k) \to X$ induce mutually inverse isomorphisms of $\mathfrak{h}^0(X)$ with $\mathbb{1}$. Moreover, if Y is connected of the same dimension d with a rational point $y \in Y(k)$, it is possible to establish an isomorphism:

$$\mathfrak{h}^{2d}(X) \simeq \mathfrak{h}^{2d}(Y).$$

(2) Let C be an irreducible smooth projective curve over k. Applying the previous item to C we get a decomposition:

$$\mathfrak{h}(C) \simeq \mathfrak{h}^0(C) \oplus \mathfrak{h}^1(C) \oplus \mathfrak{h}^2(C) \tag{2.26}$$

where $\mathfrak{h}^1(C) = \big(C, \Delta_C - p_0(C) - p_2(C)\big)$. If the genus of C is non-zero, decomposition (2.26) depends on the choice of the (equivalence class of the) point x, since changing x, the projectors $p_0(C)$ and $p_2(C)$ will be different.

(3) In particular, taking C to be the projective line $\mathbb{P}^1$, the algebraic cycle $\Delta_{\mathbb{P}^1}$ is rationally equivalent to $x \times \mathbb{P}^1 + \mathbb{P}^1 \times x$ for any point $x \in \mathbb{P}^1(k)$. Hence, formula (2.26) simplifies to:

$$\mathfrak{h}(\mathbb{P}^1) \simeq \mathfrak{h}^0(\mathbb{P}^1) \oplus \mathfrak{h}^2(\mathbb{P}^1),$$

Moreover, such a decomposition is canonical, since every two points of $\mathbb{P}^1$ are rationally equivalent. The reduced motive $\mathfrak{h}^2(\mathbb{P}^1)$ is called the *Lefschetz motive* and it is isomorphic to the motive $\mathbb{1}(-1) = \big(\mathrm{Spec}(k), \mathrm{id}, -1\big)$.

We define an additive structure on $\mathsf{Mot}(k)$ as follows: let $M = e\mathfrak{h}(X)(r)$ and $M' = e'\mathfrak{h}(X')(r')$ be motives.

(1) If $r = r'$ we define

$$M \oplus M' = \left(e \coprod e'\right) \mathfrak{h} \left(X \coprod X'\right)(r)$$

(2) If $r \neq r'$, suppose $r < r'$, then we have

$$M' = e'\mathfrak{h}(X')(r) \otimes \mathfrak{h}^2(\mathbb{P}^1)^{\otimes(r'-r)} = q\mathfrak{h}(X' \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1)(r)$$

for a suitable projector q, where in the right hand side the factor $\mathbb{P}^1$ appears $r' - r$ times. We can perform then the direct sum as in the previous item.

Finally, let X be an equi-dimensional variety of dimension d. We put:

$$\big(e\mathfrak{h}(X)(r)\big)^\vee := {}^{\mathrm{t}}e\mathfrak{h}(X)(d-r)$$

and extend it to an auto-duality on SmProj/k by additivity.

It can be proved that, with the structure defined above, the category $\mathsf{Mot}(k)$ of Chow motives is a rigid, $\mathbb{Q}$-linear, pseudo-abelian tensor category.

Definition 2.3.6. We define the *Tate motive* as $\mathfrak{h}^2(\mathbb{P}^1)^\vee$, the dual of the Lefschetz one. It is canonically identified with $\mathbb{1}(1)$.

2.3.7. Manin's identity principle – As a consequence of the definition of motives, we can interpret the Chow groups of any variety in terms of motives. Namely, we have the following isomorphisms:

$$\mathrm{CH}^i(X) \simeq \mathrm{Corr}^i\left(\mathrm{Spec}(k), X\right) \simeq \mathrm{Hom}_{\mathsf{Mot}(k)}\big(\mathbb{1}, \mathfrak{h}(X)(i)\big). \tag{2.27}$$

For any cycle $\xi \in \mathrm{CH}^i(X)$ we will still denote the morphism $\mathbb{1} \to \mathfrak{h}(X)(i)$ as ξ and, if X is equi-dimensional, we will denote as ${}^{\mathrm{t}}\xi : \mathfrak{h}(X)(\dim(X) - i) \to \mathbb{1}$ its transpose. Taking inspiration from this point of view, we *define* the i-th Chow group of a motive as

$$\mathrm{CH}^i(M) := \mathrm{Hom}_{\mathsf{Mot}(k)}(\mathbb{1}, M(i)) \tag{2.28}$$

which extends CH^* to a $\mathbb{Z}$-graded, $\mathbb{Q}$-linear tensor functor $\mathrm{CH}^* : \mathsf{Mot}(k) \longrightarrow \mathsf{sVec}_{\mathbb{Q}}$

The following result is a specialization of the Yoneda Lemma to our framework.

Lemma 2.3.8. *The functor*

$$\omega : \mathsf{Mot}(k) \longrightarrow \mathrm{Fun}\left(\mathsf{SmProj}/k, \mathsf{Set}\right) \tag{2.29}$$

is fully faithful where ω_M is defined for any $M \in \mathsf{Mot}(k)$ as:

$$\omega_M : \mathsf{SmProj}/k \longrightarrow \mathsf{Set} \tag{2.30}$$

$$Y \longmapsto \bigoplus_r \mathrm{Hom}_{\mathsf{Mot}(k)}\left(\mathfrak{h}(Y), M(r)\right) \tag{2.31}$$

while ω_f acts by composition with f.

Proof. The functor $M \longmapsto \mathrm{Hom}_{\mathsf{Mot}(k)}(-, M)$ is fully faithful by Yoneda Lemma. Notice that, for every $M, N \in \mathsf{Mot}(k)$

$$\mathrm{Hom}_{\mathsf{Mot}(k)}(N, M) = \mathrm{Hom}_{\mathsf{Mot}(k)}(\mathbb{1}, M \otimes N^\vee) = \mathrm{CH}^0(M \otimes N^\vee)$$

We know that any motive N is a direct factor of $\mathfrak{h}(Y)(n)$ for some $Y \in \mathsf{SmProj}/k$ and $n \in \mathbb{Z}$ and that $\mathrm{CH}^0\left(M \otimes \mathfrak{h}(Y)(n)\right)$ is isomorphic to $\mathrm{Hom}_{\mathsf{Mot}(k)}\left(\mathfrak{h}(Y), M(r)\right)$ for some r. This proves the claim. □

The following consequences can be deduced from the previous lemma.

Proposition 2.3.9 (Manin's Identity Principle). *(1) Let $f : M \to N$ be a morphism of motives, f is an isomorphism if and only if $\omega_f(Y) : \omega_M(Y) \to \omega_N(Y)$ is so for every $Y \in \mathsf{SmProj}/k$.*

(2) Let $f, g : M \rightrightarrows N$ be morphisms, $f = g$ if and only if $\omega_f(Y) = \omega_g(Y)$ for any $Y \in \mathsf{SmProj}/k$.

(3) Given morphisms $M_1 \xrightarrow{i_1} M \xrightarrow{p_2} M_2$ in $\mathsf{Mot}(k)$, there exist $p_1 : M \to M_1$ and $i_2 : M_2 \to M$ such that M is a direct sum of M_1 and M_2 via (i_1, i_2, p_1, p_2), if and only if the sequence

$$0 \longrightarrow \mathrm{Hom}\left(\mathfrak{h}(Y), M_1(*)\right) \xrightarrow{\omega_{i_1}(Y)} \mathrm{Hom}\left(\mathfrak{h}(Y), M(*)\right) \xrightarrow{\omega_{p_2}(Y)} \mathrm{Hom}\left(\mathfrak{h}(Y), M_2(*)\right) \longrightarrow 0.$$

is exact for any $Y \in \mathsf{SmProj}/k$.

Example 2.3.10. As a first example, we apply Manin's Identity Principle to give an explicit formula for the motive of the n-dimensional projective space $\mathbb{P}^n$. Let X be a smooth projective variety of dimension d, recall that there exists an isomorphism of graded algebras:

$$\mathrm{CH}^*(X \times \mathbb{P}^n) \simeq \mathrm{CH}^*(X)[t]/(t^{n+1}) \tag{2.32}$$

where $t \in \mathrm{CH}^1(\mathbb{P}^n)$ is the class of a hyperplane in $\mathbb{P}^n$, see [Ful98, Theorem 3.3]. Manin's principle yields an isomorphism of motives

$$\mathfrak{h}(\mathbb{P}^n) = \bigoplus_{s=0}^{n} \mathbb{1}(-s). \tag{2.33}$$

In fact, for any $Y \in \mathsf{SmProj}/k$ equi-dimensional of dimension d we have

$$\mathrm{Hom}\left(\mathfrak{h}(Y), \mathfrak{h}(\mathbb{P}^n)(r)\right) = \mathrm{CH}^{d+r}(Y \times \mathbb{P}^n) = \bigoplus_{s=0}^{n} \mathrm{CH}^{d+r-s}(Y) = \mathrm{Hom}\left(\mathfrak{h}(Y), \bigoplus_{s=0}^{n} \mathbb{1}(r-s)\right)$$

We will see more applications of this principle to in Section 2.6.1.

2.4 An example: the motive of an abelian variety

In this section we collect some important results about motives of abelian varieties. As we will see, one of the fundamental tools is given by the *Fourier-Mukai transform*, developed by S. Mukai in the context of derived categories in [Muk81] and translated by A. Beauville in the framework of Chow groups, [Bea83].

2.4.1. Chow groups of abelian varieties – A functorial decomposition for the motive of an abelian variety finds its roots in the paper [Bea86], where A. Beauville provided a decomposition of the Chow groups of an abelian variety in "eigenspaces".

We fix an abelian variety A of dimension g over k, moreover, we define for any i and j in $\mathbb{Z}$ the subgroup of $\mathrm{CH}^j(A)$:

$$\mathrm{CH}^j_i(A) := \{a \in \mathrm{CH}^j(A) \mid [n]_A^*(a) = n^{2j-i}(a) \text{ for all } n \in \mathbb{Z}\} \tag{2.34}$$

where we denote by $[n]_A : A \to A$ the morphism which sends a point $x \in A(k)$ to $x + \ldots + x$, the sum iterated n-times. The following is the main result of the aforementioned article.

Theorem 2.4.2 ([Bea83]). *With the notation introduced above, for any $j \in \mathbb{Z}$ the following decomposition holds:*

$$\mathrm{CH}^j(A) = \bigoplus_{i=j-g}^{j} \mathrm{CH}^j_i(A). \tag{2.35}$$

The proof uses duality theory for abelian varieties and in particular, as already said, the Fourier-Mukai transform, which we briefly describe in the following.

If A is an abelian variety and $A^\vee$ its dual we denote by $\mathcal{P}$ a *Poincaré bundle* on $A \times A^\vee$. Let $\mathrm{ch}(\mathcal{P}) \in \mathrm{CH}^*(A \times A^\vee)$ be the *Chern character* of the poincaré bundle. If pr_A and $\mathrm{pr}_{A^\vee}$ are the projections of $A \times A^\vee$ in A and $A^\vee$ respectively, the *Fourier-Mukai transform* of the Chow group of A is defined as:

$$\begin{aligned} \mathcal{F}_A : \mathrm{CH}(A) &\to \mathrm{CH}(A^\vee) \\ a &\mapsto (\mathrm{pr}_{A^\vee})_*\big(\mathrm{ch}(\mathcal{P}) \cdot \mathrm{pr}_A^*(a)\big) \end{aligned} \tag{2.36}$$

Example 2.4.3. Let us consider Theorem 2.4.2 in the case $p = 1$. For any line bundle M on an abelian variety A, it is possible to decompose its square $M^{\otimes 2}$ as a tensor product of a *symmetric* part L_+ and an *anti-symmetric* one L_-:

$$M^{\otimes 2} \simeq L_- \otimes L_+$$

where $[n]^*([L_+]) = [L_+]^{\otimes n^2}$ and $[n]^*([L_-]) = [L_-]^{\otimes n}$, see [MG, p. 21].

If we put $m := c_1(M)$, $l_+ := c_1(L_+)$ and $l_- := c_1(L_-)$ we get in $\mathrm{CH}^1(A)$ the formula:

$$2 \cdot m = l_- + l_+.$$

Hence, since we can invert 2, formula (2.35), for $p = 1$, simplifies to:

$$\mathrm{CH}^1(A) = \mathrm{CH}^1_0(A) \oplus \mathrm{CH}^1_1(A). \tag{2.37}$$

Notice that an analogous formula does not hold in general if we replace $\mathrm{CH}^1(A)$ by $\mathrm{CH}^1(A)_{\mathbb{Z}}$.

2.4.4. – In his article [She74] A.M. Shermenev gave a decomposition of the motive of an abelian variety, which made use of the description of the $\mathfrak{h}^1$ in terms of Jacobians of curves. A canonical and functorial decomposition has been established by C.Deninger and J.P. Murre in [DM91] where they followed Beauville's approach in order to give a result analogous to Theorem 2.4.2 in the context of motives. Moreover, their result is more general than Shermenev's one, since it holds for abelian schemes.

Let S be a smooth quasi-projective connected variety over k, it is possible to generalize the constructions of the category of motives in a relative setting, over S. For any X and Y smooth projective schemes over S, the group of *relative correspondences* from X to Y is the Chow group of the fiber product $X \times_S Y$ (see [DM91, Section 1]). If X is a smooth projective scheme over S, we denote by $\mathfrak{h}_S(X)$ the *relative motive of* X over S.

Theorem 2.4.5 ([DM91, Theorem 3.1])**.** *Let A be an abelian scheme over a smooth quasi-projective variety S over k. There is a unique decomposition:*

$$\Delta_{A/S} = \sum_{i=0}^{2g} \pi_i \in \mathrm{CH}^g(A \times_S A^\vee).$$

of the diagonal as a sum of orthogonal projectors in $\mathrm{CH}^g(A \times_S A)$, such that $(n \times_S \mathrm{id}_{A^\vee})^*(\pi_i) = n^i \pi_i$. Moreover, for every integer n, if we denote by ${}^t\Gamma_n$ the transpose of the graph of the multiplication by n in $\mathrm{CH}(A \times_S A)$:

$${}^t\Gamma_n \circ \pi_i = n^i \pi_i = \pi_i \circ {}^t\Gamma_n$$

As a corollary, we get the announced decomposition for the motive of A over S.

Corollary 2.4.6. *Let $\mathfrak{h}^i_S(A)$ be the relative Chow motive determined by π_i, then we have a decomposition:*

$$\mathfrak{h}_S(A) = \bigoplus_{i=0}^{2g} \mathfrak{h}^i_S(A) \tag{2.38}$$

which is functorial. If $f : A \to B$ is a homomorphism of abelian schemes, it induces a morphism of relative motives

$$f^* : \mathfrak{h}^i_S(B) \to \mathfrak{h}^i_S(A)$$

for all i. Moreover, $[n]^*$ acts on $\mathfrak{h}^i_S(A)$ as multiplication by n^i.

2.4.7. – K. Künnemann, in his paper [Kün94], gave an explicit description of the projectors π_i, involved in the decomposition (2.38), in terms of the *Pontrjagin product* $*$ and used it to prove the following theorem:

Theorem 2.4.8 ([Kün94, Theorem 3.3.1])**.** *Let A be an abelian scheme over a smooth quasi-projective k-variety S. For any $n > 2g$ the motive $\mathfrak{h}^n_S(A)$ is 0. Moreover, for any n there exist isomorphisms*

$$\mathsf{S}^n(\mathfrak{h}^1_S(A)) \simeq \mathfrak{h}^n_S(A). \tag{2.39}$$

2.4.9. The category of abelian motives – We define the category of *abelian motives*, denoted by $\mathsf{Mot}(k)^{\mathrm{ab}}$, as the smallest rigid pseudo-abelian $\mathbb{Q}$-linear tensor subcategory of $\mathsf{Mot}(k)$ containing the motives of abelian varieties over a finite separable extension of k.

2.5 Motives modulo an adequate equivalence relation

In this subsection we illustrate how the results in section 1.4 are linked to the classical theory of adequate equivalence relations, as illustrated for example in [And04b, Sections 3.1, 3.2] and [MNP13, Section 2.1]. The following fundamental result holds:

Lemma 2.5.1 (Jannsen [Jan00]). *There exists a bijection between tensor ideals $\mathcal{I}$ of the category $\mathsf{Mot}(k)$ of Chow motives and adequate equivalence relations $\sim$ on the algebraic cycles. If $\mathcal{I}$ is the monoidal ideal corresponding to the equivalence relation $\sim$, the following equality holds*

$$\left(\mathsf{Mot}(k)/\mathcal{I}\right)^{\natural} = \mathsf{Mot}_{\sim}(k)$$

where $\mathsf{Mot}_{\sim}(k)$ is the category of motives with respect to the equivalence relation $\sim$.

Proof. See [And04b, Lemma 4.4.1.1.]. □

Remark 2.5.2. For the previous lemma to be true it is crucial that $\mathsf{Mot}(k)$ is a rigid tensor category. Indeed, the same result does not hold for the category of effective motives (see [And04b, Remark 4.4.1.2, 1)]).

Example 2.5.3. (1) In Example 1.4.6, (1) we have defined the tensor nilradical of a tensor category $\sqrt[\otimes]{0}$ which, in the category of motives, it corresponds to the *smash nilpotence equivalence* relation $\sim_{\otimes\text{nil}}$, defined by V. Voevodsky in [Voe95]. Hence we have:

$$\left(\mathsf{Mot}(k)/\sqrt[\otimes]{0}\right)^{\natural} = \mathsf{Mot}_{\otimes\text{nil}}(k)$$

(2) Analogously, in Example 1.4.6, (2) we have defined $\mathcal{N}$, the maximal tensor ideal of a tensor category. In our case, it corresponds to the coarsest equivalence relation on the algebraic cycles $\sim_{\text{num}}$, the *numerical equivalence* (see [And04b, p. 22]). Hence we have

$$\left(\mathsf{Mot}(k)/\mathcal{N}\right)^{\natural} = \mathsf{Mot}_{\text{num}}(k)$$

(3) For any field K, if $H : \mathsf{SmProj}/k \longrightarrow \mathsf{sVec}_K$ is a Weil cohomology theory and $\mathcal{I}$ is the kernel of H, we obtain

$$\left(\mathsf{Mot}(k)/\mathcal{I}\right)^{\natural} = \mathsf{Mot}_{\text{hom}}(k),$$

where the right hand side denotes the category of motives with respect to homological equivalence (see [And04b, p. 27.]).

We conclude this section with the theorem of semi-simplicity for the category of numerical motives, which has been conjectured by A. Grothendieck and belongs to the group of the so-called standard conjectures. It has been proved in the early 90's by U. Jannsen in his paper [Jan92].

Theorem 2.5.4 (Jannsen). *The category of numerical motives $\mathsf{Mot}_{\text{num}}(k)$ is abelian and semi-simple, with* Hom*-sets of finite dimension over $\mathbb{Q}$.*

Proof. As already stated, this result is Proposition 1.4.10 in the framework of motives, with $K = \mathbb{Q}$ and $\mathcal{C} = \mathsf{Mot}(k)$, taking as $H : \mathsf{Mot}(k) \longrightarrow \mathsf{sVec}_L$ a Weil cohomology theory, for some field extension $\mathbb{Q} \subset L$. □

2.6 Blow-ups and projective bundles

In this section we study two important examples: projective bundles and blow-ups. The results we will present were proved for the first time by J. Mánin in [Mán68].

2.6.1. Projective Bundles – Let $S \in \mathsf{SmProj}/k$, $\mathcal{E}$ a locally free sheaf of rank $r+1$ on S, the *projectivized bundle* of $\mathcal{E}$ is defined as

$$\mathbb{P}_S(\mathcal{E}) := \mathcal{P}roj(\mathsf{S}^\bullet(\mathcal{E})) \tag{2.40}$$

Putting $X := \mathbb{P}_S(\mathcal{E}) \xrightarrow{\pi} S$ we can consider the tautological line bundle $\mathcal{O}_X(1)$ and we denote as $\xi := c_1(\mathcal{O}_X(1)) \in \mathrm{CH}^1(X)$ its first Chern class. The following theorem holds

Theorem 2.6.2. *The Chow group $\mathrm{CH}^*(X)$ is a free module over $\mathrm{CH}^*(S)$ via π^* with basis $(1, \xi, \dots, \xi^r)$ and multiplication given by*

$$\xi^{r+1} = \sum_{j=0}^{r} (-1)^{r-j} c_{r-j+1}(\mathcal{E}) \xi^j \tag{2.41}$$

where $c_i(\mathcal{E}) \in \mathrm{CH}^i(S)$ are the Chern classes of $\mathcal{E}$.

Proof. See [Ful98, p. 3.2]. □

For any i we put $\overline{\xi^i} : \mathfrak{h}(X)(-i) \to \mathfrak{h}(X)$ for the morphism $\delta_{X*}(\xi^i) \in \mathrm{Corr}^i(X, X)$.

Theorem 2.6.3 (Manin)**.** *The map*

$$\sum_{i=0}^{r} \overline{\xi^i} \circ \pi^* : \bigoplus_{i=0}^{r} \mathfrak{h}(S)(-i) \to \mathfrak{h}(X), \tag{2.42}$$

is an isomorphism of motives.

Proof. We apply Manin's identity principle. Let T be a smooth projective variety over k, the thesis is equivalent to proving that the induced morphism

$$\bigoplus_{i=0}^{r} \mathrm{Hom}_{\mathsf{Mot}(k)} \big(\mathfrak{h}(T), \mathfrak{h}(S)(* - i)\big) \to \mathrm{Hom}_{\mathsf{Mot}(k)} \big(\mathfrak{h}(T), \mathfrak{h}(X)(*)\big),$$

is an isomorphism. We can translate this in terms of Chow groups as

$$\sum_i \mathrm{pr}_X^*(\xi^i) \cdot (\mathrm{id}_T \times \pi)^* : \bigoplus_{i=0}^{r} \mathrm{CH}^{*-i}(T \times S) \to \mathrm{CH}^*(T \times X)$$

where pr_X is the projection of $T \times X$ on the second factor. It is an isomorphism, by Theorem 2.6.2 applied to the projective bundle $T \times X \to T \times S$, since the Chern classes of $\mathrm{pr}_X^* \mathcal{E}$ on $T \times S$ are simply $\mathrm{pr}_X^*(c_i(\mathcal{E}))$. □

2.6.4. Blow Up – Let X be a smooth projective variety over k and let us consider $Y \subset X$ a non-singular subvariety of codimension r. Moreover, let

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ {\scriptstyle \pi_Y}\downarrow & \ulcorner & \downarrow{\scriptstyle \pi_X} \\ Y & \xrightarrow[i]{} & X \end{array}$$

be a blow-up diagram, *i.e.* let X' be the blow-up of X along Y and Y' be the exceptional divisor. In this situation we put $\mathcal{N} := {\mathcal{N}_{X/Y}}^\vee$ and $\mathcal{N}' := {\mathcal{N}_{X'/Y'}}^\vee$ for the co-normal bundles of $Y \to X$ and $Y' \to X'$ respectively. Moreover, we define:

$$\begin{aligned} \eta &:= c_r(\mathcal{N}) \in \mathrm{CH}^r(Y), \\ \eta' &:= c_1(\mathcal{N}') \in \mathrm{CH}^1(Y'). \end{aligned} \tag{2.43}$$

Let us denote by $\mathcal{K}$ the kernel of the canonical epimorphism:

$$\pi_Y^*(\mathcal{N}) \to \mathcal{O}_{Y'}(1) \to 0$$

and we put $\kappa := c_{r-1}(\mathcal{K}) \in \mathrm{CH}^{r-1}(Y')$ for its r^{th} Chern class. We define morphisms:

$$\begin{aligned} \alpha : \mathrm{CH}^{*-r}(Y) &\to \mathrm{CH}^*(X) \oplus \mathrm{CH}^{*-1}(Y') \\ y &\mapsto \big(i_*(y), -\pi_Y^*(y)\kappa\big) \end{aligned}$$

and

$$\begin{aligned} \beta : \mathrm{CH}^*(X) \oplus \mathrm{CH}^{*-1}(Y') &\to \mathrm{CH}^*(X') \\ (x, y') &\mapsto \pi_X^*(x) + i'_*(y'). \end{aligned}$$

Theorem 2.6.5. *The sequence*

$$0 \longrightarrow \mathrm{CH}^{*-r}(Y) \xrightarrow{\alpha} \mathrm{CH}^*(X) \oplus \mathrm{CH}^{*-1}(Y') \xrightarrow{\beta} \mathrm{CH}^*(X') \longrightarrow 0 \tag{2.44}$$

is split exact in Ab. *A right inverse to* α *is:*

$$\begin{aligned} \alpha' : \mathrm{CH}^*(X) \oplus \mathrm{CH}^{*-1}(Y') &\to \mathrm{CH}^{*-r}(Y) \\ (x, y') &\mapsto -\pi_{Y*}(y'). \end{aligned}$$

Proof. See [Ful98, Proposition 6.7, (e)]. □

Now we define a, b and a' morphisms of motives that will give us a sequence analogous to (2.44) in the framework of motives.

$$\begin{aligned} a &:= (i_*, -\overline{\kappa} \circ \pi_Y^*) : \mathfrak{h}(Y)(-r) \to \mathfrak{h}(X) \oplus \mathfrak{h}(Y')(-1) \\ b &:= \pi_X^* \oplus i'_* : \mathfrak{h}(X) \oplus \mathfrak{h}(Y')(-1) \to \mathfrak{h}(X') \\ a' &:= 0 \oplus \pi_{Y*} : \mathfrak{h}(X) \oplus \mathfrak{h}(Y')(-1) \to \mathfrak{h}(Y)(-r) \end{aligned}$$

Corollary 2.6.6. *The sequence*

$$0 \longrightarrow \mathfrak{h}(Y)(-r) \xrightarrow{a} \mathfrak{h}(X) \oplus \mathfrak{h}(Y')(-1) \xrightarrow{b} \mathfrak{h}(X') \longrightarrow 0$$

is exact and splits in the category $\mathsf{Mot}(k)$.

Proof. We apply Manin's identity principle. For any $T \in \mathsf{SmProj}/k$, the sequence

$$0 \longrightarrow \mathrm{CH}^{*-r}(T \times Y) \xrightarrow{\alpha_T} \mathrm{CH}^*(T \times X) \oplus \mathrm{CH}^{*-1}(T \times Y) \xrightarrow{\beta_T} \mathrm{CH}^*(T \times X') \longrightarrow 0$$

is the same as sequence (2.44) for the blow-up diagram

$$\begin{array}{ccc} T \times Y' & \xrightarrow{\mathrm{id}_T \times i'} & T \times X' \\ \downarrow{\scriptstyle \mathrm{id}_T \times \pi_Y} & \ulcorner & \downarrow{\scriptstyle \mathrm{id}_T \times \pi_X} \\ T \times Y & \xrightarrow[\mathrm{id}_T \times i]{} & T \times X \end{array}$$

This implies the claim. □

We conclude this chapter with an explicit formula for the motive of the blow up of a smooth projective variety.

Corollary 2.6.7. *Let* $i : Y \to X$ *be a closed embedding of codimension* r *in* SmProj/k, *then we have*

$$\mathfrak{h}(\mathrm{Bl}_Y(X)) \simeq \mathfrak{h}(X) \oplus \left(\bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(-i) \right) \tag{2.45}$$

Proof. Using the above notation, by Theorem 2.6.6, we have that:

$$\mathfrak{h}(X') \simeq \left(X \coprod (Y' \times \mathbb{P}^1), \mathrm{id} - aa' \right).$$

But then, one can show that $\mathrm{id} - aa'$ is the identity on X and that it acts as $\big(1 - p_0(Y')\big) \otimes p_0(\mathbb{P}^1)$ on $Y \times \mathbb{P}^1$. Therefore:

$$\mathfrak{h}(X') \simeq \left(X \coprod (Y' \times \mathbb{P}^1), \mathrm{id} - aa' \right) \simeq \mathfrak{h}(X) \otimes \big(Y', 1 - p_0(Y')\big) \otimes \mathbb{1} \simeq \mathfrak{h}(X) \oplus \left(\bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(-i) \right)$$

which completes the proof. □

Chapter 3

Finite dimensional motives in the sense of Kimura and O'Sullivan

We are ready to introduce the concept of Kimura-finiteness in the framework of motives. In fact, all the machinery needed has been already developed in an abstract context in Section 1.5.

3.1 Finite dimensional motives

Recall that the category of Chow motives is a rigid pseudo-abelian $\mathbb{Q}$-linear tensor category. In particular the Definition 1.5.2 of even and odd objects holds, therefore we have the following:

Definition 3.1.1. A motive $M \in \mathsf{Mot}(k)$ is said to be finite dimensional if there exist two motives M_+ and M_-, with M_+ an even object and M_- an odd object in $\mathsf{Mot}(k)$, such that:

$$M \simeq M_+ \oplus M_-. \tag{3.1}$$

If M is a finite dimensional motive, the Kimura-dimension of M is defined as:

$$\mathrm{kim}(M) := \mathrm{kim}(M_+) + \mathrm{kim}(M_-),$$

where $M = M_+ \oplus M_-$ is a given decomposition in even and odd objects.

Remark 3.1.2. We recall briefly all the properties collected in Theorem 1.5.22, that hold for finite dimensional motives as well. Direct sums and tensor products of Kimura-finite motives are Kimura-finite, direct summands of a Kimura-finite motive are Kimura-finite and the dual of any Kimura-finite motive is so.

3.2 Kimura's conjecture

Following Definition 1.5.20, we introduce the category of Kimura-finite motives.

Definition 3.2.1. We denote by $\mathsf{Mot}(k)^{\mathrm{kim}}$ the strictly full subcategory of $\mathsf{Mot}(k)$ of Kimura-finite motives. It is a rigid $\mathbb{Q}$-linear tensor subcategory of $\mathsf{Mot}(k)$.

Conjecture 3.2.2 (Kimura, O'Sullivan)**.** *Every Chow motive is Kimura- finite. In other words*

$$\mathsf{Mot}(k)^{\mathrm{kim}} = \mathsf{Mot}(k).$$

Remark 3.2.3. If M is a finite dimensional motive decomposable as $M_+ \oplus M_-$, thanks to Proposition 1.5.17 we know that M_+ and M_- are unique up to isomorphism. However, as already pointed out in point (2) of Remark 1.5.18, the decomposition is non-canonical, as we show with the following example.

Example 3.2.4. Let A be an abelian variety of dimension g, we have seen in 2.4.4 that there is a decomposition

$$\mathfrak{h}(A) \simeq \bigoplus_{i=0}^{2g} \mathfrak{h}^i(A). \tag{3.2}$$

Moreover, Theorem 2.4.8 tells us that $\mathfrak{h}^1(A)$ is an odd motive and for any i the following isomorphism holds:

$$\mathfrak{h}^i(A) \simeq \mathsf{S}^i(\mathfrak{h}^1(A)).$$

By Proposition 1.5.4, (5) the object $\mathfrak{h}^i(A)$ is even for even i and it is odd if i is odd.

To sum up, if we define:

$$\mathfrak{h}_+(A) := \bigoplus_{i=0}^{g} \mathfrak{h}^{2i}(A), \qquad \mathfrak{h}_-(A) := \bigoplus_{i=0}^{g-1} \mathfrak{h}^{2i+1}(A),$$

we obtain a decomposition:

$$\mathfrak{h}(A) \simeq \mathfrak{h}_+(A) \oplus \mathfrak{h}_-(A), \tag{3.3}$$

in an even and an odd parts. Hence, the motive of an abelian variety is finite dimensional. A decomposition such as (3.3) is not unique in general. For example, if $g = 1$, the abelian variety A is an elliptic curve and formula (3.2) in this case reduces to equation (2.26). Since the genus of A is 1 the assertion follows by the discussion in Example 2.3.5, (2).

Theorem 3.2.5. *The subcategory* $\mathsf{Mot}(k)^{\mathrm{kim}} \subset \mathsf{Mot}(k)$ *contains the category* $\mathsf{Mot}(k)^{\mathrm{ab}}$ *of abelian motives and is closed under the formation of: direct sums, tensor products, direct summands and duals. Moreover, if S is a smooth projective variety with $\mathfrak{h}(S)$ finite dimensional motive and $\mathcal{E}$ is a locally free sheaf on S, then the motive $\mathfrak{h}(\mathbb{P}_S(\mathcal{E}))$ of the associated projective bundle is finite dimensional.*

Proof. The first assertion follows from formula (3.3). Stability under direct sums, tensor products, direct summands and duals is ensured by Theorem 1.5.22. Finally, the decomposition:

$$\mathfrak{h}\big(\mathbb{P}_S(\mathcal{E})\big) = \bigoplus_{i=0}^{r} \mathfrak{h}(S)(-i),$$

illustrated in Theorem 2.6.3, for a smooth projective variety S and a locally free sheaf $\mathcal{E}$ of rank $r+1$ on it, implies that the motive of $\mathbb{P}_S(\mathcal{E})$ is finite dimensional as soon as S is so. □

Theorem 3.2.6. *Let X be a smooth projective k-variety, $Y \subset X$ a closed subvariety that is smooth over k. Let $B = \mathrm{Bl}_Y(X)$ be the blow-up of X along Y. Then the Chow motive of B is finite dimensional if and only if the Chow motives of both X and Y are finite dimensional.*

Proof. The proof relies on the decomposition of Corollary 2.6.7

$$\mathfrak{h}(B) \simeq \mathfrak{h}(X) \oplus \left(\bigoplus_{i=1}^{r-1} \mathfrak{h}(Y)(-i) \right) \tag{3.4}$$

and the stability conditions of Theorem 3.2.5. □

We conclude this section with a result of birational geometry which gives a condition for Kimura-finiteness of the motive of a smooth projective variety X to be invariant under birational equivalence.

Corollary 3.2.7. *Let k be an algebraically closed field of characteristic 0, and let $d \geq 0$ be an integer.*

(1) Assume for all smooth k-varieties Y with $\dim(Y) \leq d-2$ the Chow motive of Y is finite dimensional. Then for a smooth k-variety X of dimension d, finite dimensionality of the Chow motive $\mathfrak{h}(X)$ only depends on the birational equivalence class of X.

(2) In particular, if X_1 and X_2 are smooth k-varieties of dimension at most 3 that are birationally equivalent, $\mathfrak{h}(X_1)$ is finite dimensional if and only if $\mathfrak{h}(X_2)$ is finite dimensional.

Proof. The proof is an immediate consequence of Theorem 3.2.6 and of the weak factorization theorem for smooth projective varieties, that we recall below. □

Theorem 3.2.8 (Weak factorization)**.** *Let k be an algebraically closed field of characteristic 0, let us consider $\varphi : X_1 \dashrightarrow X_2$ a birational morphism between smooth projective k-varieties and let $U \subset X_1$ be an open subset in which φ is an isomorphism. Then φ can be factored as a sequence of blow-ups and blow-downs with smooth projective irreducible centers disjoint from U. Namely, there exists a sequence of birational maps between smooth projective varieties:*

$$X_1 = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_i}{\dashrightarrow} V_i \overset{\varphi_i}{\dashrightarrow} \cdots \overset{\varphi_l}{\dashrightarrow} V_l = X_2.$$

such that φ_i is an isomorphism on U for every i, the composition $\varphi_l \circ \ldots \circ \varphi_1$ is φ and either $\varphi_i : V_{i-1} \dashrightarrow V_i$ or $\varphi_i^{-1} : V_i \dashrightarrow V_{i-1}$ is a morphism obtained by blowing up a smooth projective irreducible center disjoint from U.

Proof. See [Abr+99] for a proof of a more general statement. □

3.3 Some results about cohomology of finite-dimensional motives

For the category of Chow motives, the simplifying assumption made at the beginning of Section 1.5 is satisfied: for H we can take any Weil cohomology theory.

Theorem 3.3.1. *(1) The category $\mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}(k)$ is an abelian semisimple $\mathbb{Q}$-linear tensor category. The functor $\mathsf{Mot}^{\mathrm{kim}}(k) \longrightarrow \mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}(k)$ is full and conservative.*

(2) If $M \in \mathsf{Mot}^{\mathrm{kim}}(k)$ and $\overline{M}$ denotes its image in $\mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}(k)$ then M is even (resp. odd) if and only if $\overline{M}$ is even (resp. odd).

Proof. (1) Applying Lemma 1.5.21 with $\mathcal{T} = \mathsf{Mot}(k)$ we obtain by Lemma 1.4.9 that $\mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}(k)$ is a semi-simple category. Moreover, again by Lemma 1.5.21, the ideal $\mathcal{N}(M,M)$ in $\mathrm{Hom}_{\mathsf{Mot}(k)}(M,M)$ is nilpotent. Therefore, every idempotent in $\mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}$ lifts to an idempotent of $\mathsf{Mot}(M,M)$, which implies that $\mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}$ is pseudo-abelian, hence abelian, again by Lemma 1.4.9. Finally, thanks to nilpotence of $\mathcal{N}(M,M)$, every endomorphism of M which induces the identity on $\overline{M}$ is an automorphism, which implies that $\mathsf{Mot}^{\mathrm{kim}} \longrightarrow \mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}$ is conservative.

(2) If M is even, $\overline{M}$ is even, too, as $\mathsf{Mot}(k) \longrightarrow \mathsf{Mot}_{\mathrm{num}}(k)$ is a fibre functor. Conversely, since the functor $\mathsf{Mot}^{\mathrm{kim}}(k) \longrightarrow \mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}(k)$ is conservative, $\Lambda^r(\overline{M}) = \overline{\Lambda^r(M)} = 0$ implies that $\Lambda^r(M)$ is equal to 0 .

□

Corollary 3.3.2. *Let $H\colon \mathsf{Mot}(k) \longrightarrow \mathsf{sVec}_L$, for L some field of characteristic 0, be a Weil cohomology.*

(1) A motive M in $\mathsf{Mot}^{\mathrm{kim}}(k)$ is even (odd) if and only if $H(M)$ is even (odd).

(2) If M is Kimura-finite then the Kimura dimension of M equals $\dim_L(H(M))$.

(3) If $f\colon M \to N$ is a homomorphism between Kimura-finite motives then f is an isomorphism if and only if the induced map $H(f)$ is an isomorphism.

Proof. (1) If M is even, since H is a tensor functor, $H(M)$ is even too. Conversely, let M be a motive and $r \in \mathbb{N}$ with $\Lambda^r(H(M)) = H(\Lambda^r(M)) = 0$, then $\Lambda^r(M)$ is 0 in $\mathsf{Mot}_{\mathrm{num}}(k)$ and the claim follows from point 2 of Theorem 3.3.1. The proof if M is odd is similar.

(2) We may assume by additivity that M is either even or odd. Let us suppose that M is even, then $\mathrm{kim}(M) \geq \dim_L(H(M))$ since H is a tensor functor. Conversely let $d := \dim_L(H(M))$ then, using the previous argument, $\Lambda^{d+1}(M) = 0$, hence $\mathrm{kim}(M) \leq \dim_L(H(M))$.

(3) Let $f : M \to N$ be a morphism of Kimura-finite motives, if $H(f)$ is an isomorphism so is $\overline{f} : \overline{M} \to \overline{N}$ in $\mathsf{Mot}_{\mathrm{num}}(k)$ and the assertion is true since $\mathsf{Mot}^{\mathrm{kim}}(k) \longrightarrow \mathsf{Mot}^{\mathrm{kim}}_{\mathrm{num}}(k)$ is conservative.

□

Let $M \in \mathsf{Mot}(k)$ be a motive, f be an endomorphism of M and $n \in \mathbb{N}$, one can prove that the following equality holds (see [AK02, p. 7.2.5.])

$$\mathrm{Tr}(\Lambda^n(\mathrm{id}_M + f)) = \sum_{i=0}^{n} \mathrm{Tr}(\Lambda^i(f)).$$

Hence, if we extend the scalars to $\mathbb{Q}(t)$ for some indeterminate t:

$$\mathrm{Tr}(\Lambda^n(\mathrm{id}_M - tf)) = \sum_{i=0}^{n} (-1)^i \,\mathrm{Tr}(\Lambda^i(f)) t^i, \tag{3.5}$$

Analogously for symmetric powers we get that

$$\mathrm{Tr}(\mathsf{S}^n(\mathrm{id}_M - tf)) = \sum_{i=0}^{n} \mathrm{Tr}(\mathsf{S}^i(f)) t^i. \tag{3.6}$$

Definition 3.3.3. Let M be an even motive of dimension d, in such a case $\Lambda^d(M)$ is invertible with respect to tensor product and we define :

$$P_f(t) := \Lambda^d(\mathrm{id}_M - tf) \in \mathbb{Q}[t]$$

which is called the *characteristic polynomial* of f. Analogously, if M is odd, we define the *characteristic polynomial* of f as

$$P_f(t) := \mathsf{S}^d(\mathrm{id}_M - tf)$$

.

Remark 3.3.4. Notice that, if $H : \mathsf{Mot}(k) \longrightarrow \mathsf{sVec}_L$ is a Weil cohomology theory and M is a motive either even or odd, the characteristic polynomial of f coincides with the characteristic polynomial of $H(f)$ in the classical sense, thanks to formulas (3.5) and (3.6), since $\mathrm{Tr}(g) = \mathrm{Tr}(H(g))$ for every endomorphism $g : M \to M$. In particular, the characteristic polynomial of $H(f)$ has rational coefficients.

We conclude with a motivic version of a classical result, the *Cayley-Hamilton Theorem.*

Proposition 3.3.5 (Cayley-Hamilton-O'Sullivan)**.** *Let $M \in \mathsf{Mot}(k)$ be a motive either even or odd, $f : M \to M$ an endomorphism of M and $P_f(t)$ its characteristic polynomial, then $P_f(f) = 0$.*

Proof. The claim follows from the classical Cayley-Hamilton Theorem applied to $H(f)$, for some Weil cohomology theory H, by the previous remark. □

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