Arakelov intersection theory applied to torsors of semi-stable elliptic curves

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Contents

Introduction

Let K be a number field, and let E be an elliptic curve over K. We consider the Weil-Châtelet group of equivalence classes of K-torsors of E. Suppose that C is such a K-torsor, and let N be its index. That is, N is the minimal integer such that there exists a field L/K of degree N such that C has an L-rational point. It turns out, that there exists a function

$$B(N) = [K : \mathbb{Q}]N \log N + O(N),$$

depending only on E, such that this splitting field L may be chosen such that

$$\log \left| N_{K/\mathbb{Q}}(\Delta_{L/K}) \right| \leq B(N).$$

In the first chapter we will explore the theory of Green functions: real-valued functions on Riemann surfaces that appear when one attempts to invert the Laplacian operator. Next, we will introduce admissible line bundles. These take the place of line bundles in the Arakelov-theoretical divisor theory. Using the Green functions we will show that admissible line bundles exist. Next, we will look at a result of Faltings [5] that allows us to assign in a natural way metrics to the determinant of cohomology of admissible line bundles. Finally, we will derive an inequality on these metrics, using a result of Elkies (1.3.4).

The next chapter serves as an introduction to the theory of arithmetic surfaces. An arithmetic surface is a regular, integral, projective flat scheme $X \to S$ over a Dedekind scheme S of dimension 1. Given a curve C over the fraction field of S, we may try to construct an arithmetic surface over S such that its generic fiber is isomorphic to C, called a regular model of C over S. This is not always possible: when one tries to construct such an arithmetic surface in a naive way it will probably have singularities. Under some nice assumptions it turns out that these singularities can be resolved.

On surfaces over a field we have an intersection theory of divisors. When we try to generalize this to arithmetic surfaces over $\operatorname{Spec} O_K$, with O_K the ring of integers of a number field, the intersection number will not behave as nicely under linear equivalence. The problem here is that the base scheme $\operatorname{Spec} O_K$ is not 'compact' anymore. We can solve this issue by adding some 'points at infinity' to $\operatorname{Spec} O_K$; these are points corresponding to infinite places of K, whereas the closed points of S correspond to the finite places of K. In the third chapter we will introduce an intersection theory, the Arakelov intersection theory, based on this concept. Two well-known theorems from 'classical' intersection theory on surfaces, the Riemann-Roch theorem and the adjunction formula, have an Arakelov-theoretical analogue.

In the final chapter we will take a closer look at Arakelov intersection theory on arithmetic surfaces with generic fiber of genus 1. A semi-stable elliptic curve has a regular model of which the geometric fibers look rather nicely. Using this information Hriljac [7] has given an upper bound for the discriminant of the splitting field of torsors of semi-stable elliptic curves over a number field. We will inspect this upper bound more closely to arrive at the inequality listed at the beginning of this introduction.

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1 Complex geometry

In this section we will establish the complex geometrical theory behind Arakelov intersection theory. If X is a compact Riemann surface together with a smooth volume form μ with $\int_X \mu = 1$, then we can define a Laplace operator Δ_μ on the set of smooth functions on X such that $(\Delta_\mu f)\mu = \frac{i}{\pi}\partial\overline{\partial} f$. This operator is not injective as it annihilates all constant functions. But we can salvage an inverse in some cases, by convolution with a so-called Green function. If X has positive genus then there is a canonical way to define a volume form on X, and the corresponding Green function plays a crucial role in Arakelov intersection theory. We will also look at so-called admissible line bundles; these are line bundles on X with a Hermitian metric that satisfies some nice properties. As it will turn out these admissible line bundles correspond to Arakelov divisors, just as Weil divisors correspond to line bundles in the algebraic geometrical case. In the last paragraph we will define the Faltings metric on the determinant of cohomology of every admissible line bundle, and this Faltings metric will be used later on to state the Arakelov-theoretical Riemann-Roch theorem.

1.1 Currents

Let X be a differentiable manifold of dimension n. Let $\mathcal{E}^p$ denote the sheaf of smooth real-valued p-forms on X. For every open $U \subset X$, let $\mathcal{E}^p_c(U) \subset \mathcal{E}^p(U)$ denote the subset of smooth p-forms with compact support on U. Notice that $\mathcal{E}^p_c$ is not a sheaf in general. For open subsets $U \subset V \subset X$ we have natural inclusions $\mathcal{E}^p_c(U) \hookrightarrow \mathcal{E}^p_c(V)$.

Definition 1.1.1. A *current of degree* p on X is an $\mathbb{R}$-linear form on $\mathcal{E}^{n-p}_c(X)$ that is continuous in the sense of distributions; that is: if $\{\omega_i\}_{i\geq 0}$ is a sequence of forms in $\mathcal{E}^p_c(X)$ with support in a fixed compact subset of X, such that on every coordinate chart of X all derivatives of all coefficients of the ω_i converge uniformly to 0, then $\lim_{i\to\infty} T(\omega_i) = T(\omega)$. We denote the set of currents of degree p on X by $\mathcal{D}^p(X)$. This set is an $\mathbb{R}$-vector space in a natural way.

If $P \in X$ is any point, then the *Dirac delta* at P is the current of degree n defined by

$$\delta_P(f) = f(P).$$

More generally, if $D = \sum_{P\in X} n_P P$ is a formal sum of points on X (a Weil divisor if X is a Riemann surface), then we define the Dirac delta at D to be the degree n current

$$\delta_D = \sum_{P\in X} n_P \delta_P.$$

Now suppose that X is oriented, so we can take integrals. If $\alpha \in \mathcal{E}^p(X)$ is a differential form, then we define the current $[\alpha]$ of degree p by

$$[\alpha](\phi) = \int_X \alpha \wedge \phi \quad \text{for all } \phi \in \mathcal{E}^{n-p}_c(X).$$

Notice that $[\alpha] = [\beta]$ for two p-forms α, β if and only if $\alpha = \beta$. Hence we have an embedding $\mathcal{E}^p(X) \hookrightarrow \mathcal{D}^p(X)$, so we can view differential forms as currents. Some operations on differential forms extend to operations on currents in a natural way, as we will see later on. A current T of degree p is said to be *represented* on an open subset $U \subset X$ by a p-form $\alpha \in \mathcal{E}^p(U)$ if

$$T(\phi) = \int_U \alpha \wedge \phi \quad \text{for all } \phi \in \mathcal{E}_c^{n-p}(U) \subset \mathcal{E}_c^{n-p}(X).$$

Suppose now that X is a compact differentiable manifold. Let α be a differential p-form, and β a differential $(n-p-1)$-form with compact support. Then $d(\alpha\wedge\beta)$ is an n-form with compact support and it is equal to $d\alpha\wedge\beta+(-1)^p\alpha\wedge d\beta$. Taking the integral over X and applying Stokes' theorem yields

$$0 = \int_X d(\alpha \wedge \beta) = \int_X d\alpha \wedge \beta + (-1)^p \int_X \alpha \wedge d\beta$$

so $\int_X d\alpha \wedge \beta = (-1)^{p+1} \int_X \alpha \wedge d\beta$, or using the notation introduced above:

$$[d\alpha](\beta) = (-1)^{p+1}[\alpha](d\beta).$$

This notion of taking exterior derivatives can be extended in a natural way to all currents of degree p: if T is a current of degree p, we define dT to be the current of degree $p+1$ given by

$$dT(\phi) = (-1)^{p+1}T(d\phi) \quad \text{for all } \phi \in \mathcal{E}_c^{n-p-1}(X).$$

Analogously to real-valued currents, we can introduce *complex-valued* currents. A complex-valued current of degree p is an $\mathbb{R}$-linear map $\mathcal{E}_c^{n-p}(X) \to \mathbb{C}$ such that its real and imaginary parts are real-valued currents. If ϕ is a complex-valued differential form we can write $\phi = \operatorname{Re}\phi + i \operatorname{Im}\phi$ with $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$ real-valued differential forms, and we define $T(\phi) = T(\operatorname{Re}\phi) + iT(\operatorname{Im}\phi)$. Now T is a $\mathbb{C}$-linear form on the space of complex differential forms of degree $n-p$ with compact support. The definitions of Dirac delta currents, represented currents and the exterior differential carry over without any problems to the complex-valued case. Every current on a complex manifold will be assumed to be a complex-valued current from now on.

Now suppose that X is a compact complex manifold. In a similar way as for the differential d we have the identities $[\partial\alpha](\beta) = (-1)^{p+1}[\alpha](\partial\beta)$ and $[\overline{\partial}\alpha](\beta) = (-1)^{p+1}[\alpha](\overline{\partial}\beta)$ for every complex-valued differential p-form α and $n-p-1$-form β. Thus we can extend the holomorphic and antiholomorphic differentials of differential forms to the notion of holomorphic and antiholomorphic differentials of currents of degree p by setting

$$\partial T(\phi) = (-1)^{p+1}T(\partial\phi) \quad \text{and} \quad \overline{\partial}T(\phi) = (-1)^{p+1}T(\overline{\partial}\phi).$$

We will use the rest of this section to compute the degree 2 current $\partial\overline{\partial}[\log|f|]$ on a Riemann surface X; where f is a meromorphic function on X, and $[\log|f|]$ is the degree 0 current given by

$$[\log|f|](\phi) = \int_{X\setminus \operatorname{Supp} f} \log|f| \cdot \phi.$$

Define the operator $\overline{d} = \partial - \overline{\partial}$. One easily verifies that $d\overline{d} = -2\partial\overline{\partial}$, that $df \wedge \overline{d}g = dg \wedge \overline{d}f$ for all smooth functions f, g, and that $fd\overline{d}g - gd\overline{d}f = d(f\overline{d}g -$

$g\bar{d}f$). If $P \in X$ is a point and z is a local coordinate on an open neighbourhood U centered at P, then we can define local coordinates r, θ on $U \setminus \{P\}$ by setting $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$. We find that for every smooth function h the following identity holds:

$$\bar{d}h = \frac{1}{ir}\frac{\partial h}{\partial \theta}dr + ir\frac{\partial h}{\partial r}d\theta.$$

Lemma 1.1.2. *Let z be a chart centered at P, and for $r > 0$ sufficiently small let $C_P(r)$ be the circle around P given by $|z| = r$. Let h be a smooth function around P and let f be of the form $f = k \log |z| + g$, with g a smooth function around P. Then*

$$\lim_{r \downarrow 0} \int_{C_P(r)} f\bar{d}h = 0$$

and

$$\lim_{r \downarrow 0} \int_{C_P(r)} h\bar{d}f = 2\pi i k h(P).$$

Proof. Introduce polar coordinates r, θ by setting $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$. On $C(r)$ we have $dr = 0$, so we find that

$$f\bar{d}h = (k \log r + g) \cdot ir\frac{\partial h}{\partial r}d\theta.$$

As g and $\frac{\partial h}{\partial r}$ are smooth functions around P and as $r \log r$ tends to 0 as $r \downarrow 0$ we find that

$$\lim_{r \downarrow 0} \int_{C_P(r)} f\bar{d}h = \lim_{r \downarrow 0} \int_0^{2\pi} ir(k \log r + g)\frac{\partial h}{\partial r}d\theta = 0.$$

For the second equation: we have $f = k \log r + g$, so

$$h\bar{d}f = ir\frac{\partial f}{\partial r}d\theta = ir\left(\frac{k}{r} + \frac{\partial g}{\partial r}\right),$$

so

$$\lim_{r \downarrow 0} \int_{C_P(r)} h\bar{d}f = \lim_{r \downarrow 0} \left(ik \int_0^{2\pi} h\, d\theta + ir \int_0^{2\pi} h\frac{\partial g}{\partial r}\, d\theta\right)$$

The first summand tends to $ik \cdot 2\pi h(P) = 2\pi i k \cdot h(P)$ as $r \downarrow 0$, and the second summand tends to 0 as $r \downarrow 0$, proving the second equation. $\square$

Let f be a meromorphic function on X, and consider the current $[\log |f|]$ of degree 0 given by

$$[\log |f|](\phi) = \int_{X \setminus \operatorname{Supp} f} \log |f| \cdot \phi.$$

For every P in the support of f, pick a local coordinate z_P centered at P, and for r sufficiently small, let $D_P(r)$ be the open disc around P given by $|z_P| < r$, and let $C_P(r)$ be the circle around P, oriented counterclockwise, given by $|z_P| = r$. Let $S(r)$ be the union of all $D_P(r)$, with P ranging over the support of f. We find that

$$[\log |f|](\phi) = \lim_{r \downarrow 0} \int_{X \setminus S(r)} \log |f| \cdot \phi.$$

Let us compute the current $\partial\overline{\partial}[\log|f|]$. For h smooth on X we have

$$\partial\overline{\partial}[\log|f|](h) = \lim_{r\downarrow 0}\int_{X\setminus S(r)} \log|f|\partial\overline{\partial}h = -\tfrac{1}{2}\lim_{r\downarrow 0}\int_{X\setminus S(r)} \log|f|d\overline{d}h.$$

We have

$$d(\log|f|\overline{d}h - h\overline{d}\log|f|) = \log|f|d\overline{d}h - hd\overline{d}\log f = \log|f|d\overline{d}h$$

since $\partial\overline{\partial}\log|f| = 0$, so

$$\partial\overline{\partial}[\log|f|](h) = -\tfrac{1}{2}\lim_{r\downarrow 0}\int_{X\setminus S(r)} d(\log|f|\overline{d}h - h\overline{d}\log|f|).$$

Using Stokes' theorem we find that this equals

$$\tfrac{1}{2}\sum_{P\in\operatorname{Supp} f}\lim_{r\downarrow 0}\int_{D_P(r)} (\log|f|\overline{d}h - h\overline{d}\log|f|).$$

The factor (-1) comes from the fact that the orientation of the circles obtained from the orientation of X is clockwise, but the orientation of the $C_P(r)$ is counterclockwise. Around $P \in \operatorname{Supp} f$ the function f is of the form $f = z^{v_P(f)}g$ with g a nonzero holomorphic function. Hence $\log|f|$ is of the form $v_P(f)\log|z| + \log|g|$, and $\log|g|$ is a smooth function around P. Using the previous lemma we find that

$$\partial\overline{\partial}[\log|f|](h) = \tfrac{1}{2}\sum_{P\in\operatorname{Supp} f} -(2\pi i v_P(f)h(P)) = -\pi i\cdot\delta_{\operatorname{div} f}(h).$$

We have therefore proven the following theorem:

Theorem 1.1.3. *Let X be a Riemann surface, and let f be a meromorphic function on X. Then*

$$\partial\overline{\partial}[\log|f|] = -\pi i\delta_{\operatorname{div} f}. \qquad \square$$

1.2 Green functions on Riemann surfaces

Throughout this section, we will let X denote a compact connected Riemann surface and μ a volume form on X such that $\int_X \mu = 1$. Define the *Laplacian* Δ_μ *with respect to* μ to be the unique operator on the set $\mathcal{E}^0(X)$ of smooth complex-valued functions on X such that

$$(\Delta_\mu f)\mu = \frac{i}{\pi}\partial\overline{\partial}f \quad \text{for all } f \in \mathcal{E}^0(X).$$

This operator is not invertible: its kernel consists of the harmonic functions on X, and as X is compact, these functions are the constant functions on X. However, if we let $W \subset \mathcal{E}^0(X)$ denote the subspace of functions f with $\int f\mu = 0$, we get a decomposition $\mathcal{E}^0(X) = \mathbb{C}\oplus W$, and hence Δ_μ gives an injective map $W \to \mathcal{E}^0(X)$. In this section we will give an inverse to this map, using the following theorem on currents.

Theorem 1.2.1 ([2, Theorem 2.2])**.** *Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. For every current T of degree 2 on X, there exists a unique current $G_\mu T$ of degree 0 on X such that*

$$\frac{i}{\pi}\partial\bar{\partial}G_\mu T = T - T(1)\cdot[\mu] \quad \textit{and} \quad G_\mu T(\mu) = 0.$$

Moreover, if T is represented on $U \subset X$ by a 2-form $\phi \in \mathcal{E}^2(U)$, then $G_\mu T$ is represented on U by a function $f \in \mathcal{E}^0(U)$. There is a unique smooth function

$$g_\mu : (X \times X) \setminus \Delta \to \mathbb{R}$$

having a logarithmic (hence integrable) singularity along $\Delta \subset X \times X$, such that for all $\phi \in \mathcal{E}^2(X)$ the current $G_\mu[\phi]$ is represented by the function $G_\mu\phi \in \mathcal{E}^0(X)$ given by

$$G_\mu\phi(P) = \int_{Q\in X\setminus\{P\}} g_\mu(P,Q)\phi(Q).$$

Definition 1.2.2. Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. The function g_μ occurring in the previous theorem is called the *Green function* associated to μ.

If μ is real then the proof of theorem 1.2.1 in [2] shows that the Green function g_μ is real as well.

Lemma 1.2.3 ([2, Lemma 2.3])**.** *Let X be a compact and connected manifold, let $P \in X$ be a point, let μ be a smooth $(1,1)$-form of X such that $\int_X \mu = 1$, and let $g_{P,\mu}$ let the smooth function on $X \setminus \{P\}$ given by $g_{P,\mu}(Q) = g_\mu(P,Q)$. Let $[g_{P,\mu}]$ be the degree 0 current on X defined by*

$$[g_{P,\mu}](\phi) = \int_{X\setminus\{P\}} g_{P,\mu}\phi \quad \textit{for all } \phi \in \mathcal{E}^2(X).$$

Then $[g_{P,\mu}]$ is the unique current satisfying

$$\frac{i}{\pi}\partial\bar{\partial}[g_{P,\mu}] = \delta_P - [\mu] \quad \textit{and} \quad [g_{P,\mu}](\mu) = 0.$$

The current $[g_{P,\mu}]$ gives us a way to invert the Laplace operator Δ_μ. Let $f \in W$; so $f \in \mathcal{E}^0(X)$ is a smooth function on X and $[\mu](f) = \int_X f\mu = 0$. We then have:

$$[g_{P,\mu}]((\Delta_\mu f)\mu) = [g_{P,\mu}](\tfrac{1}{\pi}\partial\bar{\partial}f) = \tfrac{1}{\pi}\partial\bar{\partial}[g_{P,\mu}](f) = \delta_P(f) - [\mu](f) = f(P).$$

So we retrieve f from $\Delta_\mu f$ as the function $P \mapsto [g_{P,\mu}]((\Delta_\mu f)\mu)$.

Let's take a closer look at the Green function, especially around the diagonal.

Proposition 1.2.4 ([1, Proposition 1.1])**.** *Let X be a compact connected Riemann surface, and let μ be a smooth $(1,1)$-form on X such that $\int_X \mu = 1$. Then $g_\mu(P,Q) = g_\mu(Q,P)$ for all $P \neq Q$.* □

Let $P \in X$, and let z be a local coordinate around P. Let l_P be a smooth function on $X \setminus \{P\}$ that is given on some open neighbourhood U of P by

$l_P(Q) = \log|z(P) - z(Q)|$. By applying Stokes' theorem (cf. the proof of theorem 1.1.3) we find that

$$\partial\bar{\partial}[l_P] = -\pi i \delta_P \quad \text{on } U,$$

and from lemma 1.2.3 we know that $\partial\bar{\partial}[g_{P,\mu}] = -\pi i(\delta_P - [\mu])$, so $\partial\bar{\partial}[g_{P,\mu} - l_P]$ is represented by a smooth $(1,1)$-form on X. Using theorem 1.2.1 we find that $g_{P,\mu} - l_P$ can be extended to a smooth function on X. We therefore find that on $U \times U$ the Green function g_μ can be written as

$$g_\mu(P,Q) = \log|z(P) - z(Q)| + h(P,Q)$$

with h a smooth function on $U \times U$.

For every Weil divisor $D = \sum_P n_P P$ on X, we define the Green function $g_{D,\mu}$ to be

$$g_{D,\mu} = \sum_P n_P g_{P,\mu}.$$

If f is a meromorphic function on X with divisor $\operatorname{div} f$, then we obtain the following identity, which will be useful later on.

Theorem 1.2.5. *Let $v_\mu(f) := -\int_X \log|f|\mu$. Then*

$$[g_{\operatorname{div} f,\mu}] = [\log|f|] - [v_\mu(f)].$$

In particular, on $X \setminus \operatorname{Supp} f$, we have

$$g_{\operatorname{div} f,\mu} = \log|f| - v_\mu(f)$$

as smooth functions.

Proof. Write $\operatorname{div}(f) = \sum_P n_P P$. We have

$$\partial\bar{\partial}[g_{\operatorname{div} f,\mu}] = -\pi i \sum_P n_P(\delta_P - [\mu]) = -\pi i \delta_{\operatorname{div} f}$$

by lemma 1.2.3, and

$$\partial\bar{\partial}([\log|f|] - [v_\mu(f)]) = \partial\bar{\partial}[\log|f|] = -\pi i \delta_{\operatorname{div} f}$$

by theorem 1.1.3. Moreover, we have $[g_{\operatorname{div} f,\mu}](\mu) = 0$, again by lemma 1.2.3, and $[\log|f|](\mu) - [v_\mu(f)](\mu) = 0$, by definition of $v_\mu(f)$. Using theorem 1.2.1 we find the desired equality. □

1.3 The Arakelov-Green function

Let X be a compact and connected Riemann surface of genus $g \geq 1$. We assign to the g-dimensional complex vector space $\Omega^1_X(X)$ a hermitian inner product

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_X \omega \wedge \bar{\eta}.$$

Let $\omega_1, \ldots, \omega_g$ be an orthonormal basis of Ω^1_X with respect to this inner product, and define the smooth $(1,1)$-form μ as

$$\mu = \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \bar{\omega}_k.$$

Using the fact that the chosen basis is orthonormal, we find that $\int_X \mu = 1$. Moreover, the form μ does not depend on the choice of orthonormal basis, as the following lemma shows.

Lemma 1.3.1. *Let $\{\omega_1, \ldots, \omega_g\}$ and $\{\eta_1, \ldots, \eta_g\}$ be two orthonormal bases of Ω^1_X. Then*

$$\frac{i}{2g}\sum_{k=1}^{g} \eta_k \wedge \bar{\eta}_k = \frac{i}{2g}\sum_{k=1}^{g} \omega_k \wedge \bar{\omega}_k.$$

Proof. Using the identity $\eta_k = \sum_{l=1}^{g} \langle \eta_k, \omega_l \rangle \omega_l$ we find:

$$\begin{aligned}
\sum_{k=1}^{g} \eta_k \wedge \bar{\eta}_k &= \sum_{k=1}^{g}\sum_{l=1}^{g} \langle \eta_k, \omega_l \rangle \omega_l \wedge \bar{\eta}_k \\
&= \sum_{l=1}^{g}\sum_{k=1}^{g} \overline{\langle \omega_l, \eta_k \rangle} \omega_l \wedge \bar{\eta}_k \\
&= \sum_{l=1}^{g} \omega_l \wedge \overline{\sum_{k=1}^{g} \langle \omega_l, \eta_k \rangle \eta_k} \\
&= \sum_{l=1}^{g} \omega_l \wedge \bar{\omega}_l
\end{aligned}$$

where the last equality follows from the identity $\omega_l = \sum_{k=1}^{g} \langle \omega_l, \eta_k \rangle \eta_k$. ☐

Lemma 1.3.2. *Let $\omega_1, \ldots, \omega_g$ be an orthonormal basis for $\Omega^1_X(X)$. Then $\mu = \frac{i}{2g}\sum_{k=1}^{g} \omega_k \wedge \bar{\omega}_k$ is a volume form.*

Proof. Suppose that μ vanishes at $P \in X$. Let z be a local coordinate for X around P. Then every ω_k can be written around P as $\omega_k(z) = f_k(z)dz$, with f_k a holomorphic function. Then μ can be written as

$$\mu(z) = \frac{i}{2g}\sum_{k=1}^{g} |f_k(z)|^2 dz \wedge d\bar{z}.$$

We therefore see that f_k vanishes at P for all $k = 1, \ldots, g$, and hence that all ω_k vanish at P, so as $\omega_1, \ldots, \omega_g$ form a basis for $\Omega^1_X(X)$ we see that every holomorphic 1-form on X vanishes at P. We will show that this is not the case.

Let $K = \operatorname{div} \omega$ be a canonical divisor on X, induced by a meromorphic differential form ω. Using Riemann-Roch on the divisors 0 and P we find that

$$l(P) - l(K - P) = l(0) - l(K) + 1;$$

where $l(D) = \dim H^0(X, O_X(D))$ for every divisor D of X. Since X has positive genus, $H^0(X, O_X(P))$ can only contain the constant functions, as any nonconstant global section of $O_X(P)$ would induce an isomorphism with the Riemann sphere. Therefore we see that $l(P) = l(0) = 1$, and hence that $l(K - P) = l(K) - 1$, so $H^0(X, O_X(K - P))$ is a proper subspace of $H^0(X, O_X(K))$. Take $f \in H^0(X, O_X(K)) \setminus H^0(X, O_X(K - P))$. We see that $\operatorname{div} f\omega = K + \operatorname{div} f \geq 0$, so $f\omega$ is a holomorphic 1-form. Moreover we have $f \notin H^0(X, O_X(K - P))$, so $\operatorname{div} f\omega - P$ is not effective, showing that $f\omega$ does not have a zero at P. We have found a holomorphic 1-form with no zero at P, completing our proof. ☐

Definition 1.3.3. The form μ defined above is called the *canonical* $(1,1)$*-form* on X. We define the *Arakelov-Green function* of X to be the smooth real-valued function $g_{\mathrm{Ar}} := g_\mu$ on $(X \times X) \setminus \Delta$.

Some notation: for every point $P \in X$ we will let $g_{P,\mathrm{Ar}}$ denote the smooth function $Q \mapsto g_{\mathrm{Ar}}(P,Q)$ on $X \setminus \{P\}$, and for every divisor $D = \sum_P n_P P$ we let $g_{D,\mathrm{Ar}} = \sum_P n_P g_{P,\mathrm{Ar}}$.

The following theorem gives a useful upper bound that we will use later in this thesis.

Theorem 1.3.4 (Elkies, [7, p. 218])**.** *Let X be a Riemann surface of genus 1. There exists a constant c such that for all $n \geq 2$ and every n-tuple of pairwise different points $P_1, \ldots, P_n \in X$ the inequality*

$$\sum_{i \neq j} g_{\mathrm{Ar}}(P_i, P_j) \leq \frac{n \log n}{2} + nc$$

holds.

1.4 Admissible line bundles

Definition 1.4.1. Let X be a complex manifold, and let E be a holomorphic vector bundle on X. A *Hermitian metric* $\langle \cdot, \cdot \rangle$ assigns for every $P \in X$ a Hermitian inner product $\langle \cdot, \cdot \rangle_P$ on the fiber E_P, in such a way that for any two local sections $s, t \in E(U)$ ($U \subset X$ open) the complex-valued function $\langle s, t \rangle : P \mapsto \langle s(P), t(P) \rangle_P$ on U is smooth. A *Hermitian line bundle* is a holomorphic line bundle equipped with a Hermitian metric.

If $\langle \cdot, \cdot \rangle$ is a Hermitian metric on E, then we can assign to any local section $s \in E(U)$ a norm $\|s\| = \langle s, s \rangle^{1/2}$.

Suppose now that L is a holomorphic line bundle on X. Then giving a Hermitian metric on L is equivalent to giving for every local section $s \in L(U)$ a function $\|s\| : U \to \mathbb{R}_{\geq 0}$, such that for every holomorphic function f on U the identity $\|f \cdot s\| = |f| \|s\|$ holds, and for every locally generating section $s \in L(U)$ the function $\|s\|$ is positive-valued and smooth. We will use this method of defining a Hermitian metric on a holomorphic line bundle from now on.

If L and M are two holomorphic line bundles on X with Hermitian metrics $\|\cdot\|_L$ and $\|\cdot\|_M$, then $L \otimes M$ is again a line bundle, and it has a Hermitian metric $\|\cdot\|_{L\otimes M}$ given by $\|s \otimes t\|_{L \otimes M} = \|s\|_L \cdot \|t\|_M$.

Let L be a holomorphic line bundle on X. For every non-zero meromorphic section s of L we define a divisor $\mathrm{div}_L(s) = \sum_P n_P P$ of X as follows: if $P \in X$ is any point and t a local generating section of L, then we can write $s = ft$ around P, with f a meromorphic function around P; we define $n_P = \mathrm{ord}_P(f)$. For every non-zero meromorphic function f on X we see that $\mathrm{div}_L(fs) = \mathrm{div}(f) + \mathrm{div}_L(s)$; and if L is of the form $L = O_X(D)$ with D a divisor of X we have

$$\mathrm{div}_{O_X(D)} f = D + \mathrm{div}\, f$$

for all meromorphic functions f on X.

Suppose that we have a Hermitian line bundle L on X. We take a non-zero meromorphic section s of L and consider the current

$$-\tfrac{i}{\pi}\partial\overline{\partial}[\log \|s\|] + \delta_{\mathrm{div}_L\, s}.$$

By 1.1.3, this current does not depend on the choice of s. This leads us to the following definition.

Definition 1.4.2. Let X be a Riemann surface, equipped with a holomorphic line bundle L with a Hermitian metric $\|\cdot\|$. The *curvature* of $\|\cdot\|$ is the degree 2 current $\mathrm{curv}_{\|\cdot\|}$ defined by

$$\mathrm{curv}_{\|\cdot\|} = -\tfrac{i}{\pi}\partial\overline{\partial}[\log \|s\|] + \delta_{\mathrm{div}_L\, s},$$

with s any non-zero meromorphic section of L.

Definition 1.4.3. Let X be a compact and connected Riemann surface of positive genus, with a Hermitian line bundle L with metric $\|\cdot\|$. Then $\|\cdot\|$ is called *admissible* if $\mathrm{curv}_{\|\cdot\|} = (\deg L)[\mu]$. An *admissible line bundle* is a holomorphic line bundle equipped with an admissible metric.

If L and M are equipped with an admissible metric, then clearly the induced metric on $L \otimes M$ is admissible. Every two admissible line bundles are closely related.

Proposition 1.4.4. *If $\|\cdot\|$ and $\|\cdot\|'$ are two admissible metrics on a holomorphic line bundle L, then they are equal up to multiplication by a positive real number.*

Proof. Let s be a non-zero meromorphic section of L. Then:

$$0 = \mathrm{curv}_{\|\cdot\|} - \mathrm{curv}_{\|\cdot\|} = -\frac{i}{\pi}\partial\overline{\partial}[\log \|s\| - \log \|s\|'].$$

Using theorem 1.2.1 with $T = [\log \|s\| - \log \|s\|']$ shows that the current $[\log \|s\| - \log \|s\|']$ is represented by a constant function. Hence $\|s\|$ and $\|s\|'$ differ only up to multiplication with a positive real-valued constant. ☐

The previous proposition limits the number of admissible metrics a holomorphic line bundle can have. On the other hand, the Arakelov-Green function allows us to define an admissible metric on every holomorphic line bundle on a compact connected Riemann surface, as we will see now. As $\mathrm{Cl}(X) \cong \mathrm{Pic}(X)$ every line bundle on X is of the form $\mathcal{O}_X(D)$, with D a divisor on X. We define a smooth metric $\|\cdot\|_{O_X(D)}$ by setting $\log \|1\|_{O_X(D)} = g_{D,\mathrm{Ar}}$. Using lemma 1.2.3 one easily verifies that this indeed defines a smooth metric on X, and that this metric is admissible.

Definition 1.4.5. Let X be a compact connected Riemann surface, and let $D \in \mathrm{Div}(X)$ be a divisor. The metric on $O_X(D)$ defined above is called the *canonical (admissible) metric* on $O_X(D)$.

We will also put a metric on Ω^1_X, as follows. Let Δ be the diagonal inclusion $X \to X \times X$, and let $O_{X\times X}(-\Delta)$ be the sheaf of holomorphic functions vanishing on the diagonal. We will construct an isomorphism $\Delta^* O_{X\times X}(-\Delta) \xrightarrow{\sim} \Omega^1_X$, called the *adjunction isomorphism*. Let U be a chart on X with coordinate z; we define a homomorphism

$$\begin{aligned} O_{U\times U}(-\Delta) &\to \Delta_*(\Omega^1_X)|_{U\times U} \\ (z_1 - z_2)\cdot f(z_1, z_2) &\mapsto f(z,z)dz \end{aligned}$$

If w is another coordinate then $w = h(z)$ for some invertible holomorphic function h. Using a Taylor expansion we find:

$$\begin{aligned} (w_1 - w_2)f(w_1, w_2) &= (h(z_1) - h(z_2))f(h(z_1), h(z_2)) \\ &= (z_1 - z_2)\left(\sum_{k=1}^{\infty} \frac{h^{(k)}(z_2)}{k!}\cdot (z_1 - z_2)^{k-1}\right) f(h(z_1), h(z_2)) \end{aligned}$$

and this is mapped to $h'(z)f(h(z), h(z))dz = f(w,w)dw$, so we see that this does not depend on the choice of coordinate. As Δ^* and Δ_* are adjoints we get a homomorphism $\Delta^* O_{U\times U}(-\Delta) \to \Omega^1_U$ sending $((z_1 - z_2)\cdot f(z_1, z_2))|_\Delta$ to $f(z,z)dz$, and this is even an isomorphism. By gluing we get the adjunction isomorphism. We can define a Hermitian metric on $O_{X\times X}(-\Delta)$ as follows: outside the diagonal we have

$$\log \|1\|(P,Q) = -g_{\mathrm{Ar}}(P,Q).$$

Let U be a chart on X with coordinate z. Then $O_{U\times U}(-\Delta)$ is generated by $z_1 - z_2$. We can write $g_{\mathrm{Ar}}(z_1, z_2) = \log|z_1 - z_2| + f(z_1, z_2)$ with f a smooth function on $U \times U$. Therefore we have the equality

$$\log\|z_1 - z_2\| = \log|z_1 - z_2| + \log\|1\| = \log|z_1 - z_2| - g_{\mathrm{Ar}} = -f(z_1, z_2),$$

so $\|z_1 - z_2\|$ extends to a smooth function on $U \times U$. We get a well-defined smooth Hermitian metric on $O_{U\times U}(-\Delta)$, and this induces a smooth Hermitian metric $\|\cdot\|_{\mathrm{Ar}}$ on Ω^1_X by requiring that the adjunction isomorphism is an isometry. Concretely, if U is a chart of X with coordinate z, then

$$\log\|dz\|_{\mathrm{Ar}}(P) = \log\|z_1 - z_2\|(P,P) = \lim_{Q\to P}(\log|z(P) - z(Q)| - g_{\mathrm{Ar}}(P,Q)).$$

Theorem 1.4.6 ([1])**.** *The metric $\|\cdot\|_{\mathrm{Ar}}$ on Ω^1_X is admissible.*

1.5 Determinant of cohomology

In this section we will define the determinant of cohomology of coherent sheaves on projective schemes over a Dedekind ring or a field. We will only use the determinant of cohomology of line bundles over Riemann surfaces, but the theory generalizes very nicely into a scheme-theoretic language.

Let A be a commutative ring, and let M be a projective A-module of rank r. We define the *determinant* of M to be the locally free A-module of rank 1

$$\det M = \bigwedge^r M.$$

Lemma 1.5.1 ([2, 7.1])**.** *Suppose that*

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

is an exact sequence of finitely generated projective A-modules, with M' and M'' of rank r and s, respectively, then there is a canonical isomorphism of A-modules

$$\det M' \otimes_A \det M'' \xrightarrow{\sim} \det M$$
$$(x_1 \wedge \cdots \wedge x_r) \otimes (y_1 \wedge \cdots \wedge y_s) \mapsto \alpha x_1 \wedge \cdots \wedge \alpha x_r \wedge \tilde{y}_1 \wedge \cdots \wedge \tilde{y}_s,$$

where $\tilde{y}_i$ denotes an arbitrary element of M such that $\beta \tilde{y}_i = y_i$.

If A is a Dedekind ring, we can extend this definition to define the determinant of every finitely generated A-module. A finitely generated A-module is projective if and only if it is torsion-free. If M is a finitely generated A-module, then we can fit M in a short exact sequence $0 \to E \to F \to M \to 0$ with E and F finitely generated projective A-modules, and we can define the determinant of M to be the invertible A-module

$$\det M = \det F \otimes (\det E)^{\vee}.$$

This definition is, up to canonical isomorphism, independent of the choice of exact sequence [2, 7.2]. Every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finitely generated A-modules gives rise to a canonical isomorphism

$$\det M' \otimes_A \det M'' \xrightarrow{\sim} \det M.$$

Let A be a Dedekind ring, let $X = \operatorname{Spec} A$, and let $\mathcal{F}$ be a coherent X-module. Then $\mathcal{F} = \tilde{M}$, where $M = \mathcal{F}(X)$ is a finitely generated A-module. We can define the *determinant* of $\mathcal{F}$ to be the invertible sheaf

$$\det \mathcal{F} = (\det M)^{\sim}.$$

As in the case of modules over a ring, a short exact sequence $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ of coherent X-modules induces a canonical isomorphism

$$\det \mathcal{G} \cong \det \mathcal{F} \otimes_{O_X} \det \mathcal{H}.$$

If $f : X \to Y$ is a morphism of topological spaces, then the direct image functor f_* is a functor $\mathbf{Ab}(X) \to \mathbf{Ab}(Y)$. The functor f_* is left exact and the category $\mathbf{Ab}(X)$ has enough injectives so the right derived functors $R^i f_*$ ($i \geq 0$) of f_* exist. If $f : X \to Y$ is a morphism of ringed spaces, then the functors $R^i f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ coincide on $\mathbf{Mod}(X)$ with the right derived functors of $f_* : \mathbf{Mod}(X) \to \mathbf{Mod}(Y)$. See [6, III.8] for more details.

Suppose that A is a Noetherian ring, let $Y = \operatorname{Spec} A$, and let $f : X \to Y$ be a projective morphism of schemes. For every quasi-coherent O_X-module $\mathcal{F}$ we have $R^i f_* \mathcal{F} \cong (H^i(X, \mathcal{F}))^{\sim}$ (see [6, III.8.5]). If X has dimension n then, by Grothendieck's vanishing theorem [6, III.2.7], we have $R^i f_* \mathcal{F} = 0$ for all $i > n$. If $\mathcal{F}$ is coherent then every $H^i(X, \mathcal{F})$ is a finitely generated A-module, so $R^i f_* \mathcal{F}$ is a coherent O_Y-module [6, III.5.2].

The facts stated in the previous paragraphs allow us to state the following definition.

Definition 1.5.2. Let A be a Dedekind ring or a field, let $f : X \to \operatorname{Spec} A$ be a projective morphism, and let $\mathcal{F}$ be a coherent O_X-module. The *determinant of cohomology* of $\mathcal{F}$ is the line bundle $\det Rf_* \mathcal{F}$ on $\operatorname{Spec} A$ defined by

$$\det Rf_* \mathcal{F} = \bigotimes_{i=0}^{n} (\det R^i f_* \mathcal{F})^{\otimes (-1)^i},$$

where $n = \dim X$.

Similarly, if X is a projective scheme over a field k, then we can define for every coherent O_X-module $\mathcal{F}$ the determinant of cohomology of $\mathcal{F}$ to be the one-dimensional k-vector space

$$\det H(X, \mathcal{F}) = \bigotimes_{i=0}^{n} (\det H^i(X, \mathcal{F}))^{\otimes(-1)^i}.$$

Lemma 1.5.1 generalizes to the determinant of cohomology in a natural way.

Lemma 1.5.3. *Suppose that A is a Dedekind ring or a field, let $f : X \to \operatorname{Spec} A$ be a projective morphism, and let*

$$0 \to \mathcal{F} \xrightarrow{\alpha} \mathcal{G} \xrightarrow{\beta} \mathcal{H} \to 0$$

be a short exact sequence of coherent O_X-modules. Then there is a canonical isomorphism of line bundles on $\operatorname{Spec} A$

$$\det Rf_*\mathcal{G} \cong \det Rf_*\mathcal{F} \otimes_A \det Rf_*\mathcal{H}.$$

Similarly, if k is a field and $f : X \to \operatorname{Spec} k$ is a projective morphism, then every short exact sequence of coherent O_X-modules $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ induces a canonical isomorphism $\det H(X, \mathcal{G}) \cong \det H(X, \mathcal{F}) \otimes_k \det H(X, \mathcal{H})$.

Proof. Consider the long exact sequence of cohomology

$$\dots \xrightarrow{\delta^{i-1}} R^i f_*\mathcal{F} \xrightarrow{\alpha^i} R^i f_*\mathcal{G} \xrightarrow{\beta^i} R^i f_*\mathcal{H} \xrightarrow{\delta^i} \dots$$

This induces short exact sequences

$$0 \to \ker \alpha^i \to R^i f_*\mathcal{F} \xrightarrow{\alpha^i} \operatorname{im} \alpha^i \to 0.$$

As $\operatorname{im} \alpha^i = \ker \beta^i$ we find a natural isomorphism $\det R^i f_*\mathcal{F} \cong \det \ker \alpha^i \otimes_A \det \ker \beta^i$. Similarly, we get natural isomorphisms $\det R^i f_*\mathcal{G} \cong \det \ker \beta^i \otimes_A \det \ker \delta^i$ and $\det R^i f_*\mathcal{H} \cong \det \ker \delta^i \otimes_A \det \ker \alpha^{i+1}$. This induces natural isomorphisms

$$\begin{aligned}
Rf_*\mathcal{F} \otimes_A Rf_*\mathcal{H} &= \left(\bigotimes_{i=0}^{n} (\det R^i f_*\mathcal{F})^{\otimes(-1)^i} \right) \otimes \left(\bigotimes_{i=0}^{n} (\det R^i f_*\mathcal{H})^{\otimes(-1)^i} \right) \\
&\cong \bigotimes_{i=0}^{n} (\det \ker \alpha^i \otimes \det \ker \beta^i \otimes \det \ker \delta^i \otimes \det \ker \alpha^{i+1})^{\otimes(-1)^i} \\
&\cong \bigotimes_{i=0}^{n} (\det \ker \beta^i \otimes \det \ker \delta^i)^{\otimes(-1)^i} \\
&\cong \bigotimes_{i=0}^{n} (\det R^i f_*\mathcal{G})^{\otimes(-1)^i} \\
&= \det Rf_*\mathcal{G}.
\end{aligned}$$

The proof of the second statement is almost completely similar. □

1.6 Faltings metrics on admissible line bundles

Let X be a compact Riemann surface of genus $g \geq 1$. The Jacobian $\mathrm{Jac}(X)$ of X can be written as

$$\mathrm{Jac}(X) \cong \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g),$$

where τ is a complex valued $g \times g$-matrix that is symmetric with positive-definite imaginary part. Define on $\mathbb{C}^g$ a *theta-function* $\vartheta(z;\tau)$ as follows:

$$\vartheta(z;\tau) = \sum_{n \in \mathbb{Z}^g} \exp(\pi i({}^t n)\tau n + 2\pi i({}^t n)z).$$

It is an entire function in the variable z. The theta-function is not invariant under translation by elements from the lattice $\mathbb{Z}^g + \tau\mathbb{Z}^g$ and therefore does not descend to a function on $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$: for all $m \in \mathbb{Z}^g$ we have the identities

$$\begin{aligned} \vartheta(z+m;\tau) &= \vartheta(z;\tau) \\ \vartheta(z+\tau m;\tau) &= \exp(-\pi i({}^t m)\tau m - 2\pi i({}^t m)z) \cdot \vartheta(z;\tau). \end{aligned}$$

However, translation does leave the order of zeroes of ϑ invariant, and therefore defines a divisor Θ_0 on $\mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$. We define a hermitian metric $\|\cdot\|_{\Theta_0}$ on $O(\Theta_0)$ by setting

$$\|1\|_{\Theta_0}(x+iy) = \sqrt[4]{\det(\mathrm{Im}(\tau))} \cdot \exp(-\pi({}^t y)(\mathrm{Im}\,\tau)^{-1}y) \cdot |\vartheta(\tau, x+iy)|.$$

One easily checks that this metric is well-defined and is invariant under translation by elements of $\mathbb{Z}^g + \tau\mathbb{Z}^g$. The metric is uniquely determined by the two properties in the following proposition.

Proposition 1.6.1 ([5]). *The metric $\|\cdot\|_{\Theta_0}$ is characterized by the following two properties:*

- *The curvature form of $\|\cdot\|_{\Theta_0}$ is*

$$\mathrm{curv}_{\|\cdot\|_{\Theta_0}} = \frac{i}{2} \sum_{1 \leq k,l \leq g} (\mathrm{Im}\,\tau)^{-1}_{k,l} dz_k \wedge d\bar{z}_l.$$

- $\frac{1}{g!}\int_{\mathbb{C}^g/(\mathbb{Z}^g+\tau\mathbb{Z}^g)} \|1\|^2_{\Theta_0}\nu^g = 2^{-g/2}$.

A *Theta characteristic* of X is a divisor class $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\mathcal{L}^{\otimes 2} = K \in \mathrm{Pic}(X)$. As $\mathrm{Pic}^0(X) \cong \mathrm{Jac}(X) = \mathbb{C}^g/(\mathbb{Z}^g + \tau\mathbb{Z}^g)$ via the Abel-Jacobi map, we see that $\mathrm{Pic}^0(X)$ is a divisible group, and this implies that the set of Theta characteristics is non-empty. Also, $\mathrm{Pic}^0(X)[2]$ acts on the set of Theta characteristics of X in a natural way, and this makes the set of Theta-characteristics into a $\mathrm{Pic}^0(X)[2]$-torsor. We therefore see that there are 2^{2g} Theta characteristics.

Recall that the Abel-Jacobi map identifies $\mathrm{Jac}(X)$ and $\mathrm{Pic}^0(X)$. Define the divisor $\Theta \subset \mathrm{Pic}_{g-1}(X)$ to be the divisor consisting of the classes of line bundles of degree $g-1$ that admit a global section. The divisors Θ and Θ_0 are related by the following theorem.

Theorem 1.6.2 (Riemann). *There exists a Theta characteristic $\mathcal{L}$ of X such that under the induced isomorphism*

$$\mathrm{Pic}_{g-1}(X) \xrightarrow{\sim} \mathrm{Pic}_0(X) = \mathrm{Jac}(X) : \mathcal{M} \mapsto \mathcal{M} \otimes \mathcal{L}^{\otimes -1}$$

the divisor $\Theta \subset \mathrm{Pic}_{g-1}(X)$ corresponds with the divisor $\Theta_0 \subset \mathrm{Jac}(X)$.

Let L be an admissible line bundle, and let P be a point on X. We have an exact sequence

$$0 \to L(-P) \to L \to L[P] \to 0,$$

where $L[P]$ is the skyscraper sheaf on P associated to the fiber of L above P. The admissible metric on L induces a metric on $L[P]$, and if we equip $O_X(-P)$ with its canonical metric we also obtain a metric on $L(-P)$. We have $H^0(X, L[P]) = L[P]$ and $H^1(X, L[P]) = 0$, so the long exact sequence of cohomology of the above exact sequence gives an isomorphism on the determinants of cohomology

$$\det H(X, L) \cong \det H(X, L(-P)) \otimes_{\mathbb{C}} L[P].$$

We have now acquired enough tools to tackle the following theorem by Faltings.

Theorem 1.6.3 ([5, Theorem 1]). *Let X be a compact Riemann surface of genus $g > 0$. There is a way to assign for every admissible line bundle L on X a metric on the one-dimensional complex vector space $\det H(X, L)$, such that*

1. *For every isometric isomorphism $L \xrightarrow{\sim} M$ of admissible line bundles, the induced isomorphism $\det H(X, L) \to \det H(X, M)$ is an isometry;*
2. *If the metric on L is changed by a factor $\alpha > 0$, the metric on $\det H(X, L)$ is changed by a factor $\alpha^{\chi(L)}$, where*
$$\chi(L) = \dim H^0(X, L) - \dim H^1(X, L);$$
3. *For every admissible line bundle L and every point $P \in X$, the isomorphism $\det H(X, L) \cong \det H(X, L(-P)) \otimes_{\mathbb{C}} L[P]$ induced by the exact sequence*
$$0 \to L(-P) \to L \to L[P] \to 0$$
is an isometry.
4. *The metric on $\det H(X, \Omega_X^1)$ is the metric induced by the canonical inner product on Ω_X^1 (see 1.3) via the natural isomorphism $\det H(X, \Omega_X^1) \cong \bigwedge^g \Omega_X^1(X)$ (Serre duality).*

Notice that the first three items determine the metrics up to a common scalar factor. Item 4 then fixes this scalar factor. If we want to prove the theorem then it suffices to prove that there exist metrics satisfying the first three items.

Proof. The proof consists of two parts. First we will show that metrics can be put on the line bundles of the form $O(D)$, with D an Arakelov divisor of X, such that 2 and 3 hold. Next, we will show that these metrics also satisfy 1, and then our proof is complete, since $\operatorname{Pic} X \cong \operatorname{Cl} X$.

For the first part of the proof, we will proceed as follows. We start out by picking any metric on the determinant $\det H(X, O_X)$ of the trivial line bundle O_X on X with the canonical metric. Now item 3 should allow us to put in a recursive way metrics on the determinants $\det H(X, O_X(D))$, where D is any Weil divisor on X and $O_X(D)$ is equipped with the canonical admissible metric, by adding or subtracting points. We still need to check that this gives well-defined metrics; that is, the metrics obtained in this way do not depend on the order in which we add or subtract points.

Suppose that we are given a divisor D, and two distinct points $P, Q \in X$ together with a metric on $\det H(X, O(D - P - Q))$. Using item 3, we can then put a metric on $\det H(X, O(D))$ in two ways: by first taking the induced metric on $\det H(X, O(D-P))$ and then the induced metric on $\det H(X, O(D))$, or by going via $\det H(X, O(D-Q))$ instead. We have isomorphisms

$$\begin{aligned}\det H(X, O(D)) &\cong \det H(X, O(D-P)) \otimes O(D)[P] \\ &\cong \det H(X, O(D-P-Q)) \otimes O(D-P)[Q] \otimes O(D)[P], \\ &\cong \det H(X, O(D-P-Q)) \otimes O(-P)[Q] \\ &\quad \otimes O(D)[Q] \otimes O(D)[P]\end{aligned}$$

and similarly,

$$\det H(X, O(D)) \cong \det H(X, O(D-P-Q)) \otimes O(-Q)[P] \otimes O(D)[P] \otimes O(D)[Q],$$

both inducing a metric on $\det H(X, O(D))$.

On $O(-P)[Q]$ the canonical norm is defined by

$$\log \|1\|_{O(-P)[Q]} = g_{\mathrm{Ar}}(-P, Q) = -g_{\mathrm{Ar}}(P, Q)$$

and on $O(-Q)[P]$ the canonical norm is defined by

$$\log \|1\|_{O(-Q)[P]} = g_{\mathrm{Ar}}(-Q, P) = -g_{\mathrm{Ar}}(Q, P).$$

As g_{Ar} is symmetric we see that the two induced metrics on $\det H(X, O(D))$ are equal, and therefore the metrics are well-defined.

For the second part, we proceed as follows. Suppose that we have an isometry $O(D_1) \cong O(D_2)$ of two admissible line bundles. After adding or subtracting points we may assume that the degree of D_1 and D_2 equals $g-1$. In this case $\dim H^0(X, O(D_i)) = \dim H^1(X, O(D_i))$ by Riemann-Roch, so the metrics on $\det H(X, O(D_i))$ do not change if we scale the metrics on $O(D_i)$. We can therefore move to the case where $O(D_1) \cong O(D_2)$ as line bundles, and we need to show that this isomorphism induces an isometry on determinants.

Faltings proceeds as follows. For r sufficiently large, there exists a divisor E of degree $r+g-1$ such that D_1 and D_2 are both of the form $E - P_1 - \cdots - P_r$. One can construct a line bundle N on X^r such that the fiber of N above a point $(P_1, \ldots, P_r)$ is the determinant $\det H(X, O(E - P_1 - \cdots - P_r))$ (see [3][p. 297] for a detailed construction). The metrics on the $\det H(X, O(E - P_1 - \cdots - P_r))$ give rise to a metric on N.

Another metric on N is obtained as follows. Consider the morphism

$$\phi : X^r \to \mathrm{Pic}_{g-1}(X)$$

that sends $(P_1, \ldots, P_r)$ to the class of $O(E - P_1 - \cdots - P_r)$. It turns out that we have a canonical isomorphism

$$\phi^*(-\Theta) \cong N,$$

and the metric on $O(\Theta_0)$ defined earlier induces a metric on N via this isomorphism and the one in 1.6.2. One can compare the curvature forms of the two metrics on N that we have defined, something that is done in more detail in [8, Chapter VI], and it turns out that these curvature forms are in fact equal. This implies that the two metrics are equal up to a constant scalar factor, so the metric on N induced by the metrics on the $\det H(X, O(E - P_1 - \cdots - P_r))$ is the pullback of a metric on $\mathrm{Pic}_{g-1}(X)$ over the morphism ϕ. This shows that the Faltings metric on $\det H(X, O(E - P_1 - \cdots - P_r))$ only depends on the image of $O(E - P_1 - \cdots - P_r)$ in $\mathrm{Pic}_{g-1}(X)$, and we have proven the second part. $\square$

Definition 1.6.4. If L is an admissible line bundle on a Riemann surface X, we call the corresponding metric on $\det H(X, L)$ in the previous theorem the *Faltings metric* on $\det H(X, L)$.

It will be useful to know how the Faltings metrics behave with respect to exact sequences of the form

$$0 \to L(-P_1 - \cdots - P_r) \to L \to \bigoplus_{i=1}^{r} L[P_i] \to 0,$$

where L is an admissible line bundle on X, and $P_1, \ldots, P_r$ are distinct points on X. Taking determinants yields a natural isomorphism

$$\det H(X, L) \cong \det H(X, L(-P_1 - \cdots - P_r)) \otimes \bigotimes_{i=1}^{r} L[P_i].$$

We have Faltings metrics on $\det H(X, L)$ and $H(X, L(-P_1 - \cdots - P_r))$, and the metric on L defines a metric on $L[P_i]$. The isomorphism on determinants we found is usually not an isometry.

Proposition 1.6.5. *The isomorphism*

$$\det H(X, L) \xrightarrow{\sim} \det H(X, L(-P_1 - \cdots - P_r)) \otimes \bigotimes_{i=1}^{r} L[P_i]$$

has norm α, where

$$\log \alpha = \sum_{i<j} g_{\mathrm{Ar}}(P_i, P_j).$$

Proof. Consider the exact sequence

$$0 \to L(-P_1 - \cdots - P_r) \to L(-P_1 - \cdots - P_{r-1}) \to L(-P_1 - \cdots - P_{r-1})[P_r] \to 0;$$

by definition of the Faltings metrics we get an isometry

$$\begin{aligned} \det H(X, L(-P_1 - \cdots - P_{r-1}) \cong \det H(X, L(-P_1 - \cdots - P_r)) \\ \otimes L(-P_1 - \cdots - P_{r-1})[P_r], \end{aligned}$$

and the metric on $L(-P_1 - \cdots - P_{r-1})[P_r]$ is given by the isometry

$$L(-P_1 - \cdots - P_{r-1})[P_r] \cong L[P_r] \otimes O(-P_1 - \cdots - P_{r-1})[P_r].$$

The metric on $O(-P_1 - \cdots - P_{r-1})[P_r]$ is given by

$$\log \|1\| = g_{\mathrm{Ar}}(-P_1 - \cdots - P_{r-1}, P_r) = -\sum_{i<r} g_{\mathrm{Ar}}(P_i, P_r),$$

so the natural isomorphism

$$O(-P_1 - \cdots - P_{r-1})[P_r] \to \mathbb{C}$$

has norm $\exp(\sum_{i<r} g_{\mathrm{Ar}}(P_i, P_r))$, and the induced isomorphism

$$\det H(X, L(-P_1 - \cdots - P_{r-1}) \xrightarrow{\sim} \det H(X, L(-P_1 - \cdots - P_r)) \otimes L[P_r]$$

has the same norm. The proposition now follows by induction. $\square$

Suppose that L is an admissible line bundle of degree $g-1$ with no global sections. By Riemann-Roch we find that $H^1(X,L)=0$, so we have a natural isomorphism $\det H(X,L)\cong\mathbb{C}$. We let $\lambda(L)$ denote the norm of 1 under the Faltings metric on $\det H(X,L)\cong\mathbb{C}$. In this case, the natural isomorphism $\det H(X,L)\xrightarrow{\sim}\mathbb{C}$ has norm $\lambda(L)^{-1}$.

Corollary 1.6.6. *If L is an admissible line bundle of degree $r+g-1$ with $r\geq 1$, and $P_1,\ldots,P_r$ are distinct points on X such that $L(-P_1-\cdots-P_r)$ has no global sections, then the isomorphism*

$$\det H^0(X,L)=\det H(X,L)\xrightarrow{\sim}\bigotimes_{i=1}^{r}L[P_i]$$

induced by the isomorphism

$$H^0(X,L)\xrightarrow{\sim}\bigoplus_{i=1}^{r}L[P_i]:s\mapsto(s(P_1),\ldots,s(P_r))$$

has norm α, where

$$\log\alpha=\sum_{i<j}g_{\mathrm{Ar}}(P_i,P_j)-\log\lambda(L(-P_1-\cdots-P_r)).$$

1.7 Faltings metrics on elliptic curves

Let X be a compact connected Riemann surface of genus $g=1$. Recall the function λ defined in the previous section, that assigns to an admissible bundle of degree $g-1=0$ with no global sections a positive real number $\lambda(L)$, defined to be the norm of 1 under the Faltings metric on $\det H(X,L)\cong\mathbb{C}$. In this section, we will compute the constant $\lambda(\Omega^1_X(Q-R))$ for points Q,R on X such that $\Omega^1_X(Q-R)$ has no global sections.

The *Dedekind η-function* is the function η on the upper half plane of $\mathbb{C}$ given by the following product:

$$\eta(\tau)=e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi i\tau n}).$$

We also define

$$\|\eta\|(\tau)=\sqrt[4]{\operatorname{Im}\tau}\cdot|\eta(\tau)|.$$

Let $\tau=a+bi\in\mathbb{C}$ with $b>0$, and consider the elliptic curve $X=\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. The following lemma by Faltings gives an explicit formula for the metric on Ω^1_X. The proof of this theorem uses the theory of modular forms and is omitted.

Theorem 1.7.1 ([5, p. 417])**.** *Let z be the coordinate on $X=\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. The metric on Ω^1_X is given by*

$$\left\|dz/\sqrt{b}\right\|=\frac{1}{2\pi\|\eta\|(\tau)^2};$$

Let $Q \in X$ be any point, and let $R \in X$ be another point such that the admissible line bundle $\Omega^1_X(Q-R) = \Omega^1_X \otimes O_X(Q-R)$ has no global sections. Then $H^1(X, \Omega^1_X(Q)) = H^0(X, O(-Q)) = 0$, and $\dim H^0(X, \Omega_X(Q)) = 1$ by Riemann-Roch. We have an exact sequence

$$0 \to \Omega^1_X(Q-R) \to \Omega^1_X(Q) \to \Omega^1_X(Q)[R] \to 0$$

inducing an isomorphism on determinants

$$H^0(X, \Omega_X(Q)) \xrightarrow{\sim} \Omega^1_X(Q)[R] : \omega \mapsto \omega(R)$$

of norm $\lambda(\Omega_X(Q-R))^{-1}$. Moreover, the natural isomorphism $O_X[R] \xrightarrow{\sim} O_X(Q)[R]$ has norm $\exp g(Q,R)$ by definition of the canonical admissible metrics on O_X and $O(Q)$. Tensoring this isomorphism with $\Omega^1_X[R]$ shows that the natural isomorphism $\Omega^1_X[R] \xrightarrow{\sim} \Omega^1_X(Q)[R]$ has norm $\exp g(Q,R)$.

Theorem 1.7.2. *Let Q and R be any two points in X such that $\Omega^1_X(Q-R)$ has no global sections. Then*

$$\lambda(\Omega^1_X(Q-R)) = \frac{2\pi \|\eta\|(\tau)^2}{\exp g(Q,R)}.$$

Proof. Consider the exact sequence

$$0 \to \Omega^1_X \to \Omega^1_X(Q) \to \Omega^1_X(Q)[Q] \to 0.$$

This exact sequence induces an isometry

$$\det H(X, \Omega^1_X(Q)) \xrightarrow{\sim} \det H(X, \Omega^1_X) \otimes \Omega^1_X(Q)[Q],$$

and by using the fact that the residue map $\Omega^1_X(Q)[Q] \xrightarrow{\sim} \mathbb{C}$ is an isometry, we obtain a natural isometry

$$\det H(X, \Omega^1_X(Q)) \xrightarrow{\sim} \det H(X, \Omega^1_X).$$

Now consider the exact sequence

$$0 \to \Omega^1_X(Q-R) \to \Omega^1_X(Q) \to \Omega^1_X(Q)[R] \to 0.$$

We obtain an isometry

$$\det H(X, \Omega^1_X(Q)) \xrightarrow{\sim} \det H(X, \Omega^1_X(Q-R)) \otimes \Omega^1_X(Q)[R].$$

We have chosen R in such a way that $H^0(X, \Omega^1_X(Q-R)) = 0$, and the natural isomorphism $\det H(X, \Omega^1_X(Q-R)) \xrightarrow{\sim} \mathbb{C}$ has norm $\lambda(\Omega^1_X(Q-R))^{-1}$. We therefore obtain a natural isomorphism $\det H(X, \Omega^1_X(Q)) \xrightarrow{\sim} \Omega^1_X(Q)[R]$ of norm $\lambda(\Omega^1_X(Q-R))^{-1}$. The canonical isomorphism

$$\Omega^1_X(Q)[R] \xrightarrow{\sim} \Omega^1_X[R]$$

has norm $\exp(g(Q,R))^{-1}$: it is obtained by tensoring the inverse of the natural isomorphism $O_X[R] \to O_X(Q)[R]$ (which has norm $\exp(g(Q,R))$) with $\Omega^1_X[R]$. Finally, we have a natural isomorphism $\det H(X, \Omega^1_X) = H^0(X, \Omega^1_X) \to \Omega^1_X[R]$ given by $\omega \mapsto \omega(R)$. We will compute the norm of this isomorphism. Let $\omega =$

$dz/\sqrt{b}$; one easily checks that ω forms an orthonormal basis for the canonical inner product on $H^0(X, \Omega^1_X)$, and the Faltings metric on $H^0(X, \Omega^1_X)$ is defined in such a way that the norm of ω equals 1. By the previous theorem, the norm of $\omega(R)$ in $\Omega^1_X[R]$ is equal to

$$\frac{1}{2\pi \|\eta\|(\tau)^2},$$

and the norm of the isomorphism $\det H(X, \Omega_X) \xrightarrow{\sim} \Omega_X[R]$ is equal to this value. We get a commutative diagram of natural isomorphisms:

$$\begin{tikzcd}
\det H(X, \Omega^1_X(Q)) \arrow[r] \arrow[d] & \Omega^1_X(Q)[R] \arrow[d] \\
\det H(X, \Omega^1_X) \arrow[r] & \Omega^1_X[R].
\end{tikzcd}$$

By comparing the norms of the isomorphisms the theorem follows. □

Using Elkies' upper bound, we can give a lower bound for $\lambda(L)$ for all admissible line bundles L on X of degree 0. First of all, notice that if L_1 and L_2 are two admissible line bundles of degree 0 on X, isomorphic as line bundles on X (so the isomorphism need not be an isometry!), then the Faltings metrics on $\det H(X, L_1)$ and $\det H(X, L_2)$ agree. This follows from the second item of 1.6.3. We therefore see that $\lambda(L_1) = \lambda(L_2)$. In other words, for every admissible line bundle L on X of degree 0 with no global sections, the value $\lambda(L)$ depends only on the isomorphism class of L as a line bundle.

Let L be an admissible line bundle on X with no global sections, and let O be the zero of X. We consider the map

$$X \to \mathrm{Pic}^0(X) : Q \mapsto [L(O - Q)].$$

This map is a translation of the surjective Abel-Jacobi map, so there exists a point $Q \in X$ with $[L(O - Q)] = [\Omega^1_X]$. Moreover, we have $Q \neq O$, since Ω^1_X has global sections, and L has not. We therefore see that $L \cong \Omega^1_X(Q - O)$ as line bundles on X, and hence we see that

$$\lambda(L) = \lambda(\Omega^1_X(Q - O)) = \frac{2\pi \|\eta\|(\tau)^2}{\exp g(Q, O)}.$$

Let c be a constant as in Elkies' theorem. We then see that

$$g(Q, O) \leq \tfrac{1}{2} \log 2 + c,$$

so we obtain the following theorem.

Theorem 1.7.3. *Let X be a Riemann surface of the form $X = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with $\mathrm{Im}\,\tau > 0$, and let c be a constant as in 1.3.4. For every admissible line bundle L of degree 0 with no global sections we have*

$$\lambda(L) \geq 2\pi \|\eta\|(\tau)^2 \exp(-\tfrac{1}{2} \log 2 - c).$$

Suppose that $D > 0$ is a positive divisor of degree n on X. Using Riemann-Roch we see that $H^0(X, O_X(D))$ has dimension n and $H^1(X, O_X(D)) = 0$, so we get

$$\det H(X, O_X(D)) = \Lambda^n H^0(X, O_X(D)).$$

We can use the lower bound for $\lambda(O_X(D))$ in 1.7.3 to give the lower bound 1.7.5 for the Faltings metric $\|\cdot\|_{\mathrm{Fal}}$ on $\det H(X, O_X(D))$.

Let $\langle\cdot,\cdot\rangle_{O_X(D)}$ denote the canonical admissible metric on $O_X(D)$. This induces an inner product $(\cdot,\cdot)_D$ on $H^0(X, O_X(D))$ given by

$$(f, g)_D = \int_X \langle f, g\rangle_{O_X(D)}\mu,$$

where μ is the canonical volume form on X. We let $f_1, \ldots, f_n$ be an orthonormal basis of $H^0(X, O_X(D))$ with respect to this inner product. For every n-tuple of points $(P_1, \ldots, P_n)$ we consider the homomorphism

$$H^0(X, L) \to \bigoplus_{i=1}^{n} L[P_i] : f \mapsto (f(P_1), \ldots, f(P_n)).$$

Taking the n-th external power induces a homomorphism

$$\det H(X, L) \to \bigotimes_{i=1}^{n} L[P_i],$$

and we let $\det(f_i(P_j))$ denote the image of $f_1 \wedge \cdots \wedge f_n$ under this homomorphism. We let $U \subset X^n$ denote the open subset of points $(P_1, \ldots, P_n)$ such that $P_1, \ldots, P_n$ are pairwise different, and $O_X(D - P_1 - \cdots - P_n)$ has no global sections. The complement $X^n \setminus U$ is a closed subset of codimension 1. For all $(P_1, \ldots, P_n) \in U$ we have

$$\|\det(f_i(P_j))\| = \|f_1 \wedge \cdots \wedge f_n\|_{\mathrm{Fal}} \cdot \frac{\exp(\sum_{i<j} g_{\mathrm{Ar}}(P_i, P_j))}{\lambda(L(-P_1 - \cdots - P_r))},$$

By squaring both sides, using 1.7.3 and Elkies' inequality, we find that

$$\|\det(f_i(P_j))\|^2 \leq \|f_1 \wedge \cdots \wedge f_n\|^2_{\mathrm{Fal}} \cdot \frac{\exp(\frac{n \log n}{2} + nc + \log 2 + 2c)}{4\pi^2\|\eta\|(\tau)^4}$$

By using the following lemma we can deduce a formula for $\|f_1 \wedge \cdots \wedge f_n\|^2_{\mathrm{Fal}}$.

Lemma 1.7.4.

$$\int_{X^n} \|\det(f_i(P_j))\|^2 \mu(P_1) \wedge \ldots \mu(P_n) = n!.$$

Proof. For simplicity, we write $\langle\cdot,\cdot\rangle$ for the Hermitian metric $\langle\cdot,\cdot\rangle_{O_X(D)}$, and $(\cdot,\cdot)$ for the inner product $(\cdot,\cdot)_D$ on $H^0(X, O_X(D))$. We can rewrite $\det(f_i(P_j))$ as follows:

$$\begin{aligned}
\det(f_i(P_j)) &= (f_1(P_1), \ldots, f_1(P_n)) \wedge \cdots \wedge (f_n(P_1), \ldots, f_n(P_n)) \\
&= \sum_{\sigma} \mathrm{sgn}(\sigma) f_{\sigma 1}(P_1) \wedge \cdots \wedge f_{\sigma n}(P_n),
\end{aligned}$$

where σ ranges over the permutation group of $\{1, \ldots, n\}$. We now have

$$\begin{aligned}
\|\det(f_i(P_j))\|^2 &= \langle \det(f_i(P_j)), \det(f_i(P_j))\rangle \\
&= \sum_{\sigma,\tau} \mathrm{sgn}(\sigma\tau)\langle f_{\sigma 1}(P_1) \wedge \cdots \wedge f_{\sigma n}(P_n), f_{\tau 1}(P_1) \wedge \cdots \wedge f_{\tau n}(P_n)\rangle \\
&= \sum_{\sigma,\tau} \mathrm{sgn}(\sigma\tau) \prod_{i=1}^{n} \langle f_{\sigma i}(P_i), f_{\tau i}(P_i)\rangle
\end{aligned}$$

Taking the integral, we find

$$\begin{aligned}\int_{X^n} \|\det(f_i(P_j))\|^2 \mu(P_1) \wedge \cdots \wedge \mu(P_n) &= \sum_{\sigma,\tau} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{n} \int_X \langle f_{\sigma i}(P_i), f_{\tau i}(P_i)\rangle \mu(P_i) \\ &= \sum_{\sigma,\tau} \operatorname{sgn}(\sigma\tau) \prod_{i=1}^{n} (f_{\sigma i}, f_{\tau i})\end{aligned}$$

and as $f_1, \ldots, f_n$ is an orthonormal basis, we see that

$$\prod_{i=1}^{n} \langle f_{\sigma i}, f_{\tau i}\rangle = \prod_{i=1}^{n} \delta_{\sigma i, \tau i} = \delta_{\sigma,\tau}$$

so the integral becomes equal to the order of the permutation group of $\{1, \ldots, n\}$, completing the proof. □

If we use this lemma together with the inequality before it, we find as a result the following theorem.

Theorem 1.7.5. *Suppose that $D > 0$ is a positive divisor on the elliptic curve $X = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ ($\tau \in \mathbb{C}, \operatorname{Im}\tau > 0$) of degree n. Let $f_1, \ldots, f_n$ be an orthonormal basis of $H^0(X, O_X(D))$ with respect to the inner product $(\cdot,\cdot)_D$. Let c be a constant as in 1.3.4, and let $\|\cdot\|_{\mathrm{Fal}}$ be the Faltings metric on $\det H(X, O_X(D)) = \Lambda^n H^0(X, O_X(D))$. Then the following inequality holds:*

$$\|f_1 \wedge \cdots \wedge f_n\|^2_{\mathrm{Fal}} \geq \frac{4\pi^2 \|\eta\|(\tau)^4 n!}{\exp(\frac{n \log n}{2} + nc + \log 2 + 2c)}.$$

2 Arithmetic surfaces and Néron models

This chapter will serve as a brief overview of the theory of arithmetic surfaces. An arithmetic surface is a regular scheme that can be viewed as a family of curves over varying fields. For example, if K is a number field with ring of integers O_K, and E/K is an elliptic curve, we may want to reduce the curve modulo the primes of O_K, and store the resulting curves in an arithmetic surface. However, doing this the naive way will often result in singularities. These singularities can be resolved under some assumptions. Next, we will look at semi-stable arithmetic surfaces. The notion of semi-stability here is a generalization of the notion of semi-stability of elliptic curves over local fields or number fields. We will look how semi-stable arithmetic surfaces behave under extensions of fields. Finally we will introduce group schemes and Néron models of abelian varieties.

Let k be a field. An *affine variety over* k is the affine k-scheme associated to a finitely generated k-algebra. An *algebraic variety over* k is a k-scheme X that has a finite open covering by affine k-varieties. A *projective variety over* k is a projective k-scheme. A *curve over* k is an algebraic variety over k whose irreducible components are one-dimensional.

2.1 Arithmetic surfaces and regular models

In this section, let S be a Dedekind scheme of dimension 1. That is, S is a normal integral locally Noetherian scheme of dimension 1. Let $K = \kappa(S)$ be its fraction field. In our scenario S will often be the spectrum of the ring of integers of a number field. The generic point of S will be denoted by η.

Definition 2.1.1. A *fibered surface* is an integral, projective, flat S-scheme $X \to S$ of dimension 2. An *arithmetic surface* is a regular fibered surface.

One can view an arithmetic surface as a family of (not necessarily smooth) curves X_s over the fields $\kappa(s)$, where s ranges over the closed points of S, together with a 'generic' curve X_η over the field $\kappa(\eta)$. Later on, we will see that there is an interesting intersection theory on regular surfaces over the spectrum of the ring of integers of a number field, which we will use to prove the main inequality in this thesis.

Some properties of X_η are inherited by the closed fibers.

Proposition 2.1.2. *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme S of dimension 1. Let $s \in S$. Then the following properties are true.*

1. *The fiber X_s is a projective curve over $\kappa(s)$, and we have $p_a(X_s) = p_a(X_\eta)$.*
2. *If X_η is geometrically connected then the same holds for X_s.*
3. *If X_η is geometrically regular, then the canonical homomorphism $O_S \to \pi_* O_X$ is an isomorphism.*
4. *Suppose that X is regular. Then the morphisms $X \to S$ and $X_s \to \operatorname{Spec} \kappa(s)$ are l.c.i.'s, and we have the relation $\omega_{X_s/\kappa(s)} = \omega_{X/S}|_{X_s}$ between the canonical sheaves ([9, 6.4.7]).*

Irreducible Weil divisors on a fibered surface come in two shapes. If D is an irreducible Weil divisor then the composition $D \to S$ is a proper morphism, so it either is a constant morphism or a surjective one. In the first case D is an irreducible component of a closed fiber of $X \to S$, and in the second case D is the closure in X of a closed point on the generic fiber of $X \to S$.

Theorem 2.1.3 ([9, 8.3.4])**.** *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme of dimension 1.*

- *Let $P \in X_\eta$ be a closed point. Then its closure $\overline{\{P\}}$ in X is an irreducible closed subset of X, finite and surjective to S.*
- *Let $D \subset X$ be an irreducible closed subset of dimension 1. Then D is either an irreducible component of a closed fiber, or $D = \overline{\{P\}}$, where P is a closed point of X_η.*

Example 2.1.4. Suppose that $S = \operatorname{Spec} A$ is affine, and let $X \to S$ be an arithmetic surface. Let P be a point on the generic fiber of X, and let $L = \kappa(P)$, and $D = \overline{\{P\}}$. Then $D \to S$ is finite and surjective, and $D \times_S \operatorname{Spec} K = \operatorname{Spec} \kappa(P) = \operatorname{Spec} L$. This implies that D is affine, that $A \subset O_D(D) \subset L$, that $O_D(D)$ is finitely generated as an A-module, and that $O_D(D) \otimes_A K = L$. In other words, $O_D(D)$ is an A-order in L.

In particular, if $\kappa(P) = K$ then D is the image of a section $S \to X$ of the morphism $X \to S$. In this case, we see that for every $s \in S$, $X_s \cap D$ is reduced to a point $p \in X_s(\kappa(s))$, and X_s is smooth at p (see [9, 9.1.32]), so p belongs to a single irreducible component of X_s which is of multiplicity 1 in X_s. We obtain a natural map $X_\eta(K) \to X_s(\kappa(s))$ sending $P \in X_\eta(K)$ to $\overline{\{P\}} \cap X_s$.

Definition 2.1.5. An irreducible Weil divisor D of X is called *horizontal* if it is the closure of a closed point of the generic fiber of X, and *vertical* if it is an irreducible component of a closed fiber. More generally, a Weil divisor is called *horizontal* (resp. *vertical*) if all its irreducible components are.

One question we will consider is the following: if C is an algebraic curve over K, then does there exist an arithmetic surface $X \to S$ such that its generic fiber is isomorphic to C? For example, suppose that $E = \operatorname{Proj} \mathbb{Q}[T_1, T_2, T_3]/(F)$ is an elliptic curve over $\mathbb{Q}$, given as the zero locus of an irreducible homogeneous polynomial $F \in \mathbb{Z}[T_1, T_2, T_3]$ in the projective plane. We want to construct an arithmetic surface $X \to \operatorname{Spec} \mathbb{Z}$ with generic fiber $X_\eta = E$. A logical first candidate would be the scheme $X = \operatorname{Proj} \mathbb{Z}[T_1, T_2, T_3]/(F)$. This scheme is integral, projective, and flat ([9, 4.3.10]) over $\operatorname{Spec} \mathbb{Z}$, but unfortunately it is not necessarily regular. For example, let $F = T_2^2 T_3 - T_1^3 - p^2 T_3^3$. An affine open subset of X is then $\operatorname{Spec} \mathbb{Z}[x, y]/(y^2 - x^3 - p^2)$. Consider the point $P \in X$ given by the maximal ideal (x, y, p) of X. The local ring $O_{X,P}$ has maximal ideal $\mathfrak{m} = (x, y, p)$, and $\mathfrak{m}^2 = (x^2, xy, xp, y^2 = x^3 + p^2, yp, p^2) = (x^2, xy, xp, yp, p^2)$, and we see that $\dim \mathfrak{m}/\mathfrak{m}^2 > 2$, as x, y, p are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. Therefore P is a singular point, and $X \to \operatorname{Spec} \mathbb{Z}$ is not an arithmetic surface.

One can, however, show that singularities such as the one in the previous paragraph can be resolved under some nice assumptions.

Definition 2.1.6. Let X be a reduced locally Noetherian scheme. A proper birational morphism $\pi : Z \to X$ with Z regular is called a *desingularization of* X. If, moreover, π is an isomorphism above every regular point of X, we call π a *desingularization in the strong sense.* A *minimal desingularization of* X is a desingularization $Z \to X$ such that every other desingularization $Z' \to X$ factors uniquely through $Z' \to Z \to X$.

Proposition 2.1.7. *Let $X \to S$ be a fibered surface over an affine Dedekind scheme S of dimension 1. Then every desingularization $Y \to X$ is an arithmetic surface $Y \to S$.*

Proof. As $Y \to X$ is birational, Y is an irreducible scheme of dimension 2. Moreover Y is regular, hence normal, so Y is integral. The morphism $Y \to X$ is proper and birational and hence surjective, and $X \to S$ is surjective too, so $Y \to S$ is a non-constant morphism from an integral scheme to a Dedekind scheme, and therefore flat ([9, 4.3.10]). By flatness and surjectivity of $Y \to S$ all its fibers have dimension 1, the morphism $Y \to S$ is proper since $Y \to X$ and $X \to S$ are. The morphism $Y \to S$ is a proper flat morphism to an affine Dedekind scheme with fibers of dimension 1 and Y regular, and therefore $Y \to S$ is projective ([9, 8.3.16]). $\square$

By its universal property, a minimal desingularization, if it exists, is unique up to a unique isomorphism. If a normal fibered surface has a desingularization, then it has a minimal desingularization, as the following theorem will state.

Let $X \to S$ be an arithmetic surface. We call a prime divisor E on X an *exceptional divisor* if there exists an arithmetic surface $Y \to S$ and a morphism $f : X \to Y$ of S-schemes, such that $f(E)$ is reduced to a point and $f : X \setminus E \to Y \setminus f(E)$ is an isomorphism. Such a morphism f is then called a *contraction* of E.

Theorem 2.1.8 ([9, 9.3.32])**.** *Let $Y \to S$ be a normal fibered surface. If Y admits a desingularization, then it admits a minimal desingularization. More precisely, if $X \to Y$ is a desingularization such that no exceptional divisor of X is contained in the exceptional locus of $X \to Y$, then it is a minimal desingularization.*

The proof of this theorem shows that, given a desingularization $Z \to Y$ of a normal fibered surface $Y \to S$, the minimal desingularization of $Y \to S$ is obtained by successive contractions of exceptional divisors in the exceptional locus of $Z \to Y$. In particular we see that if $Z \to Y$ is a desingularization in the strong sense, then the minimal desingularization of $Y \to S$ is a desingularization in the strong sense.

If S is an *excellent* scheme (see below), then fibered surfaces over S have desingularizations.

Theorem 2.1.9 ([9, 10.3.45])**.** *Let S be an excellent scheme. Suppose that $X \to S$ is a fibered surface. Then X admits a desingularization in the strong sense.*

An excellent scheme is a scheme with an affine open covering $\{U_i\}$, such that each $O_X(U_i)$ is an excellent ring. The definition of an excellent ring goes beyond the scope of this thesis. It can be found in [9, 8.2.35]. Examples of excellent rings are:

- complete, Noetherian local rings (and in particular fields);
- Dedekind domains of characteristic 0;
- all localizations of excellent rings;
- all finitely generated algebras over excellent rings.

The most important thing to remember here is that the ring of integers of a number field is excellent.

Definition 2.1.10. An arithmetic surface $X \to S$ is *relatively minimal* if it has no exceptional divisors. We say that $X \to S$ is *minimal* if every birational map of regular fibered S-surfaces $Y \dashrightarrow X$ is a birational morphism.

Proposition 2.1.11 ([9, 9.3.24])**.** *Let $X \to S$ be an arithmetic surface. If $X \to S$ is minimal, then it is relatively minimal. Conversely, if $X \to S$ is relatively minimal and $p_a(X_\eta) \geq 1$, then $X \to S$ is minimal.*

Proposition 2.1.12 ([9, 9.3.26])**.** *Let $X \to S$ be an arithmetic surface with $p_a(X_\eta) \geq 1$. Let $K_{X/S}$ be a canonical divisor. Then $X \to S$ is minimal if and only if $K_{X/S} \cdot C \geq 0$ for every vertical prime divisor C of X.*

Definition 2.1.13. Let C be a normal, connected, projective curve over K. A *model of C over S* is a normal fibered surface $X \to S$ together with an isomorphism of K-schemes $X_\eta \cong K$. If, moreover, X is regular (resp. regular and minimal), we call $X \to S$ a *regular model of C* (resp. a *minimal regular model of C*). A *morphism* of two models $X \to X'$ of two models of C is a morphism of S-schemes $X \to X'$ compatible with the isomorphisms $X_\eta \cong C \cong X'_\eta$.

If a regular model exists then it is usually not unique: we can obtain new regular models by blowing up closed points or contracting exceptional divisors. Therefore we can not say a lot about the structure of these regular models. Minimal regular models are much more interesting. In the next section we will see a classification of the geometric fibers of minimal regular models of semi-stable elliptic curves over number fields.

Let us reformulate the question stated earlier in this section. Given a Dedekind scheme S of dimension 1, and a normal, connected, projective curve C over the fraction field of S, does there always exist a regular model of C over S? And does a minimal regular model exist? The next theorem gives a positive answer to these questions under the circumstances we will encounter in the next chapters.

Proposition 2.1.14 ([9, 10.1.8])**.** *Suppose that S is affine. Let C be a smooth projective curve over $\kappa(S)$. Then C admits a regular model over S. If, moreover, $p_a(C) \geq 1$, then C admits a minimal regular model over S.*

In general the minimal regular model is not stable under base change. The following proposition shows that the minimal regular model does behave nicely under étale base changes and completions.

Proposition 2.1.15 ([9, 10.1.17])**.** *Let S be a Dedekind scheme of dimension 1. Let C be a smooth projective curve over $K = \kappa(S)$ of positive genus, admitting a minimal regular model X over S. Let S' be a Dedekind scheme of dimension 1 that is étale over S or equal to $\operatorname{Spec} \widehat{O}_{S,s}$, with $s \in S$ a closed point. Let $K' = \kappa(S')$. Then $X \times_S S'$ is the minimal regular model of $C_{K'}$ over S', where $C_{K'}$ is the K'-curve obtained by base change $\operatorname{Spec} K' \to \operatorname{Spec} K$.*

This theorem shows that minimal regular models can be computed one closed point at a time. It often allows us to reduce proving statements about minimal regular models in general to proving these statements for minimal regular models of curves over local fields only. The following propositions allow us to reduce even further to the case where the residue field of this local field is separably closed. We say that a local ring $(R, \mathfrak{m})$ *dominates* a local ring $(S, \mathfrak{n})$ if $R \subset S$ and $R \cap \mathfrak{n} = \mathfrak{m}$.

Proposition 2.1.16 ([9, 10.3.32])**.** *Let O_K be a dvr with field of fractions K and residue field k, and let k' be a (not necessarily finite) algebraic extension of k. Then there exists a dvr O_L that dominates O_K, with residue field k', ramification index $e(O_L/O_K) = 1$, and such that L is separable algebraic over K.*

Proposition 2.1.17 ([9, 10.3.33])**.** *Let O_K be a dvr with residue field k, and let O_L be a dvr that dominates O_K, with field of fractions L algebraic over K. Then the following properties are true.*

1. *For every projective scheme $X \to \operatorname{Spec} O_L$, there exists a discrete valuation subring $O_{K'}$ of O_L with K' finite over K, and a projective scheme $X' \to \operatorname{Spec} O_{K'}$ such that $X \cong X' \times_{\operatorname{Spec} O_{K'}} O_L$.*
2. *Suppose moreover that L/K is separable, that $e(O_L/O_K) = 1$ and that the residue field of O_L is separable algebraic over k. Let C be a smooth projective curve over K. Then the formation of the minimal regular model of C over O_K (if they exist) commutes with the base change $\operatorname{Spec} O_L \to \operatorname{Spec} O_K$.*

2.2 Semi-stability

Let X be a reduced curve over a field k, and let $\pi : X' \to X$ be its normalization. This morphism is finite by [9, 4.1.27]. We define a coherent sheaf $\mathcal{S}$ on X using the exact sequence

$$0 \to O_X \to \pi_* O_{X'} \to \mathcal{S} \to 0.$$

As π is an isomorphism above the regular points of X, we see that $\mathcal{S}$ is supported on the singular points of X. For every $P \in X$ define $\delta_P = \operatorname{length}_{O_{X,P}} \mathcal{S}_P$. We see that $\delta_P = 0$ if and only if P is normal (and hence regular).

Definition 2.2.1. Let X be a reduced curve over an algebraically closed field k, and let $\pi : X' \to X$ be the normalization. A closed point $P \in X$ is an *ordinary multiple point* if $\delta_P = m_P - 1$, where $m_P = \#(\pi^{-1}(P))$. If, moreover, $m_x = 2$ we call x an *ordinary double point* or a *node*.

Definition 2.2.2. An algebraic curve C over an algebraically closed field k is called *semi-stable* if it is reduced, and if its singular points are ordinary double points. A curve C over an arbitrary field k is *semi-stable* if its extension $C_{\bar{k}}$ to the algebraic closure $\bar{k}$ of k is semi-stable. We call an S-scheme $f : X \to S$ *semi-stable* if f is flat, and if for every $s \in S$ the fiber X_s is a semi-stable curve over $\kappa(s)$.

Semi-stable schemes are stable under base change.

Proposition 2.2.3 ([9, 10.3.15(a)])**.** *If $X \to S$ is semi-stable, and $S' \to S$ a morphism, then $X \times_S S' \to S'$ is semi-stable.*

This proposition, together with propositions 2.1.15, 2.1.16 and 2.1.17, helps us with classifying the geometric fibers of semi-stable minimal regular arithmetic surfaces. Let S be a one-dimensional Dedekind scheme with perfect residue fields, and suppose that $X \to S$ is a minimal arithmetic surface. Let $s \in S$ be a closed point. Then the base change $X \times_S \operatorname{Spec} \widehat{O}_{S,s}$ of $X \to S$ over $\operatorname{Spec} \widehat{O}_{S,s}$ is minimal by 2.1.15. The residue field of $\widehat{O}_{S,s}$ is $\kappa(s)$, and by 2.1.16 there exists a dvr O_L that dominates O_K, has residue field $\overline{\kappa(s)}$, ramification index 1 over $\widehat{O}_{S,s}$, and fraction field L separable algebraic over K. Using 2.1.17 we now see that the base change $X \times_{\operatorname{Spec} \widehat{O}_{S,s}} O_L \to \operatorname{Spec} O_L$ is still minimal and semi-stable, and its closed fiber is the geometric fiber of $X \to S$ over s.

Theorem 2.2.4 ([9, 7.5.4])**.** *Let X be a reduced projective curve over a field k. Let $\Gamma_1, \dots, \Gamma_n$ be the irreducible components of X. Then*

$$p_a(X) + n - 1 = \sum_i p_a(\Gamma_i') + \sum_{P \in X} [\kappa(P) : k]\delta_P,$$

where Γ_i' is the normalization of Γ_i.

Example 2.2.5. We will classify the geometric fibers of semi-stable minimal regular models of genus 1 smooth, projective, geometrically connected curves over a Dedekind ring with perfect residue fields. As proven in the above discussion we may restrict ourselves to the case where K is a complete local field with ring of integers O_K and with algebraically closed residue field k, and C is a smooth geometrically connected projective curve of genus 1 over K. Let X be its minimal regular model over O_K. We consider the special fiber X_s. Let $\Gamma_1, \dots, \Gamma_n$ be its irreducible components. Using 2.2.4 we find the following identity:

$$n = \sum_i p_a(\Gamma_i') + \#\{\text{singular points of } X\}.$$

If $n = 1$ then we see that either X_s is smooth, or X_s has one node, and the normalization of X_s is $\mathbb{P}^1_k$. Suppose that $n \geq 2$. Consider the graph $\mathcal{G}$ with nodes $\Gamma_1, \dots, \Gamma_n$, and an edge between Γ_i and Γ_j if and only if $\Gamma_i \cap \Gamma_j \neq \emptyset$. As X_s is connected, $\mathcal{G}$ is connected too. Every edge of this graph corresponds to a singular point of X_s. We therefore see that $\mathcal{G}$ has at least $n-1$ edges, and at most n edges. Suppose that $\mathcal{G}$ has $n-1$ edges. The graph $\mathcal{G}$ then has at least two nodes of degree 1. By looking at the above formula, we see that either X_s has a component with a singular point, or all components of X_s are smooth and exactly one component has positive arithmetic genus. In any case, we see that there is a node in $\mathcal{G}$ of degree 1 such that the corresponding component Γ_i is smooth with arithmetic genus 0, and therefore isomorphic to $\mathbb{P}^1_k$. It intersects only one other component transversally, so its (classical) self-intersection is -1, and by Castelnuovo's criterion we find that Γ_i is an exceptional divisor, which contradicts the minimality of $X \to \operatorname{Spec} O_K$. We find that $\mathcal{G}$ has n edges, and every component is isomorphic to $\mathbb{P}^1_k$. Every vertex of $\mathcal{G}$ has degree 2: if not, $\mathcal{G}$ has a vertex of degree 1, and then the corresponding component is an exceptional divisor. We therefore see that $\mathcal{G}$ is a cyclic graph, so X_s is an n-gon of $\mathbb{P}^1_k$'s intersecting transversally.

So we find that X_s is one of the following:

- A smooth elliptic curve over k, or
- An n-gon of $\mathbb{P}^1_k$'s meeting transversally, where an 1-gon is understood to be an irreducible curve with a single node with normalization $\mathbb{P}^1_k$

Let $S' \to S$ be a morphism of Dedekind schemes of dimension 1. Suppose that $X \to S$ is a semi-stable arithmetic surface. The base change $X \times_S S'$ is again semi-stable, but it need not be an arithmetic surface anymore. We can try to solve this issue by taking the minimal desingularization $X' \to X \times_S S'$. It turns out that this minimal desingularization is still semi-stable. The geometric fibers of X' may not be isomorphic to those of X anymore, but they can still be described rather nicely.

Theorem 2.2.6. *Let $\lambda : S' \to S$ be a finite dominant morphism of one-dimensional Dedekind schemes, let $X \to S$ be a semi-stable arithmetic surface, and $X' \to X \times_S S'$ be the minimal desingularization. Then $X' \to S'$ is semi-stable. Suppose $s' \in S'$ is a closed point; write $s = \lambda(s')$, and let $e = e(s'/s) = e(\widehat{O}_{S',s'}/\widehat{O}_{S,s})$. Then the geometric fiber of X' above s' is obtained from the geometric fiber of X above s by replacing each double point by a chain of $e-1$ projective lines, of multiplicity 1, meeting transversally.*

Proof. This follows from 10.3.21, 10.3.22 and 10.3.25 in [9]. □

In the semi-stable case, the canonical sheaf behaves nicely with respect to base changes.

Proposition 2.2.7 ([8, V.5.5])**.** *Let O_K be a Dedekind ring with characteristic 0 and perfect residue fields, and denote its field of fractions by K. Let $S = \operatorname{Spec} O_K$ and let $X \to S$ be a semi-stable arithmetic surface. Suppose that L/K is a finite extension. We let O_L denote the integral closure of O_K in L, and define $S' = \operatorname{Spec} O_L$. We let $X' \to S'$ be the minimal desingularization of the base change $X \times_S S' \to S'$, and let $r : X' \to X$ be the natural projection. Then there exists a canonical isomorphism*

$$r^*\omega_{X/S} \xrightarrow{\sim} \omega_{X'/S'}$$

on X'.

Corollary 2.2.8. *Suppose that we are in the situation of 2.2.7, and suppose that $p_a(X_\eta) \geq 1$. If $X \to S$ is minimal then $X' \to S'$ is minimal.*

Proof. We need to show that $\omega_{X'/S'}$ is numerically effective. If this is not the case, then there exists an effective vertical divisor C on X' such that $\omega_{X'/S'} \cdot C < 0$. By the projection formula ([9, 9.2.12]) and proposition 2.2.7 we find that

$$r_*C \cdot \omega_{X/S} = C \cdot r^*\omega_{X/S} = C \cdot \omega_{X'/S'} < 0$$

so $\omega_{X/S}$ is not an effective vertical divisor on X. This contradicts with the fact that $X \to S$ is minimal. □

2.3 Group schemes

In this section we will repeat some of the basic definitions and properties of group schemes.

Definition 2.3.1. Let S be a scheme. A *group scheme* over S is a scheme $p : G \to S$, together with the following morphisms of S-schemes:

- multiplication: $m : G \times_S G \to G$;
- unit section: $e : S \to G$;
- inverse: $\operatorname{inv} : G \to G$,

such that the following identities hold:

- associativity: $m \circ (m \times \mathrm{id}_G) = m \circ (\mathrm{id}_G \times m) : G \times_S G \times_S G$;
- right-identity: $m \circ (\mathrm{id}_G \times e) = \mathrm{id}_G : G \times_S S \to G$;
- right-inverse: $m \circ (\mathrm{id}_G \times \mathrm{inv}) \circ \Delta_{G/S} = e \circ p : G \to G$.

Let $p_1 : G_1 \to S$ and $p_2 : G_2 \to S$ be two S-schemes, and let m_1 and m_2 denote their multiplication morphisms. A *homomorphism* of group schemes over S is a morphism $f : G_1 \to G_2$ of S-schemes, such that $f \circ m_1 = m_2 \circ (f \times f) : G_1 \times G_1 \to G_2$.

For any S-scheme T, these axioms give $G(T)$ the structure of a group. In fact, by Yoneda's lemma, giving the group scheme structure on an S-scheme G is equivalent to giving a group structure $G(T)$ for every S-scheme T, such that for every S-morphism $T_1 \to T_2$ the induced map $G(T_2) \to G(T_1)$ is a group homomorphism. Similarly, giving a homomorphism $G_1 \to G_2$ of group schemes over S is equivalent to giving, for every S-scheme T, a group homomorphism $G_1(T) \to G_2(T)$, functorial in T. One can now define subgroup schemes in a natural way. If $G \to S$ is a group scheme over S and $S' \to S$ is a morphism then the pullback $G \times_S S'$ is a group scheme over S'.

A group scheme over a field k that is moreover of finite type over k is called an *algebraic group* over k. An *Abelian variety* over k is an algebraic group that is geometrically integral and proper over k.

Proposition 2.3.2. *Abelian varieties are smooth.*

Proof. We may without loss of generality assume that k is algebraically closed, and we therefore need to show that A is regular. As A is Noetherian, it suffices to show that A is regular at its closed points, and by [9, 4.2.21] it has a regular closed point, since A is (geometrically) reduced. Let $x \in A(k)$ be such a point, and let $y \in A(k)$ be another closed point. Consider the point $z = y - x \in A(k)$. Then the isomorphism

$$A \to A \times_k \operatorname{Spec} k \xrightarrow{\mathrm{id}_A \times z} A \times_k A \xrightarrow{m} A$$

sends x to y, so y is regular too. We see that every closed point of A is regular, so A is regular, hence smooth. $\square$

Example 2.3.3.
- An elliptic curve is an abelian variety.
- Consider the scheme $\mathbb{G}_a = \operatorname{Spec} \mathbb{Z}[X]$. For every scheme T we have a natural isomorphism $\mathbb{G}_a(T) \cong O_T(T)$, thus giving $\mathbb{G}_a(T)$ a group structure, and for every morphism $T_1 \to T_2$ of schemes the induced map $\mathbb{G}_a(T_2) \to \mathbb{G}_a(T_1)$ is the homomorphism $O_{T_2}(T_2) \to O_{T_1}(T_1)$. This gives $\mathbb{G}_a$ the structure of a group scheme over $\operatorname{Spec} \mathbb{Z}$. For every scheme S we define the *additive group* over S as the S-group scheme $\mathbb{G}_{a,S} := \mathbb{G}_a \times_{\mathbb{Z}} S$.
- In a similar way we can define the structure of a group scheme on $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[X, 1/X]$ by noticing that for every scheme T one has $\mathbb{G}_m(T) = O_T(T)^*$. For every scheme S we call the S-group scheme $\mathbb{G}_{m,S} := \mathbb{G}_m \times_{\mathbb{Z}} S$ the *multiplicative group* over S.

The following theorem states that the group structure on the scheme $\mathbb{G}_{m,k}$, with k a field, is in fact unique. The analogous result holds for the additive group over a field, but a proof is not given as it is very similar.

Proposition 2.3.4. *Let k be a field, and let T be the scheme* $\operatorname{Spec} k[X, 1/X]$ *endowed with the structure of a group scheme. Then $T \cong \mathbb{G}_{m,k}$ as group schemes.*

Proof. After a change of variables we may assume that the homomorphism $k[X] \to k$ induced by the unit section $e : T \to T$ sends X to 1.

The multiplication morphism $m : T \times_k T \to T$ corresponds to a homomorphism

$$\tilde{m} : k[X, 1/X] \to k[X, 1/X] \otimes_k k[X, 1/X] = k[X_1, 1/X_1, X_2, 1/X_2].$$

Let $P(X_1, X_2)$ be the image of X under this homomorphism. As X is a unit in $k[X, 1/X]$, we see that $P(X_1, X_2)$ is a unit in $k[X_1, 1/X_1, X_2, 1/X_2]$, and it is therefore of the form $aX_1^r X_2^s$, with $a \in k^*$ and $r, s \in \mathbb{Z}$. The right-identity in the group scheme axioms translates to the identity $P(X_1, 1) = X_1$, so $a = 1$ and $s = 1$. Similarly, we have $P(1, X_2) = X_2$, so $r = 1$. We therefore see that $P(X_1, X_2) = X_1 X_2$. The group structure on T agrees with the one on $\mathbb{G}_{m,k}$, and the proof is complete. □

Let G be an algebraic group over a field k. The *identity component G^0 of G* is the connected component of G containing the unit element of G.

Theorem 2.3.5 ([9, 10.2.18])**.** *Let X be an algebraic variety over a field k.*

- *There exists a unique scheme $\pi_0(X)$, finite étale over k, and a morphism $f : X \to \pi_0(X)$, such that the following universal property is satisfied: every k-morphism $X \to Z$ to a finite étale k-scheme Z factors in a unique way as*
$$X \to \pi_0(X) \to Z.$$
- *The morphism $f : X \to \pi_0(X)$ is surjective, and for every $P \in X$ the fiber $X_{f(P)}$ is the connected component of P in X. In particular f induces a bijection between the connected components of X and the points of $\pi_0(X)$.*

The scheme $\pi_0(X)$ is called the *scheme of connected components of X*.

Corollary 2.3.6 ([9, 10.2.21])**.** *Let k be a field.*

- *Let X/k be an algebraic variety. The set of rational points $\pi_0(X)(k)$ corresponds to the connected components of X that are geometrically connected. A connected component containing a k-rational point is geometrically connected.*
- *Let G be an algebraic group over k. Then G^0 is an open algebraic subgroup of G, and the scheme $\pi_0(G)$ is a finite étale algebraic group over k.*

2.4 Néron models

Throughout this section, S denotes a Dedekind scheme of dimension 1, and $K = K(S)$ its function field.

Definition 2.4.1. Let A be an Abelian variety over K. The *Néron model of A over S* is a scheme $N \to S$ which is smooth, separated and of finite type, together with an isomorphism $N_\eta \cong A$, that satisfies the following universal property: for every smooth S-scheme X the canonical map

$$\operatorname{Mor}_S(X, N) \to \operatorname{Mor}_K(X_\eta, A)$$

induced by the isomorphism $N_\eta \cong A$ is bijective.

Usually the isomorphism $N_\eta \cong A$ is omitted. The universal property characterizes the Néron model up to a unique isomorphism. It also gives the Néron model the structure of a group scheme over S. By the universal property of the Néron model, the canonical map

$$N(S) \to A(K)$$

is a group isomorphism. Let $s \in S$ be a closed point. Then the natural map

$$N(S) \to N(\kappa(s))$$

is a group homomorphism too; see [9, 10.2.25]. Therefore we obtain a natural group homomorphism

$$A(K) \to N(\kappa(s)).$$

Theorem 2.4.2 ([10])**.** *The Néron model of every abelian variety over K exists.*

When E is an elliptic curve over K the Néron model can be described as follows.

Theorem 2.4.3 ([9, 10.2.14])**.** *Let E be an elliptic curve over K with minimal regular model X over S. The smooth locus of $X \to S$ is the Néron model of E over S.*

In this case, we have a reduction map $E(K) \to N(\kappa(s))$, and composing this with the inclusion $N \to X$ we get a map $E(K) \to X(\kappa(s))$. This is the reduction map from example 2.1.4.

3 Arakelov intersection theory

Intersection theory of divisors on surfaces is a powerful tool in algebraic geometry. If $X \to \operatorname{Spec}(k)$ is a smooth projective surface over a field k then we can assign to every two Weil divisors an intersection number. This intersection number is invariant under linear equivalence. Similarly, we can define intersection numbers of divisors on arithmetic surfaces; see [9, Section 9.1]. However, this intersection number does not behave nicely with respect to linear equivalence of divisors. For example: if $X \to \operatorname{Spec}\mathbb{Z}$ is an arithmetic surface, then every closed point $s \in \operatorname{Spec}\mathbb{Z}$ is a principal divisor, and hence its pullback X_s is a principal divisor. However, the intersection number of a horizontal divisor $\overline{\{P\}}$ with X_s is equal to $[\kappa(P) : \mathbb{Q}]$ and in particular positive. The problem here is that the base $\operatorname{Spec}\mathbb{Z}$ is not 'compact' anymore. In this chapter we will overcome this issue by 'compactifying' $\operatorname{Spec}\mathbb{Z}$ by adding a so-called 'fiber at infinity'; the Riemann surface $\mathcal{X}$ obtained from the generic fiber of X by the inclusion $\mathbb{Q} \to \mathbb{C}$. By taking into account this fiber at infinity we will find a way to make the intersection theory work with a suitable notion of linear equivalence.

In this chapter we fix a number field K, and we denote its ring of integers by O_K, and define $S = \operatorname{Spec} O_K$. We write S_{fin} for the set of closed points of $\operatorname{Spec} O_K$, or, equivalently, for the maximal ideals of O_K. We let S_∞ denote a maximal set of pairwise non-conjugate embeddings $K \to \mathbb{C}$, or equivalently, the set of equivalence classes of archimedean absolute values on K. For each $\sigma \in S_\infty$ we let K_σ denote the completion of K under the absolute value induced by σ (so $K_\sigma = \mathbb{R}$ if σ is a real embedding, and $K_\sigma = \mathbb{C}$ if σ is a complex embedding), and define $\epsilon_\sigma = [\overline{K}_\sigma : K_\sigma] \in \{1, 2\}$.

3.1 Arakelov divisors on arithmetic surfaces

Let $X \to S$ be an arithmetic surface. For every $\sigma \in S_\infty$ we can take the base change $X \times_\sigma \mathbb{C}$ of X over $\sigma : \operatorname{Spec}\mathbb{C} \to \operatorname{Spec} K$ and take its analytification $(X \times_\sigma \mathbb{C})(\mathbb{C})$ to get a Riemann surface $\mathcal{X}_\sigma$. We extend the notion of Weil divisors on X by also adding such Riemann surfaces as 'divisors at infinity'.

Definition 3.1.1. Let $p : X \to S$ be an arithmetic surface. An *Arakelov divisor* of X is an element of the abelian group

$$\widehat{\operatorname{Div}} X = \operatorname{Div} X \oplus \bigoplus_{\sigma \in S_\infty} \mathbb{R} \cdot \mathcal{X}_\sigma.$$

So every Arakelov divisor D can be written in a unique way as $D = D_{\text{fin}} + D_\infty$, where D_{fin} is a Weil divisor of X and D_∞ is a formal sum $\sum_{\sigma \in S_\infty} \alpha_\sigma \cdot \mathcal{X}_\sigma$ with $\alpha_\sigma \in \mathbb{R}$. An Arakelov divisor D with $D_\infty = 0$ will be called a *finite* divisor, and one with $D_{\text{fin}} = 0$ will be called an *infinite* divisor. A *horizontal* (resp. *vertical*) Arakelov divisor is a horizontal (resp. vertical) finite Arakelov divisor.

Arakelov divisors extend the Weil divisors we know from classical algebraic geometry. The following definition gives the Arakelov-theoretical analogue of principal Weil divisors.

Definition 3.1.2. Assume that $X \to S$ is an arithmetic surface with generic fiber of positive genus. Let $f \in \kappa(X)^*$ be a nonzero rational function. Define

$(f)_{\mathrm{Ar}} = (f)_{\mathrm{fin}} + (f)_\infty$, where $(f)_{\mathrm{fin}} \in \mathrm{Div}(X)$ is the usual principal Weil divisor of f, and

$$(f)_\infty = \sum_{v \in S_\infty} v_\sigma(f) \cdot \mathcal{X}_\sigma,$$

with

$$v_\sigma(f) = -\int_{\mathcal{X}_\sigma} \log |f|_\sigma \cdot \mu_\sigma,$$

where μ_σ is the canonical volume form on $\mathcal{X}_\sigma$. An Arakelov divisor of this form is called a *principal divisor*, and the quotient of $\widehat{\mathrm{Div}}X$ modulo the subgroup of principal divisors, the *Arakelov class group*, is denoted by $\widehat{\mathrm{Cl}}X$. Two Arakelov divisors are called *linearly equivalent* if their difference is a principal divisor.

Recall that in classical algebraic geometry divisors and invertible sheaves are closely related. If a scheme is a Noetherian, integral, separated and regular (e.g. an arithmetic surface), then its divisor class group and Picard group are isomorphic. We will now introduce in an analogous way the Arakelov-Picard group of an arithmetic surface with positive genus and show that it is again isomorphic to the Arakelov class group.

If L is a line bundle on X, and $\sigma : K \to \mathbb{C}$ is a complex embedding, then we can pull back L to obtain a line bundle on $X \times_\sigma \mathbb{C}$, and then its analytification to get a holomorphic line bundle L_σ on the Riemann surface $\mathcal{X}_\sigma$.

For example, suppose that $L = O_X(D)$, with $D = \overline{\{P\}}$ an irreducible horizontal divisor, and let $\tau_1, \ldots, \tau_n : \kappa(P) \to \mathbb{C}$ be the complex embeddings of $\kappa(P)$ extending σ. Then the τ_i induce embeddings $X(\kappa(P)) \to X(\mathbb{C})$; let $P_1, \ldots, P_n$ be the images of P under these embeddings; define $D_\sigma = P_1 + \cdots + P_n$. Then $L_\sigma = O_{\mathcal{X}_\sigma}(D_\sigma)$.

More generally, by setting $D_\sigma = 0$ for vertical divisors D, we can extend bilinearly to define D_σ for all Weil divisors D on X. We see that $(O_X(D))_\sigma = O_{\mathcal{X}_\sigma}(D_\sigma)$.

Definition 3.1.3. Let X be an arithmetic surface with generic fiber of positive genus. An *admissible line bundle* L on X is a line bundle on X together with admissible metrics on L_σ for every $\sigma \in S_\infty$.

If L and M are two admissible line bundles on X then their tensor product is again admissible, and the dual of an admissible line bundle is admissible too.

Definition 3.1.4. Let X be an arithmetic surface with generic fiber of positive genus, and let L and M be admissible line bundles on X. An *isomorphism* of admissible line bundles is an isomorphism $L \to M$ of line bundles such that the induced isomorphisms $L_\sigma \to M_\sigma$ are isometries. The *Arakelov-Picard group* of X is the group $\widehat{\mathrm{Pic}}X$ of isomorphism classes of admissible line bundles on X, where the group operation is taking tensor products.

Given a Weil divisor D on X we can construct a line bundle $O_X(D)$ on X, and this induces an isomorphism $\mathrm{Cl}\, X \xrightarrow{\sim} \mathrm{Pic}\, X$. In a similar way, we can assign to every Arakelov divisor D an admissible line bundle $O_X(D)$, and this will give an isomorphism $\widehat{\mathrm{Cl}}X \xrightarrow{\sim} \widehat{\mathrm{Pic}}X$.

Let D be an Arakelov divisor on X. Write $D = D_{\mathrm{fin}} + \sum_{\sigma \in S_\infty} a_\sigma \cdot \mathcal{X}_\sigma$. The Weil divisor D_{fin} gives rise to a line bundle $O_X(D_{\mathrm{fin}})$. For every $\sigma \in S_\infty$ we let D_σ denote the pull-back of D_{fin} to $\mathcal{X}_\sigma$; then the pull-back of $O_X(D_{\mathrm{fin}})$ to $\mathcal{X}_\sigma$

is $O_{\mathcal{X}_\sigma}(D_\sigma)$. We define an admissible metric on $O_{\mathcal{X}_\sigma}(D_\sigma)$ by multiplying the canonical admissible metric defined in 1.4.5 by $\exp(-a_\sigma)$. This gives $O_X(D)$ the structure of an admissible line bundle.

Conversely, given an admissible line bundle L on X, there exists a Weil divisor D_{fin} on X such that $L \cong O_X(D_{\text{fin}})$. The canonical metrics on the $O_{X_\sigma}(D_\sigma)$ induce canonical metrics on the L_σ, with σ ranging over the infinite places, and the admissible metrics on the L_σ differ from these canonical metric by a positive real scalar multiple (see proposition 1.4.4). We arrive at the following theorem.

Theorem 3.1.5 ([1])**.** *The homomorphism* $\widehat{\mathrm{Div}}X \to \widehat{\mathrm{Pic}}X$ *sending* D *to* $O_X(D)$ *induces a canonical isomorphism* $\widehat{\mathrm{Cl}}X \xrightarrow{\sim} \widehat{\mathrm{Pic}}X$. □

The previous theorem allows us to use the language of Arakelov divisors and admissible line bundles interchangably, something we will gladly do from now on.

One important admissible line bundle on an arithmetic surface is the *canonical line bundle*. It takes the role of the canonical divisor in Arakelov-theoretical variants of classical geometric results such as the Riemann-Roch theorem and the adjunction formula. It is defined as follows.

On X we have a canonical sheaf ω_X that is an invertible sheaf and a 1-dualizing sheaf for $X \to S$. If $\sigma : K \to \mathbb{C}$ is an embedding, then the pullback $\omega_{X,\sigma}$ on $\mathcal{X}_\sigma$ is isomorphic to $\Omega^1_{\mathcal{X}_\sigma}$, and we can therefore equip $\omega_{X,\sigma}$ with the admissible metric $\|\cdot\|_{\mathrm{Ar}}$. In this way we give ω_X the structure of an admissible line bundle.

3.2 Intersection numbers of Arakelov divisors

In classical algebraic geometry, one often looks at the intersection number of two divisors lying on a non-singular projective surface X over a field. A nice property is that these intersection numbers stay the same when one replaces the divisors by linearly equivalent ones. We wish to develop a similar theory for divisors lying on arithmetic surfaces. This is possible in some sense [9, Section 9.1], but the intersection number does not behave nicely when we consider linear equivalence, as the example in the introduction to this chapter already shows.

Again, fix an arithmetic surface $X \to S$ (keep in mind that S is the spectrum of the ring of integers of a number field, as stated in the beginning of this chapter). Suppose that D_1 and D_2 are two effective Weil divisors on X without a common component, and let $x \in X$ be any point. Let f_1 and f_2 be local equations for D_1 and D_2 around x. We define the *local intersection number* of D_1 and D_2 at x to be

$$i_x(D_1, D_2) = \mathrm{length}_{O_{X,x}} O_{X,x}/(f, g).$$

This is a non-negative integer, see [9, Section 9.1.1]. Also i_x is symmetric and bilinear, so we can extend i_x bilinearly to define $i_x(D_1, D_2)$ for any two Weil divisors with no common components. If $s \in S$ is a closed point, we define for every two Weil divisors D_1, D_2 without common component

$$i_s(D_1, D_2) = \sum_{x \in X_s \text{ closed}} i_x(D_1, D_2)[\kappa(x) : \kappa(s)].$$

Now suppose that $\sigma \in S_\infty$ is an infinite place, and let D_1 and D_2 be two distinct irreducible Weil divisors on X. We set

$$i_\sigma(D_1, D_2) = -\epsilon_\sigma \cdot g_\sigma(D_{1,\sigma}, D_{2,\sigma}),$$

where g_σ is the Arakelov-Green function on the Riemann surface $\mathcal{X}_\sigma$. We see that i_σ is symmetric and bilinear. We also see that $i_\sigma(D_1, D_2) = 0$ if D_1 or D_2 is vertical.

Theorem 3.2.1. *[1] There exists a unique symmetric bilinear pairing*

$$(- \cdot -)_{\mathrm{Ar}} : \widehat{\mathrm{Div}} X \times \widehat{\mathrm{Div}} X \to \mathbb{R}$$

that satisfies the following properties:

- *If D_1 and D_2 are finite divisors of X with no common components, then*
$$(D_1 \cdot D_2)_{\mathrm{Ar}} = \sum_{s \in S_{\mathrm{fin}}} i_s(D_1, D_2) \cdot \log \#\kappa(s) + \sum_{\sigma \in S_\infty} i_\sigma(D_1, D_2);$$
- *If D is a finite divisor and $\sigma \in S_\infty$ then*
$$(D \cdot X_\sigma)_{\mathrm{Ar}} = \epsilon_\sigma \cdot \deg_K(D|_{\mathcal{X}_\eta}) = \epsilon_\sigma \cdot \deg_K((D_{\mathrm{hor}})|_{X_\eta});$$
- *For every two $\sigma, \tau \in S_\infty$:*
$$(\mathcal{X}_\sigma \cdot \mathcal{X}_\tau)_{\mathrm{Ar}} = 0;$$
- *If D_1 and D_1' are linearly equivalent as Arakelov divisors then*
$$(D_1 \cdot D_2)_{\mathrm{Ar}} = (D_1' \cdot D_2)_{\mathrm{Ar}} \quad \textit{for all } D_2 \in \widehat{\mathrm{Div}}(X).$$

The symmetric bilinear pairing in the theorem is called the *Arakelov intersection product* or *Arakelov intersection number.*

Remark 3.2.2. Let D_1 be a vertical divisor on X, contained in a closed fiber X_s, and let D_2 be a finite divisor on X. The Arakelov intersection number $(D_1 \cdot D_2)_{\mathrm{Ar}}$ is $\log \#\kappa(s)$ times the 'classical' intersection number of Weil divisors $i_s(D_1, D_2)$. Because of this, a lot of statements about intersection numbers with vertical divisors can be translated easily to the language of Arakelov intersection numbers.

The following proposition shows that the Arakelov intersection product behaves nicely under base change. Suppose that L/K is a finite extension, and let $S' = \operatorname{Spec} O_L$. The base change $X \times_S S'$ is not necessarily regular anymore, but this can be resolved by fixing a desingularization $X' \to X \times_S S'$. We let $r : X' \to X$ be the composition. We can define pullbacks r^*D of Arakelov divisors $D \in \widehat{\mathrm{Div}} X$ as follows: for a finite divisor D the pullback r^*D is the same as the pullback of the Cartier divisor D. The pullback of an infinite fiber $\mathcal{X}_\sigma$ is $\sum_\tau \mathcal{X}_\tau$, with τ ranging over the embeddings $L \to \mathbb{C}$ extending σ.

Proposition 3.2.3 ([4, Proposition 1.3.7])**.** *For every two Arakelov divisors D_1, D_2 on X, the equality*

$$(r^*D_1 \cdot r^*D_2)_{\mathrm{Ar}} = [L : K](D_1 \cdot D_2)_{\mathrm{Ar}}$$

holds.

3.3 The Faltings-Riemann-Roch theorem

In this section, we will give an Arakelov-theoretical analogue of the Riemann-Roch theorem for surfaces. If $X \to \operatorname{Spec} k$ is a smooth projective surface over an algebraically closed field k, then for every divisor D of X we have defined the Euler characteristic $\chi(D)$ as

$$\chi(D) = \dim H^0(X, O_X(D)) - \dim H^1(X, O_X(D)) + \dim H^2(X, O_X(D)).$$

The Riemann-Roch theorem for surfaces tells us that the following identity holds:

$$\chi(D) = \tfrac{1}{2} D \cdot (D - K) + \chi(0),$$

where K is a canonical divisor on X. In order to give the Arakelov-theorerical analogue of this statement, we will define the Euler characteristic $\chi(L)$ for every admissible line bundle L.

Let M be a finitely generated $\mathbb{Z}$-module, and suppose we have a Haar measure on $M \otimes_{\mathbb{Z}} \mathbb{R}$. We then define the *absolute Euler characteristic*

$$\chi(M, \mathbb{Z}) = -\log(\operatorname{vol}(M \otimes_{\mathbb{Z}} \mathbb{R}/M)/\#M_{\text{torsion}}).$$

The ring of integers O_K of K has a natural Euclidean measure on

$$O_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \prod_{\sigma \in S_\infty} K_\sigma.$$

Proposition 3.3.1. *The absolute Euler characteristic of O_K with the natural Euclidean measure on $O_K \times_{\mathbb{Z}} \mathbb{R}$ is*

$$\chi(O_K, \mathbb{Z}) = s \log 2 - \log \sqrt{|\Delta_K|}$$

where Δ_K is the discriminant of K over $\mathbb{Q}$, and s is the number of complex places in S_∞.

Proof. Let $\sigma_1, \dots, \sigma_r$ be the real embeddings in S_∞, and let $\sigma_{r+1}, \dots, \sigma_{r+s}$ be the complex embeddings in S_∞. Let $n = r + 2s$ be the degree of the extension $K/\mathbb{Q}$. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ we have an isometry

$$O_K \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{r+2s}.$$

The embedding $O_K \to O_K \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^n$ sends an element $\alpha \in O_K$ to

$$(\sigma_1(\alpha), \dots, \sigma_r(\alpha), \operatorname{Re}\sigma_{r+1}(\alpha), \operatorname{Im}\sigma_{r+1}(\alpha), \dots, \operatorname{Re}\sigma_{r+s}(\alpha), \operatorname{Im}\sigma_{r+s}(\alpha)).$$

Let $\alpha_1, \dots, \alpha_n$ be a $\mathbb{Z}$-basis of O_K. The volume of a fundamental domain of the lattice O_K in $O_K \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ is equal to

$$\left| \det \begin{bmatrix} \sigma_1(\alpha_1) & \cdots & \sigma_r(\alpha_1) & \operatorname{Re}\sigma_{r+1}(\alpha_1) & \operatorname{Im}\sigma_{r+1}(\alpha_1) & \cdots & \operatorname{Re}\sigma_{r+s}(\alpha_1) & \cdots & \operatorname{Im}\sigma_{r+s}(\alpha_1) \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ \sigma_1(\alpha_n) & \cdots & \sigma_r(\alpha_n) & \operatorname{Re}\sigma_{r+1}(\alpha_n) & \operatorname{Im}\sigma_{r+1}(\alpha_n) & \cdots & \operatorname{Re}\sigma_{r+s}(\alpha_n) & \cdots & \operatorname{Im}\sigma_{r+s}(\alpha_n) \end{bmatrix} \right|.$$

Now notice that

$$\begin{pmatrix} z & \bar{z} \end{pmatrix} = \begin{pmatrix} \operatorname{Re} z & \operatorname{Im} z \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \quad \text{for all } z \in \mathbb{C}$$

so if we let M denote the block diagonal matrix with on the diagonal r blocks of the form $\begin{pmatrix}1\end{pmatrix}$ followed by s blocks of the form

$$\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$$

we see that by multiplying the previous large matrix with M on the right gives us the matrix

$$\begin{bmatrix}\sigma_1(\alpha_1) & \cdots & \sigma_r(\alpha_1) & \sigma_{r+1}(\alpha_1) & \bar{\sigma}_{r+1}(\alpha_1) & \cdots & \sigma_{r+s}(\alpha_1) & \cdots & \bar{\sigma}_{r+s}(\alpha_1)\\ \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \ldots & \vdots\\ \sigma_1(\alpha_n) & \cdots & \sigma_r(\alpha_n) & \sigma_{r+1}(\alpha_n) & \bar{\sigma}_{r+1}(\alpha_n) & \cdots & \sigma_{r+s}(\alpha_n) & \cdots & \bar{\sigma}_{r+s}(\alpha_n)\end{bmatrix}.$$

The absolute value of the determinant of this matrix is $\sqrt{|\Delta_K|}$, so we find

$$\mathrm{vol}(O_K\otimes_{\mathbb{Z}}\mathbb{R}/O_K) = |\det M|^{-1}\cdot\sqrt{|\Delta_K|} = \left|\det\left(\begin{smallmatrix}1 & 1\\ i & -i\end{smallmatrix}\right)\right|^{-s}\cdot\sqrt{|\Delta_K|} = 2^{-s}\cdot\sqrt{|\Delta_K|},$$

completing the proof. □

Suppose that K is a number field, and let M be an O_K-module. We define the *relative Euler characteristic* as

$$\chi(M, O_K) = \chi(M,\mathbb{Z}) - \mathrm{rank}_{O_K}(M)\cdot\chi(O_K,\mathbb{Z}).$$

Let $X\to S$ be an arithmetic surface with generic fiber of positive genus, and let L be an admissible line bundle on X. For every $\sigma\in S_\infty$ we have defined a Faltings metric on $\det H(\mathcal{X}_\sigma, L_\sigma)$. This metric defines metrics on $\det H^0(\mathcal{X}_\sigma, L_\sigma)$ and $\det H^1(\mathcal{X}_\sigma, L_\sigma)$, defined up to a common scalar, and hence we get Haar measures on $H^0(\mathcal{X}_\sigma, L_\sigma)$ and $H^1(\mathcal{X}_\sigma, L_\sigma)$, again defined up to a common scalar. We have $H^i(\mathcal{X}_\sigma, L_\sigma) = H^i(X, L)\otimes_{O_K}\overline{K}_\sigma$, so we get Haar measures on $H^i(X,L)\otimes_{O_K}K_\sigma$, by using the isomorphism $H^i(X,L)\otimes_{O_K}\overline{K}_\sigma = H^i(X,L)\otimes_{O_K}K_\sigma\oplus H^i\otimes_{O_K}iK_\sigma$ if K_σ is real. Now we get Haar measures on

$$H^i(X,L)\otimes_{\mathbb{Z}}\mathbb{R}\cong\prod_{\sigma\in S_\infty}H^i(X,L)\otimes_{O_K}K_\sigma,$$

still defined up to a common scalar. The quantity

$$\chi(H^0(X,L),O_K) - \chi(H^1(X,L),O_K)$$

does not depend on the choice of the common scalar factor, and this allows us to state the following definition.

Definition 3.3.2. Let $X\to S$ be an arithmetic surface with generic fiber of positive genus, and let L be an admissible line bundle on X. The real number

$$\chi(L) := \chi(H^0(X,L),O_K) - \chi(H^1(X,L),O_K)$$

defined in the previous paragraph is called the *Euler characteristic* of L. Similarly, if D is an Arakelov divisor on X, we let $\chi(D)$ denote the Euler characteristic of the admissible line bundle $O_X(D)$.

Theorem 3.3.3 (Faltings-Riemann-Roch, [5, Theorem 3])**.** *Let $X\to S$ be an arithmetic surface with generic fiber of positive genus, and let D be an Arakelov divisor on X. Then*

$$\chi(D) = \tfrac{1}{2}(D\cdot D-\omega_X)_{\mathrm{Ar}} + \chi(O_X).$$

3.4 The adjunction formula

The adjunction formula in algebraic geometry is the following: if $X \to \operatorname{Spec} k$ is a smooth surface over an algebraically closed field, and C is a nonsingular curve of genus g on X, then

$$(C + K) \cdot C = 2g - 2,$$

where K is a canonical divisor on X. The Arakelov-theoretical version of the adjunction formula gives a formula for the intersection number $(D + \omega_X \cdot D)_{\mathrm{Ar}}$ for every irreducible horizontal divisor D.

Let $X \to S$ be an arithmetic surface with generic fiber of genus ≥ 1, let $D = \overline{\{P\}}$ be an irreducible horizontal divisor, and define $L = \kappa(P)$. By 2.1.3, the morphism $D \to S$ is finite and surjective. In particular it is affine, so D is affine; let $R = O_D(D)$. The inclusion $\operatorname{Spec} L \to D$ of the generic point is dominant, as is $D \to S$, so on the level of rings we get inclusions $O_K \subset R \subset L$, and R is finite as O_K-module. Moreover, we see that $R \otimes_{O_K} K = L$ so $R \subset O_L$ is an O_K-order in L. We define the *relative discriminant* $\Delta_{R/K}$ of R to be the ideal of O_K generated by the elements of the form $\operatorname{disc}_{L/K}(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in R \subset O_L$.

Theorem 3.4.1 (Adjunction formula, [3, Proposition 4.1])**.** *Let $X \to S$ be an arithmetic surface with generic fiber of positive genus, and let $D = \overline{\{P\}}$ be an irreducible horizontal divisor. Define $L = \kappa(P)$, let n be the degree of L/K, and let $R = O_D(D)$ be the O_K-order in L corresponding to D (see 2.1.4). For every $\sigma \in S_\infty$ write $P_{\sigma,1}, \ldots, P_{\sigma,n}$ for the n points on $\mathcal{X}_\sigma$ defined by the n embeddings $L \to \mathbb{C}$ extending σ. Then*

$$(D + \omega_X \cdot D)_{\mathrm{Ar}} = \log \left| N_{K/\mathbb{Q}}(\Delta_{R/K}) \right| - \sum_{\sigma \in S_\infty} \epsilon_\sigma \sum_{i \neq j} g_\sigma(P_{\sigma,i}, P_{\sigma,j}).$$

If D is the image of a section $s : S \to X$ of $p : X \to S$, then the field L in the theorem is equal to K and $D = \operatorname{Spec} O_K$, and the adjunction formula takes the following form:

Corollary 3.4.2. *Let $X \to S$ be an arithmetic surface, and let $D = \overline{\{P\}}$ be an irreducible horizontal divisor with $\kappa(P) = K$. Then*

$$(D + \omega_X \cdot D)_{\mathrm{Ar}} = 0.$$

4 Elliptic curves

In the final chapter we will use Arakelov intersection theory to put an estimate on splitting fields of torsors of semi-stable elliptic curves over number fields. In the first two sections we will show that, given an elliptic curve E over a number field K, the order of every locally trivial K-torsor C of E in the Tate-Shafarevich group $Ш(E/K)$ is equal to the minimal degree of a splitting field for C.

4.1 The Weil-Châtelet group

In this section we will fix a perfect field k, and let $G_k = \mathrm{Gal}(\bar{k}/k)$ denote its absolute Galois group.

Definition 4.1.1. Let A be an Abelian variety over k. A *k-torsor* of A is a nonempty algebraic scheme X over k together with a k-morphism

$$a : X \times_k A \to X,$$

called the *A-action* on X, such that the induced morphism $(\mathrm{pr}_1, a) : X \times_k A \to X \times_k X$ is an isomorphism. A morphism of A-torsors $X \to X'$ is a morphism of k-schemes compatible with the A-action on X and X', that is, a k-morphism $\phi : X \to X'$ such that the following diagram commutes:

$$\begin{tikzcd} X \times_k A \arrow[r, "(\phi,\mathrm{id}_A)"] \arrow[d] & X' \times_k A \arrow[d] \\ X \arrow[r, "\phi"] & X'. \end{tikzcd}$$

The *Weil-Châtelet group* $\mathrm{WC}(A/k)$ of A is the set of isomorphism classes of k-torsors of A.

If T is any k-scheme such that $X(T)$ is nonempty, then the morphism

$$a_T : X(T) \times A(T) \to X(T)$$

defines a $A(T)$-action on $X(T)$; the induced isomorphism $X(T) \times A(T) \to X(T) \times X(T) : (x, g) \mapsto (x, x + g)$ then forces the action to be free and transitive. Composing the inverse of this isomorphism with the projection $X(T) \times A(T) \to A(T)$ gives a map $X(T) \times X(T) \to A(T)$ sending a pair (t, t') to the unique element $(t' - t) \in A(T)$ satisfying $t + (t' - t) = t'$.

Proposition 4.1.2. *Let X be a k-torsor of A. Then X is isomorphic to A as k-schemes if and only if $X(k) \neq \emptyset$.*

Proof. One implication is trivial. Suppose that $X(k) \neq \emptyset$. Then there exists a morphism $\mathrm{Spec}\, k \to X$, fix such a morphism from now on. This induces a morphism $f : A = \mathrm{Spec}\, k \times_k A \to X \times_k A$. Composing this with the morphism $X \times_k A \to X$ gives a k-morphism $A \to X$. Conversely, the morphism $\mathrm{Spec}\, k \to X$ defines a morphism $X = \mathrm{Spec}\, k \times_k X \to X \times_k X$. Composing this with the inverse of the isomorphism $X \times_k A \to X \times_k X$ and the projection $X \times_k A \to A$ gives a k-morphism $g : X \to A$. It remains to show that $fg = \mathrm{id}_X$ and $gf = \mathrm{id}_A$.

Let T be any other k-scheme. We will denote the composition $T \to \operatorname{Spec} k \to X$ by $x \in X(T)$. Recall that we have an $A(T)$-action on $X(T)$ defined by the k-morphism $X \times_k A \to X$. The morphism $f(T) : A(T) \to X(T)$ sends $a \in A(T)$ to $x+a \in X(T)$. Conversely, one can show that the morphism $g(T) : X(T) \to A(T)$ sends $y \in X(T)$ to the unique $a \in A(T)$ with $x + a = y$. We therefore see that $f(T)g(T) = \mathrm{id}_{X(T)}$ and $g(T)f(T) = \mathrm{id}_{A(T)}$. An application of Yoneda's lemma then proves that $fg = \mathrm{id}_X$ and $gf = \mathrm{id}_A$, so X and A are isomorphic as k-schemes. □

Notice that $\mathrm{WC}(A/k)$ is called the Weil-Châtelet *group* of A over k, while it is only defined as a set of equivalence classes of k-torsors. The following theorem shows that we can put a group structure on $\mathrm{WC}(A/k)$ in a natural way.

Theorem 4.1.3. *Let A be an abelian variety over k. There is a natural well-defined bijection*

$$\mathrm{WC}(A/k) \xrightarrow{\sim} H^1(G_k, A(\bar{k}))$$

defined as follows: for a k-torsor C of A we pick a point $x_0 \in A(\bar{k})$ and let

$$[C] \mapsto [\sigma \mapsto x_0^\sigma - x_0].$$

Proof. Silverman ([12, X.3.6]) proves this for elliptic curves; the proof carries over to arbitrary abelian varieties. □

If l/k is a field extension, then for every k-torsor X of A the base change X_l of X over $\operatorname{Spec} l \to \operatorname{Spec} k$ is an l-torsor for A_l. If we fix an embedding $\bar{k} \to \bar{l}$ then we get a group homomorphism

$$\mathrm{WC}(A/k) \to \mathrm{WC}(A/l)$$

that corresponds to the homomorphism $H^1(G_k, A(\bar{k})) \to H^1(G_l, A(\bar{l}))$ induced by the restriction homomorphism $\mathrm{Gal}(\bar{l}/l) \to \mathrm{Gal}(\bar{k}/k)$ and the inclusion $A(\bar{k}) \subset A(\bar{l})$. This group homomorphism is natural; that is, it does not depend on the choice of embedding $\bar{k} \to \bar{l}$ (see [11, VII.5.3]).

Proposition 4.1.4. *The Weil-Châtelet group* $\mathrm{WC}(A/k)$ *is torsion. More precisely: if X is a k-torsor of A and l/k is a finite field extension such that X has an l-rational point then the order of the class of X in* $\mathrm{WC}(A/k)$ *divides* $[l : k]$.

Proof. Let X be a k-torsor of A. There exists a finite extension l/k such that X has an l-rational point. Consider the restriction and corestriction homomorphisms

$$H^1(G_{\bar{k}/k}, A(\bar{k})) \xrightarrow{\mathrm{Res}} H^1(G_{\bar{k}/l}, A(\bar{k})) \xrightarrow{\mathrm{Cor}} H^1(G_{\bar{k}/k}, A(\bar{k})).$$

By [11, VII.7.6] the composition is multiplication by $[l : k]$. Using the isomorphism in 4.1.3 we get homomorphisms

$$\mathrm{WC}(A/k) \to \mathrm{WC}(A/l) \to \mathrm{WC}(A/k)$$

where the composition is multiplication by $[l : k]$. However, the class of X in $\mathrm{WC}(A/k)$ gets mapped to zero by the restriction map $\mathrm{WC}(A/k) \to \mathrm{WC}(A/l)$. Therefore this class has finite order dividing $[l : k]$. □

Definition 4.1.5. Let A be an abelian variety over k, and let C be a k-torsor for A. The *period* of C is the order of $[C]$ in $\mathrm{WC}(A/k)$. The *index* of C is the smallest $m \geq 1$ such that there exists a field extension l/k of degree m such that $C(l) \neq \emptyset$.

By 4.1.4 we immediately see that the period of a k-torsor for A divides its index.

From now on we will only look at torsors of elliptic curves. In this case the period and index turn out to be closely related to degrees of divisors. Notice that (torsors of) elliptic curves are integral and regular, so we can identify Weil and Cartier divisors.

Definition 4.1.6. Let C be a regular integral curve over a field k, and let $D = \sum_P n_P P$ be a divisor of C (here P ranges over the closed points of C). The k-degree of D, denoted by $\deg_k D$, is the integer

$$\deg_k D = \sum_P n_P [k(P) : k].$$

This integer depends on the base field k. The k will be omitted if there is no ambiguity.

Consider the curve $C_{\bar{k}} = C \times_k \bar{k}$. The Galois group $G_k = \mathrm{Gal}(\bar{k}/k)$ acts on $C_{\bar{k}}$, and hence on $\mathrm{Div}(C_{\bar{k}})$.

Proposition 4.1.7. *Let E/k be an elliptic curve, and let C be a k-torsor of E. The following three integers are equal:*

- *The index of C/k;*
- *The smallest $m > 0$ such that there exists $D \in \mathrm{Div}(C)$ with $\deg_k D = m$;*
- *The smallest $m > 0$ such that there exists $D \in \mathrm{Div}(C_{\bar{k}})^{G_{\bar{k}}}$ with $\deg_{\bar{k}} D = m$.*

Proof. Denote the three integers in the lemma by m_1, m_2, m_3, respectively. Let l/k be a field extension of degree m_1 with $C(l) \neq \emptyset$, and let $P \in C(l)$ be a closed point. Let $D = \{P\}$ be the corresponding divisor of C. We then have $\deg_k D = [k(P) : k] = [l : k] = m_1$ by minimality of the degree of l/k, so we see that $m_2 \leq m_1$. Conversely, let $D \in \mathrm{Div}(C)$ be a divisor with $\deg_k D = m_2$. Using Riemann-Roch we may assume that D is effective, and by minimality of m_2 we can even assume that D is irreducible, say $D = \{P\}$. Now $[k(P) : k] = \deg_k D = m_2$, so this shows that $m_2 \leq m_1$.

Let again l/k be an extension of degree m_1 with $C(l) \neq \emptyset$, and let $P \in C(l)$ be any point. Let $D \in \mathrm{Div}(C_{\bar{k}})$ be the divisor obtained by adding all conjugates of P under the action of G_k. By the orbit-stabilizer theorem we find $\deg_{\bar{k}} D = [k(P) : k]$, so $m_3 \leq m_1$. Conversely, let $D \in \mathrm{Div}(C_{\bar{k}})^{G_{\bar{k}}}$. Using Riemann-Roch we may assume that D is effective, and the orbit of a point P in its support is at most $\deg D$. Using the orbit-stabilizer theorem again we find that $m_1 \leq [k(P) : k] \leq m_3$. □

The period of an E-torsor has a similar characterization. In order to prove it we first need the following theorem.

Theorem 4.1.8 ([12, X.3.8])**.** *Let E be an elliptic curve over k, let C be a k-torsor of E, and let $x_0 \in C(\bar{k})$ be any point. Consider the group homomorphism*

$$\mathrm{sum} : \mathrm{Div}^0(C_{\bar{k}}) \to E(\bar{k}) : \sum_x n_x x \mapsto \sum_x n_x (x - x_0).$$

This homomorphism is a homomorphism of $G_{\bar{k}}$-modules that does not depend on the choice of x_0, and there is an exact sequence

$$1 \to \bar{k}^* \to \kappa(C_{\bar{k}})^* \xrightarrow{\mathrm{div}} \mathrm{Div}^0(C_{\bar{k}}) \xrightarrow{\mathrm{sum}} E(\bar{k}) \to 0.$$

Hence it induces an isomorphism of $G_{\bar{k}}$-modules

$$\mathrm{Pic}^0(C_{\bar{k}}) \xrightarrow{\sim} E(\bar{k}),$$

and in particular, an isomorphism

$$\mathrm{Pic}^0(C_{\bar{k}})^{G_k} \xrightarrow{\sim} E(k).$$

Proposition 4.1.9. *Let E/k be an elliptic curve, and let C be a k-torsor of E. Then the following two integers are equal:*

- *The period of C;*
- *The smallest $m > 0$ such that there exists a divisor class in $\mathrm{Pic}(C_{\bar{k}})^{G_k}$ of degree m.*

Proof. Consider the short exact sequence of G_k-modules

$$0 \to \mathrm{Pic}^0(C_{\bar{k}}) \to \mathrm{Pic}(C_{\bar{k}}) \xrightarrow{\deg} \mathbb{Z} \to 0.$$

Taking an isomorphism $\mathrm{Pic}^0(C_{\bar{k}}) \cong E(\bar{k})$ as in the previous theorem and then taking Galois cohomology yields the exact sequence

$$0 \to E(k) \to \mathrm{Pic}(C_{\bar{k}})^{G_k} \to \mathbb{Z} \to \mathrm{WC}(E/k)$$

where the morphism $\mathbb{Z} \to \mathrm{WC}(E/k)$ is given by $1 \mapsto [C]$. By exactness of the sequence the desired result follows. □

We can use these propositions to find a relation between the period and index of a k-torsor of an elliptic curve.

Theorem 4.1.10. *Let E/k be an elliptic curve, and let C be a k-torsor of E. If the natural homomorphism $\mathrm{Div}(C_{\bar{k}})^{G_k} \to \mathrm{Pic}(C_{\bar{k}})^{G_k}$ is surjective, then the period of C equals the index of C.*

Proof. The first statement follows from 4.1.7 and 4.1.9. We get a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Div}(C_{\bar{k}})^{G_k} \arrow[r, "\deg"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "="] & \mathbb{Z}/(\text{index of } C)\mathbb{Z} \arrow[r] \arrow[d] & 0 \arrow[d, "="] \\
\mathrm{Pic}(C_{\bar{k}})^{G_k} \arrow[r, "\deg"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/(\text{period of } C)\mathbb{Z} \arrow[r] & 0.
\end{tikzcd}$$

If the leftmost vertical arrow is surjective, then the third vertical arrow from the left is injective; this follows from the four lemma. □

If the period and index are both equal to 1 then the converse of the last statement in this theorem holds as well, as the following proposition shows.

Proposition 4.1.11. *Let C be a k-curve of genus 1 with $C(k) \neq \emptyset$. Then the map* $\mathrm{Div}(C_{\bar{k}})^{G_{\bar{k}}} \to \mathrm{Pic}(C_{\bar{k}})^{G_{\bar{k}}}$ *is surjective.*

Proof. Fix a point $x \in C(k)$. Let $[D] \in \mathrm{Pic}(C_{\bar{k}})^{G_k}$ be a divisor class, represented by $D \in \mathrm{Div}(C_{\bar{k}})$. The divisor $D+(1-\deg_{\bar{k}} D)x \in \mathrm{Div}(C_{\bar{k}})^{G_k}$ has degree 1 so by Riemann-Roch it is linearly equivalent to an effective divisor of degree 1, that is, there exists a point $y \in C(\bar{k})$ with $[D]+(1-\deg_{\bar{k}} D)[x] = [y] \in \mathrm{Pic}(C_{\bar{k}})$. For every $\sigma \in G_k$ we have $[D]^\sigma = [D]$ by assumption and $x^\sigma = x$ since $x \in C(k)$, so

$$[y]^\sigma = [D]^\sigma + (1 - \deg D)[x]^\sigma = [D] + (1 - \deg D)[x] = [y].$$

Hence there exists some $f \in \kappa(C_{\bar{k}})^*$ with $\mathrm{div}\, f = y^\sigma - y$, and this f is contained in $L(y)$. By Riemann-Roch $L(y)$ is one-dimensional, and it clearly contains the constant functions, so f is constant, and $y^\sigma = y$. We see that

$$[D] = [y - (1 - \deg D)x] \in \mathrm{Pic}(C_{\bar{k}})$$

and that $y-(1-\deg D)x$ is in $\mathrm{Div}(C_{\bar{k}})^{G_k}$, so the homomorphism $\mathrm{Div}(C_{\bar{k}})^{G_{\bar{k}}} \to \mathrm{Pic}(C_{\bar{k}})^{G_{\bar{k}}}$ is surjective. □

4.2 The Tate-Shafarevich group

Throughout this section we will fix a number field K. We define $S = \mathrm{Spec}\, O_K$, let S_{fin} denote the set of closed points of S, and S_∞ denote the set of infinite places. For every $s \in S_{\mathrm{fin}} \cup S_\infty$ we let K_s denote the completion of K with respect to the absolute value corresponding to s.

Definition 4.2.1. Let E be an elliptic curve over K. The *Tate-Shafarevich group* $Ш(E/K)$ is the subgroup of $\mathrm{WC}(E/K)$ given by

$$Ш(E/K) = \bigcap_{s \in S_{\mathrm{fin}} \cup S_\infty} \ker\left(\mathrm{WC}(E/K) \to \mathrm{WC}(E/K_s)\right).$$

A non-trivial element of $Ш(E/K)$ is a K-curve C isomorphic to E over $\bar{K}$ (but not over K itself!), that has a K_s-rational point for every place s of K, but no K-rational point. Such a curve fails the Hasse principle, so the Tate-Shafarevich group measures the failure of the Hasse principle for K-torsors of E. It is conjectured that the Tate-Shafarevich group is finite. For more details, see [12, Section X.4].

Theorem 4.2.2. *Let E be an elliptic curve over K, and let C be a K-torsor of E representing an element of $Ш(E/K)$. Then the period and index of C are equal.*

Proof. By theorem 4.1.10 it suffices to show that $\mathrm{Div}(C_{\bar{K}})^{G_k} \to \mathrm{Pic}(C_{\bar{K}})^{G_k}$ is surjective. Consider the exact sequence

$$1 \to \bar{K}^* \to \kappa(C_{\bar{K}})^* \to \mathrm{Div}(C_{\bar{K}}) \to \mathrm{Pic}(C_{\bar{K}}) \to 0.$$

Taking Galois cohomology on the induced sequence $1 \to \bar{K}^* \to \kappa(C_{\bar{K}})^* \to \kappa(C_{\bar{K}})^*/\bar{K} \to 1$ yields the exact sequence

$$H^1(G_K, \kappa(C_{\bar{K}})^*) \to H^1(G_K, \bar{\kappa}(C_{\bar{K}})^*/\bar{K}^*) \to H^2(G_K, \bar{K}^*) = \mathrm{Br}(K).$$

By imitating the proof of Hilbert's Theorem 90 one easily finds

$$H^1(G_K, \kappa(C_{\bar{K}})^*) = 0,$$

so the homomorphism $H^1(G_K, \kappa(C_{\bar{K}})^*/\bar{K}^*) \to \mathrm{Br}(K)$ is injective. Now taking Galois cohomology on the sequence $1 \to \kappa(C_{\bar{K}})^*/\bar{K}^* \to \mathrm{Div}(C_{\bar{K}}) \to \mathrm{Pic}(C_{\bar{K}}) \to 0$ yields the exact sequence

$$\mathrm{Div}(C_{\bar{K}})^{G_K} \to \mathrm{Pic}(C)^{G_K} \to H^1(G_K, \kappa(C_{\bar{K}})^*/\bar{K}^*).$$

Composition with the injection into $\mathrm{Br}(K)$ we found earlier yields the exact sequence

$$\mathrm{Div}(C_{\bar{K}})^{G_K} \to \mathrm{Pic}(C_{\bar{K}})^{G_K} \to \mathrm{Br}(K).$$

We can do the same while replacing K with K_v for every place of K. We obtain a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Div}(C_{\bar{K}})^{G_K} \arrow[r] \arrow[d] & \mathrm{Pic}(C_{\bar{K}})^{G_K} \arrow[r] \arrow[d] & \mathrm{Br}(K) \arrow[d] \\
\prod_v \mathrm{Div}(C_{\bar{K}_v})^{G_{K_v}} \arrow[r] & \prod_v \mathrm{Pic}(C_{\bar{K}_v})^{G_{K_v}} \arrow[r] & \prod_v \mathrm{Br}(K_v)
\end{tikzcd}$$

As C represents an element of $Ш(E/K)$, we see that $C(K_v) \neq \emptyset$ for every place v of K. Therefore the homomorphisms $\mathrm{Div}(C_{\bar{K}_v})^{G_{K_v}} \to \mathrm{Pic}(C_{\bar{K}_v})^{G_{K_v}}$ are surjective by proposition 4.1.11, so by exactness the homomorphisms $\mathrm{Pic}(C_{\bar{K}_v})^{G_{K_v}} \to \mathrm{Br}(K_v)$ are zero. The homomorphism $\mathrm{Br}(K) \to \prod_v \mathrm{Br}(K_v)$ is injective [11, p. 163] so a diagram chase shows that $\mathrm{Pic}(C_{\bar{K}})^{G_K} \to \mathrm{Br}(K)$ is zero too. By exactness of the top row we find that $\mathrm{Div}(C_{\bar{K}})^{G_K} \to \mathrm{Pic}(C_{\bar{K}})^{G_K}$ is surjective. □

4.3 Semi-stable elliptic curves

Theorem 4.3.1. *Assume that K is either a non-archimedean local field or a number field, and let O_K be its ring of integers. Let E be an elliptic curve over K, and let $X \to S = \mathrm{Spec}\, O_K$ be its minimal regular model. Then E is semi-stable if and only if $X \to S$ is semi-stable.*

Proof. Both properties need to be checked on each closed fiber. By base change $\mathrm{Spec}\, \widehat{O}_{S,s}$ we reduce to the local field case, and by taking the maximal unramified extension of $\mathrm{Frac}\, O_{S,s}$ we reduce to the case where K is a non-archimedean local field with algebraically closed residue field. We can then use the classification of reduction types ([12, p. 448]) to finish the proof. □

Let E be a semi-stable elliptic curve over a non-archimedean local field or a number field K with ring of integers O_K. We are interested in the closed fibers of the Néron model N of E over $\mathrm{Spec}\, O_K$. Recall that this Néron model can be obtained by taking the minimal regular model of E over $\mathrm{Spec}\, O_K$ and removing all the singular points in the closed fibers. By base change we may assume that K is non-archimedean local with algebraically closed residue field k. Let us first look at the unit component N_s^0. If X_s is smooth, then $N_s^0 = X_s$ is an elliptic curve over $\kappa(s)$. If X_s is an n-gon of $\mathbb{P}^1_k$'s, then N_s^0 is obtained by taking one $\mathbb{P}^1_k$ and removing the two points where it intersects the other lines. We get a scheme N_s^0 isomorphic to $\mathrm{Spec}\, k[X, 1/X]$, and by 2.3.4 we see that $N_s^0 \cong \mathbb{G}_{m,k}$. It can be shown that the component group $\pi_0(N_s)(k)$ is cyclic of order n, see [9, 10.2.24].

4.4 The Arakelov projection formula

The formulae in this section will be useful in the next section. They give an Arakelov-theoretic analogue of the projection formula from the classical intersection theory ([9, 9.2.12]).

Theorem 4.4.1 (Arakelov projection formula, [4, 5.3.2])**.** *Let E and E' be two elliptic curves over a number field K, and let $f : E \to E'$ be an isogeny. Let $X \to S = \operatorname{Spec} O_K$ and $X' \to S$ be regular models of respectively E and E' over S, and suppose that f extends to an S-morphism $f : X \to X'$. For every two Arakelov divisors $D \in \widehat{\operatorname{Div}}(X)$ and $D' \in \widehat{\operatorname{Div}}(X')$ the equality*

$$(f^* D' \cdot D)_{\mathrm{Ar}} = (D' \cdot f_* D)_{\mathrm{Ar}}$$

holds.

This theorem is especially useful when X' is the minimal regular model of E' over S. In this case, X' contains the Néron model of E' over S, and by the universal property of the Néron model, every isogeny $E \to E'$ extends to a morphism $X \to X'$.

Corollary 4.4.2. *Assume we are in the situation of 4.4.1. Let N be the degree of f. Then for all Arakelov divisors D_1, D_2 on X', the equality*

$$(f^* D_1 \cdot f^* D_2)_{\mathrm{Ar}} = N \cdot (D_1 \cdot D_2)_{\mathrm{Ar}}$$

holds.

4.5 The self-intersection of a point on a semi-stable elliptic curve

We will dedicate this section to proving the following theorem. We again fix a number field K with ring of integers O_K and define $S = \operatorname{Spec} O_K$.

Theorem 4.5.1. *Let E/K be a semi-stable elliptic curve, and let C be a K-torsor of E. Notice that E (and hence C) is geometrically connected. We let $X \to S$ denote the minimal regular model of C over S, and $D = \overline{\{P\}}$ an horizontal divisor of X. We will prove the following identity:*

$$(D \cdot \omega_X)_{\mathrm{Ar}} = \frac{\deg_K D|_{X_\eta}}{12} \sum_{s \in S_{\mathrm{fin}}} \delta_s,$$

where, for each closed $s \in S$, we define δ_s as follows:

$$\delta_s = \log \# \kappa(s) \cdot \# \operatorname{Sing}(X \times_S \overline{\kappa(s)}).$$

Szpiro [13] proves this using Néron-Tate heights. A proof using only Arakelov intersection theory is given in this section. Notice that, as $X \to S$ is semi-stable, δ_s is zero if the geometric fiber of X above s is smooth, and δ_s is $\log \# \kappa(s) \cdot n$ if this geometric fiber is an n-gon of $\mathbb{P}^1$'s. First notice that the left and right hand side of 4.5.1 are linear in D, so we may assume with no loss of generality that D is irreducible.

Suppose that L/K is a finite field extension, with ring of integers O_L, and $S' = \operatorname{Spec} O_L$. Let $X' \to S'$ be the minimal desingularization of the base change $X \times_S S'$ of X over the morphism $\lambda : S' \to S$, and let $r : X' \to X$ be the natural morphism. We let r_L^* denote the horizontal part of r^*D. Now r^*D can be written as the sum of r_L^*D plus some irreducible components of closed fibers of $X' \to S'$, and the morphism r contracts these components to a point. By [9, 10.2.12(a)] (the proof generalizes with no issue to our situation) the intersection number of $r^*\omega_X$ with these divisors is zero. Moreover, by 2.2.7, we have $r^*\omega_X = \omega_{X'}$. Using 3.2.3 we obtain

$$(r_L^*D \cdot \omega_{X'})_{\mathrm{Ar}} = (r_L^*D \cdot r^*\omega_X)_{\mathrm{Ar}} = (r^*D \cdot r^*\omega_X)_{\mathrm{Ar}} = [L:K](D \cdot \omega_X)_{\mathrm{Ar}}.$$

We also have

$$\deg_L(r_L^*D)|_{X'_\eta} = \deg_K D|_{X_\eta}$$

by [9, 7.3.7(a)], and

$$\begin{aligned}
\sum_{s' \in S'_{\mathrm{fin}}} \delta_{s'} &= \sum_{s' \in S'_{\mathrm{fin}}} \log \#\kappa(s') \cdot \# \operatorname{Sing}(X' \times_{S'} \overline{\kappa(s')}) \\
&= \sum_{s \in S_{\mathrm{fin}}} \sum_{s' \in \lambda^{-1}(s)} \log \#\kappa(s') \cdot \# \operatorname{Sing}(X' \times_{S'} \overline{\kappa(s')}) \\
&= \sum_{s \in S_{\mathrm{fin}}} \sum_{s' \in \lambda^{-1}(s)} [\kappa(s') : \kappa(s)] \log \#\kappa(s) \cdot e(s'/s) \# \operatorname{Sing}(X \times_S \overline{\kappa(s)}) \\
&= [L:K] \sum_{s \in S_{\mathrm{fin}}} \delta_s,
\end{aligned}$$

since

$$\sum_{s' \in \lambda^{-1}(s)} e(s'/s)[\kappa(s') : \kappa(s)] = [L:K].$$

Also, X' is again minimal, by 2.2.8. Both sides of 4.5.1 are changed by a factor $[L:K]$ when we replace K by a finite extension L/K, so we may do so with no loss of generality. We may therefore assume that C has a K-rational point, so we may assume that C is an elliptic curve with zero $O = P$. Let p be a prime number such that for all $s \in S$ with $\operatorname{char}(\kappa(s)) = p$ the fiber X_s is smooth (such a prime number exists, since an elliptic curve has bad reduction over only finitely many primes). We extend our field further, such that all p-torsion points of E are K-rational. We let $P_1, \ldots, P_r$ ($r = p^2$) denote the p-torsion points, and identify them with the irreducible horizontal divisors of X they induce.

By the adjunction formula, we have

$$(O \cdot \omega_X)_{\mathrm{Ar}} = -(O \cdot O)_{\mathrm{Ar}},$$

so we need to show that

$$(O \cdot O)_{\mathrm{Ar}} = -\frac{1}{12} \sum_s \delta_s.$$

By the Arakelov projection formula we have

$$p^2 (O \cdot O)_{\mathrm{Ar}} = ([p]^*O \cdot [p]^*O)_{\mathrm{Ar}} = \sum_{i,j} (P_i \cdot P_j)_{\mathrm{Ar}}.$$

By symmetry all K-rational points on an elliptic curve have the same self-intersection, so we find that

$$0 = \sum_{i \neq j} (P_i \cdot P_j).$$

For each $1 \leq i, j \leq r$ with $i \neq j$, we define a vertical divisor $\Phi_{ij} = \sum_{s \in S_{\text{fin}}} \Phi_{ij,s}$ as follows: we put $\Phi_{ij,s} = 0$ if X_s is smooth, and we let $\Phi_{ij,s} \in \mathrm{Div}_s(X) \otimes \mathbb{Q}$ be a fractional vertical divisor (that is, a divisor with rational instead of integral coefficients) above s such that $(P_i - P_j + \Phi_{ij,s} \cdot V)_{\mathrm{Ar}} = 0$ for all vertical divisors $V \in \mathrm{Div}_s(X)$ above s. The following lemma shows that this is possible. In this case, every closed fiber is connected, since the generic fiber $X_\eta \cong C$ is geometrically connected (see 2.1.2).

Lemma 4.5.2. *Let $X \to S$ be an arithmetic surface, let $K = \kappa(S)$ be the function field of S, and let P_1, P_2 be two K-rational points on the generic fiber. Let D_1, D_2 be the corresponding irreducible horizontal divisors of X. Let $s \in S$ be a closed point such that X_s is connected. Then there exists a fractional vertical divisor $\Phi \in \mathrm{Div}_s(X)_{\mathbb{Q}}$ with*

$$(D_1 - D_2 + \Phi \cdot F)_{\mathrm{Ar}} = 0 \quad \text{for all vertical } F \in \mathrm{Div}_s(X).$$

Moreover, this Φ is unique up to a fractional multiple of X_s. In particular, Φ^2 does not depend on the choice of Φ.

Proof. Let $\Gamma_1, \ldots, \Gamma_r$ be the irreducible components of X_s, and let $d_1, \ldots, d_r$ be their multiplicity in X_s. The Arakelov intersection product defines a symmetric bilinear form on $\mathrm{Div}_s(X)_{\mathbb{Q}}$ (and this bilinear form is $\log \kappa(s)$ times the bilinear form induced by the 'classical' intersection product), and hence a $\mathbb{Q}$-linear map

$$\mathrm{Div}_s(X)_{\mathbb{Q}} \to (\mathrm{Div}_s(X)_{\mathbb{Q}})^\vee : F_1 \mapsto (F_2 \mapsto (F_1 \cdot F_2)_{\mathrm{Ar}}).$$

The kernel of this linear map is spanned by $X_s = \sum_i d_i \Gamma_i$; see [9, Theorem 9.1.21 and Theorem 9.1.23]. Similarly, we see that the image of this linear map is contained in, and therefore equal to, the codimension 1 subspace

$$\left\{ f \in (\mathrm{Div}_s(X)_{\mathbb{Q}})^\vee) : \sum_i d_i f(\Gamma_i) = 0. \right\}.$$

Now consider the linear form on $\mathrm{Div}_s(X)_{\mathbb{Q}}$ given by $F \mapsto (D_2 - D_1, F)$. We have:

$$\sum_i d_i (D_2 - D_1 \cdot \Gamma_i)_{\mathrm{Ar}} = (D_2 - D_1 \cdot X_s)_{\mathrm{Ar}} = (D_2 \cdot X_s)_{\mathrm{Ar}} - (D_1 \cdot X_s)_{\mathrm{Ar}} = 0$$

as P_1 and P_2 are both K-rational points; see [9, Proposition 9.1.30]. We therefore see that this linear form is in the image of the linear map defined above, and therefore there exists a $\Phi \in \mathrm{Div}_s(X)_{\mathbb{Q}}$ with $(D_2 - D_1 \cdot F)_{\mathrm{Ar}} = (\Phi \cdot F)_{\mathrm{Ar}}$ for all $F \in \mathrm{Div}_s(X)$. □

By construction we have

$$(P_i - P_j + \Phi_{ij} \cdot V)_{\mathrm{Ar}} = 0$$

for all vertical divisors V of X. In particular, we have

$$(P_i - P_j + \Phi_{ij} \cdot \Phi_{ij})_{\mathrm{Ar}} = 0. \tag{4.5.3}$$

We have another useful identity.

Lemma 4.5.4.

$$(P_i - P_j + \Phi_{ij})^2_{\mathrm{Ar}} = 0.$$

Proof. The divisor $pP_i - pP_j$ on C is principal; see [12, III.3.5], so there exists an $f \in \kappa(C)^* = \kappa(X)^*$ with $\mathrm{div}(f) = pP_i - pP_j$. As divisors on X we have $pP_i - pP_j = (f)_{\mathrm{hor}}$. We now have:

$$\begin{aligned}(P_i - P_j + \Phi_{ij} \cdot pP_i - pP_j)_{\mathrm{Ar}} &= (P_i - P_j + \Phi_{ij} \cdot (f)_{\mathrm{hor}})_{\mathrm{Ar}} \\ &= (P_i - P_j + \Phi_{ij} \cdot -(f)_{\mathrm{ver}} - (f)_{\infty})_{\mathrm{Ar}}.\end{aligned}$$

We have $(P_i - P_j + \Phi_{ij} \cdot (f)_{\mathrm{ver}})_{\mathrm{Ar}} = 0$ by construction of Φ_{ij}, and $(P_i - P_j + \Phi_{ij} \cdot (f)_{\infty})_{\mathrm{Ar}} = 0$ as P_i and P_j are both K-rational. We therefore see that

$$p(P_i - P_j + \Phi_{ij} \cdot P_i - P_j)_{\mathrm{Ar}} = (P_i - P_j + \Phi_{ij} \cdot pP_i - pP_j)_{\mathrm{Ar}} = 0.$$

As $(P_i - P_j + \Phi_{ij} \cdot \Phi_{ij})_{\mathrm{Ar}} = 0$, we now easily obtain the desired result. □

By taking sums and using 4.5.3 and 4.5.4 we now find:

$$\begin{aligned}0 &= \sum_{i \neq j} (P_i - P_j + \Phi_{ij})^2_{\mathrm{Ar}} \\ &= \sum_{i \neq j} \left((P_i - P_j)^2_{\mathrm{Ar}} + 2(P_i - P_j \cdot \Phi_{ij})_{\mathrm{Ar}} + (\Phi_{ij})^2_{\mathrm{Ar}}\right) \\ &= \sum_{i \neq j} (P_i)^2_{\mathrm{Ar}} + (P_j)^2_{\mathrm{Ar}} - 2(P_i \cdot P_j)^2_{\mathrm{Ar}} - (\Phi_{ij})^2_{\mathrm{Ar}} \\ &= 2(p^4 - p^2)(O)^2_{\mathrm{Ar}} - \sum_{i \neq j} \Phi_{ij}^2 - 2\sum_{i \neq j} (P_i \cdot P_j)_{\mathrm{Ar}} \\ &= 2(p^4 - p^2)(O)^2_{\mathrm{Ar}} - \sum_{i \neq j} (\Phi_{ij})^2_{\mathrm{Ar}}.\end{aligned}$$

It is clear that $(\Phi_{ij})^2_{\mathrm{Ar}} = (\Phi_{ji})^2_{\mathrm{Ar}}$. Dividing the above equation by $2(p^4 - p^2)$ gives the following formula for $(O \cdot O)_{\mathrm{Ar}}$:

$$(O \cdot O)_{\mathrm{Ar}} = \frac{1}{p^4 - p^2} \sum_{i < j} (\Phi_{ij})^2_{\mathrm{Ar}} = \frac{1}{p^4 - p^2} \sum_{s \in S_{\mathrm{fin}}} \sum_{i < j} (\Phi_{ij,s})^2_{\mathrm{Ar}}. \tag{4.5.5}$$

The following lemma gives a formula for $\Phi_{ij,s}$:

Lemma 4.5.6. *Let $\Gamma_0, \ldots, \Gamma_{n_s - 1}$ be the components of X_s, numbered cyclically. Suppose that P_1 and P_2 are K-rational points on the generic fiber of X, and let D_1, D_2 be the corresponding irreducible horizontal divisors. Let $0 \leq a, b < n_s$ be the unique integers such that $D_1 \cap \Gamma_a \neq \emptyset$ and $D_2 \cap \Gamma_b \neq \emptyset$. Suppose that Φ is a rational vertical divisor above s such that $(D_1 - D_2 + \Phi \cdot F) = 0$ for all vertical divisors F above s. Then*

$$(\Phi)^2_{\mathrm{Ar}} = -\frac{|a_1 - a_2|(r - |a_1 - a_2|)}{r} \cdot \log \#\kappa(s).$$

Proof (sketch). We need to show that

$$\Phi^2 = -\frac{|a_1 - a_2|(r - |a_1 - a_2|)}{r};$$

with Φ^2 the self-intersection of Φ in terms of 'classical' intersection theory. The above result then follows by definition of the Arakelov intersection product.

Write $\Phi = \sum_i q_i \Gamma_i$. We have:

$$0 = (D_1 - D_2 + \Phi) \cdot \Phi = q_a - q_b + \Phi^2.$$

Moreover, the identities

$$0 = (D_1 - D_2 + \Phi) \cdot \Gamma_i$$

with $0 \leq i < n_s$ give n_s linear equations in the variables $q_0, \ldots, q_{n_s-1}$. A purely combinatorial argument completes the proof. □

Fix a closed point $s \in S$, and define $k = \overline{\kappa(s)}$. If X_s is smooth then $\Phi_{ij,s} = 0$ so $(\Phi_{ij,s})^2_{\mathrm{Ar}} = 0$. Suppose that X_s is not smooth. Then the geometric fiber $X_k = X_s \times_S \operatorname{Spec} k$ is an n_s-gon of projective lines meeting transversally. Let N be the Néron model of X over S. The fiber N_s is obtained by removing the singular points of X_s. The component group $\pi_0(N_s)(k)$ is cyclic, and the identity component N_s^0 has $N_s^0(k) \cong k^*$. As $\operatorname{char}(k) \neq p$, the reduction map $C(K)[p] \to N_s^0(k)$ is injective ([12, VII.3.1(a)]), so we see that the p-torsion points are distributed as p packets of p points, evenly distributed over the sides of the geometric fiber X_k. That is, n_s is divisible by p, write $n_s = pm_s$, and if we number the components of X_k by $\Gamma_0, \ldots, \Gamma_{n_s-1}$ with Γ_0 the identity component, then the components $\Gamma_0, \Gamma_{m_s}, \Gamma_{2m_s}, \ldots, \Gamma_{(p-1)m_s}$ each intersect with p among the horizontal divisors $P_1, \ldots, P_r$. We therefore see that

$$\sum_{i<j} (\Phi_{ij,s})^2_{\mathrm{Ar}} = -p^2 \sum_{0 \leq a < b < p} \frac{(bm_s - am_s)(n_s - (bm_s - am_s))}{n_s} \cdot \log \#\kappa(s).$$

This can be rewritten to

$$\sum_{i<j} (\Phi_{ij,s})^2_{\mathrm{Ar}} = -n_s \sum_{0 \leq a < b < p} (b - a)(p - (b - a)) \cdot \log \#\kappa(s).$$

The following lemma can be used to further simplify the above equation.

Lemma 4.5.7. *Let n be a positive integer. The following formula holds:*

$$12 \sum_{0 \leq i < j < n} (j - i)(n - (j - i)) = n^4 - n^2.$$

Proof. The identity can be verified by expanding the sum and using the formulae for the triangular, square piramidal and square triangular numbers. □

By using the lemma, we find that

$$\sum_{i<j} (\Phi_{ij,s})^2_{\mathrm{Ar}} = -n_s \frac{p^4 - p^2}{12} \cdot \log \#\kappa(s).$$

By summing over all closed $s \in S$ we now find:

$$\sum_{i<j} (\Phi_{ij})^2_{\mathrm{Ar}} = -\sum_{s \in S_{\mathrm{fin}}} n_s \frac{p^4 - p^2}{12} \cdot \log \#\kappa(s) = -\frac{p^4 - p^2}{12} \cdot \sum_{s \in S_{\mathrm{fin}}} \delta_s.$$

By combining this result with 4.5.5 we obtain

$$(O \cdot \omega_X)_{\mathrm{Ar}} = -(O \cdot O)_{\mathrm{Ar}} = \frac{1}{12} \sum_{s \in S_{\mathrm{fin}}} \delta_s,$$

so we have completed the proof.

4.6 The Hriljac-Faltings-Riemann-Roch theorem

Let K be a number field, and let $X \to S = \operatorname{Spec} O_K$ be an arithmetic surface with generic fiber of arithmetic genus 1. Let, moreover, D be an irreducible horizontal Weil divisor of X. We will show that $H^1(X, O_X(D)) = 0$ using the following theorem.

Theorem 4.6.1 (Grauert, [6, III.12.9])**.** *Let $f : X \to Y$ be a projective morphism of Noetherian schemes, with Y integral, and let $\mathcal{F}$ be a coherent sheaf on X, flat over Y. Suppose that for some $i \geq 0$ the function*

$$Y \to \mathbb{Z}_{\geq 0} : y \mapsto \dim_{\kappa(y)} H^i(X_y, \mathcal{F}_y)$$

is constant. Then $R^i f_*(\mathcal{F})$ is locally free on Y, and for every $y \in Y$ there is a natural isomorphism

$$R^i f_*(\mathcal{F}) \otimes \kappa(y) \xrightarrow{\sim} H^i(X_y, \mathcal{F}_y). \qquad \square$$

Corollary 4.6.2. *Let $X \to S$ be an arithmetic surface with generic fiber of arithmetic genus 1. Then $H^1(X, O_X(D)) = 0$ for every irreducible horizontal divisor D on X.*

Proof. Since the morphism $X \to S$ is flat, the arithmetic genus of every fiber of $X \to S$ is equal to $p_a(X_\eta) = 1$. For every $s \in S$ the fiber X_s is a projective curve over $\kappa(s)$ and hence, using Riemann-Roch and Serre duality, we find that $H^1(X_s, O_{X_s}(D_s)) = 0$. By Grauert's theorem $R^1 f_*(O_X(D))$ is locally free on S, and for every $s \in S$ we have $R^1 f_*(O_X(D)) \otimes \kappa(s) = 0$. Therefore $R^1 f_*(O_X(D))$ is zero, and as $R^1 f_*(O_X(D)) = H^1(X, O_X(D))^\sim$ ([6, III.8.5]), we find that $H^1(X, O_X(D)) = 0$. $\square$

As $H^1(X, O_X(D)) = 0$, the Faltings metrics on the determinants of cohomology $\det H(\mathcal{X}_\sigma, O_{\mathcal{X}_\sigma}(D_\sigma))$ induce a Haar measure $\mathrm{vol}_{\mathrm{Fal}}$ on $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$. The Hriljac-Faltings-Riemann-Roch theorem gives a formula for the volume of a fundamental domain of the lattice $H^0(X, O_X(D))$ in $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$.

Theorem 4.6.3 (Hriljac-Faltings-Riemann-Roch)**.** *Let F be a fundamental domain of the lattice $H^0(X, O_X(D))$ in $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$. The volume of F is given by*

$$-\log \mathrm{vol}_{\mathrm{Fal}} F = \tfrac{1}{2}(D \cdot D - \omega_X)_{\mathrm{Ar}} + \chi(O_X) + \deg_K(D|_{X_\eta}) \cdot (s \log 2 - \log \sqrt{|\Delta_K|}).$$

Here Δ_K denotes the discriminant of K over $\mathbb{Q}$, and s is the number of pairs of conjugate complex embeddings $K \to \mathbb{C}$.

Proof. The Faltings-Riemann-Roch theorem gives the identity

$$\chi(O_X(D)) = \tfrac{1}{2}(D \cdot D - \omega_X)_{\mathrm{Ar}} + \chi(O_X).$$

As $H^1(X, O_X(D)) = 0$ by 4.6.2, we find that

$$\begin{aligned}\chi(O_X(D)) &= \chi(H^0(X, O_X(D)), O_K) \\ &= \chi(H^0(X, O_X(D)), \mathbb{Z}) - \mathrm{rank}_{O_K}(H^0(X, O_X(D)) \cdot \chi(O_K, \mathbb{Z})) \\ &= -\log \mathrm{vol}_{\mathrm{Fal}} F - \deg_K(D|_{X_\eta}) \cdot (s \log 2 - \log \sqrt{|\Delta_K|}),\end{aligned}$$

since

$$\begin{aligned}\mathrm{rank}_{O_K}(H^0(X, O_X(D))) &= \dim_K(H^0(X, O_X(D)) \otimes_{O_K} K) \\ &= \dim_K H^0(X_K, O_{X_K}(D|_{X_K})) \\ &= \deg_K(D|_{X_K}).\end{aligned}$$

Here the second equality follows from the flatness of $X \to S$ (cf. [6, III.9.3]), and the last equality from Riemann-Roch. By putting everything together again we find the formula stated in the theorem. □

4.7 An upper bound for splitting fields of principal homogeneous spaces

In this section, we will prove the following main theorem.

Theorem 4.7.1. *Let E be a semi-stable elliptic curve over a number field K. Then there exists a function*

$$B(N) = [K : \mathbb{Q}] N \log N + O(N)$$

such that for every K-torsor C of E of index N there exists a field extension L/K of degree N with $C(L) \neq \emptyset$ and

$$\log \left| N_{K/\mathbb{Q}}(\Delta_{L/K}) \right| \leq B(N).$$

Hriljac [7] proved in his article the following result, which follows from the above theorem by applying 4.2.2. Therefore, the above theorem is a generalization of Hriljac's result.

Corollary 4.7.2. *Let E be a semi-stable elliptic curve over a number field K. There exists a function $B(N) = [K : \mathbb{Q}] N \log N + O(N)$ such that the following holds: if $[C] \in Ш(E/K)$ is an element in the Tate-Shafarevich group of order N, then there exists a field extension L/K of degree N with $C(L) \neq \emptyset$ and $\log \left| N_{K/\mathbb{Q}}(\Delta_{L/K}) \right| \leq B(N)$.*

The proof of 4.7.1 is very similar to the proof of Hriljac's result. We will use the rest of this chapter to provide the proof.

Let K be a number field, and let E/K be an elliptic curve (we will first give an upper bound 4.7.4 for $\log \left| N_{K/\mathbb{Q}}(\Delta_{L/K}) \right|$ in the general case, and derive 4.7.1 in the semi-stable case). Let C be a K-torsor of E, and let N be its index. So there exists a field extension L/K of degree N such that $C(L) \neq \emptyset$; let $P \in C(L)$ be a point. We let $X \to S = \operatorname{Spec} O_K$ denote the minimal regular model of

C over O_K, and let $D = \overline{\{P\}}$ denote the irreducible horizontal divisor of X corresponding to P. We have

$$\deg_K D|_{X_\eta} = [\kappa(P) : K] = N.$$

We have a natural Haar measure on $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$, induced by the Faltings metrics on the $\det H(X, O_{\mathcal{X}_\sigma}(D_\sigma))$ (for all $\sigma \in S_\infty$), and by the Hriljac-Faltings-Riemann-Roch theorem, the volume of a fundamental domain F of the lattice $H^0(X, O_X(D))$ in $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$ is given by

$$-\log \operatorname{vol} F = \tfrac{1}{2}(D \cdot D - \omega)_{\mathrm{Ar}} + \chi(O_X) + N \cdot (s \log 2 - \log \sqrt{|\Delta_K|}),$$

where s denotes the number of conjugate pairs of complex embeddings of K.

For every $\sigma \in S_\infty$, the canonical admissible metric $\|\cdot\|_{D_\sigma}$ on $O_{\mathcal{X}_\sigma}(D_\sigma)$ gives inner products $\langle \cdot, \cdot \rangle_{D_\sigma}(P)$ on the fibers of $O_{\mathcal{X}_\sigma}$. We obtain a pairing $(\cdot, \cdot)_\sigma$ on $H^0(\mathcal{X}_\sigma, O_{\mathcal{X}_\sigma}(D_\sigma)) = H^0(X, O_X(D)) \otimes_{O_K} \overline{K}_\sigma$ given by

$$(f, g)_\sigma = \int_{\mathcal{X}_\sigma} \langle f(P), g(P) \rangle_{D_\sigma}(P) \cdot \mu_\sigma,$$

where μ_σ denotes the canonical $(1,1)$-form on $\mathcal{X}_\sigma$. One easily checks that this pairing defines an inner product on the (real or complex) vector space $H^0(X, O_X(D)) \otimes_{O_K} K_\sigma$. Let $\|\cdot\|_\sigma$ (not to be confused with $\|\cdot\|_{D_\sigma}$!) denote the norm on $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$ associated to this inner product. Consider the subset

$$T \subset H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R} = \prod_{\sigma \in S_\infty} H^0(X, O_X(D)) \otimes_{O_K} K_\sigma$$

given by

$$T = \left\{ (f_\sigma)_\sigma \in \prod_{\sigma \in S_\infty} H^0(X, O_X(D))_{O_K} K_\sigma : \sum_{\sigma \in S_\infty} \epsilon_\sigma \|f_\sigma\|_\sigma \le [K : \mathbb{Q}] \right\}.$$

This subset is easily seen to be a convex closed subset, symmetric in the origin.

Lemma 4.7.3. *For every non-zero $(f_\sigma)_\sigma \in T$ we have*

$$\sum_{\sigma \in S_\infty} \epsilon_\sigma \int_{\mathcal{X}_\sigma} \log \|f_\sigma\|_{D_\sigma} \mu_\sigma \le 0.$$

More generally, for every $a > 0$ and every $(f_\sigma)_\sigma \in aT$ we have

$$\sum_{\sigma \in S_\infty} \epsilon_\sigma \int_{\mathcal{X}_\sigma} \log \|f_\sigma\|_{D_\sigma} \mu_\sigma \le [K : \mathbb{Q}] \log a.$$

Proof. The second statement follows from the first, using the identity $\sum_\sigma \epsilon_\sigma = [K : \mathbb{Q}]$. We will prove the first inequality. Let $(f_\sigma)_\sigma \in T$ be any element. Jensen's inequality gives, for all $\sigma \in S_\infty$:

$$\int_{\mathcal{X}_\sigma} \log \|f_\sigma\|_{D_\sigma} \mu_\sigma \le \log \left(\int_{\mathcal{X}_\sigma} \|f_\sigma\|_{D_\sigma} \mu_\sigma \right).$$

Using the inequality $1 + \frac{1}{2}\log x \leq \sqrt{x}$ for all $x > 0$ we find:

$$\int_{\mathcal{X}_\sigma} \log \|f_\sigma\|_{D_\sigma} \mu_\sigma + 1 \leq \log\left(\int_{\mathcal{X}_\sigma} \|f_\sigma\| \mu_\sigma\right) + 1 \leq \left(\int_{\mathcal{X}_\sigma} \|f_\sigma\|_{D_\sigma} \mu_\sigma\right)^{1/2} = \|f_\sigma\|_\sigma.$$

We therefore see that

$$\sum_{\sigma \in S_\infty} \epsilon_\sigma \left(\int_{\mathcal{X}_\sigma} \log \|f_\sigma\|_{D_\sigma} \mu_\sigma + 1\right) \leq \sum_{\sigma \in S_\infty} \epsilon_\sigma \|f_\sigma\|_\sigma \leq [K : \mathbb{Q}],$$

and as $\sum_\sigma \epsilon_\sigma = [K : \mathbb{Q}]$ the desired result follows. □

Let $a > 0$ be the positive real number defined by

$$2^{N[K:\mathbb{Q}]} \operatorname{vol}_{\mathrm{Fal}}(F) = a^{N[K:\mathbb{Q}]} \operatorname{vol}_{\mathrm{Fal}}(T) = \operatorname{vol}_{\mathrm{Fal}}(aT).$$

Applying Minkowski's Theorem to the lattice $H^0(X, O_X(D))$ and the centrally symmetric convex closed subset T of the $N[K : \mathbb{Q}]$-dimensional $\mathbb{R}$-vector space $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$ shows that there exists a non-zero $f \in H^0(X, O_X(D))$ that lies in aT. In particular, we see that

$$\sum_{\sigma \in S_\infty} \epsilon_\sigma \int_{\mathcal{X}_\sigma} \log \|f\|_{D_\sigma} \mu_\sigma \leq [K : \mathbb{Q}] \log a.$$

The divisor $D + (f)_{\mathrm{fin}}$ is an effective divisor, as $f \in H^0(X, O_X(D))$. Therefore its horizontal part $D' := D + (f)_{\mathrm{hor}}$ is effective too, and the degree of its restriction to $X_\eta \cong C$ is N, and therefore D' is irreducible. Let P' be its point on the generic fiber, and let $L = \kappa(P)$ be its field of fractions; the degree of L/K is N. By theorem 2.1.4 there exists an O_K-order R in O_L and a morphism $\epsilon : \operatorname{Spec} R \to X$ such that $D = \epsilon(\operatorname{Spec} R)$. As R is contained in O_L, the ideal $\Delta_{R/K}$ is contained in $\Delta_{L/K}$, and hence we have:

$$\left|N_{K/\mathbb{Q}}(\Delta_{L/K})\right| \leq \left|N_{K/\mathbb{Q}}(\Delta_{R/K})\right|.$$

The adjunction formula gives us the following equality:

$$\log \left|N_{K/\mathbb{Q}}(\Delta_{R/K})\right| = (D' + \omega_X \cdot D')_{\mathrm{Ar}} + \sum_{\sigma \in S_\infty} \epsilon_\sigma \sum_{i \neq j} g_\sigma(P'_{\sigma,i}, P'_{\sigma,j}),$$

where, for every $\sigma \in S_\infty$, the points $P'_{\sigma,1}, \ldots, P'_{\sigma,N}$ are the N points on $\mathcal{X}_\sigma$ obtained by embedding P' in $\mathcal{X}_\sigma$ via the N embeddings $L \to \mathbb{C}$ extending σ.

By Elkies' theorem 1.3.4, there exist constants c_σ for every σ such that

$$\sum_{i \neq j} g_\sigma(P'_{\sigma,i}, P'_{\sigma,j}) \leq \frac{N \log N}{2} + N c_\sigma.$$

As the Riemann surface $\mathcal{X}_\sigma$ is isomorphic to $(E \times_\sigma \mathbb{C})(\mathbb{C})$, this constant depends only on E! By setting $c = \sum_\sigma \epsilon_\sigma c_\sigma$ we obtain a constant, depending only on E, such that

$$\sum_{\sigma \in S_\infty} \epsilon_\sigma \sum_{i \neq j} g_\sigma(P'_{\sigma,i}, P'_{\sigma,j}) \leq \frac{[K : \mathbb{Q}] \cdot N \log N}{2} + N c.$$

Next, we will give an upper bound for $(D' + \omega_X \cdot D')$. We have

$$\begin{aligned}((D' + (f)_{\mathrm{ver}}) \cdot (D' + (f)_{\mathrm{ver}}))_{\mathrm{Ar}} - (D' \cdot D') &= ((2D' + (f)_{\mathrm{ver}}) \cdot (f)_{\mathrm{ver}})_{\mathrm{Ar}} \\ &= (D + D' + (f)_{\mathrm{fin}} \cdot (f)_{\mathrm{ver}})_{\mathrm{Ar}} \\ &= (D + D' - (f)_{\infty} \cdot (f)_{\mathrm{ver}})_{\mathrm{Ar}} \\ &= (D + D' \cdot (f)_{\mathrm{ver}})_{\mathrm{Ar}} \\ &\geq 0\end{aligned}$$

since D and D' are both effective horizontal divisors and $(f)_{\mathrm{ver}}$ is an effective vertical divisor (it is the vertical part of the effective divisor $D + (f)_{\mathrm{fin}}$). We have

$$\begin{aligned}(D' + (f)_{\mathrm{ver}})^2_{\mathrm{Ar}} &= (D - (f)_{\infty})^2_{\mathrm{Ar}} \\ &= (D)^2_{\mathrm{Ar}} + 2\deg_K(D|_{X_\eta}) \sum_{\sigma \in S_\infty} \epsilon_\sigma \int_{\mathcal{X}_\sigma} \log \|f\|_{D_\sigma} \mu_\sigma.\end{aligned}$$

By definition of f we have

$$\sum_{\sigma \in S_\infty} \epsilon_\sigma \int_{\mathcal{X}_\sigma} \log \|f\|_{D_\sigma} \mu_\sigma \leq [K : \mathbb{Q}] \log a,$$

so we find that

$$(D' \cdot D')_{\mathrm{Ar}} \leq ((D' + (f)_{\mathrm{ver}}) \cdot (D' + (f)_{\mathrm{ver}}))_{\mathrm{Ar}} \leq (D \cdot D)_{\mathrm{Ar}} + 2N[K : \mathbb{Q}] \log a.$$

We can also give an upper bound for $(\omega_X \cdot D')$. We have:

$$(\omega_X \cdot D') = (\omega_X \cdot (D + (f)_{\mathrm{hor}}))_{\mathrm{Ar}} = (\omega_X \cdot (D - (f)_{\mathrm{ver}} - (f)_{\infty}))_{\mathrm{Ar}}.$$

Now $(\omega_X \cdot (f)_{\mathrm{ver}})_{\mathrm{Ar}}$ is non-negative, as X is minimal, see 2.1.12. The intersection $((f)_{\mathrm{Ar}} \cdot \omega)_{\mathrm{Ar}}$ is zero, as $\deg_K(\omega_X|_C) = \deg_K \Omega_{C/k} = 0$ by Riemann-Roch. We therefore see that

$$(\omega_X \cdot D') \leq (\omega_X \cdot D).$$

Notice that equality holds if $X \to S$ is semi-stable; see 4.5.1.

By combining the previous inequalities we find the following upper bound for the discriminant of L.

$$\log \left|N_{K/\mathbb{Q}}(\Delta_{L/K})\right| \leq (D + \omega_X \cdot D)_{\mathrm{Ar}} + 2N[K : \mathbb{Q}] \log a + \frac{N[K : \mathbb{Q}] \log N}{2} + Nc.$$

By definition of a we have

$$2N[K : \mathbb{Q}] \log a = 2N[K : \mathbb{Q}] \log 2 + 2 \log \mathrm{vol}_{\mathrm{Fal}}(F) - 2 \log \mathrm{vol}_{\mathrm{Fal}}(T).$$

Now apply Hriljac-Faltings-Riemann-Roch on $\log \mathrm{vol}_{\mathrm{Fal}}(F)$ and substitute everything in the above inequality; we find that:

$$\begin{aligned}\log \left|N_{K/\mathbb{Q}}(\Delta_{L/K})\right| \leq\ & 2(D \cdot \omega_X)_{\mathrm{Ar}} + 2N[K : \mathbb{Q}] \log 2 - 2\chi(O_X) \\ & - 2N(s \log 2 - \log \sqrt{|\Delta_K|}) - 2 \log \mathrm{vol}_{\mathrm{Fal}}(T) \\ & + \frac{[K : \mathbb{Q}] N \log N}{2} + Nc.\end{aligned}$$

It remains to bound $\log \mathrm{vol}_{\mathrm{Fal}}(T)$.

Let $\sigma \in S_\infty$ be an infinite place of K. We have encountered two volumes on $H^0(X, O_X(D)) \otimes_{O_K} K_\sigma$: the Faltings volume $\mathrm{vol}_{\mathrm{Fal}}^\sigma$ induced by the Faltings metric on $H^0(\mathcal{X}_\sigma, O_{\mathcal{X}_\sigma}(D_\sigma))$, and the volume $\mathrm{vol}_{\mathrm{Ar}}^\sigma$, we will call it the *Arakelov volume*, induced by the inner product

$$(f, g)_\sigma = \int_{\mathcal{X}_\sigma} \langle f(P), g(P) \rangle_{D_\sigma} \mu_\sigma.$$

Let $f_1^\sigma, \ldots, f_N^\sigma$ be an orthonormal basis with respect to this inner product. We let P^σ be the parallelepiped spanned by $f_1^\sigma, \ldots, f_N^\sigma$ if σ is real, and by $f_1^\sigma, \ldots, f_N^\sigma, i f_1^\sigma, \ldots, i f_N^\sigma$ if σ is complex. We see that $\mathrm{vol}_{\mathrm{Ar}}^\sigma(P^\sigma) = 1$, so the Arakelov and Faltings volumes are related as follows:

$$\mathrm{vol}_{\mathrm{Fal}}^\sigma = \mathrm{vol}_{\mathrm{Fal}}^\sigma(P^\sigma) \cdot \mathrm{vol}_{\mathrm{Ar}}^\sigma .$$

The Faltings and Arakelov volumes on the $H^0(X, O_X(D)) \otimes_{O_K} K_\sigma$ induce volumes $\mathrm{vol}_{\mathrm{Fal}}$ and $\mathrm{vol}_{\mathrm{Ar}}$ on $H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$, and these are related by the following equation:

$$\mathrm{vol}_{\mathrm{Fal}} = \left(\prod_\sigma \mathrm{vol}_{\mathrm{Fal}}(P^\sigma) \right) \mathrm{vol}_{\mathrm{Ar}} .$$

One can show that the Arakelov volume of the subset $T \subset H^0(X, O_X(D)) \otimes_{\mathbb{Z}} \mathbb{R}$ is given by the following formula:

$$\mathrm{vol}_{\mathrm{Ar}}(T) = \frac{V_N^r V_{2N}^s [K:\mathbb{Q}]^{N[K:\mathbb{Q}]} (N!)^r ((2N)!)^s}{(N[K:\mathbb{Q}])! 2^{Ns}},$$

where r (resp. s) denotes the number of real (resp. complex) places of K, and

$$V_n = \frac{\pi^{n/2}}{\frac{n}{2} \Gamma(\frac{n}{2})}$$

is the volume of the unit ball in $\mathbb{R}^n$. A proof is given in [7, p. 223]. It only contains the computation of some integrals, along with some combinatorics, so it will be omitted from here.

It remains to bound $\mathrm{vol}_{\mathrm{Fal}}(P^\sigma)$ for all $\sigma \in S_\infty$. We have

$$\mathrm{vol}_{\mathrm{Fal}}(P^\sigma) = (\| f_1^\sigma \wedge \cdots \wedge f_N^\sigma \|_{\mathrm{Fal}}^\sigma)^{\epsilon_\sigma}$$

where $\|\cdot\|_{\mathrm{Fal}}^\sigma$ denotes the Faltings metric on $\Lambda^n(H^0(X, O_X(D)) \otimes_{O_K} K_\sigma)$. By 1.7.5 we have

$$\| f_1^\sigma \wedge \cdots \wedge f_N^\sigma \|_{\mathrm{Fal}}^\sigma \geq \frac{2\pi \|\eta\|(\tau_\sigma)^2 \sqrt{N!}}{\exp(\frac{N \log N}{2} + N c_\sigma + \log 2 + 2 c_\sigma)^{1/2}}$$

where τ_σ is a complex number with positive imaginary part, such that the Riemann surface $\mathcal{X}_\sigma$ is isomorphic to $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau_\sigma)$.

We finally find the following upper bound for $-2\log \mathrm{vol}_{\mathrm{Fal}}(T)$:

$$\begin{aligned}
-2\log \mathrm{vol}_{\mathrm{Fal}}(T) &= \sum_{\sigma\in S_\infty} -2\log \mathrm{vol}_{\mathrm{Fal}}(P^\sigma) - 2\log \mathrm{vol}_{\mathrm{Ar}}(T) \\
&= \sum_{\sigma\in S_\infty} -2\epsilon_\sigma \log \|f_1^\sigma \wedge \cdots \wedge f_N^\sigma\|_{\mathrm{Fal}}^\sigma - 2\log \mathrm{vol}_{\mathrm{Ar}}(T) \\
&\leq \sum_{\sigma\in S_\infty} \epsilon_\sigma \Bigg(\tfrac{N\log N}{2} + Nc_\sigma + \log 2 + 2c_\sigma \\
&\quad - 2\epsilon_\sigma \log(2\pi\|\eta\|(\tau_\sigma)^2\sqrt{N!}) \Bigg) - 2\log \mathrm{vol}_{\mathrm{Ar}}(T) \\
&= [K:\mathbb{Q}]\left(\tfrac{N\log N}{2} + \log 2\right) + (N+2)c - [K:\mathbb{Q}]\log N! \\
&\quad - 2\sum_{\sigma\in S_\infty} \epsilon_\sigma \log(2\pi\|\eta\|(\tau_\sigma)^2) - 2\log \mathrm{vol}_{\mathrm{Ar}}(T).
\end{aligned}$$

By combining the inequalities found in this section, we find the following upper bound for the discriminant of L/K:

$$\begin{aligned}
\log\left|N_{K/\mathbb{Q}}(\Delta_{L/K})\right| &\leq 2(D\cdot\omega)_{\mathrm{Ar}} - 2\chi(O_X) + (2N+1)[K:\mathbb{Q}]\log 2 + N\log|\Delta_K| \\
&\quad + [K:\mathbb{Q}](N\log N - \log N!) + (2N+2)c - 2r\log V_N \\
&\quad - 2s\log V_{2N} - 2N[K:\mathbb{Q}]\log[K:\mathbb{Q}] - 2r\log N! - 2s\log(2N)! \\
&\quad + 2\log(N[K:\mathbb{Q}])! - 2\sum_{\sigma\in S_\infty} \log(2\pi\|\eta\|(\tau_\sigma)^2). \qquad (4.7.4)
\end{aligned}$$

If $X \to S$ is semi-stable, then we have by 4.5.1:

$$(D\cdot\omega)_{\mathrm{Ar}} = \frac{N}{12}\sum_{s\in S_{\mathrm{fin}}} \delta_s,$$

where δ_s equals $\log \#\kappa(s)$ times the number of singularities in the geometric fiber of $X \to S$ above s. In particular, this value does not depend on X, but only on E. Similarly, we have an equation for $\chi(O_X)$, by Faltings ([5, Theorem 7]):

$$\chi(O_X) = \tfrac{1}{12}\sum_{s\in S_{\mathrm{fin}}\cup S_\infty} \delta_s,$$

where δ_s is the same as earlier for s finite, and for $\sigma\in S_\infty$:

$$\delta_\sigma = -\epsilon_\sigma \log\left((2\pi)^{12}\|\eta\|(\tau_\sigma)^{24}\right).$$

Therefore $\chi(O_X)$, too, is an invariant of E.

One might be interested in how this upper bound behaves when N becomes large. Most terms are either $O(1)$ or $O(N)$, but there are some terms that need further investigation. Let us begin with the term $[K:\mathbb{Q}](N\log N - \log N!)$. This term is $O(N)$; this follows from Stirling's approximation of $\log N!$. We are left with the following term:

$$-2r\log V_N - 2s\log V_{2N} - 2r\log N! - 2s\log(2N)! + 2\log(N[K:\mathbb{Q}])!.$$

We have

$$\log V_N = \log\left(\frac{\pi^{N/2}}{\Gamma(\frac{N}{2}+1)}\right) = -\tfrac{N}{2}\log\tfrac{N}{2} + O(N) = -\tfrac{N}{2}\log N + O(n),$$

by Stirling's approximation. By applying Stirling's approximation to every other term in a similar way we get

$$rN \log N + 2sN \log N - 2rN \log N - 4sN \log N + 2[K : \mathbb{Q}]N \log N + O(N)$$

and as $r + 2s = [K : \mathbb{Q}]$ we are left with

$$[K : \mathbb{Q}]N \log N + O(N).$$

We therefore see that the upper bound 4.7.4 is of the form

$$[K : \mathbb{Q}]N \log N + O(N).$$

This proves the main theorem of this section.

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