A Giraud-type Theorem for Model Topoi

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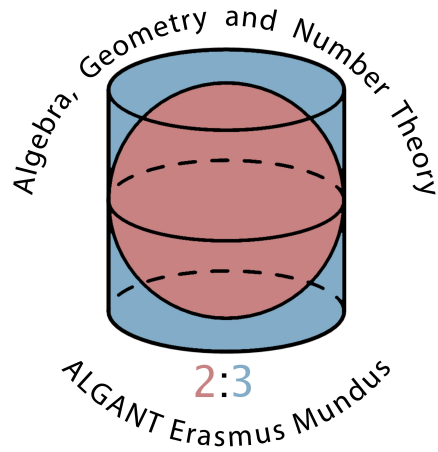
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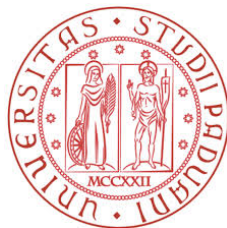
A Giraud-type Theorem for Model Topoi

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To those who still treasure epoché and yearn for ataraxia.

Acknowledgments.

Contents

Introduction.

This work is concerned with the formulation of a notion of *model topos*, intended as a model-categorical version of the classical concept of *Grothendieck topos*. Such a definition will be sensible enough to establish a *Giraud-type Theorem* for model topoi, as stated in the unpublished note *Toposes and Homotopy Toposes* by Charles Rezk ([Rzk1]). Our work is indeed an attempt of providing an account to the first part of that preprint. In general, we follow Rezk's statements and proofs, trying to fill in some non-trivial details still missing in his abovementioned exposition (see the abstract of [Rzk1] and the introductions to the various chapters below).

Classically, a *Grothendieck topos* is defined as a category which is equivalent to the category of *sheaves* $\mathrm{Sh}(\mathscr{C}, \tau)$ on a *Grothendieck site* $(\mathscr{C}, \tau)$. Here, a *Grothendieck site* consists of a small category $\mathscr{C}$ equipped with a *Grothendieck topology* τ and a *sheaf* on a Grothendieck site $(\mathscr{C}, \tau)$ is a particular kind of functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ (a *presheaf* on $\mathscr{C}$) satisfying suitable amalgamation conditions for matching families of its sections. Such a notion of sheaves appears, for instance, in [SGA3], Exposé IV and is well-known to many sorts of mathematicians, including (algebraic) geometers, topologists, analysts and logicians.

Despite their wide applications to the various fields of Mathematics, Grothendieck topoi are very interesting also when considered as purely categorical objects[1]. Indeed, they enjoy a lot of nice and desirable categorical properties. As categories, they are, for example, complete and cocomplete, cartesian and locally cartesian closed, locally presentable and they admit a subobject classifier. However, at least at first sight, Grothendieck topoi seem rather hard to detect. Indeed, to prove that a category $\mathscr{E}$ is a Grothendieck topos one has to build a Grothendieck topology τ on a small category $\mathscr{C}$ and exhibit an equivalence of categories $\mathscr{E} \simeq \mathrm{Sh}(\mathscr{C}, \tau)$. It would then be important to have a (quicker) criterion to establish if a category is a Grothendieck topos, by checking whether it satisfies some internal properties or not.

This task is accomplished by the so-called *Giraud's Theorem*. One of the possible statements of such a result says that a cocomplete and finitely complete category $\mathscr{E}$ is a Grothendieck topos if and only if it has a small set of generators and it satisfies *weak descent*, the latter being a request about certain compatibility conditions between colimits and pullbacks in $\mathscr{E}$. This version of Giraud's theorem is proven in [Rzk1]. Here, however, the definition of a Grothendieck topos is not given using the concept of sheaves for a Grothendieck topology. Indeed, Rezk defines a category $\mathrm{Sh}(\mathscr{C})$ of sheaves on a site to be a replete and reflective subcategory of $\mathrm{PSh}(\mathscr{C})$ (for a small category $\mathscr{C}$) admitting a left exact reflector.

Albeit being equivalent to the more traditional one, such a definition shows clearly the role played by the notion of a Grothendieck topos in a more general categorical framework. Indeed, declare that a category has *(categorical) small presentation* if it is equivalent to a reflective subcategory $\mathrm{PSh}(\mathscr{C})_S$ of *S-local objects* in $\mathrm{PSh}(\mathscr{C})$, for some *set S* of maps in $\mathrm{PSh}(\mathscr{C})$. Here, given a class T of arrows in a category $\mathscr{D}$, a T−local object is an object W in $\mathscr{D}$ whose associated representable functor $\mathrm{Hom}(-, W)$ sends elements of T into isomorphisms of sets (that is, upon "homming", W sees elements of T as isomorphisms). Subcategories of the form $\mathrm{PSh}(\mathscr{C})_S$ are actual localizations of $\mathrm{PSh}(\mathscr{C})$ and, up to equivalences of categories, they exhaust all locally presentable categories. On the other hand, Grothendieck topoi are (again up to equivalences) exactly all those categories admitting a small presentation with a left exact

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[1] Which is exactly what we will do in this work.

reflector.

Such a perspective on the notion of a Grothendieck topos has the great advantage to make a model-theoretical generalization of the topos-theoretical setting relatively straightforward and transparent. Passing to the model-categorical world, given a small category $\mathscr{C}$ one substitutes the presheaf category $\mathrm{PSh}(\mathscr{C})$ with the category $\mathrm{sPsh}(\mathscr{C})$ of *simplicial presheaves*, endowed with the *projective model structure*. One then says that a model category has *(homotopical) small presentation* if it is Quillen equivalent to $\mathrm{sPsh}(\mathscr{C})_S$, where $\mathscr{C}$ is a small category, S is a small set of maps in $\mathrm{sPsh}(\mathscr{C})$ and $\mathrm{sPsh}(\mathscr{C})_S$ denotes the left Bousfield localization of $\mathrm{sPsh}(\mathscr{C})$ with respect of S. At this level, there are already a lot of homotopical (that is, model-categorical) counterparts to classical results in ordinary category theory. For example, as $\mathrm{PSh}(\mathscr{C})$ is the free cocompletion of $\mathscr{C}$ under colimits, so $\mathrm{sPsh}(\mathscr{C})$ is the free cocompletion of $\mathscr{C}$ under homotopy colimits (what is called the *universal homotopy theory* on $\mathscr{C}$ in [Dug1]). Similarly, as every locally presentable category admits categorical small presentation, so every combinatorial model category admits homotopical small presentation ([Dug2]).

For technical reasons, one actually needs to consider a slight variation on the above setting, taking, for a small *simplicial* category $\mathbb{C}$, the model category $\mathrm{sPsh}(\mathbb{C})$ of *simplicial* functors $\mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ and its left Bousfield localizations. Coherently, one talks about *small simplicial presentation* for a model category. With this machinery at hand, a *model site* is defined to be a pair $(\mathbb{C}, S)$, where $\mathbb{C}$ is a small simplicial category and S is a small set of maps in $\mathrm{sPsh}(\mathbb{C})$ such that the (simplicial) left adjoint to the inclusion of $\mathrm{sPsh}(\mathbb{C})_S$ is *homotopically left exact*, i.e. it preserves *homotopy pullback squares*. A *model topos* is then a model category which is *Quillen equivalent* to $\mathrm{sPsh}(\mathbb{C})_S$ for some model site $(\mathbb{C}, S)$.

Within this model-categorical generalization of topos Theory, one can finally get also the announced *Giraud's Theorem for model topoi*: a model category $\mathscr{E}$ is a model topos if and only if it has small simplicial presentation and it satisfies *descent*. This latter property is, of course, the homotopical correction of weak descent for ordinary categories and consequently it says that in $\mathscr{E}$ homotopy colimits are well-behaved with respect to homotopy pullbacks. As in the case of its categorical counterpart, this property is exactly what is needed to turn a model category presented via generators and relations (i.e. as a Bousfield localization of a category of simplicial presheaves) into a model topos, which explains its preeminent role in Giraud's Theorem.

Organization of the work.

This work is divided in essentially two parts.

The first block (from Chapter 1 to Chapter 3) covers some elements from ordinary Topos Theory. In Chapter 1, we define Grothendieck topoi (up to equivalences of categories) as (replete and) reflective subcategories of presheaves categories with left exact reflector and we explore some of their properties, such as the existence of an orthogonal factorization system in every Grothendieck topos. We also consider the property of having small presentation for a category and show that every locally presentable category verifies it. Chapter 2 introduces the notion of weak descent for a cocomplete and finitely complete category $\mathscr{D}$. We then prove Giraud's Theorem for Grothendieck topoi: every Grothendieck topos has a small set of generators and satisfies weak descent and these two properties being both true in $\mathscr{D}$ imply that $\mathscr{D}$ actually is a Grothendieck topos. Some of the results in this chapter will be rather technical. Finally, in Chapter 3 we define Grothendieck (pre)topologies and sheaves on a Grothendieck site and establish the equivalence between our notion of a Grothendieck topos and the more usual one which is given using those concepts.

The second part of our work (from Chapter 4 to Chapter 6) is concerned with the model-categorical counterparts to the main results in the first chapters. In Chapter 4, we revise all the notions from the theory of model categories that we are going to use, focusing on definitions and statements rather than proofs, for which we essentially defer to the authoritative references given by [Hir1] and [Hov]. Once the needed concepts are settled, in Chapter 5 we start homotopifying the categorical notions we need. First of all, we develop some of the theory of localizations for model categories. Then, following [Dug1] and [Dug2], we try to explain in which sense simplicial presheaves on a small category $\mathscr{I}$ play an analogue role to that of presheaves on $\mathscr{I}$, namely, they are the *universal homotopy theory* built from $\mathscr{I}$. Furthermore, we define what it means for a model category to have small (simplicial) presentation and prove that every

(simplicial) combinatorial model category verifies this property. Finally, the core of our work comes in the last chapter, where we define model topoi and, using the descent properties for a model category, we give a model-categorical version of Giraud's Theorem.

Prerequisites and Conventions.

Throughout our work, we will freely use the basic tools from Category Theory, so that the reader is assumed to be acquainted with them. Although this assertion is deliberately vague, we do mean to include (co)ends and Kan extensions among the concepts the reader should be familiar with. Nevertheless, we shall recall some categorical notions from time to time, in order to fix terminology and point out some properties we may need. The author's suggestions for a background in Category Theory keep on being the classical works [McL] and [Bor1].

In dealing with simplicial (model) categories, we will also need some machinery from Enriched Category Theory. Again, we will review some concepts and results every now and then, but we will not give a full account of the subject, for which we refer to [Kel]. For the reader who is not comfortable with this branch of Category Theory, it would be enough to know that a simplicial category is a category having, for each pair of objects, a simplicial set (and not just a set) of morphisms. These morphisms can be composed via a composition law which is a map of simplicial sets. Most of the basic results in ordinary Category Theory carry over to the simplicial setting once all the concepts around are substituted by their enriched version (so that "functors" become "simplicial functors", "natural transformations" become "simplicial natural transformations" and so on).

A little smattering of classical (simplicial) homotopy theory may prove also useful, especially to understand some of the examples about topological spaces and simplicial sets as well as the classical model structures on both categories. The material present in §2.4 and in Chapter 3 of [Hov] should be by far enough for these purposes.

As for conventions, unless differently stated, in this work by a *category* $\mathscr{C}$ we will mean a *locally small category*, that is, for every couple of objects $A, B \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(A, B) = \mathscr{C}(A, B)$ is a set, with respect to some fixed Grothendieck Universe $\mathcal{U}$. We will call $\mathcal{U}-$sets simply sets or small sets and $\mathcal{U}-$classes simply classes. From time to time we will actually need to consider possibly large categories and in those cases we shall try to point out the issue. We will denote categories using capital script letters like $\mathscr{C}$, $\mathscr{D}$ or $\mathscr{E}$. The letters $\mathscr{M}$ and $\mathscr{N}$ will usually denote model categories. The symbol "$\cong$" will be reserved to indicate isomorphisms (between objects in a category or between categories). If $\mathscr{C}$ and $\mathscr{D}$ are categories, we will write $\mathscr{C} \simeq \mathscr{D}$ to mean that they are equivalent.

Given a category $\mathscr{D}$, $\mathrm{Mor}(\mathscr{D})$ will denote the class of morphisms in $\mathscr{D}$, while $\mathrm{Arr}(\mathscr{D})$ will be the arrow category of $\mathscr{D}$, so that $\mathrm{Ob}(\mathrm{Arr}(\mathscr{D})) = \mathrm{Mor}(\mathscr{D})$. If $\mathscr{C}$ is a small category and $\mathscr{D}$ is a category, the category of all functors from $\mathscr{C}$ to $\mathscr{D}$ will be denoted by $\mathscr{D}^{\mathscr{C}}$. When $\mathscr{D} = \mathbf{Set}$, the category $\mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ will usually be displayed as $\mathrm{PSh}(\mathscr{C})$ and its objects will be called *presheaves on $\mathscr{C}$*. In this case, we will indicate the Yoneda embedding $\mathscr{C} \longrightarrow \mathbf{Set}^{\mathscr{C}^{\mathrm{op}}}$ as

$$\mathbf{y} \colon \mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C}), \quad A \mapsto \mathscr{C}(-, A).$$

Chapter 1

Grothendieck Topoi.

> The introduction of the cipher 0 or the group concept was general nonsense, too, and for a thousand years or two mathematics was more or less stagnating, because nobody was around to make such childish steps.
>
> ———————————————
> Alexander Grothendieck,
> *Letter to Ronald Brown, 05/05/1982*

We begin our work by introducing the concept of *Grothendieck topos*, defined as a category which is equivalent to the category of sheaves on a site. Despite the terminology, our definition may appear unconventional, at least at a first sight. For us, indeed, a category of sheaves on a site will simply be any reflective subcategory of a presheaf category with a left exact reflector (see Section 1.1). This is tantamount to say that a category of sheaves is a left exact localization of a presheaf category $\mathrm{PSh}(\mathscr{C})$ (for a small category $\mathscr{C}$) identifiable with a full subcategory of *S-local objects* in $\mathrm{PSh}(\mathscr{C})$, where S is a small set of morphisms of presheaves (see Section 1.2). We will defer the proof that our notion of Grothendieck topos coincides with the more traditional one (given in terms of Grothendieck topologies) to Chapter 3. We will instead see how to deduce from our purely categorical description of Grothendieck topoi, some of their basic properties, such as the existence of a functorial orthogonal factorization system consisting of the classes of all monomorphisms and of all regular epimorphisms (see Section 1.3).

This chapter is essentially an expanded version of Section 1 in [Rzk1]. In particular, Section 1.2 below, although containing well-established results, arises from our attempt of understanding and working on Remark 1.7 of [Rzk1].

1.1 An coventional definition.

Let us start by recalling a couple of general notions that we are going to need immediately.

Definition 1.1.1. Let $\mathscr{B}$ be a category and $\mathscr{D}$ a subcategory.

1. We say that $\mathscr{D}$ is *replete (in $\mathscr{B}$)* if for any X in $\mathscr{D}$ and any isomorphism $f\colon X \xrightarrow{\cong} Y$ in $\mathscr{B}$, both f and Y are in $\mathscr{D}$.

2. We say that $\mathscr{D}$ is a *reflective subcategory of $\mathscr{B}$* if it is full and the inclusion functor of $\mathscr{D}$ into $\mathscr{B}$ has a left adjoint, which will be called the *reflector (of $\mathscr{B}$ into $\mathscr{D}$)*.

Remark 1.1.2. (i) The definition of a replete subcategory is symmetric in the sense that a subcategory $\mathscr{D} \subseteq \mathscr{B}$ is replete if and only if, for each $Y \in \mathscr{D}$ and any isomorphism $f\colon X \xrightarrow{\cong} Y$, both f and X are in $\mathscr{D}$.

(ii) If $\mathscr{D}$ is a *full* subcategory of $\mathscr{B}$, then $\mathscr{D}$ is replete if and only if each object in $\mathscr{B}$ which is isomorphic to an object in $\mathscr{D}$ is itself in $\mathscr{D}$.

(iii) Let $P\colon \mathscr{A} \longrightarrow \mathscr{B}$ be a functor between categories and declare that P is an *isofibration* if, for all objects $A \in \mathscr{A}$ and any isomorphism $\varphi\colon P(A) \overset{\cong}{\Rightarrow} B$ (for some $B \in \mathscr{B}$), there exists an isomorphism $\psi\colon A \overset{\cong}{\Rightarrow} A'$ such that $P(\psi) = \varphi$. Then a subcategory $\mathscr{D}$ of $\mathscr{B}$ is replete if and only if the inclusion of $\mathscr{D}$ in $\mathscr{B}$ is an isofibration.

Definition 1.1.3.   1. A *pseudo-site* is a pair $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$, where $\mathscr{C}$ is a small category and $\mathrm{Sh}(\mathscr{C})$ is a replete and reflective subcategory of $\mathrm{PSh}(\mathscr{C})$. An object of $\mathrm{Sh}(\mathscr{C})$ is called a *pseudo-sheaf* on the pseudo-site $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$.

   2. A *site* is a pseudo-site $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$ such that the left adjoint $a\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C})$ to the inclusion functor $i\colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$ is left exact. An object of $\mathrm{Sh}(\mathscr{C})$ is called a *sheaf* on the site $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$.

   3. A *presentable category* is a category $\mathscr{D}$ which is equivalent to the category of pseudo-sheaves on some pseudo-site.

   4. A *(Grothendieck) topos* is a category $\mathscr{E}$ which is equivalent to the category of sheaves on some site.

Example 1.1.4. Let (X, τ) be a topological space and let $\mathrm{Op}(X)$ be the category associated to the poset $(\tau, \subseteq)$. Let now $F \in \mathrm{PSh}(X) := \mathrm{PSh}(\mathrm{Op}(X))$ and take any open subset U of X and any covering $(U_i)_{i \in I}$ of U, where each U_i is open and contained in U. We have a map of sets

$$e\colon F(U) \to \prod_{i \in I} F(U_i)$$

sending $s \in F(U)$ to the element $(s_{|U_i})_{i \in I}$ in the product.[1] We also have two maps

$$\prod_{i \in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I^2} F(U_i \cap U_j)$$

induced by the family of maps

$$F(U_i \cap U_j \subseteq U_i)\colon F(U_i) \to F(U_i \cap U_j) \quad \text{and} \quad F(U_i \cap U_j \subseteq U_j)\colon F(U_j) \to F(U_i \cap U_j) \tag{1.1}$$

respectively. Since $p \circ e = q \circ e$, we can actually see e as a map

$$e\colon F(U) \to \mathrm{eq}\left(\prod_{i \in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I^2} F(U_i \cap U_j) \right).$$

One defines the category $\mathrm{Sh}(X)$ of *sheaves over* X as the full subcategory of $\mathrm{PSh}(X) = \mathrm{PSh}(\mathrm{Op}(X))$ given by those $F\colon \mathrm{Op}(X)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ such that, for all open subsets U of X and every open covering $(U_i)_{i \in I}$ of U as above, the map e is an isomorphism:

$$e\colon F(U) \overset{\cong}{\longrightarrow} \mathrm{eq}\left(\prod_{i \in I} F(U_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I^2} F(U_i \cap U_j) \right). \tag{1.2}$$

It is a classical result in sheaf theory (see, for example, [McMo] §II.5, Corollary 4) that the inclusion of $\mathrm{Sh}(X)$ in $\mathrm{PSh}(X)$ admits a left adjoint a, called the *sheafification functor*, which commutes with finite limits, so that $\mathrm{Sh}(X)$ is a Grothendieck topos.

Example 1.1.5. Let $\mathscr{T}$ be a small category with finite coproducts, so that $\mathscr{T}^{\mathrm{op}}$ is an *algebraic theory*, i.e. a small category admitting finite products.[2] Let $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ be the category of *algebras over* $\mathscr{T}^{\mathrm{op}}$, i.e the full subcategory of $\mathrm{PSh}(\mathscr{T})$ consisting of those presheaves which preserve finite products. Clearly $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ is a replete subcategory of $\mathrm{PSh}(\mathscr{T})$. Actually, it can be shown (see [ARV] Proposition 6.17) that $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ is a reflective subcategory of $\mathrm{PSh}(\mathscr{T})$ (closed under sifted colimits). The proof of this statement goes along the following lines. Firstly, one realises that the Yoneda embedding $\mathbf{y}\colon \mathscr{T} \longrightarrow \mathrm{PSh}(\mathscr{T})$ is such that, for all $A \in \mathscr{T}$, $\mathbf{y}(A) \in \mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$. Thus, we can co-restrict the Yoneda embedding to get a functor

$$\mathbf{y}_{\mathscr{T}^{\mathrm{op}}}\colon \mathscr{T} \longrightarrow \mathrm{Alg}(\mathscr{T}^{\mathrm{op}}).$$

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[1] As it is customary in this context, $s_{|U_i}$ denotes the restriction of s to $F(U_i)$, i.e. $s_{|U_i} = F(U_i \subseteq U)(s)$.

[2] Here we take the definition of an algebraic theory as given in [AdRo] so that, in particular, we do not require algebraic theories to have a (essentially) countable set of objects.

Since $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ is cocomplete, one can construct a functor $a\colon \mathrm{PSh}(\mathscr{T}) \longrightarrow \mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ as the left Kan extension of $\mathbf{y}_{\mathscr{T}^{\mathrm{op}}}$ along the (usual) Yoneda embedding $\mathbf{y}$ (see Definition 2.2.14 and Remark 2.2.15 below). Such an a is then a left adjoint for the functor $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})(\mathbf{y}_{\mathscr{T}^{\mathrm{op}}}(-), \bullet)\colon \mathrm{Alg}(\mathscr{T}^{\mathrm{op}}) \longrightarrow \mathrm{PSh}(\mathscr{T})$ which sends $F \in \mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ to the presheaf $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})(\mathbf{y}_{\mathscr{T}^{\mathrm{op}}}(-), F)$. By Yoneda's lemma, this functor $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})(\mathbf{y}_{\mathscr{T}^{\mathrm{op}}}(-), \bullet)$ is isomorphic to the inclusion of $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ into $\mathrm{PSh}(\mathscr{T})$. Therefore, $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ is a presentable category.

Example 1.1.6. Every Grothendieck topos is a presentable category by definition, but the converse does not hold. An example is provided by the category of algebras over $\mathscr{T}^{\mathrm{op}}$, where $\mathscr{T}^{\mathrm{op}}$ is given by the full subcategory of $\mathbf{Ab}^{\mathrm{op}}$ consisting of the objects $\mathbb{Z}^n$, for $n \in \mathbb{N}$. By Example 1.1.5, we know that $(\mathscr{T}^{\mathrm{op}}, \mathrm{Alg}(\mathscr{T}^{\mathrm{op}}))$ is a pseudo-site. The point now is that $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ is equivalent to the category of abelian groups (see [ARV] Example 1.11). Indeed, given an abelian group G, we can define an algebra $\widehat{G}$ over $\mathscr{T}^{\mathrm{op}}$ as

$$\widehat{G} := \mathbf{Ab}(-, G)_{|\mathscr{T}^{\mathrm{op}}},$$

the restriction of the representable functor associated to G to our subcategory $\mathscr{T}^{\mathrm{op}}$. We then get a functor $\widehat{(-)}\colon \mathbf{Ab} \longrightarrow \mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ which is fully faithful. Moreover, given an algebra F over $\mathscr{T}^{\mathrm{op}}$, we can consider the abelian group G whose underlying set is $F(\mathbb{Z})$ and whose multiplication is given by the image under F of the diagonal morphism $\mathbb{Z} \to \mathbb{Z}^2$ in $\mathscr{T}$. Since F preserves products, F is isomorphic to $\widehat{G}$. Therefore, $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ is equivalent to $\mathbf{Ab}$, so that, in particular, it is a non-trivial abelian category (i.e. an abelian category which is not equivalent to the terminal (abelian) category $\mathbf{1}$). This prevents $\mathrm{Alg}(\mathscr{T}^{\mathrm{op}})$ from being a Grothendieck topos because a Grothendieck topos needs to be cartesian closed (see Theorem 1.1.11 below) and the only cartesian closed category having a zero object (up to equivalence) is the terminal category.[3]

The following result characterises pseudo-sheaves among presheaves.

Lemma 1.1.7. *Let $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$ be a pseudo-site and let $i\colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$ be the inclusion, with left adjoint $a\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C})$.*

1. *For every $F \in \mathrm{Sh}(\mathscr{C})$, the counit map $\varepsilon_F\colon a(i(F)) \Longrightarrow F$ is an isomorphism.*

2. *The following are equivalent, for $F \in \mathrm{PSh}(\mathscr{C})$:*

    (2a) $F \in \mathrm{Sh}(\mathscr{C})$;

    (2b) for all $\tau\colon A \Longrightarrow B$ in $\mathrm{PSh}(\mathscr{C})$, if $a(\tau)\colon a(A) \Longrightarrow a(B)$ is an isomorphism in $\mathrm{Sh}(\mathscr{C})$, then $\mathrm{PSh}(\tau, F)\colon \mathrm{PSh}(\mathscr{C})(B, F) \to \mathrm{PSh}(\mathscr{C})(A, F)$ is an isomorphism in $\mathbf{Set}$;

    (2c) the unit map $\eta_F\colon F \Longrightarrow i(a(F))$ is an isomorphism in $\mathrm{PSh}(\mathscr{C})$.

Proof. 1. This follows as the inclusion functor is fully faithful (see [McL] §IV.3, Theorem 1).

2. We prove that $(2a) \Longrightarrow (2b) \Longrightarrow (2c) \Longrightarrow (2a)$.

    - $(2a) \Longrightarrow (2b)$. Take $\tau\colon A \Longrightarrow B$ in $\mathrm{PSh}(\mathscr{C})$ such that $a(\tau)\colon a(A) \Longrightarrow a(B)$ is an isomorphism of pseudo-sheaves. In the commutative diagram

    $$
    \begin{array}{ccc}
    \mathrm{Sh}(\mathscr{C})(a(B), F) & \xrightarrow{\;-\circ a(\tau)\;} & \mathrm{Sh}(\mathscr{C})(a(A), F) \\
    {\scriptstyle \cong}\Big\downarrow & & \Big\downarrow{\scriptstyle \cong} \\
    \mathrm{PSh}(\mathscr{C})(B, F) & \xrightarrow[\mathrm{PSh}(\tau, F)]{} & \mathrm{PSh}(\mathscr{C})(A, F)
    \end{array}
    $$

    the top row is an isomorphism of sets, as $a(\tau)$ is an isomorphism, so also the bottom row is such.

---

[3] If $\mathscr{D}$ is a cartesian closed category with a zero object 0, then, for all objects $X \in \mathscr{D}$,

$$0 \cong 0 \times X \cong X,$$

where the first isomorphism follows because $- \times X$ commutes with colimits (being a left adjoint), while the second is due to 0 being terminal.

- (2b) $\implies$ (2c). By the triangular identities for adjunctions, $\varepsilon_{a(F)} \circ a(\eta_F) = 1_{a(F)}$. Since $a(F)$ is a pseudo-sheaf, $\varepsilon_{a(F)}$ is an isomorphism, hence also $a(\eta_F)$ is such. By hypothesis, we have an isomorphism of sets

$$- \circ \eta_F \colon \text{PSh}(\mathscr{C})(a(F), F) \xrightarrow{\cong} \text{PSh}(\mathscr{C})(F, F),$$

where we wrote $a(F)$ instead of $a(i(F))$. This means that there is a natural transformation $\sigma \colon a(F) \implies F$ such that $\sigma \circ \eta_F = 1_F$. But now the diagram

$$
\begin{array}{ccc}
\text{PSh}(\mathscr{C})(a(F), F) & \xrightarrow{\ -\circ\eta_F\ } & \text{Sh}(\mathscr{C})(F, F) \\
{\scriptstyle \eta_F \circ -}\big\downarrow & & \big\downarrow{\scriptstyle \eta_F \circ -} \\
\text{Sh}(\mathscr{C})(a(F), a(F)) & \xrightarrow[\varphi]{} & \text{PSh}(\mathscr{C})(F, a(F))
\end{array}
$$

commutes. Here φ is the adjoint isomorphism ($i(aF) = aF$) and commutativity follows as, for all $\beta \in \text{PSh}(\mathscr{C})(a(F), F)$,

$$\varphi(\eta_F \circ \beta) = (\eta_F \circ \beta) \circ \eta_F = \eta_F \circ (\beta \circ \eta_F),$$

where the first equality follows by the description of φ in terms of the unit of the adjunction[4]. Taking $\beta = \sigma$, we get $\varphi(\eta_F \circ \sigma) = \varphi(1_{a(F)})$, so $\eta_F \circ \sigma = 1_{a(F)}$, i.e. η_F is an isomorphism.

- (2c) $\implies$ (2a). This follows trivially, as $\text{Sh}(\mathscr{C})$ is closed under isomorphisms in $\text{PSh}(\mathscr{C})$.

$\square$

Proposition 1.1.8. *A presentable category is complete and cocomplete.*

Proof. Any reflective and replete subcategory $\mathscr{D}$ of a complete and cocomplete category $\mathscr{B}$ is itself complete and cocomplete[5]. Limits in $\mathscr{D}$ coincide with those in $\mathscr{B}$, whereas the colimit of a functor $F \colon \mathscr{I} \longrightarrow \mathscr{D}$ is given by

$$a(\text{colim}\, iF),$$

where $i \colon \mathscr{D} \longrightarrow \mathscr{B}$ is the inclusion functor and $a \colon \mathscr{B} \longrightarrow \mathscr{D}$ is its left adjoint. This suffices to give the desired result. $\square$

Presentable categories are rich enough to give Adjoint Functor Theorems the simplest formulation they can have. Namely, we have the following

Proposition 1.1.9. *Let $\mathscr{E}$ be a presentable category. A functor $F \colon \mathscr{E}^{\text{op}} \longrightarrow \mathbf{Set}$ is representable by an object of $\mathscr{E}$ if and only if it commutes with limits (computed in $\mathscr{E}^{\text{op}}$).*

Proof. The property of commuting with limits is clearly necessary for a functor to be representable. To prove that it is also sufficient, we start by considering first the case when $\mathscr{E} = \text{PSh}(\mathscr{C})$ for a small category $\mathscr{C}$. Let then $F \colon \text{PSh}(\mathscr{C})^{\text{op}} \longrightarrow \mathbf{Set}$ be a limit-preserving functor and define a presheaf X on $\mathscr{C}$ by setting $X := F\mathbf{y}$, where $\mathbf{y} \colon \mathscr{C} \longrightarrow \text{PSh}(\mathscr{C})$ is the Yoneda embedding. Now, we have that

$$X \cong \text{colim}_{\text{El}(X)}\, \mathbf{y}U,$$

where $\text{El}(X)$ is the category of elements of X (objects in $\text{El}(X)$ are pairs (x_A, A), where A is an object of $\mathscr{C}$ and $x_A \in X(A)$, whereas morphisms $(x_A, A) \to (x_B, B)$ in $\text{El}(X)$ are morphisms $s \colon A \to B$ in $\mathscr{C}$ such that $(Xs)(x_B) = x_A$) and $U \colon \text{El}(X) \longrightarrow \mathscr{C}$ is the forgetful functor sending an object (x_A, A) of $\text{El}(X)$ to $A \in \mathscr{C}$. Therefore, using the hypothesis on F, we have the following chain of natural isomorphisms

$$F(X) \cong F(\text{colim}_{\text{El}(X)}\, \mathbf{y}U) \cong \lim_{\text{El}(X)} F(\mathbf{y}U) \cong \lim_{\text{El}(X)} \text{PSh}(\mathscr{C})(\mathbf{y}U, X),$$

where the last isomorphism is given by Yoneda's Lemma, as, for all object (x_A, A) of $\text{El}(X)$, one has

$$(F(\mathbf{y}U))(A) = (F\mathbf{y})(A) \cong \text{PSh}(\mathscr{C})(\mathscr{C}(-, A), F\mathbf{y}) = \text{PSh}(\mathscr{C})((\mathbf{y}U)(x_A, A), X).$$

---

[4] Suppose given an adjunction $L \colon \mathscr{C} \rightleftarrows \mathscr{D} \colon R$ (left adjoint on the left) with adjunction isomorphism $\varphi = \varphi_{X,A} \colon \mathscr{D}(LX, A) \cong \mathscr{C}(X, RA)$ and unit $\eta = \eta_X$, for $X \in \mathscr{C}$ and $A \in \mathscr{D}$. Then, for any morphism $f \colon LX \to A$ in $\mathscr{D}$, one has $\varphi_{X,A}(f) = R(f) \circ \eta_X$.

[5] See [Bor1], Proposition 3.5.4 and Proposition 3.5.5 (note that the author there assumes every reflective category to be replete by definition).

Again by Yoneda's Lemma, for all $(x_A, A) \in \mathrm{El}(X)$, we have a uniquely determined morphism (natural transformation) $u_{(x_A, A)} \colon (\mathbf{y}U)(x_A, A) \Longrightarrow X$ and these morphisms form the (x_A, A)-th coordinate of an element $u \in \lim_{\mathrm{El}(X)} \mathrm{PSh}(\mathscr{C})(\mathbf{y}U, X) \in \mathbf{Set}$ (just by definition of the arrows in $\mathrm{El}(X)$). We can then define a map of sets, for all $Y \in \mathrm{PSh}(\mathscr{C})$, given by

$$\varphi = \varphi_Y \colon \mathrm{PSh}(\mathscr{C})(Y, X) \longrightarrow F(Y), \quad \tau \mapsto (F\tau)(u).$$

It is easy to see that such a map is natural in $Y \in \mathrm{PSh}(\mathscr{C})$. We claim that actually φ_Y is an isomorphism for all $Y \in \mathrm{PSh}(\mathscr{C})$. Indeed, this is evidently true when $Y = \mathbf{y}(C)$ for $C \in \mathscr{C}$ (this is Yoneda's Lemma). On the other hand, an arbitrary $Y \in \mathrm{PSh}(\mathscr{C})$ is a colimit of representables and both F and $\mathrm{PSh}(\mathscr{C})(-, X)$ preserves limits (seen as functors from $\mathscr{E}^{\mathrm{op}}$ and from $\mathscr{C}^{\mathrm{op}}$ respectively). We then conclude that φ is an isomorphism.

Assume now that $\mathscr{E} = \mathrm{Sh}(\mathscr{C})$ is the subcategory of pseudo-sheaves for a pseudo-site $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$ and let $a \colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C})$ be the left adjoint to the inclusion of $\mathrm{Sh}(\mathscr{C})$ into $\mathrm{PSh}(\mathscr{C})$. Given a functor $F \colon \mathrm{Sh}(\mathscr{C})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ which commutes with limits, by what we have already proved, $Fa \colon \mathrm{PSh}(\mathscr{C})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable by a presheaf X. If $\tau \colon Y \Longrightarrow Z$ is a morphism of presheaves such that $a(\tau)$ is an isomorphism, then $\mathrm{PSh}(\mathscr{C})(\tau, X)$ is also an isomorphism, as it is the composite $\varphi_Y^{-1} F(a\tau) \varphi_Z$, where $\varphi \colon \mathrm{PSh}(\mathscr{C})(-, X) \Longrightarrow Fa$ is the natural transformation defined above. Therefore, by Lemma 1.1.7, $X \in \mathrm{Sh}(\mathscr{C})$. Since, for all pseudo-sheaf W, $\mathrm{Sh}(\mathscr{C})(W, X) \cong F(a(iW)) \cong F(W)$, we get that F is represented by the pseudo-sheaf X. This completes the proof. $\square$

Corollary 1.1.10. *Let $\mathscr{E}$ be a presentable category and $\mathscr{D}$ an arbitrary category. A functor $L \colon \mathscr{E} \longrightarrow \mathscr{D}$ has a right adjoint if and only if it preserves small colimits.*

Proof. Again the condition of preserving colimits is well-known to be necessary. Suppose then that a functor L as in the statement preserves all small colimits. The existence of a right adjoint $R \colon \mathscr{D} \longrightarrow \mathscr{E}$ for L is equivalent to the existence, for all $Y \in \mathscr{D}$ of an object $RY \in \mathscr{E}$ such that the functor

$$\mathscr{D}(L(-), Y) \colon \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

is represented by RY. Since L preserves colimits, $\mathscr{D}(L(-), Y)$ clearly preserves limits and we can conclude by Proposition 1.1.9 above. $\square$

As a further corollary, we can show the following

Theorem 1.1.11. *A Grothendieck topos $\mathscr{E}$ is a cartesian closed category.*

Proof. In view of the above Corollary, it is enough to prove that, for each fixed $Y \in \mathscr{E}$, the functor $(-) \times Y \colon \mathscr{E} \longrightarrow \mathscr{E}$ preserves colimits. This is clearly true when $\mathscr{E} = \mathbf{Set}$ (as $\mathbf{Set}$ is a cartesian closed category) and therefore it is also true for $\mathscr{E} = \mathrm{PSh}(\mathscr{C})$ for any small category $\mathscr{C}$ (because colimits in $\mathrm{PSh}(\mathscr{C})$ are computed objectwise). To show the required property for a general Grothendieck topos $\mathscr{E}$, it is enough to prove it for $\mathscr{E} = \mathrm{Sh}(\mathscr{C})$, where $\mathrm{Sh}(\mathscr{C})$ is such that $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$ is a site. Given $Y \in \mathrm{Sh}(\mathscr{C})$, the fact that $(-) \times Y \colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C})$ preserves colimits is now an immediate consequence of left-exactness of the left adjoint a to the inclusion $i \colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$. Indeed, if $F \colon \mathscr{J} \longrightarrow \mathrm{Sh}(\mathscr{C})$ is a functor with $\mathscr{J}$ small, then we have

$$(\mathrm{colim}_{j \in \mathscr{J}} F(j)) \times Y \cong a\left(\mathrm{colim}_{j \in \mathscr{J}} (iF)(j)\right) \times Y \overset{(\dagger)}{\cong} a\left(\mathrm{colim}_{j \in \mathscr{J}} (iF)(j)\right) \times aiY \cong$$

$$\cong a\left((\mathrm{colim}_{j \in \mathscr{J}} (iF)(j)) \times iY\right) \overset{(\ddagger)}{\cong} a\left(\mathrm{colim}_{j \in \mathscr{J}} ((iF)(j) \times iY)\right) \cong \mathrm{colim}_{j \in \mathscr{J}} (F(j) \times Y),$$

where the first and the last isomorphisms are given by how colimits are computed in $\mathrm{Sh}(\mathscr{C})$, $(\dagger)$ holds because $Y \in \mathrm{Sh}(\mathscr{C})$ and $(\ddagger)$ follows as the endofunctor $- \times iY \colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$ preserves colimits. $\square$

1.2   Locally presentable categories.

Our Lemma 1.1.7 above produces some criterions to recognise whether a presheaf over a category $\mathscr{C}$ belongs to a subcategory of pseudo-sheaves. Actually, a careful examination of both the statement and the proof of Lemma 1.1.7 shows that an identical result carries over to arbitrary reflective and replete subcategories $\mathscr{D}$ of a category $\mathscr{C}$. Namely, we can make the following

Remark 1.2.1. If $i\colon \mathscr{D} \longrightarrow \mathscr{C}$ is the inclusion of a replete, reflective subcategory with left adjoint $a\colon \mathscr{C} \longrightarrow \mathscr{D}$, then the counit of the adjunction is always an isomorphism, while an object X of $\mathscr{C}$ actually belongs to $\mathscr{D}$ precisely when the $X-$th component of the unit is also an isomorphism. This, in turn, is the case if and only if, for all morphisms $f\colon A \to B$ in $\mathscr{C}$ such that $a(f)$ is an isomorphism, the map of sets given by

$$\mathscr{C}(f, X)\colon \mathscr{C}(B, X) \to \mathscr{C}(A, X)$$

is a bijection.

With this observation in mind, we can now prove a result which shows the *local nature* of reflective subcategories. Before doing this, let us first agree on what we mean by a localization of a category at a class of morphisms.

Definition 1.2.2. Let $\mathscr{C}$ be a category and S a class of morphisms in $\mathscr{C}$. A *(nonstrict) localization of $\mathscr{C}$ at S* is a pair

$$(\mathscr{C}[S^{-1}],\ \gamma\colon \mathscr{C} \longrightarrow \mathscr{C}[S^{-1}]),$$

where $\mathscr{C}[S^{-1}]$ is a (possibly large) category and γ is a functor satisfying the following properties:

L1. for each morphism $s \in S$, $\gamma(s)$ is invertible in $\mathscr{C}[S^{-1}]$;

L2. given any functor $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ such that, for all $s \in S$, $F(s)$ is invertible in $\mathscr{B}$, there exist a functor $F_S\colon \mathscr{C}[S^{-1}] \longrightarrow \mathscr{B}$ and a natural isomorphism $\psi\colon F \overset{\cong}{\Longrightarrow} F_S\gamma$;

L3. given any category $\mathscr{B}$ the functor between (possibly large) functor categories

$$- \circ \gamma\colon \mathscr{C}[S^{-1}]^{\mathscr{B}} \longrightarrow \mathscr{C}^{\mathscr{B}}$$

is fully faithful. (So that, in particular, for every functor $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ as in L2. the functor F_S is essentially unique).

We can now give the following

Proposition 1.2.3. *Let $i\colon \mathscr{D} \longrightarrow \mathscr{C}$ be the inclusion of a reflective subcategory of $\mathscr{C}$ and let $a\colon \mathscr{C} \longrightarrow \mathscr{D}$ denote the left adjoint. Consider the class S consisting of all those morphisms f in $\mathscr{C}$ such that $a(f)$ is invertible in $\mathscr{D}$. Then the following hold.*

1. *$(\mathscr{D},\ a\colon \mathscr{C} \longrightarrow \mathscr{D})$ is a localization of $\mathscr{C}$ at S.*

2. *Assume in addition that $\mathscr{D}$ is replete[6] (see Definition 1.1.1) and let $\mathscr{C}_S$ be the full subcategory of $\mathscr{C}$ given by those objects $X \in \mathscr{C}$ such that, for all $s \in S$, $\mathscr{C}(s, X)$ is an isomorphism. Then $\mathscr{D} = \mathscr{C}_S$.*

Proof. We denote by η and ε the unit and the counit of the adjunction $a \dashv i$ respectively.

1. We check the three conditions in the Definition 1.2.2 of localization for the pair $(\mathscr{D},\ a)$.

   L1. This is trivially true by definition of the class S.

   L2. Let us first observe that, for all $X \in \mathscr{C}$, $\eta_X\colon X \to i(a(X))$ belongs to S. Indeed, $a(\eta_X)$ is an isomorphism, as it has a left inverse $\varepsilon_{a(X)}$ which is an isomorphism. Suppose then given a functor $F\colon \mathscr{C} \longrightarrow \mathscr{B}$ such that, for all $s \in S$, $F(s)$ is invertible in $\mathscr{B}$. Setting $F_S := F_{|\mathscr{D}}$ (the restriction of F to the subcategory $\mathscr{D}$), we get a natural isomorphism $\psi\colon F \Longrightarrow F_S a$ given by $F\eta$.

---

[6] This is not a strict requirement at all, as, if $\mathscr{D}$ is not replete, then closing up $\mathscr{D}$ for isomorphisms gives a full replete subcategory $\mathscr{D}'$ of $\mathscr{C}$ which is easily seen to be again reflective in $\mathscr{C}$.

L3. Let $\mathscr{B}$ be a category and $F, G \colon \mathscr{D} \longrightarrow \mathscr{B}$ be functors. Assume given $\tau, \sigma \colon F \Longrightarrow G$ natural transformations such that $\tau_a = \sigma_a \colon Fa \Longrightarrow Ga$. Then, using naturality of τ and of σ, we get, for any $A \in \mathscr{D}$,

$$\tau_A = G(\varepsilon_A)\tau_{a(i(A))}F(\varepsilon_A)^{-1} = G(\varepsilon_A)\sigma_{a(i(A))}F(\varepsilon_A)^{-1} = \sigma_A,$$

so that $\tau = \sigma$. On the other hand, suppose given a natural transformation $\tau \colon Fa \Longrightarrow Ga$. Then the natural transformation $\sigma \colon F \Longrightarrow G$ defined, for $A \in \mathscr{D}$, by $\sigma_A := G(\varepsilon_A)\tau_{i(A)}F(\varepsilon_A)^{-1}$ is such that, for all $X \in \mathscr{C}$, $\sigma_{a(X)} = \tau_X$, by naturality of τ. Therefore, $\sigma_a = \tau$ and L3. is established.

2. This is essentially the content of Remark 1.2.1 above.

$\square$

The second claim of the previous Proposition can be restated by saying that a replete and reflective subcategory is the subcategory of *S-local objects* with respect to the class S of morphisms which are inverted by the reflector. Indeed, we give the following

Definition 1.2.4. Let $\mathscr{C}$ be a category and S a class of morphisms in $\mathscr{C}$. An object $X \in \mathscr{C}$ is called *S-local* if, for every morphism $s \colon A \to B$ in S, the map of sets

$$\mathscr{C}(s, X) \colon \mathscr{C}(B, X) \to \mathscr{C}(A, X)$$

is an isomorphism. We denote by $\mathscr{C}_S$ the full subcategory of $\mathscr{C}$ spanned by the $S-$local objects.

Note that $\mathscr{C}_S$ is clearly a replete subcategory of $\mathscr{C}$.

As usual, we are particularly interested in the specialisation of the above notion to categories of presheaves.

Definition 1.2.5. A category $\mathscr{D}$ is said to *admit small presentation* if it is equivalent to $\mathrm{PSh}(\mathscr{C})_S$ for some small category $\mathscr{C}$ and some *small set S* of morphisms in $\mathrm{PSh}(\mathscr{C})$.

Our concern for these particular kinds of subcategories of presheaves categories relies on the following remarkable result, whose proof is the goal of the whole section.

Theorem 1.2.6. *Categories admitting a small presentation are exactly the locally presentable categories. More precisely, the following hold.*

1. *Let $\mathscr{C}$ be a small category and $S \subseteq \mathrm{Mor}(\mathrm{PSh}(\mathscr{C}))$ a small set of morphisms in $\mathrm{PSh}(\mathscr{C})$. Then $\mathrm{PSh}(\mathscr{C})_S$ is a reflective subcategory of $\mathrm{PSh}(\mathscr{C})$, closed under limits in $\mathrm{PSh}(\mathscr{C})$ and cocomplete. Furthermore, $\mathrm{PSh}(\mathscr{C})_S$ is a locally presentable category.*

2. *Every locally presentable category is equivalent to $\mathrm{PSh}(\mathscr{C})_S$ for some small category $\mathscr{C}$ and some small subset S of arrows in $\mathrm{PSh}(\mathscr{C})$, i.e. every locally presentable category admits small presentation*

Remark 1.2.7. In the situation of Theorem 1.2.6 (1), we get in particular that $(\mathscr{C}, \mathrm{PSh}(\mathscr{C})_S)$ is a pseudo-site.

For the sake of a self-contained exposition, we will recall the needed notions and results to prove Theorem 1.2.6. We shall follow the exposition of [AdRo].

We begin with a set-theoretic notion.

Definition 1.2.8. Let λ be a cardinal.

1. The *cofinality* of λ is the smallest ordinal[7] α such that there is a map $f \colon \alpha \to \lambda$ with unbounded range. The cofinality of λ is denoted by $\mathrm{cof}(\lambda)$.

2. λ is called a *regular* cardinal if $\lambda = \mathrm{cof}(\lambda)$.

---

[7] Recall that an *ordinal* is a set which is well-ordered by the membership relation $\in$. Such an ordinal α is a *limit ordinal* if it is not empty and $\alpha = \cup \alpha$. The class of all ordinals is well-ordered by the relation $\alpha < \delta$ if and only if $\alpha \in \delta$, for ordinals α and δ.

For the rest of this section, λ will always denote a *regular* cardinal.

Definition 1.2.9. 1. A category $\mathscr{D}$ is called $\lambda-$*filtered* provided that every small subcategory of $\mathscr{D}$ having less than λ morphisms admits a compatible cocone in $\mathscr{D}$, i.e. given any small subcategory $\mathscr{I}$ of $\mathscr{D}$ such that $|\operatorname{Mor}(\mathscr{I})| < \lambda$, there exist an object $B \in \mathscr{D}$ and a family of morphisms

$$(f_i \colon A_i \to B)_{A_i \in \operatorname{Ob}(\mathscr{I})},$$

such that, for all morphisms $s \colon A_i \to A_j$ in $\mathscr{I}$, $f_j \circ s = f_i$. A functor from a $\lambda-$filtered category is called a $\lambda-$*filtered diagram* (or a $\lambda-$*filtered functor*). A λ-*filtered colimit* is a colimit of a $\lambda-$filtered functor.

2. An object A of a category $\mathscr{D}$ is called $\lambda-$*presentable* if the Hom-functor $\operatorname{Hom}_{\mathscr{D}}(A, -)$ preserves $\lambda-$filtered colimits. An object A of $\mathscr{D}$ is *presentable* if it is $\lambda-$presentable for some regular cardinal λ.

3. A category $\mathscr{D}$ is *locally $\lambda-$presentable* if it is cocomplete and has a *small* set $\mathcal{A}$ of $\lambda-$presentable objects such that every object in $\mathscr{D}$ is a (small) $\lambda-$filtered colimit of objects in $\mathcal{A}$. A category $\mathscr{D}$ is *locally presentable* if it is locally $\lambda-$presentable for some regular cardinal λ.

4. A *locally finitely presentable* category is a locally $\aleph_0-$presentable category.

Remark 1.2.10. Let us list some basic facts about presentable objects and locally presentable categories.

(i) An $\aleph_0-$filtered category (resp. an $\aleph_0-$filtered colimit) is a filtered category (resp. a filtered colimit) in the usual sense.

(ii) We say that a poset $(I, \leq)$ is $\lambda-$*directed* if every subset of I having cardinality smaller than λ has an upper bound in I. Similarly, one defines $\lambda-$*directed diagrams* and $\lambda-$*directed colimits*. It can be proven that any $\lambda-$filtered colimit can be realised as a $\lambda-$directed colimit, so that we could have given Definitions 1.2.9.2. and 1.2.9.3. above using $\lambda-$directed colimits instead of $\lambda-$filtered colimits.

(iii) Suppose given a functor $F \colon \mathscr{I} \longrightarrow \mathscr{D}$, where $\mathscr{D}$ is any category and $|\operatorname{Mor}(\mathscr{I})| < \lambda$ (such a functor is also called a $\lambda-$*small* diagram in $\mathscr{D}$). If $F(i)$ is a $\lambda-$presentable object in $\mathscr{D}$ for all objects $i \in \mathscr{I}$ and if $\operatorname{colim} F$ exists in $\mathscr{D}$, then $\operatorname{colim} F$ is a $\lambda-$presentable object. In particular, every object in a locally $\lambda-$presentable category is presentable.

(iv) The condition on the subset $\mathcal{A}$ in the definition of a locally $\lambda-$presentable category $\mathscr{D}$ is satisfied exactly when both the following properties are verified:

   (a) every object of $\mathscr{D}$ is a $\lambda-$directed colimit of $\lambda-$presentable objects

and

   (b) there exists, up to isomorphisms, only a small set of $\lambda-$presentable objects. More precisely, consider the class of $\lambda-$presentable objects in $\mathscr{D}$ and the equivalence relation on it which identifies two elements if they are isomorphic objects in $\mathscr{D}$. Then any class obtained by choosing a representative for every equivalence class is actually a small set.

We shall denote by $\mathbf{Pres}_\lambda(\mathscr{D})$ the full (small) subcategory generated by any of these essentially unique sets of $\lambda-$presentable objects in a locally λ-presentable category $\mathscr{D}$.

(v) Let $\mathscr{D}$ be a category and $\mathscr{C}$ be a small subcategory of $\mathscr{D}$. For every object $K \in \mathscr{D}$, we denote by $(\mathscr{C} \downarrow K)$ the full subcategory of the slice category $(\mathscr{D} \downarrow K)$ having as objects all the morphisms $X \to K$ in $\mathscr{D}$, where $X \in \mathscr{C}$. The *canonical diagram* of K with respect to $\mathscr{C}$ is the (obvious) forgetful functor $U \colon (\mathscr{C} \downarrow K) \longrightarrow \mathscr{D}$. We say that K is a *canonical colimit of $\mathscr{C}$-objects*, if

$$(K, \ (f \colon U(f \colon X \to K) \to K)_{f \in (\mathscr{C} \downarrow K)})$$

is a colimit of U. Now, if $\mathscr{D}$ is a locally λ-presentable category, then for each object K of $\mathscr{D}$, the canonical diagram with respect to $\mathbf{Pres}_\lambda(\mathscr{D})$ is λ-filtered and K is its canonical colimit (see [AdRo], Proposition 1.22 for a proof).

(vi) A locally presentable category is complete (and cocomplete).

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We now gather the most fundamental examples of locally (finitely) presentable categories for us.

Example 1.2.11. 1. **Set** is locally finitely presentable. Indeed, every finite set is finitely presentable, each set is a directed colimit of its finite subsets and there exists, up to isomorphisms, only a small set of finite subsets (the first infinite ordinal ω).

2. For every small category $\mathscr{C}$, $\mathbf{Set}^{\mathscr{C}}$ (and hence also $\mathrm{PSh}(\mathscr{C})$) is locally finitely presentable. In particular, the category $\mathbf{sSet}$ of simplicial sets is locally presentable.

The linking bridge between categories admitting small presentations and locally presentable categories is given by the following concept.

Definition 1.2.12. Let $\mathscr{D}$ be a category.

1. An object $K \in \mathscr{D}$ is said to be *orthogonal* to a morphism $m \colon A \to A'$ provided that for each morphism $f \colon A \to K$ in $\mathscr{D}$ there exists a unique morphism $f' \colon A' \to K$ such that $f' \circ m = f$, as in the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\ \forall f\ } & K \\
{\scriptstyle m}\searrow & & \nearrow {\scriptstyle \exists! f'} \\
& A' &
\end{array}
$$

2. For each class S of morphisms in $\mathscr{D}$ we denote by $S^{\perp}$ the full (replete) subcategory of $\mathscr{D}$ spanned by those objects K which are orthogonal to each morphism in S.

3. A full subcategory of $\mathscr{D}$ is called an *orthogonality class* (resp. a *small orthogonality class*) provided that it has the form $S^{\perp}$ for a class (resp. a small set) S of morphisms of $\mathscr{D}$.

The crucial (and quite trivial) observation we need is given by the following

Remark 1.2.13. By the very definition of being orthogonal to a morphism, a (small) orthogonality class in a category $\mathscr{D}$ is given precisely by $\mathscr{D}_S$ (see Definition 1.2.4) for some class (resp. small set) S of morphisms in $\mathscr{D}$. In other words, $S^{\perp} = \mathscr{D}_S$.

Lemma 1.2.14. *An orthogonality class $S^{\perp}$ in a category $\mathscr{D}$ is closed under all limits existing in $\mathscr{D}$. In particular, if $\mathscr{D}$ is complete, then so is $S^{\perp}$.*

Proof. Suppose that $F \colon \mathscr{I} \longrightarrow S^{\perp}$ is a functor having a limit

$$
(L, \ (\pi_i \colon L \to F(i))_{i \in \mathscr{I}})
$$

in $\mathscr{D}$. Let $f \colon A \to L$ be a morphism in $\mathscr{D}$ and $s \colon A \to A'$ be a morphism in S. Since each $F(i)$ belongs to $S^{\perp}$, for every $i \in \mathscr{I}$, we obtain a unique morphism $f_i \colon A' \to F(i)$ such that $\pi_i f = f_i s$. Uniqueness of such a f_i guarantees that we get a compatible cone

$$
(f_i \colon A' \to F(i))_{i \in \mathscr{I}}
$$

over F with vertex A'. By definition of limit, there exists a unique $f' \colon A' \to L$ such that, for all $i \in \mathscr{I}$, $\pi_i f' = f_i$, i.e. $f's = f$. This proves that $L \in S^{\perp}$. $\qquad\square$

Orthogonality classes $S^{\perp}$ associated to classes S of morphisms between presentable objects deserve a special name

Definition 1.2.15. Let $\mathscr{D}$ be a category. An orthogonality class $S^{\perp}$ in $\mathscr{D}$ is called a λ-*orthogonality class* if every morphism in S has $\lambda-$presentable domain and codomain.

The following observation is somewhat technical but fundamental for what comes next.

Lemma 1.2.16. *In a locally λ-presentable category $\mathscr{D}$:*

(i) *for any regular cardinal $\mu \leq \lambda$, every μ-orthogonality class in $\mathscr{D}$ is a small orthogonality class;*

(ii) *for every small orthogonality class, there is a regular cardinal μ such that this is a μ-orthogonality class.*

Proof. The proof of these properties relies on the facts that each object in a locally λ-presentable category is presentable and that there exists an essentially unique small set of λ-presentable objects (see Remark 1.2.10 (iii) and (iv)).

Indeed, for (i), let $S^\perp$ be a μ-orthogonality class in $\mathscr{D}$. If $\mu \leq \lambda$, then each $s \in S$ has λ-presentable domain and codomain[8], so $S^\perp$ is a small orthogonality class because there is an essentially unique small set of λ-presentable objects.[9]

To show (ii), if $S^\perp$ is a small orthogonality class, let $\mathrm{Ob}(S)$ be the *small set* of objects of $\mathscr{D}$ which are either domains or codomains of some morphism in S. Since every object in $\mathscr{D}$ is presentable, we can pick, for all $A \in \mathrm{Ob}(S)$ a regular cardinal μ_A such that A is μ_A-presentable. Let C be the *small set* given by all these cardinals and let μ be the first regular cardinal not smaller than $\sup C$. Then each $A \in \mathrm{Ob}(S)$ is $(\max\{\lambda, \mu\})$-presentable, so that $S^\perp$ is a $(\max\{\lambda, \mu\})$-orthogonality class. $\square$

Proposition 1.2.17. *Each λ-orthogonality class $S^\perp$ of a category $\mathscr{D}$ is closed under $\lambda-$directed colimits (hence also under λ-filtered colimits, see Remark 1.2.10 (ii)).*

Proof. Write $S = \{s_i \colon A_i \to A_i'\}_{i \in I}$, where every A_i and every A_i' $(i \in I)$ are λ-presentable objects. Let $(T, \leq)$ be a λ-directed poset and $F \colon (T, \leq) \longrightarrow S^\perp$ be a functor having a colimit in $\mathscr{D}$, say

$$(K, \ (k_t \colon F(t) \to K)_{t \in T}).$$

We need to show that $K \in S^\perp$. For every $i \in I$ and every morphism $f \colon A_i \to K$, there exist a $t \in T$ and a factorization $f = k_t f_1$ of f (since A_i is $\lambda-$presentable) and there exists $f_1' \colon A_i' \to F(t)$ such that $f_1 = f_1' s_i$ (because $F(t) \in S^\perp$). Therefore, $f' = k_t f_1' \colon A_i' \to K$ satisfies $f = f' s_i$. We now show that such a f' is unique. Assume $f'' \colon A_i' \to K$ also verifies $f = f'' s_i$. Since $(T, \leq)$ is λ-directed, there exists $t_0 \in T$ such that $f' = k_{t_0} f_1$ and $f'' = k_{t_0} f_2$ for some $f_1, f_2 \colon A_i' \to K_{t_0}$. Since A_i' is a λ-presentable object and since we have $k_{t_0} f_1 s_i = k_{t_0} f_2 s_i$, there is $t \in T$ with $t_0 \leq t$ such that $F(t_0 \leq t) \colon K_{t_0} \to K_t$ fulfils $F(t_0 \leq t) f_1 s_i = F(t_0 \leq t) f_2 s_i$. Since $F(t) \in S^\perp$, it must be $F(t_0 \leq t) f_1 = F(t_0 \leq t) f_2$. It follows that $f' = k_t F(t_0 \leq t) f_1 = f''$. $\square$

λ-orthogonality classes in locally presentable categories have the key property we need to give us part of Theorem 1.2.6. Namely,

Theorem 1.2.18. *Let $\mathscr{D}$ be a locally presentable category. Then the following hold.*

1. *Every λ-orthogonality class in $\mathscr{D}$ is a reflective subcategory of $\mathscr{D}$ closed under λ-directed colimits.*

2. *Every reflective subcategory of $\mathscr{D}$ closed under λ-directed colimits is locally presentable.*

We will omit the proof of this theorem, as it is quite technical. The interested reader is referred to [AdRo], Theorem 1.39[10] or Theorem 2.48. We notice instead that we get the following

Corollary 1.2.19. *Every small orthogonality class of a locally presentable category is locally presentable.*

Proof. This is clear in view of Lemma 1.2.16. $\square$

Remark 1.2.13 and Example 1.2.11 allow us to conclude now that 1. of Theorem 1.2.6 holds.

The second part of Theorem 1.2.6 follows from the next result (together with Remark 1.2.13 again). Before stating it, let us introduce the following notation. Given a regular cardinal λ and a small category $\mathscr{C}$ we denote by $\mathbf{Cont}_\lambda(\mathscr{C})$ the full subcategory of $\mathbf{Set}^{\mathscr{C}}$ given by those functors $\mathscr{C} \longrightarrow \mathbf{Set}$ preserving all $\lambda-$small limits existing in $\mathscr{C}$ (i.e. all limits indexed by a small category having less than λ morphisms).

Theorem 1.2.20 (Representation Theorem, [AdRo], Theorem 1.46). *Let λ be a regular cardinal. For a category $\mathscr{D}$ the following are equivalent:*

---

[8] Every μ-presentable object is also λ-presentable

[9] Observe that, if $K \in \mathscr{D}$ is orthogonal to $s \colon A \to A'$ in $\mathscr{D}$ and if both $i \colon A \to B$ and $j \colon A' \to B'$ are isomorphisms, then K is also orthogonal to $jsi^{-1} \colon B \to B'$.

[10] The reader should be aware that in [AdRo] our Theorem 1.2.18 has a stronger formulation, claiming also that each reflective subcategory of $\mathscr{D}$ which is closed under λ-directed colimits is a λ-orthogonality class in $\mathscr{D}$. However, this statement is not true, since it fails for $\lambda = \aleph_0$, as proved in J. Jürjens, *On a problem of Gabriel and Ulmer*, Journal of Pure and Applied Algebra, 2001. Vol. 158, pp. 183-196.

1. $\mathscr{D}$ is locally $\lambda-presentable$;

2. $\mathscr{D}$ is equivalent to $\mathbf{Cont}_\lambda(\mathscr{C})$ for some small category $\mathscr{C}$;

3. $\mathscr{D}$ is equivalent to a λ-orthogonality class in $\mathbf{Set}^{\mathscr{C}}$ for some small category $\mathscr{C}$;

4. $\mathscr{D}$ is equivalent to a replete and reflective subcategory of $\mathbf{Set}^{\mathscr{C}}$ closed under λ-directed colimits for some small category $\mathscr{C}$.

Moreover, $\mathscr{C}$ in 2.···4. above can be chosen to be $(\mathbf{Pres}_\lambda(\mathscr{D}))^{\mathrm{op}}$, where $\mathbf{Pres}_\lambda(\mathscr{D})$ is the full (small) subcategory generated by any of the essentially unique sets of $\lambda-presentable$ objects (see Remark 1.2.10 (iv)).

Proof. • 1. $\implies$ 2. Let $\mathscr{D}$ be a locally λ-presentable category and set $\mathscr{C} := \mathbf{Pres}_\lambda(\mathscr{D})$. We can consider the following *canonical functor*

$$E\colon \mathscr{D} \longrightarrow \mathrm{PSh}(\mathscr{C})$$

which sends each object K of $\mathscr{D}$ to the restriction $\mathrm{Hom}_{\mathscr{D}}(-,K)_{|\mathscr{C}^{\mathrm{op}}}$ of the contravariant Hom-functor of $\mathscr{D}$ to $\mathscr{C}^{\mathrm{op}}$ and each morphism $f\colon K \to K'$ to $\mathrm{Hom}_{\mathscr{D}}(-,f)$, seen as a natural transformation $E(K) \implies E(K')$. Recall now that:

(i) $\mathscr{C}$ is made of λ-presentable objects of $\mathscr{D}$ (just by definition of $\mathbf{Pres}_\lambda(\mathscr{D})$);

(ii) for each object K of $\mathscr{D}$, the canonical diagram with respect to $\mathscr{C}$ is λ-filtered and its colimit is K (see Remark 1.2.10(v)).

We get that property (i) above is equivalent to fully faithfulness of E, whereas property (ii) means that E preserves λ-directed colimits (see [AdRo], Proposition 1.26). Therefore, if $\mathscr{D}'$ is the full, replete subcategory of $\mathrm{PSh}(\mathscr{C})$ given by all presheaves which are isomorphic to $E(K)$ for some $K \in \mathscr{D}$, we have that $\mathscr{D}$ is equivalent to $\mathscr{D}'$. Our proof will be complete if we prove that $\mathscr{D}'$ is precisely $\mathbf{Cont}_\lambda(\mathscr{C}^{\mathrm{op}})$. To this end, note first that every functor in $\mathscr{D}'$ preserves λ-small limits (i.e. it belongs to $\mathbf{Cont}_\lambda(\mathscr{C}^{\mathrm{op}})$). This is because, for every object $K \in \mathscr{D}$, $E(K)$ preserves λ-small limits in $\mathscr{C}^{\mathrm{op}}$, as $\mathrm{Hom}_{\mathscr{D}}(-,K)$ preserves limits in $\mathscr{D}^{\mathrm{op}}$ and $\mathscr{C}^{\mathrm{op}}$ is closed under λ-small limits in $\mathscr{D}^{\mathrm{op}}$ (see Remark 1.2.10 (iii)). Conversely, let $H\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be a functor preserving λ-small limits. Since E preserves λ-directed colimits (hence, also λ-filtered colimits), $\mathscr{D}'$ is closed under λ-filtered colimits in $\mathrm{PSh}(\mathscr{C})$, so proving that H is a λ-filtered colimit of functors in $\mathscr{D}'$ is enough to conclude that $H \in \mathscr{D}'$. To show this fact, note that the Yoneda embedding

$$\mathbf{y}\colon \mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C})$$

clearly satisfies $\mathbf{y}(\mathscr{C}) \subseteq \mathscr{D}'$ and H, being a presheaf, is a canonical colimit of objects of $\mathbf{y}(\mathscr{C})$. We need to verify that this colimit is λ-filtered. Yoneda's Lemma implies that each λ-small subcategory of $(\mathbf{y}(\mathscr{C}) \downarrow H)$ is of the form $((\mathbf{y}F)(\mathscr{I}) \downarrow H)$, for some functor $F\colon \mathscr{I} \longrightarrow \mathscr{C}$, where $\mathscr{I}$ is λ-small. Now, given any such a functor F, let C be a colimit of F in $\mathscr{C}$, with colimiting cocone $(c_i\colon F(i) \to C)_{i \in \mathscr{I}}$ (such a colimit exists because $\mathscr{C} = \mathbf{Pres}_\lambda(\mathscr{D})$ is closed under λ-small colimits by Remark 1.2.10 (iii)). By the hypothesis of preserving λ-small limits, $H(C)$ is a limit of $HF^{\mathrm{op}}\colon \mathscr{I}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, with limiting cone $(H(c_i)\colon H(C) \to H(F(i)))_{i \in \mathscr{I}}$. Now, the category $((\mathbf{y}F)(\mathscr{I}) \downarrow H)$ gives a compatible cone $1 \implies HF$ over HF: each object $a_i\colon \mathrm{Hom}_{\mathscr{C}}(-,F(i)) \longrightarrow H$ in $((\mathbf{y}F)(\mathscr{I}) \downarrow H)$ corresponds to a unique element $x_{a_i} \in H(F(i))$ and a morphism

$$\mathrm{Hom}_{\mathscr{C}}(-,F(i)) \implies \mathrm{Hom}_{\mathscr{C}}(-,F(j))$$

in $((\mathbf{y}F)(\mathscr{I}) \downarrow H)$ corresponds to a unique morphism $f_{ji}\colon F(i) \to F(j)$ in $\mathscr{C}$ such that $H(f_{ji})(x_{b_j}) = x_{a_i}$ (where $x_{b_j} \in H(F(j))$ represents $b_j\colon \mathrm{Hom}_{\mathscr{C}}(-,F(j)) \implies H$). Therefore, this cone over HF factors uniquely through a map of sets $1 \to H(C)$, i.e. there is a unique $x \in H(C)$ such that $(H(c_i))(x) = x_{a_i}$ for all $i \in \mathscr{I}$ and all $x_{a_i} \in H(F(i))$ corresponding to $a_i\colon \mathrm{Hom}_C(-,F(i)) \implies H$. This element $x \in H(C)$ gives rise to an object $c_x\colon \mathbf{y}(C) = \mathrm{Hom}_{\mathscr{C}}(-,C) \implies H$ of $(\mathbf{y}(\mathscr{C}) \downarrow H)$ which is the vertex of a compatible cocone for $((\mathbf{y}F)(\mathscr{I}) \downarrow H)$ whose a_i-th component (for an object $a_i\colon \mathrm{Hom}_{\mathscr{C}}(-,F(i)) \longrightarrow H$ in $((\mathbf{y}F)(\mathscr{I}) \downarrow H)$) is the morphism in $(\mathbf{y}(\mathscr{C}) \downarrow H)$ given by

$$\mathrm{Hom}_{\mathscr{C}}(-,c_i)\colon a_i \longrightarrow c_x.$$

This shows that $(\mathbf{y}(\mathscr{C}) \downarrow H)$ is λ-filtered, as required.

- 2. $\implies$ 3. Using the definition of colimit, it is easy to notice that $\mathbf{Cont}_\lambda(\mathscr{C}^{\mathrm{op}}) = S^\perp$, where S consists of all the canonical natural transformations

$$s\colon \operatorname{colim}_{i \in \mathscr{I}}(\operatorname{Hom}_{\mathscr{C}}(F(i), \,-)) \implies \operatorname{Hom}_{\mathscr{C}}(\lim_{i \in \mathscr{I}} F(i), \,-)$$

  for all functors $F\colon \mathscr{I} \longrightarrow \mathscr{C}^{\mathrm{op}}$ from a small category $\mathscr{I}$ having less than λ morphisms. Since the domain of each s is λ-presentable (it is a λ-small colimit of λ-presentable objects) and the codomain is finitely presentable, $S^\perp$ is a λ-orthogonality class in $\operatorname{PSh}(\mathscr{C})$.

- 3. $\implies$ 4. $\implies$ 1. This is Theorem 1.2.18 above.

$\square$

Finally, we get

Corollary 1.2.21. *Theorem 1.2.6 holds.*

Proof. Given a small category $\mathscr{C}$, for each small set S of morphisms of presheaves on $\mathscr{C}$, $\operatorname{PSh}(\mathscr{C})_S$ is a small orthogonality class by Remark 1.2.13. Theorem 1.2.18 shows that $(\mathscr{C}, \operatorname{PSh}(\mathscr{C})_S)$ is a pseudo-site and Corollary 1.2.19 implies that $\operatorname{PSh}(\mathscr{C})_S$ is locally finitely presentable. This proves point 1. of Theorem 1.2.6, as already remarked. Now, if $\mathscr{D}$ is a locally $\lambda-$presentable category, then Theorem 1.2.20 gives that $\mathscr{D}$ is equivalent to a $\lambda-$orthogonality class of $\operatorname{PSh}(\mathscr{C})$ (for some small category $\mathscr{C}$). Since $\operatorname{PSh}(\mathscr{C})$ is locally finitely presentable, it is also $\lambda-$presentable, because λ, being a regular cardinal, is not smaller than $\aleph_0$. Thus, every $\lambda-$orthogonality class of $\operatorname{PSh}(\mathscr{C})$ is a small orthogonality class by Lemma 1.2.16 and we can conclude that $\mathscr{D}$ admits small presentation by Remark 1.2.13. $\square$

Remark 1.2.22. It is proved in [Bor3] Proposition 3.4.16 that every Grothendieck topos is locally presentable. Thus, every Grothendieck topos is equivalent to $\operatorname{PSh}(\mathscr{C})_S$ for some small set $S \subseteq \operatorname{Mor}(\operatorname{PSh}(\mathscr{C}))$. In this case, moreover, the inclusion of $\operatorname{PSh}(\mathscr{C})_S$ in $\operatorname{PSh}(\mathscr{C})$ has a left exact left adjoint.

1.3 Factorizations in a Topos.

In this section we are going to prove that every Grothendieck topos admits a (regular epi, mono)-factorization. We will also show some properties of regular epimorphisms with respect to composition and pullbacks.

We start by recalling the needed notions.

Definition 1.3.1. Let $p\colon X \to Y$ be an arrow in a category $\mathscr{D}$.

- p is a *regular epimorphism* if it is the coequalizer of a pair of arrows in $\mathscr{D}$ with target X.

- p is an *effective epimorphism* if the diagram

$$X \times_Y X \rightrightarrows X \xrightarrow{p} Y \tag{1.3}$$

is a coequalizer, where the parallel arrows are the two projections from the pullback.

Note that regular and effective epimorphisms are indeed epimorphisms and being an effective epimorphism implies being a regular epimorphism.

Example 1.3.2. (i) In the category of sets, every epimorphism is regular and effective. This follows as, for every surjection $p\colon X \to Y$, Y is isomorphic to the quotient of X by the equivalence relation on X given by $X \times_Y X \subseteq X \times X$. This equivalence relation identifies two elements of X when they have the same image under p. Since coequalizers in $\mathrm{PSh}(\mathscr{C})$ are computed pointwise (for a small category $\mathscr{C}$), in every category of presheaves all epimorphisms are regular and effective.

(ii) In an abelian category, every epimorphism is a regular epimorphism as it is the cokernel of its kernel (and (co)kernels are (co)equalizers).

(iii) In the category of commutative rings, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism, but it is *not* a regular epimorphism: the condition of being a coequalizer for $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is equivalent to the fact that every morphism $\mathbb{Z} \to A$ to a commutative ring A can be extended (uniquely) to a morphism $\mathbb{Q} \to A$ and this is manifestly false. Note also that, despite being a monomorphism and an epimorphism, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is not an isomorphism.

Proposition 1.3.3. *Let $\mathscr{D}$ be a category.*

(i) Every regular epimorphism is orthogonal to every monomorphism. Namely, given any commutative solid diagram

where p is a regular epimorphism and i is a monomorphism, there exists a unique dotted filler k making the whole diagram commute.

(ii) If $h\colon X \to Y$ is an arrow in $\mathscr{D}$ which is both a monomorphism and a regular epimorphism, then h is an isomorphism.

(iii) Given an arrow $h\colon X \to Y$ in $\mathscr{D}$, there is, up to isomorphism, at most one factorization of h into a regular epimorphism p followed by a monomorphism i.

Proof. (i) Let $s, t\colon U \to A$ be arrows having p as their coequalizer. We then obtain

$$ifs = gps = gpt = ift$$

which gives $fs = ft$ because i is a monomorphism. As p is the coequalizer of s and t, this implies that there exists a unique $k\colon B \to X$ such that $kp = f$. Since $ikp = if = gp$ and p is an epimorphism, we also get $ik = g$.

(ii) A two-sided inverse of h is given as the diagonal filler k in the commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{\ id_X\ } & X \\
{\scriptstyle h}\downarrow & {\scriptstyle k}\nearrow & \downarrow{\scriptstyle h} \\
Y & \xrightarrow[\ id_Y\ ]{} & Y
\end{array}
$$

(iii) Suppose there are two factorizations of h as $h = ip$ and $h = i'p'$ with $p\colon X \to Z$, $p'\colon X \to Z'$ regular epimorphisms and $i\colon Z \to Y$, $i'\colon Z' \to Y$ monomorphisms. Consider the dotted lifting l in the following solid commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ p\ } & Z \\
{\scriptstyle p'}\downarrow & {\scriptstyle l}\nearrow & \downarrow{\scriptstyle i} \\
Z' & \xrightarrow[\ i'\ ]{} & Y
\end{array}
$$

Since $il = i'$ is a monomorphism, then so is l. On the other hand, if $s, t\colon U \to X$ are arrows such that p is their coequalizer, then l is the coequalizer of $p's$ and $p't$. Thus, l is a monomorphism and a regular epimorphism, hence an isomorphism by (ii) above. $\qquad\square$

Here is the announced factorization result for topoi.

Proposition 1.3.4. *Let $\mathscr{E}$ be a Grothendieck topos and $f\colon X \to Y$ a map in $\mathscr{E}$. Then there exists an essentialy unique factorization $f = ip$ where i is a monomorphism and p is a regular epimorphism. Furthermore, p is the coequalizer of the pair of projections $X \times_Y X \rightrightarrows X$.*

Proof. The result is clearly true in the category of sets, see Example 1.3.2 (i). The Proposition is then also valid in every category of presheaves as regular epimorphisms and monomorphisms in $\mathrm{PSh}(\mathscr{C})$ (where $\mathscr{C}$ is a small category) are given by those natural transformations which are objectwise such. Suppose now that $\mathscr{E} = \mathrm{Sh}(\mathscr{C})$ is a category of sheaves on a small category $\mathscr{C}$ and denote as usual by $i\colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$ the inclusion and by $a\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C})$ its left adjoint. We can then consider the factorization of $i(f)$ in $\mathrm{PSh}(\mathscr{C})$ as

$$
i(X) \times_{i(Y)} i(X) \rightrightarrows i(X) \xrightarrow{q} A \xrightarrow{j} i(Y),
$$

where q is the coequalizer of the two projections and j is a monomorphism. Using left exactness of a and the fact that $ai \cong \mathrm{Id}$, we get the diagram

$$
X \times_Y X \rightrightarrows X \xrightarrow{a(q)} a(A) \xrightarrow{a(j)} Y
$$

in $\mathrm{Sh}(\mathscr{C})$ where $a(q)$ is the coequalizer of the two projections from $X \times_Y X$ (because a commutes with colimits and finite limits) and $a(j)$ is a monomorphism (again because a is left exact). Thus $f = a(j)a(q)$ and $a(q)$ is an effective epimorphism. $\qquad\square$

Remark 1.3.5. A close inspection to the proof of Proposition 1.3.4 shows that actually the (regular epi, mono)-factorization in a Grothendieck topos is *functorial*. This essentially means that, given a commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & Y \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle h} \\
X' & \xrightarrow[\ f'\ ]{} & Y'
\end{array}
$$

in a Grothendieck topos $\mathscr{E}$, if we choose (regular epi, mono)-factorizations $f = ip$ and $f' = i'p'$ of f and f' respectively, then we get a unique arrow $t\colon \mathrm{cod}(p) = \mathrm{dom}(i) \to \mathrm{cod}(p') = \mathrm{dom}(i')$ making the

following diagram commute

$$
\begin{array}{ccccc}
 & & \xrightarrow{\quad f \quad} & & \\
X & \xrightarrow{\;p\;} & \operatorname{cod}(p) & \xrightarrow{\;i\;} & Y \\
{\scriptstyle k}\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle h} \\
X' & \xrightarrow{\;p'\;} & \operatorname{cod}(p') & \xrightarrow{\;i'\;} & Y' \\
 & & \xrightarrow{\quad f' \quad} & &
\end{array}
$$

This is trivial in the category of sets and hence in categories of presheaves. Since the (regular epi, mono)-factorizations in a category of sheaves $\mathrm{Sh}(\mathscr{C})$ are built (up to isomorphism) from the analogous factorizations in $\mathrm{PSh}(\mathscr{C})$, using the isomorphism $ai \cong \mathrm{Id}$ (i is the inclusion of $\mathrm{Sh}(\mathscr{C})$ in $\mathrm{PSh}(\mathscr{C})$ and a is the reflector) we get the result also for categories of sheaves.

As an immediate by-product of Proposition 1.3.4 we obtain the

Corollary 1.3.6. *In a Grothendieck topos every regular epimorphism is an effective epimorphism.*

Another consequence of Proposition 1.3.4 is given by the following

Corollary 1.3.7. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in a Grothendieck topos $\mathscr{E}$.*

(i) If gf is a regular epimorphism, then so is g.

(ii) If f and g are regular epimorphisms, then so is gf.

Proof. (i) Suppose gf is a regular epimorphism and consider a (regular epi, mono)-factorization of g as $g = ip$, with $p\colon Y \to W$. By part (i) of Proposition 1.3.3, there is a dotted lifting $r\colon Z \to W$ in the solid commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\;pf\;} & W \\
{\scriptstyle gf}\downarrow & \nearrow{\scriptstyle r} & \downarrow{\scriptstyle i} \\
Z & \xrightarrow{\;id_Z\;} & Z
\end{array}
$$

Thus, in particular, $ir = id_Z$, so that i is a monomorphism with a right inverse, hence an isomorphism. This proves that g is a regular epimorphism.

(ii) Write $gf = ip$ where p is a regular epimorphism and i is a monomorphism. As above, by Proposition 1.3.3 (i) there is a (unique) $r\colon Y \to \operatorname{cod}(p)$ such that $rf = p$ and $ir = g$. Since g is a regular epimorphism, then i is a regular epimorphism as well by the first point above. Therefore, i is both a monomorphism and a regular epimorphism, so it is an isomorphism. It follows that gf is a regular epimorphism. $\qquad\square$

Finally, we discuss the behaviour of regular epimorphisms with respect to pullbacks in a topos.

Proposition 1.3.8. *Let $\mathscr{E}$ be a Grothendieck topos. Suppose given a pullback square*

$$
\begin{array}{ccc}
U & \xrightarrow{\;q\;} & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
V & \xrightarrow{\;p\;} & Y
\end{array}
\tag{1.4}
$$

If f is a monomorphism or a regular epimorphism or an isomorphism, then so is g. If p is a regular epimorphism, then also the converse holds.

Proof. Using Proposition 1.3.3, the case of isomorphism follows from the other two.

For the monomorphism part, it is well-known that, given any category $\mathscr{D}$, the pullback of a monomorphism along any map in $\mathscr{D}$ is still a monomorphism. We then need to show that if p is a regular

epimorphism and g is a monomorphism, then f is a monomorphism. This is easy in the category of sets: if x, $x' \in X$ are such that $f(x) = f(x')$, pick $v \in V$ such that $p(v) = f(x)$. Then, since U is a pullback, there are unique $u, u' \in U$ such that $g(u) = v$, $g(u') = v$ and $q(u) = x$, $q(u') = x'$; as g is mono, $u = u'$, so $x = x'$. Thus the result holds for sets, hence also for presheaves. Let us then assume $\mathscr{E} = \mathrm{Sh}(\mathscr{C})$, where $\mathscr{C}$ is a small category, with inclusion functor $i\colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$ and left adjoint a. Given the pullback square (1.4) with p regular epimorphism, take a (regular epi, mono)-factorization $i(p) = dv$ of $i(p)$ in $\mathrm{PSh}(\mathscr{C})$. Consider then the following diagram in $\mathrm{PSh}(\mathscr{C})$

$$
\begin{array}{ccccc}
& & \overset{i(q)}{\longrightarrow} & & \\
iU & \overset{u}{\longrightarrow} & X' & \overset{b}{\longrightarrow} & iX \\
\downarrow{\scriptstyle i(g)} & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle i(f)} \\
iV & \underset{v}{\longrightarrow} & Y' & \underset{d}{\longrightarrow} & iY \\
& & \underset{i(p)}{\longrightarrow} & &
\end{array}
$$

Here X' is the pullback of $Y' \overset{d}{\to} iY \overset{i(f)}{\leftarrow} iX$ and the map u is induced on the pullback by the arrows $vi(g)$ and $i(q)$. Since the outer square is a pullback (because i preserves limits), the left-hand square is a pullback as well. As $i(g)$ is a mono and v is a regular epimorphism, we then get that f' is a monomorphism, by what we already proved in $\mathrm{PSh}(\mathscr{C})$. Now p is isomorphic to $ai(p)$ and $ai(p) = a(d)a(v)$, where $a(d)$ is mono because d is mono and a is left exact. Also, $a(v)$ is a regular epimorphism, as v is such and a commutes with colimits. Since p is a regular epi by hypothesis, Proposition 1.3.3 (iii) gives that $a(d)$ must be an isomorphism. It follows that $ai(f)$ is isomorphic to $a(f')$, thus it is a monomorphism. Hence also f is such.

For the regular epimorphism case, if p and g are regular epi in a Grothendieck topos, then $pg = fq$ is a regular epi and hence f is a regular epi, by Corollary 1.3.7. Thus we are left to show that if f is a regular epimorphism in (1.4), then so is g. This is immediate for the category of sets and hence for presheaves, so we need to prove the result for $\mathscr{E} = \mathrm{Sh}(\mathscr{C})$ (where $\mathscr{C}$ is a small category). Keeping the usual notations for the inclusion of $\mathrm{Sh}(\mathscr{C})$ in $\mathrm{PSh}(\mathscr{C})$ and for its left adjoint, let us take a (regular epi, mono)-factorization $i(f) = db$ of $i(f)$ in $\mathrm{PSh}(\mathscr{C})$ and consider the diagram of presheaves

$$
\begin{array}{ccccc}
& & \overset{i(g)}{\longrightarrow} & & \\
iU & \overset{u}{\longrightarrow} & V' & \overset{v}{\longrightarrow} & iV \\
\downarrow{\scriptstyle i(q)} & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle i(p)} \\
iX & \underset{b}{\longrightarrow} & Y' & \underset{d}{\longrightarrow} & iY \\
& & \underset{i(f)}{\longrightarrow} & &
\end{array}
$$

Here V' is the pullback of $Y' \overset{d}{\to} iY \overset{i(p)}{\leftarrow} iV$ and u is the morphism induced into the pullback by the maps $bi(q)$ and $i(g)$. As in the case of monomorphisms, we then get that u is the pullback along p' of the regular epimorphism b, thus it is a regular epimorphism itself. It follows that $a(u)$ is a regular epi too. Since f is isomorphic to $a(db) = a(d)a(b)$ and f is a regular epi in $\mathrm{Sh}(\mathscr{C})$, we get that $a(d)$ is also such by Corollary 1.3.7. Since $a(d)$ is a monomorphism as well (because d is such), it has to be an isomorphism. As a preserves pullbacks, it follows that $a(v)$ is an isomorphism, hence $ai(g) = a(v)a(u)$ is a regular epimorphism. We conclude that also g is a regular epi. $\qquad\square$

Corollary 1.3.9. *In a Grothendieck topos $\mathscr{E}$, a morphism $f\colon X \to Y$ is a regular epimorphism if and only there is a pullback square*

$$
\begin{array}{ccc}
U & \overset{q}{\longrightarrow} & X \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
V & \underset{p}{\longrightarrow} & Y
\end{array}
$$

where p is a regular epimorphism and g admits a section.

Proof. If f is a regular epi, it is enough to take $p = f$, as then $g\colon X \times_Y X \to X$ has an obvious section given by the map $X \to X \times_Y X$ induced by the identity of X. Conversely, if we are given a pullback square as the one in the statement of the Corollary, let $s\colon V \to U$ be a section of g. Then $id_V = gs$ and so g is a regular epimorphism by Corollary 1.3.7. Proposition 1.3.8 now implies that f is a regular epimorphism as well. $\qquad\square$

Chapter 2

The descent properties of a Topos.

> The author is told with distressing regularity that "there are no theorems in category theory" - which typically means that the speaker does not know any theorems in category theory.
>
> Emily Riehl,
> *Category Theory in Context.* ([Rie])

We are now going to introduce the properties of *weak descent* for a cocomplete and finitely complete category $\mathscr{D}$. Roughly speaking, these properties encode some commutativity conditions between colimits and pullbacks in $\mathscr{D}$ so that their respective behaviour is nice enough (see Section 2.1). We will show that every Grothendieck topos $\mathscr{E}$ satisfies weak descent and the main point we will make is that, actually, when we are given a cocomplete and finitely complete category having a small set of generators, also the converse holds (see Section 2.2). This result will be our version of Giraud's Theorem. Its conceptual importance relies on the fact that it allows to characterise Grothendieck topoi somehow internally. Indeed, given a category $\mathscr{E}$, in order to decide whether or not it is a Grothendieck topos, one has a criterion which just consists in checking whether some properties of $\mathscr{E}$ are satisfied or not. In particular, one is relieved from the task of finding explicitly a site $(\mathscr{C},\ \mathrm{Sh}(\mathscr{C}))$ and an equivalence of categories $\mathscr{E} \simeq \mathrm{Sh}(\mathscr{C})$, as these data are already provided in full generality by the proof of Giraud's Theorem itself.

In addition to the material presented in Section 2 of [Rzk1], we have provided here a detailed proof of the fact that every Grothendieck topos admits weak descent. Our proof of Proposition 2.1.14 below is also not present in the original work by Rezk, where it is left as an exercise (cf. Proposition 2.4 of [Rzk1]).

2.1   The notion of weak descent.

Definition 2.1.1. Let $\mathscr{E}$ be a cocomplete and finitely complete category. We say that $\mathscr{E}$ *has weak descent* (or that $\mathscr{E}$ *verifies weak descent*) if the following properties hold in $\mathscr{E}$.

(P1a) Let I be a small set and let $\{X_i\}_{i \in I} \subseteq \mathrm{Ob}(\mathscr{E})$ be a small set of objects in $\mathscr{E}$. Set $X := \coprod_{i \in I} X_i$ and consider any map $f \colon Y \to X$ in $\mathscr{E}$. For every $i \in I$, let Y_i denote the pullback of

$$X_i \to X \xleftarrow{f} Y,$$

where $X_i \to X$ is the coprojection of X_i into the coproduct. Then the morphism

$$\coprod_{i \in I} Y_i \to Y,$$

induced by the pullback projections $Y_i \to Y$, is an isomorphism.

(P1b) Let I be a small set and let $\{f_i \colon Y_i \to X_i\}_{i \in I}$ be a small set of morphisms in $\mathscr{E}$. Set also

$$Y := \coprod_{i \in I} Y_i, \quad X := \coprod_{i \in I} X_i \quad \text{and} \quad f := \coprod_{i \in I} f_i \colon Y \to X.$$

Then, for all $i \in I$, the morphism $Y_i \to X_i \times_X Y$ induced by the pair of arrows

$$f_i \colon Y_i \to X_i, \quad Y_i \to Y$$

is an isomorphism. (Here $Y_i \to Y$ is of course the coprojection of Y_i into the coproduct).

(P2a) Let

$$X_1 \xleftarrow{k_1} X_0 \xrightarrow{k_2} X_2$$

be a diagram in $\mathscr{E}$ (also called a *span* in $\mathscr{E}$) and let X denote its colimit object (which is a pushout). Given any morphism $f \colon Y \to X$, consider, for all $i \in \{0, 1, 2\}$, $Y_i := X_i \times_X Y$ so as to get the span in $\mathscr{E}$

$$Y_1 \leftarrow Y_0 \to Y_2,$$

where the morphisms are induced by k_1 and k_2. Then, the induced arrow

$$\mathrm{colim}(Y_1 \leftarrow Y_0 \to Y_2) \to Y$$

is an isomorphism.

(P2b) Suppose given a commutative diagram

$$
\begin{array}{ccccc}
Y_1 & \xleftarrow{\;h_1\;} & Y_0 & \xrightarrow{\;h_2\;} & Y_2 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_2} \\
X_1 & \xleftarrow{\;k_1\;} & X_0 & \xrightarrow{\;k_2\;} & X_2
\end{array}
\tag{2.1}
$$

in $\mathscr{E}$, where both squares are pullback squares. Set

$$Y := \mathrm{colim}(Y_1 \xleftarrow{h_1} Y_0 \xrightarrow{h_2} Y_2), \quad X := \mathrm{colim}(X_1 \xleftarrow{k_1} X_0 \xrightarrow{k_2} X_2) \quad \text{and} \quad f := \mathrm{colim}(f_1,\ f_0,\ f_2).$$

Then, for all $i \in \{0, 1, 2\}$, the morphisms

$$g_i \colon Y_i \to X_i \times_X Y$$

induced by f_i and the canonical arrows $Y_i \to Y$ are regular epimorphisms.

Remark 2.1.2. The weak descent properties are invariant under equivalences of categories. More precisely, if $F \colon \mathscr{D} \longrightarrow \mathscr{E}$ is (part of) an equivalence of categories, then $\mathscr{D}$ satisfies weak descent if and only if $\mathscr{E}$ does. This is just because a functor which is part of an equivalence of categories preserves (and creates) all limits and colimits.

Remark 2.1.3. Given a cocomplete and finitely complete category $\mathscr{E}$, properties (P1a) and (P1b) of Definition 2.1.1 for $\mathscr{E}$ imply what follows. Let $\{X_i\}_{i \in I}$ be a small set of objects in $\mathscr{E}$ (indexed by a small set I) and let $X := \coprod_{i \in I} X_i$. Then there is an equivalence of categories

$$\mathscr{E}/X \rightleftarrows \prod_{i \in I} \mathscr{E}/X_i \tag{2.2}$$

given by the pair of functors

$$P \colon \mathscr{E}/X \longrightarrow \prod_{i \in I} \mathscr{E}/X_i \quad \text{and} \quad S \colon \prod_{i \in I} \mathscr{E}/X_i \longrightarrow \mathscr{E}/X$$

defined on objects by

$$P \colon (Y \xrightarrow{f} X) \mapsto (X_i \times_X Y \to X_i)_{i \in I} \quad \text{and} \quad S \colon (Y_i \xrightarrow{f_i} X_i)_{i \in I} \mapsto \left(\coprod_{i \in I} Y_i \xrightarrow{\coprod_{i \in I} f_i} \coprod_{i \in I} X_i \right)$$

and acting on morphisms in the obvious manner. Indeed, the requirements that $S \circ P \cong \mathrm{Id}$ and $P \circ S \cong \mathrm{Id}$ are easily seen to follow from (P1a) and (P1b) respectively. For instance, if $f \colon Y \to X$ is an element of $\mathscr{E}/X$, set, for all $i \in I$, $Y_i := X_i \times_X Y$ and denote by π_{X_i} and by $\pi_Y^{(i)}$ the pullback projections into X_i

and Y respectively. Thus, each π_{X_i} is the pullback of the coprojection $X_i \to X$ along f and we have $P(f) = (\pi_{X_i})_{i \in I}$. By (P1a) the canonical morphism

$$\pi_f \colon \coprod_{i \in I} Y_i \to Y$$

whose cocomponents are the pullback projections $\pi_Y^{(i)}$ (for all $i \in I$) is an isomorphism. By its very definition, π_f verifies

$$f \circ \pi_f = \coprod_{i \in I} \pi_{X_i} = S(P(f)).$$

This means exactly that π_f is an isomorphism $S(P(f)) \to f$ in $\mathscr{E}/X$. Naturality in f of such an isomorphism follows readily by the definitions of S and P on morphisms and by the construction of π_f.

Properties (P1a) and (P2b) of Definition 2.1.1 above are closely related to the behaviour of the pullback functor with respect to colimits.

More precisely, recall that, given a category $\mathscr{E}$ admitting pullbacks, we can define, for every fixed morphism $f \colon Y \to X$, a *change of base* functor

$$f^* \colon \mathscr{E}/X \longrightarrow \mathscr{E}/Y, \tag{2.3}$$

which acts by pulling back along f, i.e. sending $u \colon A \to X$ in $\mathscr{E}/X$ to the pullback projection $A \times_X Y \to Y$ as in the pullback square

$$
\begin{array}{ccc}
A \times_X Y & \longrightarrow & Y \\
\downarrow & & \downarrow f \\
A & \stackrel{u}{\longrightarrow} & X
\end{array}
$$

(Of course, to give the definition of such a functor f^* one needs to fix a pullback functor on $\mathscr{E}$).

Remark 2.1.4. Given a category $\mathscr{E}$, it is well known that, for all objects $X \in \mathscr{E}$, the forgetful functor $U \colon \mathscr{E}/X \longrightarrow \mathscr{E}$ creates colimits (see [McL] Lemma §V.6). In other words, if $F \colon \mathscr{I} \longrightarrow \mathscr{E}/X$ is a functor from a small category $\mathscr{I}$ and $\operatorname{colim} UF$ exists in $\mathscr{E}$, then a colimit of F is the morphism $\operatorname{colim} UF \to X$ induced on the colimit by the family of morphisms $F(i) \colon U(F(i)) \to X$, for $i \in \mathscr{I}$.

Proposition 2.1.5. *Let $\mathscr{E}$ be a cocomplete and finitely complete category.*[1] *Then $\mathscr{E}$ satisfies (P1a) if and only if, for every morphism $f \colon Y \to X$ in $\mathscr{E}$ (between any two objects Y and X of $\mathscr{E}$), the change of base functor f^* preserves coproducts.*

Proof. Assume first that every change of base functor preserves coproducts. Then, given any small set of objects $(X_i)_{i \in I}$ of $\mathscr{E}$ (where I is a small set), if $X := \coprod_{i \in I} X_i$, a coproduct of the family of coprojections $(X_i \to X)_{i \in I}$ in $\mathscr{E}/X$ is $\operatorname{id}_X \colon X \to X$ and $f^*(\operatorname{id}_X) = \operatorname{id}_Y$, for any morphism $f \colon Y \to X$ in $\mathscr{E}$. On the other hand, applying first f^* to each of the coprojections $X_i \to X$ and then taking the coproduct of $(f^* \colon Y_i := X_i \times_X Y \to Y)_{i \in I}$ in $\mathscr{E}/Y$ gives precisely the morphism $t \colon \coprod_{i \in I} Y_i \to Y$ of $\mathscr{E}$ appearing in (P1a). The hypothesis that f^* preserves coproducts says exactly that t is an isomorphism $t \to \operatorname{id}_Y$ in $\mathscr{E}/Y$, hence it is also an isomorphism $\coprod_{i \in I} Y_i \to Y$ in $\mathscr{E}$.

Conversely, assume that (P1a) holds in $\mathscr{E}$ and take any morphism $f \colon Y \to X$ in $\mathscr{E}$. Suppose also given a small set of objects $(i_\alpha \colon A_\alpha \to X)_{\alpha \in \Lambda}$ of $\mathscr{E}/X$. Its coproduct in $\mathscr{E}/X$ is the morphism $i \colon \coprod_{\alpha \in \Lambda} A_\alpha \to X$ in $\mathscr{E}$ induced by the i_α ($\coprod_{\alpha \in \Lambda} A_\alpha$ is of course the coproduct in $\mathscr{E}$). Applying f^* to i, one then gets the object

$$f^*(i) \colon \left(\coprod_{\alpha \in \Lambda} A_\alpha \right) \times_X Y \to Y$$

of $\mathscr{E}/Y$ which fits into the following pullback square in $\mathscr{E}$

$$
\begin{array}{ccc}
\left(\coprod_{\alpha \in \Lambda} A_\alpha \right) \times_X Y & \stackrel{f^*(i)}{\longrightarrow} & Y \\
\downarrow & & \downarrow f \\
\left(\coprod_{\alpha \in \Lambda} A_\alpha \right) & \stackrel{i}{\longrightarrow} & X
\end{array}
$$

---

[1] Actually, it is enough to require that $\mathscr{E}$ has all small coproducts and all pullbacks.

On the other hand, one can first apply f^* to every i_α and get $f^*(i_\alpha)\colon A_\alpha \times_X Y \to Y$ fitting into the pullback square in $\mathscr{E}$

$$
\begin{array}{ccc}
A_\alpha \times_X Y & \xrightarrow{\;f^*(i_\alpha)\;} & Y \\
\downarrow & & \downarrow{\scriptstyle f} \\
A_\alpha & \xrightarrow{\;i_\alpha\;} & X
\end{array}
$$

Then one can take the coproducts in $\mathscr{E}/Y$ of the $(f^*(i_\alpha))_{\alpha \in \Lambda}$ and get the object

$$
\sum_\alpha f^*(i_\alpha)\colon \coprod_{\alpha \in \Lambda} (A_\alpha \times_X Y) \to Y
$$

of $\mathscr{E}/Y$ which makes the following diagram commutative in $\mathscr{E}$

$$
\begin{array}{ccc}
\coprod_{\alpha \in \Lambda} (A_\alpha \times_X Y) & \xrightarrow{\;\sum_\alpha f^*(i_\alpha)\;} & Y \\
\downarrow & & \downarrow{\scriptstyle f} \\
\left(\coprod_{\alpha \in \Lambda} A_\alpha\right) & \xrightarrow{\;i\;} & X
\end{array}
$$

(Here the notation $\sum_\alpha f^*(i_\alpha)$ is used to suggest that the cocomponents of such a morphism are the morphisms $f^*(i_\alpha)$). We need to prove that the canonical morphism in $\mathscr{E}$

$$
\sum_\alpha (i_\alpha \times_X \mathrm{id}_Y)\colon \coprod_{\alpha \in \Lambda} (A_\alpha \times_X Y) \to \left(\coprod_{\alpha \in \Lambda} A_\alpha\right) \times_X Y
$$

is an isomorphism in $\mathscr{E}$: it will then automatically be an isomorphism in $\mathscr{E}/Y$ as, by the very definition of all the morphisms involved, $f^*(i) \circ (\sum_\alpha (i_\alpha \times_X \mathrm{id}_Y)) = \sum_\alpha f^*(i_\alpha)$. To simplify the notation a little bit, let us set $A := \coprod_{\alpha \in \Lambda} A_\alpha$. Now, applying (P1a) to the objects A_α of $\mathscr{E}$ and to the morphism

$$
\pi_A\colon A \times_X Y \to A
$$

given by the pullback projection, we get an isomorphism (in $\mathscr{E}$)

$$
\psi\colon \coprod_{\alpha \in \Lambda} A_\alpha \times_A (A \times_X Y) \xrightarrow{\;\cong\;} A \times_X Y
$$

whose α-cocomponent is the morphism

$$
A_\alpha \times_A (A \times_X Y) \to A \times_X Y.
$$

But now, we have, for all $\alpha \in \Lambda$, commutative squares

$$
\begin{array}{ccccc}
A_\alpha \times_A (A \times_X Y) & \longrightarrow & A \times_X Y & \xrightarrow{\;f^*(i)\;} & Y \\
\downarrow & & \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle f} \\
A_\alpha & \xrightarrow{\;i_\alpha\;} & A & \xrightarrow{\;i\;} & X
\end{array}
$$

where the inner squares are pullbacks. It follows that the outer square is a pullback as well, i.e. there is, for every $\alpha \in \Lambda$, an isomorphism

$$
\varphi_\alpha\colon A_\alpha \times_A (A \times_X Y) \xrightarrow{\;\cong\;} A_\alpha \times_X Y,
$$

induced by the morphisms

$$
A_\alpha \times_A (A \times_X Y) \to A_\alpha, \quad A_\alpha \times_A (A \times_X Y) \to A \times_X Y \to Y.
$$

Hence, we have an isomorphism

$$
\coprod_{\alpha \in \Lambda} \varphi_\alpha\colon \coprod_{\alpha \in \Lambda} A_\alpha \times_A (A \times_X Y) \xrightarrow{\;\cong\;} \coprod_{\alpha \in \Lambda} (A_\alpha \times_X Y).
$$

The composite

$$\psi \circ \left(\coprod_{\alpha \in \Lambda} \varphi_\alpha \right)^{-1} : \coprod_{\alpha \in \Lambda} (A_\alpha \times_X Y) \cong \coprod_{\alpha \in \Lambda} A_\alpha \times_A (A \times_X Y) \cong A \times_X Y$$

gives us the desired isomorphism. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

A weakened version of Proposition 2.1.5 above holds for pushouts as well, namely

Lemma 2.1.6. *Let $\mathscr{E}$ be a cocomplete and finitely complete category.[2] If, for any $f : Y \to X$ in $\mathscr{E}$, the change of base functor f^* preserves pushouts, then $\mathscr{E}$ satisfies (P2a).*

Proof. Let

$$X_1 \xleftarrow{k_1} X_0 \xrightarrow{k_2} X_2$$

be a span in $\mathscr{E}$ and consider a pushout square in $\mathscr{E}$

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\ k_2\ } & X_2 \\
{\scriptstyle k_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
X_1 & \xrightarrow{\ f_1\ } & X
\end{array}
$$

which is a diagram D in $\mathscr{E}/X$ as well. Given any $f : Y \to X$ in $\mathscr{E}$, applying first f^* to D, we get the commutative diagram in $\mathscr{E}$

$$
\begin{array}{ccc}
Y \times_X X_0 & \xrightarrow{\ f^*(k_2)\ } & Y \times_X X_2 \\
{\scriptstyle f^*(k_1)}\downarrow & & \downarrow{\scriptstyle f^*(f_2)} \\
Y \times_X X_1 & \xrightarrow{\ f^*(f_1)\ } & Y
\end{array}
$$

and we can then take its pushout in $\mathscr{E}/Y$

$$
\begin{array}{ccc}
(Y \times_X X_0 \to Y) & \xrightarrow{\ f^*(k_2)\ } & (Y \times_X X_2 \to Y) \\
{\scriptstyle f^*(k_1)}\downarrow & & \downarrow \\
(Y \times_X X_1 \to Y) & \xrightarrow{\qquad\qquad} & (Z \xrightarrow{t} Y)
\end{array}
$$

By Remark 2.1.4, here Z is the pushout of

$$Y \times_X X_1 \xleftarrow{\mathrm{id}_Y \times_X k_1} Y \times_X X_0 \xrightarrow{\mathrm{id}_Y \times_X k_2} Y \times_X X_2$$

in $\mathscr{E}$ and t is the morphism induced by the pullback projections

$$Y \times_X X_1 \to Y, \quad Y \times_X X_2 \to Y.$$

On the other hand, we can first do the pushout in $\mathscr{E}/X$ of the diagram D, which is just $\mathrm{id}_X : X \to X$, because D is already a pushout square in $\mathscr{E}$ (where the pushout object is X). Then, we can apply f^* to obtain the commutative diagram

$$
\begin{array}{ccc}
(Y \times_X X_0 \to Y) & \xrightarrow{\qquad f^*(k_2) \qquad} & (Y \times_X X_2 \to Y) \\
{\scriptstyle f^*(k_1)}\downarrow & & \downarrow \\
(Y \times_X X_1 \to Y) & \xrightarrow{\qquad\qquad} & (Y \times_X X \to Y) = (Y \xrightarrow{\mathrm{id}_Y} Y)
\end{array}
$$

Since f^* preserves pushouts, $t : Z \to Y$ is an isomorphism in $\mathscr{E}$ (as well as in $\mathscr{E}/Y$, since of course $\mathrm{id}_Y \circ t = t$), as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

As a consequence, we get the following

---

[2]Again, it is enough to require that $\mathscr{E}$ has all pushouts and all pullbacks.

Corollary 2.1.7. *Let $\mathscr{E}$ be a cocomplete and finitely complete category. If, for any $f\colon Y \to X$ in $\mathscr{E}$, the change of base functor f^* preserves (small) colimits, then $\mathscr{E}$ satisfies (P1a) and (P2a).*

Remark 2.1.8. The property of small colimits of being preserved under all change of base functors f^* is often formulated by saying that colimits are *stable under pullbacks* in $\mathscr{E}$ or that colimits are *universal* in $\mathscr{E}$.

We can now start proving that a Grothendieck topos admits weak descent. We have already shown in Proposition 1.1.8 that a Grothendieck topos is complete and cocomplete, hence to show that it admits weak descent, it is enough to verify that properties (P1a)$\cdots$(P2b) hold in a Grothendieck topos. We begin with the following

Proposition 2.1.9. *Let $f\colon Y \to X$ be a morphism in a Grothendieck topos $\mathscr{E}$. Then the change of base functor $f^*\colon \mathscr{E}/X \longrightarrow \mathscr{E}/Y$ preserves small colimits.*

Proof. As usual, it is enough to prove the statement when $\mathscr{E}$ is **Set**, $\mathrm{PSh}(\mathscr{C})$ or $\mathrm{Sh}(\mathscr{C})$ for $\mathscr{C}$ a small category.

The claim is true when $\mathscr{E} = \mathbf{Set}$ because in this case, if $f\colon Y \to X$, the change of base functor $f^*\colon \mathbf{Set}/X \to \mathbf{Set}/Y$ has a right adjoint $\prod_f\colon \mathbf{Set}/Y \to \mathbf{Set}/X$ (see [McMo] §1.9 Theorem 3). Indeed, we have the equivalence of categories

$$\mathbf{Set}/Y \xrightarrow{\simeq} \mathbf{Set}^Y, \quad (A \xrightarrow{t} Y) \mapsto (t^{-1}(\{y\}))_{y \in Y},$$

where on the right hand side the set Y is meant as a discrete category. Under such an equivalence, the right adjoint $\prod_f$ is given by

$$\prod_f\colon \mathbf{Set}^Y \longrightarrow \mathbf{Set}^X, \quad (B_y)_{y \in Y} \mapsto \left(\prod_{y \in f^{-1}(\{x\})} B_y \right)_{x \in X}.$$

When $\mathscr{E} = \mathrm{PSh}(\mathscr{C})$ (for a small category $\mathscr{C}$) the statement follows immediately from its validity for the category of sets and Remark 2.1.4. Indeed, pullbacks and colimits are computed objectwise in $\mathrm{PSh}(\mathscr{C})$ and the natural transformations which are isomorphisms are precisely the ones that are isomorphisms componentwise.

Finally, assume that $\mathscr{E} = \mathrm{Sh}(\mathscr{C})$ and denote by a the left adjoint to the inclusion functor $i\colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$. Fix also $f\colon Y \to X$ in $\mathscr{E}$. Let $F\colon \mathscr{I} \longrightarrow \mathrm{Sh}(\mathscr{C})/X$ be a functor from a small category $\mathscr{I}$ and let $U\colon \mathrm{Sh}(\mathscr{C})/X \longrightarrow \mathrm{Sh}(\mathscr{C})$ be the forgetful functor. By Remark 2.1.4 it is enough to show that we have a canonical isomorphism

$$\mathrm{colim}(UF \times_X Y) \cong (\mathrm{colim}\, UF) \times_X Y$$

in $\mathrm{Sh}(\mathscr{C})$ (this meaning, of course, that the canonical arrow $\mathrm{colim}(UF \times_X Y) \to (\mathrm{colim}\, UF) \times_X Y$ is an isomorphism). Now, considering the morphism $if\colon iY \longrightarrow iX$ in $\mathrm{PSh}(\mathscr{C})$, since $(if)^*$ preserves small colimits, we know that we have a canonical isomorphism

$$\mathrm{colim}(iUF \times_{iX} iY) \cong (\mathrm{colim}\, iUF) \times_{iX} iY.$$

Since a preserves colimits and pullbacks, by Lemma 1.1.7 and using the definition of colimits in $\mathrm{Sh}(\mathscr{C})$, we get the chain of canonical isomorphisms

$$\mathrm{colim}(UF \times_X Y) = a(\mathrm{colim}(iUF \times_{iX} iY)) \cong (a(\mathrm{colim}\, iUF)) \times_{aiX} aiY$$

$$\cong (\mathrm{colim}(aiUF)) \times_{aiX} aiY \cong (\mathrm{colim}\, UF) \times_X Y,$$

as required. $\qquad\square$

By Corollary 2.1.7, we then get that any Grothendieck Topos verifies properties (P1a) and (P2a).

Lemma 2.1.10. *In a Grothendieck topos $\mathscr{E}$, property (P1b) of Definition 2.1.1 holds.*

Proof. Let us first assume that $\mathscr{E} = \mathbf{Set}$. Take then a small set I and a family of maps $(f_i\colon Y_i \to X_i)_{i \in I}$. Set $X := \coprod_{i \in I} X_i$, $Y := \coprod_{i \in I} Y_i$ and $f := \coprod_{i \in I} f_i$. In $\mathbf{Set}$, the coprojections of a summand into a coproduct are monomorphisms and, as in any category, the pullback of a monomorphism is again a monomorphism. Hence, for any fixed $i \in I$, in the pullback square

$$
\begin{array}{ccc}
X_i \times_X Y & \longrightarrow & Y \\
\downarrow & & \downarrow f \\
X_i & \longrightarrow & X
\end{array}
$$

both the bottom and the top rows are monomorphisms. Thus, the map

$$t\colon Y_i \to X_i \times_X Y,$$

induced on the pullback by f_i and by the mono $Y_i \rightarrowtail Y$, is a monomorphism. Now, to prove that t is an isomorphism, it is enough to show that, for all sets Z, the induced map

$$t \circ -\colon \operatorname{Hom}_{\mathbf{Set}}(Z, Y_i) \to \operatorname{Hom}_{\mathbf{Set}}(Z, X_i \times_X Y)$$

is bijective, which amounts to prove its surjectivity, as $t \circ -$ is automatically injective when t is a monomorphism. Actually, since any set Z is a coproduct of copies of the terminal object 1 and the contravariant Hom functor sends coproducts to products, it is enough to prove that

$$t \circ -\colon \operatorname{Hom}_{\mathbf{Set}}(1, Y_i) \to \operatorname{Hom}_{\mathbf{Set}}(1, X_i \times_X Y)$$

is surjective. But now, the definition of pullback yields the isomorphism

$$\operatorname{Hom}_{\mathbf{Set}}(1, X_i \times_X Y) \simeq \operatorname{Hom}_{\mathbf{Set}}(1, X_i) \times_{\operatorname{Hom}_{\mathbf{Set}}(1, X)} \operatorname{Hom}_{\mathbf{Set}}(1, Y).$$

Thus, a *global element* $1 \to X_i \times_X Y$ of $X_i \times_X Y$[3] corresponds to a couple of global elements

$$1 \to X_i, \quad 1 \to Y$$

such that the following diagram commutes

$$
\begin{array}{ccc}
1 & \longrightarrow & Y \\
\downarrow & & \downarrow f \\
X_i & \longrightarrow & X
\end{array}
$$

Since 1 is a *connected object* in $\mathbf{Set}$[4], the global element $1 \to Y$ factors through the coprojections into Y of one of the summands, which must be Y_i, as the diagram above commutes. Hence, we get a global element w of Y_i and a commutative diagram

where j_{X_i} and j_{Y_i} are the coprojections into the coproduct. This proves that

$$t \circ -\colon \operatorname{Hom}_{\mathbf{Set}}(1, Y_i) \to \operatorname{Hom}_{\mathbf{Set}}(1, X_i \times_X Y)$$

is surjective, as required.

---

[3] Given a category $\mathscr{C}$ with a terminal object 1, a *global element* of an object A of $\mathscr{C}$ is an arrow $1 \to A$.

[4] Given a category $\mathscr{C}$, an object A of $\mathscr{C}$ is said to be *connected* if the Hom functor $\operatorname{Hom}_{\mathscr{C}}(A, -)\colon \mathscr{C} \longrightarrow \mathbf{Set}$ preserves coproducts. So, actually, 1 is the only connected object in the category of sets.

Since (P1b) holds for **Set** and limits, colimits and isomorphisms in presheaves categories are constructed or checked objectwise, it carries over to the case $\mathcal{E} = \mathrm{PSh}(\mathscr{C})$, for a small category $\mathscr{C}$.

Assume now that $\mathcal{E} = \mathrm{Sh}(\mathscr{C})$ and let a be the left adjoint to the inclusion functor $\iota\colon \mathrm{Sh}(\mathscr{C}) \to \mathrm{PSh}(\mathscr{C})$. Given a small set I and a family of morphisms $(f_i\colon Y_i \to X_i)_{i \in I}$ of sheaves, using the same notation as in the case of sets for Y, X and f, we have, for a fixed $i \in I$, the pullback diagram in $\mathrm{Sh}(\mathscr{C})$ given by

$$
\begin{array}{ccc}
X_i \times_X Y & \longrightarrow & Y \\
\downarrow & & \downarrow {\scriptstyle f} \\
X_i & \longrightarrow & X
\end{array}
$$

On the other hand, we can also construct the following pullback diagram in $\mathrm{PSh}(\mathscr{C})$

$$
\begin{array}{ccc}
\iota X_i \times_{\iota X} \iota Y & \longrightarrow & \iota Y \\
\downarrow & & \downarrow {\scriptstyle \iota f} \\
\iota X_i & \longrightarrow & \iota X
\end{array}
$$

and we know that we have the canonical isomorphism

$$
\iota Y_i \cong \iota X_i \times_{\iota X} \iota Y,
$$

given by the arrow on the pullback induced by the morphisms $\iota(f_i)$ and $\iota Y_i \to \iota Y$. Applying now the left adjoint a and recalling that $a \circ \iota \cong \mathrm{Id}$ and that a is left exact, we have the chain of canonical isomorphisms

$$
Y_i \cong a(\iota Y_i) \cong a(\iota X_i \times_{\iota X} \iota Y) \cong a(\iota X_i) \times_{a(\iota X)} a(\iota Y) \cong X_i \times_X Y,
$$

giving the required canonical isomorphism. □

Proposition 2.1.11. *Every Grothendieck topos has weak descent.*

Proof. By Proposition 2.1.9 and Corollary 2.1.7, every Grothendieck topos $\mathcal{E}$ verifies (P1a) and (P2a) of Definition 2.1.1, while Lemma 2.1.10 says that in $\mathcal{E}$ also (P1b) holds. Thus, we only need to prove that in every Grothendieck topos also (P2b) is satisfied. As usual, using the facts that (co)limits in presheaves categories are computed pointwise and that the left adjoint to the inclusion of a category of sheaves is left exact, it is easy to see that it is enough to verify (P2b) when $\mathcal{E} = \mathbf{Set}$. The proof of this statement is rather technical and surprisingly elaborated, so we state it separately as the following Lemma. □

Lemma 2.1.12. *Property (P2b) of Definition 2.1.1 holds for $\mathcal{E} = \mathbf{Set}$.*[5]

Proof. We stick to the notations of Definition 2.1.1. So we are given a commutative diagram

$$
\begin{array}{ccccc}
Y_1 & \xleftarrow{\; h_1 \;} & Y_0 & \xrightarrow{\; h_2 \;} & Y_2 \\
\downarrow {\scriptstyle f_1} & & \downarrow {\scriptstyle f_0} & & \downarrow {\scriptstyle f_2} \\
X_1 & \xleftarrow{\; k_1 \;} & X_0 & \xrightarrow{\; k_2 \;} & X_2
\end{array}
\tag{2.4}
$$

in **Set**, where both squares are pullback. Setting

$$
Y := \mathrm{colim}(Y_1 \xleftarrow{h_1} Y_0 \xrightarrow{h_2} Y_2), \quad X := \mathrm{colim}(X_1 \xleftarrow{k_1} X_0 \xrightarrow{k_2} X_2) \quad \text{and} \quad f := \mathrm{colim}(f_1,\, f_0,\, f_2).
$$

we must prove that, for all $i \in \{0, 1, 2\}$, the morphisms

$$
g_i\colon Y_i \to X_i \times_X Y
$$

induced by f_i and the canonical arrows $Y_i \to Y$ are regular epimorphisms.

We start by showing that g_1 is regular epi. In the category of sets every epimorphism is regular, so we need to prove that g_1 is surjective. Let then $(x_1, y) \in X_1 \times_X Y$, so that $[x_1] = f(y)$ in the pushout X (recall that X is a quotient of $X_1 \coprod X_2$). Now, we have either

---

[5]The author is totally indebted to Zhen Lin Low for the proof of this result.

- $y = [y_1]$ for some $y_1 \in Y_1$

or

- $y = [y_2]$ for some $y_2 \in Y_2$.

In this second case, $[x_1] = f(y) = f([y_2]) = [f_2(y_2)]$, so that $f_2(y_2) = k_2(x_0)$ for some $x_0 \in X_0$ (again by definition of the pushout X). Thus, since $Y_0 \cong Y_2 \times_{X_2} X_0$ by hypothesis and (y_2, x_0) is an element of this pullback, there is a unique $y_0 \in Y_0$ such that $f_0(y_0) = x_0$ and $h_2(y_0) = y_2$. Then, if $y_1 := h_1(y_0)$, we have

$$[y_1] = [h_1(y_0)] = [h_2(y_0)] = [y_2] = y.$$

This shows that we can always assume

$$\exists y_1 \in Y_1 \ ([y_1] = y). \tag{2.5}$$

The point now is that we want to choose this y_1 in such a way that $f_1(y_1) = x_1$. To do this, we are going to define a metric on X_1 which measures the distance of $a, b \in X_1$ in terms of a sequence of pairs of elements in X_0 which links a and b via k_1 and k_2. Namely, we claim that there exists a unique function

$$d \colon X_1 \times X_1 \to \mathbb{N} \cup \{\infty\}$$

(where $\infty \notin \mathbb{N}$ and, for all $n \in \mathbb{N}$, $n < \infty$) with the following properties, for $a, b \in X_1$:

(i) $a = b \iff d(a, b) = 0$;

(ii) for any $n \in \mathbb{N} \setminus \{0\}$, if $r_1, \ s_1, \ \ldots, \ r_n, \ s_n \in X_0$ are such that

$$\forall j \in \{1, \ \ldots, \ n\} \ (k_2(r_j) = k_2(s_j)) \ \text{ and } \ \forall j \in \{1, \ \ldots, \ n-1\} \ (k_1(r_{j+1}) = k_1(s_j)), \tag{2.6}$$

then $d(k_1(r_1), \ k_1(s_n)) \le n$;

(iii) if $d(a, b) = n \in \mathbb{N} \setminus \{0\}$, then there are elements $r_1, \ s_1, \ \ldots, \ r_n, \ s_n \in X_0$ such that

$$(*) \quad \text{property (2.6) holds and } k_1(r_1) = a, \ k_1(s_n) = b.$$

Indeed, set, for all $(a, b) \in X_1 \times X_1$ with $a \ne b$,

$$A_{(a,b)} = \{n \in \mathbb{N} \setminus \{0\} : \text{ there are } r_1, \ s_1, \ \ldots, \ r_n, \ s_n \in X_0 \text{ such that } (*) \text{ holds}\}.$$

Then, the function $d \colon X_1 \times X_1 \to \mathbb{N} \cup \{\infty\}$ defined by

$$d(a, b) := \begin{cases} 0 & \text{if } a = b \\ \min A_{(a,b)} & \text{if } a \ne b \text{ and } A_{(a,b)} \ne \emptyset \\ \infty & \text{if } a \ne b \text{ and } A_{(a,b)} = \emptyset \end{cases}$$

clearly verifies (i)$\cdots$(iii) above and every other function with those properties must coincide with d. By the above description of d we also get that d is actually a metric on X_1.

Let us now consider again $y_1 \in Y_1$ as in (2.5). If $f_1(y_1) = x_1$, then we are done, because we get $(x_1, y) = g_1(y_1)$. If $f(y_1) \ne x_1$, then $d(f_1(y_1), x_1) > 0$ and $d(f_1(y_1), x_1) < \infty$. Indeed, as $[x_1] = [f_1(y_1)]$, there are $n \in \mathbb{N} \setminus \{0\}$ and $r_1, s_1, \ \ldots, \ r_n, \ s_n$ such that

$$f_1(y_1) = k_1(r_1) \sim k_2(r_1) = k_2(s_1) \sim k_1(s_1) = k_1(r_2) \sim \ldots \sim k_1(s_n) = x_1,$$

where $\sim$ denotes the equivalence relation generated by $\{(k_1(x_0), \ k_2(x_0)) \in X_1 \times X_1 : \ x_0 \in X_0\}$ which gives the pushout X as $X_1 \coprod X_2 / \sim$. Hence, by (iii) above, there are $r := r_1, \ s := s_1 \in X_0$ such that $k_1(r) = f_1(y_1)$, $k_2(r) = k_2(s)$ and $d(k_1(s), x_1) < d(f_1(y_1), x_1)$. Since $Y_0 \cong Y_1 \times_{X_1} X_0$, there must be a unique $r' \in Y_0$ such that $f_0(r') = r$ and $h_1(r') = y_1$. Thus,

$$f_2(h_2(r')) = k_2(f_0(r')) = k_2(r) = k_2(s),$$

so there is a unique $s' \in Y_0$ satisfying $f_0(s') = s$ and $h_2(s') = h_2(r')$ ($Y_0 \cong Y_2 \times_{X_2} X_0$). Therefore, $f_1(h_1(s')) = k_1(f_0(s')) = k_1(s)$ with $h_1(s') \in Y_1$, $k_1(s) \in X_1$ and

$$y_1 = h_1(r') \sim h_2(r') = h_2(s') \sim h_1(s'),$$

so that $[h_1(s')] = [y_1] = [y]$. Since $d(k_1(s), x_1) < d(f_1(y_1), x_1) < \infty$, by induction on $d(f_1(y_1), x_1)$ we can conclude that there is always some choice of $y_1 \in Y_1$ with $[y_1] = y$ and $f_1(y_1) = x_1$. This shows that $g_1 \colon Y_1 \to X_1 \times_X Y$ is surjective.

By symmetry, since the above argument works in an identical fashion when considering g_2, we get that $g_2 \colon Y_2 \to X_2 \times_X Y$ is an epimorphism as well.

It remains to show that $g_0 \colon Y_0 \to X_0 \times_X Y$ is surjective. Let $(x_0, y) \in X_0 \times_X Y$ so that $[k_1(x_0)] = f(y)$. By the above discussion, there is $y_1 \in Y_1$ such that $f_1(y_1) = k_1(x_0)$ and $[y_1] = y$. As Y_0 is a pullback, there is a unique $y_0 \in Y_0$ such that $h_2(y_0) = y_1$ and $f_0(y_0) = x_0$. This gives surjectivity of g_0 and completes the proof. $\qquad\square$

We can restate the weak descent property using the following notion

Definition 2.1.13. Let $\mathscr{C}$ and $\mathscr{E}$ be categories. A natural transformation $\tau \colon X \Longrightarrow Y$ between functors $X, Y \colon \mathscr{C} \longrightarrow \mathscr{E}$ is called *equifibered* if, for each morphism $s \colon C \to C'$ in $\mathscr{C}$, the square

$$
\begin{array}{ccc}
X(C) & \xrightarrow{\ X(s)\ } & X(C') \\
{\scriptstyle \tau_C}\downarrow & & \downarrow{\scriptstyle \tau_{C'}} \\
Y(C) & \xrightarrow[\ Y(s)\ ]{} & Y(C')
\end{array}
\tag{2.7}
$$

is a pullback square[6].

Proposition 2.1.14. *Let $\mathscr{E}$ be a category with weak descent and $\mathscr{C}$ a small category.*

1. *Let $X \colon \mathscr{C} \longrightarrow \mathscr{E}$ be a functor. Set $\bar{X} := \operatorname{colim} X$ and fix an arrow $f \colon \bar{Y} \to \bar{X}$ in $\mathscr{E}$. Define a functor $Y \colon \mathscr{C} \longrightarrow \mathscr{E}$ by*
$$
Y(C) := X(C) \times_{\bar{X}} \bar{Y}, \qquad C \in \mathscr{C}
$$
   and by the obvious action on morphisms of $\mathscr{C}$. Then the arrow
$$
\operatorname{colim} Y \to \bar{Y}
$$
   induced by the family of projections $(X(C) \times_{\bar{X}} \bar{Y} \to \bar{Y})_{C \in \mathscr{C}}$ is an isomorphism.

2. *Let $\tau \colon X \Longrightarrow Y$ be an equifibered natural transformation of functors $X, Y \colon \mathscr{C} \longrightarrow \mathscr{E}$. Set*
$$
\bar{X} := \operatorname{colim} X, \quad \bar{Y} := \operatorname{colim} Y, \quad \bar{\tau} := \operatorname{colim}(\tau) \colon \bar{X} \to \bar{Y}.
$$
   Then for each object $C \in \mathscr{C}$, the obvious map
$$
g = g_C \colon X(C) \to Y(C) \times_{\bar{Y}} \bar{X}
$$
   is a regular epimorphism. If $\mathscr{C}$ is a groupoid with at most one arrow between any two objects, then each g_C is an isomorphism.

Proof. To begin with, note that, with respect to Definition 2.1.1, we have

- (P1a) is equivalent to 1. and (P1b) is equivalent 2. when $\mathscr{C}$ is a discrete category (the latter because every natural transformation between functors defined on a discrete category is trivially equifibered);

- (P2a) is equivalent to 1. and (P2b) is equivalent to 2. when $\mathscr{C}$ is the category of cospans, i.e. $\mathscr{C} = \ \bullet \longrightarrow \bullet \longleftarrow \bullet\ $.

The idea is then that 1. and 2. of the Proposition must hold for general (small) colimits as they hold for coproducts and pushouts, from which every colimit can be built. We show how to deduce this for 1. and for 2. separately.

---

[6] This means of course that there is a pullback in $\mathscr{E}$ of the cospan $(Y(s), \tau_{C'})$ and that such a pullback is isomorphic to the span $(\tau_C, X(s))$.

1. Recall than that, given a functor $X \colon \mathscr{C} \longrightarrow \mathscr{E}$ (from a small category $\mathscr{C}$) a colimit (object) for X is a coequalizer of

$$A_X := \coprod_{h \in \mathrm{Mor}(\mathscr{C})} X(\mathrm{dom}(h)) \underset{l_X}{\overset{k_X}{\rightrightarrows}} \coprod_{C \in \mathscr{C}} X(C) =: B_X$$

Here the $h-$th components of k_X and l_X are $i_{\mathrm{dom}(h)}$ and $i_{\mathrm{cod}(h)}X(h)$ respectively, where, for $C \in \mathscr{C}$, i_C is the $C-$th coprojection into the rightmost coproduct. Equivalently, $\mathrm{colim}\, X$ is the pushout of the diagram

$$A_X \xleftarrow{\nabla} A_X \coprod A_X \xrightarrow{(k_X,\, l_X)} B_X \tag{2.8}$$

Now, in the situation of part 1. the pushout of the diagram

$$A_X \times_{\bar{X}} \bar{Y} \xleftarrow{\nabla \times_{\bar{X}} \bar{Y}} (A_X \coprod A_X) \times_{\bar{X}} \bar{Y} \xrightarrow{(k_X,\, l_X)\times_{\bar{X}}\bar{Y}} B_X \times_{\bar{X}} \bar{Y}$$

is (isomorphic to) $\bar{Y}$. On the other hand, if $Y := X(-) \times_{\bar{X}} \bar{Y}$, we get, using analogous definitions to the case of X, that $\mathrm{colim}\, Y$ is the pushout

$$A_Y \xleftarrow{\nabla} A_Y \coprod A_Y \xrightarrow{(k_Y,\, l_Y)} B_Y \tag{2.9}$$

Using Lemma 2.1.5, we deduce that

$$A_Y = \coprod_{h \in \mathrm{Mor}(\mathscr{C})} \left(X(\mathrm{dom}(h)) \times_{\bar{X}} \bar{Y} \right) \cong \left(\coprod_{h \in \mathrm{Mor}(\mathscr{C})} X(\mathrm{dom}(h)) \right) \times_{\bar{X}} \bar{Y} = A_X \times_{\bar{X}} \bar{Y}$$

$$B_Y = \coprod_{C \in \mathrm{Mor}(\mathscr{C})} \left(X(C) \times_{\bar{X}} \bar{Y} \right) \cong \left(\coprod_{C \in \mathrm{Mor}(\mathscr{C})} X(C) \right) \times_{\bar{X}} \bar{Y} = B_X \times_{\bar{X}} \bar{Y}$$

and these isomorphisms are compatible with the coprojections into the coproducts. Under these isomorphisms, the maps k_Y and l_Y become $k_X \times_{\bar{X}} \bar{Y}$ and $l_X \times_{\bar{X}} \bar{Y}$ respectively. Therefore, the pushout of (2.1.5) (which is $\mathrm{colim}\, Y$) is (isomorphic to) the pushout of

$$A_X \times_{\bar{X}} \bar{Y} \xleftarrow{\nabla} (A_X \times_{\bar{X}} \bar{Y}) \coprod (A_X \times_{\bar{X}} \bar{Y}) \xrightarrow{(k_X \times_{\bar{X}} \bar{Y},\, l_X \times_{\bar{X}} \bar{Y})} B_X \times_{\bar{X}} \bar{Y}$$

Again by Lemma 2.1.5, we have that

$$(A_X \times_{\bar{X}} \bar{Y}) \coprod (A_X \times_{\bar{X}} \bar{Y}) \cong \left(A_X \coprod A_X \right) \times_{\bar{X}} \bar{Y}$$

and under this isomorphism $(k_X \times_{\bar{X}} \bar{Y},\, l_X \times_{\bar{X}} \bar{Y})$ becomes $(k_X,\, l_X) \times_{\bar{X}} \bar{Y}$, so that $\mathrm{colim}\, Y \cong \bar{Y}$ as required.

2. Call a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathscr{E}$ a *quasi-pullback square*[7] if the induced map

$$A \to B \times_D C$$

is a regular epimorphism. Of course, each pullback square is a quasi-pullback square. With this terminology, we thus need to prove that, for all $C \in \mathscr{C}$, there is a quasi-pullback square in $\mathscr{E}$ given by

$$\begin{array}{ccc} X(C) & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ Y(C) & \longrightarrow & \bar{Y} \end{array}$$

---

[7]As far as the author knows, this terminology is not standard. We have just maken it up for the present exposition.

Note that the usual pasting lemma for pullbacks gives the same result for quasi-pullbacks. Namely, given a commutative diagram in $\mathscr{E}$ of the form

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & D & \longrightarrow & F
\end{array}
$$

if both squares are quasi-pullbacks, then also the outer square is such (and it is also true that, if the right and the outer square are quasi-pullback squares, then so is the left one). Let us now take $C \in \mathscr{C}$ and consider the commutative diagram in $\mathscr{E}$ given by

$$
\begin{array}{ccccc}
X(C) & \longrightarrow & \coprod\limits_{C\in\mathscr{C}} X(C) & \longrightarrow & \bar{X} \\
\downarrow & & \downarrow & & \downarrow \\
Y(C) & \longrightarrow & \coprod\limits_{C\in\mathscr{C}} Y(C) & \longrightarrow & \bar{Y}
\end{array}
\tag{2.10}
$$

Here the left square is a pullback by property (P1b) of weak descent. So to get our result that the outer rectangle is a quasi-pullback, it is enough to show that the right square is such. We will do this using the description in (2.8) of colimits as pushouts. We have a commutative diagram in $\mathscr{E}$ given by

$$
\begin{array}{ccccc}
A_X & \xleftarrow{\nabla} & A_X \coprod A_X & \xrightarrow{(k_X,\, l_X)} & B_X \\
\downarrow & & \downarrow & & \downarrow \\
A_Y & \xleftarrow[\nabla]{} & A_Y \coprod A_Y & \xrightarrow[(k_Y,\, l_Y)]{} & B_Y
\end{array}
\tag{2.11}
$$

Here, as in part 1. above, we used the notations

$$
A_X = \coprod_{h\in\mathrm{Mor}(\mathscr{C})} X(\mathrm{dom}(h)), \qquad B_X = \coprod_{C\in\mathscr{C}} X(C)
$$

and similarly for Y. Moreover, the vertical arrows are induced by the natural transformation $\tau\colon X \Longrightarrow Y$; for example, the leftmost vertical arrow is given by

$$
\coprod_{h\in\mathrm{Mor}(\mathscr{C})} \tau(\mathrm{dom}(h)).
$$

We want to show that both squares in (2.11) are pullback squares. In this way, we will get that the right square in (2.10) is a quasi-pullback by property (P2b) of weak descent, thus completing our proof. Now, using Proposition 2.1.5, we see that the two squares in (2.11) are pullbacks if each of the following squares is a pullback

$$
\begin{array}{ccccccccc}
A_X & \xrightarrow{id_{A_X}} & A_X & \quad & A_X & \xrightarrow{k_X} & B_X & \quad & A_X & \xrightarrow{l_X} & B_X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
A_Y & \xrightarrow[id_{A_Y}]{} & A_Y & & A_Y & \xrightarrow[k_Y]{} & B_Y & & A_Y & \xrightarrow[l_Y]{} & B_Y
\end{array}
\tag{2.12}
$$

(Recall that the codiagonal $\nabla\colon A \coprod A \to A$ for an object $A \in \mathscr{E}$ is the map on the coproduct induced by the identity on A). The leftmost square is clearly a pullback. As for the second one, recall that, for all $h \in \mathrm{Mor}(\mathscr{C})$, we have, by definition of k_X, $k_X \circ i_h = i_{\mathrm{dom}(h)}$, where

$$
i_h\colon X(\mathrm{dom}(h)) \to A_X, \qquad i_{\mathrm{dom}(h)}\colon X(\mathrm{dom}(h)) \to B_X
$$

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are the coprojections into the coproducts and similarly for k_Y. Furthermore, for each $h \in \mathrm{Mor}(\mathscr{C})$, the commutative square

$$
\begin{array}{ccc}
X(\mathrm{dom}(h)) & \xrightarrow{i_{\mathrm{dom}(h)}} & B_X \\
\downarrow{\scriptstyle \tau(\mathrm{dom}(h))} & & \downarrow \\
Y(\mathrm{dom}(h)) & \xrightarrow[i_{\mathrm{dom}(h)}]{} & B_Y
\end{array}
\tag{2.13}
$$

is a pullback square by (P1b) of weak descent. Then, by Proposition 2.1.5, taking the coproducts over $h \in \mathrm{Mor}(\mathscr{C})$ of the left vertices of all the squares (2.13) above, we get that the second square in (2.12) is a pullback. We then just need to show that

$$
\begin{array}{ccc}
A_X & \xrightarrow{l_X} & B_X \\
\downarrow & & \downarrow \\
A_Y & \xrightarrow[l_Y]{} & B_Y
\end{array}
$$

is a pullback square. For each $h \in \mathrm{Mor}(\mathscr{C})$, the arrow l_X satisfies, by definition, $l_X \circ i_h = i_{\mathrm{dom}(h)} \circ X(h)$, where i_h and $i_{\mathrm{dom}(h)}$ are coprojections into coproducts as above. Fix now $D \in \mathscr{C}$ and let h be an arrow with codomain D. Then, since τ is equifibered by hypothesis, we have that

$$
X(\mathrm{dom}(h)) \cong Y(\mathrm{dom}(f)) \times_{Y(D)} X(D)
$$

and hence, using as usual Proposition 2.1.5, we get a pullback square

$$
\begin{array}{ccc}
\coprod_{h \in \mathrm{Mor}(\mathscr{C}): \ \mathrm{cod}(h)=D} X(\mathrm{dom}(h)) & \longrightarrow & X(D) \\
\downarrow & & \downarrow \\
\coprod_{h \in \mathrm{Mor}(\mathscr{C}): \ \mathrm{cod}(h)=D} Y(\mathrm{dom}(h)) & \longrightarrow & Y(D)
\end{array}
$$

Since the left square in (2.10) is a pullback, the pasting lemma implies that we have a pullback square

$$
\begin{array}{ccc}
\coprod_{h \in \mathrm{Mor}(\mathscr{C}): \ \mathrm{cod}(h)=C} X(\mathrm{dom}(h)) & \longrightarrow & \coprod_{C \in \mathscr{C}} X(C) \\
\downarrow & & \downarrow \\
\coprod_{h \in \mathrm{Mor}(\mathscr{C}): \ \mathrm{cod}(h)=C} Y(\mathrm{dom}(h)) & \longrightarrow & \coprod_{C \in \mathscr{C}} Y(C)
\end{array}
\tag{2.14}
$$

Summing over all $D \in \mathscr{C}$ the left vertices of all the squares of the form (2.14) above, we get by Proposition 2.1.5 that the commutative diagram

$$
\begin{array}{ccc}
A_X & \xrightarrow{l_X} & B_X \\
\downarrow & & \downarrow \\
A_Y & \xrightarrow[l_Y]{} & B_Y
\end{array}
$$

is a pullback square. This proves that, for each $C \in \mathscr{C}$, the map

$$
g_C \colon X(C) \to Y(C) \times_{\bar{Y}} \bar{X}
$$

is a regular epimorphism.

Assume now that $\mathscr{C}$ is a groupoid with at most one arrow between any of its objects. We need to show that each of the arrow g_C is an isomorphism. Observe first that the groupoid condition for $\mathscr{C}$ implies that each functor defined on $\mathscr{C}$ with values in $\mathscr{E}$ sends arrows in $\mathscr{C}$ (which are isomorphisms) to isomorphisms in $\mathscr{E}$. For each connected component of $\mathscr{C}$, choose now a representative object in $\mathscr{C}$ and take the set $\mathcal{R}$ of all these chosen representatives. Denote by

$$\varepsilon \colon \mathcal{R} \longrightarrow \mathscr{C}$$

the inclusion functor of $\mathcal{R}$, seen as a discrete subcategory of $\mathscr{C}$. For each $C \in \mathscr{C}$, consider the category $(C \downarrow \varepsilon)$ of objects over ε; its objects are arrows $C \to \varepsilon(R)$ in $\mathscr{C}$ for $R \in \mathcal{R}$. This category $(C \downarrow \varepsilon)$ is not empty because C is in the same connected component of (exactly) one $R \in \mathcal{R}$. This means that there is a (finite) zig-zag of arrows connecting C and R and, since every arrow in $\mathscr{C}$ is an isomorphism, we can invert some of the arrows of such a zig-zag to get a morphism $C \to R$. Moreover, $(C \downarrow \varepsilon)$ is a connected category. For, if $f \colon C \to \varepsilon(R)$ and $g \colon C \to \varepsilon(S)$ are objects in $(C \downarrow \varepsilon)$, then R and S are in the same connected component of $\mathscr{C}$, thus they must be equal. Since for any two objects in $\mathscr{C}$ there is at most one arrow between them, this implies that $f = g$. Since $C \in \mathscr{C}$ was arbitrary, we get that the inclusion ε is a cofinal functor. Therefore, we have

$$\bar{X} = \operatorname{colim} X = \coprod_{R \in \mathcal{R}} X(C), \qquad \bar{Y} = \operatorname{colim} Y = \coprod_{R \in \mathcal{R}} Y(C).$$

By property (P1b) of weak descent, we then obtain that, for all $R \in \mathcal{R}$, the arrow

$$X(R) \to Y(R) \times_{\bar{Y}} \bar{X}$$

is invertible. But now, for each $C \in \mathscr{C}$, there is an arrow $C \to R$ in $\mathscr{C}$ for some $R \in \mathcal{R}$ and hence a commutative diagram

$$
\begin{array}{ccc}
X(C) & \xrightarrow{\ g_C\ } & Y(C) \times_{\bar{Y}} \bar{X} \\
\downarrow & & \downarrow \\
X(R) & \xrightarrow[\ g_R\ ]{} & Y(R) \times_{\bar{Y}} \bar{X}
\end{array}
$$

Here all the morphisms except the upper horizontal arrow are iso and thus also that one is such.

The proof is then complete.

$\square$

Remark 2.1.15. As it is pointed out in its proof, we could actually formulate Proposition 2.1.14 by saying that, for a cocomplete and finitely complete category $\mathscr{E}$, admitting weak descent is *equivalent* to verifying properties 1. and 2. of Proposition 2.1.14.

We can improve the second point of Proposition 2.1.14 if τ is a monomorphism.

Proposition 2.1.16. *In the situation of part 2. of Proposition 2.1.14, suppose that $\tau_C \colon X(C) \to Y(C)$ is a monomorphism in $\mathscr{E}$ for all $C \in \mathscr{C}$. Then $\bar{\tau} = \operatorname{colim}(\tau) \colon \bar{X} \to \bar{Y}$ is a monomorphism and the arrows $X(C) \to Y(C) \times_{\bar{Y}} \bar{X}$ are isomorphisms for all $C \in \mathscr{C}$.*

Proof. By part 2. of Proposition 2.1.14, each morphism $g \colon X(C) \to Y(C) \times_{\bar{Y}} \bar{X}$ is a regular epimorphism. Since the composition of g with the projection $Y(C) \times_{\bar{Y}} \bar{X} \to Y(C)$ is τ_c, which is assumed to be a monomorphism, g is a monomorphism as well, thus it is an isomorphism by Proposition 1.3.3 (ii). Now, for each $C \in \mathscr{C}$, we have a commutative diagram

$$
\begin{array}{ccccc}
X(C) & \xrightarrow{\ id\ } & X(C) & \longrightarrow & \bar{X} \\
{\scriptstyle id}\downarrow & & {\scriptstyle \tau}\downarrow & & {\scriptstyle \bar{\tau}}\downarrow \\
X(C) & \xrightarrow[\ \tau\ ]{} & Y(C) & \longrightarrow & \bar{Y}
\end{array}
$$

where the rightmost horizontal morphisms are the canonical arrows into the colimits. In this diagram, the right square is a pullback by the part already proved and the left square is a pullback as τ is mono. It follows that also the outer square is a pullback, i.e. for all $C \in \mathscr{C}$ it holds that

$$X(C) \times_{\bar{Y}} \bar{X} \cong X(C)$$

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and the isomorphism is given by the projection. Similarly, we get that

$$X(C) \times_{\bar{X}} (\bar{X} \times_{\bar{Y}} \bar{X}) \cong X(C) \times_{\bar{Y}} \bar{X}$$

for all $C \in \mathscr{C}$. But now

$$\bar{X} \times_{\bar{Y}} \bar{X} \cong \operatorname{colim}_{C \in \mathscr{C}} (X(C) \times_{\bar{X}} (\bar{X} \times_{\bar{Y}} \bar{X})) \cong \operatorname{colim}_{C \in \mathscr{C}} X(C) \cong \bar{X},$$

where the first isomorphism is given by part 1. of Proposition 2.1.14. This shows that $\bar{X}$ is isomorphic to the pullback of $\bar{\tau}$ along itself. Thus $\bar{\tau}$ is a monomorphism. $\qquad\square$

We conclude this section by proving that every Grothendieck Topos has a subobject classifier.

Recall that given a category $\mathscr{D}$ and an object X of $\mathscr{D}$, one can define an equivalence relation on the class of monic arrows $m\colon S \rightarrowtail X$ by declaring that two monics m, m' with codomain X are equivalent if there is an isomorphism $\psi\colon \operatorname{dom}(m) \to \operatorname{dom}(m')$ such that $m'\psi = m$. A *subobject* of X is then an equivalence class of monics with target X. By a common abuse of language, we say often that $m\colon S \rightarrowtail X$ (or even S) is a subobject of X, meaning always the equivalence class of m. One denotes with $\operatorname{Sub}_{\mathscr{D}}(X)$ the class of subobjects of X and says that $\mathscr{D}$ is *well-powered* if $\operatorname{Sub}_{\mathscr{D}}(X)$ is a small set for all $X \in \mathscr{D}$. If $\mathscr{D}$ is well-powered and has pullbacks, we get a functor

$$\operatorname{Sub}_{\mathscr{D}}\colon \mathscr{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$$

which sends an arrow $f\colon Y \to X$ in $\mathscr{D}$ to the map

$$\operatorname{Sub}_{\mathscr{D}}(f)\colon \operatorname{Sub}_{\mathscr{D}}(X) \to \operatorname{Sub}_{\mathscr{D}}(Y)$$

taking a monic $m\colon S \rightarrowtail X$ to its pullback arrow along f.

Definition 2.1.17. Let $\mathscr{D}$ be a finitely complete category with terminal object 1. A *subobject classifier* for $\mathscr{D}$ is a monic

$$\text{true}\colon 1 \rightarrowtail \Omega$$

such that to every monic $m\colon S \rightarrowtail X$ in $\mathscr{D}$ there is a unique arrow $\varphi\colon X \to \Omega$ turning the following diagram into a pullback square

The morphism φ is called the *characteristic map* of the subobject m.

The reason for the name "subobject classifier" is given by the following

Proposition 2.1.18. *A finitely complete category $\mathscr{D}$ (with small Hom sets) has a subobject classifier if and only if there are an object $\Omega \in \mathscr{D}$ and an isomorphism*

$$\theta_X\colon \operatorname{Sub}_{\mathscr{D}}(X) \cong \operatorname{Hom}_{\mathscr{D}}(X, \Omega)$$

natural in $X \in \mathscr{D}$. When this holds, $\mathscr{D}$ is well-powered, hence, by the above natural isomorphism, the functor $\operatorname{Sub}_{\mathscr{D}}\colon \mathscr{D}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable.

Proof. See [McMo] §1.3, Proposition 1. $\qquad\square$

Note that in the category of sets, every subobject of $X \in \mathbf{Set}$ has a representative given by the inclusion into X of one of its subsets. Thus $\mathbf{Set}$ is well-powered. A subobject classifier for $\mathbf{Set}$ is given by $\Omega := 2$ and by a map $1 \rightarrowtail 2$ which "picks the value true" (for example, one can take the inclusion of 1 into 2). As announced, the same properties hold in any Grothendieck topos.

Proposition 2.1.19. *Given a topos $\mathscr{E}$, the class $\operatorname{Sub}_{\mathscr{E}}(X)$ is a small set for all $A \in \mathscr{E}$ and the functor $\operatorname{Sub}_{\mathscr{E}}\colon \mathscr{E}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable. Thus, $\mathscr{E}$ has a subobject classifier.*

Proof. Subobjects in presheaves categories have representatives given by subfunctors, so $\mathrm{PSh}(\mathscr{C})$ is well-powered when $\mathscr{C}$ is a small category. If $\mathrm{Sh}(\mathscr{C})$ is a category of sheaves, every monomorphism of sheaves is a monomorphism of the underlying presheaves, because $\mathrm{Sh}(\mathscr{C})$ is closed under limits in $\mathrm{PSh}(\mathscr{C})$. Hence, also $\mathrm{Sh}(\mathscr{C})$ is well-powered.

In order to show that $\mathrm{Sub}_{\mathscr{E}}$ is representable, it is enough to prove that it sends colimits in $\mathscr{E}$ to limits in **Set**, by Proposition 1.1.9. Thus we need to show that, for any functor $X\colon \mathscr{I} \longrightarrow \mathscr{E}$ from a small category $\mathscr{I}$, we have

$$\mathrm{Sub}_{\mathscr{E}}(\mathrm{colim}_{i\in\mathscr{I}} X(i)) \cong \lim_{i\in\mathscr{I}} \mathrm{Sub}_{\mathscr{E}}(X(i))$$

Set $\bar{X} := \mathrm{colim}\, X$. Given a subobject $m\colon A \rightarrowtail \bar{X}$ of $\bar{X}$ in $\mathscr{E}$, for each $i \in \mathscr{I}$, we can pullback m along the colimiting arrow $X(i) \to \bar{X}$. This gives rise to an element $(A \times_{\bar{X}} X(i) \to X(i))_{i\in\mathscr{I}}$ in $\lim_{i\in\mathscr{I}} \mathrm{Sub}_{\mathscr{E}}(X(i))$. Conversely, if we are given an element $(A(i) \to X(i))_{i\in\mathscr{I}}$ of $\lim_{i\in\mathscr{I}} \mathrm{Sub}_{\mathscr{E}}(X(i))$, we can define a functor $A\colon \mathscr{I} \longrightarrow \mathscr{E}$ sending an object $i \in \mathscr{I}$ to A_i and an arrow $h\colon i \to j$ to the unique morphism $A(h)\colon A_i \to A_j$ in $\mathscr{E}$ making the following diagram commute

$$
\begin{array}{ccc}
A_i & \longrightarrow & X_i \\
{\scriptstyle A(h)}\big\downarrow & & \big\downarrow{\scriptstyle X(h)} \\
A_j & \longrightarrow & X_j
\end{array}
$$

(Such an $A(h)$ exists as $(A(i) \to X(i))_{i\in\mathscr{I}} \in \lim_{i\in\mathscr{I}} \mathrm{Sub}_{\mathscr{E}}(X(i))$). In this way, $(A(i) \to X(i))_{i\in\mathscr{I}}$ forms an equifibered natural transformation $\tau\colon A \Longrightarrow X$ which is a monomorphism. By Proposition 2.1.16, we get that the induced arrow $\mathrm{colim}(\tau)\colon \mathrm{colim}\, A \to \mathrm{colim}\, X$ is a subobject of $\mathrm{colim}\, X$. The assignments

$$(A \rightarrowtail \mathrm{colim}\, X) \mapsto (A \times_{\bar{X}} X(i) \to X(i))_{i\in\mathscr{I}}$$

and

$$(A(i) \to X(i))_{i\in\mathscr{I}} \mapsto (\mathrm{colim}(\tau)\colon \mathrm{colim}\, A \to \mathrm{colim}\, X)$$

are mutually inverse, completing the proof. $\qquad\square$

2.2 Giraud's Theorem

In this section we are going to state and prove the main result of the chapter, namely a theorem by Giraud which characterises Grothendieck topoi internally, as categories satisfying suitable properties. We first need to recall a categorical notion.

Definition 2.2.1. Let $\mathscr{E}$ be a category. A class $\mathcal{C}$ of objects in $\mathscr{E}$ is said to *generate* $\mathscr{E}$ (or to be a *class of generators* for $\mathscr{E}$) if, for any pair $f, g \colon X \to Y$ of arrows in $\mathscr{E}$, we have that $f = g$ if and only if $ft = gt$ for all $C \in \mathcal{C}$ and all $t \colon C \to X$. If $\mathcal{C} = \{C\}$ for a single object $C \in \mathscr{E}$, we say that C is a *generator* for $\mathscr{E}$.

Remark 2.2.2. Suppose $\mathscr{E}$ is a category with small coproducts. Then a *small set* $\mathcal{C}$ of objects in $\mathscr{E}$ generates $\mathscr{E}$ if and only if for all objects X in $\mathscr{E}$ the obvious arrow

$$p_X \colon \coprod_{h \colon C \to X, \; C \in \mathcal{C}} C \to X$$

is an epimorphism.

Lemma 2.2.3. *Let $\mathscr{E}$ be a Grothendieck topos. Then $\mathscr{E}$ has a small set of generators.*

Proof. **Set** has a small set of generators given by $\{1\}$. Let then $\mathscr{C}$ be a small category and let $\mathscr{E} := \operatorname{Sh}(\mathscr{C})$ be a category of sheaves over $\mathscr{C}$. As usual, let $i \colon \operatorname{Sh}(\mathscr{C}) \longrightarrow \operatorname{PSh}(\mathscr{C})$ be the inclusion functor and write $a \colon \operatorname{PSh}(\mathscr{C}) \longrightarrow \operatorname{Sh}(\mathscr{C})$ for the left adjoint. Take $F \in \operatorname{Sh}(\mathscr{C})$ and recall that iF (as any presheaf) can be written as a colimit of representable functors, namely

$$iF \cong \operatorname{colim}_{(C, \; e) \in \operatorname{El}(iF)} \mathscr{C}(-, C),$$

where $\operatorname{El}(iF)$ is the category of elements of iF. Since a preserves colimits, we get

$$F \cong aiF \cong a\left(\operatorname{colim}_{(C, \; e) \in \operatorname{El}(iF)} \mathscr{C}(-, C)\right) \cong \operatorname{colim}_{(C, \; e) \in \operatorname{El}(iF)} \left(a\mathscr{C}(-, C)\right).$$

By the universal property of the colimit, this implies in particular that the small set consisting of all $a\mathscr{C}(-, C)$ for $C \in \mathscr{C}$ generates $\operatorname{Sh}(\mathscr{C})$. $\qquad\square$

We are now ready to state the

Theorem 2.2.4 (Giraud's Theorem, cf. [Rzk1], Theorem 2.8)**.** *Let $\mathscr{E}$ be a cocomplete and finitely complete category. Then $\mathscr{E}$ is a Grothendieck topos if and only if $\mathscr{E}$ has weak descent and is generated by a small set of objects.*

Observe that Propositions 1.1.8 and 2.1.11 as well as Lemma 2.2.3 imply that a Grothendieck topos satisfies the properties of Giraud's Theorem.

Before giving a proof of Giraud's Theorem, we state and prove the following

Corollary 2.2.5. *Let $\mathscr{E}$ be a Grothendieck topos. Then the following hold.*

(i) *For each object X of $\mathscr{E}$, the slice category $\mathscr{E}/X$ is a Grothendieck topos.*

(ii) *For every small category $\mathscr{D}$, the functor category $\mathscr{E}^{\mathscr{D}}$ is a Grothendieck topos.*

Proof. (of Corollary 2.2.5) Both for (i) and for (ii) we just check that the hypotheses of Giraud's theorem are satisfied. For sake of simplicity, we assume that $\mathscr{E}$ has a single generator $C \in \mathscr{E}$, the general case being similar.

(i) Since $\mathscr{E}$ is cocomplete and finitely complete, then so is $\mathscr{E}/X$. It is also clear that $\mathscr{E}/X$ has weak descent, basically by Remark 2.1.4. Finally, $\mathscr{E}(C, X)$ is a small set of generators for $\mathscr{E}$. Therefore, $\mathscr{E}/X$ verifies (i) and (ii) of Theorem 2.2.4.

(ii) It is again immediate that $\mathscr{E}^{\mathscr{D}}$ is cocomplete and finitely complete and that it satisfies weak descent. For each $D \in \mathscr{D}$ define an object F_D in $\mathscr{E}^{\mathscr{D}}$ by setting, for all $D' \in \mathscr{E}$,

$$F_D(D') := \coprod_{\mathscr{D}(D, D')} C.$$

If $f\colon D' \to D''$ is an arrow in $\mathscr{D}$, then $F_D(f)$ is the morphism sending the summand of $F_D(D')$ corresponding to $\sigma\colon D \to D'$ to the summand of $F_D(D'')$ corresponding to $f \circ \sigma$. It is not difficult to see that, for all $D \in \mathscr{D}$ and all $G \in \mathscr{E}^{\mathscr{D}}$, one has

$$\mathscr{E}^{\mathscr{D}}(F_D,\ G) \cong \mathscr{E}(C,\ GD).$$

Since C is a generator for $\mathscr{E}$, this implies that $\{F_D\}_{D \in \mathscr{D}}$ is a small set of generators for $\mathscr{E}^{\mathscr{D}}$.

$\square$

Remark 2.2.6. By Proposition 2.1.9, given a topos $\mathscr{E}$ and a morphism $f\colon Y \to X$ in $\mathscr{E}$, the change of base functor

$$f^*\colon \mathscr{E}/X \longrightarrow \mathscr{E}/Y$$

preserves all small colimits. But we have just seen that the slice category $\mathscr{E}/X$ is again a topos. Since every Grothendieck topos is a presentable category (see Definition 1.1.3), Corollary 1.1.10 implies that f^* has a right adjoint. It has also a *left* adjoint, given by composition with f. In fact, one can prove (see [McMo] §1.9) that, given any category $\mathscr{D}$ with pullbacks and any arrow $f\colon C \to B$ in $\mathscr{D}$, the change of base functor f^* has a left adjoint given by composition with f. If, moreover, $\mathscr{D}/B$ is cartesian closed, than each such f^* also has a right adjoint.

The rest of this section is devoted to show Giraud's Theorem. We shall need a lot of auxiliary results that hold in every category with weak descent; some of these we have already proved for topoi.

Proposition 2.2.7. *Let $\mathscr{E}$ be a category with weak descent and suppose given a pullback square in $\mathscr{E}$*

$$
\begin{array}{ccc}
U & \xrightarrow{\ q\ } & X \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\
V & \xrightarrow[\ p\ ]{} & Y
\end{array}
$$

where p is a regular epimorphism. Then q is a regular epimorphism as well and f is an isomorphism if and only if g is such.

Proof. Consider the commutative diagram

$$
\begin{array}{ccccc}
W \times_Y X & \rightrightarrows & U & \xrightarrow{\ q\ } & X \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\
W & \rightrightarrows & V & \xrightarrow[\ p\ ]{} & Y
\end{array}
$$

where the bottom row is a coequalizer diagram. The top row is obtained from the bottom one by pulling back along f and is then a coequalizer by part 1. of Proposition 2.1.14. Therefore, q is a regular epimorphism. If g is an isomorphism, then so is h, since it is obtained from g by pullback. It follows that also f is an isomorphism, as it is the map induced by (h, g) on the colimits of the diagrams $W \times_Y X \rightrightarrows U$ and $W \rightrightarrows V$. $\square$

Proposition 2.2.8. *Let $\mathscr{E}$ be a category with weak descent and let $f\colon X \to Y$ be a map in $\mathscr{E}$. Then there exists an essentially unique factorization $f = ip$ where i is a monomorphism and p is a regular epimorphism. Furthermore, p is the coequalizer of the pair of projections $X \times_Y X \rightrightarrows X$. In particular, all regular epimorphisms are effective epimorphisms.*

Proof. We have a diagram

$$X \times_Y X \rightrightarrows X \xrightarrow{\ p\ } I \xrightarrow{\ i\ } Y$$

where p is the coequalizer of the projections from $X \times_Y X$ and i is the unique arrow $I \to Y$ in $\mathscr{E}$ such that $ip = f$. So we only need to check that i is a monomorphism. Equivalently, we want to prove that the arrow t in the following diagram is an isomorphism

$$
\begin{array}{ccccc}
X \times_I (I \times_Y I) & \longrightarrow & I \times_Y I & \longrightarrow & I \\
\downarrow & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle i} \\
X & \xrightarrow[\ p\ ]{} & I & \xrightarrow[\ i\ ]{} & Y
\end{array}
$$

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Since p is a regular epimorphism and the left hand square is a pullback, Proposition 2.2.7 implies that t is an isomorphism if and only if the arrow $(X \times_I (I \times_Y I)) \to X$ is such. Under the isomorphism $(X \times_I (I \times_Y I)) \cong X \times_Y I$, this arrow $(X \times_I (I \times_Y I)) \to X$ becomes the projection $j \colon X \times_Y I \to X$. So we need to show that j is an isomorphism. Let us consider the following diagram

$$
\begin{array}{ccccccc}
X \times_Y X \times_Y X & \rightrightarrows & X \times_Y X & \xrightarrow{q} & X \times_Y I & \xrightarrow{j} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow f \\
X \times_Y X & \rightrightarrows & X & \xrightarrow{p} & I & \xrightarrow{i} & Y
\end{array}
$$

Here the top row is obtained from the bottom one by pullback along f and using the pasting lemma for pullbacks. Since the middle square is a pullback, by Proposition 2.1.14, q is a coequalizer of the pair of arrows shown. Now, it is immediate to see that the diagram

$$
X \times_Y X \times_Y X \xrightarrow[\pi_2]{\pi_1} X \times_Y X \xrightarrow{jq} X
$$

is a *split fork* in $\mathscr{E}$. This means that we have arrows

$$
X \xrightarrow{\ s\ } X \times_Y X \xrightarrow{\ t\ } X \times_Y X \times_Y X
$$

satisfying

$$
\pi_1 t = id, \qquad jqs = id, \qquad \pi_2 t = sjq.
$$

It follows readily that jq is a coequalizer of π_1 and π_2 as q: if $h \colon X \times_Y X \to Z$ is a morphism in $\mathscr{E}$ such that $h\pi_1 = h\pi_2$, then there is a unique $k \colon X \times_Y I \to Z$ such that $kq = h$, so that kqs verifies

$$
kqsjq = kq\pi_2 t = h\pi_2 t = h\pi_1 t = h.
$$

We conclude that j must be an isomorphism, so that i is a monomorphism. By Proposition 1.3.3, the (regular epi, mono)-factorization of f is unique up to within isomorphism. This concludes the proof. $\qquad \square$

Corollary 2.2.9. *Let $\mathscr{E}$ be a category with weak descent. Given a small category $\mathscr{J}$, let $\tau \colon F \Longrightarrow G$ be a natural transformation between functors $F, G \colon \mathscr{J} \longrightarrow \mathscr{E}$. If τ is an objectwise regular epimorphism, then the induced arrow*

$$
\mathrm{colim}(\tau) \colon \mathrm{colim}\, F \to \mathrm{colim}\, G
$$

is a regular epimorphism as well.

Proof. By Proposition 2.2.8 and the fact that $\mathscr{E}^{\mathscr{J}}$ has weak descent, τ is the coequalizer of the two projections $F \times_G F \rightrightarrows F$. Since colimits commute among them, we have

$$
\mathrm{colim}\, \tau = \mathrm{colim}(\mathrm{coeq}(F \times_G F \rightrightarrows F)) \cong \mathrm{coeq}(\mathrm{colim}(F \times_G F \rightrightarrows F)),
$$

so $\mathrm{colim}\, \tau$ is a regular epi. $\qquad \square$

The following result has the same proof of Corollary 1.3.7, so we omit it.

Corollary 2.2.10. *Let $f \colon X \to Y$ and $g \colon Y \to Z$ be morphisms in a category $\mathscr{E}$ having weak descent.*

(i) *If gf is a regular epimorphism, then so is g.*

(ii) *If f and g are regular epimorphisms, then so is gf.*

The hierarchy of epimorphisms in a category with weak descent is simplified by the following

Proposition 2.2.11. *Let $\mathscr{E}$ be a category with weak descent. Then every epimorphism in $\mathscr{E}$ is a regular epimorphism.*

Proof. Since every map in $\mathscr{E}$ admits a (regular epi, mono)-factorization, by Corollary 2.2.10 above, it is enough to show that if $f: A \to B$ is an epimorphism and a monomorphism, then it is a regular epimorphism. Consider the diagram

$$
\begin{array}{ccccc}
A & \xleftarrow{\ id\ } & A & \xrightarrow{\ f\ } & B \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\
B & \xleftarrow{\ f\ } & A & \xrightarrow{\ f\ } & B
\end{array}
$$

Here (f, id, id) is an equifibered natural transformation between the diagrams given by the rows, because f is a monomorphism. The map induced by (f, id, id) between the colimits of the rows is (isomorphic to) the identity of B, so that property (P2b) of weak descent implies that $f: A \to B \cong B \times_B B$ is a regular epimorphism, as required. $\qquad\square$

Remark 2.2.12. Proposition 2.2.11 above and Proposition 1.3.3 imply that in a category with weak descent (so, in particular, in a Grothendieck topos) an arrow is an isomorphism if and only if it is both mono and epi. In other words, every category with weak descent is *balanced*.

We also need the following technical result.

Lemma 2.2.13. *Let $\mathscr{J}$ be a small category and let $\mathscr{E}$ be a category with weak descent. Consider a pullback square in $\mathscr{E}^{\mathscr{J}}$ of the form*

$$
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{\ f\ } & B
\end{array}
$$

where both f and g are equifibered natural transformations. Then the canonical arrow

$$
\operatorname{colim} A \to (\operatorname{colim} X) \times_{\operatorname{colim} B} (\operatorname{colim} Y)
$$

is a regular epimorphism.

Proof. Write $\bar{A}$, $\bar{B}$, $\bar{X}$, $\bar{Y}$ for the colimits of A, B, X and Y respectively. Consider the functors

$$
X' := B(-) \times_{\bar{B}} \bar{X}, \quad Y' := B(-) \times_{\bar{B}} \bar{Y}, \quad A' := B(-) \times_{\bar{B}} (\bar{X} \times_{\bar{B}} \bar{Y}).
$$

Note that, for all $j \in \mathscr{J}$, we have

$$
A'(j) \cong (B(j) \times_{\bar{B}} \bar{X}) \times_{\bar{B}} \bar{Y} = X'(j) \times_{\bar{B}} \bar{Y} \cong X'(j) \times_{B(j)} (B(j) \times_{\bar{B}} \bar{Y}) = X'(j) \times_{B(j)} Y'(j).
$$

By the first part of Proposition 2.1.14, we obtain immediately that $\operatorname{colim} X' \cong \bar{X}$, $\operatorname{colim} Y' \cong \bar{Y}$ and $\operatorname{colim} A' \cong \bar{X} \times_{\bar{B}} \bar{Y}$. Now, by Corollary 2.2.9, the Lemma will be proved if we show that, for all $j \in \mathscr{J}$, the map $A(j) \to A'(j)$ is a regular epimorphism. Such a map is given as the composite

$$
A(j) \cong X(j) \times_{B(j)} Y(j) \to X(j) \times_{B(j)} Y'(j) \to X'(j) \times_{B(j)} Y'(j) \cong A'(j). \tag{2.15}
$$

But now the arrows $X(j) \to X'(j)$ and $Y(j) \to Y'(j)$ are regular epimorphisms for all $j \in \mathscr{J}$, by the hypothesis that both f and g are equifibered and using the second part of the providential Proposition 2.1.14. Since regular epimorphisms are preserved under pullbacks (Proposition 2.2.7) and are closed under composition (Corollary 2.2.10), we then get that also the morphism (2.15) above is a regular epimorphism, as needed. $\qquad\square$

With all these preliminary results at hand, the proof of Giraud's theorem will follow from the next two Propositions. Before stating them, let us first recall a notion from general Category Theory that we are going to need immediately.

Definition 2.2.14. Let $F: \mathscr{A} \longrightarrow \mathscr{B}$ and $G: \mathscr{A} \longrightarrow \mathscr{D}$ be functors. A pair (K, α), where $K: \mathscr{B} \longrightarrow \mathscr{D}$ is a functor and $\alpha: G \Longrightarrow KF$ is a natural transformation, is a *left Kan extension* of G along F if it satisfies the following universal property. Given any pair (H, β) where $H: \mathscr{B} \longrightarrow \mathscr{D}$ is a functor and $\beta: G \Longrightarrow HF$ is a natural transformation, there exists a unique natural transformation $\gamma: K \Longrightarrow H$ such that $\gamma_F \circ \alpha = \beta$.

Remark 2.2.15. (a) As usual when dealing with universal properties, a left Kan extension of G along F, if it exists, is unique up to within a unique isomorphism. One then commonly talks about *the* left Kan extension of G along F and denotes it by $\mathrm{Lan}_F(G)$.

(b) It is a well-known fact (see [Bor1] Theorem 3.7.2) that if $\mathscr{A}$ is a small category and $\mathscr{D}$ is a cocomplete category, then for any couple of functors $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G\colon \mathscr{A} \longrightarrow \mathscr{D}$, $\mathrm{Lan}_F(G)$ exists and can be computed objectwise as follows. For all $B \in \mathscr{B}$, let $\mathrm{El}(B)$ be the category of elements of the functor $\mathscr{B}(F-, B)\colon \mathscr{A} \longrightarrow \mathbf{Set}$ and let $U_B\colon \mathrm{El}(B) \longrightarrow \mathscr{A}$ be the forgetful functor. Then

$$(\mathrm{Lan}_F(G))(B) = \mathrm{colim}_{\mathrm{El}(B)}\, GU_B. \tag{2.16}$$

Alternatively, one can construct $\mathrm{Lan}_F(G)$ as the coend

$$(\mathrm{Lan}_F(G))(B) = \int^{A \in \mathscr{A}} \mathscr{B}(FA, B) \cdot GA, \tag{2.17}$$

where $\mathscr{B}(FA, B) \cdot GA$ is the copower $\coprod_{\mathscr{B}(FA, B)} GA$.

(c) Given a functor $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ between small categories, we get, for all categories $\mathscr{D}$, an induced functor

$$F^* = (-) \circ F\colon \mathscr{D}^{\mathscr{B}} \longrightarrow \mathscr{D}^{\mathscr{A}}$$

Now, if we fix a category $\mathscr{D}$ and if every functor $G\colon \mathscr{A} \longrightarrow \mathscr{D}$ has a left Kan extension $\mathrm{Lan}_F(G)$, then the assignment

$$\mathrm{Lan}_F(-)\colon \mathscr{D}^{\mathscr{A}} \longrightarrow \mathscr{D}^{\mathscr{B}}, \quad G \mapsto \mathrm{Lan}_F(G)$$

defines a left adjoint of F^*. Thus, we have natural isomorphisms

$$\mathscr{D}^{\mathscr{A}}(\mathrm{Lan}_F(G),\ K) \cong \mathscr{D}^{\mathscr{B}}(G,\ KF)$$

for all $G \in \mathscr{D}^{\mathscr{A}}$ and for all $K \in \mathscr{D}^{\mathscr{B}}$.

Remark 2.2.16. Given functors $F\colon \mathscr{A} \longrightarrow \mathscr{B}$ and $G\colon \mathscr{A} \longrightarrow \mathscr{D}$, one can dualize Definition 2.2.14 by considering the *right Kan extension* of G along F as a pair $(K,\ \beta)$, where $K\colon \mathscr{B} \longrightarrow \mathscr{D}$ is a functor and $\beta\colon KF \Longrightarrow G$ is a natural transformation which is universal among such pairs. This right Kan extension, if it exists, is again unique up to a unique isomorphism and is denoted by $\mathrm{Ran}_F(G)$. Right Kan extensions satisfy the dual properties of those listed in Remark 2.2.15 for left Kan extensions.

We can now prove the following

Proposition 2.2.17. *Let $\mathscr{E}$ be a cocomplete and finitely complete category which has weak descent and a small set of generators $\mathcal{C}$. Let $\mathscr{C}$ be the full small subcategory of $\mathscr{E}$ spanned by $\mathcal{C}$. Then there is an adjoint pair*

$$\mathrm{PSh}(\mathscr{C}) \underset{R}{\overset{L}{\rightleftarrows}} \bot\ \mathscr{E}\ ,$$

where (L is the left adjoint and) R sends $X \in \mathscr{E}$ to the functor $\mathscr{E}(-, X) \in \mathrm{PSh}(\mathscr{C})$. Moreover, the counit $\varepsilon\colon LR \longrightarrow \mathrm{Id}_{\mathscr{E}}$ is a natural isomorphism, so that $\mathscr{E}$ is equivalent to a full (replete) subcategory of $\mathrm{PSh}(\mathscr{C})$.

Proof. Let $\mathbf{y}\colon \mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C})$ be the Yoneda embedding and $i\colon \mathscr{C} \longrightarrow \mathscr{E}$ the inclusion. We then set

$$L := \mathrm{Lan}_{\mathbf{y}}(i)\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathscr{E}$$

By the description of $\mathrm{Lan}_{\mathbf{y}}(i)(F)$ (for $F \in \mathrm{PSh}(\mathscr{C})$) in terms of the colimit (2.16), it is immediate to see that L is left adjoint to R (see also [McMo] §1.5, Theorem 2). Note also that, using again (2.16), we get

$$\mathrm{Lan}_{\mathbf{y}}(i)(F) = \mathrm{colim}_{(C,\ x \in F(C))}\, C$$

where $C \in \mathscr{C}$ and the colimit is taken over the category of elements of F. Fixing $X \in \mathscr{E}$ and applying this to $F = \mathscr{E}(-, X)$ one sees that $(LR)(X)$ is the coequalizer in

$$\left(\coprod_{C' \to C \to X} C' \right) \rightrightarrows \left(\coprod_{C \to X} C \right) \overset{g}{\to} (LR)(X)$$

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Here the first coproduct is over all sequences of maps $C' \to C \to X$ in $\mathscr{E}$ with $C, C' \in \mathscr{C}$ (these sequences are just the arrows in the category of elements of $\mathscr{E}(-, X)$), whereas the second coproduct is made over all elements of $\mathscr{E}(C, X)$, for $C \in \mathscr{C}$. The $X-$th component of the counit, $\varepsilon_X \colon (LR)(X) \to X$, is the map obtained using the universal property of the coequalizer $(LR)(X)$ from the morphism

$$ q \colon \coprod_{d \colon C \to X} C \to X $$

whose $d-$th component is d itself. Set $U := \coprod_{C \to X} C$; since $\mathscr{E}$ is generated by $\mathscr{C}$, the map $q \colon U \to X$ above is an epimorphism (see Remark 2.2.2), hence it is an effective epimorphism by Propositions 2.2.11 and 2.2.7. Thus, q is the coequalizer of $U \times_X U \rightrightarrows U$.

Consider now the small set S of all commutative squares in $\mathscr{E}$

$$
\begin{array}{ccc}
D & \xrightarrow{\;d\;} & C \\
{\scriptstyle d'}\big\downarrow & & \big\downarrow{\scriptstyle c} \\
C' & \xrightarrow{\;c'\;} & X
\end{array}
\tag{2.18}
$$

where D, C, C' are in $\mathscr{C}$ and take $\coprod_S D$, where D is the northwest corner of a commutative square as the one in (2.18) above. By weak descent, there is an isomorphism

$$ U \times_X U \cong \left(\coprod_{C \to X} C \right) \times_X \left(\coprod_{C' \to X} C' \right) \cong \coprod_{(C \to X,\ C' \to X)} (C \times_X C'). $$

Under this identification, we get an epimorphism

$$ p \colon \coprod_S D \to U \times_X U. $$

Its cocomponent corresponding to a commutative square like (2.18) is the map $D \to C \times_X C'$ induced on the pullback by d and d'.

We need this p to prove that ε_X is an epimorphism. Indeed, since, by its definition, $\varepsilon_X \circ g = q$ and q is an epimorphism, we get that ε_X is epi as well. We show that ε_X is an isomorphism by proving that it admits a retraction. To this end, we just need to show that $g \colon U \to (LR)(X)$ coequalizes the pair of projections $U \times_X U \rightrightarrows U$ because in this case, there would be a unique $r \colon X \to (LR)(X)$ such that $rq = g$ and then $\varepsilon_X rq = \varepsilon_X g = q$. Thus, we would get $\varepsilon_X r = id_X$, as q is an epimorphism.

So we are left to prove that g coequalizes $U \times_X U \rightrightarrows U$. Given $c \in \mathscr{E}(C, X)$ for $C \in \mathscr{C}$, denote by $g_c \colon C \to (LR)(X)$ the $c-$th cocomponent of g (recall that $U = \coprod_{C \to X} C$). Now, by definition of g, we have that g coequalizes the pair of arrows

$$ \coprod_{C' \to C \to X} C' \rightrightarrows \coprod_{C \to X} C. $$

This means that $g_c c' = g_{cc'}$ for every sequence of maps $C' \xrightarrow{c'} C \xrightarrow{c} X$ in $\mathscr{E}$ with $C, C' \in \mathscr{C}$. Therefore, for every commutative square (2.18) we have

$$ g_c d = g_{cd} = g_{c'd'} = g_{c'} d'. $$

This implies that the two possible composite arrows

$$ \coprod_S D \xrightarrow{\;p\;} U \times_X U \rightrightarrows U \xrightarrow{\;g\;} (LR)(X) $$

are the same. Since p is an epimorphism, we conclude that g coequalizes $U \times_X U \rightrightarrows U$, as required. This completes the proof. $\qquad \square$

Proposition 2.2.18. *Let $\mathscr{C}$ be a small category. Assume $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$ is a pseudo-site. Thus, $\mathrm{Sh}(\mathscr{C})$ is a replete and reflective subcategory of $\mathrm{PSh}(\mathscr{C})$ and we denote by $a\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C})$ the reflector. Suppose in addition that the Yoneda embedding $\mathbf{y}\colon \mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C})$ factors through $\mathrm{Sh}(\mathscr{C})$ and that its corestriction $\mathscr{C} \longrightarrow \mathrm{Sh}(\mathscr{C})$ is a full and faithful embedding. Then, if $\mathrm{Sh}(\mathscr{C})$ has weak descent, a is left exact.*

Proof. Recall from Definition 1.1.3 that we call an object of $\mathrm{Sh}(\mathscr{C})$ a pseudo-sheaf. By Lemma 1.1.7, pseudo-sheaves are exactly those presheaves X such that the unit morphism $X \Longrightarrow iaX$ is an isomorphism (here, as usual, $i\colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$ is the inclusion functor). Note that, by hypothesis, every representable presheaf is a pseudo-sheaf.

Since $\mathrm{Sh}(\mathscr{C})$ (as any replete and reflective subcategory) is closed in $\mathrm{PSh}(\mathscr{C})$ under limits, we get that the terminal object $1 \in \mathrm{PSh}(\mathscr{C})$ is automatically a pseudo-sheaf and $1 \cong a1$ is the terminal object also in $\mathrm{Sh}(\mathscr{C})$. So, a is left exact if and only if it preserves pullbacks. Thus, we need to show that, for any pullback diagram

$$
\begin{array}{ccc}
X \times_B Y & \twoheadrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & B
\end{array}
$$

in $\mathrm{PSh}(\mathscr{C})$, the canonical arrow

$$a(X \times_B Y) \Longrightarrow aX \times_{aB} aY \text{ is an isomorphism.} \tag{2.19}$$

We will show this through a sequence of specific cases.

(i) *Property (2.19) holds if X, Y and B are pseudo-sheaves.* This is clear because $\mathrm{Sh}(\mathscr{C})$ is closed under limits in $\mathrm{PSh}(\mathscr{C})$, so in this case $X \times_B Y$ is already a pseudo-sheaf.

(ii) *Property (2.19) holds if Y and B are pseudo-sheaves.* Every presheaf is a colimit (in $\mathrm{PSh}(\mathscr{C})$) of pseudo-sheaves because every representable presheaf is a pseudo-sheaf. Write then $X \cong \mathrm{colim}\, U$ for a functor $U\colon \mathscr{J} \longrightarrow \mathrm{Sh}(\mathscr{C}) \subseteq \mathrm{PSh}(\mathscr{C})$ from a small category $\mathscr{J}$. For each $j \in \mathscr{J}$, we have a commutative diagram

$$
\begin{array}{ccccc}
U(j) \times_B Y & \longrightarrow & X \times_B Y & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
U(j) & \longrightarrow & X & \longrightarrow & B
\end{array}
$$

where both squares are pullbacks. Now, Proposition 2.1.14 in $\mathrm{PSh}(\mathscr{C})$ gives $(\mathrm{colim}\, U) \times_B Y \cong X \times_B Y$. Since $U(j)$ is a pseudo-sheaf for all $j \in \mathscr{J}$, by part (i) we get that $aU(j) \times_{aB} aY \cong a(U(j) \times_B Y)$. We can use again Proposition 2.1.14 in $\mathrm{Sh}(\mathscr{C})$ to see that

$$\mathrm{colim}(aU \times_{aB} aY) \cong (\mathrm{colim}\, aU) \times_{aB} aY.$$

Hence, using that a commutes with colimits and with finite limits, we get

$$a(X \times_B Y) \cong a(\mathrm{colim}(U \times_B Y)) \cong \mathrm{colim}(a(U \times_B Y)) \cong \mathrm{colim}(aU \times_{aB} aY) \cong aX \times_{aB} aY,$$

as required.

(iii) *Property (2.19) holds if B is a pseudo-sheaf.* The proof in this case is exactly as the one in (ii) above, except that we can drop the hypothesis that Y is a presheaf by using (ii). Indeed, with the same notation as in (ii), we still get that, for $j \in \mathscr{J}$, $aU(j) \times_{aB} aY \cong a(U(j) \times_B Y)$ because $U(j)$ and B are pseudo-sheaves and we have (ii) at hand.

(iv) *The functor a preserves products.* This follows immediately from (iii) because the terminal presheaf 1 is a pseudo-sheaf.

(v) *The functor a preserves monomorphisms.* Let $X \Longrightarrow Y$ be a monomorphism of presheaves. Write $Y = \mathrm{colim}\, V$ for a functor $V\colon \mathscr{J} \longrightarrow \mathrm{Sh}(\mathscr{C}) \subseteq \mathrm{PSh}(\mathscr{C})$. Set also $U := V(-) \times_Y X$ and let

$\tau\colon U \Longrightarrow V$ be the evident natural transformation. Note that, being a pullback of a monomorphism, τ is mono. For each arrow $i \to j$ in $\mathscr{J}$ we have

$$aV(i) \times_{aV(j)} aU(j) \overset{(iii)}{\cong} a(V(i) \times_{V(j)} U(j)) = a(V(i) \times_{V(j)} (V(j) \times_Y X)) \cong a(V(i) \times_Y X) \cong aU(i),$$

so that $a\tau\colon aU \Longrightarrow aV$ is an equifibered natural transformation between pseudo-sheaves and each arrow $a\tau(j)\colon aU(j) \to aV(j)$ is a monomorphism (as $aU(j) \times_{aV(j)} aU(j) \cong a(U(j) \times_{V(j)} U(j)) \cong aU(j))$). By Proposition 2.1.16 applied to $\mathrm{Sh}(\mathscr{C})$, we get that $\mathrm{colim}\, a\tau\colon \mathrm{colim}\, aU \Longrightarrow \mathrm{colim}\, aV$ is a monomorphism. Under the isomorphisms $\mathrm{colim}\, aU \cong aX$ and $\mathrm{colim}\, aV \cong aY$, this morphism corresponds to $aX \Longrightarrow aY$, which is then mono.

(vi) *Property* (2.19) *holds for general presheaves X, Y and B.* Note that the obvious map $\kappa\colon X \times_B Y \Longrightarrow X \times Y$ is a monomorphism, as it is an equalizer. The composite map

$$a(X \times_B Y) \overset{\sigma}{\Longrightarrow} aX \times_{aB} aY \Longrightarrow aX \times aY$$

is isomorphic to $a\kappa$ by (iv) above and it is then an isomorphism by (v). It follows that σ is a monomorphism, so we only need to show that it is a regular epimorphism as well. Write $B = \mathrm{colim}\, W$ for a functor W from a small category $\mathscr{J}$ into pseudo-sheaves. Consider the two functors from $\mathscr{J}$ into presheaves defined by

$$U(-) := W(-) \times_B X \qquad \text{and} \qquad V(-) := W(-) \times_B Y$$

Thanks to the ubiquitous Proposition 2.1.14, we get

$$X \cong \mathrm{colim}\, U, \quad Y \cong \mathrm{colim}\, V, \quad \mathrm{colim}(U \times_W V) \cong X \times_B Y.$$

Using (iii) we see that both the maps $aU \Longrightarrow aW$ and $aV \Longrightarrow aW$ (induced by the evident arrows $U \Longrightarrow W$ and $V \Longrightarrow W$) are equifibered natural transformations. Therefore, using Lemma 2.2.13, we get that the map

$$a(X \times_B Y) \cong a(\mathrm{colim}(U \times_W V)) \cong \mathrm{colim}(aU \times_{aW} aV) \Longrightarrow aX \times_{aB} aY$$

is a regular epimorphism, as required.

The proof is now complete. $\qquad\square$

Finally, we obtain the

Proof. (of Giraud's Theorem 2.2.4) By Proposition 2.2.17, every category with weak descent is equivalent to a category of pseudo-sheaves satisfying the hypotheses of Proposition 2.2.18. We can then conclude using that same Proposition. $\qquad\square$

We end this chapter by giving the notion of morphisms of Grothendieck topoi.

Definition 2.2.19. Let $\mathscr{E}$ and $\mathscr{E}'$ be topoi. A *geometric morphism* $f\colon \mathscr{E} \to \mathscr{E}'$ from $\mathscr{E}$ to $\mathscr{E}'$ is a functor

$$f^*\colon \mathscr{E}' \longrightarrow \mathscr{E}$$

which preserves small colimits and finite limits. The right adjoint $\mathscr{E} \longrightarrow \mathscr{E}'$ to such a functor (which exists by Corollary 1.1.10) is denoted by f_*.

Let us collect here some examples of geometric morphisms.

Example 2.2.20. 1. Given a site $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}))$, the left adjoint to the inclusion of $\mathrm{Sh}(\mathscr{C})$ into $\mathrm{PSh}(\mathscr{C})$ is a geometric morphism $\mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$.

2. Let $\mathscr{E}$ be a Grothendieck topos and take a morphism $g\colon X \to Y$ in $\mathscr{E}$. By Proposition 2.1.9, the change of base functor

$$f^*\colon \mathscr{E}/Y \longrightarrow \mathscr{E}/X, \quad (B \to Y) \mapsto (B \times_Y X \to X)$$

is a geometric morphism $\mathscr{E}/X \longrightarrow \mathscr{E}/Y$. If $Y = 1$, the terminal object in $\mathscr{E}$, then we denote the right adjoint $f_*\colon \mathscr{E}/X \longrightarrow \mathscr{E}$ to the change of base functor by sect_X. For each $g\colon A \to X$, we call $\mathrm{sect}_X(g) \in \mathscr{E}$ the *object of sections* of g. Note that $\mathrm{sect}_X(X \times Y \to X)$ is canonically isomorphic to the exponential object Y^X in $\mathscr{E}$ (both objects have the same universal property).

3. Given any topos $\mathscr{E}$, there is a geometric morphism $\pi\colon \mathscr{E} \longrightarrow \mathbf{Set}$ where

$$\pi^*\colon \mathbf{Set} \longrightarrow \mathscr{E}, \qquad S \mapsto \coprod_S 1$$

The right adjoint $\pi_*\colon \mathscr{E} \longrightarrow \mathbf{Set}$ is the *global section functor* sending each object $X \in \mathscr{E}$ to the set of its global elements, $\mathscr{E}(1, X)$. Note that, when $\mathscr{E}$ is the category of sheaves on a topological space (T, τ), $\pi_*(F) \cong F(T)$.

4. For any small category $\mathscr{C}$ and any object $C \in \mathscr{C}$, there is a geometric morphism $f_C\colon \mathbf{Set} \longrightarrow \mathrm{PSh}(\mathscr{C})$ given by evaluation at C:
$$f_C^*\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathbf{Set}, \quad X \mapsto X(C).$$

5. Let (X, τ) be a topological space and $x \in X$. There is a geometric morphism $\mathbf{Set} \longrightarrow \mathrm{Sh}(X)$ given by taking stalks at x. Namely, the stalk functor

$$\mathrm{Stalk}_x\colon \mathrm{Sh}(X) \longrightarrow \mathbf{Set}, \quad F \mapsto F_x := \mathrm{colim}_{x \in U \in \tau} F(U)$$

has a right adjoint given by the *skyscraper sheaf* construction. This is the functor

$$\mathrm{Sky}_x\colon \mathbf{Set} \longrightarrow \mathrm{Sh}(X), \quad A \mapsto \mathrm{Sky}_x(A),$$

where $\mathrm{Sky}_x(A)$ is the sheaf on X sending an open subset U of X to A if $x \in U$ and to 1 otherwise (see [McMo] §II.6 Lemma 7).

6. Let $f\colon X \to Y$ be a continuous map of topological spaces (X, τ) and (Y, σ). Then f gives rise to a well-known geometric morphism $\mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(Y)$, still denoted by f, given by the *inverse image functor*, i.e. the functor sending a sheaf F over Y to the sheafification of the presheaf on X given by
$$\tau \ni U \mapsto \mathrm{colim}_{f(U) \subseteq V \in \sigma} F(V).$$

The right adjoint is the *direct image* which sends a sheaf G over X to the sheaf over Y defined, for each open subsets V of Y, by $G(f^{-1}(V))$ (see [McMo] §II.9).

Chapter 3

Grothendieck Topologies.

As we have already mentioned, the notion of Grothendieck topos that we gave in Definition 1.1.3 is quite unorthodox. In this chapter, we establish the definitions of *Grothendieck (pre)topologies* and of *sheaves* with respect to such a Grothendieck topology τ on a small category $\mathscr{C}$ (see Section 3.1). We then show that every category $\mathrm{Sh}(\mathscr{C}, \tau)$ consisting of those sheaves is a Grothendieck topos in our sense and viceversa, thus proving that our Definition 1.1.3 is just another way of describing the usual notion of a Grothendieck topos (see Section 3.2). We will follow the account of [McMo] (Chapter III) and refer to it for some of the proofs in the classical context.

3.1 Grothendieck sites.

Let $\mathscr{C}$ be a small category. Recall that

$$\mathbf{y} \colon \mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C})$$

denotes the Yoneda embedding. Thus, for all $C \in \mathscr{C}$, $\mathbf{y}(C) = \mathscr{C}(-, C)$.

Lemma 3.1.1. *Let C be an object of $\mathscr{C}$. Let $\mathcal{S}_C$ be the set made of all those subsets $F \subseteq \cup_{C' \in \mathscr{C}} \mathscr{C}(C', C)$ which are right ideals with respect to the composition in $\mathscr{C}$, i.e. given a composable pair (f, g) of arrows in $\mathscr{C}$, if f is in F, then fg is in F as well. Then there is a bijection*

$$\mathrm{Sub}_{\mathrm{PSh}(\mathscr{C})}(\mathbf{y}(C)) \cong \mathcal{S}_C,$$

where $\mathrm{Sub}_{\mathrm{PSh}(\mathscr{C})}(\mathbf{y}(C))$ is the set of subobjects of $\mathbf{y}(C)$.

Proof. A subobject of $\mathbf{y}(C)$ can be uniquely represented by a subfunctor $S \rightarrowtail \mathbf{y}(C)$ of $\mathbf{y}(C)$ (so that $S(C') \subseteq \mathscr{C}(C', C)$ and S acts on arrows of $\mathscr{C}$ as $\mathbf{y}(C)$). Given such a subfunctor $S \rightarrowtail \mathbf{y}(C)$, the set $F_S := \cup_{C' \in \mathscr{C}} S(C')$ is in $\mathcal{S}$. Viceversa, if $F \in \mathcal{S}$, we get a subfunctor S_F of $\mathbf{y}(C)$ by setting, for $C' \in \mathscr{C}$,

$$S_F(C') := \{f \in F \colon \mathrm{dom}(f) = C'\}.$$

The assignments $S \mapsto F_S$ and $F \mapsto S_F$ are mutually inverse. $\qquad\square$

Definition 3.1.2. A *sieve* over an object $C \in \mathscr{C}$ is a subfunctor $S \rightarrowtail \mathbf{y}(C)$. The sieve $\mathbf{y}(C) \rightarrowtail \mathbf{y}(C)$ given by the identity of $\mathbf{y}(C)$ is called the *maximal sieve* on C.

By the above Lemma 3.1.1, we can equivalently consider a sieve over C as an element of $\mathcal{S}_C$. We shall often confuse these two descriptions of a sieve and freely use them.

Given a sieve $S \rightarrowtail \mathbf{y}(C)$ over $C \in \mathscr{C}$ and an arrow $f \in \mathscr{C}(C', C)$, we define the *pullback sieve* of S along f as the sieve $f^{-1}S \rightarrowtail \mathbf{y}(C')$ over C' fitting in a pullback square

$$
\begin{array}{ccc}
f^{-1}S & \longrightarrow & \mathbf{y}(C') \\
\downarrow & & \downarrow {\scriptstyle \mathbf{y}(f)} \\
S & \longrightarrow & \mathbf{y}(C)
\end{array}
$$

In other words, if $S = \{f_i \colon X_i \to C\}_{i \in I}$, then

$$f^{-1}S = \{g \colon X_i \to C' \colon\ fg \in S\}.$$

Definition 3.1.3. Let $\mathscr{C}$ be a small category and let $\mathcal{P}(\mathrm{Mor}(\mathscr{C}))$ denote the power set of $\mathrm{Mor}(\mathscr{C})$. A *Grothendieck topology* on $\mathscr{C}$ is a function

$$\tau \colon \mathrm{Ob}(\mathscr{C}) \longrightarrow \mathcal{P}(\mathcal{P}(\mathrm{Mor}(\mathscr{C}))),$$

such that, for each $C \in \mathscr{C}$, $\tau(C)$ is a set of sieves over C and the following properties are satisfied, for all $C \in \mathscr{C}$:

(i) the maximal sieve over C belongs to $\tau(C)$;

(ii) (*stability axiom*) if $S \in \tau(C)$, then $f^{-1}(S) \in \tau(D)$ for each arrow $f \colon D \to C$;

(iii) (*transitivity axiom*) if $T \in \tau(C)$ and S is a sieve over C such that, for each morphism $f \colon D \to C$ in T, $f^{-1}(S) \in \tau(D)$, then $S \in \tau(C)$.

The elements of $\tau(C)$ for $C \in \mathscr{C}$ are called *covering sieves* of C.

Let us look at some examples of Grothendieck topologies.

Example 3.1.4.   1. For every small category $\mathscr{C}$, the function assigning to each $C \in \mathscr{C}$ the set

$$\{\mathbf{y}(C) \to \mathbf{y}(C)\}$$

consisting only of the maximal sieve is clearly a Grothendieck topology on $\mathscr{C}$. It is called the *trivial topology* on $\mathscr{C}$ as it is the minimal topology allowed by the axioms of Definition 3.1.3.

2. The assignment

$$\mathrm{Ob}(C) \ni C \mapsto \tau(C) := \{S \in \mathcal{P}(\mathrm{Mor}(C)) \colon\ S \neq \emptyset \text{ and } S \text{ is a sieve over } C\} \qquad (3.1)$$

is a Grothendieck topology on $\mathscr{C}$ if and only if, given any pair of arrows f, g in C having common codomain, there are morphisms h and k with the same domain and such that $\mathrm{cod}(h) = \mathrm{dom}(f)$, $\mathrm{cod}(k) = \mathrm{dom}(g)$ and $f \circ h = g \circ k$:

$$
\begin{array}{ccc}
\exists T & \xrightarrow{\ \exists h\ } & D \\
{\scriptstyle \exists k} \downarrow & & \downarrow {\scriptstyle \forall f} \\
E & \xrightarrow[\ \forall g\ ]{} & C
\end{array}
\qquad (3.2)
$$

If this is the case, τ is called the *atomic topology* on $\mathscr{C}$ and it is the maximal admissible topology on a small category $\mathscr{C}$. We remark that τ always verifies (i) and (iii) of Definition 3.1.3, even if $\mathscr{C}$ does not satisfy (3.2). In fact, such a condition is equivalent to the stability axiom for τ. Note also that property (3.2) is automatically true in every category $\mathscr{C}$ with pullbacks (even though, of course, it is a weaker property than admitting pullbacks of every cospan).

3. Let (X, σ) be a topological space and consider the small category $\mathrm{Op}(X)$ given by the poset $(\sigma, \subseteq)$. Since, for every $U, V \in \mathrm{Op}(X)$, there is a unique arrow $V \to U$ in $\mathrm{Op}(X)$ precisely when $V \subseteq U$, a sieve over U can be identified with a family S of open subsets of X contained in U which is *downward closed*. This means that $(V' \subseteq V\ \wedge\ V \in S) \implies V' \in S$. We can define a Grothendieck topology $\tau = \tau_\sigma$ on $\mathrm{Op}(X)$ by declaring that, for $U \in \mathrm{Op}(X)$, a sieve S over U is in $\tau(U)$ if and only if U is a union of the open subsets belonging to S. This topology τ is called the *open cover topology* on $\mathrm{Op}(X)$.

Sometimes, one is naturally provided with a notion of coverings for an object $C \in \mathscr{C}$, the collection of which, however, does not form a sieve over C. Still, if these coverings satisfy some compatibility requirements, one can build a Grothendieck topology out of them.

Definition 3.1.5. Let $C \in \mathscr{C}$, where $\mathscr{C}$ is a small category. A *sink* over C is a set $\{C_i \to C\}_{i \in I}$ of arrows in $\mathscr{C}$ with common target given by C.

Definition 3.1.6. Let $\mathscr{C}$ be a small category admitting pullbacks. A *basis (for a topology)* on $\mathscr{C}$ (or a *(Grothendieck) pretopology* on $\mathscr{C}$) is a function

$$\beta \colon \mathrm{Ob}(\mathscr{C}) \longrightarrow \mathcal{P}(\mathcal{P}(\mathrm{Mor}(\mathscr{C})))$$

such that, for all $C \in \mathrm{Ob}(\mathscr{C})$, $\beta(C)$ is a family of sinks over C and the following properties are satisfied:

(i') if $f \colon C' \to C$ is an isomorphism in $\mathscr{C}$, then $\{f\} \in \beta(C)$;

(ii') if $\{f_i \colon C_i \to C\}_{i \in I} \in \beta(C)$, then, for all morphism $g \colon D \to C$, every family $P = \{\pi_{2_i} \colon C_i \times_C D \to D\}_{i \in I}$ of pullbacks of the arrows f_i along g belongs to $\beta(D)$;

(iii') if $\{f_i \colon C_i \to C\}_{i \in I} \in \beta(C)$ and, for all $i \in I$, $\{g_{ij} \colon D_{ij} \to C_i\}_{j \in I_i}$ is a sink over C_i belonging to $\beta(C_i)$, then the family of composites

$$\{f_i \circ g_{ij} \colon D_{ij} \to C\}_{i \in I, \ j \in I_i}$$

is an element of $\beta(C)$.

Note that, due to (i') of Definition 3.1.3, if there are $C \in \mathrm{Ob}(\mathscr{C})$ and an arrow $g \in \mathrm{Mor}(\mathscr{C})$ such that $\mathrm{cod}(g) = C$ and $g \neq id_C$, a topology on $\mathscr{C}$ is not a pretopology on $\mathscr{C}$. Nevertheless, we have the following

Proposition 3.1.7. *Let β be a pretopology on a small category $\mathscr{C}$ admitting pullbacks. Then we get a topology on $\mathscr{C}$ as the function τ which associates to each $C \in \mathrm{Ob}(\mathscr{C})$ the set $\tau(C)$ of sieves on C such that, for all sieves S over C,*

$$S \in \tau(C) \iff \exists R \in \beta(C) \text{ such that } R \subseteq S. \tag{3.3}$$

This Grothendieck topology on $\mathscr{C}$ is called the topology generated by β and is denoted as (β).

Proof. This is a routine check. Indeed, clearly the maximal sieve is in $\tau(C)$ for each $C \in \mathscr{C}$. If $S \in \tau(C)$ and g is an arrow in $\mathscr{C}$ with target C, let $D := \mathrm{dom}(g)$ and take $R \in \beta(C)$ such that $R \subseteq S$. Let also $T' \in \beta(D)$ be the sink over D given by the pullbacks of f along g,

$$\mathrm{dom}(f) \times_C D \to D,$$

where f ranges in R. Then, by definition of pullback, $T' \subseteq g^{-1}(S)$ and then $g^{-1}(S) \in \tau(D)$. This shows that τ verifies the stability axiom of Definition 3.1.3. Similarly, one shows that the transitivity axiom is satisfied as well. $\qquad\square$

There is a kind of converse to the above result, given by the next

Proposition 3.1.8. *Let τ be a Grothendieck topology on a small category $\mathscr{C}$ with pullbacks. Consider the function sending each $C \in \mathrm{Ob}(\mathscr{C})$ to the set $\beta(C)$ of sinks over C such that, for each sink R over C,*

$$R \in \beta(C) \iff (R) \in \tau(C) \tag{3.4}$$

Here (R) is the sieve generated by R, i.e. the smallest sieve over C containing the sink R. Then β is a basis on $\mathscr{C}$ which generates the given topology τ. Furthermore, for each basis γ on $\mathscr{C}$ generating τ and for all $C \in \mathrm{Ob}(\mathscr{C})$, $\gamma(C) \subseteq \beta(C)$. In other words, β is the maximal basis on $\mathscr{C}$ generating β.

Proof. If R is a sink over $C \in \mathscr{C}$, then (R) can be explicitely described as

$$(R) = \{g \in \mathrm{Mor}(\mathscr{C}) : \mathrm{cod}(g) = C \ \wedge \ \exists f \in R \ \exists h \in \mathscr{C}(\mathrm{dom}(g), \, \mathrm{dom}(f)) \ (g = f \circ h)\}. \tag{3.5}$$

In other words, (R) is the set of morphisms in $\mathscr{C}$ which have C as range and factor through some arrow in R. Using this characterization of (R), it is not difficult to show that β defined as in (3.4) is a pretopology on $\mathscr{C}$. Indeed, (i') of Definition 3.1.6 is clearly satisfied as the sieve generated by an isomorphism with

target C is the maximal sieve over C. To show (ii') for β, let $R = \{f_i \colon C_i \to C\}_{i \in I} \in \beta(C)$ and take an arrow $g \colon D \to C$ in $\mathscr{C}$. Consider the family $Q = \{\pi_{2_i} \colon C_i \times_C D \to D\}_{i \in I}$ of pullbacks of each f_i along g; for all $k \in g^{-1}((R))$ we have that

$$\operatorname{cod}(k) = D \quad \text{and} \quad \exists i \in I \ \exists h \colon \operatorname{dom}(k) \to C_i \quad (g \circ k = f_i \circ h).$$

By definition of pullback, for each $k \in g^{-1}((R))$ there is a (unique) morphism

$$t \colon \operatorname{dom}(k) \to C_i \times_C D$$

such that $\pi_{2_i} \circ t = k$. Thus, $g^*((R)) \subseteq (Q)$ and then, since $g^*((R)) \in \tau(C)$ by the stability axiom, $(Q) \in \tau(C)$, so that $Q \in \beta(C)$. This shows that property (ii') of Definition 3.1.6 holds for β. One proves similarly that β verifies (iii') as well, so it is a basis on $\mathscr{C}$. Clearly, β generates τ and is maximal among the bases doing so. $\qquad\square$

Grothendieck (pre)topologies are used to define the classical notion of sites. In our context, we will call them Grothendieck sites to distinguish them from the namesake concept introduced in Definition 1.1.3.

Definition 3.1.9. A *Grothendieck site* is a pair $(\mathscr{C},\ \tau)$, where $\mathscr{C}$ is a small category and τ is a Grothendieck topology on $\mathscr{C}$.

3.2 Grothendieck topoi and sheaves on a Grothendieck site.

In this section we establish the announced equivalence between our Definition 1.1.3 of a Grothendieck topos and the usual one, given in terms of sheaves on a Grothendieck site. Let us point out what we mean by the latter.

Definition 3.2.1. Let $(\mathscr{C},\ \tau)$ be a Grothendieck site. A presheaf $F \in \mathrm{PSh}(\mathscr{C})$ is called a *sheaf (on the site $(\mathscr{C},\ \tau)$)* if, for every $C \in \mathscr{C}$ and every covering sieve $s\colon S \rightarrowtail \mathbf{y}(C)$, the map of sets

$$\mathrm{PSh}(\mathscr{C})(s, F)\colon \mathrm{PSh}(\mathscr{C})(\mathbf{y}(C), F) \to \mathrm{PSh}(\mathscr{C})(S, F)$$

is an isomorphism. In other words, every natural transformation $S \Longrightarrow F$ can be uniquely extended to a natural transformation $\mathbf{y}(C) \Longrightarrow F$. We denote by $\mathrm{Sh}(\mathscr{C},\ \tau)$ the full subcategory of $\mathrm{PSh}(\mathscr{C})$ made of sheaves on the site $(\mathscr{C},\ \tau)$.

A more concrete description of the sheaf property can be given using the notion of matching families.

Let us first introduce the following notation. Suppose given a small category $\mathscr{C}$ and a presheaf $P \in \mathrm{PSh}(\mathscr{C})$. For any $f\colon D \to C$ and any $x \in P(C)$, we set

$$x \cdot f := (P(f))(x).$$

Note that $x \cdot (f \circ g) = (x \cdot f) \cdot g$, whenever the composition $f \circ g$ makes sense.

Definition 3.2.2. Let $(\mathscr{C},\ \tau)$ be a Grothendieck site and let $P \in \mathrm{PSh}(\mathscr{C})$. Given $C \in \mathscr{C}$ and $S \in \tau(C)$, a *matching family (of elements of P) for S* is a function

$$\mathbf{x}\colon S \longrightarrow \bigcup_{f \in S} P(\mathrm{dom}(f)), \qquad (f\colon D \to C) \mapsto x_f \in P(D),$$

such that

$$\forall E \in \mathscr{C} \quad \forall g \in \mathscr{C}(E, D) \quad (x_f \cdot g = x_{f \circ g}). \tag{3.6}$$

An *amalgamation* of a matching family $\mathbf{x} = (x_f)_{f \in S}$ for S is an element $x \in P(C)$ such that

$$\forall f \in S \quad (x \cdot f = x_f). \tag{3.7}$$

We can then reinterpret the sheaf condition as follows.

Proposition 3.2.3. *Let $(\mathscr{C},\ \tau)$ be a Grothendieck site. A presheaf $P \in \mathrm{PSh}(\mathscr{C})$ is a sheaf on $(\mathscr{C},\ \tau)$ if and only if, for all objects $C \in \mathscr{C}$ and for all covering sieves $S \in \tau(C)$, every matching family $\mathbf{x}$ for S has a unique amalgamation.*

Proof. Fix $C \in \mathscr{C}$ and $S \in \tau(C)$ and let $\mathrm{Match}(S, P)$ be the (small) set of matching families for S. Then there is a bijection

$$\mathrm{Match}(S, P) \cong \mathrm{PSh}(S, P), \tag{3.8}$$

where on the right-hand side, S is seen as a subfunctor of P. This bijection associates to each $\mathbf{x} = (x_f)_{f \in S} \in \mathrm{Match}(S, P)$, the natural transformation $\alpha_{\mathbf{x}}\colon S \Longrightarrow P$ such that, for all $D \in \mathscr{C}$, $\alpha_{\mathbf{x}}(D)\colon SD \to PD$ sends $g \in SD$ to x_g. An inverse for this map is the function sending a natural transformation $\alpha\colon S \Longrightarrow P$ to the matching family for S given as $(\alpha_{\mathrm{dom}(g)}(g))_{g \in S}$.

Using the identification (3.6), the Proposition is an immediate consequence of Yoneda's lemma: if P is a presheaf verifying the condition on matching families, take a matching family $\alpha\colon S \Longrightarrow P$ for S and let $x \in PC$ be its unique amalgamation. Then, α factors through the inclusion of S into $\mathbf{y}(C)$, via the unique $\gamma\colon \mathbf{y}(C) \Longrightarrow P$ such that $\gamma_C(id_C) = x$. Thus, P is a sheaf. Viceversa, if P is a sheaf, the unique amalgamation for a matching family $\alpha \in \mathrm{PSh}(C)(S, P)$ is exactly $\gamma_C(id_C)$ (for the unique factorization γ of α through $S \rightarrowtail \mathbf{y}(C)$). $\square$

The characterization of sheaves in terms of matching families resembles the usual glueing axioms for sheaves on a topological space which provide unique amalgamations for compatible families of sections over a covering of the space itself.

One can also show that the sheaf condition can be expressed in terms of a certain map being an equalizer for all $C \in \mathscr{C}$ and all covering sieves S over C. We do this in the special case when $\mathscr{C}$ has pullbacks, so that we can consider bases for a Grothendieck topology. In this way, we will get immediately that, for a topological space (X, σ), $\mathrm{Sh}(X)$ as defined in Example 1.1.4 is the same as $\mathrm{Sh}(\mathrm{Op}(X), \tau_\sigma)$, where $\mathrm{Op}(X)$ is the poset $(\sigma, \subseteq)$ and τ_σ is the open covering topology on $\mathrm{Op}(X)$ (see Example 3.1.4). We refer the interested reader to [McMo] Chapter III (4.3) for the more general case.

Recall from the last section that, when $\mathscr{C}$ is a small category with pullbacks, specifying a Grothendieck topology on $\mathscr{C}$ is equivalent to give a (maximal) basis which generates it (see Propositions 3.1.8 and 3.1.7).

Given a small category $\mathscr{C}$ with pullbacks, let then $(\mathscr{C}, \tau)$ be a Grothendieck site and let β be a basis generating the Grothendieck topology τ. Given an object $C \in \mathscr{C}$ and a sink $R = \{f_i : C_i \to C\}_{i \in I}$, if $f_i, f_j \in R$, let us denote by $\pi^1_{ij} : C_i \times_C C_j \to C_i$ and $\pi^2_{ij} : C_i \times_C C_j \to C_j$ the projections from the pullback.

Proposition 3.2.4. *Let $(\mathscr{C}, \tau)$ be a Grothendieck site where $\mathscr{C}$ has pullbacks and let β be a basis generating the Grothendieck topology τ. A presheaf $P \in \mathrm{PSh}(\mathscr{C})$ is a sheaf on the Grothendieck site $(\mathscr{C}, \tau)$ if and only if, for any sink $\{f_i : C_i \to C\} \in \beta(C)$, the natural map*

$$e \colon P(C) \to \mathrm{eq}\left(\prod_{i \in I} P(C_i) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{(i,j) \in I^2} P(C_i \times_C C_j) \right) \tag{3.9}$$

is an isomorphism. Here, for $x \in P(C)$, $(x_i)_{i \in I} \in \prod_{i \in I} P(C_i)$ and $(i,j) \in I^2$, we have

$$e(x) := (x \cdot f_i)_{i \in I}, \quad [p((x_i)_{i \in I})]_{(i,j)} = x_i \cdot \pi^1_{ij}, \quad [q((x_i)_{i \in I})]_{(i,j)} = x_j \cdot \pi^2_{ij}.$$

Proof. See [McMo], Chapter 3, Proposition 4.1. $\qquad\square$

Remark 3.2.5. The map e in (3.9) above is precisely the map e in (1.1), taking $\mathscr{C}$ to be $\mathrm{Op}(X)$ and τ to be the open covering topology τ_σ for some topological space (X, σ). Note indeed that a basis generating the Grothendieck topology τ_σ is given associating to each open U in X, the family of all sets $\{U_i\}_{i \in I}$ such that each U_i is contained in U and $\cup_{i \in I} U_i = U$. Thus, $\mathrm{Sh}(X) = \mathrm{Sh}(\mathrm{Op}(X), \tau_\sigma)$, as announced.

Given the definition of sheaves on a Grothendieck site, we are now ready to prove that they exhaust (up to equivalences of categories) all Grothendieck topoi.

Theorem 3.2.6. *Let $(\mathscr{C}, \tau)$ be a Grothendieck site. Then the pair $(\mathscr{C}, \mathrm{Sh}(\mathscr{C}, \tau))$ is a site in the sense of Definition 1.1.3 and $\mathrm{Sh}(\mathscr{C}, \tau)$ is a Grothendieck topos. The left adjoint $a \colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C}, \tau)$ to the inclusion of $\mathrm{Sh}(\mathscr{C}, \tau)$ in $\mathrm{PSh}(\mathscr{C})$ is called sheafification functor.*

This result is shown in [McMo] Chapter III.5. We outline here the most important steps in the proof.

Note first that $\mathrm{Sh}(\mathscr{C}, \tau)$ is clearly a replete full subcategory of $\mathscr{C}$, so that one needs to find a left exact reflector $a \colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C}, \tau)$. To build a, one first constructs an intermediate functor

$$(-)^+ \colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C}), \qquad P \mapsto P^+.$$

For each $P \in \mathrm{PSh}(\mathscr{C})$ the presheaf P^+ sends $C \in \mathscr{C}$ to

$$P^+(C) := \mathrm{colim}_{S \in \tau(C)} \mathrm{PSh}(\mathscr{C})(S, P) = \mathrm{colim}_{S \in \tau(C)} \mathrm{Match}(S, P), \tag{3.10}$$

where the colimits is taken over the poset given by the set $\tau(C)$ ordered by reverse inclusion. This is a directed poset because the intersection of two covering sieves is a covering sieve. Therefore, the functor $(-)^+$ commutes with finite limits. For each $P \in \mathrm{PSh}(\mathscr{C})$ and all $C \in \mathscr{C}$, since the identity map $id_{\mathbf{y}(C)} \colon \mathbf{y}(C) \Longrightarrow \mathbf{y}(C)$ is a covering sieve, we get a map

$$\eta_P(C) \colon P(C) \cong \mathrm{PSh}(\mathscr{C})(\mathbf{y}(C), X) \to P^+(C)$$

The functions $(\eta_X(C))_{C \in \mathscr{C}}$ give rise to a natural transformation $\eta_X \colon P \Longrightarrow P^+$ and these natural transformations assemble together into a map of functors $\eta \colon \mathrm{Id}_{\mathrm{PSh}(\mathscr{C})} \Longrightarrow (-)^+$. This plus contruction has the following properties:

- for all $P \in \mathrm{PSh}(\mathscr{C})$, P^+ is a *separated presheaf*, i.e. a presheaf G such that, for all covering sieves $s\colon S \rightarrowtail \mathbf{y}(C)$ over $C \in \mathscr{C}$, the map

$$\mathrm{PSh}(\mathscr{C})(s, G)\colon \mathrm{PSh}(\mathscr{C})(\mathbf{y}(C), G) \to \mathrm{PSh}(\mathscr{C})(S, G)$$

  is a monomorphism;

- if P is a separated presheaf, then P^+ is a sheaf;

- if F is a sheaf on $(\mathscr{C},\ \tau)$, then, for all $P \in \mathrm{PSh}(\mathscr{C})$, the map

$$\mathrm{PSh}(\mathscr{C})(\eta_P, F)\colon \mathrm{PSh}(\mathscr{C})(P^+, F) \to \mathrm{PSh}(\mathscr{C})(P, F)$$

  is an isomorphism.

These properties imply that $(-)^+$, seen as a functor onto the full (and replete) subcategory of $\mathrm{PSh}(\mathscr{C})$ given by separated presheaf, is a left exact reflector for that subcategory. The unit of this adjunction is given by the natural transformation η. Furthermore, one gets the required left adjoint a to the inclusion $\mathrm{Sh}(\mathscr{C},\ \tau) \longrightarrow \mathrm{PSh}(\mathscr{C})$ by setting

$$a := (-)^+ \circ (-)^+\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{Sh}(\mathscr{C},\ \tau),$$

which is left exact, as $(-)^+$ is such.

We can immediately prove the converse to the above result with the following

Theorem 3.2.7 (cf. [Rzk1], Proposition 3.8.)**.** *Let $(\mathscr{C},\ \mathrm{Sh}(\mathscr{C}))$ be a site in the sense of Definition 1.1.3. Then there exists a Grothendieck topology τ on $\mathscr{C}$ such that $\mathrm{Sh}(\mathscr{C}) = \mathrm{Sh}(\mathscr{C},\ \tau)$ (and this is a strict, set-theoretical equality).*

As a consequence, we get the

Corollary 3.2.8. *Every Grothendieck topos is equivalent to the category of sheaves over a Grothendieck site.*

We now prove Theorem 3.2.7.

Proof. (Of Theorem 3.2.7). Let us denote by

$$a\colon \mathrm{PSh}(\mathscr{C}) \rightleftarrows \mathrm{Sh}(\mathscr{C}) \colon i$$

the adjoint pair associated to $\mathrm{Sh}(\mathscr{C}) \subseteq \mathrm{PSh}(\mathscr{C})$, so that $i\colon \mathrm{Sh}(\mathscr{C}) \longrightarrow \mathrm{PSh}(\mathscr{C})$ is the inclusion and a is the reflector.

We define a Grothendieck topology τ on $\mathscr{C}$ by declaring that, for all $C \in \mathscr{C}$, a sieve $s\colon S \rightarrowtail \mathbf{y}(C)$ over C is a covering sieve (i.e. is in $\tau(C)$) precisely when $a(s)$ is an isomorphism. Note that (i) of the Definition 3.1.3 of a Grothendieck topology is trivial, whereas the stability axiom (ii) follows because a is left exact. As for the transitivity axiom, suppose given a sieve $s\colon S \rightarrowtail \mathbf{y}(C)$ over C and assume there is a covering sieve $t\colon T \rightarrowtail \mathbf{y}(C)$ over C such that, for all $C' \in \mathscr{C}$ and all $f \in T(C')$, the pullback sieve $f^{-1}s\colon f^{-1}S \rightarrowtail \mathbf{y}(C')$ is in $\tau(C')$. In other words, for all $(C',\ f) \in \mathrm{El}(T)$ (the category of elements of T), we have that $a(f^{-1}s)\colon a(f^{-1}S) \to a(\mathbf{y}(C))$ is an isomorphism. Note that, since a is left exact, $a(f^{-1}s)$ is the pullback of $a(s)$ along $a(\mathbf{y}(f))$. Since a commutes with colimits and with finite limits, we then have a chain of canonical isomorphisms

$$aT \cong a\left(\mathrm{colim}_{(C',\ f)\in\mathrm{El}(T)} \mathbf{y}(C)\right) \cong \mathrm{colim}_{(C',\ f)\in\mathrm{El}(T)} a(\mathbf{y}(C)) \cong \mathrm{colim}_{(C',\ f)\in\mathrm{El}(T)} a(f^{-1}S) \cong$$

$$\cong \mathrm{colim}_{(C',\ f)\in\mathrm{El}(T)}(a(\mathbf{y}(C')) \times_{a(\mathbf{y}(C))} a(S)) \cong \left(\mathrm{colim}_{(C',\ f)\in\mathrm{El}(T)} a(\mathbf{y}(C'))\right) \times_{a(\mathbf{y}(C))} a(S) \cong$$

$$\cong a(T) \times_{a(\mathbf{y}(C))} a(S),$$

where the penultimate isomorphism is due to Proposition 2.1.14 applied to the category with weak descent $\mathrm{Sh}(\mathscr{C})$. Hence, in the commutative diagram

$$
\begin{array}{ccc}
a(T) \times_{a(\mathbf{y}(C))} a(S) & \longrightarrow & a(T) \\
\downarrow & & \downarrow {\scriptstyle a(t)} \\
a(S) & \xrightarrow[a(s)]{} & a(\mathbf{y}(C))
\end{array}
$$

the upper horizontal arrow is an isomorphism. Since, by the hypothesis $t \in \tau(C)$, $a(t)$ is an isomorphism, we get that also the left vertical arrow is an isomorphism, being a pullback of an iso. It follows that $a(s)$ is invertible as well, i.e. $s \in \tau(C)$.

Therefore, τ is a Grothendieck topology on $\mathscr{C}$. Let us denote by

$$a_\tau \colon \mathrm{PSh}(\mathscr{C}) \rightleftarrows \mathrm{Sh}(\mathscr{C}, \ \tau) \colon\! i_\tau$$

the adjoint pair associated to $\mathrm{Sh}(\mathscr{C}, \ \tau) \subseteq \mathrm{PSh}(\mathscr{C})$. Our goal is to prove that $\mathrm{Sh}(\mathscr{C}) = \mathrm{Sh}(\mathscr{C}, \ \tau)$. By definition of τ, given an object $C \in \mathscr{C}$ and a covering sieve $s \colon S \rightarrowtail \mathbf{y}(C)$ over C, $a(s)$ is an isomorphism. Thus, if $F \in \mathrm{Sh}(\mathscr{C})$, the map

$$\mathrm{PSh}(\mathscr{C})(s, F) \colon \mathrm{PSh}(\mathscr{C})(\mathbf{y}(C), F) \to \mathrm{PSh}(\mathscr{C})(S, F)$$

is an isomorphism by the second part of Lemma 1.1.7. This shows that $\mathrm{Sh}(\mathscr{C}) \subseteq \mathrm{Sh}(\mathscr{C}, \ \tau)$. In order to prove the other inclusion, by Lemma 1.1.7 again, it suffices to show that, for any morphism $\delta \colon X \Longrightarrow Y$ in $\mathrm{PSh}(\mathscr{C})$,

$$\text{if } a(\delta) \text{ is an isomorphism, then } a_\tau(\delta) \text{ is an isomorphism.} \qquad (*)$$

We now claim that it is enough to prove $(*)$ above just when δ is a monomorphism or when δ is a regular epimorphism. Indeed, let $\delta \in \mathrm{PSh}(\mathscr{C})$ be such that $a(\delta)$ is an isomorphism. We can factor $\delta = ip$, where i is a monomorphism and p is a regular epi. Since a is left exact, $a(i)$ is a monomorphism in $\mathrm{Sh}(\mathscr{C})$. But $a(\delta)$ is an isomorphism, hence $a(i)$ is an isomorphism as well (invertibility of $a(\delta)$ implies that $a(i)$ is a regular epi and we have just seen that it is also a monomorphism). Thus, also $a(p)$ is invertible. This shows that we can reduce the proof of $(*)$ to the cases where δ is either mono or regular epi.

(i) *Property $(*)$ holds if δ is a monomorphism.* If $\delta \colon X \Longrightarrow Y$, write Y as a colimit of representable presheaves, say $Y = \mathrm{colim}_{j \in \mathscr{J}} V(j)$, for some small category $\mathscr{J}$ and some functor $V \colon \mathscr{J} \longrightarrow \mathrm{PSh}(\mathscr{C})$ landing into the representable presheaves. Consider the functor

$$U(-) := V(-) \times_Y X \colon \mathscr{J} \longrightarrow \mathrm{PSh}(\mathscr{C})$$

Since δ is mono, the maps $\delta_j \colon U(j) \to V(j)$ given by the pullback projections are sieves over $V(j)$, for each $j \in \mathscr{J}$. Since a commutes with pullbacks, we have moreover that each $a(\delta_j)$ is an isomorphism (as it is a pullback of the isomorphism $a(\delta)$). This means that $\delta_j \in \tau$. Note now that *every* $s \in \tau$ has the property that $a_\tau(s)$ is an isomorphism. Indeed, for all $F \in \mathrm{Sh}(\mathscr{C}, \ \tau)$ the map

$$\mathrm{Sh}(\mathscr{C}, \ \tau)(a_\tau(s), F) \colon \mathrm{Sh}(\mathscr{C}, \ \tau)(a_\tau(\mathbf{y}(C)), F) \to \mathrm{Sh}(\mathscr{C}, \ \tau)(a_\tau(S), F)$$

fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Sh}(\mathscr{C}, \ \tau)(a_\tau(\mathbf{y}(C)), F) & \xrightarrow{\ \cong\ } & \mathrm{PSh}(\mathscr{C})(\mathbf{y}(C), F) \\
{\scriptstyle \mathrm{Sh}(\mathscr{C}, \ \tau)(a_\tau(s), F)} \downarrow & & \downarrow {\scriptstyle \mathrm{PSh}(\mathscr{C})(s, F)} \\
\mathrm{Sh}(\mathscr{C}, \ \tau)(a_\tau(S), F) & \xrightarrow[\ \cong\ ]{} & \mathrm{PSh}(\mathscr{C})(S, F)
\end{array}
$$

where the horizontal arrows are the isomorphisms coming from the adjunction $(a_\tau, \ i_\tau)$, while the right vertical arrow is an isomorphism because F is a sheaf on the site $(\mathscr{C}, \ \tau)$. In particular, each $a_\tau(\delta_j)$ is an isomorphism. Since a_τ preserves colimits and, by weak descent, $X \cong \mathrm{colim}_{j \in \mathscr{J}} U(j)$ (see Proposition 2.1.14), we get that $a_\tau(\delta)$ is an isomorphism.

(ii) *Property $(*)$ holds if δ is a regular epimorphism.* Since a_τ preserves colimits, $a_\tau(\delta)$ is a regular epimorphism. Let us then show that $a(\delta)$ is a monomorphism as well. Let $\theta \colon X \to X \times_Y X$ be the map induced by the identity of X on the pullback of δ along itself. This map is a monomorphism, by its very definition (composing it with one of the projections from the pullback gives the identity on X, which is a monomorphism). Since a preserves pullbacks and $a(\delta)$ is an isomorphism, we must have that $a(\theta)$ is iso. By the first case above, it follows that $a_\tau(\theta)$ is an isomorphism as well and thus $a_\tau(\delta)$ is a monomorphism (as a_τ is left exact).

$\square$

Remark 3.2.9. Having proved Theorem 3.2.7 and its Corollary, we can then say that categories of sheaves on a Grothendieck site (i.e. the categories which, up to equivalences, are usually called Grothendieck topoi) are equivalently and precisely

1. the left exact, replete and reflective subcategories of presheaves categories (where left exact means that the reflector is left exact);

2. the categories $\mathrm{PSh}(\mathscr{C})_S$ of $S-$local objects for a small set S of arrows in $\mathrm{PSh}(\mathscr{C})$ which are left exact (see Section 1.2).

We have thus described the usual notion of a Grothendieck topos in purely categorical terms. This characterisation should make the model-theoretical translation of the notion of a topos (what we shall call a *model topos*) transparent.

Chapter 4

Model Categories.

> There is no scorn more profound, or on the whole more justifiable, than that of the men who make for the men who explain. Exposition, criticism, appreciation, is work for second-rate minds.[1]

> Godfrey H. Hardy,
> *A Mathematician's Apology.*

Reaching our goal of giving a model categorical analogue to the notion of Grothendieck topoi and to Giraud's Theorem will require quite a bit of technical machinery from Model Category Theory. In this chapter, we will record all the model categorical notions needed in the following. For the sake of a self-contained work, we shall give all the definitions and the statements of the results that we are going to use. However, most (if not all) proofs of the main theorems will be omitted. Nevertheless, we shall give precise references for those results and invite the interested reader to go through them.

4.1 Getting started.

4.1.1 The homotopy category of a model category.

Definition 4.1.1.   1. A *model category* is a 4-uple

$$(\mathscr{M}, \; \mathcal{W}, \; \mathrm{Fib}(\mathscr{M}), \; \mathrm{Cof}(\mathscr{M})),$$

where $\mathscr{M}$ is a category, $\mathcal{W}^2$, $\mathrm{Fib}(\mathscr{M})$ and $\mathrm{Cof}(\mathscr{M})$ are classes of arrows in $\mathscr{M}$ (called the classes of *weak equivalences*, *fibrations* and *cofibrations* respectively) satisfying the following set of axioms.

(M1) $\mathscr{M}$ is complete and cocomplete.

(M2) *(Two-out-of-three axiom)* Given composable arrows f, g in $\mathscr{M}$, if two of f, g and gf are in $\mathcal{W}$, then so is the third.

(M3) *(Retract axiom)* If f and g are morphisms in $\mathscr{M}$ such that f is a retract of g (in the category of arrows of $\mathscr{M}$) and g is in $\mathcal{W}$, $\mathrm{Fib}(\mathscr{M})$ or $\mathrm{Cof}(\mathscr{M})$, then so is f.

(M4) *(Lifting axiom)* Suppose given a solid commutative diagram in $\mathscr{M}$

$$
\begin{array}{ccc}
A & \xrightarrow{\;f\;} & X \\
{\scriptstyle i}\downarrow & {\scriptstyle k}\nearrow & \downarrow{\scriptstyle p} \\
B & \xrightarrow[\;g\;]{} & Y
\end{array}
$$

Then there is a dotted filler $k\colon B \to X$ if i is a cofibration and p is a *trivial fibration* (i.e. an element in $\mathcal{W} \cap \mathrm{Fib}(\mathscr{M})$) or if i is a *trivial cofibration* (i.e. an element in $\mathcal{W} \cap \mathrm{Cof}(\mathscr{M})$) and p is a fibration.

---

[1] Still, we are very glad to be second-rate minds.
[2] For sake of notational coherence, we should have written $\mathcal{W}(\mathscr{M})$ as well, but we prefer ease over coherence in this case.

(M5) *(Factorization Axiom)* Every map g in $\mathscr{M}$ admits two functorial factorizations:

      (a) $g = qi$, where i is a cofibration and q is a trivial fibration;

      (b) $g = pj$, where j is a trivial cofibration and p is a fibration.

If $(\mathscr{M},\ \mathcal{W},\ \mathrm{Fib}(\mathscr{M}),\ \mathrm{Cof}(\mathscr{M}))$ is a model category, we set $\mathrm{TrFib}(\mathscr{M}) := \mathrm{Fib}(\mathscr{M}) \cap \mathcal{W}$ and $\mathrm{TrCof}(\mathscr{M}) := \mathrm{Cof}(\mathscr{M}) \cap \mathcal{W}$.

2. Given a complete and cocomplete category $\mathscr{M}$, a triple $(\mathcal{W},\ \mathrm{Fib}(\mathscr{M}),\ \mathrm{Cof}(\mathscr{M}))$ of subclasses of arrows in $\mathscr{M}$ is called a *model category structure* on $\mathscr{M}$ if it satisfies the axioms (M2)$\cdots$(M5) above. The model category $(\mathscr{M},\ \mathcal{W},\ \mathrm{Fib}(\mathscr{M}),\ \mathrm{Cof}(\mathscr{M}))$ is called the model category *associated* to the model category structure given by $(\mathcal{W},\ \mathrm{Fib}(\mathscr{M}),\ \mathrm{Cof}(\mathscr{M}))$.

Example 4.1.2. Every complete and cocomplete category $\mathscr{M}$ admits three model category structures given by declaring one of the three distinguished classes of maps to be the class of isomorphisms in $\mathscr{M}$ and the other two to consist of all maps in $\mathscr{M}$.

Remark 4.1.3.   (i) We will commonly denote a model category $(\mathscr{M},\ \mathcal{W},\ \mathrm{Fib}(\mathscr{M}),\ \mathrm{Cof}(\mathscr{M}))$ just by $\mathscr{M}$, leaving the classes of maps implicit, at least as long as no risk of confusion arises. Anyway, it is understood that the classes of weak equivalences, fibrations and cofibrations are part of the constituent data of a model category.

(ii) If $g\colon X \to Y$ is an arrow in $\mathscr{M}$, we may write:

      – $g\colon X \xrightarrow{\sim} Y$, if g is a weak equivalence in $\mathscr{M}$;

      – $g\colon X \twoheadrightarrow Y$, if g is a fibration in $\mathscr{M}$;

      – $g\colon X \rightarrowtail Y$, if g is a cofibration in $\mathscr{M}$.

Remark 4.1.4. If $(\mathscr{M},\ \mathcal{W},\ \mathrm{Fib}(\mathscr{M}),\ \mathrm{Cof}(\mathscr{M}))$ is a model category, then $\mathscr{M}^{\mathrm{op}}$ has a model category structure as well, with weak equivalences given by those of $\mathscr{M}$, class of cofibrations given by $\mathrm{Fib}(\mathscr{M})$ and class of fibrations given by $\mathrm{Cof}(\mathscr{M})$. Therefore, we have a *duality principle for model categories*, so that every statement that is true for all model categories implies a dual statement in which cofibrations are replaced by fibrations and fibrations are replaced by cofibrations (and every other categorical notion is dualized).

Example 4.1.5. Let S be a set and, for each $s \in S$, let $\mathscr{M}_s$ be a model category. Then $\prod_{s \in S} \mathscr{M}_s$ admits a model category structure where the weak equivalences, the fibrations and the cofibrations are the arrows which are componentwise such.

Example 4.1.6. Let $\mathscr{M}$ be a model category and let $X \in \mathscr{M}$. Denote by

$$U\colon \mathscr{M}/X \longrightarrow \mathscr{M} \quad \text{and} \quad V\colon X/\mathscr{M} \longrightarrow \mathscr{M}$$

the obvious forgetful functors from the category of objects over X and under X respectively. Then:

- $\mathscr{M}/X$ admits a model category structure where an arrow is a weak equivalence, a fibration or a cofibration if and only if its image under U is a weak equivalence, a fibration or a cofibration in $\mathscr{M}$ respectively;

- $X/\mathscr{M}$ admits a model category structure where an arrow is a weak equivalence, a fibration or a cofibration if and only if its image under V is a weak equivalence, a fibration or a cofibration in $\mathscr{M}$ respectively.

Definition 4.1.7. Let $\mathscr{M}$ be a category.

1. Let $i\colon A \to B$ and $p\colon X \to Y$ be maps in $\mathscr{M}$. If, for every commutative solid diagram of the form

there is a dotted filler $k\colon B \to X$, we say that (i,p) is a *lifting-extension pair*, that i has the *left lifting property (LLP)* with respect to p and that p has the *right lifting property (RLP)* with respect to i.

2. If $\mathcal{S}$ is a class of maps in $\mathcal{M}$, we denote by $\mathrm{RLP}(\mathcal{S})$ (respectively by $\mathrm{LLP}(\mathcal{S})$) the class of maps which have the right (resp. the left) lifting property with respect to each arrow in $\mathcal{S}$.

Proposition 4.1.8. *Let* $(\mathcal{M},\ \mathcal{W},\ \mathrm{Fib}(\mathcal{M}),\ \mathrm{Cof}(\mathcal{M}))$ *be a model category.*

(a) *We have the following equalities of subclasses of arrows in* $\mathcal{M}$:

$$\mathrm{Cof}(\mathcal{M}) = \mathrm{LLP}(\mathrm{TrFib}(\mathcal{M})), \qquad \mathrm{TrCof}(\mathcal{M}) = \mathrm{LLP}(\mathrm{Fib}(\mathcal{M})),$$

$$\mathrm{Fib}(\mathcal{M}) = \mathrm{RLP}(\mathrm{TrCof}(\mathcal{M})), \qquad \mathrm{TrFib}(\mathcal{M}) = \mathrm{RLP}(\mathrm{Cof}(\mathcal{M})).$$

(b) $\mathcal{W}$, $\mathrm{Cof}(\mathcal{M})$ *and* $\mathrm{Fib}(\mathcal{M})$ *contain all isomorphisms in* $\mathcal{M}$ *and are closed under compositions of their elements.*

(c) $\mathrm{TrFib}(\mathcal{M})$ *and* $\mathrm{Fib}(\mathcal{M})$ *are closed under pullbacks and products, whereas* $\mathrm{TrCof}(\mathcal{M})$ *and* $\mathrm{Cof}(\mathcal{M})$ *are closed under pushouts and coproducts.*

Proof. Part (a) is Proposition 7.2.3 in [Hir1], whereas (b) and (c) follow formally by the characterisations of (trivial) fibrations and (trivial) cofibrations as classes of maps having a lifting property. $\square$

Definition 4.1.9. Let $\mathcal{M}$ be a model category and let $*$ and $\emptyset$ be the terminal and the initial object in $\mathcal{M}$ respectively.

1. An object $X \in \mathcal{M}$ is called *fibrant* if the unique morphism $X \to *$ is a fibration in (the model category structure on) $\mathcal{M}$. The full subcategory of $\mathcal{M}$ generated by the fibrant objects in $\mathcal{M}$ is denoted by $\mathcal{M}_f$

2. An object $A \in \mathcal{M}$ is called *cofibrant* if the unique morphism $\emptyset \to A$ is a cofibration in (the model category structure on) $\mathcal{M}$. The full subcategory of $\mathcal{M}$ spanned by the cofibrant objects in $\mathcal{M}$ is denoted by $\mathcal{M}_c$

3. An object $W \in \mathcal{M}$ is called *cofibrant-fibrant* (or *fibrant-cofibrant*) if it is both fibrant and cofibrant. The full subcategory of $\mathcal{M}$ spanned by the cofibrant-fibrant objects in $\mathcal{M}$ is denoted by $\mathcal{M}_{cf}$.

We will also need the following notion that will be used thoroughly in the rest of our work.

Definition 4.1.10. Let $\mathcal{M}$ be a model category.

1. If X and Y are objects in $\mathcal{M}$, we say that X and Y are *weakly equivalent*, if there is a finite zig-zag of weak equivalences connecting them. More precisely, this means that there is a natural number $n \in \mathbb{N} \setminus \{0\}$ and there are objects $W_1, \ldots, W_n$ of $\mathcal{M}$ for which there exists a diagram in $\mathcal{M}$

$$X \xrightarrow{\sim} W_1 \xleftarrow{\sim} W_2 \xleftarrow{\sim} \cdots \xrightarrow{\sim} W_n \xleftarrow{\sim} Y,$$

where each arrow is a weak equivalence in $\mathcal{M}$ and can point either to the left or to the right. We shall use the notation $X \approx Y$, to indicate that X and Y are weakly equivalent.

2. If $\mathscr{C}$ is a category an $F, G \colon \mathscr{C} \longrightarrow \mathcal{M}$ are functors, then we say that F and G are *naturally weakly equivalent* if there is a natural number $n \in \mathbb{N} \setminus \{0\}$ and there are functors $W_1, \ldots, W_n$ from $\mathscr{C}$ to $\mathcal{M}$ such that, for every object $A \in \mathscr{C}$, there is a natural zig-zag of weak equivalences

$$F(A) \xrightarrow{\sim} W_1(A) \xleftarrow{\sim} W_2(A) \xleftarrow{\sim} \cdots \xrightarrow{\sim} W_n(A) \xleftarrow{\sim} G(A)$$

connecting $F(A)$ to $G(A)$. We shall again use the notation $F \approx G$ to mean that F and G are naturally weakly equivalent.

Definition 4.1.11. Let $\mathcal{M}$ be a model category.

1. Given $X \in \mathcal{M}$, a *fibrant approximation* to X is a pair $(RX,\ r_X \colon X \to RX)$ where RX is a fibrant object in $\mathcal{M}$ and r_X is a weak equivalence.

2. Given $X \in \mathcal{M}$, a *cofibrant approximation* to X is a pair $(QX,\ q_X \colon QX \to X)$ where QX is a cofibrant object in $\mathcal{M}$ and q_X is a weak equivalence.

3. A *functorial fibrant approximation* in $\mathscr{M}$ is a pair

$$(R\colon \mathscr{M} \longrightarrow \mathscr{M},\ r\colon \operatorname{Id}_{\mathscr{M}} \Longrightarrow R),$$

such that, for all $X \in \mathscr{M}$, $(RX,\ r_X)$ is a fibrant approximation to X. In particular, we may consider R as a functor taking values in $\mathscr{M}_f$.

4. A *functorial cofibrant approximation* in $\mathscr{M}$ is a pair

$$(Q\colon \mathscr{M} \longrightarrow \mathscr{M},\ q\colon Q \Longrightarrow \operatorname{Id}_{\mathscr{M}})$$

such that, for all $X \in \mathscr{M}$, $(QX,\ q_X)$ is a cofibrant approximation to X. In particular, we may consider Q as a functor taking values in $\mathscr{M}_c$.

Remark 4.1.12. We shall commonly denote a fibrant approximation $(RX,\ r_X)$ to $X \in \mathscr{M}$ simply by RX, leaving the weak equivalence implicit. Such a weak equivalence, however, is always understood to be given. Similar remarks apply to cofibrant approximations to X and to functorial fibrant and cofibrant approximations in $\mathscr{M}$.

Remark 4.1.13. Using the two-out-of-three property for weak equivalences, it is clear that, given an arrow $g\colon X \to Y$ in a model category $\mathscr{M}$ with chosen functorial approximations R and Q, g is a weak equivalence in $\mathscr{M}$ if and only if Rg is a weak equivalence or, equivalently, if and only if Qg is such.

The functorial factorization axiom (M5) of Definition 4.1.1 implies the existence of a functorial fibrant approximation $R_{\mathscr{M}}$ and of a functorial cofibrant approximation $Q_{\mathscr{M}}$ on $\mathscr{M}$. Indeed, one can simply factorize, for each object $X \in \mathscr{M}$, the unique arrow $X \to *$ as a trivial cofibration $r_X\colon X \overset{\sim}{\rightarrowtail} RX$ followed by a fibration $RX \twoheadrightarrow *$ and the unique arrow $\emptyset \to X$ as a cofibration $\emptyset \rightarrowtail QX$ followed by a trivial fibration $q_X\colon QX \overset{\sim}{\twoheadrightarrow} X$.

Definition 4.1.14. Given a model category $\mathscr{M}$, we shall refer to the functorial approximations $R = R_{\mathscr{M}}$ and $Q = Q_{\mathscr{M}}$ given by the functorial factorization in $\mathscr{M}$ as the *canonical (functorial) fibrant approximation* and the *canonical (functorial) cofibrant approximation* in $\mathscr{M}$ respectively.

Remark 4.1.15. The canonical fibrant and cofibrant approximations in $\mathscr{M}$ verify the following property: for all $X \in \mathscr{M}$, r_X is a trivial cofibration in $\mathscr{M}$ (and not just a weak equivalence) and q_X is a trivial fibration in $\mathscr{M}$ (and not just a weak equivalence).

Every two (functorial) fibrant or cofibrant approximations in $\mathscr{M}$ can be connected via a zig-zag of (natural) weak equivalences. Namely, we have the following, fundamental

Proposition 4.1.16. *Let $\mathscr{M}$ be a model category and let R and Q be the canonical fibrant and cofibrant approximations in $\mathscr{M}$ respectively. Fix an object $X \in \mathscr{M}$. Then the following hold.*

(i) *Let $R'X$ and $R''X$ be fibrant approximations to X. Then there is a zig-zag of weak equivalences*

$$R'X \overset{\sim}{\leftarrow} RX \overset{\sim}{\rightarrow} R''X$$

as maps in $X/\mathscr{M}$.

(ii) *Let $Q'X$ and $Q''X$ be cofibrant approximations to X. Then there is a zig-zag of weak equivalences*

$$Q'X \overset{\sim}{\rightarrow} QX \overset{\sim}{\leftarrow} Q''X$$

as maps in $\mathscr{M}/X$.

(iii) *Let $R',\ R''\colon \mathscr{M} \longrightarrow \mathscr{M}$ be functorial fibrant approximations in $\mathscr{M}$. Then R' is naturally weakly equivalent to R''.*

(iv) *Let $Q',\ Q''\colon \mathscr{M} \longrightarrow \mathscr{M}$ be functorial cofibrant approximations in $\mathscr{M}$. Then Q' is naturally weakly equivalent to Q''.*

Proof. All the claims are proven in several steps in [Hir1], §8.1. $\square$

Remark 4.1.17. There is a natural concept of *equivalence* among zig-zags of weak equivalences between two fixed objects X and Y in a model category $\mathscr{M}$ (see Definition 4.1.10). Essentially, one says that two zig-zags of weak equivalences between X and Y are equivalent if one can be obtained from the other by finite iterations of the following operations:

(i) composing consecutive arrows;

(ii) removing adjacent equal maps which point in different directions.

If any two zig-zags of weak equivalences between X and Y are equivalent, then one says that there is an *essentially unique* zig-zag of weak equivalences between X and Y. It can be proven that the zig-zags of weak equivalences of Proposition 4.1.16 are indeed essentially unique (see Chapter 14 of [Hir1] for details).

We now present the two archetipical examples of model categories.

Example 4.1.18. We let **Top** denote a category of topological spaces and continuous maps among them satisfying the following properties:

- **Top** is complete and cocomplete;

- **Top** contains all the CW complexes;

- for all $Y \in \textbf{Top}$, there is a functor $Y^{(-)} \colon \textbf{Top} \longrightarrow \textbf{Top}$ such that, for all $X, Z \in \textbf{Top}$, the underlying set of Y^Z is the set of all continuous maps from Z to Y and there is a natural isomorphism of sets

$$\textbf{Top}(X \times Z, Y) \cong \textbf{Top}(X, Y^Z).$$

For example, it is well-known that the category of compactly generated Hausdorff spaces satisfy these properties (see [Ste]).[3]

There is a model category structure on **Top**, called the *Quillen model structure* on **Top**, given as follows:

(i) the weak equivalences are the *weak homotopy equivalences*, i.e. those maps $f \colon X \to Y$ of topological spaces which induce isomorphisms on all homotopy groups (for every choice of the basepoint) and on the set of path components;

(ii) the fibrations are the *Serre fibrations*, i.e. those maps $p \colon E \to X$ of spaces which have the right lifting property with respect to the inclusions $I^n \hookrightarrow I^{n+1}$, for all $n \in \mathbb{N}$ and with I being the unit interval $[0, 1]$ ($I^0 := \{0\}$);

(iii) the cofibrations are the maps with the left lifting property with respect to all Serre fibrations which are also weak homotopy equivalences. Equivalently, the cofibrations are the retracts of relative cell complexes (see Section 4.2.1).

For this model structure on **Top**, every space is a fibrant object and cofibrant objects are retracts of cell complexes. In particular, CW complexes are cofibrant objects.

Example 4.1.19. Let **sSet** be the category of simplicial sets, that is $\textbf{sSet} := \mathrm{PSh}(\Delta)$, where Δ is the simplex category whose objects are the ordinals $[n] = n + 1$, for $n \in \mathbb{N}$. There is a model category structure on **sSet**, called again the *(Kan-)Quillen model structure* on **sSet**, given as follows:

(i) the weak equivalences are the morphisms $f \colon X \to Y$ of simplicial sets such that their *geometric realization* $|f| \colon |X| \to |Y|$ (see Example 4.1.39 below) is a weak homotopy equivalence;

(ii) the fibrations are the *Kan fibrations*, i.e. those maps $p \colon E \to B$ of simplicial sets which have the right lifting property with respect to every inclusion $\Lambda^k[n] \hookrightarrow \Delta[n]$, for all $n \in \mathbb{N}$ and all $0 \le k \le n$. (Recall that $\Delta[n]$ is the representable presheaf $\Delta(-, [n])$ for $[n] \in \Delta$ and $\Lambda^k[n]$ is the $(n, k)-$th *horn*, i.e. the subsimplicial set of $\Delta[n]$ given by the union of all faces except the $k-$th one);

(iii) the cofibrations are the monomorphisms.

For this model category structure, every simplicial set is a cofibrant object and the fibrant objects are exactly the *Kan complexes*.

---

[3] The category of all topological spaces fails to satisfy the third property listed above. However, such a property is only needed if we want to consider **Top** as a *simplicial model category* (see Section 4.2.3), whereas it is irrelevant if we just need to treat **Top** as a plain *model category*.

Definition 4.1.20. Let $\mathscr{M}$ be a model category with class of weak equivalences given by $\mathcal{W}$. The *(Quillen) homotopy category* of $\mathscr{M}$ is the localization $\mathscr{M}[\mathcal{W}^{-1}]$ of $\mathscr{M}$ at $\mathcal{W}$ (see Definition 1.2.2). We shall denote such a localization by $\mathrm{Ho}(\mathscr{M})$ (or by $\mathrm{Ho}(\mathscr{M}, \mathcal{W})$, if necessary), whereas $\gamma\colon \mathscr{M} \longrightarrow \mathrm{Ho}(\mathscr{M})$ will indicate the localization functor (which is the identity on the objects of $\mathscr{M}$).

Remark 4.1.21. It may be worth pointing out how the localization $\mathscr{M}[\mathcal{W}^{-1}]$ can be constructed in general. The objects of such a localization are the same as those of $\mathscr{M}$, whilst arrows from X to Y are given by equivalence classes of zig-zags of maps in $\mathscr{M}$

$$X \to W_1 \leftarrow W_2 \leftarrow \cdots \to W_n \leftarrow Y,$$

where the arrows pointing to the left are in $\mathcal{W}$. The equivalence relation is the one generated by the identifications already explained in Remark 4.1.17: we can substitute composable arrows with their compositions and delete or add pairs of arrows in $\mathcal{W}$ of the form $\overset{w}{\to} \bullet \overset{w}{\leftarrow}$.

As we remarked in the Definition 1.2.2 of localization, the category $\mathrm{Ho}(\mathscr{M})$ may well be, a priori, a large category. In other words, if one requires a category to be, by definition, locally $\mathcal{U}-$small with respect to a fixed Grothendieck universe $\mathcal{U}$, $\mathrm{Ho}(\mathscr{M})$ might not exist without passing to a larger universe $\mathcal{U}' \ni \mathcal{U}$. This is *not* the case for the homotopy category of a model category. To see this, one needs to introduce the *homotopy relation* among maps in $\mathscr{M}$.

Definition 4.1.22. Let $\mathscr{M}$ be a model category and let $f, g\colon A \to X$ be arrows in $\mathscr{M}$.

1. A *cylinder object* for A is a factorization

$$A \coprod A \rightarrowtail \mathrm{Cyl}(A) \overset{\sim}{\to} A$$

   of the fold map $\nabla\colon A \coprod A \to A$ into a cofibration $i_0 \coprod i_1\colon A \coprod A \to \mathrm{Cyl}(A)$ followed by a weak equivalence $s\colon \mathrm{Cyl}(A) \to A$. We may abuse of language and use the term *cylinder object* for A to indicate just the object $\mathrm{Cyl}(A)$, leaving the cofibration $i_0 \coprod i_1$ and the weak equivalence s implicit.

2. A *path object* for X is a factorization

$$X \overset{\sim}{\to} \mathrm{Path}(X) \twoheadrightarrow X \times X$$

   of the diagonal map $\Delta\colon X \to X \times X$ into a weak equivalence $r\colon X \to \mathrm{Path}(X)$ followed by a fibration $(p_0, p_1)\colon \mathrm{Path}(X) \to X \times X$. We may abuse of language and use the term *path object* for X to indicate just the object $\mathrm{Path}(X)$, leaving the weak equivalence r and the fibration (p_0, p_1) implicit.

3. A *left homotopy* from f to g is a map $H\colon \mathrm{Cyl}(A) \to X$ for some cylinder object $\mathrm{Cyl}(A)$ for A such that $H i_0 = f$ and $H i_1 = g$. We say that f and g are *left homotopic*, written $f \overset{l}{\sim} g$, if there is a left homotopy from f to g.

4. A *right homotopy* from f to g is a map $H\colon A \to \mathrm{Path}(X)$ for some path object $\mathrm{Path}(X)$ for X such that $p_0 H = f$ and $p_1 H = g$. We say that f and g are *right homotopic*, written $f \overset{r}{\sim} g$, if there is a right homotopy from f to g.

5. We say that f and g are *homotopic*, written $f \sim g$, if they are both left and right homotopic.

6. The arrow f is a *homotopy equivalence* if there is a map $h\colon X \to A$ such that $hf \sim 1_A$ and $fh \sim 1_X$.

Remark 4.1.23. Although we adopted the notation $\mathrm{Cyl}(A)$ for a cylinder object for $A \in \mathscr{M}$, we do not mean to suggest that this is a functor of $A \in \mathscr{M}$, or that there is any favourite choice for such a cylinder object. An analogous consideration applies to path objects $\mathrm{Path}(A)$. However, using the functorial factorization axiom for $\mathscr{M}$, we *can* get functorial cylinder and path objects for A, obtained by applying the functorial factorizations to the fold and the diagonal maps $A \coprod A \to A$ and $A \to A \times A$ respectively. These functorial cylinder and path objects have the additional properties that $s\colon \mathrm{Cyl}(A) \to A$ is a trivial fibration (and not just a weak equivalence) and $r\colon X \to \mathrm{Path}(X)$ is a trivial cofibration (and not just a weak equivalence).

Proposition 4.1.24. *Let $\mathscr{M}$ be a model category and let $f, g\colon A \to X$ be arrows in $\mathscr{M}$.*

(i) If $f \overset{l}{\sim} g$ and $h\colon X \to Y$ is any morphism in $\mathcal{M}$, then $hf \overset{l}{\sim} hg$. Dually, if $f \overset{r}{\sim} g$ and $k\colon B \to A$ is any arrow in $\mathcal{M}$, then $fk \overset{r}{\sim} gk$.

(ii) If X is fibrant, $f \overset{l}{\sim} g$ and $k\colon B \to A$ is an arrow in $\mathcal{M}$, then $fk \overset{l}{\sim} gk$. Dually, if A is cofibrant, $f \overset{r}{\sim} g$ and $h\colon X \to Y$ is a morphism in $\mathcal{M}$, then $hf \overset{r}{\sim} hg$.

(iii) If A is a cofibrant object in $\mathcal{M}$, then being left homotopic is an equivalence relation on $\mathcal{M}(A,X)$. Dually, when X is fibrant, being right homotopic is an equivalence relation on $\mathcal{M}(A,X)$.

(iv) If A is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, if $\mathrm{Path}(X)$ is any path object for X, then there is a right homotopy $K\colon A \to \mathrm{Path}(X)$ from f to g. Dually, if X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$ and there is a left homotopy from f to g using any cylinder object for A.

Proof. This is a shortened version of Proposition 1.2.5 in [Hov]. $\qquad\square$

Corollary 4.1.25. *Let $\mathcal{M}$ be a model category.*

(i) *Given a cofibrant object A in $\mathcal{M}$ and a fibrant object X in $\mathcal{M}$, the left homotopy and the right homotopy relation coincide and are equivalence relations on $\mathcal{M}(A,X)$.*

(ii) *The homotopy relation $\sim$ on the morphisms in $\mathcal{M}_{cf}$ is an equivalence relation and is compatible with composition. Therefore, there is a category $\mathcal{M}_{cf}/\sim$ with the same objects as those of $\mathcal{M}_{cf}$ and with Hom-sets given, for $A, X \in \mathcal{M}_{cf}$, by $\mathcal{M}(A,X)/\sim$.*

Finally, we get the announced

Theorem 4.1.26. *Let $\mathcal{M}$ be a model category, $\mathrm{Ho}(\mathcal{M})$ its homotopy category and $\gamma\colon \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M})$ the localization functor. Let also Q and R denote the canonical cofibrant and fibrant replacement functor on $\mathcal{M}$ respectively. Then the following hold.*

1. *For all $X, Y \in \mathcal{M}$, there are natural isomorphisms*

$$\mathcal{M}(QRX, QRY)/\sim \;\cong\; \mathrm{Ho}(\mathcal{M})(\gamma X, \gamma Y) \;\cong\; \mathcal{M}(RQX, RQY)/\sim \,.$$

   In addition, there is a natural isomorphism

$$\mathrm{Ho}(\mathcal{M})(\gamma X, \gamma Y) \;\cong\; \mathcal{M}(QX, RY)/\sim \,.$$

   If X is cofibrant and Y is fibrant, there is a natural isomorphism $\mathrm{Ho}(\mathcal{M})(\gamma X, \gamma Y) \cong \mathcal{M}(X,Y)/\sim$. In particular, $\mathrm{Ho}(\mathcal{M})$ is a locally small category.

2. *The localization functor $\gamma\colon \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M})$ identifies left or right homotopic maps.*

3. *A morphism $f\colon A \to B$ in $\mathcal{M}$ is a weak equivalence in $\mathcal{M}$ if and only if γf is an isomorphism in $\mathrm{Ho}(\mathcal{M})$.*

Proof. See [Hov], Theorem 1.2.10. $\qquad\square$

Remark 4.1.27. From the last part of Theorem 4.1.26 above, it follows in particular that two objects X and Y in $\mathcal{M}$ are weakly equivalent (see Definition 4.1.10) if and only if they are isomorphic as objects in the homotopy category $\mathrm{Ho}(\mathcal{M})$.

4.1.2 Quillen pairs and derived functors.

Before defining morphisms of model categories, we state the following

Proposition 4.1.28 (Ken Brown's Lemma)**.** *Let $\mathcal{M}$ be a model category. Assume that $\mathcal{D}$ is a category equipped with a distinguished subcategory which satisfies the two-out-of-three property and call the arrows in such a subcategory weak equivalences in $\mathcal{D}$.*

(i) *Suppose $F\colon \mathcal{M} \longrightarrow \mathcal{D}$ is a functor that sends trivial cofibrations between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{D}$. Then F takes all weak equivalences between cofibrant objects in $\mathcal{M}$ to weak equivalences in $\mathcal{D}$.*

(ii) Suppose $F\colon \mathscr{M} \longrightarrow \mathscr{D}$ is a functor that sends trivial fibrations between fibrant objects in $\mathscr{M}$ into weak equivalences in $\mathscr{D}$. Then F takes all weak equivalences between fibrant objects in $\mathscr{M}$ to weak equivalences in $\mathscr{D}$.

Proof. We prove part (i) as part (ii) follows by duality. Let $f\colon A \to B$ be a weak equivalence of cofibrant objects. We can factor the map $(f, 1_B)\colon A \coprod B \to B$ in $\mathscr{M}$ into a cofibration $q\colon A \coprod B \rightarrowtail C$ followed by a trivial fibration $p\colon C \overset{\sim}{\twoheadrightarrow} B$. Since both A and B are cofibrant objects, both the coprojection arrows $i_1\colon A \to A \coprod B$ and $i_2\colon B \to A \coprod B$ are cofibrations. Using the two-out-of-three axiom for weak equivalences, qi_1 and qi_2 are trivial cofibrations of cofibrant objects. Thus, $F(qi_1)$ and $F(qi_2)$ are weak equivalences in $\mathscr{D}$. Since $pqi_2 = 1_B$, $F(p)$ is a weak equivalence and hence also $F(f) = F(pqi_1)$ is such. $\qquad\square$

Proposition 4.1.29. *Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{N} \qquad \perp$$

be an adjoint pair (where F is the left adjoint). Then the following are equivalent:

 (i) *F preserves cofibrations and trivial cofibrations;*

 (ii) *G preserves fibrations and trivial fibrations;*

 (iii) *F preserves cofibrations and G preserves fibrations;*

 (iv) *F preserves trivial cofibrations and G preserves trivial fibrations.*

Proof. This follows easily from the characterizations of (trivial) (co)fibrations in terms of lifting properties (see Proposition 4.1.8) and the fact that, given an adjunction $F \dashv G$ as above, if i is a map in $\mathscr{M}$ and p is a map in $\mathscr{N}$, then (Fi, p) is a lifting pair (see Definition 4.1.7) in $\mathscr{N}$ if and only if (i, Gp) is a lifting pair in $\mathscr{M}$. See Proposition 8.5.3 of [Hir1]. $\qquad\square$

We can then give the following

Definition 4.1.30. Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and let

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{N} \qquad \perp$$

be an adjoint pair. We say that (F, G) is a *Quillen pair* from $\mathscr{M}$ to $\mathscr{N}$, that F is a *left Quillen functor* and that G is a *right Quillen functor*, if one of the equivalent conditions of Proposition 4.1.29 is satisfied.

Quillen functors pass to homotopy categories in a sense that is made precise by the following definition.

Definition 4.1.31. Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and let $\mathscr{D}$ be a category. Let also $F\colon \mathscr{M} \longrightarrow \mathscr{D}$ and $G\colon \mathscr{M} \longrightarrow \mathscr{N}$ be functors. Denote by $\gamma_{\mathscr{M}}\colon \mathscr{M} \longrightarrow \mathrm{Ho}(\mathscr{M})$ and by $\gamma_{\mathscr{N}}\colon \mathscr{N} \longrightarrow \mathrm{Ho}(\mathscr{N})$ the localization functors.

1. A *left derived functor* of F is a right (!) Kan extension (see Remark 2.2.16)

$$(\mathrm{L}F\colon \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathscr{D}, \ \varepsilon\colon \mathrm{L}F \circ \gamma_{\mathscr{M}} \Longrightarrow F)$$

   of F along $\gamma_{\mathscr{M}}$.

2. A *right derived functor* of F is a left (!) Kan extension (see Definition 2.2.14)

$$(\mathrm{R}F\colon \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathscr{D}, \ \varepsilon\colon F \Longrightarrow \mathrm{R}F \circ \gamma_{\mathscr{M}})$$

   of F along $\gamma_{\mathscr{M}}$.

3. A *total left derived functor* of G is a left derived functor of $\gamma_{\mathscr{N}} \circ G$. We set

$$\mathbb{L}(G) := \mathrm{L}(\gamma_{\mathscr{N}} \circ G)\colon \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{N}).$$

4. A *total right derived functor* of G is a right derived functor of $\gamma_{\mathscr{N}} \circ G$. We set

$$\mathbb{R}(G) := \mathrm{R}(\gamma_{\mathscr{N}} \circ G)\colon \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{N}).$$

We shall commonly use the terms *total left derived functor* or *total right derived functor* to refer only to the functor $\mathbb{L}(G)$ or to the functor $\mathbb{R}(G)$, thus omitting the associated natural transformations.

Theorem 4.1.32. *Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and suppose that*

is a Quillen pair. Then:

1. *the total left derived functor $\mathbb{L}(F)\colon \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{N})$ of F exists;*

2. *the total right derived functor $\mathbb{R}(G)\colon \mathrm{Ho}(\mathscr{N}) \longrightarrow \mathrm{Ho}(\mathscr{M})$ of G exists.*

Proof. Let $Q\colon \mathscr{M} \longrightarrow \mathscr{M}_c \subseteq \mathscr{M}$ be any fixed functorial cofibrant replacement in $\mathscr{M}$ and consider the functor

$$FQ\colon \mathscr{M} \longrightarrow \mathscr{N}.$$

Since F is a left Quillen functor, Ken Brown's Lemma (see Proposition 4.1.28) implies that FQ sends weak equivalences in $\mathscr{M}$ into weak equivalences in $\mathscr{N}$. Thus, by the universal property of $\mathrm{Ho}(\mathscr{M})$, $\gamma_{\mathscr{N}} \circ FQ$ factors in a unique way through $\gamma_{\mathscr{M}}\colon \mathscr{M} \longrightarrow \mathrm{Ho}(\mathscr{M})$ as $\mathbb{L}(F) \circ \gamma_{\mathscr{M}}$. The natural transformation

$$\mathbb{L}(F) \circ \gamma_{\mathscr{M}} \Longrightarrow \gamma_{\mathscr{N}} \circ F$$

is given in the X−th component (for $X \in \mathscr{M}$) by the equivalence class in $\mathrm{Ho}(\mathscr{N})$ of the morphism $FQX \to FX$. Similarly, the total right derived functor of G is constructed as the composite

$$GR\colon \mathscr{N} \longrightarrow \mathscr{M},$$

where $R\colon \mathscr{N} \longrightarrow \mathscr{N}_f \subseteq \mathscr{N}$ is any fixed functorial fibrant approximation in $\mathscr{N}$ and the natural transformation

$$\gamma_{\mathscr{M}} \circ G \Longrightarrow \mathbb{R}(G) \circ \gamma_{\mathscr{N}}$$

is given using the map $GY \to GRY$, for $Y \in \mathscr{N}$. $\qquad \square$

Remark 4.1.33. 1. In the situation of Theorem 4.1.32, we will sometimes abuse of notation and indicate by $\mathbb{L}(F)$ (or by $\mathbb{R}(G)$) both the total left derived functor (or the total right derived functor) of F (respectively of G) and its lifting to the actual model categories given by the composite FQ (respectively GR). We shall call these liftings *point-set (left or right) derived functors*. The proof of Theorem 4.1.32 above shows that $\gamma_{\mathscr{N}} \circ FQ = \mathbb{L}(F) \circ \gamma_{\mathscr{M}}$ (respectively $\gamma_{\mathscr{M}} \circ GR = \mathbb{R}(G) \circ \gamma_{\mathscr{N}}$).

2. In the proof of Theorem 4.1.32 above, different choices of the functorial cofibrant and fibrant approximations in $\mathscr{M}$ and $\mathscr{N}$ respectively lead to isomorphic total left and right derived functors at the level of the homotopy categories, thanks to Proposition 4.1.16, Remark 4.1.17 and Remark 4.1.27. However, the *liftings of the total derived functors* to the overlying model categories obtained by choosing two different functorial approximations are only *naturally weakly equivalent*. Thus, at the level of model categories, (point-set) derived functors can be considered (defined) only up to essentially unique zig-zags of weak equivalences.

Proposition 4.1.34. *Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and suppose that*

is a Quillen pair. Then there is an adjunction between the homotopy categories

$$\mathrm{Ho}(\mathscr{M}) \underset{\mathbb{R}(G)}{\overset{\mathbb{L}(F)}{\rightleftarrows}} \perp \mathrm{Ho}(\mathscr{N})$$

which we call the derived adjunction of (F, G).

Proof. Using the natural isomorphisms in 4.1.26, it is enough to find, for $X \in \mathscr{M}$ and $Y \in \mathscr{N}$, natural bijections

$$\mathscr{N}(FQX, RY)/\sim \,\cong\, \mathscr{M}(QX, GRY)/\sim, \tag{4.1}$$

where $\sim$ is the homotopy relation and we have denoted by Q and R the canonical functorial cofibrant and fibrant replacements both in $\mathscr{M}$ and in $\mathscr{N}$. If φ is the adjunction isomorphism for the Quillen pair (F, G), then $\varphi_{QX, RY}$ respects the homotopy relation, hence it gives the desired isomorphism in 4.1 (see [Hov], Lemma 1.3.10 for details). $\qquad\square$

Remark 4.1.35. Let us consider the derived adjunction $(\mathbb{L}(F), \mathbb{R}(G))$ as above. For $X \in \mathscr{M}$ and $Y \in \mathscr{N}$, let $q_X \colon QX \overset{\sim}{\to} X$ and $r_Y \colon Y \overset{\sim}{\to} RY$ be the weak equivalences given by the functorial factorizations of $\emptyset \to X$ and of $Y \to *$ in $\mathscr{M}$ and in $\mathscr{N}$ respectively. Finally, let η and ε be the unit and the counit of the Quillen pair (F, G). Then:

- the unit of the derived adjunction has $X-$th component given by the composite

$$X \xrightarrow{q_X^{-1}} QX \xrightarrow{G(r_{FQX}) \circ \eta_{QX}} GRFQX \tag{4.2}$$

  in $\mathrm{Ho}(\mathscr{M})$;

- the counit of the derived adjunction has $Y-$th component given by the composite

$$FQGRY \xrightarrow{\varepsilon_{RY} \circ F(q_{GRY})} RY \xrightarrow{r_Y^{-1}} Y \tag{4.3}$$

  in $\mathrm{Ho}(\mathscr{N})$.

Taking into account the homotopy theories (categories) they define, the right notion of equivalence between model categories is given by pairs of Quillen functors inducing an equivalence of categories once derived. The property of a derived adjunction to be an equivalence of categories can be checked before passing to the homotopy categories. Indeed, we give the following

Definition 4.1.36. Let $(F, G, \varphi) \colon \mathscr{M} \longrightarrow \mathscr{N}$ be a Quillen pair, where F is the left adjoint and φ is the adjunction isomorphism. We say that (F, G) is a *Quillen equivalence* if, for all cofibrant objects X of $\mathscr{M}$ and for all fibrant objects Y in $\mathscr{N}$, an arrow $f \colon FX \to Y$ is a weak equivalence in $\mathscr{N}$ if and only if the adjoint arrow $\varphi_{X,Y}(f) \colon X \to GY$ is a weak equivalence in $\mathscr{M}$.

As announced, we have the following

Proposition 4.1.37. *The following are equivalent for a Quillen pair $(F, G, \varphi) \colon \mathscr{M} \longrightarrow \mathscr{N}$:*

(i) (F, G, φ) is a Quillen equivalence;

(ii) the composite

$$X \xrightarrow{\eta_X} GFX \xrightarrow{G(r_{FX})} GRFX$$

is a weak equivalence for all cofibrant $X \in \mathscr{M}$ and the composite

$$FQRY \xrightarrow{F(q_{GY})} FQY \xrightarrow{\varepsilon_Y} Y$$

is a weak equivalence for all fibrant $Y \in \mathscr{N}$;

(iii) the derived adjunction of (F, G, φ) is an equivalence of categories $\mathrm{Ho}(\mathscr{M}) \simeq \mathrm{Ho}(\mathscr{N})$.

Proof. This is Proposition 1.3.13 in [Hov]. $\qquad\square$

Definition 4.1.38. We say that model categories $\mathscr{M}$ and $\mathscr{N}$ are *Quillen equivalent* if there exists a zig-zag

$$\mathscr{M} \longrightarrow \mathscr{M}_1 \longleftarrow \mathscr{M}_2 \longrightarrow \cdots \longleftarrow \mathscr{M}_n \longrightarrow \mathscr{N}$$

of Quillen equivalences of model categories, where the displayed arrows represent the left adjoint of the Quillen equivalence and can point either to the left or to the right.

Here is a fundamental example of Quillen equivalence in Algebraic Topology.

Example 4.1.39. There is an adjoint pair

$$\mathbf{sSet} \underset{\mathrm{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \bot \quad \mathbf{Top} \tag{4.4}$$

defined as follows:

(i) the functor $\mathrm{Sing}\colon \mathbf{Top} \longrightarrow \mathbf{sSet}$ is called the *singular complex* functor and sends a space $X \in \mathbf{Top}$ to the simplicial set given by $\mathbf{Top}(\Delta^{(-)}, X)$, where, for $n \in \mathbb{N}$, Δ^n is the standard $n-$simplex in $\mathbb{R}^n$;

(ii) the functor $|\cdot|\colon \mathbf{sSet} \longrightarrow \mathbf{Top}$ is called the *geometric realization* functor and sends a simplicial set K to the space

$$\mathrm{colim}_{(a,[n]) \in \mathrm{El}(K)} \Delta^n,$$

where $\mathrm{El}(K)$ is the category of elements of K. Such a functor can be realized as the left Kan extension (see Definition 2.2.14) of the Yoneda embedding

$$\mathbf{y}\colon \Delta \longrightarrow \mathbf{sSet}$$

along the embedding

$$\Delta \longrightarrow \mathbf{Top}$$

which sends $[n]$ to Δ^n.

If we give $\mathbf{Top}$ and $\mathbf{sSet}$ the Quillen model structures of Examples 4.1.18 and of 4.1.19 respectively, then the adjunction (4.4) is a Quillen equivalence. Furthermore, the following properties hold:

(a) the geometric realization commutes with finite limits;

(b) for each space X, $\mathrm{Sing}(X)$ is a Kan complex;

(c) for each simplicial set K, $|K|$ is a CW-complex.

We end this section by recording the following result that will be needed in Chapter 6 (see Corollary 6.2.4).

Proposition 4.1.40. *Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and suppose that*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \bot \quad \mathscr{N}$$

is a Quillen pair. Fix an object $X \in \mathscr{N}$. Then the following statements hold.

1. *The evident induced adjunction of overcategories*

$$\mathscr{M}/GX \underset{G^*}{\overset{F^*}{\rightleftarrows}} \bot \quad \mathscr{N}/X$$

is a Quillen pair, when $\mathscr{M}/GX$ and $\mathscr{N}/X$ are endowed with the model category structures of Example 4.1.6.

2. If X is fibrant in $\mathcal{N}$ and (F, G) is a Quillen equivalence, then (F^*, G^*) is a Quillen equivalence as well.

Proof. The functors F^* sends the object $A \to GX$ of $\mathcal{M}/GX$ to the adjunct map $FA \to X$ which is an object of $\mathcal{N}/X$, whereas G^* takes $Y \to X$ in $\mathcal{N}/X$ to $GY \to GX$ in $\mathcal{M}/GX$. It is immediate to see that the adjunction isomorphism for $F \dashv G$ also gives natural isomorphisms

$$\mathcal{N}/X(F^*(A \to GX),\ Y \to X) \cong \mathcal{M}/GX(A \to GX,\ G^*(Y \to X))$$

for $(A \to GX) \in \mathcal{M}/GX$ and $(Y \to X) \in \mathcal{N}/X$. Thus, $F^* \dashv G^*$. Now, an arrow

$$k\colon (A \to GX) \to (B \to GX)$$

in $\mathcal{M}/GX$ is an arrow $k\colon A \to B$ in $\mathcal{M}$ making the relevant triangle commute. By definition of the model structure on $\mathcal{M}/GX$, k is a cofibration in $\mathcal{M}/GX$ if and only if it is such in $\mathcal{M}$. But $F^*(k) = F(k)$ and this is a cofibration in $\mathcal{N}$, hence also in $\mathcal{N}/X$, because F is left Quillen. The dual argument shows that G^* sends fibrations to fibrations, so that $(F^*,\ G^*)$ is a Quillen pair by Proposition 4.1.29. This proves part 1.

Assume now that X is fibrant and that (F, G) is a Quillen equivalence. Let $A \to GX$, $Y \to X$ be a cofibrant object and a fibrant object in $\mathcal{M}/GX$ and in $\mathcal{N}/X$ respectively. This means that the object A is cofibrant in $\mathcal{M}$ and $Y \to X$ is a fibration in $\mathcal{N}$ (see Example 4.1.6). Since X is fibrant by hypothesis, we then get that also Y is such. Consider then a map

$$h\colon F^*(A \to GX) \to (Y \to X)$$

in $\mathcal{N}/X$ which, as before, is a map $h\colon FA \to Y$ in $\mathcal{N}$ rendering the evident triangle commutative. Such a morphism is a weak equivalence in $\mathcal{N}/X$ if and only if it is such in $\mathcal{N}$. But A is cofibrant in $\mathcal{M}$ and Y is fibrant in $\mathcal{N}$, so that $h\colon FA \to X$ is a weak equivalence in $\mathcal{N}$ if an only if its adjunct map $A \to GY = G^*(h)$ is a weak equivalence in $\mathcal{M}$ because (F, G) is a Quillen equivalence. We conclude that (F^*, G^*) is a Quillen equivalence. $\qquad\square$

4.2 Remarkable classes of model categories.

In this section, we discuss some examples of nice model categories which we are going to deal with in the next chapters. In particular, we will see how it may be possible to get a model category structure on a complete and cocomplete category $\mathscr{M}$ by generating cofibrations out of a *set* of maps as retracts of transfinite compositions of pushouts of maps in that sets.

4.2.1 Cofibrantly generated model categories.

In this subsection we will give a recipe to prove that a category admits a model structure, minimizing the conditions that one needs to check. Our rather technical journey to reach this goal starts with the following

Definition 4.2.1. Let γ be a cardinal and let $\mathscr{C}$ be a category.

1. An ordinal α is $\gamma-$*filtered* if it is $\gamma-$filtered as a category (see Definition 1.2.9).

2. A $\gamma-$*sequence* in $\mathscr{C}$ is a functor $F\colon \gamma \longrightarrow \mathscr{C}$ such that, for all limit ordinals $\alpha < \gamma$, the induced arrow
$$\mathrm{colim}_{\beta<\alpha} F(\beta) \to F(\alpha)$$
is an isomorphism in $\mathscr{C}$. The map
$$F(0) \to \mathrm{colim}_{\alpha<\gamma} F(\alpha)$$
is called the *(transfinite) composition* of the $\gamma-$sequence F.

3. Suppose that $\mathcal{D}$ is a subclass of arrows in $\mathscr{C}$. Let also $F\colon \gamma \longrightarrow \mathscr{C}$ be a $\gamma-$sequence in $\mathscr{C}$ with the property that, for all $\beta < \gamma$ such that $\beta + 1 < \gamma$, the morphism $F(\beta) \to F(\beta+1)$ is in $\mathcal{D}$. A *transfinite composition* of maps of $\mathcal{D}$ is a composition $F(0) \to \mathrm{colim}_{\alpha<\gamma} F(\alpha)$ for some $\gamma-$sequence F as the one above.

The reader is invited to compare the following Definition with the notion of a $\kappa-$presentable object as given in Definition 1.2.9.

Definition 4.2.2. Suppose given a cocomplete category $\mathscr{C}$ and a subclass $\mathcal{D} \subseteq \mathrm{Mor}(\mathscr{C})$ of morphisms in $\mathscr{C}$. Let also A be an object in $\mathscr{C}$ and κ a cardinal.

1. We say that A is $\kappa-$*small relative to* $\mathcal{D}$ if, for all $\kappa-$filtered ordinals δ and all $\delta-$sequences $F\colon \delta \longrightarrow \mathscr{C}$ such that $F(\beta) \to F(\beta+1) \in \mathcal{D}$ for $\beta + 1 < \delta$, the canonical map of sets
$$\mathrm{colim}_{\beta<\delta} \mathscr{C}(A, F(\beta)) \to \mathscr{C}(A, \mathrm{colim}_{\beta<\delta} F(\beta))$$
is an isomorphism. A is said to be *small relative to* $\mathcal{D}$ if it is $\kappa-$small relative to $\mathcal{D}$ for some cardinal κ. If $\mathcal{D} = \mathrm{Mor}(\mathscr{C})$ and A is small relative to $\mathrm{Mor}(\mathscr{C})$, we simply say that A is *small*.

2. A is called *finite (relative to* $\mathcal{D}$) if it is $\kappa-$small (relative to $\mathcal{D}$) for some finite cardinal κ.

We are going to combine transfinite compositions and smallness of some objects to prove that specific kinds of sets of maps give rise to functorial weak factorization systems in a cocomplete category.

Definition 4.2.3. Let $\mathcal{I}$ be a class of morphisms in a category $\mathscr{C}$.

1. An arrow in $\mathscr{C}$ is $\mathcal{I}-$*injective* if it is in $\mathrm{RLP}(\mathcal{I})$ (see Definition 4.1.7). We set $\mathrm{Inj}(\mathcal{I}) := \mathrm{RLP}(\mathcal{I})$.

2. An arrow in $\mathscr{C}$ is an $\mathcal{I}-$*cofibration* if it is in $\mathrm{LLP}(\mathrm{Inj}(\mathcal{I}))$. We set $\mathrm{Cof}(\mathcal{I}) := \mathrm{LLP}(\mathrm{Inj}(\mathcal{I})) = \mathrm{LLP}(\mathrm{RLP}(\mathcal{I}))$.

Definition 4.2.4. Let $\mathscr{C}$ be a cocomplete category and let $\mathcal{I}$ be a *set* of morphisms in $\mathscr{C}$. A *relative $\mathcal{I}-$cell complex* is a transfinite composition of pushouts of elements in $\mathcal{I}$, i.e. $f\colon A \to B$ is a relative $\mathcal{I}-$cell complex if and only if there are an ordinal δ and a $\delta-$sequence $F\colon \delta \longrightarrow \mathscr{C}$ such that f is the composition of F and, moreover, for every $\beta < \delta$ such that $\beta + 1 < \delta$, there is a pushout square

$$
\begin{array}{ccc}
C(\beta) & \longrightarrow & F(\beta) \\
{\scriptstyle g_\beta} \downarrow & & \downarrow \\
D(\beta) & \longrightarrow & F(\beta+1)
\end{array}
$$

with $g_\beta \in \mathcal{I}$. The collection of relative $\mathcal{I}-$cell complexes will be denoted by $\mathrm{Cell}(\mathcal{I})$. An object A of $\mathscr{C}$ is an $\mathcal{I}-cell\ complex$ if the map $\emptyset \to A$ is in $\mathrm{Cell}(\mathcal{I})$. (Here $\emptyset$ denotes the initial object in $\mathscr{C}$).

Definition 4.2.5. Let $\mathscr{C}$ be a cocomplete category and let $\mathcal{I}$ be a set of arrows in $\mathscr{C}$. We say that $\mathcal{I}$ *permits the small object argument* if the domains of the elements of $\mathcal{I}$ are small relative to $\mathrm{Cell}(\mathcal{I})$.

Finally, we can state the announced factorization result.

Theorem 4.2.6 (The small object argument)**.** *Let $\mathscr{C}$ be a cocomplete category and assume that $\mathcal{I}$ is a set of morphisms in $\mathscr{C}$ which permits the small object argument. Then there is a functorial factorization of every arrow in $\mathscr{C}$ into a relative $\mathcal{I}-$cell complex followed by an $\mathcal{I}-$injective arrow.*

Proof. See [Hov], Theorem 2.1.14. □

Definition 4.2.7. A model category $\mathscr{M}$ is called a *cofibrantly generated model category* if there are *sets* $\mathcal{I}$ and $\mathcal{J}$ of arrows in $\mathscr{M}$ such that:

(i) $\mathcal{I}$ and $\mathcal{J}$ permit the small object argument;

(ii) $\mathrm{Fib}(\mathscr{M}) = \mathrm{Inj}(\mathcal{J})$;

(iii) $\mathrm{TrFib}(\mathscr{M}) = \mathrm{Inj}(\mathcal{I})$.

The sets $\mathcal{I}$ and $\mathcal{J}$ are called the set of *generating cofibrations* and the set of *generating trivial cofibrations* respectively.

Remark 4.2.8. (i) Suppose given a cofibrantly generated model category $\mathscr{M}$ with generating cofibrations and generating trivial cofibrations given by $\mathcal{I}$ and $\mathcal{J}$ respectively. Then we have

$$\mathrm{Cof}(\mathscr{M}) = \mathrm{Cof}(\mathcal{I}) \quad \text{and} \quad \mathrm{TrCof}(\mathscr{M}) = \mathrm{Cof}(\mathcal{J}).$$

(ii) Note that, in the situation of Definition 4.2.7 above, we are *not* requiring that the functorial factorizations existing a priori on the model category $\mathscr{M}$ are the ones given by the small object argument applied to $\mathcal{I}$ and $\mathcal{J}$.

The point now is that, under suitable hypotheses, we can use the small object argument to construct functorial factorizations for a cofibrantly generated model category structure on a bare category $\mathscr{C}$. Namely, we have the following recognition result, due to Daniel Kan.

Theorem 4.2.9. *Let $\mathscr{C}$ be a complete and cocomplete category. Suppose that $\mathcal{W}$ is a subcategory of $\mathscr{C}$ and that $\mathcal{I}$, $\mathcal{J}$ are sets of arrows in $\mathscr{C}$. Then there is a cofibrantly generated model structure on $\mathscr{C}$ having $\mathcal{W}$ as the class of weak equivalences, $\mathcal{I}$ as the set of generating cofibrations and $\mathcal{J}$ as the set of generating trivial cofibrations if and only if the following conditions are satisfied:*

1. *$\mathcal{W}$ has the two-out-of-three property and is closed under retracts (in the arrow category of $\mathscr{C}$);*

2. *$\mathcal{I}$ and $\mathcal{J}$ permit the small object arguments;*

3. *$\mathrm{Cell}(\mathcal{J}) \subseteq \mathcal{W} \cap \mathrm{Cof}(\mathcal{I})$ and $\mathrm{Inj}(\mathcal{I}) \subseteq \mathcal{W} \cap \mathrm{Inj}(\mathcal{J})$;*

4. *at least one of the inclusion in 3. above is an equality.*

Proof. See [Hov], Theorem 2.1.19. We just remark that, when conditions $1. \cdots 4.$ are met for a 4-uple $(\mathscr{C},\ \mathcal{W},\ \mathcal{I},\ \mathcal{J})$ as in the statement of the Theorem, then the cofibrantly generated model structure on $\mathscr{C}$ can be constructed so as to have functorial factorizations given by the small object argument applied to $\mathcal{I}$ and to $\mathcal{J}$. □

There is an improvement of the notion of a cofibrantly generated model category which we will need later on.

Definition 4.2.10. A model category $\mathscr{M}$ is a *combinatorial model category* if it is a cofibrantly generated model category and its underlying category is locally $\lambda-$presentable, for some regular cardinal λ (see Definition 1.2.9).

Example 4.2.11. For any $n \in \mathbb{N}$, let D^n and S^n denote the unit disk in $\mathbb{R}^n$ and the $n-$sphere in $\mathbb{R}^{n+1}$ respectively. Set also $S^{-1} := \emptyset$ and let I be the closed unit interval $[0,1]$ in $\mathbb{R}$. Then the category **Top** of topological spaces endowed with the Quillen model structure (see Example 4.1.18) forms a cofibrantly generated model category where:

- the set $\mathcal{I}$ of generating cofibrations is given by all the boundary inclusion $S^{n-1} \to D^n$, for $n \in \mathbb{N}$;

- the set $\mathcal{J}$ of generating trivial cofibrations is given by the maps

$$D^n \to D^n \times I, \quad x \mapsto (x,0), \quad n \in \mathbb{N}.$$

This is proven in §2.4 of [Hov]. However, **Top** is *not* a combinatorial model category (for any cofibrantly generated model structure on it) because it is not locally presentable (see, for example, [AdRo] §1.B).

Example 4.2.12. The Kan-Quillen model category structure on simplicial sets (see Example 4.1.19) is a cofibrantly generated model category structure such that:

- the set $\mathcal{I}$ of generating cofibrations is given by the canonical inclusions $\partial\Delta[n] \to \Delta[n]$, for $n \in \mathbb{N}$. (Recall that $\partial\Delta[n]$ is the boundary simplicial set of $\Delta[n]$ whose nondegenerate $k-$simplices (for $k \leq n$) corresponds to the non-identity, injective and monotone maps $[k] \to [n]$);

- the set $\mathcal{J}$ of generating trivial cofibrations is given by the canonical inclusion $\Lambda^k[n] \to \Delta[n]$, for $n \in \mathbb{N} \setminus \{0\}$ and $0 \leq k \leq n$ (see Example 4.1.19).

A proof can be found in Chapter 3 of [Hov]. Since **sSet** is a locally finitely presentable category (see Example 1.2.11), the Quillen model structure turns **sSet** into a combinatorial model category.

Example 4.2.13. Let R be a ring and let $R - \mathbf{Mod}$ be the category of (say) left $R-$modules. There is a cofibrantly generated model category structure on $\mathrm{Ch}(R - \mathbf{Mod})$, the category of (unbounded) chain complexes of $R-$modules, called the *standard model structure*, where the weak equivalences are the quasi-isomorphisms of chain complexes (i.e. those chain maps inducing isomorphisms on homology groups) and the fibrations are the chain maps which are epimorphisms. This is Theorem 2.3.11 in [Hov]. Note that the homotopy category of such a model category is the classical *derived category* of R.

4.2.2 Proper model categories.

In every model category, we can prove the following stability result for weak equivalences under pushouts and pullbacks.

Proposition 4.2.14. *Let $\mathscr{M}$ be a model category and suppose that*

$$S := \begin{pmatrix} \begin{array}{ccc} A & \xrightarrow{\ i\ } & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{\ p\ } & D \end{array} \end{pmatrix}$$

is a commutative square in $\mathscr{M}$.

(i) *If S is a pushout square, A and B are cofibrant objects, f is a weak equivalence and i is a cofibration, then g is a weak equivalence. Thus, every pushout of a weak equivalence between cofibrant objects along a cofibration is a weak equivalence.*

(ii) *If S is a pullback square, C and D are fibrant objects, g is a weak equivalence and p is a fibration, then f is a weak equivalence. Thus, every pullback of a weak equivalence between fibrant objects along a fibration is a weak equivalence.*

Proof. See [Rdy], Theorem B. $\qquad\square$

Those model categories where the thesis of Proposition 4.2.14 above holds without the hypotheses on cofibrancy or fibrancy of objects have been given specific names.

Definition 4.2.15. Let $\mathscr{M}$ be a model category.

1. We say that $\mathscr{M}$ is a *left proper* model category if every pushout of a weak equivalence along a cofibration is again a weak equivalence.

2. We say that $\mathscr{M}$ is a *right proper* model category if every pullback of a weak equivalence along a fibration is again a weak equivalence.

3. If $\mathscr{M}$ is both a left proper and a right proper model category, we say that $\mathscr{M}$ is a *proper* model category.

Remark 4.2.16. By Proposition 4.2.14, we get immediately that if in a model category $\mathscr{M}$ all objects are cofibrant (respectively fibrant), then $\mathscr{M}$ is a left proper (respectively right proper) model category. In particular, **sSet** with the Kan-Quillen model structure (see Example 4.1.19) is a left proper model category and **Top** with the Quillen model structure (see Example 4.1.18) is a right proper model category. It can actually be proven (see Theorems 13.1.11 and 13.1.13 of [Hir1]) that both **sSet** and **Top** with the Quillen model structure are *proper* model categories.

The primary relevance of left and right proper model categories relies in the simplification they bring to the theory of *homotopy pushouts* and of *homotopy pullbacks* (see Section 4.5 below).

Definition 4.2.17. Let $\mathscr{M}$ be a right proper model category and let E be a functorial factorization of every map $f \colon X \to Y$ as

$$X \xrightarrow{i_f} E(f) \xrightarrow{p_f} Y,$$

where i_f is a trivial cofibration and p_f is a fibration.

1. The *homotopy pullback* of a cospan $X \xrightarrow{g} Z \xleftarrow{h} Y$ is the pullback of $E(g) \xrightarrow{p_g} Z \xleftarrow{p_h} E(h)$.

2. A commutative square in $\mathscr{M}$

$$
\begin{array}{ccc}
W & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z
\end{array}
$$

   is called a *homotopy Cartesian square* if the canonical map from W to the homotopy pullback of the cospan $X \to Z \leftarrow Y$ is a weak equivalence.

Definition 4.2.18. Let $\mathscr{M}$ be a left proper model category and let E be a functorial factorization of every map $f \colon X \to Y$ as

$$X \xrightarrow{i_f} E(f) \xrightarrow{p_f} Y,$$

where i_f is a cofibration and p_f is a trivial fibration.

1. The *homotopy pushout* of a span $X \xleftarrow{g} Z \xrightarrow{h} Y$ is the pushout of $E(g) \xleftarrow{i_g} Z \xrightarrow{i_h} E(h)$.

2. A commutative square in $\mathscr{M}$

$$
\begin{array}{ccc}
W & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z
\end{array}
$$

   is called a *homotopy cocartesian square* if the canonical map from the homotopy pushout of the span $X \leftarrow W \to Y$ to Z is a weak equivalence.

These notions of homotopy pullbacks and pushouts are the right homotopical corrections of the ordinary pullbacks and pushouts, as they are *homotopy invariant*, in the sense made precise by the following

Proposition 4.2.19. *(i) Let $\mathscr{M}$ be a right proper model category with a functorial factorization E as in Definition 4.2.17 and suppose given a commutative diagram in $\mathscr{M}$*

$$
\begin{array}{ccccc}
Y_1 & \xrightarrow{h_1} & Y_0 & \xleftarrow{h_2} & Y_2 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_2} \\
X_1 & \xrightarrow{k_1} & X_0 & \xleftarrow{k_2} & X_2
\end{array}
$$

where the vertical maps are weak equivalences. Then the induced map of homotopy pullbacks

$$E(h_1) \times_{Y_0} E(h_2) \to E(k_1) \times_{X_0} E(k_2)$$

is a weak equivalence.

(ii) Let $\mathscr{M}$ be a left proper model category with a functorial factorization E as in Definition 4.2.18 and suppose given a commutative diagram in $\mathscr{M}$

$$
\begin{array}{ccccc}
Y_1 & \xleftarrow{\ h_1\ } & Y_0 & \xrightarrow{\ h_2\ } & Y_2 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_2} \\
X_1 & \xleftarrow{\ k_1\ } & X_0 & \xrightarrow{\ k_2\ } & X_2
\end{array}
$$

where the vertical maps are weak equivalences. Then the induced map of homotopy pushouts

$$E(h_1) \amalg_{Y_0} E(h_2) \to E(k_1) \amalg_{X_0} E(k_2)$$

is a weak equivalence.

Proof. See Propositions 13.3.4 and 13.5.3 of [Hir1]. □

As a consequence, we get the following

Proposition 4.2.20. *Let $\mathscr{M}$ be a model category and suppose given a commutative diagram in $\mathscr{M}$*

$$
\begin{array}{cc}
\begin{array}{ccc}
A & \longrightarrow & B \\
\swarrow{\scriptstyle \alpha} & & \swarrow{\scriptstyle \beta} \\
A' \longrightarrow B' & & \\
C & \longrightarrow & D \\
\swarrow{\scriptstyle \gamma} & & \swarrow{\scriptstyle \delta} \\
C' \longrightarrow D' & &
\end{array}
\end{array}
\qquad (4.5)
$$

where α, β, δ and γ are weak equivalences.

(i) *If $\mathscr{M}$ is a right proper model category, then the front square is a homotopy Cartesian square if and only if the back square is a homotopy Cartesian square.*

(ii) *If $\mathscr{M}$ is a left proper model category, then the front square is a homotopy cocartesian square if and only if the back square is a homotopy cocartesian square.*

Proof. We prove the first part, the second being dual. Let P be the homotopy pullback of $C \to D \leftarrow B$ and let P' be the homotopy pullback of $C' \to D' \leftarrow B'$. Then there is a commutative square in $\mathscr{M}$

$$
\begin{array}{ccc}
A & \longrightarrow & P \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \eta} \\
A' & \longrightarrow & P'
\end{array}
$$

where η is induced by β, γ and δ. By Proposition 4.2.19, η is a weak equivalence and α is a weak equivalence by hypothesis. By the two-out-of-three property for weak equivalences, we conclude. □

Finally, we see how several computing processes for the homotopy pullback give naturally weakly equivalent results.

Proposition 4.2.21. *Let*

$$X \xrightarrow{g} Z \xleftarrow{h} Y$$

be a cospan in a right proper model category $\mathscr{M}$. Suppose given factorizations

$$X \xrightarrow{j_g} W_g \xrightarrow{q_g} Z \quad and \quad Y \xrightarrow{j_h} W_h \xrightarrow{q_h} Z$$

of, respectively, g and h, such that j_g, j_h are weak equivalences and q_g, q_h are fibrations. Then the homotopy pullback of the cospan

$$X \xrightarrow{g} Z \xleftarrow{h} Y$$

is naturally weakly equivalent to each of $W_g \times_Z W_h$, $W_g \times_Z Y$ and $X \times_Z W_h$.

Proof. This is Proposition 13.3.7 in [Hir1]. □

Remark 4.2.22. (i) Proposition 4.2.21 implies that, up to natural zig-zags of weak equivalences, the homotopy pullback of a cospan in a right proper model category does not depend upon the choice of the functorial factorization E made in Definition 4.2.17.

(ii) A dual result to Proposition 4.2.21 holds for homotopy pushouts in left proper model categories.

4.2.3 Simplicial model categories.

The category **sSet** of simplicial sets (as every presheaf category) admits a symmetric and closed monoidal structure with respect to the product bifunctor

$$(-) \times (?) \colon \mathbf{sSet} \times \mathbf{sSet} \longrightarrow \mathbf{sSet}.$$

In particular, this means that **sSet** admits an *internal Hom functor*, i.e. there is a bifunctor

$$\mathrm{Map}(-, ?) \colon \mathbf{sSet}^{\mathrm{op}} \times \mathbf{sSet} \longrightarrow \mathbf{sSet}$$

such that, for all K, L, $N \in \mathbf{sSet}$, there are natural isomorphisms

$$\mathbf{sSet}(K \times L, \ N) \cong \mathbf{sSet}(K, \ \mathrm{Map}(L, N)).$$

For each K, $L \in \mathbf{sSet}$, we can explicitly define $\mathrm{Map}(K, L)$ as the simplicial set whose $n-$simplices, for $n \in \mathbb{N}$, are given by

$$\mathrm{Map}(K, L)_n := \mathbf{sSet}(K \times \Delta[n], L) \tag{4.6}$$

Note also that the unit for the product on **sSet** is given by $\Delta[0]$, which is the terminal object in **sSet**.

Definition 4.2.23. A *simplicial category* $\mathbb{C}$ is a simplicially enriched category, i.e. a category enriched over the symmetric and closed monoidal category $(\mathbf{sSet}, \ \times, \ \Delta[0])$.

Given such a simplicial category $\mathbb{C}$ and objects X, Y, Z of $\mathbb{C}$, we shall denote by $\mathrm{Map}_{\mathbb{C}}(X, Y) \in \mathbf{sSet}$ the enriching Hom-object and, when needed, we will denote by

$$c = c_{XYZ} \colon \mathrm{Map}_{\mathbb{C}}(Y, Z) \times \mathrm{Map}_{\mathbb{C}}(X, Y) \to \mathrm{Map}_{\mathbb{C}}(X, Z)$$

the enriching composition map.

We shall not establish here the basic concepts in the theory of enriched categories. We shall instead use them freely: we refer the reader to the bedrock monograph [Kel]. Anyway, we recall that, to any simplicial category $\mathbb{C}$, we can associate an ordinary category $\mathscr{C}$, called the *underlying category* of $\mathbb{C}$, such that:

- $\mathrm{Ob}(\mathscr{C}) = \mathrm{Ob}(\mathbb{C})$;

- for every $X, Y \in \mathrm{Ob}(\mathscr{C})$,

$$\mathscr{C}(X, Y) := \mathrm{Map}_{\mathbb{C}}(X, Y)_0 \cong \mathbf{sSet}(\Delta[0], \ \mathrm{Map}_{\mathbb{C}}(X, Y)).$$

The composition rule in $\mathscr{C}$ is induced by the enriching composition maps in $\mathbb{C}$.

Definition 4.2.24. Let $\mathbb{C}$ be a simplicial category and let $\mathscr{C}$ be its underlying ordinary category. Given objects $X, Y \in \mathbb{C}$, a *map* (or an *arrow* or a *morphism*) from X to Y is a map $f \colon X \to Y$ in $\mathscr{C}$, i.e. an element of $\mathscr{C}(X, Y) = \mathrm{Map}_{\mathbb{C}}(X, Y)_0$. Given maps $f \colon X \to Y$ and $g \colon Y \to Z$ in $\mathbb{C}$, their *composition* is the composite map $g \circ f$ (or, simply, gf) in $\mathscr{C}$.

Remark 4.2.25. We may not bother to distinguish between a simplicial category $\mathbb{C}$ and its underlying category $\mathscr{C}$, unless real risks of ambiguity arise. In particular, we might write sentences like: "Let $\mathscr{C}$ be a simplicial category". However, what we actually mean is that there is a simplicial category $\mathbb{C}$ (which we suppose given) whose underlying category is (isomorphic to) $\mathscr{C}$. In the same way, we shall not usually distinguish between a simplicial functor (a **sSet**$-$enriched functor) and its underlying ordinary functor.

Remark 4.2.26. Let $\mathbb{C}$ be a simplicial category and X an object in $\mathbb{C}$. Let also $g\colon Y \to Z$ be a map in $\mathbb{C}$ and denote by i_g the corresponding map of simplicial sets $\Delta[0] \to \mathrm{Map}_{\mathbb{C}}(Y,Z)$ obtained via the Yoneda isomorphism

$$\mathrm{Map}_{\mathbb{C}}(Y,Z)_0 \cong \mathbf{sSet}(\Delta[0],\ \mathrm{Map}_{\mathbb{C}}(Y,Z)).$$

Then any such map g gives rise to maps of simplicial sets

$$g_*\colon \mathrm{Map}_{\mathbb{C}}(X,Y) \to \mathrm{Map}_{\mathbb{C}}(X,Z) \quad \text{and} \quad g^*\colon \mathrm{Map}_{\mathbb{C}}(Z,X) \to \mathrm{Map}_{\mathbb{C}}(Y,X)$$

given as the compositions

$$\mathrm{Map}_{\mathbb{C}}(X,Y) \xrightarrow{\cong} \Delta[0] \times \mathrm{Map}_{\mathbb{C}}(X,Y) \xrightarrow{i_g \times 1} \mathrm{Map}_{\mathbb{C}}(Y,Z) \times \mathrm{Map}_{\mathbb{C}}(X,Y) \xrightarrow{c_{XYZ}} \mathrm{Map}_{\mathbb{C}}(X,Z)$$

and

$$\mathrm{Map}_{\mathbb{C}}(Z,X) \xrightarrow{\cong} \mathrm{Map}_{\mathbb{C}}(Z,X) \times \Delta[0] \xrightarrow{1 \times i_g} \mathrm{Map}_{\mathbb{C}}(Z,X) \times \mathrm{Map}_{\mathbb{C}}(Y,Z) \xrightarrow{c_{YZX}} \mathrm{Map}_{\mathbb{C}}(Y,X)$$

respectively. In this way, we get a bifunctor

$$\mathrm{Map}_{\mathscr{C}}(-,?)\colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathbf{sSet}, \quad (X,Y) \mapsto \mathrm{Map}_{\mathbb{C}}(X,Y).$$

Actually, it can be shown (see [Kel], §1.6), that such a bifunctor is the underlying ordinary functor of an enriched bifunctor

$$\mathrm{Map}_{\mathbb{C}}\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \longrightarrow \mathbf{sSet}, \quad (X,Y) \mapsto \mathrm{Map}_{\mathbb{C}}(X,Y),$$

where $\mathbf{sSet}$ has the enriched structure explained in Example 4.2.27 below. In the following, if there is no risk of misunderstanding, we may confuse among the two bifunctors and denote both of them as $\mathrm{Map}_{\mathbb{C}}$.

Example 4.2.27. $\mathbf{sSet}$ is a simplicial category. More precisely, there is a simplicial category $\mathbb{SS}$ having simplicial sets as objects and such that, for all simplicial set K, L,

$$\mathrm{Map}_{\mathbb{SS}}(K,L) = \mathrm{Map}(K,L)$$

(see (4.6)), so that the underlying category of $\mathbb{SS}$ is (isomorphic to) $\mathbf{sSet}$.

Example 4.2.28. $\mathbf{Top}$, the category of topological spaces, is a simplicial category (recall our convention in Example 4.1.18). More precisely, there exists a simplicial category $\mathbb{T}$ whose objects are topological spaces and such that, for topological spaces X, Y, we have

$$\mathrm{Map}_{\mathbb{T}}(X,Y) = \mathbf{Top}(X \times \Delta^{(-)}, Y).$$

(See Example 4.1.39). In particular, the underlying category of $\mathbb{T}$ is (isomorphic to) $\mathbf{Top}$.

Before assembling together the notions of simplicial category and that of model category, we need a couple of further concepts.

Definition 4.2.29. Let $\mathbb{C}$ be a simplicial category. We say that $\mathbb{C}$ is *tensored* (over $\mathbf{sSet}$) if, for all $A \in \mathbb{C}$, there is a $\mathbf{sSet}$−adjunction

$$\mathbf{sSet} \underset{\mathrm{Map}_{\mathbb{C}}(A,-)}{\overset{A \otimes (-)}{\rightleftarrows}} \bot \ \mathbb{C}$$

Dually, we say that $\mathbb{C}$ is *cotensored* if, for all $B \in \mathbb{C}$, there is a $\mathbf{sSet}$−adjunction

$$\mathbf{sSet} \underset{\mathrm{Map}_{\mathbb{C}}(-,B)}{\overset{B^{(-)}}{\rightleftarrows}} \bot \ \mathbb{C}^{\mathrm{op}}$$

Remark 4.2.30. Thus, the existence of tensors and cotensors for a simplicial category $\mathbb{C}$ is equivalent to the existence, for every $K \in \mathbf{sSet}$ and for all $A, B \in \mathbb{C}$, of natural $\mathbf{sSet}$−enriched isomorphisms

$$\mathrm{Map}_{\mathbb{C}}(A \otimes K,\ B) \cong \mathrm{Map}(K,\ \mathrm{Map}_{\mathbb{C}}(A,B)) \quad \text{and} \quad \mathrm{Map}_{\mathbb{C}}(A,\ B^K) \cong \mathrm{Map}(K,\ \mathrm{Map}_{\mathbb{C}}(A,B)) \qquad (4.7)$$

respectively. At the level of the underlying category $\mathscr{C}$, this gives natural isomorphisms

$$\mathscr{C}(A \otimes K,\ B) \cong \mathbf{sSet}(K,\ \mathrm{Map}_{\mathbb{C}}(A,B)) \quad \text{and} \quad \mathscr{C}(A,\ B^K) \cong \mathbf{sSet}(K,\ \mathrm{Map}_{\mathbb{C}}(A,B)) \qquad (4.8)$$

Remark 4.2.31. Let $\mathscr{M}$ be a simplicial category (see Remark 4.2.25) which is tensored and cotensored and let $\mathscr{C}$ be an arbitrary small category. Then $\mathscr{M}^{\mathscr{C}}$ is a simplicial, tensored and cotensored category as well. The tensor and the cotensor are defined objectwise: if $F \colon \mathscr{C} \longrightarrow \mathscr{M}$ is a functor and K is a simplicial set, then, for $\alpha \in \mathscr{C}$, $(F \otimes K)(\alpha) := F(\alpha) \otimes K$ and $(F^K)(\alpha) := F(\alpha)^K$, where the tensor and the cotensor on the right sides are the ones existing on $\mathscr{M}$. If $F, G \in \mathscr{M}^{\mathscr{C}}$, then the enriching simplicial object $\mathrm{Map}_{\mathscr{M}^{\mathscr{C}}}(F, G) \in \mathbf{sSet}$ has set of $n-$simplices given by $\mathscr{M}^{\mathscr{C}}(F \otimes \Delta[n], G)$.

Definition 4.2.32. A *simplicial model category* is a model category $\mathscr{M}$ which is also a simplicial category (see Remark 4.2.25) $\mathbb{M}$ satisfying the following axioms:

(SM1) $\mathbb{M}$ is tensored and cotensored over $\mathbf{sSet}$;

(SM2) if $i \colon A \to B$ is a cofibration in $\mathscr{M}$ and $p \colon X \to Y$ is a fibration in $\mathscr{M}$, then the map of simplicial sets

$$\mathrm{Map}_{\mathscr{M}}(B, X) \to \mathrm{Map}_{\mathscr{M}}(A, X) \times_{\mathrm{Map}_{\mathscr{M}}(A,Y)} \mathrm{Map}_{\mathscr{M}}(B, Y) \tag{4.9}$$

is a fibration which is a trivial fibration if either i or p is a weak equivalence.

Example 4.2.33. For each $X, K \in \mathbf{sSet}$, set

$$X \otimes K := X \times K, \qquad X^K := \mathrm{Map}(K, X).$$

These assignments turn $\mathbf{sSet}$ into a simplicial category which is tensored and cotensored. In fact, with the Kan-Quillen model structure of Example 4.1.19, $\mathbf{sSet}$ is a simplicial model category (see [Hov], Theorem 3.1.1 for axiom (SM2)).

Example 4.2.34. Let $X \in \mathbf{Top}$ and $K \in \mathbf{sSet}$. Define

$$X \otimes K := X \times |K|, \qquad X^K := X^{|K|}$$

where $|K|$ is the geometric realization of K (see Example 4.1.18 and Example 4.1.39). These assignments turn $\mathbf{Top}$ into a simplicial category which is tensored and cotensored. In fact, with the Quillen model structure of Example 4.1.18, $\mathbf{Top}$ is a simplicial model category.

Remark 4.2.35. Let $\mathscr{M}$ be a simplicial model category (see Remark 4.2.25). Then, from the axioms (SM1) and (SM2), we get that the following hold.

1. If A is a cofibrant object in $\mathscr{M}$, then there is a Quillen pair

$$\mathbf{sSet} \quad \overset{A \otimes (-)}{\underset{\mathrm{Map}_{\mathscr{M}}(A,-)}{\rightleftarrows}} \quad \mathscr{M}$$

2. If X is a fibrant object in $\mathscr{M}$, then there is a Quillen pair

$$\mathbf{sSet} \quad \overset{X^{(-)}}{\underset{\mathrm{Map}_{\mathscr{M}}(-,X)}{\rightleftarrows}} \quad \mathscr{M}^{\mathrm{op}}$$

3. If A is a cofibrant object and X is a fibrant object in $\mathscr{M}$, then the simplicial set $\mathrm{Map}_{\mathscr{M}}(A, X)$ is a Kan complex.

We end this section by gathering together the properties of the model structures on $\mathbf{sSet}$ and on $\mathbf{Top}$ that we have discovered so far.

Remark 4.2.36. 1. The category $\mathbf{sSet}$ of simplicial sets admits a simplicial, combinatorial and proper model category structure for which every object is cofibrant.

2. The category $\mathbf{Top}$ of topological spaces (recall Example 4.1.18 for our convention on topological spaces) admits a simplicial, cofibrantly generated and proper model category structure for which every object is fibrant.

4.3 Functor Categories.

Given a model category $\mathscr{M}$ and a small category $\mathscr{C}$, the obvious guess for weak equivalences in $\mathscr{M}^{\mathscr{C}}$ (the natural transformations which are objectwise weak equivalences in $\mathscr{M}$) can not be shown, in general, to form the class of weak equivalences for some model category structure on $\mathscr{M}^{\mathscr{C}}$. However, imposing some conditions either on the base model category $\mathscr{M}$ or on the exponent category $\mathscr{C}$, we can indeed prove the existence of a model category structure on the category of functors from $\mathscr{C}$ to $\mathscr{M}$ with weak equivalences given pointwise. When such model structures on functor categories exist, it makes also sense to ask whether the colimit or the limit functor are Quillen functors and in some circumstances it turns out that this is indeed the case. In particular, when $\mathscr{M}$ is a cofibrantly generated model category, the *projective model structure* is available on $\mathscr{M}^{\mathscr{C}}$ (for any small category $\mathscr{C}$) and the colimit functor is a left Quillen functor. In fact, this is a key property of cofibrantly generated model categories, which explain their preeminent role in the theory of model categories. Paraphrasing a famous claim made by Mac Lane[4], we could well say that cofibrantly generated model categories are defined to get the projective model structure and the projective model structure is defined so as to turn the colimit into a Quillen functor.

4.3.1 The projective and the injective model structure.

Definition 4.3.1. Let $\mathscr{M}$ be a model category and let $\mathscr{C}$ be a small category. We say that a natural transformation $\tau\colon F \Longrightarrow G$ between functors $F, G\colon \mathscr{C} \longrightarrow \mathscr{M}$ is

(i) a *natural weak equivalence* (or a *pointwise weak equivalence*) if, for all $i \in \mathscr{C}$, $\tau_i\colon F(i) \longrightarrow G(i)$ is a weak equivalence in $\mathscr{M}$;

(ii) a *pointwise fibration* if, for all $i \in \mathscr{C}$, $\tau_i\colon F(i) \longrightarrow G(i)$ is a fibration in $\mathscr{M}$;

(iii) a *pointwise cofibration* if, for all $i \in \mathscr{C}$, $\tau_i\colon F(i) \longrightarrow G(i)$ is a cofibration in $\mathscr{M}$.

Definition 4.3.2. Let $\mathscr{M}$ be a model category and $\mathscr{C}$ a small category.

1. The *projective model structure* (or the *Bousfield-Kan model structure*) on $\mathscr{M}^{\mathscr{C}}$, if it exists, is the model category structure where weak equivalences and fibrations are the pointwise weak equivalences and the pointwise fibrations respectively.

2. The *injective model structure* (or the *Heller model structure*) on $\mathscr{M}^{\mathscr{C}}$, if it exists, is the model category structure where weak equivalences and cofibrations are the pointwise weak equivalences and the pointwise cofibrations respectively.

Proposition 4.3.3. *Let $\mathscr{M}$ be a model category and let $\mathscr{C}$ be a small category. Denote by*

$$c\colon \mathscr{M} \longrightarrow \mathscr{M}^{\mathscr{C}}$$

the constant functor, taking an object $X \in \mathscr{M}$ to the constant $\mathscr{C}-$diagram at X.

1. Assume that the projective model structure exists on $\mathscr{M}^{\mathscr{C}}$. Then there is a Quillen pair

2. Assume that the injective model structure exists on $\mathscr{M}^{\mathscr{C}}$. Then there is a Quillen pair

---

[4] "As Eilenberg-Mac Lane first observed, "category" has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformations"." ([McL], §I.4)

Proof. The existence of the adjunctions (for any small category $\mathscr{C}$) is equivalent to (co)completeness of $\mathscr{M}$. The fact that c is a right (respectively, a left) Quillen functor for the projective (respectively, for the injective) model structure on $\mathscr{M}^{\mathscr{C}}$ is obvious from the definition of such a model structure. $\qquad\square$

Here are the needed existence theorems for the projective and the injective model structures.

Theorem 4.3.4. *Let $\mathscr{M}$ be a cofibrantly generated model category and let $\mathscr{C}$ be a small category. Then:*

1. *$\mathscr{M}^{\mathscr{C}}$ is a cofibrantly generated model category with respect to the projective model structure;*

2. *if, in addition, $\mathscr{M}$ is a combinatorial model category (see Definition 4.2.10), then so is $\mathscr{M}^{\mathscr{C}}$ with the projective model structure;*

3. *if, in addition, $\mathscr{M}$ is left or right proper (see Definition 4.2.15), then so is $\mathscr{M}^{\mathscr{C}}$ with the projective model structure.*

4. *if, in addition, $\mathscr{M}$ is a simplicial model category (see Remark 4.2.25), then so is $\mathscr{M}^{\mathscr{C}}$ with the projective model structure and with the simplicial structure of Remark 4.2.31.*

Proof. The first part of the result is Theorem 11.6.1 in [Hir1]. The second claim follows because, if $\mathscr{D}$ is a locally presentable category, then, for any small category $\mathscr{C}$, $\mathscr{D}^{\mathscr{C}}$ is locally presentable as well (see Corollary 1.54 in [AdRo]). The third point is Remark A.2.8.4 in [Lur] and, finally, the last part of the Theorem can be found in [Hir1], §11.7. $\qquad\square$

Theorem 4.3.5. *Let $\mathscr{M}$ be a combinatorial model category and let $\mathscr{C}$ be a small category. Then:*

1. *$\mathscr{M}^{\mathscr{C}}$ is a combinatorial model category with respect to the injective model structure;*

2. *if, in addition, $\mathscr{M}$ is left or right proper, then so is $\mathscr{M}^{\mathscr{C}}$ with the injective model structure;*

3. *if, in addition, $\mathscr{M}$ is a simplicial model category, then so is $\mathscr{M}^{\mathscr{C}}$ with the injective model structure and with the simplicial structure of Remark 4.2.31.*

Proof. See Proposition A.2.8.2 and Remark A.2.8.4 in [Lur]. $\qquad\square$

Thus, given a combinatorial model category $\mathscr{M}$, there are at least two model structures that we can put on functor categories $\mathscr{M}^{\mathscr{C}}$. It turns out that they are Quillen equivalent.

Proposition 4.3.6. *Let $\mathscr{M}$ be a combinatorial model category and $\mathscr{C}$ any small category. Then there is a Quillen equivalence*

$$(\mathscr{M}^{\mathscr{C}})_{\boldsymbol{proj}} \underset{\mathrm{Id}}{\overset{\mathrm{Id}}{\rightleftarrows}} \perp \ (\mathscr{M}^{\mathscr{C}})_{\boldsymbol{inj}}$$

Proof. Since the two model structures share the same weak equivalences, one just needs to check that a projective cofibration is a pointwise cofibration and, viceversa, that an injective fibration is a pointwise fibration. The first half is Proposition 11.6.3 in [Hir1], whereas the second follows from the first. For, if $\tau\colon F \Longrightarrow G$ is a fibration in the injective model structure, then it has the right lifting property with respect to all trivial pointwise cofibrations, hence also to all trivial cofibrations for the projective model structure. $\qquad\square$

When the projective or the injective model structure exists (for cofibrantly generated or combinatorial model categories), they allow us to lift Quillen adjunctions to functor categories, as stated by the following

Proposition 4.3.7. *Let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \perp \ \mathscr{N}$$

be a Quillen pair between model categories and let $\mathscr{C}$ be a small category. Assume that:

(i) *either both $\mathscr{M}$ and $\mathscr{N}$ are cofibrantly generated*

(ii) *or both $\mathscr{M}$ and $\mathscr{N}$ are combinatorial model categories.*

Then there are Quillen pairs

$$(\mathscr{M}^{\mathscr{C}})_{\mathbf{proj}} \overset{F \circ -}{\underset{G \circ -}{\rightleftarrows}} \bot \ (\mathscr{N}^{\mathscr{C}})_{\mathbf{proj}} \quad or \quad (\mathscr{M}^{\mathscr{C}})_{\mathbf{inj}} \overset{F \circ -}{\underset{G \circ -}{\rightleftarrows}} \bot \ (\mathscr{N}^{\mathscr{C}})_{\mathbf{inj}}$$

respectively which are Quillen equivalences if (F, G) was a Quillen equivalence to start with.

Proof. From the definitions of the projective and the injective model structures, it is clear that $(F \circ -, G \circ -)$ is a Quillen pair. The part on Quillen equivalences follows because every fibrant or cofibrant object in either the projective or the injective model structure is also such objectwise. □

Remark 4.3.8. Since **sSet** is a simplicial, combinatorial and proper model category (see Remark 4.2.36), for any small category $\mathscr{C}$, $\mathbf{sSet}^{\mathscr{C}}$ is a simplicial, combinatorial and proper model category both with the projective and the injective model structure. Unless differently stated, we shall however always implicitly assume that $\mathbf{sSet}^{\mathscr{C}}$, when seen as a model category, is endowed with the *projective model structure.*

4.3.2 The Reedy model structure.

There is a quite peculiar property of the indexing small category $\mathscr{C}$ that ensures the existence of a model structure on $\mathscr{M}^{\mathscr{C}}$ for *any* model category $\mathscr{M}$. We introduce such a property in the following

Definition 4.3.9. A *Reedy category* is a 4-uple

$$(\mathscr{C}, \ \mathscr{C}_+, \ \mathscr{C}_-, \ \deg),$$

where:

- $\mathscr{C}$ is a category;

- $\mathscr{C}_+$ and $\mathscr{C}_-$ are two subcategories of $\mathscr{C}$ containing all objects of $\mathscr{C}$ (this kind of subcategories are sometimes called *wide subcategories* or *lluf subcategories*[5]);

- $\deg \colon \mathrm{Ob}(\mathscr{C}) \to \omega$ is a function into the first limit ordinal ω and is called the *degree function.*

These data need to satisfy the following axioms:

(R1) for every arrow f_+ in $\mathscr{C}_+$ which is not an identity morphism, $\deg(\mathrm{cod}(f_+)) > \deg(\mathrm{dom}(f_+))$;

(R2) for every arrow f_- in $\mathscr{C}_-$ which is not an identity morphism, $\deg(\mathrm{cod}(f_-)) < \deg(\mathrm{dom}(f_-))$;

(R3) for every arrow $f \in \mathscr{C}$, there exists a unique factorization $f = f_+ f_-$, where $f_- \in \mathscr{C}_-$ and $f_+ \in \mathscr{C}_+$.

Remark 4.3.10. As usual, we shall commonly refer to a Reedy category $(\mathscr{C}, \ \mathscr{C}_+, \ \mathscr{C}_-, \ \deg)$ mentioning only the underlying category $\mathscr{C}$ and assuming, implicitly, that all the other data are given and available. We may also rephrase conditions (R1) and (R2) by saying simply that every arrow in $\mathscr{C}_+$ *raises degree* and every arrow in $\mathscr{C}_-$ *lowers degree.*

Definition 4.3.11. Let $\mathscr{C}$ be a Reedy category.

1. If $\mathscr{C}_+$ is the discrete lluf subcategory of $\mathscr{C}$, we say that $\mathscr{C}$ is an *inverse category.*

2. If $\mathscr{C}_-$ is the discrete lluf subcategory of $\mathscr{C}$, we say that $\mathscr{C}$ is a *direct category.*

Remark 4.3.12. 1. If $\mathscr{C}$ is a Reedy category, then so is $\mathscr{C}^{\mathrm{op}}$ by setting $(\mathscr{C}^{\mathrm{op}})_+ := (\mathscr{C}_-)^{\mathrm{op}}$ and $(\mathscr{C}^{\mathrm{op}})_- := (\mathscr{C}_+)^{\mathrm{op}}$. The degree function is the same as that of $\mathscr{C}$.

2. If $\mathscr{C}$ and $\mathscr{D}$ are Reedy categories, then $\mathscr{C} \times \mathscr{D}$ is a Reedy category as well, with $(\mathscr{C} \times \mathscr{D})_+ = \mathscr{C}_+ \times \mathscr{D}_+$ and $(\mathscr{C} \times \mathscr{D})_- = \mathscr{C}_- \times \mathscr{D}_-$. The degree function is given by $\deg(C, D) = \deg(C) + \deg(D)$, for $C \in \mathscr{C}$ and $D \in \mathscr{D}$.

We will need in particular the following examples of Reedy categories.

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[5] The term "lluf" is "full" spelled backwards. Note that, as a full subcategory is completely determined by its class of objects, so a lluf subcategory is totally understood when its class of arrows is specified. Moreover, the only full lluf subcategory of a category is that category itself.

Example 4.3.13. Let $\mathscr{I}$ be the pushout category, $\mathscr{I} = b \leftarrow a \rightarrow c$ (where we do not display identity morphisms). $\mathscr{I}$ admits a direct category structure so that $\mathscr{I}_+ := \mathscr{I}$ and $\mathscr{I}_-$ is the discrete lluf subcategory of $\mathscr{I}$. In this case, $\deg(a) = 0$ and $\deg(b) = 1 = \deg(c)$.

Example 4.3.14. There is another Reedy structure on $\mathscr{I}$ given by taking $\mathscr{I}_+$ as $a \rightarrow c$ and $\mathscr{I}_-$ as $b \leftarrow a$, so that $\deg(b) = 0$, $\deg(a) = 1$ and $\deg(c) = 2$.

Example 4.3.15. There is a a Reedy structure on the simplex category Δ given by defining Δ_+ as the lluf subcategory of Δ consisting of injective maps and Δ_- as the lluf subcategory consisting of surjective maps. The degree of $[n] \in \Delta$ is $n \in \omega$.

In order to describe the model structure on $\mathscr{M}^{\mathscr{C}}$ for a Reedy category $\mathscr{C}$ and a model category $\mathscr{M}$, we have to introduce some further notations.

Definition 4.3.16. Let $\mathscr{C}$ be a Reedy category and let α be an object of $\mathscr{C}$.

1. The *latching category* of $\mathscr{C}$ at α is the category $\partial(\mathscr{C}_+ \downarrow \alpha)$ defined as the full subcategory of $(\mathscr{C}_+ \downarrow \alpha)$ consisting of all the objects except 1_α.

2. The *matching category* of $\mathscr{C}$ at α is the category $\partial(\alpha \downarrow \mathscr{C}_-)$ defined as the full subcategory of $(\alpha \downarrow \mathscr{C}_-)$ consisting of all the objects except 1_α.

Given a *small* Reedy category $\mathscr{C}$ and an object $\alpha \in \mathscr{C}$, the forgetful functors $U \colon \partial(\mathscr{C}_+ \downarrow \alpha) \longrightarrow \mathscr{C}$ and $V \colon \partial(\alpha \downarrow \mathscr{C}_-) \rightarrow \mathscr{C}$ induce functors

$$U^* \colon \mathscr{M}^{\mathscr{C}} \longrightarrow \mathscr{M}^{\partial(\mathscr{C}_+ \downarrow \alpha)} \quad \text{and} \quad V^* \colon \mathscr{M}^{\mathscr{C}} \longrightarrow \mathscr{M}^{\partial(\alpha \downarrow \mathscr{C}_-)}$$

for any (model) category $\mathscr{M}$.

Definition 4.3.17. Let $\mathscr{C}$ be a small Reedy category and let $\mathscr{M}$ be a model category. Fix also $\alpha \in \mathscr{C}$.

1. The *latching functor* at α is the functor $L_\alpha \colon \mathscr{M}^{\mathscr{C}} \longrightarrow \mathscr{M}$ given as the composition

$$L_\alpha \colon \mathscr{M}^{\mathscr{C}} \xrightarrow{\ U^*\ } \mathscr{M}^{\partial(\mathscr{C}_+ \downarrow \alpha)} \xrightarrow{\ \mathrm{colim}\ } \mathscr{M} \tag{4.10}$$

For a functor $F \colon \mathscr{C} \longrightarrow \mathscr{M}$, the object

$$L_\alpha F = \mathrm{colim}_{\partial(\mathscr{C}_+ \downarrow \alpha)} U^* F = \mathrm{colim}_{(\beta \to \alpha) \in \partial(\mathscr{C}_+ \downarrow \alpha)} F_\beta \tag{4.11}$$

is called the *latching object* of F at α. The induced canonical arrow

$$l_\alpha \colon L_\alpha F \to F_\alpha \tag{4.12}$$

is called the *latching map* of F at α.

2. The *matching functor* at α is the functor $M_\alpha \colon \mathscr{M}^{\mathscr{C}} \longrightarrow \mathscr{M}$ given as the composition

$$M_\alpha \colon \mathscr{M}^{\mathscr{C}} \xrightarrow{\ V^*\ } \mathscr{M}^{\partial(\alpha \downarrow \mathscr{C}_-)} \xrightarrow{\ \lim\ } \mathscr{M} \tag{4.13}$$

For a functor $F \colon \mathscr{C} \longrightarrow \mathscr{M}$, the object

$$M_\alpha F = \lim_{\partial(\alpha \downarrow \mathscr{C}_-)} V^* F = \lim_{(\alpha \to \beta) \in \partial(\alpha \downarrow \mathscr{C}_-)} F_\beta \tag{4.14}$$

is called the *matching object* of F at α. The induced canonical arrow

$$m_\alpha \colon F_\alpha \to M_\alpha F \tag{4.15}$$

is called the *matching map* of F at α.

We can now state the existence theorem of a model structure for functor categories with exponent given by a small Reedy category.

Theorem 4.3.18. *Let $\mathscr{C}$ be a small Reedy category and $\mathscr{M}$ a model category. Then $\mathscr{M}^{\mathscr{C}}$ admits a model category structure by declaring that an arrow $\tau \colon F \Longrightarrow G$ in $\mathscr{M}^{\mathscr{C}}$ is*

(i) a weak equivalence if and only if it is a pointwise weak equivalence;

(ii) a (Reedy) cofibration if and only if, for all $\alpha \in \mathscr{C}$, the relative latching map

$$F_\alpha \amalg_{L_\alpha F} L_\alpha G \longrightarrow G_\alpha \tag{4.16}$$

is a cofibration in $\mathscr{M}$.

(iii) a (Reedy) fibration if and only if, for all $\alpha \in \mathscr{C}$, the relative matching map

$$F_\alpha \longrightarrow G_\alpha \times_{M_\alpha G} M_\alpha F \tag{4.17}$$

is a fibration in $\mathscr{M}$.

Proof. The most difficult axioms to check among those for a model category structure are the lifting and the factorization ones (see Definition 4.1.1). The idea is to construct the needed maps and factorizations inductively on the degree of objects of $\mathscr{C}$. See [Hir1], §15.3.16 for a complete proof. □

Definition 4.3.19. Given a model category $\mathscr{M}$ and a small Reedy category $\mathscr{C}$, the model category structure described in Theorem 4.3.18 above is called the *Reedy model structure* on $\mathscr{M}^{\mathscr{C}}$.

Remark 4.3.20. Let $\mathscr{C}$ be a small Reedy category. If $\mathscr{M}$ is a left or right proper model category, then so is $\mathscr{M}^{\mathscr{C}}$ when endowed with the Reedy model structure. If $\mathscr{M}$ is a simplicial model category (see Remark 4.2.25), then so is $\mathscr{M}^{\mathscr{C}}$ when endowed with the Reedy model structure and with the simplicial structure of Remark 4.2.31. See [Hir1], Theorem 15.3.4.

Remark 4.3.21. Suppose that $\mathscr{C}$ is a small direct category. Then, for all $\alpha \in \mathscr{C}$, the matching category $\partial(\alpha \downarrow \mathscr{C}_-)$ is empty. It follows that, given any model category $\mathscr{M}$, a Reedy fibration in $\mathscr{M}^{\mathscr{C}}$ is precisely a pointwise fibration. In particular, if $\mathscr{M}$ is a cofibrantly generated model category, then the projective and the Reedy model structures on $\mathscr{M}^{\mathscr{C}}$ coincide. Dually, if $\mathscr{C}$ is a small inverse category, then a Reedy cofibration in $\mathscr{M}^{\mathscr{C}}$ is precisely a pointwise cofibration, so that, when $\mathscr{M}$ is a combinatorial model category, the injective and the Reedy model structures on $\mathscr{M}^{\mathscr{C}}$ coincide (see Definition 4.2.7, Definition 4.2.10 and Section 4.3.1).

If we are given an arbitrary small Reedy category, Remark 4.3.21 can be rearranged and takes the form of the following two results.

Proposition 4.3.22. *Let $\mathscr{C}$ be a small Reedy category and let $\mathscr{M}$ be a model category. Fix a natural transformation $\tau \colon F \Longrightarrow G$ of functors from $\mathscr{C}$ to $\mathscr{M}$.*

(i) *If τ is a Reedy cofibration, then, for every object $\alpha \in \mathscr{C}$, both the map $\tau_\alpha \colon F_\alpha \to G_\alpha$ and the induced map of latching objects $L_\alpha F \to L_\alpha G$ are cofibrations in $\mathscr{M}$.*

(ii) *If τ is a Reedy fibration, then, for every object $\alpha \in \mathscr{C}$, both the map $\tau_\alpha \colon F_\alpha \to G_\alpha$ and the induced map of matching objects $M_\alpha F \to M_\alpha G$ are fibrations in $\mathscr{M}$.*

Proof. This is [Hir1], Proposition 15.3.11. □

Remark 4.3.23. From Proposition 4.3.22 above, it follows in particular that, if F is Reedy cofibrant in $\mathscr{M}^{\mathscr{C}}$ (for a small Reedy category $\mathscr{C}$), then, for all $\alpha \in \mathscr{C}$, both F_α and $L_\alpha F$ are cofibrant objects in $\mathscr{M}$. A dual result holds for Reedy fibrant diagrams.

Theorem 4.3.24. *Let $\mathscr{M}$ be a combinatorial model category and let $\mathscr{C}$ be a small Reedy category. Then there are Quillen equivalences*

$$(\mathscr{M}^{\mathscr{C}})_{inj} \underset{Id}{\overset{Id}{\rightleftarrows}} \perp (\mathscr{M}^{\mathscr{C}})_{Reedy} \underset{Id}{\overset{Id}{\rightleftarrows}} \perp (\mathscr{M}^{\mathscr{C}})_{proj}$$

Proof. It is an immediate consequence of Proposition 4.3.22 above, since all the model structures on $\mathscr{M}^{\mathscr{C}}$ have the same weak equivalences. □

Example 4.3.25. Let $\mathscr{I} = b \leftarrow a \rightarrow c$ be the pushout category (where we do not display identity morphisms) and let $\mathscr{M}$ be a model category.

(i) Endow $\mathscr{I}$ with the direct category structure of Example 4.3.13 and consider the associated Reedy model structure on $\mathscr{M}^{\mathscr{I}}$. Let $F\colon \mathscr{I} \longrightarrow \mathscr{M}$ be a functor. The formulas for the latching objects give $L_b F \cong L_c F \cong F_a$ and $L_a F \cong \emptyset$ (the initial object of $\mathscr{M}$). It follows that the latching maps l_b and l_c can be identified with the maps $F_a \to F_b$ and $F_a \to F_c$ respectively, whilst l_a is the map from the initial object to F_a. Therefore, a span

$$F_b \leftarrow F_a \to F_c$$

is cofibrant for this Reedy model category structure on $\mathscr{M}^{\mathscr{I}}$ exactly when all objects in the span are cofibrant and all maps are cofibrations in $\mathscr{M}$.

(ii) Endow $\mathscr{I}$ with the Reedy category structure of Example 4.3.14 and consider the associated Reedy model structure on $\mathscr{M}^{\mathscr{I}}$. Let $F\colon \mathscr{I} \longrightarrow \mathscr{M}$ be a functor. In this case, $L_a F \cong L_b F \cong \emptyset$ and $L_c F \cong F_a$, whereas the latching maps are the obvious ones. Thus, a span

$$F_b \leftarrow F_a \to F_c$$

is cofibrant for this Reedy model category structure on $\mathscr{M}^{\mathscr{I}}$ exactly when all objects in the span are cofibrant and at least one of the two maps is a cofibration in $\mathscr{M}$.

Example 4.3.26. With the notations of Example 4.3.25 above, let $\mathscr{I}^{\mathrm{op}}$ be the pullback category. We then get what follows.

(i) Endow $\mathscr{I}^{\mathrm{op}}$ with the inverse category structure given as the dual of the direct structure of Example 4.3.13. Then a cospan
$$F_b \to F_a \leftarrow F_c$$
is Reedy fibrant exactly when all objects in the cospan are fibrant and all maps are fibrations in $\mathscr{M}$.

(ii) Endow $\mathscr{I}^{\mathrm{op}}$ with the Reedy structure which is the dual of the one in Example 4.3.14. Then a cospan
$$F_b \to F_a \leftarrow F_c$$
is Reedy fibrant exactly when all objects in the cospan are fibrant and at least one of the two maps is a cofibration in $\mathscr{M}$.

Example 4.3.27. Let $\mathscr{M}$ be a model category and let Δ be the simplex category. The category $\mathscr{M}^{\Delta}$ is called the category of *cosimplicial objects* in $\mathscr{M}$, while the category $\mathscr{M}^{\Delta^{\mathrm{op}}}$ is called the category of *simplicial objects* in $\mathscr{M}$. Both of these categories admit a Reedy model structure induced by the Reedy structures on Δ and on Δ^{op} of Example 4.3.15.

We saw in Proposition 4.3.3 that when the projective or the injective model structures are available on a functor category, the colimit or the limit functor are left or right Quillen functors respectively. For the Reedy model structure, this is not quite true in general. However, we can describe exactly the properties that a Reedy category $\mathscr{C}$ needs to have in order to turn the colimit or the limit functors into Quillen functors.

Definition 4.3.28. Let $\mathscr{C}$ be a Reedy category.

1. We say that $\mathscr{C}$ *has cofibrant constants* if, for all $\alpha \in \mathscr{C}$, the latching category $\partial(C_+ \downarrow \alpha)$ is either empty or connected.

2. We say that $\mathscr{C}$ *has fibrant constants* if, for all $\alpha \in \mathscr{C}$, the matching category $\partial(\alpha \downarrow C_-)$ is either empty or connected.

Example 4.3.29. (i) The simplex category Δ is a Reedy category with fibrant constants.

(ii) Every direct (respectively inverse) category is a Reedy category with fibrant (respectively cofibrant) constants. In particular, the pushout category $\mathscr{I}$ (respectively, the pullback category $\mathscr{I}^{\mathrm{op}}$) with the direct category structure of Example 4.3.13 (respectively, with the inverse category structure given by the dual of the direct structure of Example 4.3.13) is a Reedy category with fibrant (respectively cofibrant) constants.

(iii) The pushout category $\mathscr{I}$ (respectively, the pullback category $\mathscr{I}^{\mathrm{op}}$) with the Reedy structure of Example 4.3.14 (respectively, with the Reedy structure given by the dual of the Reedy structure of Example 4.3.14) is a Reedy category with fibrant (respectively cofibrant) constants.

As announced, we have the following

Theorem 4.3.30. *Let $\mathscr{C}$ be a Reedy category.*

1. *$\mathscr{C}$ has fibrant constants if and only if, for all model categories $\mathscr{M}$,*

$$\mathrm{colim} \colon (\mathscr{M}^{\mathscr{C}})_{Reedy} \longrightarrow \mathscr{M}$$

is a left Quillen functor.

2. *$\mathscr{C}$ has cofibrant constants if and only if, for all model categories $\mathscr{M}$,*

$$\lim \colon (\mathscr{M}^{\mathscr{C}})_{Reedy} \longrightarrow \mathscr{M}$$

is a right Quillen functor.

Proof. This is Theorem 15.10.8 of [Hir1]. □

4.3.3 Model structures for simplical functors.

In this paragraph, we briefly explore generalizations of the projective and the injective model structure to categories of simplicial functors.

Let $\mathbb{C}$ and $\mathbb{D}$ be simplicial categories with $\mathbb{C}$ small. Then there exists a simplicial category $[\mathbb{C}, \mathbb{D}]$ whose objects are simplicial functors (that is, **sSet**$-$enriched functors); for simplicial functors $F, G \colon \mathbb{C} \longrightarrow \mathbb{D}$, the enriching object $\mathrm{Map}_{[\mathbb{C}, \mathbb{D}]}(F, G)$ is defined as the **sSet**$-$enriched end

$$\mathrm{Map}_{[\mathbb{C}, \mathbb{D}]}(F, G) := \int_{n \in \mathbb{C}} \mathrm{Map}_{\mathbb{D}}(F(n), G(n))$$

(See [Kel], §2.1 and §2.2.). In particular, the underlying category of $[\mathbb{C}, \mathbb{D}]$ is the category $[\mathbb{C}, \mathbb{D}]_O$ of simplicial functors and simplicial (i.e. **sSet**$-$enriched) natural transformations among them. A simplicial functor $F \colon \mathbb{C} \longrightarrow \mathbb{D}$ gives rise to an underlying ordinary functor

$$F_O \colon \mathscr{C} \longrightarrow \mathscr{D}$$

between the underlying categories of $\mathbb{C}$ and $\mathbb{D}$ respectively. Such an F_O sends an object $A \in \mathscr{C}$ into $FA \in \mathscr{D}$ and a map $g \colon A \to B \in \mathrm{Map}_{\mathbb{C}}(A, B)_0 = \mathscr{C}(A, B)$ to $F_0(g) \in \mathrm{Map}_{\mathbb{D}}(FA, FB)_0$ (see Definition 4.2.24). From now on, we will not use different notations to distinguish between $[\mathbb{C}, \mathbb{D}]$ and its underlying category or between a simplicial functor $F \colon \mathbb{C} \longrightarrow \mathbb{D}$ and its underlying ordinary functor F_O (see Remark 4.2.25).

Suppose now that $\mathbb{D}$ is *tensored* and *cotensored* by (simplicial) functors $A \otimes (-) \colon \textbf{sSet} \longrightarrow \mathbb{D}$ and $B^{(-)} \colon \textbf{sSet} \longrightarrow \mathbb{D}^{\mathrm{op}}$ for all $A, B \in \mathbb{D}$ (see Definition 4.2.29). Then also $[\mathbb{C}, \mathbb{D}]$ is tensored and cotensored: tensors and cotensors are defined objectwise (see also Remark 4.2.31). For example, when $F \in [\mathbb{C}, \mathbb{D}]$ is a simplicial functor, for each $K \in \textbf{sSet}$ we put

$$(F \otimes K)(n) := F(n) \otimes K, \qquad n \in \mathbb{C}.$$

Given simplicial sets K and L, the needed map

$$(F \otimes (-))_{KL} \colon \mathrm{Map}(K, L) \to \mathrm{Map}_{[\mathbb{C}, \mathbb{D}]}(F \otimes K, F \otimes L) = \int_{n \in \mathbb{C}} \mathrm{Map}_{\mathbb{D}}(F(n) \otimes K, F(n) \otimes L)$$

is induced by the universal property of the enriched end from the maps

$$(F(n) \otimes (-))_{KL} \colon \mathrm{Map}(K, L) \to \mathrm{Map}_{\mathbb{D}}(F(n) \otimes K, F(n) \otimes L),$$

when $n \in \mathbb{C}$.

We can now state the enriched versions of Theorems 4.3.4 and 4.3.5.

Theorem 4.3.31. *Let $\mathbb{C}$ be a small simplicial category and let $\mathbb{D}$ be a simplicial combinatorial model category (see Definition 4.2.32). Consider the simplicial tensored and cotensored category $[\mathbb{C}, \mathbb{D}]$. The following hold.*

1. *$[\mathbb{C}, \mathbb{D}]$ is a simplicial combinatorial model category with respect to the projective model structure for which a (simplicial) natural transformation $\tau \colon F \Longrightarrow G$ between simplicial functors F and G is a weak equivalence (respectively a fibration) if, for every $n \in \mathbb{C}$, the component $\tau_n \colon F(n) \to G(n)$ is a weak equivalence (respectively a fibration) in (the underlying category of) $\mathbb{D}$.*

2. *$[\mathbb{C}, \mathbb{D}]$ is a simplicial combinatorial model category with respect to the injective model structure for which a (simplicial) natural transformation $\tau \colon F \Longrightarrow G$ between simplicial functors F and G is a weak equivalence (respectively a cofibration) if, for every $n \in \mathbb{C}$, the component $\tau_n \colon F(n) \to G(n)$ is a weak equivalence (respectively a cofibration) in (the underlying category of) $\mathbb{D}$.*

Furthermore, if $\mathbb{D}$ is a left or right proper model category, then so is $[\mathbb{C}, \mathbb{D}]$ with either the projective or the injective model structure.

Proof. This follows from Remark A.2.8.4, Example A.3.2.18, Theorem A.3.3.2 and Remark A.3.3.4 of [Lur]. □

Remark 4.3.32. We will mainly use the above Theorem 4.3.31 in the special case where $\mathbb{D} = \mathbf{sSet}$ (see Remark 4.2.36). As in Remark 4.3.8, unless differently stated, we will always implicitly assume that, for a small simplicial category $\mathbb{C}$, $[\mathbb{C}, \mathbf{sSet}]$ is endowed with the *projective model structure*.

4.4 Function complexes in model categories.

In this section, we will see how to define, for each model category $\mathscr{M}$, a *(derived) mapping space* bifunctor $\mathrm{map}^{\mathrm{h}} \colon \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathbf{sSet}$ assigning to every pair (X, Y) of objects in $\mathscr{M}$ a simplicial set (actually, a Kan complex) $\mathrm{map}^{\mathrm{h}}(X, Y)$ which, in degree 0, is given by $\mathscr{M}(QX, RY)$ for a cofibrant approximation QX and a fibrant approximation RY of X and Y respectively. Although this bifunctor does not turn $\mathscr{M}$ into a simplicial model category, it serves as a homotopical correction to the Hom sets of a model category $\mathscr{M}$: each $\mathrm{map}^{\mathrm{h}}(X, Y)$ is a simplicial set whose set of path components is naturally isomorphic to $\mathrm{Ho}(\mathscr{M})(X, Y)$ and such a mapping space can be used to detect weak equivalences. Furthermore, this mapping space behaves well with respect to Quillen adjunctions and Quillen equivalence. We begin this section by introducing the notions of *cosimplicial* and *simplicial resolutions* which are needed to define the bifunctor $\mathrm{map}^{\mathrm{h}}$.

4.4.1 (Co)simplicial resolutions and frames.

Let $\mathscr{M}$ be a model category and let $X \in \mathscr{M}$. We will denote the constant cosimplicial object at X by $cc_* X$ and the constant simplicial object by $cs_* X$ (see Example 4.3.27).

Definition 4.4.1. Let $\mathscr{M}$ be a model category and let $X \in \mathscr{M}$.

1. A *cosimplicial resolution* of X is a cofibrant approximation $\widetilde{X} \to cc_* X$ in the Reedy model structure on $\mathscr{M}^{\Delta}$. If the weak equivalence $\widetilde{X} \to cc_* X$ is a Reedy trivial fibration, we say that the cosimplicial resolution is a *fibrant cosimplicial resolution*.

2. A *simplicial resolution* of X is a fibrant approximation $cs_* X \to \widehat{X}$ in the Reedy model structure on $\mathscr{M}^{\Delta^{\mathrm{op}}}$. If the weak equivalence $cs_* X \to \widehat{X}$ is a Reedy trivial cofibration, we say that the simplicial resolution is a *cofibrant simplicial resolution*.

3. A *functorial (fibrant) cosimplicial resolution* on $\mathscr{M}$ is a pair (F, ι) in which $F \colon \mathscr{M} \longrightarrow \mathscr{M}^{\Delta}$ is a functor and $\iota \colon F \Longrightarrow cc_*$ is a natural transformation such that $\iota_Y \colon FY \to cc_* Y$ is a (fibrant) cosimplicial resolution of Y.

4. A *functorial (cofibrant) simplicial resolution* on $\mathscr{M}$ is a pair (G, μ) in which $G \colon \mathscr{M} \longrightarrow \mathscr{M}^{\Delta^{\mathrm{op}}}$ is a functor and $\mu \colon cs_* \Longrightarrow G$ is a natural transformation such that $\mu_Y \colon cs_* Y \to GY$ is a (cofibrant) simplicial resolution of Y.

Remark 4.4.2. Let $\mathscr{M}$ be a model category and fix $X \in \mathscr{M}$. Suppose that $\widetilde{X} \to cc_* X$ and $cs_* X \to \widehat{X}$ are cosimplicial and simplicial resolutions of X respectively. By Proposition 4.3.22 and Remark 4.3.23, $\widetilde{X}_0 \to X$ is a cofibrant approximation to X and $X \to \widehat{X}_0$ is a fibrant approximation to X. Furthermore, from the definitions it follows that $\widetilde{X}_1$ is a cylinder object for $\widetilde{X}_0$, whereas $\widehat{X}_1$ is a path object for $\widehat{X}_0$ (see Definition 4.1.22).

Since $\mathscr{M}^{\Delta}$ and $\mathscr{M}^{\Delta^{\mathrm{op}}}$ are model categories, they have canonical functorial fibrant and cofibrant approximations which give functorial simplicial and cosimplicial resolutions on $\mathscr{M}$.

Theorem 4.4.3. *Let $\mathscr{M}$ be a model category, let (Q_{Reedy}, q_{Reedy}) be the canonical cofibrant approximation on $\mathscr{M}^{\Delta}$ and let (R_{Reedy}, r_{Reedy}) be the canonical fibrant approximation on $\mathscr{M}^{\Delta^{\mathrm{op}}}$ (see Definition 4.1.14).*

1. *The pair $(Q_{Reedy} \circ cc_*, \ (q_{Reedy})_{cc_*})$ is a functorial fibrant cosimplicial resolution (F, ι) on $\mathscr{M}$.*

2. *The pair $(R_{Reedy} \circ cs_*, \ (r_{Reedy})_{cs_*})$ is a functorial cofibrant simplicial resolution (G, μ) on $\mathscr{M}$.*

Cofibrant and fibrant resolutions on a model category are essentially unique, as witnessed by the following

Proposition 4.4.4. *Let $\mathscr{M}$ be a model category.*

1. *Any two (co)simplicial resolutions of $X \in \mathscr{M}$ are connected by an essentially unique zig-zag (see Remark 4.1.17) of weak equivalences.*

2. *Any two functorial (co)simplicial resolutions on $\mathscr{M}$ are connected by an essentially unique zig-zag (see Remark 4.1.17) of weak equivalences.*

Proof. See Proposition 16.1.17 and Proposition 16.1.18 of [Hir1]. $\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

A way to build (co)simplicial resolutions is provided by the following notion.

Definition 4.4.5. Let $\mathscr{M}$ be a model category and let X be an object of $\mathscr{M}$.

1. A *cosimplicial frame* on X is a cosimplicial object X^* in $\mathscr{M}$ together with a weak equivalence $X^* \to cc_* X$ in the Reedy model structure on $\mathscr{M}^\Delta$ such that

    (i) $X_0^* \to X$ is an isomorphism;
    (ii) if X is a cofibrant object of $\mathscr{M}$, then X^* is a cofibrant object in $\mathscr{M}^\Delta$.

2. A *simplicial frame* on X is a simplicial object X_* in $\mathscr{M}$ together with a weak equivalence $cs_* X \to X_*$ such that

    (i) $X \to (X_*)_0$ is an isomorphism;
    (ii) if X is a fibrant object of $\mathscr{M}$, then X_* is a fibrant object in $\mathscr{M}^{\Delta^{\mathrm{op}}}$.

As usual, we may refer to a cosimplicial or to a simplicial frame on X by mentioning only the object X^* or X_* respectively, leaving the weak equivalences $X^* \to cc_* X$ and $cs_* X \to X_*$ implicit.

Remark 4.4.6. Here is a recipe to construct cosimplicial resolutions for an object X in a model category $\mathscr{M}$ (see Proposition 16.6.6 in [Hir1]):

(i) Take a cofibrant approximation $QX \to X$ to X in $\mathscr{M}$.

(ii) Pick a cosimplicial frame $(QX)^* \to cc_*(QX)$ on QX.

(iii) The composite map $(QX)^* \to cc_*(QX) \to cc_* X$ is a cosimplicial resolution of X.

There is of course a dual result for simplicial resolutions, involving fibrant approximations and simplicial frames.

Using the Reedy structure on Δ and on Δ^{op} (see Remark 4.3.12 and Example 4.3.15), it is possible to prove the following

Theorem 4.4.7. *Let $\mathscr{M}$ be a model category. Then the functorial factorizations on $\mathscr{M}$ provides a functorial cosimplicial frame and a functorial simplicial frame on $\mathscr{M}$. More precisely, there are functors*

$$(-)^\circ \colon \mathscr{M} \longrightarrow \mathscr{M}^\Delta \quad and \quad (-)_\circ \colon \mathscr{M} \longrightarrow \mathscr{M}^{\Delta^{\mathrm{op}}}$$

and natural transformations

$$i \colon (-)^\circ \Longrightarrow cc_*(-) \quad and \quad j \colon cs_* \Longrightarrow (-)_\circ$$

such that:

1. *for all $X \in \mathscr{M}$, $i_X \colon X^\circ \to cc_* X$ and $cs_* X \to X_\circ$ are cosimplicial and simplicial frames on X respectively;*

2. *for all $X \in \mathscr{M}$, the map i_X is a Reedy trivial fibration (and not just a weak equivalence), whereas the map j_X is a Reedy trivial cofibration (and not just a weak equivalence).*

Proof. This follows from Proposition 16.6.8 of [Hir1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 4.4.8. There is an analogous result to Proposition 4.4.4 for frames (see Theorem 16.6.10 of [Hir1]). Thus, any two functorial cosimplicial frames on $\mathscr{M}$ are connected by an essentially unique zig-zag of weak equivalences and the same is true for simplicial frames.

Definition 4.4.9. For any model category $\mathscr{M}$, we will refer to the functorial cosimplical and simplicial frames induced by the functorial factorization on $\mathscr{M}$ of Theorem 4.4.7 as the *canonical* cosimplicial frame and simplicial frame respectively.

Definition 4.4.10. A *framed model category* is a triple $(\mathscr{M}, (-)^*, (-)_*)$, where $\mathscr{M}$ is a model category, while $(-)^* \colon \mathscr{M} \longrightarrow \mathscr{M}^\Delta$ and $(-)_* \colon \mathscr{M} \longrightarrow \mathscr{M}^{\Delta^{\mathrm{op}}}$ are functorial cosimplicial and simplicial frames on $\mathscr{M}$ respectively.

Hence, any model category is canonically a framed model category. We shall also need the following

Example 4.4.11. Let $\mathscr{M}$ be a simplicial model category (see Remark 4.2.25). Then we get a framed model category structure on $\mathscr{M}$ by defining, for each object $X \in \mathscr{M}$,

$$X^* := X \otimes \Delta[-], \qquad X_* := X^{\Delta[-]},$$

where $(-) \otimes (?)$ and $(-)^{(?)}$ denote the tensor and the cotensor bifunctors on $\mathscr{M}$ (see Proposition 16.6.23 of [Hir1]). We will call these frames on $\mathscr{M}$ the *standard* frames on $\mathscr{M}$.

4.4.2 Homotopy Function Complexes

As announced, we can use cosimplicial and simplicial resolutions to define, for each pair of objects in $\mathscr{M}$ a simplicial set of maps among them.

Definition 4.4.12. Let $\mathscr{M}$ be a model category and let X, Y be objects in $\mathscr{M}$.

1. A *left homotopy function complex* from X to Y is a triple

$$(\widetilde{X}, \ RY, \ \mathscr{M}(\widetilde{X}, \ RY)),$$

where $\widetilde{X}$ is a cosimplicial resolution of X, RY is a fibrant approximation to Y and $\mathscr{M}(\widetilde{X}, \ RY)$ is the simplicial set whose set of $n-$simplices (for $n \geq 0$) is given by $\mathscr{M}(\widetilde{X}_n, RY)$.

2. A *right homotopy function complex* from X to Y is a triple

$$(QX, \ \widehat{Y}, \ \mathscr{M}(QX, \ \widehat{Y})),$$

where QX is a cofibrant approximation to X, $\widehat{Y}$ is a simplicial resolution of Y, and $\mathscr{M}(QX, \ \widehat{Y})$ is the simplicial set whose set of $n-$simplices (for $n \geq 0$) is given by $\mathscr{M}(QX, \ \widehat{Y}_n)$.

3. A *two-sided homotopy function complex* from X to Y is a triple

$$(\widetilde{X}, \ \widehat{Y}, \ \mathrm{diag}\, \mathscr{M}(\widetilde{X}, \ \widehat{Y})),$$

where $\widetilde{X}$ is a cosimplicial resolution of X, $\widehat{Y}$ is a simplicial resolution of Y and $\mathrm{diag}\, \mathscr{M}(\widetilde{X}, \ \widehat{Y})$ is the simplicial set whose set of $n-$simplices (for $n \geq 0$) is given by $\mathscr{M}(\widetilde{X}_n, \ \widehat{Y}_n)$.[6]

We will commonly refer to a homotopy function complex (left, right or two-sided) by mentioning only the simplicial set appearing in the triple which defines it, leaving the other ingredients implicit.

Remark 4.4.13. We gather here some facts about homotopy function complexes.

(i) Given objects X and Y in $\mathscr{M}$, any homotopy function complex (left, right or two-sided) is a Kan complex.

(ii) Suppose given objects X and Y in $\mathscr{M}$. Then for every couple (f, g), where $f \colon X^\bullet \to \widetilde{X}$ is a map of cosimplicial resolutions of X and $g \colon RY \to R'Y$ is a map of fibrant resolutions of Y, we get an induced weak equivalence of simplicial sets $\mathscr{M}(\widetilde{X}, RY) \to \mathscr{M}(X^\bullet, R'Y)$ sending an $n-$simplex $s \colon \widetilde{X}_n \to RY$ to $g \circ s \circ f_n$. Here a map of cosimplicial resolutions is simply a map in $\mathscr{M}^\Delta$, while a map $g \colon RY \to R'Y$ of fibrant resolutions $Y \to RY$ and $Y \to R'Y$ is just a map in the undercategory $Y/\mathscr{M}$. Now, a *change of left homotopy function complexes map* is any morphism of simplicial sets $h \colon \mathscr{M}(\widetilde{X}, RY) \to \mathscr{M}(X^\bullet, R'Y)$ which is induced by a couple of maps (f, g) as above. Then, any two left homotopy function complexes from X to Y are connected by an essentially unique zig-zag of change of left homotopy function complexes maps. In particular, any two left homotopy function complexes from X to Y are weakly equivalent. The same kind of remark applies to right and to two-sided homotopy function complexes (with the corresponding notions of change of homotopy function complexes maps).

(iii) We can define *functorial* left, right and two-sided homotopy function complexes on a model category $\mathscr{M}$ in the obvious way as functors $\mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathbf{sSet}$ sending each pair of objects (X, Y) to a left, right or two-sided homotopy function complex from X to Y (see [Hir1], §17.5). Similarly, one defines change of functorial left, right or two-sided homotopy function complex maps and proves that any two functorial left, right or two-sided homotopy complexes are connected by an essentially unique zig-zag of such maps.

Using *any* functorial cosimplicial or simplicial resolution as well as *any* functorial fibrant or cofibrant replacement on a model category $\mathscr{M}$, we can see that every model category admits functorial left, right and two-sided homotopy complexes. Up to homotopy, all these left, right and two-sided homotopy function complexes can be identified, as witnessed by the following

---

[6] Setting, for all $n, m \in \mathbb{N}$, $\mathscr{M}(\widetilde{X}, \ \widehat{Y})_{n,m} := \mathscr{M}(\widetilde{X}_n, \ \widehat{Y}_m)$, we get a bisimplicial set (i.e. a simplicial simplicial set) and $\mathrm{diag}\, \mathscr{M}(\widetilde{X}, \ \widehat{Y})$ is its *diagonal*.

Theorem 4.4.14. *Let $\mathcal{M}$ be a model category. For every ordered pair*

$$(\mathrm{map}^h{}_1(-,?)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathbf{sSet}, \ \mathrm{map}^h{}_2(-,?)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathbf{sSet})$$

where $\mathrm{map}^h{}_1$ and $\mathrm{map}^h{}_2$ are functorial left, right or two-sided homotopy function complexes, there is a uniquely defined (up to homotopy) and natural (up to homotopy) pointwise homotopy equivalence

$$h_{1,2}\colon \mathrm{map}^h{}_1(-,?) \Longrightarrow \mathrm{map}^h{}_2(-,?).$$

Furthermore, given any other functorial left, right or two-sided homotopy function complex $\mathrm{map}^h{}_3(-,?)$, if

$$h_{1,3}\colon \mathrm{map}^h{}_1(-,?) \Longrightarrow \mathrm{map}^h{}_3(-,?) \quad and \quad h_{2,3}\colon \mathrm{map}^h{}_2(-,?) \Longrightarrow \mathrm{map}^h{}_3(-,?)$$

are the corresponding pointwise homotopy equivalences, then there is a homotopy

$$h_{2,3} \circ h_{1,2} \simeq h_{1,3}.$$

Proof. See [Hir1], Theorem 17.5.30. $\qquad\square$

This justifies the following

Definition 4.4.15. Let $\mathcal{M}$ be a model category.

1. For any pair (X,Y) of objects in $\mathcal{M}$, a *homotopy function complex* from X to Y is any left, right or two-sided homotopy function complex from X to Y. We will denote such a homotopy function complex by $\mathrm{map}^h(X,Y)$.

2. A *(functorial) homotopy function complex on $\mathcal{M}$* (or a *(derived) mapping space on $\mathcal{M}$*) is any functorial left, right or two-sided homotopy function complex on $\mathcal{M}$. We will denote such a functorial homotopy function complex by $\mathrm{map}^h(-,?)\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \longrightarrow \mathbf{sSet}$.

3. Given an arrow $g\colon Y \to X$ in $\mathcal{M}$ and an object $B \in \mathcal{M}$, we will denote by $g_*\colon \mathrm{map}^h(B,Y) \to \mathrm{map}^h(B,X)$ the morphism $\mathrm{map}^h(1_B, g)$ induced by g, for some homotopy function complex $\mathrm{map}^h(-,?)$ on $\mathcal{M}$. Dually, we will denote by $g^*\colon \mathrm{map}^h(X,B) \to \mathrm{map}^h(Y,B)$ the morphism $\mathrm{map}^h(g, 1_B)$.

Example 4.4.16. Let $\mathcal{M}$ be a simplicial model category (see Remark 4.2.25). If A is a cofibrant object in $\mathcal{M}$ and X is a fibrant object in $\mathcal{M}$, then the enriching simplicial set $\mathrm{Map}_{\mathcal{M}}(A,X)$ is a homotopy function complex $\mathrm{map}^h(A,X)$. This follows from Example 4.4.11.

As a consequence of Theorem 4.4.14 above, we get the

Corollary 4.4.17. *Let $\mathcal{M}$ be a model category, let B be an object of $\mathcal{M}$ and let $g\colon Y \to X$ be a morphism in $\mathcal{M}$.*

(i) *If $g_*\colon \mathrm{map}^h(B,Y) \to \mathrm{map}^h(B,X)$ is a weak equivalence for some homotopy function complex $\mathrm{map}^h(-,?)$, then $g_*\colon \mathrm{map}^{h'}(B,Y) \to \mathrm{map}^{h'}(B,X)$ is a weak equivalence for any other homotopy function complex $\mathrm{map}^{h'}(-,?)$.*

(ii) *If $g^*\colon \mathrm{map}^h(X,B) \to \mathrm{map}^h(Y,B)$ is a weak equivalence for some homotopy function complex $\mathrm{map}^h(-,?)$, then $g^*\colon \mathrm{map}^{h'}(X,B) \to \mathrm{map}^{h'}(Y,B)$ is a weak equivalence for any other homotopy function complex $\mathrm{map}^{h'}(-,?)$.*

Homotopy function complexes are to model categories $\mathcal{M}$ as ordinary Hom-set bifunctors are to ordinary categories $\mathcal{D}$. We try to explain this motto with the following few results.

Firstly, we can use homotopy function complexes to detect weak equivalences, exactly as we can use Hom-sets to detect isomorphisms.

Proposition 4.4.18. *Let $\mathcal{M}$ be a model category and $\mathrm{map}^h(-,?)$ a homotopy function complex on $\mathcal{M}$. Then the following are equivalent, for a morphism $g\colon X \to Y$ in $\mathcal{M}$.*

1. *g is a weak equivalence.*

2. *For every object B in $\mathcal{M}$, the induced map $g_*\colon \mathrm{map}^h(B,X) \to \mathrm{map}^h(B,Y)$ is a weak equivalence.*

3. *For every cofibrant object B in $\mathscr{M}$, the induced map $g_*\colon \operatorname{map}^{\mathrm{h}}(B, X) \to \operatorname{map}^{\mathrm{h}}(B, Y)$ is a weak equivalence.*

4. *For every object Z in $\mathscr{M}$, the induced map $g^*\colon \operatorname{map}^{\mathrm{h}}(Y, Z) \to \operatorname{map}^{\mathrm{h}}(X, Z)$ is a weak equivalence.*

5. *For every fibrant object Z in $\mathscr{M}$, the induced map $g^*\colon \operatorname{map}^{\mathrm{h}}(Y, Z) \to \operatorname{map}^{\mathrm{h}}(X, Z)$ is a weak equivalence.*

Proof. This is [Hir1], Theorem 17.7.7. □

Homotopy function complexes on a model category $\mathscr{M}$ can also be thought of as higher counterparts to the Hom-sets of the homotopy category on $\mathscr{M}$, in a sense made precise by the following

Proposition 4.4.19. *Let $\mathscr{M}$ be a model category with homotopy function complex $\operatorname{map}^{\mathrm{h}}(-, ?)$ and homotopy category $\operatorname{Ho}(\mathscr{M})$. Then there are natural isomorphisms*

$$\pi_0 \operatorname{map}^{\mathrm{h}}(X, Y) \cong \operatorname{Ho}(\mathscr{M})(X, Y),$$

for $X, Y \in \mathscr{M}$.

Proof. By Theorem 4.4.14 we can take $\operatorname{map}^{\mathrm{h}}(X, Y)$ to be $\mathscr{M}(\widetilde{X}, RY)$ for a (functorial) cosimplicial resolution of X and a (functorial) fibrant approximation of Y in $\mathscr{M}$. By Remark 4.4.2, $\widetilde{X}_0$ is a cofibrant approximation to X and $\widetilde{X}_1$ is a cylinder object for X. Hence, by Remark 4.1.24, two $0-$simplexes $s, t\colon \widetilde{X}_0 \to RY$ of $\operatorname{map}^{\mathrm{h}}(X, Y)$ are in the same path-component of $\operatorname{map}^{\mathrm{h}}(X, Y)$ (i.e. represent the same element in $\pi_0 \operatorname{map}^{\mathrm{h}}(X, Y)$) if and only if they are (left) homotopic. This means that $\pi_0 \operatorname{map}^{\mathrm{h}}(X, Y) = \pi_0 \mathscr{M}(\widetilde{X}, RY)$ is isomorphic to the set $\mathscr{M}(QX, RY)/ \sim$, where $\sim$ is the (left) homotopy relation (see Definition 4.1.22). Theorem 4.1.26 says that this latter set is naturally isomorphic to $\operatorname{Ho}(\mathscr{M})(X, Y)$, as required. □

Finally, homotopy function complexes have the expected behaviour with respect to Quillen functors and Quillen equivalences.

Proposition 4.4.20. *Let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \bot \ \mathscr{N}$$

be a Quillen pair between model categories $\mathscr{M}$ and $\mathscr{N}$. Let $\operatorname{map}^{\mathrm{h}}{}_{\mathscr{M}}$ (respectively $\operatorname{map}^{\mathrm{h}}{}_{\mathscr{N}}$) be a homotopy function complex on $\mathscr{M}$ (respectively on $\mathscr{N}$) and denote by $\mathbb{L}F$ (resp. by $\mathbb{R}G$) a point-set left derived functor of F (resp. a point-set right derived functor of G). Thus, $\mathbb{L}F = F \circ Q$ and $\mathbb{R}G = G \circ R$, where Q is a functorial cofibrant replacement on $\mathscr{M}$ and R is a functorial fibrant replacement on $\mathscr{N}$ (see Remark 4.1.33). Then the following statements hold.

(i) *For all $X \in \mathscr{M}$ and $Y \in \mathscr{N}$, we have*

$$\operatorname{map}^{\mathrm{h}}{}_{\mathscr{N}}((\mathbb{L}F)X, Y) \approx \operatorname{map}^{\mathrm{h}}{}_{\mathscr{M}}(X, (\mathbb{R}G)Y)$$

(see Definition 4.1.10).

(ii) *If (F, G) is a Quillen equivalence, then, for all $X, X' \in \mathscr{M}$ and all $Y, Y' \in \mathscr{N}$, we have*

$$\operatorname{map}^{\mathrm{h}}{}_{\mathscr{M}}(X, X') \approx \operatorname{map}^{\mathrm{h}}{}_{\mathscr{N}}((\mathbb{L}F)X, (\mathbb{L}F)X') \quad \text{and} \quad \operatorname{map}^{\mathrm{h}}{}_{\mathscr{N}}(Y, Y') \approx \operatorname{map}^{\mathrm{h}}{}_{\mathscr{M}}((\mathbb{R}G)Y, (\mathbb{R}G)Y').$$

Proof. The first part is a consequence of the following observation together with Theorem 4.4.14 and Proposition 4.4.18. Given a cofibrant object W in $\mathscr{M}$ and a fibrant object Z in $\mathscr{N}$, let $\widetilde{W}$ be a (functorial) cofibrant resolution of W. Then, since the Quillen pair (F, G) lifts to a Quillen pair

$$(\mathscr{M}^{\Delta})_{\mathbf{Reedy}} \underset{G \circ -}{\overset{F \circ -}{\rightleftarrows}} \bot \ (\mathscr{N}^{\Delta})_{\mathbf{Reedy}}$$

(see Example 4.3.15, Theorem 4.3.18 and Proposition 15.4.1 of [Hir1]), we get that $\widetilde{FW}$ is a (functorial) cosimplicial resolution of the cofibrant object FW, so that we have a (natural) isomorphism

$$\mathscr{N}(\widetilde{FW}, Z) \cong \mathscr{M}(\widetilde{W}, GZ).$$

Now, if (F, G) is also a Quillen equivalence, then we have that, for all cofibrant object W of $\mathscr{M}$, the composite map

$$W \to GFW \to GRFW$$

is a weak equivalence in $\mathscr{M}$ (see Proposition 4.1.37). Therefore, using Theorem 4.4.18, we get the following chain of naturally weakly equivalent simplicial sets

$$\mathrm{map}^{\mathrm{h}}{}_{\mathscr{M}}(X, X') \approx \mathrm{map}^{\mathrm{h}}{}_{\mathscr{M}}(QX, QX') \approx \mathrm{map}^{\mathrm{h}}{}_{\mathscr{M}}(QX, GRFQX') \overset{(\dagger)}{\approx} \mathrm{map}^{\mathrm{h}}{}_{\mathscr{N}}(FQX, RFQX') \approx$$

$$\approx \mathrm{map}^{\mathrm{h}}{}_{\mathscr{N}}(FQX, FQX') = \mathrm{map}^{\mathrm{h}}{}_{\mathscr{N}}((\mathbb{L}F)X, (\mathbb{L}F)X'),$$

where $(\dagger)$ follows from the first part because QX is cofibrant and $RFQX'$ is fibrant. Using the fact that, for all fibrant object Z in $\mathscr{N}$, the composite map

$$FQGZ \to FGZ \to Z$$

is a weak equivalence in $\mathscr{N}$, one sees also that $\mathrm{map}^{\mathrm{h}}{}_{\mathscr{N}}(Y, Y') \approx \mathrm{map}^{\mathrm{h}}{}_{\mathscr{M}}((\mathbb{R}G)Y, (\mathbb{R}G)Y')$. The proof is then complete. $\qquad\square$

4.5 Homotopy (co)limits.

Given a model category $\mathscr{M}$ and a small category $\mathscr{I}$, (co)completeness of $\mathscr{M}$ ensures the existence of all (co)limits of functors $\mathscr{I} \longrightarrow \mathscr{M}$. Hence, upon choosing a (co)limit for any object of $\mathscr{M}^{\mathscr{I}}$, one can always consider the colimit and limit functors, together with their adjoints as in

$$\mathscr{M}^{\mathscr{I}} \underset{c}{\overset{\mathrm{colim}}{\rightleftarrows}} \perp \mathscr{M} \quad \text{and} \quad \mathscr{M} \underset{\mathrm{lim}}{\overset{c}{\rightleftarrows}} \perp \mathscr{M}^{\mathscr{I}} \,,$$

where c is the diagonal (constant) functor.

However, the (co)limit functor is not, in general, *homotopy invariant* (or *homotopical*), i.e. given a natural transformation $\tau \colon F \Longrightarrow G \colon \mathscr{I} \longrightarrow \mathscr{M}$ which is a pointwise weak equivalence, it is not true that the induced arrow

$$\mathrm{colim}\,\tau \colon \mathrm{colim}\,F \longrightarrow \mathrm{colim}\,G \quad (\text{or } \lim\tau \colon \lim F \longrightarrow \lim G)$$

is a weak equivalence in $\mathscr{M}$. For example, if $\mathscr{I}$ is the pushout category

$$\bullet \longleftarrow \bullet \longrightarrow \bullet$$

and $\mathscr{M}$ is **Top** (with the conventions and the Quillen model structure of Example 4.1.18), then we can consider the following diagrams in $\mathscr{M}$

$$F = \ast \longleftarrow S^n \longrightarrow D^{n+1} \quad \text{and} \quad G = \ast \longleftarrow S^n \longrightarrow \ast$$

(here $S^n \longrightarrow D^{n+1}$ is the boundary inclusion of the n-sphere into the $(n+1)$-unit disk). We have an obvious natural weak equivalence $\tau \colon F \Longrightarrow G$ which is the identity on $\ast$ and on S^n and collapses the whole D^{n+1} to a point. However, $\mathrm{colim}\,F \cong S^{n+1}$ and $\mathrm{colim}\,G \cong \ast$, so that the induced map $\mathrm{colim}\,F \longrightarrow \mathrm{colim}\,G$ is not a weak equivalence.

We will thus define homotopy (co)limits as suitable deformations ("homotopy correction") of the ordinary (co)limits so that they become homotopy invariant. This will be done by emulating what happens with Quillen functors: although they need not to preserve all weak equivalences, their compositions with functorial cofibrant or fibrant approximations do so. Even if, in general, it might be meaningless to ask whether the colimit or the limit functors are Quillen functors (because the functor category $\mathscr{M}^{\mathscr{I}}$ may not admit a model category structure), using the theory of *homotopical categories* it is still possible to consider generalizations of cofibrant and fibrant approximations whose compositions with the colimit or the limit functor are homotopical.

4.5.1 Homotopical categories.

In this first subsection, we follow "the Blue Beast" [DHKS][7] and introduce some concepts and results in the theory of homotopical categories that we need to define a meaningful notion of homotopy (co)limits. However, before starting, a remark is needed. In what follows, we shall also consider functor categories of the form $\mathscr{N}^{\mathscr{M}}$ where $\mathscr{M}$ is not necessarily small, even if, in general, these categories are not locally small. This is mainly because we will be also interested in the case where $\mathscr{M}$ is a complete and cocomplete category and the smallness condition for such categories would imply that they need to be posets (admitting arbitrary infima and suprema, see [McL], §V.2 Proposition 3). Therefore, the smallness request would trivialize our discussion too much. We then ask the reader to forgive our sloppiness about size problems for functor categories, reassured by the fact that a proper, formal treatment of this issue is possible, enlarging the working universe (see, for example, the discussion in §8.1 of [DHKS]).

Definition 4.5.1. A *homotopical category* is a pair $(\mathscr{H}, \mathcal{W})$, where $\mathscr{H}$ is a category and $\mathcal{W}$ is a subclass of the class of all morphisms in $\mathscr{H}$ having the following properties:

(i) for all objects A of $\mathscr{H}$, $id_A \in \mathcal{W}$;

---

[7] Apparently, Daniel Kan used that expression to refer to its work [DHKS], as reported by a memorial note in his honour jointly written by Clark Barwick, Michael Hopkins, Haynes Miller and Ieke Moerdijk. The author is grateful to Matan Prasma for pointing this fact out and letting him know about the abovementioned note which he eagerly read.

(ii) $\mathcal{W}$ satisfies the *two-out-of-six* property, i.e. if f, g, h are morphisms in $\mathscr{H}$ such that the two compositions gf and hg exist and are in $\mathcal{W}$, then also f, g, h and hgf belong to $\mathcal{W}$.

The elements of the distinguished class $\mathcal{W}$ are called *weak equivalences*.

As usual, we will commonly leave the class of weak equivalences in a homotopical category implicit.

Remark 4.5.2. Assuming that both gf and hg are identity morphisms or that at least one among f, g and h is an identity morphism, one readily proves that, if $(\mathscr{H}, \mathcal{W})$ is a homotopical category, then $\mathcal{W}$ *contains all isomorphisms* in $\mathscr{H}$ and has the *two-out-of-three* property. In particular, $\mathcal{W}$ is a *subcategory* of $\mathscr{H}$.

Definition 4.5.3. Let $\mathscr{H}$ and $\mathscr{L}$ be homotopical categories.

1. A functor $F\colon \mathscr{H} \longrightarrow \mathscr{L}$ is called a *homotopical functor* if it *preserves weak equivalences*, i.e. for any weak equivalence $f\colon A \to B$ in $\mathscr{H}$, $F(f)\colon F(A) \to F(B)$ is a weak equivalence in $\mathscr{L}$.

2. Given (not necessarily homotopical) functors, $F, G\colon \mathscr{H} \longrightarrow \mathscr{L}$, a *natural weak equivalence* from F to G is a natural transformation $\tau\colon F \Longrightarrow G$ such that, for all $A \in \mathscr{H}$, the A−th component τ_A of τ is a weak equivalence in $\mathscr{L}$.

3. Two homotopical functors $F, G\colon \mathscr{H} \longrightarrow \mathscr{L}$ are *naturally weakly equivalent* if there is a zig-zag of natural weak equivalences connecting them.

4. A homotopical functor $F\colon \mathscr{H} \longrightarrow \mathscr{L}$ is a *homotopical equivalence (of homotopical categories)* if there is a homotopical functor $G\colon \mathscr{L} \longrightarrow \mathscr{H}$ such that the composite functors $F \circ G$ and $G \circ F$ are naturally weakly equvalent to $\mathrm{Id}_{\mathscr{L}}$ and $\mathrm{Id}_{\mathscr{H}}$ respectively.

5. We will denote by $(\mathscr{L}^{\mathscr{H}})_W$ the full subcategory of the functor category $\mathscr{L}^{\mathscr{H}}$ given by homotopical functors. (So, in particular, morphisms between two homotopical functors are given by *all* natural transformations between them).

We have the following, unsurprising result, whose proof is immediate and will thus be omitted.

Proposition 4.5.4. *Let $\mathscr{L}$ be a homotopical category.*

(i) *If $\mathscr{C}$ is an ordinary category, then the functor category $\mathscr{L}^{\mathscr{C}}$ is a homotopical category with weak equivalences given by natural weak equivalences.*

(ii) *If $\mathscr{H}$ is a homotopical category, then the category $(\mathscr{L}^{\mathscr{H}})_W$ is a homotopical category with weak equivalences given by natural weak equivalences.*

Model categories fit perfectly into the theory of homotopical categories, since

Proposition 4.5.5. *Let $(\mathcal{M}, \mathcal{W}, \mathrm{Cof}(\mathcal{M}), \mathrm{Fib}(\mathcal{M}))$ be a model category. Then $(\mathcal{M}, \mathcal{W})$ is a homotopical category.*

Proof. We have to prove that weak equivalences in a model category satisfy the 2-out-of-6 property. Let then

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

be composable morphisms in $\mathcal{M}$ with gf and hg weak equivalences. If $\gamma\colon \mathcal{M} \longrightarrow \mathrm{Ho}(\mathcal{M})$ is the localization functor, we have that $\gamma(gf)$ is an isomorphism in $\mathrm{Ho}(\mathcal{M})$. Since $\gamma(f)(\gamma(gf))^{-1}$ is an inverse for $\gamma(g)$, g is a weak equivalence in $\mathcal{M}$ (see Theorem 4.1.26). Using the two-out-of-three property, we then get that also f, h and hgf are weak equivalences, as required. $\square$

Remark 4.5.6. Combining Propositions 4.5.4 and 4.5.5 we get, in particular, that for any model category $\mathcal{M}$ and for any small category $\mathscr{I}$, the functor category $\mathcal{M}^{\mathscr{I}}$ is a homotopical category where weak equivalences are those natural transformations that are objectwise weak equivalences in $\mathcal{M}$. For a model category $\mathcal{M}$ and a small category $\mathscr{I}$, we will always consider $\mathcal{M}^{\mathscr{I}}$ as a homotopical category with respect to this class of pointwise weak equivalences.

Definition 4.5.7. Let $(\mathscr{H}, \mathcal{W})$ be a homotopical category. The *homotopy category* $\mathrm{Ho}(\mathscr{H})$ of $(\mathscr{H}, \mathcal{W})$ is the (possibly large) localization $\mathscr{H}[\mathcal{W}^{-1}]$ of $\mathscr{H}$ with respect to $\mathcal{W}$ (see Definition 1.2.2).

Remark 4.5.8. Using the homotopy category of a homotopical category, given a functor $F\colon \mathscr{H} \longrightarrow \mathscr{L}$ between homotopical categories, we can define the notions of *left or right derived functor*, *total left or right derived functor* and *point-set left or right derived functor* for F exactly as we did for functors between model categories (see Definition 4.1.31 and Remark 4.1.33).

As we have already remarked, to deal with the bad homotopical behaviour of limits and colimits, we need to find a way to study functors between homotopical categories which are not necessarily globally homotopical, but become such when restricted to suitable subcategories of their domain. More precisely, we give the following

Definition 4.5.9. Let $\mathscr{H}$ be a homotopical category and $\mathscr{H}_0$ a full subcategory of $\mathscr{H}$. We say that $\mathscr{H}_0$ is a *left* (resp. *right*) *deformation retract* of $\mathscr{H}$ if there is a pair (Q, q) (respectively (R, r)), where $Q\colon \mathscr{H} \longrightarrow \mathscr{H}$ (respectively $R\colon \mathscr{H} \longrightarrow \mathscr{H}$) is a homotopical functor sending $A \in \mathscr{H}$ to $Q(A) \in \mathscr{H}_0$ (respectively sending $A \in \mathscr{H}$ to $R(A) \in \mathscr{H}_0$) and

$$q\colon Q \Longrightarrow id_{\mathscr{H}} \quad (\text{respectivley } r\colon id_{\mathscr{H}} \Longrightarrow R)$$

is a natural weak equivalence. We call the pair (Q, q) (respectively the pair (R, r)) a *left deformation* (respectively a *right deformation*) of $\mathscr{H}$ into $\mathscr{H}_0$.

Definition 4.5.10. Let $F\colon \mathscr{H} \longrightarrow \mathscr{L}$ be a functor between homotopical categories. We say that F is *left deformable* (respectively *right deformable*) if there is a left (respectively a right) deformation retract $\mathscr{H}_0$ of $\mathscr{H}$ such that the restriction of F to $\mathscr{H}_0$ is homotopical. In this case a left (respectively a right) deformation of $\mathscr{H}$ into $\mathscr{H}_0$ is called a *left* (respectively *right*) *F-deformation* and $\mathscr{H}_0$ is called an $F-$*deformation retract*.

Clearly, every homotopical functor is both left and right deformable. Using cofibrant and fibrant replacement functors for a model category $\mathscr{M}$ and applying Ken Brown's Lemma (see Proposition 4.1.28) we get immediately the following result, which provides a plethora of further examples for deformable functors.

Theorem 4.5.11. *Let $\mathscr{M}$ be a model category and let $\mathscr{M}_c$ and $\mathscr{M}_f$ denote the full subcategory of $\mathscr{M}$ consisting of cofibrant and fibrant objects respectively. Then the following properties hold.*

1. *$\mathscr{M}_c$ is a left deformation retract of $\mathscr{M}$, whereas $\mathscr{M}_f$ is a right deformation retract of $\mathscr{M}$.*

2. *Every functor between model categories preserving weak equivalences between cofibrant (resp. fibrant) objects is left (resp. right) deformable. In particular, every left (resp. right) Quillen functor $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ is left (resp. right) deformable and $\mathscr{M}_c$ (resp. $\mathscr{M}_f$) is a left (resp. right) $F-$deformation retract.*

The above result comes as no surprise: left and right deformations for a homotopical category are defined exactly so as to generalise the properties of functorial cofibrant and fibrant replacements for a model category respectively. Similarly, deformable functors can be thought of as the homotopical abstraction of Quillen functors from the model-categorical setting. For example, we have the following

Proposition 4.5.12. *Let $F\colon \mathscr{H} \longrightarrow \mathscr{L}$ be a left (respectively right) deformable functor. Then F admits a point-set left (respectively right) derived functor, hence also a total left (respectively right) derived functor, $\mathbb{L}F$ (respectively, $\mathbb{R}F$). If (Q, q) (respectively (R, r)) is a left $F-$deformation (respectively a right $F-$deformation), then $\mathbb{L}F = FQ$ (respectively $\mathbb{R}G = GR$).*

Proof. This follows using a completely similar argument to the one given in the proof of Theorem 4.1.32. $\square$

Now, our definition of homotopy (co)limits should not just give us homotopical replacements of ordinary (co)limits, but should as well encode in itself a homotopical counterpart to the constituent property of (co)limits to be unique up to isomorphisms. This is possible through the following notions.

Definition 4.5.13. Let $\mathscr{H}$ be a homotopical category.

1. We say that $\mathscr{H}$ is *homotopically contractible*, if the unique (homotopical) functor $\mathscr{H} \longrightarrow \mathbf{1}$, where $\mathbf{1}$ is the terminal (homotopical) category, is a homotopical equivalence of homotopical categories (see Definition 4.5.3).

2. Given a non empty set I, we say that objects X_i ($i \in I$) of $\mathscr{H}$ are *canonically weakly equivalent* (or *homotopically unique*) if the full (homotopical) subcategory of $\mathscr{H}$ spanned by those X_i's and by all the objects in $\mathscr{H}$ which are weakly equivalent[8] to any of them is homotopically contractible.

Remark 4.5.14. Using the two-out-of-three property for weak equivalences in a homotopical category $\mathscr{H}$ (see Remark 4.5.2, we get that, if $\mathscr{H}$ is homotopically contractible, then every map in $\mathscr{H}$ is a weak equivalence and any two objects in $\mathscr{H}$ are weakly equivalent (see [DHKS], §13).

Definition 4.5.15. Let $\mathscr{H}$ be a homotopical category. An object $Z \in \mathscr{H}$ is called *homotopically initial* (resp. *homotopically terminal*) if there exist functors F_0, $F_1 \colon \mathscr{H} \longrightarrow \mathscr{H}$ and a natural transformation $\tau \colon F_0 \Longrightarrow F_1$ (resp. $\tau \colon F_1 \Longrightarrow F_0$) such that:

(i) F_0 is naturally weakly equivalent (see Definition 4.5.3) to $cZ \colon \mathscr{H} \longrightarrow \mathscr{H}$, the constant functor at Z;

(ii) F_1 is naturally weakly equivalent to $id_{\mathscr{H}}$;

(iii) τ_Z is a weak equivalence in $\mathscr{H}$.

Remark 4.5.16. A motivation for the above definition may come from the following somehow unorthodox characterization of terminal and initial objects in a category $\mathscr{D}$. An object $Z \in \mathscr{D}$ is initial (respectively terminal) if and only if there is a natural transformation $\sigma \colon cZ \Longrightarrow \mathrm{Id}_{\mathscr{D}}$ (respectively $\sigma \colon \mathrm{Id}_{\mathscr{D}} \Longrightarrow cZ$), where cZ is the constant functor at Z, such that $\sigma_Z \colon Z \to Z$ is an isomorphism.

The following Proposition explains the terminology just introduced.

Proposition 4.5.17. *Let $\mathscr{H}$ be a homotopical category and let Z be a homotopically initial (resp. homotopically terminal) object in $\mathscr{H}$. Then an object $A \in \mathscr{H}$ is a homotopically initial (resp. homotopically terminal) object in $\mathscr{H}$ if and only if it is weakly equivalent to Z. Furthermore, homotopically initial (resp. homotopically terminal) objects are homotopically unique (see Definition 4.5.13).*

Proof. This follows from the definitions. See [DHKS], Proposition 38.3. □

Given a (not necessarily homotopical) functor $F \colon \mathscr{H} \longrightarrow \mathscr{L}$, where $\mathscr{H}$ and $\mathscr{L}$ are homotopical categories, we can consider the full subcategory of the overcategory $(\mathscr{L}^{\mathscr{H}} \downarrow F)$ consisting of all those natural transformations $\sigma \colon G \Longrightarrow F$ where G is a homotopical functor. We will denote such a full subcategory by $((\mathscr{L}^{\mathscr{H}})_W \downarrow F)$. Note that it naturally inherits a structure of homotopical category from $(\mathscr{L}^{\mathscr{H}})_W$. Similarly, we can define $(F \downarrow (\mathscr{L}^{\mathscr{H}})_W)$. Homotopically initial and terminal objects in these homotopical categories are given special names.

Definition 4.5.18. Let $\mathscr{H}$ and $\mathscr{L}$ be homotopical categories and let $F \colon \mathscr{H} \longrightarrow \mathscr{L}$ be a (not necessarily homotopical) functor between them. A *left approximation* (respectively *right approximation*) is a homotopically terminal (respectively initial) object in $((\mathscr{L}^{\mathscr{H}})_W \downarrow F)$ (respectively in $(F \downarrow (\mathscr{L}^{\mathscr{H}})_W)$).

A sufficient condition for the existence of left and right approximations is provided by the following

Theorem 4.5.19. *Let $\mathscr{H}$ and $\mathscr{L}$ be homotopical categories and let $F \colon \mathscr{H} \longrightarrow \mathscr{L}$ be a (not necessarily homotopical) functor between them. If F is left deformable (resp. right deformable), then there exists a left approximation (resp. a right approximation) of F.*

Proof. We prove only the left version, the other being dual. Let $\mathscr{H}_0$ be a left deformation retract with left deformation (Q, q) such that F is homotopical when restricted to $\mathscr{H}_0$. Note that FQ is a homotopical functor, so (FQ, Fq) is an object in $\mathscr{D} := ((\mathscr{L}^{\mathscr{H}})_W \downarrow F)$. We claim that actually (FQ, Fq) is a left approximation for F. Indeed, let (H, σ) be an object in $\mathscr{D}$ and consider the diagram

$$
\begin{array}{ccc}
HQ & \xrightarrow{\;\sigma_Q\;} & FQ \\
{\scriptstyle H\tau}\downarrow & & \downarrow{\scriptstyle F\tau} \\
H & \xrightarrow[\;\sigma\;]{} & F
\end{array}
$$

---

[8] Of course, as in the model categorical case, two objects of a homotopical category are *weakly equivalent* if there is a zig-zag of weak equivalences connecting them, see Definition 4.1.10.

which commutes by naturality of σ. We can then define a functor $F_1 \colon \mathscr{D} \longrightarrow \mathscr{D}$ which sends (H, σ) into $(HQ, \sigma Hq)$. Since q is a natural weak equivalence and H is a homotopical functor, Hq is a natural weak equivalence $HQ \Longrightarrow H$, thus it is a weak equivalence in the homotopical category $\mathscr{D}$ between the object $(HQ, \sigma Hq)$ and the object (H, σ). Therefore, F_1 is naturally weak equivalent to $\mathrm{Id}_{\mathscr{D}}$. If we define F_0 as the constant functor with value (FQ, Fq), we actually have that the assignment $(H, \sigma) \mapsto \sigma QR$ defines (the (H, σ)−th component of) a natural transformation $F_1 \Longrightarrow F_0$ such that

$$F_1(FQ, Fq) = (FQQ, FqFQq) \xrightarrow{Fq_Q} (FQ, Fq) = F_0(FQ, Fq)$$

is a weak equivalence in $\mathscr{D}$, as FQ is a homotopical functor. Thus, (FQ, Fq) is a left approximation for F. $\qquad\square$

Remark 4.5.20. Combining Proposition 4.5.12 and Theorem 4.5.19, we then get that, if $F \colon \mathscr{H} \Longrightarrow \mathscr{L}$ is a left deformable functor and (Q, q) is a left F−deformation, then $(FQ, \ Fq)$ is both a point-set left derived functor and a left approximation for F.

Remark 4.5.21. Let $F \colon \mathscr{H} \longrightarrow \mathscr{L}$ be a left deformable functor and let $\mathscr{I}$ be a small category. Then the functor $F^* = F \circ - \colon \mathscr{H}^{\mathscr{I}} \longrightarrow \mathscr{L}^{\mathscr{I}}$ is left deformable. Indeed, if (Q, q) is a left F−deformation for the left F−deformation retract $\mathscr{H}_0$, then $(Q^* = Q \circ -, \ q^*)$ is a left F^*-deformation for the left F^*−deformation retract $\mathscr{H}_0^{\mathscr{I}}$. Here $q^* \colon Q^* \Longrightarrow \mathrm{Id}_{\mathscr{H}^{\mathscr{I}}}$ is the natural transformation whose X−th component, for a functor $X \colon \mathscr{I} \longrightarrow \mathscr{H}$, is the natural transformation $q_{X(-)} \colon QX(-) \Longrightarrow X(-)$. In particular,

$$\mathbb{L}(F \circ -) = F^* Q^*$$

is a left point-set derived functor (and a left approximation) of $F^* = F \circ -$, which we will call the left point-set derived functor of F^* *associated to* $\mathbb{L}F = FQ$. Note that, by definition, for a functor $X \colon \mathscr{I} \longrightarrow \mathscr{H}$, $(\mathbb{L}(F \circ -))(X) = F \circ Q \circ X = (\mathbb{L}F) \circ X$. Of course, a dual remark holds for right deformable functors.

4.5.2 Definition and properties of homotopy (co)limits.

Theorem 4.5.22. *Let $\mathscr{M}$ be a model category and let $\mathscr{I}$ be a small category. Then the colimit and the limit functors*

$$\mathrm{colim} \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}, \quad \lim \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}$$

are left and right deformable respectively.

Proof. This is a particular instance of Theorem 20.5 of [DHKS]. Following [Shu], we just mention briefly how to get a left colim-deformation, the construction of a right lim-deformation being dual. Let $\Delta \mathscr{I}$ be defined as the category of simplices of the nerve $N\mathscr{I}$ of $\mathscr{I}$, i.e. $\Delta \mathscr{I} = \mathrm{El}(N\mathscr{I})$, the category of elements of $N\mathscr{I}$. There is a projection functor

$$P \colon \Delta \mathscr{I} \longrightarrow \mathscr{I}$$

which sends an n−simplex $i_0 \to i_i \cdots \to i_n$ of $N\mathscr{I}$ to i_n and we can consider the induced functor

$$P^* = (-) \circ P \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}^{\Delta \mathscr{I}}.$$

Now, the category $\Delta \mathscr{I}$ is a Reedy category (see Example 15.1.19 of [Hir1]), so we can take a functorial Reedy cofibrant replacement $(Q, \ q)$ in $(\mathscr{M}^{\Delta \mathscr{I}})_{\mathrm{Reedy}}$. The composite

$$\mathscr{M}^{\mathscr{I}} \xrightarrow{P^*} \mathscr{M}^{\Delta \mathscr{I}} \xrightarrow{Q} \mathscr{M}^{\Delta \mathscr{I}} \xrightarrow{\mathrm{Lan}_{P^*}} \mathscr{M}^{\mathscr{I}}$$

functor (see Definition 2.2.14) together with the composite natural transformation

$$\mathrm{Lan}_{P^*}(q_{P^*}) \colon \ \mathrm{Lan}_{P^*} QP^* \Longrightarrow \mathrm{Lan}_{P^*} P^* \Longrightarrow \mathrm{Id}_{\mathscr{M}^{\mathscr{I}}}$$

give the desired left colim-deformation. $\qquad\square$

Due to Theorem 4.5.22, it makes now sense (compare with [Shu], Theorem 5.4) to give the following

Definition 4.5.23. Let $\mathscr{M}$ be a model category and let $\mathscr{I}$ be a small category. Denote by $\mathrm{colim}_{\mathscr{I}}$ and $\lim_{\mathscr{I}}$ the colimit and the limit functors $\mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}$ respectively.

1. A *homotopy colimit (of shape $\mathscr{I}$) on $\mathscr{M}$* is a pair

$$(\text{hocolim}_{\mathscr{I}} := \text{colim}_{\mathscr{I}} Q, \ \text{colim}_{\mathscr{I}} q), \tag{4.18}$$

where (Q, q) is a left $\text{colim}_{\mathscr{I}}$-deformation of $\text{colim}_{\mathscr{I}}$. Given such a homotopy colimit, for $X \in \mathscr{M}^{\mathscr{I}}$, the object $\text{hocolim}\, X \in \mathscr{M}$ is called a *homotopy colimit of X*.

2. A *homotopy limit (of shape $\mathscr{I}$) on $\mathscr{M}$* is a pair

$$(\text{holim}_{\mathscr{I}} := \lim_{\mathscr{I}} R, \ \lim_{\mathscr{I}} r), \tag{4.19}$$

where (R, r) is a right $\lim_{\mathscr{I}}$-deformation of $\lim_{\mathscr{I}}$. Given such a homotopy limit, for $X \in \mathscr{M}^{\mathscr{I}}$, the object $\text{holim}\, X \in \mathscr{M}$ is called a *homotopy limit of X*.

Remark 4.5.24. Unsurprisingly, in the following we will usually refer to a homotopy colimit (of shape $\mathscr{I}$ on a model category $\mathscr{M}$) just as the functor $\text{hocolim}_{\mathscr{I}}$, leaving the companion natural transformation implicit. The same notational remark applies to homotopy limits.

Remark 4.5.25. From Definition 4.5.23, we get in particular that, for a small category $\mathscr{I}$ and a model category $\mathscr{M}$:

(i) $\text{hocolim}_{\mathscr{I}}$ is a homotopical functor from $\mathscr{M}^{\mathscr{I}}$ to $\mathscr{M}$;

(ii) $\text{hocolim}_{\mathscr{I}}$ comes equipped with a natural transformation $\text{hocolim}_{\mathscr{I}} \Longrightarrow \text{colim}_{\mathscr{I}}$;

(iii) $\text{hocolim}_{\mathscr{I}}$ is both a point-set left derived functor and a left approximation of $\text{colim}_{\mathscr{I}}$ (see Remark 4.5.20;

(iv) if $\text{hocolim}^1_{\mathscr{I}}$ and $\text{hocolim}^2_{\mathscr{I}}$ are two homotopy colimits of shape $\mathscr{I}$ on $\mathscr{M}$, then there is a zig-zag of natural weak equivalences *over* $\text{colim}_{\mathscr{I}}$ between them, i.e. $\text{hocolim}^1_{\mathscr{I}}$ and $\text{hocolim}^2_{\mathscr{I}}$ are weakly equivalent as objects in the overcategory $\left(\left(\mathscr{M}^{(\mathscr{M}^{\mathscr{I}})} \right)_W \downarrow \text{colim}_{\mathscr{I}} \right)$ (see Definition 4.5.18).

We also get the following result, which is proven in [DHKS], §20.2:

(v) there is an adjoint pair

$$\text{Ho}(\mathscr{M}^{\mathscr{I}}) \underset{c_{\mathscr{I}}}{\overset{\text{hocolim}_{\mathscr{I}}}{\rightleftarrows}} \perp \text{Ho}(\mathscr{M})$$

where $c_{\mathscr{I}}$ is the constant-diagram functor $c_{\mathscr{I}} \colon \mathscr{M} \longrightarrow \mathscr{M}^{\mathscr{I}}$ and it makes sense to consider $\text{hocolim}_{\mathscr{I}}$ and $c_{\mathscr{I}}$ at the level of the homotopy categories, as both of them are homotopical functors of homotopical categories (recall our convention on the homotopical structure on $\mathscr{M}^{\mathscr{I}}$ given in Remark 4.5.6).

Dual properties hold for $\text{holim}_{\mathscr{I}}$.

Remark 4.5.26. We defined homotopy colimits and limits for model categories in terms only of their underlying structure of homotopical categories. It follows in particular that, if $\underline{\mathscr{M}}$ and $\overline{\mathscr{M}}$ are model categories with the same underlying category and the same class of weak equivalences, any homotopy (co)limit on $\underline{\mathscr{M}}$ will be a homotopy (co)limit on $\overline{\mathscr{M}}$ and viceversa.

Left and right Quillen functors are homotopically compatible with homotopy colimits and homotopy limits respectively.

Proposition 4.5.27. *Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \perp \mathscr{N}$$

be a Quillen pair (see Definition 4.1.30). Denote by $\mathbb{L}F$ and by $\mathbb{R}G$ left and right point-set derived functors of F and of G respectively. Let also $\mathbb{L}(F \circ -)$ and $\mathbb{R}(G \circ -)$ be the left and right point-set derived functors associated to $\mathbb{L}F$ and to $\mathbb{R}G$ respectively (see Remark 4.5.21). For a small category $\mathscr{I}$, consider homotopy colimit and limit functors

$$\text{hocolim}^{\mathscr{M}}, \ \text{holim}^{\mathscr{M}} \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M} \quad and \quad \text{hocolim}^{\mathscr{N}}, \ \text{holim}^{\mathscr{N}} \colon \mathscr{N}^{\mathscr{I}} \longrightarrow \mathscr{N}$$

on $\mathscr{M}$ and on $\mathscr{N}$ respectively. Then:

1. *the functors*
$$\text{hocolim}^{\mathscr{N}} \circ \mathbb{L}(F \circ -) \quad and \quad \mathbb{L}F \circ \text{hocolim}^{\mathscr{M}}$$

are canonically weakly equivalent (see Definition 4.5.13) as homotopical functors over the canonically isomorphic functors
$$\text{colim}^{\mathscr{N}}(F \circ -) \xrightarrow{\cong} F \text{colim}^{\mathscr{M}} ;$$

2. *the functors*
$$\mathbb{R}G \circ \text{holim}^{\mathscr{N}} \quad and \quad \text{holim}^{\mathscr{M}} \circ \mathbb{R}(G \circ -)$$

are canonically weakly equivalent as homotopical functors over the canonically isomorphic functors

$$G \lim^{\mathscr{N}} \xrightarrow{\cong} \lim^{\mathscr{M}}(G \circ -) .$$

Proof. By [DHKS] §20.4, the composition of a homotopy colimit on $\mathscr{M}$ with a left approximation of F and the composition of a left approximation of $F \circ -$ with a homotopy colimit on $\mathscr{N}$ are both left approximations of the canonically isomorphic functors $\text{colim}^{\mathscr{N}}(F \circ -) \cong F \text{colim}^{\mathscr{M}}$ and dually for right approximations of G and homotopy limits in $\mathscr{M}$ and in $\mathscr{N}$. This gives us what required. $\square$

Remark 4.5.28. We will usually refer to the thesis of Proposition 4.5.27 by saying that the (point-set) left derived functor of a left Quillen functor *preserves homotopy colimits* and, dually, the (point-set) right derived functor of a right Quillen functor *preserves homotopy limits*. More generally, given a homotopical functor $L \colon \mathscr{M} \longrightarrow \mathscr{N}$ between model categories, we will say that L *preserves* (or *commutes with*) *homotopy colimits* if, for any small category $\mathscr{I}$, $L(\text{hocolim}_{\mathscr{I}})$ and $\text{hocolim}_{\mathscr{I}}(L)$ are canonically weakly equivalent as homotopical functors from $\mathscr{M}^{\mathscr{I}}$ to $\mathscr{N}$. Dualizing, we also get the notion of *preserving homotopy limits*.

When working with Quillen equivalences, we can improve Proposition 4.5.27 as follows.

Proposition 4.5.29. *Let $\mathscr{M}$ and $\mathscr{N}$ be model categories and let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \bot \; \mathscr{N}$$

be a Quillen equivalence (see Definition 4.1.36). Denote by $\mathbb{L}F$ and by $\mathbb{R}G$ left and right point-set derived functors of F and of G respectively. Let also $\mathbb{L}(F \circ -)$ and $\mathbb{R}(G \circ -)$ be the left and right point-set derived functors associated to $\mathbb{L}F$ and to $\mathbb{R}G$ respectively (see Remark 4.5.21). For a small category $\mathscr{I}$, consider homotopy colimit and limit functors

$$\text{hocolim}^{\mathscr{M}}, \text{holim}^{\mathscr{M}} \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M} \quad and \quad \text{hocolim}^{\mathscr{N}}, \text{holim}^{\mathscr{N}} \colon \mathscr{N}^{\mathscr{I}} \longrightarrow \mathscr{N}$$

on $\mathscr{M}$ and on $\mathscr{N}$ respectively. Then:

1. *the functors*
$$\text{hocolim}^{\mathscr{M}} \circ \mathbb{R}(G \circ -) \quad and \quad \mathbb{R}G \circ \text{hocolim}^{\mathscr{N}}$$

are weakly equivalent as homotopical functors $\mathscr{N}^{\mathscr{I}} \longrightarrow \mathscr{M}$;

2. *the functors*
$$\mathbb{L}F \circ \text{holim}^{\mathscr{M}} \quad and \quad \text{holim}^{\mathscr{N}} \circ \mathbb{L}(F \circ -)$$

are weakly equivalent as homotopical functors $\mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{N}$.

Proof. To avoid possible sources of confusions, we denote by $\mathbf{L}F$ and by $\mathbf{R}G$ the *total* left and right derived functors of F and of G, which are (by definition) the right and left Kan extensions of $\gamma_{\mathscr{N}} \circ \mathbb{L}F$ and of $\mathbb{R}G \circ \gamma_{\mathscr{M}}$ along $\gamma_{\mathscr{M}}$ and $\gamma_{\mathscr{N}}$ respectively. Here $\gamma_{\mathscr{M}} \colon \mathscr{M} \longrightarrow \text{Ho}(\mathscr{M})$ and $\gamma_{\mathscr{N}} \colon \mathscr{N} \longrightarrow \text{Ho}(\mathscr{N})$ are the localization functors. Similarly, we indicate by $\mathbb{L}(F \circ -)$ and by $\mathbb{R}(G \circ -)$ the *total* left and right derived functors of $F \circ -$ and of $G \circ -$ respectively.

Recall now that $\mathscr{M}_c$ (the full subcategory of $\mathscr{M}$ spanned by the cofibrant objects) is a left $F-$deformation retract and $\mathscr{N}_f$ (the full subcategory of $\mathscr{N}$ spanned by fibrant objects) is a right $G-$deformation retract (see Theorem 4.5.11). Thus, the condition for the Quillen pair (F, G) to be a Quillen equivalence (see

Definition 4.1.36) can be interpreted by saying that, for some (specific) left $F-$deformation retract $\mathscr{M}_0$ of $\mathscr{M}$ and for some (specific) right $G-$deformation retract $\mathscr{N}_0$ of $\mathscr{N}$, the following property, called the *Quillen condition*, holds: given any pair of objects $(X, A) \in \mathscr{M}_0 \times \mathscr{N}_0$, an arrow $FX \to A$ is a weak equivalence in $\mathscr{N}$ if and only if its adjunct $X \to GA$ is a weak equivalence in $\mathscr{M}$. With this formulation, the Quillen condition makes sense for any adjoint pair

$$\mathscr{H} \underset{G}{\overset{F}{\rightleftarrows}} \bot \; \mathscr{L}$$

between homotopical categories which is a *deformable adjunction*, i.e. it is such that F is left deformable and G is right deformable. Moreover, it can be proven (see [DHKS], §45) that:

- a deformable adjunction (F, G) as above satisfies the Quillen condition for *some* pair $(\mathscr{H}_0, \mathscr{L}_0)$, where $\mathscr{H}_0$ is a left $F-$deformation retract and $\mathscr{L}_0$ is a right $G-$deformation retract, if and only if it satisfies the Quillen condition for *any* such pair $(\mathscr{H}_0, \mathscr{L}_0)$;

- given a deformable adjunction (F, G), the total left derived functors of F and G gives rise to an adjoint pair

$$\mathrm{Ho}(\mathscr{H}) \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\rightleftarrows}} \bot \; \mathrm{Ho}(\mathscr{L})$$

which is an equivalence of categories if (F, G) satisfies the Quillen condition.

Now, if (F, G) is our starting Quillen equivalence, the adjoint pair

$$\mathscr{M}^{\mathscr{I}} \underset{G\circ-}{\overset{F\circ-}{\rightleftarrows}} \bot \; \mathscr{N}^{\mathscr{I}}$$

is deformable by Remark 4.5.21 and clearly satisfies the Quillen condition because (F, G) does so. Therefore, we have a (non-commutative) diagram of categories and functors

$$
\begin{array}{ccc}
\mathrm{Ho}(\mathscr{M}^{\mathscr{I}}) & \underset{\mathbf{R}(G\circ-)}{\overset{\mathbf{L}(F\circ-)}{\rightleftarrows}} \simeq & \mathrm{Ho}(\mathscr{N}^{\mathscr{I}}) \\
\Big\downarrow \mathrm{hocolim}^{\mathscr{M}} & & \Big\downarrow \mathrm{hocolim}^{\mathscr{N}} \\
\mathrm{Ho}(\mathscr{M}) & \underset{\mathbf{R}G}{\overset{\mathbf{L}F}{\rightleftarrows}} \simeq & \mathrm{Ho}(\mathscr{N})
\end{array}
$$

where it makes sense to consider $\mathrm{hocolim}^{\mathscr{M}}$ and $\mathrm{hocolim}^{\mathscr{N}}$ as functors between the homotopy categories of their domains and codomains, because both of them are homotopical functors by definition. Part 1. of Proposition 4.5.27 says that there is a (canonical) natural isomorphism

$$\mathrm{hocolim}^{\mathscr{N}} \circ \mathbf{L}(F \circ -) \cong \mathbf{L}F \circ \mathrm{hocolim}^{\mathscr{M}} .$$

It follows that also $\mathrm{hocolim}^{\mathscr{M}} \circ \mathbf{R}(G \circ -) \cong \mathbf{R}G \circ \mathrm{hocolim}^{\mathscr{N}}$, which gives the first part of the thesis (because the total right derived functor of G and of $\mathbb{R}G$ are isomorphic). The second part is dual.

$\square$

4.5.3 Computing homotopy (co)limits.

Although any two homotopy colimits of shape $\mathscr{I}$ on the model category $\mathscr{M}$ are naturally weakly equivalent, they may look quite different as functors $\mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}$. We present here some of these different features so as to point out that our definition of homotopy (co)limits comprises common and used constructions in homotopy theory that are sometimes referred to as homotopy (co)limits themselves.

Colimits and limits as Quillen functors. The first situation we consider is described by the following Corollary to Theorem 4.5.11.

Corollary 4.5.30. *Let $\mathscr{M}$ be a model category and let $\mathscr{I}$ be a small category such that the functor category $\mathscr{M}^{\mathscr{I}}$ admits a model category structure for which the colimit (resp. the limit) functor*

$$\mathrm{colim}\colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M} \quad (\textit{resp. } \lim\colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M})$$

is a left (resp. right) Quillen functor. Then, for any cofibrant replacement functor $Q\colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}^{\mathscr{I}}$ (resp. for any functorial fibrant replacement functor $R\colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}^{\mathscr{I}}$) on $\mathscr{M}^{\mathscr{I}}$, the composition

$$\mathrm{colim} \circ Q \quad (\textit{resp. } \lim \circ R)$$

is a homotopy colimit functor (resp. a homotopy lmit functor) of shape $\mathscr{I}$ on $\mathscr{M}$.

Remark 4.5.31. Here are the most common and relevant cases to which Corollary 4.5.30 applies:

(i) if $\mathscr{M}$ is a *cofibrantly generated* model category (see Definition 4.2.7), then, for all small categories $\mathscr{I}$, the colimit functor is a left Quillen functor for the projective model structure on $\mathscr{M}^{\mathscr{I}}$. If $\mathscr{M}$ is in additional a *combinatorial* model category (see Definition 4.2.10), then, for all small categories $\mathscr{I}$, the limit functor is a right Quillen functor for the injective model structure on $\mathscr{M}^{\mathscr{I}}$ (see Proposition 4.3.3 and Theorems 4.3.4 and 4.3.5);

(ii) if $\mathscr{I}$ is a Reedy category with fibrant constants (see Definitions 4.3.9 and 4.3.28), then for all model categories $\mathscr{M}$, the colimit functor is a left Quillen functor with respect to the Reedy model structure on $\mathscr{M}^{\mathscr{I}}$ (see Theorems 4.3.18 and 4.3.30). Dually, if $\mathscr{I}$ is a Reedy category with cofibrant constants, then for all model categories $\mathscr{M}$, the limit functor is a right Quillen functor with respect to the Reedy model structure on $\mathscr{M}^{\mathscr{I}}$.

A specific instance of (ii) in Remark 4.5.31 above is given by taking $\mathscr{I}$ to be either the pushout or the pullback category, endowed with one of the Reedy category structures of Examples 4.3.13 and 4.3.14 (see also Remark 4.3.12). Homotopy colimits and limits over these indexing categories deserve the obvious names

Definition 4.5.32. Let $\mathscr{I}$ be the pushout category $\mathscr{I} = b \leftarrow a \rightarrow c$ and let $\mathscr{M}$ be a model category.

1. A homotopy colimit of shape $\mathscr{I}$ on $\mathscr{M}$ is called a *homotopy pushout* on $\mathscr{M}$ and denoted as

$$(-) \amalg_{(?)}^{h} (\bullet)\colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}, \quad (X \rightarrow Y \leftarrow Z) \mapsto X \amalg_{Y}^{h} Z.$$

2. A homotopy limit of shape $\mathscr{J} := \mathscr{I}^{\mathrm{op}}$ on $\mathscr{M}$ is called a *homotopy pullback* on $\mathscr{M}$ and denoted as

$$(-) \times_{(?)}^{h} (\bullet)\colon \mathscr{M}^{\mathscr{J}} \longrightarrow \mathscr{M}, \quad (X \rightarrow Y \leftarrow Z) \mapsto X \times_{Y}^{h} Z.$$

For the following couple of results, we need to recall a piece of notation. Given a natural number n, we denote by $[n]$ the ordinal $n + 1$ regarded as a posetal category (i.e. as the category associated to the underlying poset of $n + 1$).

Proposition 4.5.33. *Let $\mathscr{M}$ be a model category and let $\mathscr{J}$ be the pullback category. Let also*

$$R\colon \mathscr{M} \longrightarrow \mathscr{M}$$

be a functorial fibrant replacement on $\mathscr{M}$ and let

$$\delta\colon \mathscr{M}^{[1]} \longrightarrow \mathscr{M}^{[2]}$$

be a functorial factorization of every arrow $f\colon X \rightarrow Z$ in $\mathscr{M}$ into a weak equivalence $w(f)\colon X \rightarrow E(f)$ followed by a fibration $p(f)\colon E(f) \rightarrow Z$, so that

$$\delta(f) = X \xrightarrow{w(f)} E(f) \xrightarrow{p(f)} Z.$$

Then the following statements hold.

(i) *The functor*

$$R' \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}^{\mathscr{I}}, \quad (X \xrightarrow{f} Z \xleftarrow{g} Y) \mapsto (E(Rf) \xrightarrow{p(Rf)} RZ \xleftarrow{p(Rg)} E(Rg))$$

is (part of) a right deformation for $\mathscr{M}^{\mathscr{I}}$ and the full subcategory of $\mathscr{M}^{\mathscr{I}}$ spanned by the image of R' is a right deformation retract of $\mathscr{M}^{\mathscr{I}}$ on which the pullback functor $\lim_{\mathscr{J}}$ is homotopical. Hence, $\lim_{\mathscr{J}} \circ R'$ is a homotopy pullback on $\mathscr{M}$.

(ii) *The functor*

$$R'' \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}^{\mathscr{I}}, \quad (X \xrightarrow{f} Z \xleftarrow{g} Y) \mapsto (RX \xrightarrow{Rf} RZ \xleftarrow{p(Rg)} E(Rg))$$

is (part of) a right deformation for $\mathscr{M}^{\mathscr{I}}$ and the full subcategory of $\mathscr{M}^{\mathscr{I}}$ spanned by the image of R'' is a right deformation retract of $\mathscr{M}^{\mathscr{I}}$ on which the pullback functor $\lim_{\mathscr{J}}$ is homotopical. Hence, $\lim_{\mathscr{J}} \circ R''$ is a homotopy pullback on $\mathscr{M}$.

A dual thesis holds for the pushout category $\mathscr{I}$, cofibrant replacement functors on $\mathscr{M}$ and functorial factorizations of a map in $\mathscr{M}$ into a cofibration followed by a weak equivalence.

Proof. By Example 4.3.26, R' and R'' are fibrant replacement functors for the Reedy model category structures on $\mathscr{M}^{\mathscr{I}}$ given by the dual of the Reedy structures of Example 4.3.13 and of Example 4.3.14 respectively. Since $\mathscr{J}$ has cofibrant constants (see Example 4.3.29), the result follows from Corollary 4.5.30 and Remark 4.5.31. □

Remark 4.5.34. In Definition 4.2.17, we already considered the notion of homotopy pullback for a cospan in a right proper model category $\mathscr{M}$, so that, a priori, there may be some ambiguity with Definition 4.5.32 above. However, keeping the same notations of Proposition 4.5.33, it is not difficult to see that the results in Section 4.2.2 imply that the functor

$$R''' \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}^{\mathscr{I}}$$

given by

$$R''' \colon (X \xrightarrow{f} Z \xleftarrow{g} Y) \mapsto (E(f) \xrightarrow{p(f)} Z \xleftarrow{p(g)} E(g))$$

is (part of) a right deformation for $\mathscr{M}^{\mathscr{I}}$ and the full subcategory of $\mathscr{M}^{\mathscr{I}}$ spanned by the image of R''' is a right deformation retract of $\mathscr{M}^{\mathscr{I}}$ on which the pullback functor $\lim_{\mathscr{J}}$ is homotopical. Hence, $\lim_{\mathscr{J}} \circ R'''$ is a homotopy pullback on $\mathscr{M}$, so that Definitions 4.2.17 and 4.5.32 are compatible. A dual result holds of course for homotopy pushouts in a left proper model category.

When dealing with right or left proper model categories, we can drastically simplify the description of homotopy pullbacks and pushouts given in Proposition 4.5.33.

Proposition 4.5.35. *Let $\mathscr{M}$ be a right proper model category and let $\mathscr{J}$ be the pullback category. Let also*

$$\delta \colon \mathscr{M}^{[1]} \longrightarrow \mathscr{M}^{[2]}$$

be a functorial factorization of every arrow $f \colon X \to Z$ in $\mathscr{M}$ into a weak equivalence $w(f) \colon X \to E(f)$ followed by a fibration $p(f) \colon E(f) \to Z$, so that

$$\delta(f) = X \xrightarrow{w(f)} E(f) \xrightarrow{p(f)} Z.$$

Then the functor

$$R \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}^{\mathscr{I}}, \quad (X \xrightarrow{f} Z \xleftarrow{g} Y) \mapsto (E(f) \xrightarrow{p(f)} Z \xleftarrow{g} Y)$$

is (part of) a right deformation for $\mathscr{M}^{\mathscr{I}}$ and the full subcategory of $\mathscr{M}^{\mathscr{I}}$ spanned by the image of R is a right deformation retract of $\mathscr{M}^{\mathscr{I}}$ on which the pullback functor $\lim_{\mathscr{J}}$ is homotopical. Hence, $\lim_{\mathscr{J}} \circ R$ is a homotopy pullback on $\mathscr{M}$. A dual thesis holds for left proper model categories, pushouts and functorial factorizations of a map in $\mathscr{M}$ into a cofibration followed by a weak equivalence.

Proof. Since R is built up using a functorial factorization of arrows in $\mathscr{M}$, using the two-out-of-three property of weak equivalences, it is easy to see that R is a homotopical functor. We have an obvious natural weak equivalence

$$r\colon \mathrm{Id}_{\mathscr{M}^{\mathscr{I}}} \Longrightarrow R, \quad r_{(X \xrightarrow{f} Z \xleftarrow{g} Y)} := (w(f),\ id_Z,\ id_Y)\colon (X \xrightarrow{f} Z \xleftarrow{g} Y) \to (E(f) \xrightarrow{p(f)} Z \xleftarrow{g} Y)$$

as in the commutative diagram

where $w(f)$ is a weak equivalence by hypothesis. Note that, up to now, we have not used the hypothesis of right properness for $\mathscr{M}$. We claim that the pullback functor $\lim_{\mathscr{J}}$ is homotopical on the full subcategory of $\mathscr{M}$ consisting of those cospans $A \to B \leftarrow C$ where $A \to B$ is a fibration. In particular, $\lim_{\mathscr{J}}$ would be homotopical on the full subcategory of $\mathscr{M}^{\mathscr{I}}$ spanned by the image of R as well. Suppose then given a solid commutative diagram in $\mathscr{M}$ as the one below

Form the pullbacks of the back and front cospans so as to get the following commutative cube in $\mathscr{M}$

By Proposition 4.2.21, the front and the back square in the cube above are homotopy Cartesian squares (see Definition 4.2.17) and therefore Proposition 4.2.20 implies that the dotted arrow h is a weak equivalence. $\qquad \square$

Definition 4.5.36. Let $\mathscr{M}$ be a model category and let $(-) \times_{(?)}^{h} (\bullet)$, $(-) \amalg_{(?)}^{h} (\bullet)$ be homotopy pullback and homotopy pushout functors on $\mathscr{M}$ respectively (see Definition 4.5.32). We say that a commutative square

is a *homotopy pullback square* (respectively a *homotopy pushout square*) if the composite map

$$A \to B \times_C D \to B \times_C^{h} D \quad (\text{respectively } B \amalg_A^{h} C \to B \amalg_A C \to D)$$

is a weak equivalence.

Remark 4.5.37. Since any two homotopy pullbacks (resp. any two homotopy pushouts) are weakly equivalent as functors under the ordinary pullback functor (resp. over the ordinary pushout functor), Definition 4.5.36 above is well-given, in the sense that a commutative square

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

in $\mathscr{M}$ is a homotopy pullback square (resp. a homotopy pushout square) if and only if the relevant map out of A (resp. into D) is a weak equivalence for *some* homotopy pullback functor $(-) \times_{(?)}^{h} (\bullet)$ (resp. for *some* homotopy pushout functor $(-) \amalg_{(?)}^{h} (\bullet)$) on $\mathscr{M}$.

As an immediate consequence of Remark 4.5.34 and of Proposition 4.5.35, we get the following

Corollary 4.5.38. *Let $\mathscr{M}$ be a right proper model category. Then a commutative square*

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

in $\mathscr{M}$ is a homotopy pullback square if and only if it is a homotopy Cartesian square (see Definition 4.2.17). A dual thesis holds for left model categories, homotopy pushout and homotopy cocartesian squares.

Homotopy (co)limits as (co)ends. We now turn to another possible description of homotopy colimits and limits in an arbitrary model category in terms of suitable (co)ends.

The first ingredient we need is a construction made on arbitrary model categories to mimic the tensors and the cotensors (see Definition 4.2.29) available in a simplicial model category (see Definition 4.2.32).

Definition 4.5.39. Let $\mathscr{M}$ be a model category.

1. Let $X \in \mathscr{M}^{\Delta}$ and $K \in \mathbf{sSet}$. We define the object $X \otimes K \in \mathscr{M}$ by

$$
X \otimes K := \mathrm{colim}_{([n],s) \in \Delta K} X([n]), \tag{4.20}
$$

   where ΔK is the category of simplices of K, i.e. the category $\mathrm{El}(K)$ of elements of K.

2. Let $Y \in \mathscr{M}^{\Delta^{\mathrm{op}}}$ and $K \in \mathbf{sSet}$. We define the object $Y^{K} \in \mathscr{M}$ by

$$
Y^{K} := \mathrm{lim}_{([n],s) \in (\Delta K)^{\mathrm{op}}} Y([n]). \tag{4.21}
$$

Remark 4.5.40. From the definitions, it is easy to see that, in the situation of Definition 4.5.39, we get bifunctors

$$
(-) \otimes (\bullet) \colon \mathscr{M}^{\Delta} \times \mathbf{sSet} \longrightarrow \mathscr{M} \quad \text{and} \quad (-)^{(\bullet)} \colon \mathscr{M}^{\Delta^{\mathrm{op}}} \times \mathbf{sSet}^{\mathrm{op}} \longrightarrow \mathscr{M}
$$

Note, for example, that, for $X \in \mathscr{M}^{\Delta}$ and $K \in \mathbf{sSet}$, $X \otimes K$ is simply $(\mathrm{Lan}_{\mathbf{y}} X)(K)$, where $\mathbf{y} \colon \Delta \longrightarrow \mathbf{sSet}$ is the Yoneda embedding (see Definition 2.2.14). Furthermore, for all $X \in \mathscr{M}^{\Delta}$ and all $Y \in \mathscr{M}^{\Delta^{\mathrm{op}}}$, we have adjoint pairs

$$
\begin{array}{ccc}
\mathbf{sSet} \xrightleftharpoons[\mathscr{M}(X,-)]{X \otimes (-)} \mathscr{M} & \quad \text{and} \quad & \mathscr{M} \xrightleftharpoons[Y^{(-)}]{\mathscr{M}(-,Y)} \mathbf{sSet}^{\mathrm{op}}
\end{array}
$$

Here $\mathscr{M}(X, -)$ is the functor sending $A \in \mathscr{M}$ to the simplicial set $\mathscr{M}(X, A)$ such that, for $n \in \mathbb{N}$, $\mathscr{M}(X, A)_{n} = \mathscr{M}(X[n], A)$ and similarly for $\mathscr{M}(-, Y)$ (compare with the analogous definitions given in Definition 4.4.12).

Recall now that in Section 4.4.1 we defined frames (see Definition 4.4.5) and framed model categories (see Definition 4.4.10). We also saw that every model category has a canonical structure of framed model category induced by its canonical functorial factorizations (see Definition 4.4.9).

Definition 4.5.41. Let $(\mathscr{M},\ (-)^*,\ (-)_*)$ be a framed model category and let $\mathscr{C}$ be a small category.

1. For any $X \in \mathscr{M}^{\mathscr{C}}$ and any $K \in \mathbf{sSet}^{\mathscr{C}^{\mathrm{op}}}$, we define the object $X^* \otimes_{\mathscr{C}} K \in \mathscr{M}$ by the coend

$$X \otimes_{\mathscr{C}}^* K := \int^{\alpha \in \mathscr{C}} (X(\alpha))^* \otimes K(\alpha) \tag{4.22}$$

2. For any $X \in \mathscr{M}^{\mathscr{C}}$ and any $K \in \mathbf{sSet}^{\mathscr{C}}$, we define the object $\mathrm{Hom}_*^{\mathscr{C}}(K, X) \in \mathscr{M}$ as the end

$$\mathrm{Hom}_*^{\mathscr{C}}(K, X) := \int_{\alpha \in \mathscr{C}} (X(\alpha))_*^{K(\alpha)} \tag{4.23}$$

Remark 4.5.42. As in Remark 4.5.40, in the situation of Definition 4.5.41, we get bifunctors

$$(-) \otimes_{\mathscr{C}}^* (\bullet) \colon \mathscr{M}^{\mathscr{C}} \times \mathbf{sSet}^{\mathscr{C}^{\mathrm{op}}} \longrightarrow \mathscr{M} \quad \text{and} \quad \mathrm{Hom}_*^{\mathscr{C}}(-, \bullet) \colon (\mathbf{sSet}^{\mathscr{C}})^{\mathrm{op}} \times \mathscr{M}^{\mathscr{C}} \longrightarrow \mathscr{M}$$

such that, for a fixed object $X \in \mathscr{M}^{\mathscr{C}}$, there are adjoint pairs

$$\mathbf{sSet}^{\mathscr{C}^{\mathrm{op}}} \underset{\mathscr{M}(X,-)}{\overset{X \otimes_{\mathscr{C}}^*(-)}{\rightleftarrows}} \perp \mathscr{M} \qquad \text{and} \qquad \mathscr{M} \underset{\mathrm{Hom}_*^{\mathscr{C}}(-,X)}{\overset{\mathscr{M}(-,X)}{\rightleftarrows}} \perp (\mathbf{sSet}^{\mathscr{C}})^{\mathrm{op}}$$

(see [Hir1], Proposition 19.2.13).

The following result says that we can think to $X \otimes_{\mathscr{C}}^* K$ and to $\mathrm{Hom}_*^{\mathscr{C}}(K, X)$ as generalizations of $\mathrm{colim}\, X$ and $\lim X$ respectively.

Proposition 4.5.43. *Let* $(\mathscr{M},\ (-)^*,\ (-)_*)$ *be a framed model category and let* $\mathscr{C}$ *be a small category. Denote by* $P \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ *and by* $P' \colon \mathscr{C} \longrightarrow \mathbf{sSet}$ *the constant functors at* $\Delta[0]$. *Then there are isomorphisms*

$$X \otimes_{\mathscr{C}}^* P \cong \mathrm{colim}\, X \quad \text{and} \quad \mathrm{Hom}_*^{\mathscr{C}}(P', X) \cong \lim X$$

which are natural in $X \in \mathscr{M}^{\mathscr{C}}$.

Proof. See [Hir1], Proposition 19.2.9. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We can now give the following

Definition 4.5.44. Let $(\mathscr{M},\ (-)^*,\ (-)_*)$ be a framed model category and let $\mathscr{C}$ be a small category. Denote by $Q \colon \mathscr{M} \longrightarrow \mathscr{M}$ and by $R \colon \mathscr{M} \longrightarrow \mathscr{M}$ the canonical cofibrant and fibrant replacement functors respectively (see Definition 4.1.14).

1. The *framed homotopy colimit (of shape $\mathscr{C}$) on $\mathscr{M}$ (with respect to the frame $((-)^*,\ (-)_*)$ on $\mathscr{M}$)* is the functor $\mathrm{hocolim}_{\mathscr{C}}^* \colon \mathscr{M}^{\mathscr{C}} \longrightarrow \mathscr{M}$ given, for all $X \in \mathscr{M}^{\mathscr{C}}$, by

$$\mathrm{hocolim}_{\mathscr{C}}^*(X) := (Q \circ X) \otimes_{\mathscr{C}}^* N(- \downarrow \mathscr{C})^{\mathrm{op}},$$

where $N(- \downarrow \mathscr{C})^{\mathrm{op}} \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ is the functor sending $\alpha \in \mathscr{C}$ to the nerve of the category $(\alpha \downarrow \mathscr{C})^{\mathrm{op}}$.

2. The *framed homotopy limit (of shape $\mathscr{C}$) on $\mathscr{M}$ (with respect to the frame $((-)^*,\ (-)_*)$ on $\mathscr{M}$)* is the functor $\mathrm{holim}_{*,\mathscr{C}} \colon \mathscr{M}^{\mathscr{C}} \longrightarrow \mathscr{M}$ given, for all $X \in \mathscr{M}^{\mathscr{C}}$, by

$$\mathrm{holim}_{*,\mathscr{C}}(X) := \mathrm{Hom}_*^{\mathscr{C}}(N(\mathscr{C} \downarrow -),\ R \circ X),$$

where $N(\mathscr{C} \downarrow -) \colon \mathscr{C} \longrightarrow \mathbf{sSet}$ is the functor sending $\alpha \in \mathscr{C}$ to the nerve of the category $(\mathscr{C} \downarrow \alpha)$.

Remark 4.5.45. Our Definition 4.5.44 above differs from the definitions of homotopy colimits and limits given in [Hir1], §19.1 by the fact that the author there does not utilize any functorial cofibrant and fibrant replacement (and does not use the attribute *framed* for homotopy (co)limits either). In other words, retaining the same notations of Definition 4.5.44, if $\mathrm{hocolim}_{\mathscr{C}}^{H}(X)$ denotes the homotopy colimits in the sense of [Hir1] for $X\colon \mathscr{C} \longrightarrow \mathscr{M}$, then $\mathrm{hocolim}_{\mathscr{C}}^{*}(X) = \mathrm{hocolim}_{\mathscr{C}}^{H}(QX)$. We needed to make this cofibrant correction, as $\mathrm{hocolim}_{\mathscr{C}}^{H}$ is not a homotopical functor on $\mathscr{M}^{\mathscr{C}}$, but only on $(\mathscr{M}_c)^{\mathscr{C}}$ (see [Hir1], Theorem 19.4.2), so that there would have been no hope to show that a framed homotopy colimit is indeed a homotopy colimit (in the sense of Definition 4.5.23) if we had to follow [Hir1]. As a consequence of our choice, the results about $\mathrm{hocolim}_{\mathscr{C}}^{H}$ that in [Hir1] are claimed to be true for pointwise cofibrant diagrams on $\mathscr{M}$ (or for maps between such diagrams) hold true for our $\mathrm{hocolim}_{\mathscr{C}}^{*}$ without such an assumption. Dual remarks apply to framed homotopy limits with respect to Hirschhorn's definition.

Remark 4.5.46. Let $(\mathscr{M}, (-)^{*}, (-)_{*})$ be a framed model category and let $\mathscr{C}$ be a small category. By Proposition 4.5.43, if $P\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ is the constant functor at $\Delta[0]$, the unique map $N(-\downarrow\mathscr{C})^{\mathrm{op}} \Longrightarrow P$ induces a map

$$X \otimes_{\mathscr{C}}^{*} N(-\downarrow\mathscr{C})^{\mathrm{op}} \Longrightarrow \mathrm{colim}_{\mathscr{C}} X$$

natural in $X \in \mathscr{M}^{\mathscr{C}}$. If Q is the canonical functorial cofibrant replacement on $\mathscr{M}$, the natural transformation $Q \circ (-) \Longrightarrow \mathrm{Id}_{\mathscr{M}^{\mathscr{C}}}$ induces then a map of functors $\mathrm{hocolim}_{\mathscr{C}}^{*}(\bullet) \Longrightarrow (\bullet) \otimes_{\mathscr{C}}^{*} N(-\downarrow\mathscr{C})^{\mathrm{op}}$. All in all, we then get a map

$$\mathrm{hocolim}_{\mathscr{C}}^{*}(X) \to \mathrm{colim}_{\mathscr{C}} X \tag{4.24}$$

which is natural in $X \in \mathscr{M}^{\mathscr{C}}$, i.e. $\mathrm{hocolim}_{\mathscr{C}}^{*}$ is a functor over $\mathrm{colim}_{\mathscr{C}}$. Similarly, we get a map

$$\lim_{\mathscr{C}} X \to \mathrm{holim}_{*,\mathscr{C}} X \tag{4.25}$$

again natural in $X \in \mathscr{M}^{\mathscr{C}}$.

The following result says that the specific choice of the frame in the definition of framed homotopy (co)limits is unessential.

Proposition 4.5.47. *Let $\mathscr{M}' = (\mathscr{M}, (-)^{*}, (-)_{*})$ and $\mathscr{M}'' = (\mathscr{M}, (-)^{\bullet}, (-)_{\bullet})$ be framed model categories with the same underlying model category $\mathscr{M}$.*

1. *For any small category $\mathscr{C}$, there is an essentially unique zig-zag of natural weak equivalences between $\mathrm{hocolim}_{\mathscr{C}}^{*}$ and $\mathrm{hocolim}_{\mathscr{C}}^{\bullet}$.*

2. *For any small category $\mathscr{C}$, there is an essentially unique zig-zag of natural weak equivalences between $\mathrm{holim}_{*,\mathscr{C}}$ and $\mathrm{holim}_{\bullet,\mathscr{C}}$.*

Proof. Keeping Remark 4.5.45 in mind, this is Theorem 19.4.3 of [Hir1]. $\square$

Remark 4.5.48. In view of Proposition 4.5.47, we will just write $\mathrm{hocolim}_{\mathscr{C}}^{*}$ and $\mathrm{holim}_{*,\mathscr{C}}$ and refer to them as framed homotopy colimit and framed homotopy limit on $\mathscr{M}$, without explicitly mentioning the cosimplicial and the simplicial frames involved.

Remark 4.5.49. Let $\mathscr{M}$ be a simplicial model category with tensor functor $(-)\otimes(?)$ and cotensor functor $(-)^{(\bullet)}$ (see Remark 4.2.25, Definition 4.2.29 and Definition 4.2.32). Consider the standard cosimplicial and simplicial frames $(-)^{*}$ and $(-)_{*}$ on $\mathscr{M}$ induced by the simplicial structure (see Example 4.4.11). It can then be proven (see Proposition 16.6.6 of [Hir1]) that, for all $A \in \mathscr{M}$ and all $K \in \mathbf{sSet}$, there are natural isomorphisms

$$A^{*} \otimes K \cong A \otimes K \quad \text{and} \quad A_{*}^{K} \cong A^{K},$$

where the left hand sides are given as in Definition 4.5.39. Therefore, for a small category $\mathscr{C}$ and a functor $X\colon \mathscr{C} \longrightarrow \mathscr{M}$, we can write

$$\mathrm{hocolim}_{\mathscr{C}}^{*}(X) = \int^{\alpha\in\mathscr{C}} QX(\alpha) \otimes N(\alpha\downarrow\mathscr{C})^{\mathrm{op}} \quad \text{and} \quad \mathrm{holim}_{*,\mathscr{C}}(X) = \int_{\alpha\in\mathscr{C}} (RX(\alpha))^{N(\mathscr{C}\downarrow\alpha)},$$

where Q and R are the canonical cofibrant and fibrant approximations respectively.

By Remarks 4.5.45 and 4.5.46, framed homotopy colimits are homotopical functors over the colimit functor and dually for framed homotopy limits. Therefore, it makes sense to ask whether they are homotopy colimits and limits in the sense of Definition 4.5.23. Indeed, we have the following

Theorem 4.5.50. *Every framed homotopy (co)limit is a homotopy (co)limit.*

The above result is elegantly proven in [Shu], §8 (see, in particular Theorem 8.5) and Appendix A, for the simplicial case, i.e. for framed homotopy (co)limits in a simplicial model category (with respect to the standard frames on it). However, using the theory of frames as well as the properties of the functors $(-) \otimes^*_{\mathscr{C}} (\bullet)$ and $\mathrm{Hom}_{*,\mathscr{C}}$ (see [Hir1], §19.2 and 19.3), one can rephrase the whole proof of Theorem 4.5.50 given in [Shu] to adapt it to general framed homotopy (co)limits, as remarked by the author himself (see again [Shu], Appendix A).

We conclude this section by stating the homotopical version of the fact that covariant and contravariant Hom-functors for an ordinary category $\mathscr{D}$ preserve limits existing in $\mathscr{D}$ or in $\mathscr{D}^{\mathrm{op}}$ respectively.

Proposition 4.5.51. *Let $\mathscr{M}$ be a model category and let $\mathrm{map}^{\mathrm{h}}(-,?) \colon \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathbf{sSet}$ be a homotopy function complex on $\mathscr{M}$ (see Definition 4.4.15). Given a small category $\mathscr{C}$, consider a functor $X \colon \mathscr{C} \longrightarrow \mathscr{M}$ and an object $A \in \mathscr{M}$. Then the following hold.*

1. *The simplicial sets*
$$\mathrm{map}^{\mathrm{h}}(\mathrm{hocolim}_{\mathscr{C}} X, A) \quad and \quad \mathrm{holim}_{\mathscr{C}} \, \mathrm{map}^{\mathrm{h}}(X, A)$$
   are naturally weakly equivalent.

2. *The simplicial sets*
$$\mathrm{map}^{\mathrm{h}}(A, \mathrm{holim}_{\mathscr{C}} X) \quad and \quad \mathrm{holim}_{\mathscr{C}} \, \mathrm{map}^{\mathrm{h}}(A, X)$$
   are naturally weakly equivalent.

Proof. Since any two homotopy (co)limits are naturally weakly equivalent, by Proposition 4.4.18 and Theorem 4.5.50 it is enough to prove the claim for framed homotopy (co)limits. Noticing that we can always assume, up to a weak equivalence, that the object $A \in \mathscr{M}$ is fibrant (for the first part) or cofibrant (for the second part), the thesis for framed homotopy (co)limits is [Hir1], Theorem 19.4.4 (see also Remark 4.5.45). $\square$

Chapter 5

Localizations and Small Presentations of Model Categories.

Da v tselom mire ne otyshchete vy podobnogo naslazhden'ia! Zdes', imenno zdes' podrazhaet Bogu chelovek. Bog predostavil sebe delo tvoren'ia, kak vysshee vsekh naslazhden'e, i trebuet ot cheloveka takzhe, chtoby on byl podobnym tvortsom blagodenstviia vokrug sebia. I eto nazyvaiut skuchnym delom!

Nikolai V. Gogol', *Mjortvyje duši.*

Having all the needed model categorical notions at our disposal, we can now start homotopifying some of the constructions and the results that we saw in Chapters 1 and 2. A general *leitmotiv* to produce homotopical versions[1] of notions and results in ordinary Category Theory consists of replacing every occurence of the category **Set** of sets with the category **sSet** of simplicial sets[2] and of "homotopically correcting" categorical concepts, for example substituting colimits with homotopy colimits[3] or left adjoints with point-set and total left derived functors of left Quillen functors (see Remark 4.1.33). So, for instance, the role played by presheaves categories in our theory of Grothendieck topoi should be now played by simplicial presheaves in the theory of model topoi. In fact, following [Dug1] we will determine in which sense, for a small category $\mathscr{C}$, the category sPsh($\mathscr{C}$) of simplicial presheaves on $\mathscr{C}$ is the *free homotopical cocompletion* of $\mathscr{C}$ in analogy to the well-known fact that PSh($\mathscr{C}$) is the free cocompletion of $\mathscr{C}$. We shall also define what a *(homotopical) small presentation* for a model category is (see Definition 1.2.5 for the corresponding notion in ordinary Category Theory) and we shall see that every combinatorial model category admits such a small presentation, thus providing the model theoretical analogue of Theorem 1.2.6. In order to do this, we will need to understand what ought to be a meaningful notion of *localization* for model categories and get some basic properties of it. This is accomplished through the theory of *(left) Bousfield localizations*, which we will explain in the first section below, quoting [Hir1] for the most important theorems.

This chapter is an expanded version of Section 5 in [Rzk1]. Our Theorem 5.3.8 below proves a claim made by Rezk about particularly nice small simplicial presentations for simplicial model categories. Such a statement appears in the proof of Theorem 6.9 in [Rzk1] but is not explicitly shown to be true there.

5.1 Localizing Model Categories.

5.1.1 Left localizations and S-local equivalences.

Localizing a category to a class of maps means finding a universal category where those maps become isomorphisms, through the action of a suitable localization functor. Localizing a *model* category to a class

---

[1] Here *homotopical* really means *model-theoretical.*
[2] The "add an s" rule.
[3] The "add an h" rule.

of maps should then mean finding a universal model category where those maps become *weak equivalences*, again through the action of an appropriate localization functor between model categories. However, a priori such a localization functor may be either a left Quillen or a right Quillen functor and we thus get the corresponding distinct (but dual) notions of *left localization* and *right localization*. Since we will only need the former one, we refrain from developing the theory of right localizations and focus only on the left version.

Definition 5.1.1. Let $\mathscr{M}$ be a model category and let S be a class of morphisms in $\mathscr{M}$. A *left localization of $\mathscr{M}$ with respect to S*, if it exists, is a couple

$$(\mathscr{M}_S, \; j\colon \mathscr{M} \longrightarrow \mathscr{M}_S) \tag{5.1}$$

where $\mathscr{M}_S$ is a model category and j is a left Quillen functor such that the total left derived functor $\mathbb{L}j\colon \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{M}_S)$ sends the images of elements of S in $\mathrm{Ho}(\mathscr{M})$ to isomorphisms in $\mathrm{Ho}(\mathscr{M}_S)$. Such a pair $(\mathscr{M}_S, \; j)$ is also required to be initial among those with the same property.

Clearly, if it exists, a left localization of $\mathscr{M}$ with respect to S is unique up to within a unique isomorphism (of pairs such as (5.1))

Remark 5.1.2. As mentioned above, we will only consider left localizations of model categories, hence, from now on, we will drop the directional adjective "left" in our exposition and just talk about a *localization of a model category $\mathscr{M}$ with respect to* a class $S \subseteq \mathrm{Mor}(\mathscr{M})$. Although this may result in a potential ambiguity with the notion of localization of an ordinary category (see Definition 1.2.2), the only categorical localization we are interested in when dealing with model categories is the homotopy category, to which we already reserved specific name and notation. Hence, in the following, all localizations of a model category $\mathscr{M}$ will be understood in the sense of Definition 5.1 above. We will also often use the phrase *"localization of $\mathscr{M}$ with respect to S"* to refer to just one of the two components of $(\mathscr{M}_S, j)$, leaving the other one implicit.

Note that, trivially, if S coincides with the class $\mathcal{W}$ of weak equivalences of a model category $\mathscr{M}$, the localization of $\mathscr{M}$ with respect to S exists and is given by $\mathrm{Id}_{\mathscr{M}}\colon \mathscr{M} \longrightarrow \mathscr{M}$. We will see in a while under which hypotheses on $\mathscr{M}$ (and on S) a localization $\mathscr{M}_S$ is guaranteed to exist.

To describe localizations and their properties, we need to consider the following notions, which is a generalization to model categories of Definition 1.2.4.

Definition 5.1.3. Let $\mathscr{M}$ be a model category and let S be a class of morphisms in $\mathscr{M}$. Let also $\mathrm{map}^{\mathrm{h}}\colon \mathscr{M}^{\mathrm{op}} \times \mathscr{M} \longrightarrow \mathbf{sSet}$ be a homotopy function complex on $\mathscr{M}$ (see Definition 4.4.15).

1. An object W of $\mathscr{M}$ is called *S-local* if it is fibrant in $\mathscr{M}$ and for every map $f\colon A \to B$ in S the induced map of homotopy function complexes

$$f^*\colon \mathrm{map}^{\mathrm{h}}(B, W) \to \mathrm{map}^{\mathrm{h}}(A, W)$$

is a weak equivalence.

2. A map $g\colon X \to Y$ in $\mathscr{M}$ is called an *$S-$local equivalence* if for every $S-$local object W the induced map of homotopy function complexes

$$g^*\colon \mathrm{map}^{\mathrm{h}}(Y, W) \to \mathrm{map}^{\mathrm{h}}(X, W)$$

is a weak equivalence.

Remark 5.1.4. Let $\mathscr{M}$ be a model category and S a class of maps in $\mathscr{M}$.

(i) As in the case of localizations, a priori some conflicts between Definition 5.1.3 and Definition 1.2.4 may arise. However, as in Remark 5.1.2, when dealing with model categories S-local objects will be always understood in the sense of Definition 5.1.3 above, unless differently stated.

(ii) By Theorem 4.4.14, the induced maps f^* and g^* that appear in Definition 5.1.3 above are weak equivalences for one choice of homotopy function complex on $\mathscr{M}$ if and only if they are weak equivalences for any such a choice. Thus the notions of S-local object and of S-local weak equivalence are independent of the chosen mapping space $\mathrm{map}^{\mathrm{h}}$. This kind of remark will apply to all the notions and the results that we will give and state in terms of homotopy function complexes.

(iii) From the definitions, it follows immediately that every element of S is an $S-$local equivalence.

Remark 5.1.5. Let S be a class of maps in a model category $\mathscr{M}$. Say that an object W in $\mathscr{M}$ is *quasi S-local* if, for every map $f \in S$, the induced map of homotopy function complexes

$$f^* \colon \operatorname{map}^{\mathrm{h}}(\operatorname{cod}(f), W) \to \operatorname{map}^{\mathrm{h}}(\operatorname{dom}(f), W)$$

is a weak equivalence (for one and hence for any homotopy function complex $\operatorname{map}^{\mathrm{h}}(-,?)$ on $\mathscr{M}$). In other words, an $S-$local object is a fibrant quasi $S-$local object and an object $W \in \mathscr{M}$ is quasi $S-$local if and only if any of its fibrant approximations is an $S-$local object. In particular, a map g in $\mathscr{M}$ is a quasi $S-$local equivalence (with the obvious meaning) if and only if it is an $S-$local equivalence. It follows that adding or dropping the fibrancy request in the definition of an $S-$local object does not result in any gain or loss of generality. However, we decided to follow the standard convention in the literature (see, for example, [Hir1] or [Low]) and insisted on asking an $S-$local object to be fibrant to start with.

Remark 5.1.6. For a model category $\mathscr{M}$ and a class S of maps in $\mathscr{M}$, being an $S-$local object is a weakly homotopy invariant property for fibrant objects. Namely, if X and Y are weakly equivalent fibrant objects, then X is $S-$local if an only if Y is $S-$local. Indeed, up to taking fibrant replacements of all the weak equivalences in a zig-zag connecting X and Y, we can suppose that X and Y are weakly equivalent through fibrant objects. Hence, it is enough to show that, if X and Y are fibrant objects and $g \colon X \to Y$ is a weak equivalence, then X is $S-$local if and only if Y is such. But this follows immediately from the fact that, for every $f \colon A \to B$ and any homotopy function complex on $\mathscr{M}$, in the commutative square

$$
\begin{array}{ccc}
\operatorname{map}^{\mathrm{h}}(B, X) & \xrightarrow{\operatorname{map}^{\mathrm{h}}(f, 1_X)} & \operatorname{map}^{\mathrm{h}}(A, X) \\
\downarrow & & \downarrow \\
\operatorname{map}^{\mathrm{h}}(B, Y) & \xrightarrow[\operatorname{map}^{\mathrm{h}}(f, 1_Y)]{} & \operatorname{map}^{\mathrm{h}}(A, Y)
\end{array}
$$

the vertical arrows are weak equivalences by Theorem 4.4.18.

Using $S-$local objects and $S-$local equivalences, we can describe when the total left derived functor of a left Quillen functor $F \colon \mathscr{M} \longrightarrow \mathscr{N}$ sends the image of an element of S in $\operatorname{Ho}(\mathscr{M})$ to an isomorphism in $\operatorname{Ho}(\mathscr{N})$ and we can do this just by examining F itself.

Theorem 5.1.7. *Let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{N} \quad \perp$$

be a Quillen pair between model categories $\mathscr{M}$ and $\mathscr{N}$ and let S be a class of maps in $\mathscr{M}$. Then the following statements are equivalent.

1. *The total left derived functor $\mathbb{L}F \colon \operatorname{Ho}(\mathscr{M}) \longrightarrow \operatorname{Ho}(\mathscr{N})$ of F takes the images in $\operatorname{Ho}(\mathscr{M})$ of elements of S into isomorphisms in $\operatorname{Ho}(\mathscr{N})$.*

2. *The left adjoint F takes cofibrant approximations to an element of S into a weak equivalence in $\mathscr{N}$.[4]*

3. *The right adjoint G takes every fibrant object of $\mathscr{N}$ into an $S-$local object of $\mathscr{M}$.*

4. *The left adjoint F takes every S-local equivalence between cofibrant objects into a weak equivalence in $\mathscr{N}$.*

Proof. The equivalence between 1. and 2. follows from the explicit description of the total left derived functor of F given in the proof of Theorem 4.1.32, together with the fact that a left Quillen functor F takes one cofibrant approximation Qg to a map g into a weak equivalence in $\mathscr{M}$ if and only if it takes any cofibrant approximation to g into a weak equivalence in $\mathscr{M}$ (see [Hir1], proposition 8.1.24).

---

[4] A *cofibrant approximation* of a map $f \colon X \to Y$ in $\mathscr{M}$ is of course a map $\tilde{f} \colon QX \to QY$ between cofibrant approximations $i_X \colon QX \to X$ and $i_Y \colon QY \to Y$ of X and Y respectively such that $f \circ i_X = i_Y \circ \tilde{f}$.

To show that 2. is equivalent to 3., let $f\colon A \to B$ be a morphism in S and take a cosimplicial resolution $\widetilde{f}\colon \widetilde{A} \to \widetilde{B}$ of f, i.e. consider a Reedy cofibrant approximation to the map $cc_* f$ in $\mathscr{M}^\Delta$, where $cc_*\colon \mathscr{M} \longrightarrow \mathscr{M}^\Delta$ is the constant diagram. Since Reedy cofibrations are pointwise such, it follows that $\widetilde{f}_0\colon \widetilde{A}_0 \to \widetilde{B}_0$ is a cofibrant approximation to f. Furthermore, since F lifts to a left Quillen functor on $\mathscr{M}^\Delta$, $F(\widetilde{f})\colon F(\widetilde{A}) \to F(\widetilde{B})$ is a cosimplicial resolution of $F(\widetilde{f}_0)$. Now, by Theorem 4.4.18, we have that $F(\widetilde{f}_0)$ is a weak equivalence in $\mathscr{N}$ if and only if, for all fibrant objects Z in $\mathscr{N}$, the map of simplicial sets $\mathscr{N}(F(\widetilde{B}), Z) \to \mathscr{N}(F(\widetilde{A}), Z)$ is a weak equivalence. This, in turn, holds if and only if $\mathscr{M}(\widetilde{B}, GZ) \to \mathscr{M}(\widetilde{A}, GZ)$ is a weak equivalence. By what we have already remarked, if F takes at least one cofibrant approximation to f into a weak equivalence, it sends all such cofibrant approximations to f into weak equivalences, hence we get the equivalence between 2. and 3.

The proof that 3. and 4. are equivalent is similar. $\qquad\square$

The class of $S-$local equivalences has the same formal properties shared by the class of weak equivalences in a model category.

Theorem 5.1.8 (cf. [Rzk1], §5.3)**.** *Let $\mathscr{M}$ be a model category and let S be a class of maps in $\mathscr{M}$. Then the following statements hold.*

(i) *Every weak equivalence in $\mathscr{M}$ is an $S-$local equivalence.*

(ii) *The class of $S-$local equivalences satisfies the two-out-of-three property (see Definition 4.1.1).*

(iii) *The class of $S-$local equivalences is closed under retracts (in the arrow category of $\mathscr{M}$).*

(iv) *Suppose given a homotopy pushout square*

$$
\begin{array}{ccc}
X & \xrightarrow{\ t\ } & X' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
Y & \longrightarrow & Y'
\end{array}
$$

in $\mathscr{M}$ (see Definition 4.5.36) where f is an $S-$local equivalence. Then g is an $S-$local equivalence.

(v) *Let $\tau\colon X \Longrightarrow Y$ be a natural transformation between functors $X, Y\colon \mathscr{J} \longrightarrow \mathscr{M}$ and assume that τ_j is an $S-$local equivalence for all $j \in \mathscr{J}$. Then $\operatorname{hocolim}\tau$ is an $S-$local equivalence as well.*

Remark 5.1.9. Since any two homotopy colimits on $\mathscr{M}$ are naturally weakly equivalent (see Remark 4.5.25), the two-out-of-three property for $S-$local equivalences implies that (v) in Theorem 5.1.8 above holds for one homotopy colimit if and only if it holds for any homotopy colimit.

Proof. (Of Theorem 5.1.8). The first part follows from Theorem 4.4.18, whereas (ii) and (iii) can be deduced by applying a functorial homotopy function complex map$^{\mathrm{h}}$ on $\mathscr{M}$ and using the two-out-of-three property and the closure under retracts for weak equivalences of simplicial sets.

For the fourth point, given a homotopy pushout square as in the statement of the Theorem, let $\mathscr{I} = b \leftarrow a \to c$ be the pushout category with the direct category structure of Example 4.3.13. Let Q be a functorial cofibrant approximation in the Reedy model structure of $\mathscr{M}^{\mathscr{I}}$ and let

$$
QY \xleftarrow{\ Qf\ } QX \xrightarrow{\ Qt\ } QX'
$$

be the result of applying Q to the span $Y \xleftarrow{\ f\ } X \xrightarrow{\ t\ } X'$ of $\mathscr{M}$. By Example 4.3.25, QY, QX and QX' are cofibrant objects in $\mathscr{M}$, while Qf and Qt are cofibrations in $\mathscr{M}$. Consider then the following solid diagram in $\mathscr{M}$

$$
\begin{array}{ccc}
& QX \longrightarrow QX' & \\
X \longrightarrow X' & & \\
f\big\downarrow \quad QY \longrightarrow P & & \\
Y \longrightarrow Y' & h &
\end{array}
$$

where $P := QY \coprod_{QX} QX'$ is cofibrant, $X' \to Y'$ is the given g and the arrows marked as weak equivalences are such by definition of Q and of the weak equivalences in $(\mathcal{M}^{\mathscr{I}})_{\text{Reedy}}$. Note that, since f is an $S-$local equivalence, by (i) and (ii) above, $Qf \colon QX \to QY$ is also an $S-$local equivalence. As $\mathscr{I}$ has fibrant constants (see Example 4.3.29), Corollary 4.5.30 and Remark 4.5.31 imply that P is a homotopy pushout for the span (f, t) and then by hypothesis we have that the dotted h is a weak equivalence. Hence, again by parts (i) and (ii), the above diagram shows that $g \colon X' \to Y'$ is an $S-$local equivalence if an only if $QX' \to P$ is such. We will thus prove this second claim. Let then W be an $S-$local object and pick a simplicial resolution $\widehat{W}$ of W. We get a pullback square

$$
\begin{array}{ccc}
\mathcal{M}(P, \widehat{W}) & \longrightarrow & \mathcal{M}(QX', \widehat{W}) \\
\downarrow & & \downarrow {\scriptstyle (Qt)^*} \\
\mathcal{M}(QY, \widehat{W}) & \xrightarrow{(Qf)^*} & \mathcal{M}(QX, \widehat{W})
\end{array}
\tag{5.2}
$$

of simplicial sets, where all objects are homotopy function complexes and $(Qf)^*$, $(Qt)^*$ are fibrations because Qf and Qt are cofibrations and W is fibrant in $\mathcal{M}$ (see [Hov], Corollary 5.4.4). Since Qf is an $S-$local equivalence we have that $(Qf)^*$ is a weak equivalence and therefore

$$\mathcal{M}(P, \widehat{W}) \to \mathcal{M}(QX', \widehat{W})$$

is a weak equivalence, because it is the pullback of a weak equivalence along a fibration $((Qt)^*)$ in the right proper model category $\mathbf{sSet}$. Since W was chosen as an arbitrary $S-$local object, this shows that $QX' \to P$ is an $S-$local equivalence and so also g needs to be such, as we already observed.

Finally, for (v), take a homotopy function complex $\mathrm{map}^{\mathrm{h}}$ on $\mathcal{M}$. Note that, if W is an $S-$local object, since $\mathrm{map}^{\mathrm{h}}(\mathrm{hocolim}_{i \in \mathscr{I}} X(i), W)$ is naturally weakly equivalent to $\mathrm{holim}_{i \in \mathscr{I}} \mathrm{map}^{\mathrm{h}}(X(i), W)$ (see Proposition 4.5.51), the map

$$(\mathrm{hocolim}_{i \in \mathscr{I}} \tau_i)^* \colon \mathrm{map}^{\mathrm{h}}(\mathrm{hocolim}_{i \in \mathscr{I}} Y(i), W) \to \mathrm{map}^{\mathrm{h}}(\mathrm{hocolim}_{i \in \mathscr{I}} X(i), W)$$

is naturally weakly equivalent to the map

$$\mathrm{holim}_{i \in \mathscr{I}} \tau_i^* \colon \mathrm{holim}_{i \in \mathscr{I}} \mathrm{map}^{\mathrm{h}}(Y(i), W) \to \mathrm{holim}_{i \in \mathscr{I}} \mathrm{map}^{\mathrm{h}}(X(i), W)$$

which is a weak equivalence, because each τ_i^* is a weak equivalence by hypothesis. $\qquad\square$

The following result gives a sufficient condition for an $S-$local equivalence to be an actual weak equivalence in $\mathcal{M}$.

Proposition 5.1.10. *Let $\mathcal{M}$ be a model category and S a class of maps in $\mathcal{M}$. Then any $S-$local weak equivalence between $S-$local objects is a weak equivalence in $\mathcal{M}$.*

Proof. Let $g \colon X \to Y$ be an $S-$local weak equivalence between $S-$local objects X and Y and let $\mathrm{map}^{\mathrm{h}}(-, ?)$ be a homotopy function complex on $\mathcal{M}$. Then the maps

$$g^* \colon \mathrm{map}^{\mathrm{h}}(Y, X) \to \mathrm{map}^{\mathrm{h}}(X, X) \quad \text{and} \quad g^* \colon \mathrm{map}^{\mathrm{h}}(Y, Y) \to \mathrm{map}^{\mathrm{h}}(X, Y)$$

are both weak equivalences of simplicial sets. By Proposition 17.7.6 of [Hir1], this is enough to conclude that g is a weak equivalence. $\qquad\square$

Definition 5.1.11. Let $\mathcal{M}$ be a model category and let S be a class of morphisms in $\mathcal{M}$.

1. An *S-localization* of an object $X \in \mathcal{M}$ is a couple $(R_S X, \ j \colon X \to R_S X)$ where $R_S X$ is an $S-$local object and j is an $S-$local equivalence.

2. An *S-localization* of a map $g \colon X \to Y$ is a triple

$$((R_S X, \ j_X \colon X \to R_S X), \ (R_S Y, \ j_Y \colon Y \to R_S Y), \ R_S g \colon R_S X \to R_S Y),$$

where $(R_S X, \ j_X \colon X \to R_S X)$ and $(R_S Y, \ j_Y \colon Y \to R_S Y)$ are $S-$localizations of X and Y respectively and $R_S g$ is a morphism in $\mathcal{M}$ such that $(R_S g) \circ j_X = j_Y \circ g$.

As usual, we will refer to $S-$localizations $(R_S X, \ j\colon X \to R_S X)$ of an object X mentioning only the $S-$local object $R_S X$ or the $S-$local equivalence j, leaving the other ingredient implicit. A similar remark applies to $S-$localizations of maps.

The existence of some kind of localizations for certain classes of model categories will give us the existence of functorial $S-$localizations. We end this section by stating a couple of results about $S-$localizations.

Proposition 5.1.12. *Let $\mathscr{M}$ be a model category and let S be a class of arrows in $\mathscr{M}$.*

(i) *Given a fibrant object X in $\mathscr{M}$ and an $S-$localization $j\colon X \to R_S X$ of X, j is a weak equivalence if and only if X is $S-$local.*

(ii) *If $R_S g\colon R_S X \to R_S Y$ is an $S-$localization of a map $g\colon X \to Y$ in $\mathscr{M}$, then g is a weak equivalence if and only if $R_S g$ is an $S-$local equivalence.*

Proof.   (i) This follows immediately from Proposition 5.1.10 and Remark 5.1.6.

(ii) By the two-out-of-three property for $S-$local equivalences (see Theorem 5.1.8), g is an $S-$local equivalence if and only if $R_S g$ is an $S-$local equivalence. We conclude using again Proposition 5.1.10.

$\square$

5.1.2   Left Bousfield localizations.

The kind of localizations we will use have the property that the weak equivalences in the localization are given exactly by the $S-$local equivalences for some subclass S of morphisms in a model category $\mathscr{M}$.

Definition 5.1.13. Let $\mathscr{M}$ be a model category and let S be a class of morphisms in $\mathscr{M}$. A *(left) Bousfield localization of $\mathscr{M}$ with respect to S*, if it exists, is the model category $\mathcal{L}_S \mathscr{M}$ with underlying category $\mathscr{M}$ and such that:

1. weak equivalences in $\mathcal{L}_S \mathscr{M}$ are the $S-$local equivalences in $\mathscr{M}$ (see Definition 5.1.3);

2. cofibrations in $\mathcal{L}_S \mathscr{M}$ are all and only the cofibrations in $\mathscr{M}$;

3. fibrations in $\mathcal{L}_S \mathscr{M}$ are given by the maps in $\mathscr{M}$ having the right lifting property with respect to those morphisms in $\mathscr{M}$ which are both cofibrations and $S-$local equivalences.

In the following, we will only consider left Bousfield localizations as defined in Definition 5.1.13 above, hence we will simply call them *Bousfield localizations*.

Remark 5.1.14. Let $\mathscr{M}$ be a model category and let S be a class of morphisms in $\mathscr{M}$. Assume that the Bousfield localization $\mathcal{L}_S \mathscr{M}$ exists. Then, from Definition 5.1.13 and from Theorem 5.1.8, it follows immediately that:

(i) there is a Quillen pair

$$\mathscr{M} \underset{\mathrm{Id}_{\mathscr{M}}}{\overset{\mathrm{Id}_{\mathscr{M}}}{\rightleftarrows}} \perp \mathcal{L}_S \mathscr{M} \ ;$$

(ii) every weak equivalence in $\mathscr{M}$ is a weak equivalence also in $\mathcal{L}_S \mathscr{M}$;

(ii) the class of trivial fibrations of $\mathcal{L}_S \mathscr{M}$ equals the class of trivial fibrations of $\mathscr{M}$.

Calling the model category described in Definition 5.1.13 a Bousfield localization would probably not be fair if we could not prove that, if such a Bousfield localization exists, it is indeed a localization (in the sense of Definition 5.1.1).[5] Luckily, we have the following results which show exactly that Bousfield localizations deserve their names.

Proposition 5.1.15. *Let $\mathscr{M}$ be a model category and let S be a class of morphisms in $\mathscr{M}$. Suppose that the Bousfield localization $\mathcal{L}_S \mathscr{M}$ of $\mathscr{M}$ with respect to S exists and let $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ be a left Quillen functor from $\mathscr{M}$ to a model category $\mathscr{N}$. If F takes every cofibrant approximation to a map in S into a weak equivalence in $\mathscr{N}$, then F is a left Quillen functor when considered as a functor from $\mathcal{L}_S \mathscr{M}$ to $\mathscr{N}$.*

---

[5] After all, in Mathematics, we could well say that *nomina nuda tenemus*.

Proof. Let $G\colon \mathscr{N} \longrightarrow \mathscr{M}$ be the right adjoint to $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ which is also a right adjoint to F when we consider it as a functor from $\mathcal{L}_S\mathscr{M}$, because $\mathcal{L}_S\mathscr{M}$ and $\mathscr{M}$ have the same underlying category. To show that $G\colon \mathscr{N} \longrightarrow \mathcal{L}_S\mathscr{M}$ is a right Quillen functor it is enough to show that it preserves trivial fibrations and fibrations between fibrant objects (see Proposition 8.5.4 of [Hir1]). The first property is clear, because $\mathrm{TrFib}(\mathcal{L}_S\mathscr{M}) = \mathrm{TrFib}(\mathscr{M})$ and $G\colon \mathscr{N} \longrightarrow \mathscr{M}$ is a right Quillen functor. On the other hand, the hypothesis on F and Theorem 5.1.7 imply that if $p\colon X \to Y$ is a fibration between fibrant objects in $\mathscr{N}$, then $Gp\colon GX \to GY$ is a fibration in $\mathscr{M}$ between S-local objects. This gives that Gp is a fibration in $\mathcal{L}_S\mathscr{M}$ (see [Hir1], Proposition 3.3.16). $\qquad\square$

Theorem 5.1.16. *Let $\mathscr{M}$ be a model category and let S be a class of morphisms in $\mathscr{M}$. Suppose that the Bousfield localization $\mathcal{L}_S\mathscr{M}$ of $\mathscr{M}$ with respect to S exists. Then*

$$(\mathcal{L}_S\mathscr{M},\ \mathrm{Id}_{\mathscr{M}}\colon \mathscr{M} \longrightarrow \mathcal{L}_S\mathscr{M})$$

is a localization of $\mathscr{M}$ with respect to S (see Definition 5.1.1).

Proof. Let $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ be a left Quillen functor into a model category $\mathscr{N}$ and assume that its total left derived functor $\mathbb{L}F\colon \mathrm{Ho}(\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{N})$ sends the images of elements of S in $\mathrm{Ho}(\mathscr{M})$ into isomorphisms in $\mathrm{Ho}(\mathscr{N})$. Then of course there is a unique functor $\bar{F}\colon \mathcal{L}_S\mathscr{M} \longrightarrow \mathscr{N}$ such that $\bar{F} \circ \mathrm{Id}_{\mathscr{M}} = F$ and such an $\bar{F}$ is F itself. The hypothesis on $\mathbb{L}F$ and Theorem 5.1.7 give that $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ sends every cofibrant approximation to an element in S into a weak equivalence in $\mathscr{N}$. By Proposition 5.1.15, this implies that $F\colon \mathcal{L}_S\mathscr{M} \longrightarrow \mathscr{N}$ is a left Quillen functor, as required. $\qquad\square$

We can describe the relations between the homotopy theories presented by $\mathcal{L}_S\mathscr{M}$ and by $\mathscr{M}$ as follows.

Proposition 5.1.17 (cf. [Rzk1], §5.3). *Let $\mathscr{M}$ be a model category and let S be a class of maps in $\mathscr{M}$. Assume that the Bousfield localization $\mathcal{L}_S\mathscr{M}$ of $\mathscr{M}$ with respect to S exists and denote by*

$$\mathscr{M} \underset{i}{\overset{a}{\rightleftarrows}} \mathcal{L}_S\mathscr{M}$$

the associated adjoint pair, so that both a and i are the identity functors on $\mathscr{M}$. Let also

$$\mathbb{R}i\colon \mathcal{L}_S\mathscr{M} \longrightarrow \mathscr{M} \quad and \quad \mathbf{R}i\colon \mathrm{Ho}(\mathcal{L}_S\mathscr{M}) \longrightarrow \mathrm{Ho}(\mathscr{M})$$

be a point-set left derived functor and the total left derived functor of i respectively (see Definition 4.1.31 and Remark 4.1.33). Then the following properties hold.

1. *$\mathbf{R}i$ is fully faithful.*

2. *$\mathbf{R}i$ induces an equivalence of categories between $\mathrm{Ho}(\mathcal{L}_S\mathscr{M})$ and the full subcategory of $\mathrm{Ho}(\mathscr{M})$ consisting of (the images under the localization functor $\gamma_{\mathscr{M}}\colon \mathscr{M} \longrightarrow \mathrm{Ho}(\mathscr{M})$ of) the S-local objects in $\mathscr{M}$.*

3. *The simplicial sets*
   $$\mathrm{map}^{\mathrm{h}}{}_{\mathcal{L}_S\mathscr{M}}(X, Y) \quad and \quad \mathrm{map}^{\mathrm{h}}{}_{\mathscr{M}}((\mathbb{R}i)X,\ (\mathbb{R}i)Y)$$
   are naturally weakly equivalent, for $X, Y \in \mathscr{M}$.

Proof. Denote by $\mathbf{L}a$ and by $\mathbb{L}a$ the total left derived functor and a point-set left derived functor of a. Write also $\mathbb{L}a = a \circ Q$ and $\mathbb{R}i = i \circ R_S$, where (Q, q) is a functorial cofibrant replacement in $\mathscr{M}$ and (R_S, r_S) is a functorial fibrant replacement in $\mathcal{L}_S\mathscr{M}$. Note that, since both a and i are the identity on $\mathscr{M}$, for an object $X \in \mathscr{M}$, we simply have $(\mathbb{L}a)(X) = QX$ and $(\mathbb{R}i)(X) = R_SX$.

1. Since $\mathbf{L}a$ is left adjoint to $\mathbf{R}i$, it suffices to show that the counit $(\mathbf{L}a)(\mathbf{R}i) \Longrightarrow \mathrm{Id}_{\mathrm{Ho}(\mathcal{L}_S\mathscr{M})}$ of this adjunction is an isomorphism. By the description of the counit of the derived adjunction given in Remark 4.1.35, this happens if and only if, for all fibrant objects Y in $\mathcal{L}_S\mathscr{M}$, the map $\varepsilon_Y \circ a_{q_{iY}}$ is a weak equivalence in $\mathcal{L}_S\mathscr{M}$, where ε_Y is the $Y-$th component of the counit of the adjunction $a \dashv i$. Since both a and i are identity functors this amounts to say that, for each fibrant object $Y \in \mathcal{L}_S\mathscr{M}$, the map $q_Y\colon QY \to Y$ is a weak equivalence in $\mathcal{L}_S\mathscr{M}$. This is of course true because q_Y is a weak equivalence in $\mathscr{M}$ (as (Q, q) is a functorial cofibrant replacement in $\mathscr{M}$) and weak equivalences in $\mathscr{M}$ are weak equivalences in $\mathcal{L}_S\mathscr{M}$ by Theorem 5.1.8.

2. Note first that, if Y is an object in $\mathscr{M}$, then $(\mathbb{R}i)(Y) = (\mathbf{R}i)Y = R_S Y$ is really an S-local object in $\mathscr{M}$. Indeed, by Theorem 5.1.7, i takes every fibrant object of $\mathcal{L}_S\mathscr{M}$ (such as $R_S Y$) into an S-local object of $\mathscr{M}$, since clearly a sends any cofibrant approximation to a map in S into a weak equivalence in $\mathcal{L}_S\mathscr{M}$ (as weak equivalences in $\mathscr{M}$ are S-local equivalences). Conversely, if A is an S-local object in $\mathscr{M}$, then the weak equivalence $A \xrightarrow{\sim_S} R_S A$ in $\mathcal{L}_S\mathscr{M}$ is an $S-$local equivalence in $\mathscr{M}$ between $S-$local objects. By Proposition 5.1.10, $\sim_S$ is then a weak equivalence in $\mathscr{M}$ and therefore A and $R_S A = (\mathbf{R}i)(A)$ are isomorphic in $\mathrm{Ho}(\mathscr{M})$. This proves that $\mathbf{R}i$ is essentially surjective on the full subcategory spanned by the S-local objects in $\mathrm{Ho}(\mathscr{M})$, so that we can conclude by the first part above.

3. Let X, Y be objects of $\mathscr{M}$. Since weak equivalences in $\mathcal{L}_S\mathscr{M}$ are sent to weak equivalences by $\mathrm{map}^{\mathrm{h}}{}_{\mathcal{L}_S\mathscr{M}}$ (see Proposition 4.4.18), the simplicial set $\mathrm{map}^{\mathrm{h}}{}_{\mathcal{L}_S\mathscr{M}}(X, Y)$ is naturally weakly equivalent to $\mathrm{map}^{\mathrm{h}}{}_{\mathcal{L}_S\mathscr{M}}(R_S X, R_S Y)$. Since $R_S Y$ is fibrant in $\mathcal{L}_S\mathscr{M}$, $\mathrm{map}^{\mathrm{h}}{}_{\mathcal{L}_S\mathscr{M}}(R_S X, R_S Y)$ is naturally weakly equivalent to $\mathcal{L}_S\mathscr{M}(\widetilde{R_S X}, R_S Y)$ for a cosimplicial resolution of $R_S X$ (see Definitions 4.4.1 and 4.4.12). Now, the Quillen pair $a \dashv i$ lifts to a Quillen pair $a \circ (-) \dashv i \circ (-)$ between the categories of cosimplicial objects of $\mathscr{M}$ and of $\mathcal{L}_S\mathscr{M}$ (see Example 4.3.27) endowed with the Reedy model structure induced by the Reedy structure on Δ (see Example 4.3.15). Hence, in particular, any functorial Reedy cofibrant approximation in $\mathscr{M}^\Delta$ will be also a functorial Reedy cofibrant approximation in $(\mathcal{L}_S\mathscr{M})^\Delta$. Since cosimplicial resolutions of an object A of a model category $\mathscr{N}$ are, by definition, Reedy cofibrant approximations of $cc_* A$ (the constant cosimplicial object at A), we can take $\widetilde{R_S X}$ to be a cosimplicial resolution of $R_S X$ in $\mathscr{M}$. Since $R_S Y$ is fibrant also in $\mathscr{M}$ (because it is an $S-$local object), we then get the following chain of naturally weakly equivalent simplicial sets (see Definition 4.1.10):

$$\mathrm{map}^{\mathrm{h}}{}_{\mathcal{L}_S\mathscr{M}}(X, Y) \approx \mathcal{L}_S\mathscr{M}(\widetilde{R_S X}, R_S Y) = \mathscr{M}(\widetilde{R_S X}, R_S Y) \approx$$

$$\approx \mathrm{map}^{\mathrm{h}}{}_{\mathscr{M}}(R_S X, R_S Y) = \mathrm{map}^{\mathrm{h}}{}_{\mathscr{M}}((\mathbb{R}i)X, (\mathbb{R}i)Y),$$

as required.

$\square$

We finally address the question of existence of Bousfield localizations.

Our existence theorem for Bousfield localizations deals with a class of model categories called *cellular* model categories.

Definition 5.1.18. A *cellular* model category is a cofibrantly generated model category (see Definition 4.2.7) $\mathscr{M}$ for which:

- cofibrations are *effective monomorphisms* (dual notion to that of effective epimorphism given in Definition 1.3);

- there are sets $\mathcal{I}$ and $\mathcal{J}$ of generating cofibrations and generating trivial cofibrations respectively such that the domains of elements of $\mathcal{J}$ are small relative to $\mathcal{I}$ (see Definition 4.2.2) and both the domains and the codomains of the elements of $\mathcal{I}$ are *compact* (in the sense of [Hir1], Definition 11.4.1).

The notion of compactness we refer to above is quite technical and we will not explain it here. For our purposes, it is enough to know what follows.

Remark 5.1.19. The category **Top** of topological spaces and the category **sSet** of simplicial sets with respect to the Quillen model category structure (see Examples 4.1.18 and 4.1.19) are both cellular model categories (see Proposition 12.1.4 of [Hir1]). Furthermore, if $\mathscr{M}$ is a cellular model category and $\mathscr{C}$ is a small category, then $\mathscr{M}^\mathscr{C}$ is a cellular model category with respect to the projective model structure (see Definition 4.3.2 and Theorem 4.3.4).

We can now state the following

Theorem 5.1.20. *Let $\mathscr{M}$ be a model category which is either*

(a) a left proper and cellular model category (see Definitions 4.2.15 and 5.1.18)

or

(b) *a left proper and combinatorial model category (see Definition 4.2.10).*

Then, for every set S of morphisms in $\mathscr{M}$, the Bousfield localization $\mathcal{L}_S\mathscr{M}$ of $\mathscr{M}$ with respect to S exists and is again a left proper, cellular model category or a left proper, combinatorial model category respectively. Furthermore, we also have that:

(i) *the fibrant objects of $\mathcal{L}_S\mathscr{M}$ are exactly the S-local objects of $\mathscr{M}$;*

(ii) *if $\mathscr{M}$ is a simplicial model category, then also $\mathcal{L}_S\mathscr{M}$ is a simplicial model category with simplicial, tensored and cotensored structures given by those of $\mathscr{M}$ (see Definitions 4.2.32 and 4.2.29).*

Proof. See [Hir1], Theorem 4.1.1 and [Bar], Theorem 4.7. □

Remark 5.1.21. We stress that Theorem 5.1.20 ensures the existence of Bousfield localizations $\mathcal{L}_S\mathscr{M}$ for a *set S* of maps in $\mathscr{M}$ and not for an arbitrary class of maps.

Remark 5.1.22. Since the only kinds of localizations that we will consider are Bousfield localizations, from now on, given a model category $\mathscr{M}$ and a class S of maps in $\mathscr{M}$, we will indicate $\mathcal{L}_S\mathscr{M}$ simply by $\mathscr{M}_S$.

Given a small category $\mathscr{I}$ and a small simplicial category $\mathbb{C}$ the model categories

$$\mathrm{sPsh}(\mathscr{I}) := \mathbf{sSet}^{\mathscr{I}^{\mathrm{op}}} \quad \text{and} \quad \mathbf{sPsh}(\mathbb{C}) := [\mathbb{C}^{\mathrm{op}}, \mathbf{sSet}] \tag{5.3}$$

with the *projective model structure* are left proper, combinatorial and simplicial model categories (see Theorems 4.3.2 and 4.3.31). We will call the elements of $\mathrm{sPsh}(\mathscr{I})$ and of $\mathbf{sPsh}(\mathbb{C})$ *simplicial presheaves on $\mathscr{I}$* and *on $\mathbb{C}$* respectively. By Theorem 5.1.20, for every set S of maps in $\mathrm{sPsh}(\mathscr{I})$ or in $\mathbf{sPsh}(\mathbb{C})$, the Bousfield localizations

$$\mathrm{sPsh}(\mathscr{I})_S \quad \text{and} \quad \mathbf{sPsh}(\mathbb{C})_S$$

exist and are again left proper, simplicial and combinatorial model categories.

5.2   Universal Homotopy Theories.

Following [Dug1], in this section we are going to see in which sense the category of simplicial presheaves on a small category $\mathscr{C}$ is the homotopical counterpart of the category of presheaves on $\mathscr{C}$.

We start by recalling a basic fact of presheaves categories that characterises them completely, up to isomorphism, via a universal property. Note that we basically used this result in the proof of Proposition 2.2.17.

Theorem 5.2.1. *Let $\mathscr{C}$ be a small category. Then the Yoneda embedding*

$$\mathbf{y}\colon \mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C})$$

exhibits $\mathrm{PSh}(\mathscr{C})$ as the free cocompletion of $\mathscr{C}$, that is, given any functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ into a cocomplete category $\mathscr{D}$, there is a unique (up to isomorphism) cocontinuous functor $L_F\colon \mathrm{PSh}(\mathscr{C}) \longrightarrow \mathscr{D}$ making the following diagram commutative

$$
\begin{array}{ccc}
\mathscr{C} & \xrightarrow{\ \mathbf{y}\ } & \mathrm{PSh}(\mathscr{C}) \\
& {\scriptstyle F}\searrow & \downarrow{\scriptstyle L_F} \\
& & \mathscr{D}
\end{array}
$$

up to an invertible natural transformation $L_F \circ \mathbf{y} \overset{\cong}{\Longrightarrow} F$. Furthermore, such an L_F can be defined as the left Kan extension of F along $\mathbf{y}$ (see Definition 2.2.14), hence it is left adjoint to the functor

$$\mathscr{D}(F(-),?)\colon \mathscr{D} \longrightarrow \mathrm{PSh}(\mathscr{C}), \quad A \mapsto (\mathscr{D}(F(-),A)\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}).$$

Remark 5.2.2. The proof of the above result (that can be found, for example, in [McMo], §1.5) relies on the basic observation that every presheaf on a small category is isomorphic to a colimit of representables[6]. More precisely, if X is a presheaf on $\mathscr{C}$ and $\mathrm{El}(X)$ is its category of elements, then

$$X \cong \mathrm{colim}_{(x,A)\in \mathrm{El}(X)} \mathbf{y}A.$$

Since an object (x,A) of $\mathrm{El}(X)$ determines and is determined uniquely by a map $\mathbf{y}A \longrightarrow X$, i.e. by an object of $(\mathbf{y} \downarrow X)$, we can also write

$$X \cong \mathrm{colim}_{(s\colon \mathbf{y}A \longrightarrow X)\in(\mathbf{y}\downarrow X)} \mathbf{y}A \tag{5.4}$$

The homotopical version of Theorem 5.2.1 above is the main result of [Dug1]. In order to state it, given a small category $\mathscr{C}$, we let

$$r\colon \mathscr{C} \longrightarrow \mathrm{sPsh}(\mathscr{C}) \tag{5.5}$$

be the composite of the Yoneda embedding $\mathbf{y}\colon \mathscr{C} \longrightarrow \mathrm{PSh}(\mathscr{C})$ with the embedding

$$\mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{sPsh}(\mathscr{C})$$

which sends a presheaf $F\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ to the simplicial presheaf mapping an object $A \in \mathscr{C}^{\mathrm{op}}$ to the discrete simplicial set having in each dimension n the set $F(A)$. Note that such a functor $\mathrm{PSh}(\mathscr{C}) \longrightarrow \mathrm{sPsh}(\mathscr{C})$ is simply the constant functor sending a presheaf F to the constant simplicial object on it, when we see $\mathrm{sPsh}(\mathscr{C})$ as the category $\mathrm{PSh}(\mathscr{C})^{\Delta^{\mathrm{op}}}$ of simplicial objects in $\mathrm{PSh}(\mathscr{C})$.

The embedding r is called the *(discrete) simplicial Yoneda embedding*. We will actually denote such an r again by $\mathbf{y}$, seen this time as a functor $\mathbf{y}\colon \mathscr{C} \longrightarrow \mathrm{sPsh}(\mathscr{C})$.

Theorem 5.2.3. *Let $\mathscr{C}$ be a small category. Then, any functor $\gamma\colon \mathscr{C} \longrightarrow \mathscr{M}$ into a model category $\mathscr{M}$ factors through $\mathrm{sPsh}(\mathscr{C})$ in the sense that there is a Quillen pair*

$$
\begin{array}{ccc}
& \mathrm{Re}=\mathrm{Re}_\gamma & \\
\mathrm{sPsh}(\mathscr{C}) & \underset{\mathrm{Sing}=\mathrm{Sing}_\gamma}{\overset{}{\rightleftarrows}} \perp & \mathscr{M}
\end{array}
$$

---

[6] We have already used this fundamental fact several times in Chapters 1 and 2.

(where $\mathrm{sPsh}(\mathscr{C})$ *has the projective model structure) and there is a natural weak equivalence*

$$\eta\colon \mathrm{Re}\circ\mathbf{y} \overset{\sim}{\Longrightarrow} \gamma,$$

as in the (non-commutative) diagram:

Moreover, the category of all such factorizations of γ *through* $\mathrm{sPsh}(\mathscr{C})$ *is contractible.*

Here the *category of factorizations of* γ *through* $\mathrm{sPsh}(\mathscr{C})$ is the category $\mathrm{Fact}(\gamma)$ having as objects triples (L, R, τ), where $L\colon \mathrm{sPsh}(\mathscr{C}) \rightleftarrows \mathscr{M} \colon R$ is a Quillen pair (left adjoint on the left) and τ is a natural weak equivalence $\tau\colon \mathrm{Re}\circ\mathbf{y} \Longrightarrow \gamma$. A morphism $(L, R, \tau) \to (L', R', \tau')$ in $\mathrm{Fact}(\gamma)$ is simply a natural transformation $\sigma\colon L \Longrightarrow L'$ between the left adjoints such that $\tau' \circ \sigma_R = \tau$.

Proof. We just sketch the main ideas of the proof. The whole point is to show that $\mathrm{Fact}(\gamma)$ is equivalent to the category $\mathrm{cos}(\gamma)$ of cosimplicial resolutions of γ. Here a cosimplicial resolution of γ is a functor $\Gamma\colon \mathscr{C} \longrightarrow \mathscr{M}^\Delta$ equipped with a natural weak equivalence $\iota\colon \Gamma \overset{\sim}{\Longrightarrow} cc_* \circ \gamma$ such that $\iota_A\colon \Gamma(A) \to cc_*(\gamma A)$ is a cosimplicial resolution of $\gamma(A)$ (see Definition 4.4.1) for all $A \in \mathscr{C}$. Since it is a well-established result (see [Hir1], Theorem 16.7.5) that $\mathrm{cos}(\gamma)$ is a contractible category, also $\mathrm{Fact}(\gamma)$ would be (non-empty and) contractible.

Thus, it is enough to show that $\mathrm{Fact}(\gamma)$ and $\mathrm{cos}(\gamma)$ are equivalent categories. Given a factorization of γ, say (L, R, τ), we get a cosimplicial resolution Γ of γ by sending $A \in \mathscr{C}$ to $\Gamma(A) := L(\mathbf{y}A \times \Delta[-])$. Here, for each $n \in \mathbb{N}$, $\mathbf{y}A \times \Delta[n]$ is the product in the category $\mathrm{sPsh}(\mathscr{C})$ of $\mathbf{y}A$ with the constant functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ at $\Delta[n]$. This assignment defines indeed a cosimplicial resolution of γ because:

(i) each $\mathbf{y}A$ is a cofibrant object in $\mathrm{sPsh}(\mathscr{C})$ with the projective model structure. This follows from the description of the sets of generating (trivial) cofibrations for the cofibrantly generated model structure on $\mathrm{sPsh}(\mathscr{C})$ (see [Hir1], Example 11.5.31);

(ii) $\mathbf{y}A \times \Delta[-]$ is a cosimplicial resolution of $\mathbf{y}A$: this follows from (i) above, Remark 4.2.31, Remark 4.4.6 and Example 4.4.11;

(iii) left Quillen functors preserve cosimplicial resolutions of cofibrant objects (see [Hir1], Proposition 16.2.1).

Viceversa, if we are given a cosimplicial resolution $\Gamma\colon \mathscr{C} \longrightarrow \mathscr{M}^\Delta$ of γ we get a factorization of γ by setting

$$L\colon \mathrm{sPsh}(\mathscr{C}) \longrightarrow \mathscr{M}, \quad F \mapsto \Gamma \otimes_\gamma F := \int^{A\in\mathscr{C}} \Gamma(A) \otimes F(A)$$

and

$$R\colon \mathscr{M} \longrightarrow \mathrm{sPsh}(\mathscr{C}), \quad X \mapsto \mathscr{M}(\Gamma(-), X),$$

where, for $G \in \mathscr{M}^\Delta$ and $K \in \mathbf{sSet}$, $G\otimes K$ is defined as in (4.20). The natural weak equivalence $L\circ\mathbf{y} \overset{\sim}{\Longrightarrow} \gamma$ is obtained, for $B \in \mathscr{C}$, through the following chain of isomorphisms

$$L(\mathbf{y}(B)) = \Gamma \otimes_\gamma \mathbf{y}B = \int^{A\in\mathscr{C}} \Gamma(A) \otimes \mathscr{C}(A,B) \overset{(\dagger)}{\cong} \int^{A\in\mathscr{C}} (\Gamma(A))_0 \cdot \mathscr{C}(A,B) \cong$$

$$\cong (\Gamma(B))_0 \overset{\sim}{\to} \gamma B,$$

where:

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- in the right hand side of the isomorphism (†), $(\Gamma(A))_0 \cdot \mathscr{C}(A, B) := \coprod_{\mathscr{C}(B,A)} (\Gamma(A))_0 \in \mathcal{M}$ and the isomorphism follows as

$$\operatorname{colim}_{([n],s) \in \Delta(\mathscr{C}(A,B))} (\Gamma(A))_n \cong \operatorname{colim}_{([0],s) \in \Delta(\mathscr{C}(A,B))} (\Gamma(A))_0 \cong (\Gamma(A))_0 \cdot \mathscr{C}(A, B),$$

  where the first isomorphism is due to the fact that the only non degenerate simplices in $\mathscr{C}(A, B)$ (seen as a discrete simplicial set) are the $0-$th simplices, whereas the second isomorphism holds because the category over which the colimit is taken is a discrete category;

- the last isomorphism follows from the coend form of the Yoneda lemma (also known as the *Ninja Yoneda Lemma*, see [Lor], Proposition 2.1);

- the last map is the weak equivalence given by the fact that $\Gamma(B)$ is a cosimplicial resolution of $\gamma(B)$.

$\square$

The analogy between the homotopical properties of $\mathrm{sPsh}(\mathscr{C})$ and the categorical properties of $\mathrm{PSh}(\mathscr{C})$ comprises also a model-categorical version of Remark 5.2.2 in the following sense. Let $\mathscr{C}$ be a small category and let X be a simplicial presheaf on $\mathscr{C}$. There is a functor

$$\Gamma \colon \mathscr{C} \times \Delta \longrightarrow \mathrm{sPsh}(\mathscr{C}), \quad (A, [n]) \mapsto \mathbf{y}A \times \Delta[n].$$

As above, here $\mathbf{y}$ is the discrete simplicial Yoneda embedding and $\mathbf{y}A \times \Delta[n]$ is the product in the category $\mathrm{sPsh}(\mathscr{C})$ of $\mathbf{y}A$ with the constant functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ at $\Delta[n]$. We can then consider the over-category $(\Gamma \downarrow X)$ whose objects are pairs

$$((A, [n]),\ \mathbf{y}A \times \Delta[n] \to X)$$

With a slight abuse of notation, we shall indicate this over-category as $(\mathscr{C} \times \Delta \downarrow X)$. Note that $(\mathscr{C} \times \Delta \downarrow X)$ is the category of elements of X when we see it not as a functor $\mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ but as a presheaf $\mathscr{C}^{\mathrm{op}} \times \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. We have a functor

$$V \colon (\mathscr{C} \times \Delta \downarrow X) \longrightarrow \mathrm{sPsh}(\mathscr{C}), \quad ((A, [n]),\ \mathbf{y}A \times \Delta[n] \to X) \mapsto \mathbf{y}A \times \Delta[n].$$

We then get a composite map

$$\operatorname{hocolim} V \longrightarrow \operatorname{colim} V \longrightarrow X$$

(for any choice of the homotopy colimit functor $\operatorname{hocolim} \colon \mathrm{sPsh}(\mathscr{C})^{(\mathscr{C} \times \Delta \downarrow X)} \longrightarrow \mathrm{sPsh}(\mathscr{C})$) and we denote $\operatorname{hocolim} V$ by $\operatorname{hocolim}(\mathscr{C} \times \Delta \downarrow X)$. Note that the map $\operatorname{colim} V \to X$ is actually an isomorphism, because it is a specific instance of (5.4), when we see the simplicial presheaf X on $\mathscr{C}$ as a presheaf over $\mathscr{C} \times \Delta$.

Proposition 5.2.4. *Let $\mathscr{C}$ be a small category and let $X \in \mathrm{sPsh}(\mathscr{C})$. Then the canonical map*

$$\operatorname{hocolim}(\mathscr{C} \times \Delta \downarrow X) \to X$$

is a weak equivalence.

Proof. See [Dug1], Proposition 2.9. $\square$

Note now that, as remarked in the proof of Theorem 5.2.3, the functor sending $A \in \mathscr{C}$ to $\mathbf{y}A \times \Delta[-]$ is a cosimplicial resolution of the simplicial Yoneda embedding. Thus, we have a natural weak equivalence $\Gamma \xRightarrow{\sim} cc_* \mathbf{y}$. Considering the functor

$$U \colon (\mathscr{C} \times \Delta \downarrow X) \longrightarrow \mathrm{sPsh}(\mathscr{C}), \quad ((A, [n]),\ \mathbf{y}A \times \Delta[n] \to X) \mapsto \mathbf{y}A,$$

such a weak equivalence of functors lifts to a natural weak equivalence $V \xRightarrow{\sim} U$. Hence, for all simplicial presheaves $X \in \mathrm{sPsh}(\mathscr{C})$, we get a zig-zag of weak equivalences

$$\operatorname{hocolim} U \xleftarrow{\sim} \operatorname{hocolim} V \xrightarrow{\sim} X$$

thanks to Proposition 5.2.4.

Remark 5.2.5. Keeping the parallelism with (5.4), we can write $\operatorname{hocolim} U$ as

$$\operatorname{hocolim}_{((A,[n]),\ \mathbf{y}A \times \Delta[n] \to X) \in (\mathscr{C} \times \Delta \downarrow X)} \mathbf{y}A, \tag{5.6}$$

so that we can rephrase the above discussion by saying that *every simplicial presheaf is (naturally weakly equivalent to) a homotopy colimit of representables*.

Corollary 5.2.6. *Let $\mathscr{C}$ be a small category and let $\mathscr{M}$ be a model category. Suppose that*

$$L\colon \mathrm{sPsh}(\mathscr{C}) \longrightarrow \mathscr{M}$$

is a homotopical functor which preserves homotopy colimits (see Remark 4.5.28). Then there is a Quillen pair

$$\mathrm{sPsh}(\mathscr{C}) \quad \perp \quad \mathscr{M}$$

such that L is naturally weakly equivalent to a (hence to any) point-set left derived functor $\mathbb{L}(\mathrm{Re})$ of Re (see Remark 4.1.33).

Proof. By Theorem 5.2.3, there are a Quillen pair $\mathrm{Re}\colon \mathrm{sPsh}(\mathscr{C}) \rightleftarrows \mathscr{M} :\mathrm{Sing}$ and a natural weak equivalence $\eta\colon \mathrm{Re}\circ\mathbf{y} \overset{\sim}{\Longrightarrow} L\circ\mathbf{y}$. Write now $\mathbb{L}(\mathrm{Re}) = \mathrm{Re}\circ Q$ for a functorial cofibrant approximation Q in $\mathrm{sPsh}(\mathscr{C})$. By Proposition 5.6 and Remark 5.2.5, for each $X \in \mathrm{sPsh}(\mathscr{C})$ there is a functor

$$U\colon \mathscr{I} \longrightarrow \mathrm{sPsh}(\mathscr{C})$$

from a small category $\mathscr{I}$ and landing into representable simplicial presheaves (i.e. into the (essential) image of $\mathbf{y}\colon \mathscr{C} \longrightarrow \mathrm{sPsh}(\mathscr{C})$) such that we have naturally weakly equivalent objects $\mathrm{hocolim}\, U \approx X$ (see Definition 4.1.10). Thus, η induces a weak equivalence $\mathrm{hocolim}_{i\in\mathscr{I}} \mathrm{Re}\, U(i) \overset{\sim}{\to} \mathrm{hocolim}_{i\in\mathscr{I}} LU(i)$. Therefore, we have the following chain of naturally weakly equivalent objects

$$LX \approx L(\mathrm{hocolim}_{i\in\mathscr{I}} U(i)) \overset{\mathrm{Hp}}{\approx} \mathrm{hocolim}_{i\in\mathscr{I}} LU(i) \overset{\sim}{\leftarrow} \mathrm{hocolim}_{i\in\mathscr{I}} \mathrm{Re}\, U(i) \overset{\sim}{\leftarrow} \mathrm{hocolim}_{i\in\mathscr{I}} \mathbb{L}(\mathrm{Re})(U(i)) \overset{(\dagger)}{\approx}$$

$$\overset{(\dagger)}{\approx} \mathbb{L}(\mathrm{Re})(\mathrm{hocolim}_{i\in\mathscr{I}} U(i)) \approx \mathbb{L}(\mathrm{Re})(X),$$

where the two sides of $(\dagger)$ are naturally weakly equivalent by Proposition 4.5.27. This gives us what required. $\qquad\square$

5.3   Homotopical small presentation.

Given a small category $\mathscr{C}$, Theorem 5.2.1 and Remark 5.2.2 say that we can think to the category $\text{PSh}(\mathscr{C})$ of presheaves over $\mathscr{C}$ as *(freely) generated by adjoining formal colimits of (diagrams of) objects in* $\mathscr{C}$. Consequently, the process of forming, for a set S of maps of morphisms in $\text{PSh}(\mathscr{C})$, the subcategory of $S-$local objects (in the sense of Definition 1.2.4) which is a localization of $\text{PSh}(\mathscr{C})$ (in the sense of Definition 1.2.2) can be understood as adding *relations* (given by the elements of S) to the generators in $\mathscr{C}$. In Section 1.2 we showed that in this way we recover exactly (up to equivalences of categories) all the locally presentable categories.

Theorem 5.2.3 and Remark 5.2.5 suggest that the homotopification of these results should start by considering, for a small category $\mathscr{C}$, the category $\text{sPsh}(\mathscr{C})$ of simplicial presheaves on $\mathscr{C}$. Indeed, $\text{sPsh}(\mathscr{C})$ can be thought of as the model category *generated by adding formal homotopy colimits of (diagrams of) elements in* $\mathscr{C}$. We then give the following Definition, which can be interpreted as a formalization of the idea of presenting a model category $\mathscr{M}$ by giving *generators* and *relations*.

Definition 5.3.1. Let $\mathscr{M}$ be a model category. A *(homotopical) small presentation* for $\mathscr{M}$ is a triple

$$(\mathscr{C},\ S,\ L\colon \text{sPsh}(\mathscr{C}) \rightleftarrows \mathscr{M} \colon U),$$

where $\mathscr{C}$ is a small category, S is a *set* of maps in $\text{sPsh}(\mathscr{C})$ and

$$\text{sPsh}(\mathscr{C}) \quad \underset{U}{\overset{L}{\perp}} \quad \mathscr{M}$$

is a Quillen pair such that:

1. the total left derived functor of L takes the images of maps of S in $\text{Ho}(\text{sPsh}(\mathscr{C}))$ into isomorphisms in $\mathscr{M}$;

2. the induced Quillen pair

$$\text{sPsh}(\mathscr{C})_S \quad \underset{U}{\overset{L}{\perp}} \quad \mathscr{M}$$

   is a Quillen equivalence (see Theorem 5.1.16 and Remark 5.1.22).

When such a homotopical small presentation for $\mathscr{M}$ exists, we will say that $\mathscr{M}$ *admits (homotopical) small presentation* or that $\mathscr{M}$ *has (homotopical) small presentation*.

Remark 5.3.2. In Definition 5.3.1 above, we added the attribute "homotopical" to the concept of small presentation for a model category $\mathscr{M}$ in order to discern that notion from the one given in Definition 1.2.5 for a general category $\mathscr{D}$. However, in the following, when dealing with model categories we will only be interested in homotopical small presentations, hence we will simply talk about *small presentations* for model categories, always meaning homotopical small presentations.

Example 5.3.3. Consider the embedding of the simplicial category Δ into the category **Top** of topological spaces sending a non-empty finite ordinal $[n]$ to the $n-$standard simplex Δ^n in $\mathbb{R}^n$. By Theorem 5.2.3, there is an associated Quillen pair

$$\text{sPsh}(\Delta) \quad \underset{\text{Sing}}{\overset{\text{Re}}{\perp}} \quad \textbf{Top}$$

Here $\text{sPsh}(\Delta)$ is (identifiable with) the category of bisimplicial sets and Re is just the usual realization $|X|$ of a bisimplicial set X (see [Hir1], §15.11). Now, this Quillen pair is not a Quillen equivalence because the representables $\Delta[n]$ need not to be contractible (whereas their realizations Δ^n are such). However, it turns

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out that, if $*$ is the terminal object in $\mathrm{sPsh}(\Delta)$ and we set $S := \{\Delta[n] \to * : \ n \in \mathbb{N}\} \subseteq \mathrm{Mor}(\mathrm{sPsh}(\Delta))$, then the induced adjoint pair

$$\mathrm{sPsh}(\Delta)_S \ \perp \ \overset{\mathrm{Re}}{\underset{\mathrm{Sing}}{\rightleftarrows}} \ \mathbf{Top}$$

is indeed a Quillen equivalence (see [Dug1], Example 5.6). Thus, $\mathbf{Top}$ admits small presentation. Of course, there is also a much simpler small presentation for $\mathbf{Top}$, given by the Quillen equivalence

$$\mathbf{sSet} \ \perp \ \overset{|\cdot|}{\underset{\mathrm{Sing}}{\rightleftarrows}} \ \mathbf{Top}$$

of Example 4.1.39. Note indeed that $\mathbf{sSet} \cong \mathrm{sPsh}(\mathbf{1})$ (where $\mathbf{1}$ is the terminal category) and, for any model category $\mathscr{M}$, $\mathscr{M} \cong \mathscr{M}_\emptyset$.

The homotopical version of Theorem 1.2.6 is the main result of [Dug2]. We state it here together with an important Corollary.

Theorem 5.3.4 ([Dug2], Theorem 1.1). *Every combinatorial model category (see Definition 4.2.10) admits small presentation.*

Corollary 5.3.5. *Let $\mathscr{M}$ be a model category and suppose that $\mathscr{M}$ is Quillen equivalent to a category $\mathscr{N}$ admitting small presentation (see Definition 4.1.38). Then $\mathscr{M}$ admits small presentation as well.*

Proof. This follows from Corollary 6.5 of [Dug1], since, for all small categories $\mathscr{C}$ and for any set S of maps in $\mathrm{sPsh}(\mathscr{C})$, the Bousfield localization $\mathrm{sPsh}(\mathscr{C})_S$ is a combinatorial model category (see Theorem 5.1.20). $\qquad\square$

We shall not give a proof of Theorem 5.3.4 here. However, we will explicitly show how to get a version of it for *simplicial combinatorial* model categories, imitating the flow of thoughts described in [Dug2]. First of all, we need the following variation of Definition 5.3.1. Recall that, given a small simplicial category $\mathbb{C}$ (see Definition 4.2.23), the simplicial category $\mathrm{sPsh}(\mathbb{C}) := [\mathbb{C}^{\mathrm{op}}, \mathbf{sSet}]$ of simplicial functors from $\mathbb{C}^{\mathrm{op}}$ to $\mathbf{sSet}$ endowed with the projective model structure is a simplicial, combinatorial and left proper model category (see Theorem 4.3.31), hence the Bousfield localization $\mathrm{sPsh}(\mathbb{C})_S$ exists for every small set S of maps in $\mathrm{sPsh}(\mathbb{C})_S$, by Theorem 5.1.20.

Definition 5.3.6. Let $\mathscr{M}$ be a model category. A *small simplicial presentation* for $\mathscr{M}$ is a triple

$$(\mathbb{C}, \ S, \ L \colon \mathrm{sPsh}(\mathbb{C}) \rightleftarrows \mathscr{M} \colon U),$$

where $\mathbb{C}$ is a small simplicial category, S is a *set* of maps in $\mathrm{sPsh}(\mathbb{C})$ and

$$\mathrm{sPsh}(\mathbb{C}) \ \perp \ \overset{L}{\underset{U}{\rightleftarrows}} \ \mathscr{M}$$

is a Quillen pair such that:

1. the total left derived functor of L takes the images of maps of S in $\mathrm{Ho}(\mathrm{sPsh}(\mathbb{C}))$ into isomorphisms in $\mathscr{M}$;

2. the induced Quillen pair

$$\mathrm{sPsh}(\mathbb{C})_S \ \perp \ \overset{L}{\underset{U}{\rightleftarrows}} \ \mathscr{M}$$

is a Quillen equivalence (see Theorem 5.1.16 and Remark 5.1.22).

When such a small simplicial presentation for $\mathscr{M}$ exists, we will say that $\mathscr{M}$ *admits small simplicial presentation* or that $\mathscr{M}$ *has small simplicial presentation*.

Using Dugger's Theorem 5.3.4, we can show the following

Proposition 5.3.7. *A model category $\mathcal{M}$ admits small presentation if and only if it admits small simplicial presentation.*

Proof. Every small ordinary category $\mathscr{C}$ can be seen as a simplicial category $\bar{\mathbb{C}}$ where the simplicial set of morphism from $X \in \mathscr{C}$ to $Y \in \mathscr{C}$ is the set $\mathscr{C}(X,Y)$ seen as a discrete simplicial set. Furthermore, in this case, $\mathrm{sPsh}(\bar{\mathbb{C}}) \cong \mathrm{sPsh}(\mathscr{C})$. We thus get that if $\mathcal{M}$ has small presentation, it has also small simplicial presentation. The reverse implication follows because, given a small simplicial category $\mathbb{C}$ and a small set S of morphisms in $\mathrm{sPsh}(\mathbb{C})$, $\mathrm{sPsh}(\mathbb{C})_S$ is a combinatorial model category by Theorem 5.1.20, hence we can use Theorem 5.3.4 to get a small presentation for $\mathrm{sPsh}(\mathbb{C})_S$. $\qquad\square$

The result we would like to prove is the following

Theorem 5.3.8. *Let $\mathbb{M}$ be a simplicial combinatorial model category. Then $\mathbb{M}$ has a small simplicial presentation*

$$\mathrm{sPsh}(\mathbb{C}) \quad \underset{U}{\overset{L}{\rightleftarrows}} \perp \quad \mathbb{M}$$

in which L and U are simplicial functors and $\mathbb{C}$ is a small, full and simplicial subcategory of $\mathbb{M}$ consisting of fibrant and cofibrant objects. Furthermore, the simplicial Yoneda embedding

$$\mathbf{y} \colon \mathbb{C} \longrightarrow \mathrm{sPsh}(\mathbb{C})$$

factors through the subcategory of S-local objects in $\mathrm{sPsh}(\mathbb{C})$.

Here the *simplicial Yoneda embedding* is the fully faithful simplicial functor

$$\mathbf{y} \colon \mathbb{C} \longrightarrow \mathrm{sPsh}(\mathbb{C}), \quad A \mapsto \mathrm{Map}_{\mathbb{C}}(-, A)$$

Note that, when $\mathbb{C}$ is an ordinary small category seen as a discrete simplicial category, $\mathbf{y} \colon \mathbb{C} \longrightarrow \mathrm{sPsh}(\mathbb{C})$ is just the discrete simplicial Yoneda embedding $\mathbf{y} \colon \mathscr{C} \longrightarrow \mathrm{sPsh}(\mathscr{C})$ discussed in Section 5.2 above.

Before we start delving into the proof of Theorem 5.3.8, we need to recall some facts about Enriched Category Theory. Essentially, what we want is the simplicial version of Theorem 5.2.1 saying that $\mathrm{sPsh}(\mathbb{C})$ is the *free simplicial cocompletion* (i.e. the free cocompletion under simplicial *weighted* colimits) of a small simplicial category $\mathbb{C}$.

Theorem 5.3.9. *Let $\mathbb{C}$ be a small simplicial category and let $\mathbb{D}$ be a cocomplete simplicial category, i.e. a simplicial category admitting all small weighted colimits (see [Kel], Chapter 3). Then, for every simplicial functor $\gamma \colon \mathbb{C} \longrightarrow \mathbb{D}$, there is a unique simplicial functor (up to isomorphisms of simplicial functors)*

$$L_{\gamma} \colon \mathrm{sPsh}(\mathbb{C}) \longrightarrow \mathbb{D}$$

such that the following diagram commutes up to an invertible simplicial natural transformation $L_{\gamma} \circ \mathbf{y} \overset{\cong}{\Longrightarrow} \gamma$:

$$
\begin{array}{ccc}
\mathbb{C} & \overset{\mathbf{y}}{\longrightarrow} & \mathrm{sPsh}(\mathbb{C}) \\
 & \gamma \searrow & \downarrow L_{\gamma} \\
 & & \mathbb{D}
\end{array}
$$

Furthermore, the functor L_{γ} can be taken as the simplicial left Kan extension $\mathrm{Lan}_{\mathbf{y}}\, \gamma$ of γ along $\mathbf{y}$ and thus has a simplicial right adjoint

$$\mathrm{Map}_{\mathbb{D}}(\gamma(-), ?) \colon \mathbb{D} \longrightarrow \mathrm{sPsh}(\mathbb{C}), \quad A \mapsto (\mathrm{Map}_{\mathbb{D}}(\gamma(-), A) \colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}).$$

Proof. See [Kel], Theorem 4.51 (and, more generally, Chapter 4). $\qquad\square$

Remark 5.3.10. The cocompleteness condition on the simplicial category $\mathbb{D}$ in Theorem 5.3.9 above is the (simplicially) enriched analogue of the usual cocompleteness for ordinary categories and, in general, it is a stronger requirement than the simple cocompleteness of the underlying category $\mathscr{D}$ of $\mathbb{D}$. However, *if* $\mathbb{D}$ is a tensored and cotensored simplicial category (see Definition 4.2.29) whose underlying category is cocomplete, *then* $\mathbb{D}$ admits all weighted colimits and thus it is cocomplete as a simplicial category (see [Kel], §3.10). In particular, every simplicial model category $\mathbb{M}$ is a cocomplete simplicial category, so that it satisfies the hypothesis of Theorem 5.3.9.

Lemma 5.3.11. *Let $\mathbb{C}$ be a small simplicial category and let $\mathbb{M}$ be a simplicial model category. Suppose given a simplicial functor $\gamma \colon \mathbb{C} \longrightarrow \mathbb{M}$ such that, for all $A \in \mathbb{C}$, $\gamma(A)$ is a cofibrant object in $\mathbb{M}$. Then the simplicial adjunction*

$$\mathrm{sPsh}(\mathbb{C}) \underset{\mathrm{Map}_{\mathbb{M}}(\gamma(-),?)}{\overset{\mathrm{Lan}_{\mathbf{y}}(\gamma)}{\rightleftarrows}} \mathbb{M}$$

of Theorem 5.3.9 is a Quillen pair.

Proof. Given a fibration $X \twoheadrightarrow Y$ in $\mathbb{M}$, the morphism

$$\mathrm{Map}_{\mathbb{M}}(\gamma(-), X) \to \mathrm{Map}_{\mathbb{M}}(\gamma(-), Y)$$

is a fibration in $\mathrm{sPsh}(\mathbb{C})$ with the projective model structure. Indeed, for every $A \in \mathbb{C}$, the map $\mathrm{Map}_{\mathbb{M}}(\gamma(A), X) \to \mathrm{Map}_{\mathbb{M}}(\gamma(A), Y)$ is a fibration of simplicial sets, as $\gamma(A)$ is cofibrant in the simplicial model category $\mathbb{M}$ (see Remark 4.2.35). For the same reasons, the right adjoint $\mathrm{Map}_{\mathbb{M}}(\gamma(-), ?)$ also preserves trivial fibrations, as required. $\qquad\square$

We also need the following technical result about combinatorial model categories from [Dug2]

Lemma 5.3.12. *Let $\mathscr{M}$ be a combinatorial model category with sets of generating cofibrations and trivial cofibrations given by $\mathcal{I}$ and $\mathcal{J}$ respectively (see Definition 4.2.7 and Definition 4.2.10). Let also λ be a regular cardinal such that the underlying category of $\mathscr{M}$ is λ-locally presentable (see Definition 1.2.9). Then there exists a cardinal $\kappa \geq \lambda$ satisfying the following properties.*

(i) *The domains and the codomains of elements of $\mathcal{I}$ and $\mathcal{J}$ are κ-presentable (see Definition 1.2.9), so that fibrations and trivial fibrations are preserved by κ-filtered colimits.*

(ii) *There exist cofibrant and fibrant replacement functors on $\mathscr{M}$ preserving κ-filtered colimits.*

(iii) *κ-filtered colimits of weak equivalences are weak equivalences, i.e. if $D \Longrightarrow D' \colon \mathscr{I} \longrightarrow \mathscr{M}$ is a natural weak equivalence between functors from a κ-filtered category $\mathscr{I}$, then the induced map $\mathrm{colim}\, D \to \mathrm{colim}\, D'$ is a weak equivalence as well.*

(iv) *There exist functorial factorizations of maps $X \to Y$ in $\mathscr{M}$ as $X \overset{\sim}{\rightarrowtail} \bar{X} \twoheadrightarrow Y$ and $X \rightarrowtail \bar{Y} \overset{\sim}{\twoheadrightarrow} Y$ such that, if X and Y are $\kappa-$presentable, then so are $\bar{X}$ and $\bar{Y}$ as well.*

Proof. Since $\mathcal{I}$ and $\mathcal{J}$ are sets and every object in a locally presentable category is presentable (see Remark 1.2.10), there is certainly a cardinal $\lambda_1 \geq \lambda$ such that property (i) holds (recall that if an object is $\mu-$presentable, it is also μ'-presentable for any (regular) cardinal $\mu' \geq \mu$). By [Dug2], Proposition 2.3 there are cardinals λ_2, λ_3 and λ_4 verifying properties (ii), (iii) and (iv) respectively. Taking κ as the maximum among these λ_j's, we get the thesis. $\qquad\square$

Remark 5.3.13. Let $\mathscr{M}$ be a combinatorial model category and let κ be a cardinal as in Lemma 5.3.12.

1. Since $\kappa \geq \lambda$ and $\mathscr{M}$ is $\lambda-$presentable, it is also κ-presentable, so that we can write every object in $\mathscr{M}$ as a κ-filtered colimit of κ-presentable objects.

2. By (iii) of Lemma 5.3.12 we get in particular that, when $\mathscr{I}$ is a small $\kappa-$filtered category,

$$\mathrm{colim}_{\mathscr{I}} \colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}$$

is a homotopy colimit functor of shape $\mathscr{I}$ on $\mathscr{M}$ (see Definition 4.5.23) because it is a homotopical functor (see Remark 4.5.6).

3. A κ-filtered colimit of fibrant objects in $\mathscr{M}$ is again fibrant. Indeed, if $F\colon \mathscr{I} \longrightarrow \mathscr{M}$ is a functor from a κ-filtered category $\mathscr{I}$ into the subcategory of fibrant objects in $\mathscr{M}$, the map $\mathrm{colim}_{i \in \mathscr{I}} F(i) \to *$ into the terminal object of $\mathscr{M}$ is a fibration if and only if it has the right lifting property with respect to every generating trivial cofibration (see Definition 4.2.7). But this last fact follows now easily because the domains and the codomains of generating trivial cofibrations are κ-presentable, so that each map $\mathrm{dom}(f) \to \mathrm{colim}_{i \in \mathscr{I}} F(i)$, for f a generating trivial cofibration, factors as

$$\mathrm{dom}(f) \to F(i_0) \to \mathrm{colim}_{i \in \mathscr{I}} F(i)$$

for some $i_0 \in \mathscr{I}$ and the map $F(i_0) \to *$ is a fibration by hypothesis.

4. If X is a κ-presentable object in $\mathscr{M}$, there is a cosimplicial resolution $\widetilde{X} \xrightarrow{\sim} cc_* X$ of X such that, for all $n \in \mathbb{N}$, $\widetilde{X}_n$ is κ-presentable. This is proven in [Dug2], Lemma 6.3.

5. Let X be a κ-presentable object in $\mathscr{M}$ and let $F\colon \mathscr{I} \longrightarrow \mathscr{M}$ be a functor from a κ-filtered category $\mathscr{I}$ and such that $F(i)$ is a fibrant object in $\mathscr{M}$, for every object $i \in \mathscr{I}$. Take then a cosimplicial resolution $\widetilde{X}$ of X such that, for every $n \in \mathbb{N}$, $\widetilde{X}_n$ is a κ-presentable object in $\mathscr{M}$. Then, the canonical map
$$\mathrm{colim}_{i \in \mathscr{I}} \mathscr{M}(\widetilde{X}, F(i)) \to \mathscr{M}(\widetilde{X}, \mathrm{colim}_{i \in \mathscr{I}} F(i))$$
is an isomorphism because it is pointwise such (recall that, for $A \in \mathscr{M}$ and $n \in \mathbb{N}$, $\mathscr{M}(\widetilde{X}, A)_n := \mathscr{M}(\widetilde{X}_n, A)$). Now, since $\mathrm{colim}_{\mathscr{I}}$ is a homotopy colimit and $\mathscr{M}(\widetilde{X}, A)$ is a homotopy function complex $\mathrm{map}^{\mathrm{h}}(X, A)$ (see Definition 4.4.15) for any fibrant object A, we then get that $\mathrm{hocolim}_{\mathscr{I}} \mathrm{map}^{\mathrm{h}}(X, F(i))$ and $\mathrm{map}^{\mathrm{h}}(X, \mathrm{hocolim}_{i \in \mathscr{I}} F(i))$ are naturally weakly equivalent (observe that, by 3. above, the object $\mathrm{colim}_{i \in \mathscr{I}} F(i)$ is fibrant in $\mathscr{M}$). All in all, we have obtained that, for any $k-$presentable object X and any functor $F\colon \mathscr{I} \longrightarrow \mathscr{M}$,

$$\mathrm{map}^{\mathrm{h}}(X, \mathrm{hocolim}_{i \in \mathscr{I}} F(i)) \approx \mathrm{hocolim}_{\mathscr{I}} \mathrm{map}^{\mathrm{h}}(X, F(i)).$$

The last ingredient we need is the following

Proposition 5.3.14 ([Dug2], Proposition 3.2)**.** *Let $\mathscr{M}$ and $\mathscr{N}$ be combinatorial model categories with $\mathscr{M}$ left proper and let $F\colon \mathscr{M} \longrightarrow \mathscr{N}$ be a left Quillen functor with right adjoint G. Assume that, for any fibrant object $X \in \mathscr{M}$, the canonical map $\mathbb{L}F(G(X)) \to X$ is a weak equivalence, where $\mathbb{L}F$ is any point-set left derived functor of F (see Remark 4.1.33). Then there exists a set S of maps in $\mathscr{M}$ such that $\mathbb{L}F$ sends elements of S into weak equivalences in $\mathscr{N}$. Moreover, the induced left Quillen functor $F\colon \mathscr{M}_S \longrightarrow \mathscr{N}$ is (part of) a Quillen equivalence.*

We can finally give the announced

Proof. (Of Theorem 5.3.8). Let κ be a cardinal as in Lemma 5.3.12 and we can safely assume that the initial and the terminal objects in $\mathscr{M}$ are κ-presentable as well. Let $\mathscr{M}_\kappa$ be the full subcategory generated by any set of representatives of isomorphism classes of κ-presentable objects in $\mathbb{M}$ (see Remark 1.2.10 and also Chapter 2 of [Dug2]). We take $\mathbb{C}$ to be the small full simplicial subcategory of $\mathbb{M}$ spanned by the fibrant-cofibrant objects in $\mathscr{M}_\kappa$, so that $\mathrm{Map}_{\mathbb{C}}(A, B) = \mathrm{Map}_{\mathbb{M}}(A, B)$, for objects $A, B \in \mathbb{C}$. Let also $\iota\colon \mathbb{C} \longrightarrow \mathbb{M}$ be the (simplicial) inclusion functor. By Theorem 5.3.9, we get a simplicial adjunction

$$\mathrm{sPsh}(\mathbb{C}) \quad \underset{U}{\overset{L}{\rightleftarrows}} \quad \mathbb{M}$$

where $L = \mathrm{Lan}_{\mathbf{y}}(\iota)$ and $U = \mathrm{Map}_{\mathbb{M}}(\iota(-), ?)$ and Lemma 5.3.11 says that (L, U) is actually a Quillen pair. By Theorem 4.3.2 and Proposition 5.3.14, to get that (L, U) extends to a Quillen equivalence from $\mathrm{sPsh}(\mathbb{C})_S$ for some small set S of maps in $\mathrm{sPsh}(\mathbb{C})$, it is enough to prove that, for all fibrant objects $X \in \mathbb{M}$, the natural map
$$(\mathbb{L}L)(U(X)) \to X$$
is a weak equivalence (see Remark 4.1.35, noticing that we can avoid taking the (point-set) right derived functor of U, as X is fibrant to start with). Since (the underlying category of) $\mathbb{M}$ is locally presentable and by our choice of κ, we can write $X \cong \mathrm{colim}_{j \in \mathscr{J}} W_j$ for a functor $W\colon \mathscr{J} \longrightarrow \mathbb{M}$ from a κ-filtered category $\mathscr{J}$ and landing into κ-presentable objects (see Remark 5.3.13). We claim that, up to weak

equivalences, we can actually assume that each W_j is a fibrant-cofibrant κ-presentable object in $\mathbb{M}$. This is because, by (iv) of Lemma 5.3.12 and since $\emptyset$ and $*$ are κ-presentable, there are functorial cofibrant and fibrant approximations $Q, R \colon \mathbb{M} \longrightarrow \mathbb{M}$ respectively such that, if A is a κ-presentable, then QA and RA are κ-presentable as well. Furthermore, if $B \in \mathbb{M}$ is a cofibrant (resp. fibrant), then RB (resp. QB) is fibrant-cofibrant. Hence, since $\operatorname{colim}_{j \in \mathscr{J}}$ is a homotopical functor (see again Remark 5.3.13), we get natural weak equivalences

$$\operatorname{colim}_{j \in \mathscr{J}} RQW_j \xleftarrow{\sim} \operatorname{colim}_{j \in \mathscr{J}} QW_j \xrightarrow{\sim} \operatorname{colim}_{j \in \mathscr{J}} W_j \cong X$$

and $\operatorname{colim}_{j \in \mathscr{J}} RQW_j$ is a fibrant object as it is a κ-filtered colimit of fibrant objects. Now, if $\mathbb{R}U = U \circ R'$ is a point-set right derived functor of U (so that R' is a functorial fibrant approximation on $\mathscr{M}$), we get the following commutative diagram in $\mathbb{M}$

$$
\begin{array}{ccccc}
\mathbb{L}L(\mathbb{R}U(\operatorname{colim}_{j \in \mathscr{J}} RQW_j)) & \xleftarrow{\sim} & \mathbb{L}L(\mathbb{R}U(\operatorname{colim}_{j \in \mathscr{J}} QW_j)) & \xrightarrow{\sim} & \mathbb{L}L(\mathbb{R}U(X)) \\
\downarrow & & \downarrow & & \downarrow \\
R'(\operatorname{colim}_{j \in \mathscr{J}} RQW_j) & \xleftarrow{\sim} & R'(\operatorname{colim}_{j \in \mathscr{J}} QW_j) & \xrightarrow{\sim} & R'(X)
\end{array}
$$

It follows that the rightmost vertical map is a weak equivalence if and only if the leftmost vertical map is a weak equivalence. But, since both $\operatorname{colim}_{j \in \mathscr{J}}(RQW_j)$ and X are fibrant to start with, those two vertical arrows are naturally weakly equivalent to the canonical maps

$$\mathbb{L}L(U(\operatorname{colim}_{j \in \mathscr{J}}(RQW_j)) \to \operatorname{colim}_{j \in \mathscr{J}} RQW_j \quad \text{and} \quad \mathbb{L}L(U(X)) \to X$$

respectively. We then see that $\mathbb{L}L(U(\operatorname{colim}_{j \in \mathscr{J}} RQW_j)) \to \operatorname{colim}_{j \in \mathscr{J}}(RQW_j)$ is a weak equivalence if and only if $\mathbb{L}L(U(X)) \to X$ is so.

Thus, we conclude that, up to relabelling $\operatorname{colim}_{j \in \mathscr{J}}(RQW_j)$ as X, we have come down to the case $X \cong \operatorname{colim}_{j \in \mathscr{J}} W_j$, where each W_j is κ-presentable and fibrant-cofibrant. Now, there is a weak equivalence

$$\operatorname{colim}_{j \in \mathscr{J}} UW_j \to U(\operatorname{colim}_{j \in \mathscr{J}} W_j) \cong UX$$

in $\mathrm{sPsh}(\mathbb{C})$. Indeed, for every object A in $\mathbb{C}$, the $A-$th component of the above map is given as

$$\operatorname{colim}(UW_j)(A) = \operatorname{colim}_{j \in \mathscr{J}} \operatorname{Map}_{\mathbb{M}}(A, W_j) \to \operatorname{Map}_{\mathbb{M}}(A, \operatorname{colim}_{j \in \mathscr{J}} W_j) \cong (UX)(A) \qquad (5.7)$$

by definition of U and of the enriching Hom-simplicial sets in $\mathbb{C}$. This latter map is a weak equivalence by Remark 5.3.13 5. because $\operatorname{colim}_{j \in \mathscr{J}}$ is a homotopical functor and $\operatorname{Map}_{\mathbb{M}}(A, W_j)$, $\operatorname{Map}_{\mathbb{M}}(A, X)$ are homotopy function complexes $\operatorname{map}^{\mathrm{h}}(A, W(j))$, $\operatorname{map}^{\mathrm{h}}(A, X)$ respectively, as A is cofibrant and $W(j)$, X are fibrant in the simplicial model category $\mathbb{M}$ (see Example 4.4.16). Since κ-filtered colimits in $\mathbf{sSet}$ are homotopy colimits[7], we then get a composite weak equivalence

$$\operatorname{hocolim}_{j \in \mathscr{J}} UW_j \xrightarrow{\sim} \operatorname{colim}_{j \in \mathscr{J}} UW_j \xrightarrow{\sim} UX$$

where the first map is a weak equivalence because it is pointwise such. Since left derived functors of left Quillen functors commute with homotopy colimits, our map $(\mathbb{L}L)(U(X)) \to X$ is then weakly equivalent to the map

$$\operatorname{hocolim}_{j \in \mathscr{J}}(\mathbb{L}L)(UW_j) \to \operatorname{colim}_{j \in \mathscr{J}}(\mathbb{L}L)(UW_j) \to X$$

Finally, since $L(UW_j) \cong W_j$ (because $UW(j)$ is representable, see [Kel], Proposition 4.23), and both $\operatorname{hocolim}_{j \in \mathscr{J}}$, $\operatorname{colim}_{j \in \mathscr{J}}$ preserves weak equivalences, $\operatorname{hocolim}_{j \in \mathscr{J}}(\mathbb{L}L)(UW_j)$ and $\operatorname{colim}_{j \in \mathscr{J}}(\mathbb{L}L)(UW_j)$ are naturally weakly equivalent to $\operatorname{hocolim}_{j \in \mathscr{J}} W_j$ and $\operatorname{colim}_{j \in \mathscr{J}} W_j$ respectively. Hence, the map

$$\operatorname{hocolim}_{j \in \mathscr{J}}(\mathbb{L}L)(UW_j) \to \operatorname{colim}_{j \in \mathscr{J}}(\mathbb{L}L)(UW_j) \to X$$

is weakly equivalent to

$$\operatorname{hocolim}_{j \in \mathscr{J}} W_j \to \operatorname{colim}_{j \in \mathscr{J}} W_j \to X$$

---

[7] Up to using Lemma 5.3.12 for the locally finitely presentable category $\mathbf{sSet}$ and redefining the cardinal κ so that it satisfies also this property, together with the other ones we need.

which is a weak equivalence as it is the composite of weak equivalences. Hence, as already remarked, by Proposition 5.3.14, we get that there is a set S of maps in $\mathrm{sPsh}(\mathbb{C})$ such that (L, U) passes to a Quillen equivalence

$$\mathrm{sPsh}(\mathbb{C})_S \;\underset{U}{\overset{L}{\rightleftarrows}}\; \mathbb{M}$$

so that we get our small simplicial presentation for $\mathbb{M}$. To conclude that $\mathbf{y}\colon \mathbb{C} \longrightarrow \mathrm{sPsh}(\mathbb{C})$ factors through the $S-$local objects, just note that each $B \in \mathbb{C}$ is a fibrant object in $\mathbb{M}$ (by our choice of $\mathbb{C}$), so $UB = \mathrm{Map}_{\mathbb{M}}(\iota(-), B) = \mathrm{Map}_{\mathbb{C}}(-, B) = \mathbf{y}B$ is a fibrant object in $\mathrm{sPsh}(\mathbb{C})_S$ because U is a right Quillen functor not only into $\mathrm{sPsh}(\mathbb{C})$ but also into $\mathrm{sPsh}(\mathbb{C})_S$. By Theorem 5.1.20, this means that each $UB = \mathbf{y}B$ is an $S-$local object in $\mathrm{sPsh}(\mathbb{C})$, as required. $\qquad\square$

Remark 5.3.15. Since, for a small ordinary category $\mathscr{I}$ and for any small set T of morphisms in $\mathrm{sPsh}(\mathscr{I})$, the category $\mathrm{sPsh}(\mathscr{I})_T$ is a simplicial combinatorial model category, we can apply Theorem 5.3.8 to get a small simplicial presentation for $\mathrm{sPsh}(\mathscr{I})_T$ in the form $\mathrm{sPsh}(\mathbb{C})_S$, where $\mathbb{C}$ is a small simplicial category which is not discrete in general (compare with the proof of Proposition 5.3.7). This is because $\mathbb{C}$ can be chosen as a full simplicial subcategory of $\mathrm{sPsh}(\mathscr{I})_S$ and, for $F, G \in \mathrm{sPsh}(\mathscr{I})_S$, $\mathrm{Map}_{\mathrm{sPsh}(\mathscr{I})_S}(F, G) = \mathrm{Map}_{\mathrm{sPsh}(\mathscr{I})}(F, G)$ is not a discrete simplicial set in general (see Remark 4.2.31 for the simplicial structure on simplicial presheaves).

Chapter 6

Giraud's Theorem for model topoi.

We have finally come to the core of our work. We give here the definition of a *model topos* and formulate a *Giraud-type theorem* for such model topoi. We can accomplish this task exactly as in the ordinary categorical framework by introducing the *descent* properties for model categories (see Section 6.1). These properties are verified in model topoi and are actually sufficient to characterise them among categories with small simplicial presentation (see Section 6.2). The reader may want to go back to Chapter 2 from time to time in order to compare the results in ordinary Category Theory exposed there with their model-categorical counterparts explained below. We conclude our work with Section 6.3. where, using Giraud's Theorem, we give a brief (and somehow sketchy) account of one of the main and most interesting examples of model topoi which can be constructed out of a Grothendieck site $(\mathscr{C}, \tau)$ (see Definition 3.1.9). Indeed, given a small category $\mathscr{C}$ and a Grothendieck (pre)topology τ on it, one can define a model category structure on $\mathrm{sPsh}(\mathscr{C})$ which, unlike the projective and the injective model structures, takes into account the added structure given by τ. We will call this model structure the *Jardine model structure* on $\mathrm{sPsh}(\mathscr{C})$ (associated to $(\mathscr{C}, \tau)$) and we will denote the associated model category by $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$. Its fibrant objects will be *homotopy sheaves* (on the Grothendieck site $(\mathscr{C}, \tau)$, i.e. simplicial presheaves satisfying a homotopical version of the classical sheaf conditions in presence of a Grothendieck (pre)topology (see (3.9)). In this way, we somehow complete our homotopified picture of classical sheaf theory: as Grothendieck topoi are categories admitting a left exact small presentations and coincide with categories of sheaves on a Grothendieck site, so model topoi are model categories admitting a (homotopically) left exact small (simplicial) presentation and include model categories which present the homotopy theory of homotopy sheaves on a Grothendieck site.

This last chapter is probably the one where our gaps-filling work with respect to [Rzk1] is more accentuated. In comparison with Section 6 of [Rzk1], we give here a slightly different formulation of the descent properties for a model category and point out how the notions of model topos and of descent are invariant under Quillen equivalences. Using another work by Rezk ([Rzk2]), we also show explicitly how to conclude that every model topos has descent (cf. [Rzk1], Proposition 6.6). Our proof of Giraud's Theorem for model topoi (see Theorem 6.2.2 below) follows closely the sketch given in [Rzk1] but tries to work out the needed details. Our Section 6.3 below explains Example 6.3 of [Rzk1]. Although a proof of the fact that the Jardine model structure on simplicial presheaves gives rise to a model topos may possibly be found in [ToVe], our use of Giraud's theorem to show that this is indeed the case is somewhat original, at least to the best of our knowledge.

6.1 Descent for model categories.

Definition 6.1.1. Let $\mathscr{M}$ be a model category and let S be a class of maps in $\mathscr{M}$. Assume that the left Bousfield localization $\mathscr{M}_S$ of $\mathscr{M}$ with respect to S exists and consider the associated Quillen pair

$$\mathscr{M} \underset{i}{\overset{a}{\underset{\perp}{\rightleftarrows}}} \mathscr{M}_S$$

(see Definition 5.1.13, Remark 5.1.14 and Remark 5.1.22). We say that $\mathscr{M}_S$ is a *(homotopical) left exact localization* (of $\mathscr{M}$ with respect to S) if the (point-set) left derived functor $\mathbb{L}a$ of a preserves homotopy pullback squares (see Definition 4.5.36). This means that, if

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
$$

is a homotopy pullback square in $\mathscr{M}$, then

$$
\begin{array}{ccc}
\mathbb{L}a(A) & \longrightarrow & \mathbb{L}a(B) \\
\downarrow & & \downarrow \\
\mathbb{L}a(C) & \longrightarrow & \mathbb{L}a(D)
\end{array}
$$

is a homotopy pullback square in $\mathscr{M}_S$.

Remark 6.1.2. As usual, we will drop the attribute *homotopical* from now on and just talk about *left exact localizations* of model categories $\mathscr{M}$. In fact, recall that in the adjunction $a \dashv i$ of Definition 6.1.1, both a and i are the identity functors on $\mathscr{M}$ (see Remark 5.1.14), so the only non-trivial kind of left exactness we can require to a is the homotopical one.

We are now ready to give the central definition of our work (compare with Definition 1.1.3).

Definition 6.1.3. 1. A *model site* is a pair $(\mathbb{C},\ S)$, where $\mathbb{C}$ is a small simplicial category and S is a *set* of maps in sPsh($\mathbb{C}$) such that the localization sPsh($\mathbb{C}$)$_S$ is left exact.

2. A *model topos* is a model category $\mathscr{M}$ which is Quillen equivalent (see Definition 4.1.38) to a left exact localization sPsh($\mathbb{C}$)$_S$ for some model site $(\mathbb{C},\ S)$.

Remark 6.1.4. Let $\mathscr{M}$ be a model topos. From Definition 6.1.3 above we get that

(i) $\mathscr{M}$ has small simplicial presentation (see also Corollary 5.3.5 and Proposition 5.3.7);

(ii) if $\mathscr{N}$ is a model category which is Quillen equivalent to $\mathscr{M}$ (see Definition 4.1.38), then $\mathscr{N}$ is a model topos as well.

Example 6.1.5. The most basic example of model topos is of course given by the category **sSet** of simplicial sets (with the Kan-Quillen model structure of Example 4.1.19) and, more generally, by sPsh($\mathbb{C}$) for any small simplicial category $\mathbb{C}$.

Example 6.1.6. Since **Top** with the Quillen model structure is Quillen equivalent to **sSet** (see Examples 4.1.18 and 4.1.39), Remark 6.1.4 says that **Top** is a model topos as well. However, it may be worth pointing out that, unlike **sSet**, **Top** is *not* a Grothendieck topos. (Here by **Top** we mean, say, the category of compactly generated and Hausdorff topological spaces).

Definition 6.1.7. Let $\mathscr{M}$ be a model category and let $\mathscr{J}$ be a small category. Suppose given a natural transformation $\tau\colon X \implies Y$ between functors $X, Y\colon \mathscr{J} \longrightarrow \mathscr{M}$. We say that τ is *(homotopically)*

equifibered if, for every arrow $j \to k$ in $\mathscr{J}$, the commutative square

$$
\begin{array}{ccc}
X(j) & \xrightarrow{\tau_j} & Y(j) \\
\downarrow & & \downarrow \\
X(k) & \xrightarrow[\tau_k]{} & Y(k)
\end{array}
$$

is a homotopy pullback square in $\mathscr{M}$.

With the concept of homotopically equifibered natural transformations at hand, we can give the following, fundamental

Definition 6.1.8 (cf. [Rzk1], §6.5)**.** Let $\mathscr{M}$ be a model category. We say that $\mathscr{M}$ *has* (or *satisfies*) *descent* if the following properties hold in $\mathscr{M}$.

(P1) Let $Y \to Z$ be a map in $\mathscr{M}$. Suppose given a functor

$$
K \colon \mathscr{I} \longrightarrow \mathrm{Arr}(\mathscr{M})/(Y \to Z),
$$

where $\mathscr{I}$ is a small category, $\mathrm{Arr}(\mathscr{M})$ is the category of arrows in $\mathscr{M}$ and $\mathrm{Arr}(\mathscr{M})/(Y \to Z)$ is the category of objects in $\mathrm{Arr}(\mathscr{M})$ over $(Y \to Z) \in \mathrm{Arr}(\mathscr{M})$. Write, for each $i \in \mathscr{I}$,

$$
K(i) = \left(
\begin{array}{ccc}
P_i & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z_i & \longrightarrow & Z
\end{array}
\right)
\tag{6.1}
$$

and assume that, for every $i \in \mathscr{I}$, the commutative square (6.1) is a homotopy pullback square. Then the induced commutative square

$$
\begin{array}{ccc}
\mathrm{hocolim}_{i \in \mathscr{I}}\, P_i & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathrm{hocolim}_{i \in \mathscr{I}}\, Z_i & \longrightarrow & Z
\end{array}
\tag{6.2}
$$

is a homotopy pullback square. Here the maps $\mathrm{hocolim}_{i \in \mathscr{I}} P_i \to Y$ and $\mathrm{hocolim}_{i \in \mathscr{I}} Z_i \to Z$ are given as the composite maps $\mathrm{hocolim}_{i \in \mathscr{I}} P_i \to \mathrm{colim}_{i \in \mathscr{I}} P(i) \to Y$ and $\mathrm{hocolim}_{i \in \mathscr{I}} Z_i \to \mathrm{colim}_{i \in \mathscr{I}} Z(i) \to Z$ respectively.

(P2) Let $Y \to Z$ be a map in $\mathscr{M}$ and suppose given a functor

$$
K \colon \mathscr{I} \longrightarrow \mathrm{Arr}(\mathscr{M})/(Y \to Z),
$$

for a small category $\mathscr{I}$. Writing, for each $i \in \mathscr{I}$, $K(i)$ as in (6.1) above, we get functors

$$
P \colon \mathscr{I} \longrightarrow \mathscr{M}, \quad i \mapsto P_i \quad \text{and} \quad Z_\bullet \colon \mathscr{I} \longrightarrow \mathscr{M}, \quad i \mapsto Z_i
$$

and a natural transformation $\tau \colon P \Longrightarrow Z_\bullet$. Assume that τ is homotopically equifibered and that the canonical maps $\mathrm{hocolim}_{i \in \mathscr{I}} P_i \to Y$, $\mathrm{hocolim}_{i \in \mathscr{I}} Z_i \to Z$ are weak equivalences. Then, for every $i \in \mathscr{I}$, the square (6.1) is a homotopy pullback square.

Remark 6.1.9. With a slight abuse of language, we will usually refer to the datum of the functor K appearing in Definition 6.1.8 by saying that we are given *functorial commutative squares* (or *functorial homotopy pullback squares*) in $\mathscr{M}$

$$
\begin{array}{ccc}
P_i & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z_i & \longrightarrow & Z
\end{array}
$$

indexed by a small category $\mathscr{I}$.

There are some observations about properties (P1) and (P2) in the definition of descent that may be worth pointing out. The reader should keep in mind Proposition 2.1.14 and Remark 2.1.15 for a comparison with the categorical notion of weak descent.

Remark 6.1.10. The notion of descent does *not* depend upon the specific choices of the homotopy colimit functors involved. This means that, for a fixed small category $\mathscr{I}$, a model category $\mathscr{M}$ satisfies (P1) and (P2) of Definition 6.1.8 with respect to *one* fixed homotopy colimit functor $\operatorname{hocolim}_{i\in\mathscr{I}}\colon \mathscr{M}^{\mathscr{I}} \longrightarrow \mathscr{M}$ if and only if it satisfies those properties for *any* such homotopy colimit functor. The reason is that the maps $\operatorname{hocolim}_{i\in\mathscr{I}} P_i \to Y$ and $\operatorname{hocolim}_{i\in\mathscr{I}} Z_i \to Z$ considered in (P1) and in (P2) factors (by their own definitions) through the colimits $\operatorname{colim}_{i\in\mathscr{I}} P_i$ and $\operatorname{colim}_{i\in\mathscr{I}} Z_i$ respectively and every two homotopy colimit functors of shape $\mathscr{I}$ are weakly equivalent as functors over $\operatorname{colim}_{i\in\mathscr{I}}$ (see Remark 4.5.25). This implies, for example, that, in the situation of (P1), if $\operatorname{hocolim}_{i\in\mathscr{I}}$ and $\operatorname{hocolim}^1_{i\in\mathscr{I}}$ are two homotopy colimit functors of shape $\mathscr{I}$, then (6.2) is a homotopy pullback square if and only if the same square where $\operatorname{hocolim}_{i\in\mathscr{I}}$ is substituted with $\operatorname{hocolim}^1_{i\in\mathscr{I}}$ is a homotopy pullback square as well, because those two squares are weakly equivalent.

Remark 6.1.11. Property (P1) of descent implies what follows. Let $\mathscr{M}$ be a model categories where (P1) holds. Suppose we are given functorial cospans

$$
\begin{array}{c}
Y \\
\downarrow \\
X_i \longrightarrow Z
\end{array}
$$

in $\mathscr{M}$ indexed by the small category $\mathscr{I}$. Then $\operatorname{hocolim}_{i\in\mathscr{I}}(X_i \times_Z^h Y) \approx (\operatorname{hocolim}_{i\in\mathscr{I}} X_i) \times_Z^h Y$ (see Definition 4.1.10). Indeed, if R is a right deformation retract for the pullback functor, we have functorial homotopy pullback squares

$$
\begin{array}{ccc}
X_i \times_Z^h Y = RX_i \times_{RZ} RY & \longrightarrow & RY \\
\downarrow & & \downarrow \\
RX_i & \longrightarrow & RZ
\end{array}
$$

where $RX_i \to RZ \leftarrow RY := R(X_i \to Z \leftarrow Y)$. We thus get the zig-zag of weak equivalences

$$
\operatorname{hocolim}_{i\in\mathscr{I}}(X_i \times_Z^h Y) \xrightarrow{\sim} (\operatorname{hocolim}_{i\in\mathscr{I}} RX_i) \times_{RZ}^h RY \xleftarrow{\sim} (\operatorname{hocolim}_{i\in\mathscr{I}} X_i) \times_Z^h Y \tag{6.3}
$$

where the right-pointing arrow is a weak equivalence by (P1), whereas the left-pointing arrow is a weak equivalence because the homotopy pullback is a homotopical functor.

Remark 6.1.12. Property (P2) of descent has the following consequence. Let $\mathscr{M}$ be a model category satisfying (P2) and suppose given a homotopically equifibered natural transformation $\tau\colon Y \Longrightarrow X$ between functors $X, Y\colon \mathscr{I} \longrightarrow \mathscr{M}$, for a small category $\mathscr{I}$. Write $\operatorname{hocolim}_{i\in\mathscr{I}} = \operatorname{colim} \circ Q^{\mathscr{I}}$, where $Q^{\mathscr{I}}$ is a left ($\operatorname{colim}_{i\in\mathscr{I}}$)-deformation. Then, for each $i \in \mathscr{I}$, the commutative square

$$
\begin{array}{ccc}
Q^{\mathscr{I}}Y(i) & \longrightarrow & \operatorname{hocolim}_{i\in\mathscr{I}} Y_i \\
{\scriptstyle Q^{\mathscr{I}}\tau_i}\downarrow & & \downarrow{\scriptstyle \operatorname{hocolim}_{i\in\mathscr{I}}\tau_i} \\
Q^{\mathscr{I}}X(i) & \longrightarrow & \operatorname{hocolim}_{i\in\mathscr{I}} X_i
\end{array}
\tag{6.4}
$$

is a homotopy pullback square. Indeed, $Q^{\mathscr{I}}\tau$ is clearly homotopically equifibered because, for every arrow $j \to k$ in $\mathscr{I}$, we have a commutative cube in $\mathscr{M}$

$$
\begin{array}{ccc}
 & Q^{\mathscr{I}}Y(j) \longrightarrow Q^{\mathscr{I}}X(j) & \\
 & \swarrow{\sim}\quad\downarrow \qquad \swarrow{\sim}\quad\downarrow & \\
Y(j) & \longrightarrow X(j) & \\
\downarrow & Q^{\mathscr{I}}Y(k) \longrightarrow Q^{\mathscr{I}}X(k) & \\
\downarrow & \swarrow{\sim}\quad\downarrow \qquad \swarrow{\sim} & \\
Y(k) & \longrightarrow X(k) &
\end{array}
$$

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and the front face is a homotopy pullback square by hypothesis. Thus the fact that (6.4) is a homotopy pullback square is just a specific instance of (P2).

We stressed in Remark 6.1.10 that our formulation of descent does not depend upon the chosen models for homotopy colimits in a model category $\mathscr{M}$. However, what we actually want is a notion of descent which does not depend upon the chosen model for the homotopy theory presented by $\mathscr{M}$. In other words, we want descent to be invariant under Quillen equivalences. Before proving that this is actually the case, we need the following result from general model category theory.

Lemma 6.1.13. *Let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{N} \qquad \perp$$

be a Quillen pair between model categories $\mathscr{M}$ and $\mathscr{N}$. Then the following hold.

(i) *Any point-set right derived functor $\mathbb{R}G$ of G (see Remark 4.1.33) preserves homotopy pullback squares.*

(ii) *If (F, G) is a Quillen equivalence, then any point-set left derived functor $\mathbb{L}F$ of F preserves homotopy pullback squares as well.*

Proof. Write $\mathbb{R}G = G \circ R$, where R is a functorial fibrant approximation in $\mathscr{N}$. We consider the right deformation for the pullback functor on $\mathscr{N}$ which sends a cospan $X \to Y \leftarrow Z$ in $\mathscr{N}$ to the cospan $R'X \twoheadrightarrow R'Y \twoheadleftarrow R'Z$ that is obtained by considering $RX \to RY \leftarrow RZ$ and functorially factoring the two maps $RX \to RY$ and $RZ \to RY$ as a weak equivalence followed by a fibration (see Proposition 4.5.33). Note that $R'X$, $R'Z$ and $R'Y$ are all fibrant objects.

(i) Suppose given a homotopy pullback square

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

in $\mathscr{N}$. Applying R to vertices and arrows of this square we thus get another homotopy pullback square, so that we have a weak equivalence $RP \overset{\sim}{\to} R'(RX) \times_{R'(RZ)} R'(RY) = RX \times_{RZ}^h RY$. Since G is a right adjoint and preserves weak equivalences between fibrant objects, we get a weak equivalence

$$GRP \overset{\sim}{\to} G(R'(RX) \times_{R'(RZ)} R'(RY)) \cong GR'(RX) \times_{GR'(RZ)} GR'(RY)$$

As G is a right Quillen functor, the pullback on the right hand side is the limit of a cospan made of fibrations between fibrant object, so that

$$\begin{array}{ccc} GR'(RX) \times_{GR'(RZ)} GR'(RY) & \longrightarrow & GR'(RY) \\ \downarrow & & \downarrow \\ GR'(RX) & \longrightarrow & GR'(RZ) \end{array}$$

is a homotopy pullback square, i.e. $GR'(RX) \times_{GR'(RZ)} GR'(RY) \to GR'(RX) \times_{GR'(RZ)}^h GR'(RY)$ is a weak equivalence. Since the homotopy pullback is a homotopical functor, we also have a weak equivalence $GRX \times_{GRZ}^h GRY \overset{\sim}{\to} GR'(RX) \times_{GR'(RZ)}^h GR'(RY)$. Therefore, the commutative triangle

$$\begin{array}{ccc} GRP & \longrightarrow & GRX \times_{GRZ}^h GRY \\ & \searrow{\scriptstyle \sim} & \downarrow{\scriptstyle \sim} \\ & & GR'(RX) \times_{GR'(RZ)}^h GR'(RY) \end{array}$$

and the two-out-of-three property for weak equivalence imply that $\mathbb{R}GP \to \mathbb{R}GX \times_{\mathbb{R}GZ}^h \mathbb{R}GY$ is a weak equivalence, as required.

(ii) Write $\mathbb{L}F$ as $F \circ Q$ for Q a functorial cofibrant replacement in $\mathscr{M}$ and consider any homotopy pullback square

$$
\begin{array}{ccc}
P & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z
\end{array}
$$

in $\mathscr{M}$. We have the following commutative cube in $\mathscr{N}$

$$
\begin{array}{ccc}
FQP & \longrightarrow & FQY \\
& RFQP \longrightarrow RFQY & \\
& & \\
FQX & \longrightarrow & FQZ \\
& RFQX \longrightarrow RFQZ &
\end{array}
$$

so that the front square is a homotopy pullback square in $\mathscr{N}$ if and only if the back square is such. Hence we need to show that $h\colon RFQP \to RFQX \times^h_{RFQZ} RFQY$ is a weak equivalence in $\mathscr{N}$. Now, $RFQX \times^h_{RFQZ} RFQY = R'RFQX \times_{R'RFQZ} R'RFQY$, so we get a map

$$Gh\colon GRFQP \to G(R'RFQX \times_{R'RFQZ} R'RFQY) \cong GR'RFQX \times_{GR'RFQZ} GR'RFQY$$

As in part (i) above, there are weak equivalences

$$GR'RFQX \times_{GR'RFQZ} GR'RFQY \overset{\sim}{\to} GR'RFQX \times^h_{GR'RFQZ} GR'RFQY$$

and

$$\mathbb{R}G\mathbb{L}FX \times^h_{\mathbb{R}G\mathbb{L}FZ} \mathbb{R}G\mathbb{L}FY \overset{\sim}{\to} GR'RFQX \times^h_{GR'RFQZ} GR'RFQY$$

and they are such that Gh is a weak equivalence if and only if the map

$$\mathbb{R}G\mathbb{L}FP \to \mathbb{R}G\mathbb{L}FX \times^h_{\mathbb{R}G\mathbb{L}FZ} \mathbb{R}G\mathbb{L}FY$$

is a weak equivalence. Consider now the following commutative square in $\mathscr{M}$

$$
\begin{array}{ccc}
QP & \longrightarrow & QX \times^h_{QZ} QY \\
\downarrow & & \downarrow \\
\mathbb{R}G\mathbb{L}FP & \longrightarrow & \mathbb{R}G\mathbb{L}FX \times^h_{\mathbb{R}G\mathbb{L}FZ} \mathbb{R}G\mathbb{L}FY
\end{array}
$$

The left vertical arrow is a weak equivalence because QP is cofibrant and (F, G) is a Quillen equivalence (see Proposition 4.1.37). The same reason, together with the fact that the homotopy pullback is a homotopical functor, gives that the right vertical arrow is a weak equivalence. Finally, since $P \to X \times^h_Z Y$ is a weak equivalence by hypothesis, so is $QP \to QX \times^h_{QZ} QY$. Therefore, we get that $\mathbb{R}G\mathbb{L}FP \to \mathbb{R}G\mathbb{L}FX \times^h_{\mathbb{R}G\mathbb{L}FZ} \mathbb{R}G\mathbb{L}FY$ is also a weak equivalence. By the above discussion, this implies that Gh is a weak equivalence as well. But h is a map between fibrant objects in $\mathscr{N}$ and then, since G is the right adjoint in a Quillen equivalence, the fact that Gh is a weak equivalence allows us to conclude that also h is a weak equivalence.[1]

$\square$

Proposition 6.1.14. *Let*

$$\mathscr{M} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{N} \quad \perp$$

be a Quillen equivalence between model categories $\mathscr{M}$ and $\mathscr{N}$. Then $\mathscr{M}$ has descent if and only if $\mathscr{N}$ has descent.

---

[1] The right adjoint $G\colon \mathscr{N} \longrightarrow \mathscr{M}$ in a Quillen equivalence *reflects weak equivalences between fibrant objects*, i.e. if $h\colon A \to B$ is an arrow between fibrant objects such that Gh is a weak equivalence, then so is h. See [Hov], Corollary 1.3.16.

Proof. We will only show that if $\mathscr{M}$ has descent then so has $\mathscr{N}$, the other implication is similar. Write $\mathbb{L}F = F \circ Q$ and $\mathbb{R}G = G \circ R$ where Q and R denotes, as usual, a functorial cofibrant approximation in $\mathscr{M}$ and a functorial fibrant approximation in $\mathscr{N}$ respectively.

(P1) Suppose given a functorial homotopy pullback square in $\mathscr{N}$

$$
\begin{array}{ccc}
P_i & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z_i & \longrightarrow & Z
\end{array}
$$

indexed by the small category $\mathscr{I}$ (see Remark 6.1.9). We need to show that

$$
h \colon \operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{N}} P_i \to \left(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{N}} Z_i \right) \times_Z^h Y
$$

is a weak equivalence in $\mathscr{N}$. By Lemma 6.1.13, we get functorial homotopy pullback squares in $\mathscr{M}$ given by

$$
\begin{array}{ccc}
\mathbb{R}GP_i & \longrightarrow & \mathbb{R}GY \\
\downarrow & & \downarrow \\
\mathbb{R}GZ_i & \longrightarrow & \mathbb{R}GZ
\end{array}
$$

Since $\mathscr{M}$ has (P1), we obtain a weak equivalence

$$
\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GP_i \xrightarrow{\sim} \left(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GZ_i \right) \times_{\mathbb{R}GZ}^h \mathbb{R}GY
$$

Applying $\mathbb{L}F$ to such a map we find a weak equivalence

$$
k \colon \mathbb{L}F\left(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GP_i \right) \xrightarrow{\sim} \mathbb{L}F\left(\left(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GZ_i \right) \times_{\mathbb{R}GZ}^h \mathbb{R}GY \right)
$$

in $\mathscr{N}$. But now this morphism is naturally weakly equivalent to our h, which is then a weak equivalence itself[2]. For the domain of k this is because $\mathbb{L}F$ commutes with homotopy colimits (see Proposition 4.5.27 and Remark 4.5.28) and because there is a natural weak equivalence $\mathbb{L}F\mathbb{R}GP_i \xrightarrow{\sim} RP_i$, for each $i \in \mathscr{I}$, since (F, G) is a Quillen equivalence (see Proposition 4.1.37). For the codomain of k, using again commutativity with homotopy colimits of $\mathbb{L}F$ as well as the properties of Quillen equivalences, we get instead

$$
\mathbb{L}F\left(\left(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GZ_i \right) \times_{\mathbb{R}GZ}^h \mathbb{R}GY \right) \overset{(\dagger)}{\approx} \left(\mathbb{L}F(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GZ_i) \right) \times_{\mathbb{L}F\mathbb{R}GZ}^h \mathbb{L}F\mathbb{R}GY \approx
$$

$$
\approx \left(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{N}} \mathbb{L}F\mathbb{R}GZ_i \right) \times_{\mathbb{L}F\mathbb{R}GZ}^h \mathbb{L}F\mathbb{R}GY \approx \left(\operatorname{hocolim}_{i \in \mathscr{I}}^{\mathscr{N}} Z_i \right) \times_Z^h Y
$$

(see Definition 4.1.10), where $(\dagger)$ follows because $\mathbb{L}F$ preserves homotopy pullback squares.

(P2) Suppose given functorial commutative squares in $\mathscr{N}$

$$
D(i) := \left(
\begin{array}{ccc}
P_i & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z_i & \longrightarrow & Z
\end{array}
\right)
$$

indexed by a small category $\mathscr{I}$ and satisfying the hypotheses of property (P2) as in Definition 6.1.8. Considering the functorial commutative squares

$$
\mathbb{R}GD(i) := \left(
\begin{array}{ccc}
\mathbb{R}GP_i & \longrightarrow & \mathbb{R}GY \\
\downarrow & & \downarrow \\
\mathbb{R}GZ_i & \longrightarrow & \mathbb{R}GZ
\end{array}
\right)
$$

---

[2] This works under the assumption that the relevant square in the homotopy category commutes. Even if we could not verify it fully, we expect this to be true because all the weak equivalences involved are built as canonical maps, whereas the zig-zags of weak equivalences we consider are essentially unique.

in $\mathscr{M}$, we get that $\mathbb{R}GP \Longrightarrow \mathbb{R}GZ_\bullet$ is equifibered. (Here $Z_\bullet$ denotes the functor $i \to Z_i$ from $\mathscr{I}$ to $\mathscr{M}$). This is because $P \Longrightarrow Z_\bullet$ is equifibered by hypothesis and $\mathbb{R}G$ preserves equifibered natural transformations, as it preserves homotopy pullback squares. Furthermore, since $\mathrm{hocolim}^{\mathscr{N}} P_i \to Y$ is a weak equivalence in $\mathscr{N}$, we have

$$\mathrm{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GP_i \approx \mathbb{R}G\left(\mathrm{hocolim}^{\mathscr{N}} P_i\right) \xrightarrow{\sim} \mathbb{R}GY$$

(see Proposition 4.5.29), so that[3] $\mathrm{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GP_i \xrightarrow{\sim} \mathbb{R}GY$ and analogously $\mathrm{hocolim}_{i \in \mathscr{I}}^{\mathscr{M}} \mathbb{R}GZ_i \xrightarrow{\sim} \mathbb{R}GZ$. By (P2) in $\mathscr{N}$, for every $i \in \mathscr{I}$, $\mathbb{R}GD(i)$ is thus a homotopy pullback square. But now, in the following commutative cube in $\mathscr{N}$

the back square is a homotopy pullback square by Lemma 6.1.13 and the diagonal arrows are weak equivalences because (F, G) is a Quillen equivalence. Therefore, also the front square is a homotopy pullback square, so that $D(i)$ is such as well.

$\square$

The rest of this section is devoted to show the following

Proposition 6.1.15. *Every model topos has descent.*

The proof of Proposition 6.1.15 is not completely straightforward and we will need some auxiliary results from [Rzk2] for it. We are going to adopt the following strategy.

(S1) We show that **sSet** has descent.

(S2) We prove that, for a small simplicial category $\mathbb{C}$, $\mathrm{sPsh}(\mathbb{C})$ has descent.

(S3) We verify that, if $\mathscr{M}$ is a model category with descent, then every left exact localization $\mathscr{M}_S$ of $\mathscr{M}$ also has descent.

Having (S1)$\cdots$(S3) at hand, we can conclude that Proposition 6.1.15 holds because having descent for a model category is invariant under Quillen equivalences (see Proposition 6.1.14).

We then proceed to validate (S1)$\cdots$(S3) separately.

(S1) We start by introducing a notion which makes sense in every model category $\mathscr{M}$.

Definition 6.1.16. Let $\mathscr{M}$ be a model category and let $\mathscr{I}$ be a small category. A functor $X\colon \mathscr{I} \longrightarrow \mathscr{M}$ is called a *homotopy colimit diagram* if the map

$$\mathrm{hocolim}_{\mathscr{I}} X \to \mathrm{colim}_{\mathscr{I}} X$$

is a weak equivalence.

Note that the definition does not depend upon the choice of the homotopy colimit of shape $\mathscr{I}$ on $\mathscr{M}$. The concept of homotopy colimit diagram plays a central role in the following result, which we borrow from [Rzk2].

---

[3] See footnote 2.

Theorem 6.1.17. *Let $\tau\colon X \Longrightarrow Y$ be a natural transformation between functors $X, Y\colon \mathscr{I} \longrightarrow \mathbf{sSet}$ from a small category $\mathscr{I}$ and assume that Y is a homotopy colimit diagram. Then the following statements hold.*

1. *If, for any $i \in \mathscr{I}$, the following square*

$$
\begin{array}{ccc}
X(i) & \longrightarrow & \operatorname{colim}_{\mathscr{I}} X \\
{\scriptstyle \tau_i}\downarrow & & \downarrow{\scriptstyle \operatorname{colim}_{\mathscr{I}} \tau} \\
Y(i) & \longrightarrow & \operatorname{colim}_{\mathscr{I}} Y
\end{array}
\tag{6.5}
$$

   is a homotopy pullback square, then X is a homotopy colimit diagram.

2. *If X is a homotopy colimit diagram and τ is equifibered, then, for all $i \in \mathscr{I}$, the square (6.5) is a homotopy pullback square.*

Proof. This is a particular instance of [Rzk2], Theorem 1.4 (see also Example 3.12 there). □

We now show how Theorem 6.1.17 implies descent for simplicial sets.

As for (P1), let us consider functorial homotopy pullback squares in $\mathbf{sSet}$ indexed by a small category $\mathscr{I}$ as in (6.2). Since $\mathbf{sSet}$ is a proper model category (see Remark 4.2.16), for every $i \in \mathscr{I}$, a homotopy pullback for the cospan $Z_i \to Z \xleftarrow{f} Y$ is given as the ordinary pullback $Z_i \times_Z E(f)$ for a (functorial) factorization

$$
Y \xrightarrow{\sim} E(f) \xtwoheadrightarrow{f'} Y
$$

of f into a weak equivalence followed by a fibration (see Proposition 4.5.35). Thus, by the hypothesis that $P_i \to Z_i \times_Z^h Y$ is a weak equivalence, we get the following commutative square in $\mathbf{sSet}$

$$
\begin{array}{ccc}
\operatorname{hocolim}_{i\in\mathscr{I}} P_i & \longrightarrow & (\operatorname{hocolim}_{i\in\mathscr{I}} Z_i) \times_Z^h Y \\
{\scriptstyle \sim}\downarrow & & \downarrow{\scriptstyle \sim} \\
\operatorname{hocolim}_{i\in\mathscr{I}} (Z_i \times_Z E(f)) & \longrightarrow & (\operatorname{hocolim}_{i\in\mathscr{I}} Z_i) \times_Z^h E(f)
\end{array}
$$

where the vertical arrows are weak equivalences because the homotopy colimit is homotopical. Hence, it is enough to show that the bottom horizontal map is a weak equivalence. Write $\operatorname{hocolim}_{i\in\mathscr{I}} = \operatorname{colim}_{i\in\mathscr{I}} \circ Q^{\mathscr{I}}$, where $Q^{\mathscr{I}}$ is a left $(\operatorname{colim}_{i\in\mathscr{I}})$-deformation. Set $\bar{Z} := \operatorname{colim}_{i\in\mathscr{I}}(Q^{\mathscr{I}} Z)(i) = \operatorname{hocolim}_{i\in\mathscr{I}} Z(i)$ and consider, for every $i \in \mathscr{I}$, the following pullback square

$$
D(i) := \left(
\begin{array}{ccc}
(Q^{\mathscr{I}} Z_\bullet)(i) \times_Z E(f) \cong (Q^{\mathscr{I}} Z)(i) \times_{\bar{Z}} (\bar{Z} \times_Z E(f)) & \longrightarrow & \bar{Z} \times_Z E(f) \\
\downarrow & & \downarrow \\
(Q^{\mathscr{I}} Z_\bullet)(i) & \longrightarrow & \bar{Z}
\end{array}
\right)
$$

Note that the right vertical map is a fibration (it is the pullback along $\bar{Z} \to Z$ of the fibration $f'\colon E(f) \twoheadrightarrow Z$), so that $D(i)$ is a homotopy pullback square (see Proposition 4.2.21 and Corollary 4.5.38). We also have an isomorphism

$$
\operatorname{colim}_{i\in\mathscr{I}} \big((Q^{\mathscr{I}} Z_\bullet)(i) \times_Z E(f)\big) \xrightarrow{\cong} \bar{Z} \times_Z E(f)
$$

because $\mathbf{sSet}$ is a Grothendieck topos, so it satisfies (categorical) weak descent (see Proposition 2.1.11, Proposition 2.1.14 and Remark 2.1.15). Now, since $Q^{\mathscr{I}} Z_\bullet$ is a homotopy colimit diagram (there is a weak equivalence $\operatorname{hocolim}_{i\in\mathscr{I}}(Q^{\mathscr{I}} Z_\bullet)(i) \xrightarrow{\sim} \operatorname{hocolim}_{i\in\mathscr{I}} Z(i)$ and the right-hand side is the colimit of $Q^{\mathscr{I}} Z_\bullet$), Theorem 6.1.17, says that we have a weak equivalence

$$
\operatorname{hocolim}_{i\in\mathscr{I}} \big((Q^{\mathscr{I}} Z_\bullet)(i) \times_Z E(f)\big) \xrightarrow{\sim} \operatorname{colim}_{i\in\mathscr{I}} \big((Q^{\mathscr{I}} Z_\bullet)(i) \times_Z E(f)\big) \cong \bar{Z} \times_Z E(f) \overset{def}{=}
$$

$$
= (\operatorname{hocolim}_{i\in\mathscr{I}} Z(i)) \times_Z E(F)
$$

Finally, the commutative diagram

$$\text{hocolim}_{i\in\mathscr{I}}\left((Q^{\mathscr{I}}Z_\bullet)(i)\times_Z E(f)\right) \xrightarrow{\ \sim\ } (\text{hocolim}_{i\in\mathscr{I}} Z(i))\times_Z E(F) \xrightarrow{\ \sim\ } (\text{hocolim}_{i\in\mathscr{I}} Z(i))\times_Z^h E(F)$$

$$\text{hocolim}_{i\in\mathscr{I}}\left(Z(i)\times_Z E(F)\right)$$

gives that our map $\text{hocolim}_{i\in\mathscr{I}}\left(Z(i)\times_Z E(F)\right) \to (\text{hocolim}_{i\in\mathscr{I}} Z(i))\times_Z^h E(F)$ is a weak equivalence, as required. Thus, **sSet** verifies (P1).

We now turn to prove that (P2) holds for **sSet**. Take then functorial commutative squares in **sSet**

$$K(i) := \begin{pmatrix} P_i \longrightarrow Y \\ \downarrow \qquad \downarrow \\ Z_i \longrightarrow Z \end{pmatrix}$$

indexed by a small category $\mathscr{I}$ as in the statement of (P2) in Definition 6.1.8, so that the induced natural transformation $\tau\colon P \Longrightarrow Z_\bullet$ is equifibered and we have weak equivalences $\text{hocolim}_{i\in\mathscr{I}} P_i \xrightarrow{\sim} Y$, $\text{hocolim}_{i\in\mathscr{I}} Z_i \xrightarrow{\sim} Z$. Writing as above $\text{hocolim}_{i\in\mathscr{I}} = \text{colim}_{i\in\mathscr{I}} \circ Q^{\mathscr{I}}$, we then get that also $Q^{\mathscr{I}}\tau$ is equifibered. Hence, Theorem 6.1.17 gives that, for each $i \in \mathscr{I}$, there is a homotopy pullback square

$$(Q^{\mathscr{I}}P)(i) \longrightarrow \text{colim}_{i\in\mathscr{I}}(Q^{\mathscr{I}}P)(i) = \text{hocolim}_{i\in\mathscr{I}} P_i$$

$$(Q^{\mathscr{I}}Z_\bullet)(i) \longrightarrow \text{colim}_{i\in\mathscr{I}}(Q^{\mathscr{I}}Z_\bullet)(i) = \text{hocolim}_{i\in\mathscr{I}} Z_i$$

We then conclude as the following commutative cube in **sSet**

$$
\begin{array}{c}
(Q^{\mathscr{I}}P)(i) \longrightarrow \text{hocolim}_{i\in\mathscr{I}} P_i \\
P_i \longrightarrow Y \\
(Q^{\mathscr{I}}Z_\bullet)(i) \longrightarrow \text{hocolim}_{i\in\mathscr{I}} Z_i \\
Z_i \longrightarrow Z
\end{array}
$$

witnesses that our starting $K(i)$ is indeed a homotopy pullback square.

(S2) Given a small simplicial category $\mathbb{C}$, descent for $\text{sPsh}(\mathbb{C})$ (with the projective model structure) follows formally from the pointwise definition of weak equivalences and from the fact that in $\text{sPsh}(\mathbb{C})$ *homotopy colimit and limits* can be *computed pointwise*. A quick explanation for this comes by noticing that the formulas for framed homotopy (co)limits in a simplicial category of Remark 4.5.49 can be expressed as weighted (**sSet**-)colimits and limits (once we see an ordinary category as a discrete simplicial category) and in the **sSet**−enriched category of **sSet**−enriched functors weighted (co)limits (exist and) can be computed objectwise (see [Kel], §3.3). Thus, fixing a functorial cofibrant approximation Q in $\text{sPsh}(\mathbb{C})$, we get, for a functor $X\colon \mathscr{I} \longrightarrow \text{sPsh}(\mathbb{C})$ from a small category $\mathscr{I}$ and for $A \in \mathbb{C}$, $(\text{hocolim}\, X)(A) \cong (QX)(A) \otimes_{\mathbb{I}} N(- \downarrow \mathscr{I})^{\text{op}}$, where $\mathbb{I}$ denotes the category $\mathscr{I}$ seen as a discrete simplicial category, $(-) \otimes_{\mathbb{I}} (?)$ denotes the colimit of the left-hand side weighted by the right-hand side and, finally, where $(QX)(A)$ is the (ordinary) functor $\mathscr{I} \longrightarrow \text{sSet}$ sending $i \in \mathscr{I}$ to $(QX)_i(A)$. Since now, for each $i \in \mathscr{I}$, $(QX)_i(A)$ is a cofibrant approximation of $(X_i)(A)$ because cofibrant objects for the projective

model structure are pointwise such, $(QX)(A) \otimes_{\mathbb{I}} N(- \downarrow \mathscr{I})^{\mathrm{op}}$ computes (up to weak equivalences) the homotopy colimit in **sSet** of $(X_{(-)})(A) \colon \mathscr{I} \longrightarrow \mathbf{sSet}$. In other words, the functor

$$A \mapsto \mathrm{hocolim}_{i \in \mathscr{I}} X_i(A)$$

is a homotopy colimit for X. The same kind of remark applies to homotopy limits in $\mathrm{sPsh}(\mathbb{C})$ as well.

(S3) Let $\mathscr{M}$ be a model category having descent and suppose that $\mathscr{M}_S$ (exists and) is a left exact localization of $\mathscr{M}$, for some class S of maps in $\mathscr{M}$ (see Definition 6.1.1). We want to prove that also $\mathscr{M}_S$ has descent. Consider the associated Quillen pair

$$\mathscr{M} \underset{i}{\overset{a}{\rightleftarrows}} \perp \; \mathscr{M}_S$$

where both functors are the identity on (the underlying category of) $\mathscr{M}$ and a is left exact. Write $(\mathbb{L}a)(-) = (a \circ Q)(-) = Q(-)$ and $(\mathbb{R}i)(-) = (i \circ R_S)(-) = R_S(-)$, for functorial cofibrant and fibrant approximations Q and R_S on $\mathscr{M}$ and on $\mathscr{M}_S$ respectively.

In order to verify (P1) for $\mathscr{M}_S$, consider functorial commutative squares

$$K(i) := \left(\begin{array}{ccc} P_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & Z \end{array} \right)$$

in $\mathscr{M}_S$ indexed by a small category $\mathscr{I}$ and assume that each $K(i)$ is a homotopy pullback square. Since $\mathbb{R}i$ preserves homotopy pullback squares (see Lemma 6.1.13), we get functorial homotopy pullback squares

$$R_S K(i) := \left(\begin{array}{ccc} R_S P_i & \longrightarrow & R_S Y \\ \downarrow & & \downarrow \\ R_S Z_i & \longrightarrow & R_S Z \end{array} \right)$$

in $\mathscr{M}$. Property (P1) for $\mathscr{M}$ now gives a weak equivalence

$$\mathrm{hocolim}_{i \in \mathscr{I}} R_S P_i \xrightarrow{\sim} (\mathrm{hocolim}_{i \in \mathscr{I}} R_S Z_i) \times^h_{R_S Z} R_S Y$$

and applying $\mathbb{L}a$ we then obtain a weak equivalence

$$k \colon \mathbb{L}a \left(\mathrm{hocolim}_{i \in \mathscr{I}} R_S P_i \right) \xrightarrow{\sim} \mathbb{L}a \left((\mathrm{hocolim}_{i \in \mathscr{I}} R_S Z_i) \times^h_{R_S Z} R_S Y \right)$$

in $\mathscr{M}_S$. Now this morphism is naturally weakly equivalent to the map

$$h \colon \mathrm{hocolim}^S_{i \in \mathscr{I}} P_i \to \left(\mathrm{hocolim}^S_{i \in \mathscr{I}} Z_i \right) \times^h_Z Y$$

which is then a weak equivalence itself[4]. (Here $\mathrm{hocolim}^S_{i \in \mathscr{I}}$ denotes a homotopy colimit of shape $\mathscr{I}$ on $\mathscr{M}_S$). For the domain this is because $\mathbb{L}a$ commutes with homotopy colimits (see Proposition 4.5.27 and Remark 4.5.28) and Q is a cofibrant approximation not only in $\mathscr{M}$ but also in $\mathscr{M}_S$ (because $\mathscr{M}_S$ has the same cofibrant objects of $\mathscr{M}$ but more weak equivalences). As for the codomain, using again that $\mathbb{L}a$ commutes with homotopy colimits, we get instead

$$\mathbb{L}a \left((\mathrm{hocolim}_{i \in \mathscr{I}} R_S Z_i) \times^h_{R_S Z} R_S Y \right) \overset{(\dagger)}{\approx} \left(\mathbb{L}a \left(\mathrm{hocolim}_{i \in \mathscr{I}} R_S Z_i \right) \times_{\mathbb{L}a(R_S Z)} \mathbb{L}a(R_S Y) \right) \approx$$

$$\approx \left(\mathrm{hocolim}^S_{i \in \mathscr{I}} \mathbb{L}a(R_S Z_i) \right) \times^h_{\mathbb{L}a(R_S Z)} \mathbb{L}a(R_S Y) = \left(\mathrm{hocolim}^S_{i \in \mathscr{I}} Q R_S Z_i \right) \times^h_{Q R_S Z} Q R_S Y \approx$$

---

[4] See footnote 2.

$$\approx \left(\operatorname{hocolim}_{i \in \mathscr{I}}^{S} Z_i \right) \times_Z^h Y,$$

(see Definition 4.1.10), where (†) follows from the fact that a is left exact.

As for (P2), suppose given again functorial commutative squares $K(i)$ in $\mathscr{M}_S$ as above and such that the induced natural transformation $\tau \colon P \implies Z_\bullet$ is equifibered and the maps $\operatorname{hocolim}_{i \in \mathscr{I}} P_i \to Y$ and $\operatorname{hocolim}_{i \in \mathscr{I}} Z_i \to Z$ are weak equivalences. Then, as in the proof of Proposition 6.1.14, we get that $R_S K(i)$ are functorial commutative squares in $\mathscr{M}$ verifying the hypotheses for (P2) in $\mathscr{M}$. Therefore, each $R_S K(i)$ is a homotopy pullback square in $\mathscr{M}$. Since a is left exact by hypothesis, we then get that, for every $i \in \mathscr{I}$,

$$\begin{array}{ccc} QR_SP_i & \longrightarrow & QR_SY \\ \downarrow & & \downarrow \\ QR_SZ_i & \longrightarrow & QR_SZ \end{array}$$

is a homotopy pullback square which is naturally weakly equivalent in $\mathscr{M}_S$ to our starting $K(i)$. Thus, $K(i)$ is a homotopy pullback square as well.

6.2 Model-categorical Giraud's Theorem.

Before stating and proving the central theorem of our work, we show a consequence of descent.

Proposition 6.2.1. *Let $\mathcal{M}$ be a model category with descent and let $\mathcal{I}$ be a small category. Suppose given a commutative square*

$$
\begin{array}{ccc}
A & \longrightarrow & Y \\
\downarrow & & \downarrow g \\
X & \underset{f}{\longrightarrow} & B
\end{array}
$$

(6.6)

in $\mathcal{M}^{\mathcal{I}}$ with f and g homotopically equifibered and assume that such a square is objectwise a homotopy pullback square in $\mathcal{M}$. Then the induced square of homotopy colimits

$$
\begin{array}{ccc}
\operatorname{hocolim}_{\mathcal{I}} A & \longrightarrow & \operatorname{hocolim}_{\mathcal{I}} Y \\
\downarrow & & \downarrow \\
\operatorname{hocolim}_{\mathcal{I}} X & \longrightarrow & \operatorname{hocolim}_{\mathcal{I}} B
\end{array}
$$

is a homotopy pullback square in $\mathcal{M}$.

Proof. We choose the model for the homotopy pullback described in part 1. of Proposition 4.5.33. Thus, if $B \to C \leftarrow D$ is a cospan in $\mathcal{M}$, its homotopy pullback is given as the limit of the cospan $RB \twoheadrightarrow RC \twoheadleftarrow RD$ obtained by functorially replacing the objects and the arrows of $B \to C \leftarrow D$ by fibrant objects and fibrations respectively. Set also $\bar{B} := \operatorname{hocolim}_{\mathcal{I}} B$ and analogously for $\bar{A}$, $\bar{X}$ and $\bar{Y}$. We must then prove that the map

$$
\bar{A} \to \bar{X} \times^h_{\bar{B}} \bar{Y}
$$

is a weak equivalence.

Writing $\operatorname{hocolim}_{\mathcal{I}} = \operatorname{colim} \circ Q^{\mathcal{I}}$ for a left $\operatorname{colim}_{\mathcal{I}}$ −deformation $Q^{\mathcal{I}}$, we define functors

$$
X', Y', A' \colon \mathcal{I} \longrightarrow \mathcal{M}
$$

by setting, for $j \in \mathcal{I}$,

$$
X'(j) := R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} R\bar{X}, \quad Y'(j) := R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} R\bar{Y},
$$
$$
A'(j) := R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} (R\bar{X} \times_{R\bar{B}} R\bar{Y})
$$

Note that $X'(j) = (Q^{\mathcal{I}} B)(j) \times^h_{\bar{B}} \bar{X}$, $Y'(j) = (Q^{\mathcal{I}} B)(j) \times^h_{\bar{B}} \bar{Y}$ and

$$
A'(j) = R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} (\bar{X} \times^h_{\bar{B}} \bar{Y}) \xrightarrow{\sim} R(Q^{\mathcal{I}} B)(j) \times^h_{R\bar{B}} (\bar{X} \times^h_{\bar{B}} \bar{Y}),
$$

where the (natural) weak equivalence is given by the fact that $A'(j)$ is a pullback of a cospan made of fibrations between fibrant objects. Note also that we have, for $j \in \mathcal{I}$, canonical isomorphisms

$$
A'(j) = R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} (R\bar{X} \times_{R\bar{B}} R\bar{Y}) \cong (R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} R\bar{X}) \times_{R\bar{B}} R\bar{Y} \cong
$$

$$
\cong \left((R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} R\bar{X}) \times_{R(Q^{\mathcal{I}} B)(j)} R(Q^{\mathcal{I}} B)(j)) \right) \times_{R\bar{B}} R\bar{Y} \cong
$$

$$
\cong (R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} R\bar{X}) \times_{R(Q^{\mathcal{I}} B)(j)} (R(Q^{\mathcal{I}} B)(j) \times_{R\bar{B}} R\bar{Y}) = X'(j) \times_{R(Q^{\mathcal{I}} B)(j)} Y'(j)
$$

and again we have a weak equivalence $X'(j) \times_{R(Q^{\mathcal{I}} B)(j)} Y'(j) \xrightarrow{\sim} X'(j) \times^h_{R(Q^{\mathcal{I}} B)(j)} Y'(j)$ for the same reason as above. Thus we obtain a weak equivalence

$$
A'(j) \xrightarrow{\sim} X'(j) \times^h_{R(Q^{\mathcal{I}} B)(j)} Y'(j).
$$

Observe now that, since f and g are homotopically equifibered, so are $Q^{\mathcal{I}} f$ and $Q^{\mathcal{I}} g$. Hence, by (P2) in $\mathcal{M}$ (see also Remark 6.1.12), we get weak equivalences

$$
(Q^{\mathcal{I}} X)(j) \xrightarrow{\sim} (Q^{\mathcal{I}} B)(j) \times^h_{\bar{B}} \bar{X} = X'(j) \quad \text{and} \quad (Q^{\mathcal{I}} Y)(j) \xrightarrow{\sim} (Q^{\mathcal{I}} B)(j) \times^h_{\bar{B}} \bar{Y} = Y'(j)
$$

for every $j \in \mathscr{I}$. The hypothesis that (6.6) is objectwise a homotopy pullback square implies that we also have a weak equivalence

$$(Q^{\mathscr{I}}A)(j) \xrightarrow{\sim} (Q^{\mathscr{I}}X)(j) \times^h_{(Q^{\mathscr{I}}B)(j)} (Q^{\mathscr{I}}Y)(j)$$

again for each $j \in \mathscr{I}$. Therefore, we obtain composite (natural) weak equivalences

$$(Q^{\mathscr{I}}A)(j) \xrightarrow{\sim} X'(j) \times^h_{R(Q^{\mathscr{I}}B)(j)} Y'(j)$$

Passing to the homotopy colimit, we deduce that there is a weak equivalence

$$k \colon \operatorname{hocolim}_{j \in \mathscr{I}}(Q^{\mathscr{I}}A)(j) \xrightarrow{\sim} \operatorname{hocolim}_{j \in \mathscr{I}} \left(X'(j) \times^h_{R(Q^{\mathscr{I}}B)(j)} Y'(j) \right)$$

Now, for each $j \in \mathscr{I}$, there is a map

$$(Q^{\mathscr{I}}A)(j) \xrightarrow{\sim} R(Q^{\mathscr{I}}A)(j) \to A'(j) = R(Q^{\mathscr{I}}B)(j) \times_{R\bar{B}} (R\bar{X} \times_{R\bar{B}} R\bar{Y})$$

induced by $R(Q^{\mathscr{I}}A)(j) \to R(Q^{\mathscr{I}}B)(j)$, $R(Q^{\mathscr{I}}A)(j) \to R\bar{X}$ and $R(Q^{\mathscr{I}}A)(j) \to R\bar{Y}$. The homotopy colimit of these maps fits into the commutative triangle

so that we get a weak equivalence

$$\operatorname{hocolim}_{j \in \mathscr{I}}(Q^{\mathscr{I}}A)(j) \xrightarrow{\sim} \operatorname{hocolim}_{j \in \mathscr{I}} A'(j) = \operatorname{hocolim}_{j \in \mathscr{I}} \left(R(Q^{\mathscr{I}}B)(j) \times_{R\bar{B}} (R\bar{X} \times_{R\bar{B}} R\bar{Y}) \right)$$

But now we have functorial homotopy pullback squares

because all the morphisms appearing are fibrations among fibrant objects. Thus, by (P1), we have

$$\operatorname{hocolim}_{j \in \mathscr{I}} A'(j) \xrightarrow{\sim} \left(\operatorname{hocolim}_{j \in \mathscr{I}} R(Q^{\mathscr{I}}B)(j) \right) \times^h_{R\bar{B}} \left(\bar{X} \times^h_{\bar{B}} \bar{Y} \right)$$

(recall that $\bar{X} \times^h_{\bar{B}} \bar{Y} = R\bar{X} \times_{R\bar{B}} R\bar{Y}$). We then obtain a composite weak equivalence

$$\operatorname{hocolim}_{j \in \mathscr{I}}(Q^{\mathscr{I}}A)(j) \xrightarrow{\sim} \left(\operatorname{hocolim}_{j \in \mathscr{I}} R(Q^{\mathscr{I}}B)(j) \right) \times^h_{R\bar{B}} \left(\bar{X} \times^h_{\bar{B}} \bar{Y} \right) \xrightarrow{\sim}$$
$$\xrightarrow{\sim} \left(\operatorname{hocolim}_{j \in \mathscr{I}} RB(j) \right) \times^h_{R\bar{B}} \left(\bar{X} \times^h_{\bar{B}} \bar{Y} \right)$$

There is a commutative diagram

Here the bottom map is the composition

$$l \colon \bar{A} \to \bar{X} \times_{\bar{B}} \bar{Y} \cong \bar{B} \times_{\bar{B}} \left(\bar{X} \times_{\bar{B}} \bar{Y} \right) \to \bar{B} \times_{R\bar{B}} \left(\bar{X} \times_{\bar{B}} \bar{Y} \right) \to \bar{B} \times^h_{R\bar{B}} \left(\bar{X} \times^h_{\bar{B}} \bar{Y} \right)$$

and the above square witnesses that it is a weak equivalence. We conclude that the composite map

$$\bar{A} \xrightarrow{\sim} \bar{B} \times^h_{R\bar{B}} \left(\bar{X} \times^h_{\bar{B}} \bar{Y} \right) \xrightarrow{\sim} R\bar{B} \times^h_{R\bar{B}} \left(\bar{X} \times^h_{\bar{B}} \bar{Y} \right) \xrightarrow{\sim} \bar{X} \times^h_{\bar{B}} \bar{Y}$$

is a weak equivalence, as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

As announced, we can now turn to the following

Theorem 6.2.2 (**Giraud's Theorem for Model Topoi**, cf. [Rzk1], Theorem 6.9)**.** *Let $\mathscr{M}$ be a model category. Then $\mathscr{M}$ is a model topos if and only if it has small simplicial presentation and satisfies descent.*

Proof. From Remark 6.1.4 and Proposition 6.1.15 it follows that the two conditions in the Theorem are necessary for a model category to be a model topos. Suppose then that $\mathscr{M}$ admits small simplicial presentation and has descent. In particular, $\mathscr{M}$ is Quillen equivalent to a simplicial and combinatorial model category (see Theorem 4.3.31 and Theorem 5.1.20). Since having small simplicial presentation and descent as well as being a model topos are invariant properties under Quillen equivalences, we can assume that $\mathscr{M}$ is a simplicial and combinatorial model category. By Theorem 5.3.8, there are a small full simplicial subcategory $\mathbb{C}$ of $\mathscr{M}$ made of fibrant-cofibrant objects and a Quillen equivalence

$$\mathrm{sPsh}(\mathbb{C})_S \ \perp \ \mathscr{M}$$

for some small set S of maps in $\mathrm{sPsh}(\mathbb{C})$. Observe that, since $\mathscr{M}$ has descent, so has $\mathrm{sPsh}(\mathbb{C})_S$. Furthermore, we also have that the simplicial Yoneda embedding $\mathbf{y}\colon \mathbb{C} \longrightarrow \mathrm{sPsh}(\mathbb{C})$ lands into $S-$local objects. We now prove that the associated Quillen pair

$$\mathrm{sPsh}(\mathbb{C}) \ \underset{i}{\overset{a}{\perp}} \ \mathrm{sPsh}(\mathbb{C})_S$$

is such that a is homotopically left exact. Setting $\mathbb{P} := \mathrm{sPsh}(\mathbb{C})$ and $\mathbb{E} := \mathrm{sPsh}(\mathbb{C})_S$, let then

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & B \end{array} \qquad (6.7)$$

be a homotopy pullback square in $\mathbb{P}$. We need to show that

$$h\colon \mathbb{L}a(P) \to \mathbb{L}a(X) \times^h_{\mathbb{L}a(B)} \mathbb{L}a(Y) \quad \text{is a weak equivalence} \qquad (*)$$

Before doing this, we need to make another observation. As in Remark 5.1.5, we say that an object W in $\mathbb{P}$ is *quasi S-local* if, for every map g in S, the induced map of homotopy function complexes

$$g^*\colon \mathrm{map}^{\mathrm{h}}(\mathrm{cod}(g), W) \to \mathrm{map}^{\mathrm{h}}(\mathrm{dom}(g), W)$$

is a weak equivalence. Now, there is a simplicially enriched analogue of Proposition 5.6, namely, given a simplicially enriched functor $X\colon \mathbb{D}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ (for a small simplicial category $\mathbb{D}$), there is a functor $U\colon \mathscr{J} \longrightarrow \mathrm{sPsh}(\mathbb{D})$ (from a small category $\mathscr{J}$) such that each $U(j)$ is of the form $\mathbf{y}A \otimes \Delta[n]$ (for some $A \in \mathbb{D}$ and $n \in \mathbb{N}$) and we have a natural weak equivalence

$$\mathrm{hocolim}_{\mathscr{J}} U \xrightarrow{\sim} X$$

(see [Rsc], Theorem 3.2 and its proof). In our case, for every object $A \in \mathbb{C}$, $\mathbf{y}A$ is an $S-$local object and, since $\mathbf{y}A$ is cofibrant in $\mathrm{sPsh}(\mathbb{C})$, $\mathbf{y}A \otimes \Delta[-]$ is a cofibrant resolution for $\mathbf{y}A$ (see Example 4.4.11). Thus, for every $n \in \mathbb{N}$, we have a weak equivalence $\mathbf{y}A \otimes \Delta[n] \to \mathbf{y}A$, so that every $\mathbf{y}A \otimes \Delta[n]$ is a quasi $S-$local object (because homotopy function complexes sends weak equivalences to weak equivalences, see Proposition 4.4.18). Therefore, by the above discussion we get that, for each $X \in \mathrm{sPsh}(\mathbb{C})$ there is a natural weak equivalence $\mathrm{hocolim}_{\mathscr{J}} U \xrightarrow{\sim} X$ where $U\colon \mathscr{J} \longrightarrow \mathrm{sPsh}(\mathbb{C})$ lands into quasi S-local objects. For the rest of the proof, we will refer to this result by saying that

$$\text{every } X \in \mathrm{sPsh}(\mathbb{C}) \text{ is a homotopy colimit of quasi S-local objects.} \qquad (**)$$

As for the categorical counterpart seen in the proof of Proposition 2.2.18, we now show that $(*)$ holds through a sequence of specific cases.

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(a) *Property* $(*)$ *holds when X, B and Y are quasi S-local objects.* Up to taking functorial fibrant approximations in $\mathbb{P}$ of the vertices in (6.7), since being a homotopy pullback square is invariant under weak equivalences of squares, we can suppose that X, B and Y are $S-$local objects. As usual, a homotopy pullback in $\mathbb{E}$ of the cospan

$$(X \to B \leftarrow Y) = (aX \to aB \leftarrow aY)$$

can be taken as the ordinary pullback of $R_S X \twoheadrightarrow R_S B \twoheadleftarrow R_S Y$ in $\mathbb{E}$ where the latter cospan is obtained by the former functorially replacing objects with fibrant ones and arrows with fibrations in $\mathbb{E} = \mathbb{P}_S$. Thus, we have a weak equivalence of cospans in $\mathbb{E}$

$$(6.8)$$

The displayed weak equivalences, being such in $\mathbb{E} = \mathbb{P}_S$, are $S-$local equivalences in $\mathbb{P}$ by definition of the Bousfield localization. Since $R_S X$, $R_S B$ and $R_S Y$ are fibrant objects in $\mathbb{E}$, they are $S-$local objects in $\mathbb{P}$ and then each $\sim_S$ is an S-local equivalence between $S-$local objects in $\mathbb{P}$, so that it is an actual weak equivalence in $\mathbb{P}$ (see Sections 5.1 and 5.2). On the other hand, fibrations in $\mathbb{E} = \mathbb{P}_S$ are such also in $\mathbb{P}$, so that the cospan $R_S X \twoheadrightarrow R_S B \twoheadleftarrow R_S Y$ can be used to compute the homotopy pullback of $X \to B \leftarrow Y$ not only in $\mathbb{E}$ but also in $\mathbb{P}$ and the hypothesis on (6.7), says that the canonical map $P \to R_S X \times_{R_S B} R_S Y$ is a weak equivalence in $\mathbb{P}$ and hence in $\mathbb{E}$ as well. This says that

is a homotopy pullback square (recall that a is an identity functor) in $\mathbb{E}$. A fortiori, also

is a homotopy pullback square in $\mathbb{E}$ (because a cofibrant approximation in $\mathbb{P}$ is such also in $\mathbb{E}$), so that h is a weak equivalence.

(b) *Property* $(*)$ *holds when B and Y are quasi S-local objects.* By $(**)$, up to rewriting every vertex in (6.7) as a homotopy colimit in $\mathbb{P}$ of quasi $S-$local objects (and relabelling again those homotopy colimits as P, Y, X and B respectively), since being a homotopy pullback square is invariant under weak equivalences of squares and $\mathbb{L}a$ is homotopical, we can assume that $X = \mathrm{hocolim}_{j \in \mathscr{J}} U(j)$ in $\mathbb{P}$, where each $U(j)$ is a quasi $S-$local object. Write $\mathrm{hocolim}_{j \in \mathscr{J}} = \mathrm{colim} \circ Q^{\mathscr{J}}$, for a left $(\mathrm{colim}_{j \in \mathscr{J}})$-deformation. Taking the usual model for the homotopy pullback in $\mathbb{P}$ (a homotopy pullback for $X \to B \leftarrow Y$ is the ordinary pullback of $RX \twoheadrightarrow RB \twoheadleftarrow RY$), consider, for each $j \in \mathscr{J}$, the following commutative diagram

$$(6.9)$$

where all objects are fibrant, all maps are fibrations and the two squares are pullback squares. In particular, the outer square is a homotopy pullback square, so (P1) in $\mathbb{P}$ gives the canonical weak

equivalence

$$\operatorname{hocolim}_{j \in \mathscr{J}} \left(Q^{\mathscr{I}} U(j) \times_B^h Y \right) \xrightarrow{\sim} \left(\operatorname{hocolim}_{j \in \mathscr{J}} RQ^{\mathscr{I}} U(j) \right) \times_{RB}^h RY$$

Now, there is a commutative square

$$
\begin{array}{ccc}
\operatorname{hocolim}_{j \in \mathscr{J}} \left(Q^{\mathscr{I}} U(j) \times_B^h Y \right) & \xrightarrow{\;\sim\;} & \left(\operatorname{hocolim}_{j \in \mathscr{J}} RQ^{\mathscr{I}} U(j) \right) \times_{RB}^h RY \\
\downarrow & & \downarrow{\scriptstyle \sim} \\
X \times_{RB}^h RY & \xrightarrow{\;\sim\;} & \left(\operatorname{hocolim}_{j \in \mathscr{J}} RU(j) \right) \times_{RB}^h RY
\end{array}
$$

where the right vertical arrow, writing $X \times_{RB}^h RY = RX \times_{RRB} RRY$, is induced by the evident maps $Q^{\mathscr{I}} U(j) \times_B^h Y = RQ^{\mathscr{I}} U(j) \times_{RB} RY \to RX \times_{RRB} RRY$. Thus, we get a composite weak equivalence

$$\operatorname{hocolim}_{j \in \mathscr{J}} \left(Q^{\mathscr{I}} U(j) \times_B^h Y \right) \xrightarrow{\sim} X \times_{RB}^h RY \xrightarrow{\sim} RX \times_{RB}^h RY.$$

Since there is a weak equivalence $X \times_B^h Y \xrightarrow{\sim} RX \times_{RB}^h RY$, we also get that the map

$$l \colon \operatorname{hocolim}_{j \in \mathscr{J}} \left(Q^{\mathscr{I}} U(j) \times_B^h Y \right) \xrightarrow{\sim} X \times_B^h Y = RX \times_{RB} RY$$

is a weak equivalence (such a map is induced by the evident map $RQ^{\mathscr{I}} U(j) \times_{RB} RY \to RX \times_{RB} RY$, where the left-hand side is $Q^{\mathscr{I}} U(j) \times_B^h Y$). On the other hand, in (6.9) the objects $RQ^{\mathscr{I}} U(j)$, RB and RY are $S-$local. Thus, from part (a) we get functorial homotopy pullback squares

$$
\begin{array}{ccc}
\mathbb{L}a \left(Q^{\mathscr{I}} U(j) \times_B^h Y \right) & \longrightarrow & \mathbb{L}a(RY) \\
\downarrow & & \downarrow \\
\mathbb{L}a(RQ^{\mathscr{I}} U(j)) & \longrightarrow & \mathbb{L}a(RB)
\end{array}
\qquad (6.10)
$$

so that (P1) in $\mathbb{E}$ gives the canonical weak equivalence

$$\operatorname{hocolim}_{j \in \mathscr{J}} \mathbb{L}a \left(Q^{\mathscr{I}} U(j) \times_B^h Y \right) \xrightarrow{\sim} \left(\operatorname{hocolim}_{j \in \mathscr{J}} \mathbb{L}a(RQ^{\mathscr{I}} U(j)) \right) \times_{\mathbb{L}a(RB)}^h \mathbb{L}a(RY)$$

This map is naturally weakly equivalent to the canonical map

$$\mathbb{L}a(X \times_B^h Y) \to \mathbb{L}a(RX) \times_{\mathbb{L}a(RB)}^h \mathbb{L}a(RY)$$

which is then a weak equivalence itself[5]. For the domain, this follows because $\mathbb{L}a$ commutes with homotopy colimits and we can apply $\mathbb{L}a$ to the weak equivalence l to get again weak equivalence. For the codomain, this follows because, using once more that $\mathbb{L}a$ commutes with homotopy colimits, we have

$$\left(\operatorname{hocolim}_{j \in \mathscr{J}} \mathbb{L}a(RQ^{\mathscr{I}} U(j)) \right) \times_{\mathbb{L}a(RB)}^h \mathbb{L}a(RY) \approx \mathbb{L}a \left(\operatorname{hocolim}_{j \in \mathscr{J}} (RQ^{\mathscr{I}} U(j)) \right) \times_{\mathbb{L}a(RB)}^h \mathbb{L}a(RY) \approx$$

$$\approx \mathbb{L}a(RX) \times_{\mathbb{L}a(RB)}^h \mathbb{L}a(RY)$$

Hence, the square

$$
\begin{array}{ccc}
\mathbb{L}a \left(X \times_B^h Y \right) & \longrightarrow & \mathbb{L}a(RY) \\
\downarrow & & \downarrow \\
\mathbb{L}a(RX) & \longrightarrow & \mathbb{L}a(RB)
\end{array}
$$

is a homotopy pullback square in $\mathbb{E}$. Since (6.7) is a homotopy pullback square in $\mathbb{P}$ (and $\mathbb{L}a$ is homotopical), this latter square is weakly equivalent to

$$
\begin{array}{ccc}
\mathbb{L}a(P) & \longrightarrow & \mathbb{L}a(Y) \\
\downarrow & & \downarrow \\
\mathbb{L}a(X) & \longrightarrow & \mathbb{L}a(B)
\end{array}
$$

so that $(*)$ is a weak equivalence.

---

[5] See footnote 2.

(c) *Property* (∗) *holds when B is a quasi S-local objects.* This is proven exactly as in (b), except for the fact that we can drop the hypothesis on Y by using the thesis of (b). Indeed, in the proof of part (b) above, we used the hypothesis that both Y and B were quasi S-local objects only when we had to conclude that (6.10) was a homotopy pullback square. However, if we now know that B alone is a quasi S-local object, we still get that (6.10) is a homotopy pullback square because $RQ^{\mathscr{I}}U(j)$ is an $S-$local object and we have (b) at hand.

(d) *Property* (∗) *holds for general X, B and Y.* Reasoning as in (b) above, we can suppose that $B = \mathrm{hocolim}_{j\in\mathscr{J}}\, W(j)$ in $\mathbb{P}$ where $W\colon \mathscr{J} \longrightarrow \mathbb{P}$ is a functor landing in quasi S-local objects. Write $\mathrm{hocolim}_{\mathscr{J}} = \mathrm{colim}_{\mathscr{J}} \circ Q^{\mathscr{J}}$ and take the same model of the homotopy pullback as in (b). We have already argued above that we just need to show that the canonical map

$$\mathbb{L}a\left(X\times_B^h Y\right) = \mathbb{L}a\left(RX\times_{RB} RY\right) \to \mathbb{L}a(RX)\times_{\mathbb{L}a(RB)}^h \mathbb{L}a(RY)$$

is a weak equivalence. Define functors $U,V\colon \mathscr{J} \longrightarrow \mathbb{P}$ by setting, for $j\in\mathscr{J}$,

$$U(j) := Q^{\mathscr{I}}W(j)\times_B^h X = RQ^{\mathscr{I}}W(j)\times_{RB} RX, \quad V(j) := Q^{\mathscr{I}}W(j)\times_B^h Y = RQ^{\mathscr{I}}W(j)\times_{RB} RY.$$

(P1) in $\mathbb{P}$ gives then canonical weak equivalences

$$\mathrm{hocolim}_{j\in\mathscr{J}}\, U(j) \xrightarrow{\sim} \left(\mathrm{hocolim}_{j\in\mathscr{J}}\, RQ^{\mathscr{I}}W(j)\right)\times_{RB}^h RX$$

and

$$\mathrm{hocolim}_{j\in\mathscr{J}}\, V(j) \xrightarrow{\sim} \left(\mathrm{hocolim}_{j\in\mathscr{J}}\, RQ^{\mathscr{I}}W(j)\right)\times_{RB}^h RY$$

The same proof as the one we used in (b) above to conclude that the map named there as l was a weak equivalence now shows that we can obtain from these arrows natural weak equivalences

$$w\colon \mathrm{hocolim}_{j\in\mathscr{J}}\, U(j) \xrightarrow{\sim} B\times_B^h X = RX \quad \text{and} \quad v\colon \mathrm{hocolim}_{j\in\mathscr{J}}\, U(j) \xrightarrow{\sim} RY$$

From the definitions of U and V, we also obtain the following commutative diagrams in $\mathbb{P}$, for each $j\in\mathscr{J}$,

Here all objects are fibrant, all morphisms are fibrations, the two squares are (homotopy) pullback squares and the outer square is a (homotopy) pullback square. Note now that the natural transformations $U(-) \implies RQ^{\mathscr{I}}W(-)$ and $V(-) \implies RQ^{\mathscr{I}}W(-)$ are homotopically equifibered because, for every arrow $j\to k$ in $\mathscr{J}$, the relevant squares are pullback squares where all maps are fibrations between fibrant objects. Proposition 6.2.1 then gives a natural weak equivalence

$$\mathrm{hocolim}_{j\in\mathscr{J}}\left(U(j)\times_{RQ^{\mathscr{I}}W(j)} V(j)\right) \xrightarrow{\sim} \left(\mathrm{hocolim}_{j\in\mathscr{J}}\, U(j)\right)\times_{\mathrm{hocolim}_{j\in\mathscr{J}}\, RQ^{\mathscr{I}}W(j)}^h \left(\mathrm{hocolim}_{j\in\mathscr{J}}\, V(j)\right)$$

Utilizing the weak equivalences w and v above and the same kind of reasoning as the one used to get the weak equivalence l in (b), we actually obtain a natural weak equivalence

$$\mathrm{hocolim}_{j\in\mathscr{J}}\left(U(j)\times_{RQ^{\mathscr{I}}W(j)} V(j)\right) \xrightarrow{\sim} RX\times_{RB}^h RY$$

which then also provides a natural weak equivalence

$$u\colon \mathrm{hocolim}_{j\in\mathscr{J}}\left(U(j)\times_{RQ^{\mathscr{I}}W(j)} V(j)\right) \xrightarrow{\sim} X\times_B^h Y$$

This map is induced by the cocone

$$U(j)\times_{RQ^{\mathscr{I}}W(j)} V(j) \cong RQ^{\mathscr{I}}W(j)\times_{RB}\left(RX\times_{RB} RY\right) \to RX\times_{RB} RY = X\times_B^h Y.$$

Since each $RQ^{\mathscr{I}}W(j)$ is an S-local object and $U(-) \implies RQ^{\mathscr{I}}W(-)$ and $V(-) \implies RQ^{\mathscr{I}}W(-)$ are homotopically equifibered, by part (b), we get that also the natural transformations $\mathbb{L}a(U(-)) \implies$

$\mathbb{L}a(RQ^{\mathscr{J}}W(-))$ and $\mathbb{L}a(V(-)) \Longrightarrow \mathbb{L}a(RQ^{\mathscr{J}}W(-))$ are homotopically equifibered. By (c) above, we have a canonical weak equivalence

$$\mathbb{L}a(U(j) \times_{RB} RY) \overset{\sim}{\to} \mathbb{L}a(U(j)) \times^h_{\mathbb{L}a(RQ^{\mathscr{J}}W(j))} \mathbb{L}a(V(j))$$

Hence, Proposition 6.2.1 for $\mathbb{E}$ provides the canonical weak equivalence

$$\operatorname{hocolim}_{j \in \mathscr{J}} \mathbb{L}a(U(j) \times_{RB} RY) \overset{\sim}{\to}$$

$$\overset{\sim}{\to} \left(\operatorname{hocolim}_{j \in \mathscr{J}} \mathbb{L}a(U(j))\right) \times^h_{\operatorname{hocolim}_{j \in \mathscr{J}} \mathbb{L}a(RQ^{\mathscr{J}}W(j))} \left(\operatorname{hocolim}_{j \in \mathscr{J}} \mathbb{L}a(V(j))\right)$$

This map is now naturally weakly equivalent to our canonical map

$$\mathbb{L}a\left(X \times^h_B Y\right) \to \mathbb{L}a(RX) \times^h_{\mathbb{L}a(RB)} \mathbb{L}a(RY)$$

which is then a weak equivalence[6]. For the domain this follows from commutativity of $\mathbb{L}a$ with homotopy colimits and using the above weak equivalence u. For the codomain, we just use again preservation of homotopy colimits by $\mathbb{L}a$ together with the weak equivalences v and w above.

$\square$

Remark 6.2.3. At a first inspection, it may look like we did not use property (P2) of descent in the proof of Giraud's Theorem above. However, its use is actually hidden in the proof of Proposition 6.2.1, which was fundamental to validate part (d).

We end this section with the following result which, as in the categorical counterpart, has a relatively easy proof when Giraud's Theorem is available.

Corollary 6.2.4. *Let $\mathscr{M}$ be a model topos and let X be a fibrant object in $\mathscr{M}$. Then the overcategory $\mathscr{M}/X$ with the model structure of Example 4.1.6 is a model topos.*

Proof. Since $\mathscr{M}$ is a model topos, there is a Quillen equivalence

$$\operatorname{sPsh}(\mathbb{C})_S \underset{U}{\overset{L}{\rightleftarrows}} \bot \ \mathscr{M}$$

for some small simplicial category $\mathbb{C}$ and a set $S \subseteq \operatorname{Mor}(\operatorname{sPsh}(\mathbb{C}))$. By Proposition 4.1.40, since X is fibrant, there is an induced Quillen equivalence of overcategories

$$\operatorname{sPsh}(\mathbb{C})_S/UX \underset{U^*}{\overset{L^*}{\rightleftarrows}} \bot \ \mathscr{M}/X$$

Now, the point is that $\operatorname{sPsh}(\mathbb{C})_S/UX$ is still a combinatorial model category: the fact that it is cofibrantly generated follows from [Hir2] whereas locally presentablility can be deduced using the characterization given in Theorem 1.2.20 (see also [CRV], Remark 3). Therefore, by the results in Section 5.3, $\operatorname{sPsh}(\mathbb{C})_S/UX$ has small simplicial presentation and hence also $\mathscr{M}/X$ has.

We then need to prove that $\mathscr{M}/X$ satisfies descent. We shall prove that $\mathscr{M}/X$ verifies property (P1), the proof for (P2) being analogous but easier. Suppose given functorial homotopy pullback squares in $\mathscr{M}/X$

(6.11)

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indexed by a small category $\mathscr{I}$. A way to obtain a Reedy fibrant replacement of the cospan

$$(Z_i \to X) \to (Z \to X) \leftarrow (Y \to X) \tag{6.12}$$

in $\mathscr{M}/X$ is given as follows. Fix a functorial factorization E of arrows f in $\mathscr{M}$ into a weak equivalence $\mathrm{dom}(f) \overset{\sim}{\to} E(\mathrm{dom}(f))$ followed by a fibration $E(\mathrm{dom}(f)) \twoheadrightarrow \mathrm{cod}(f)$. Applying E to the maps into X in the cospan (6.12), we get another cospan in $\mathscr{M}/X$

$$(EZ_i \twoheadrightarrow X) \to (EZ \twoheadrightarrow X) \leftarrow (EY \twoheadrightarrow X) \tag{6.13}$$

Note that all objects EZ_i, EZ and EY are fibrant in $\mathscr{M}$. In particular, we also get a cospan $EZ_i \to EZ \leftarrow EY$ in $\mathscr{M}$ and we can factor each of those maps using E again. This gives a cospan in $\mathscr{M}/X$

$$(EEZ_i \twoheadrightarrow X) \twoheadrightarrow (EEZ \twoheadrightarrow X) \twoheadleftarrow (EEY \twoheadrightarrow X) \tag{6.14}$$

where EEZ_i, EEZ and EEY are still fibrant objects in $\mathscr{M}$. We have then a diagram in $\mathscr{M}$

$$\tag{6.15}$$

where all objects are fibrant and all arrows are fibrations in $\mathscr{M}$. Now, a homotopy pullback of the original cospan (6.12) in $\mathscr{M}/X$ is given by the ordinary pullback of the cospan (6.14) in $\mathscr{M}/X$. Such a pullback is given by taking the pullback $EEZ_i \times_{EEZ} EEY$ in $\mathscr{M}$ and then considering the map into X fitting into the following commutative square

where all objects are fibrant and all morphisms are fibrations in $\mathscr{M}$[7]. By construction, $EEZ_i \times_{EEZ} EEY$ is a homotopy pullback of $Z_i \to Z \leftarrow Y$ in $\mathscr{M}$. Hence, the weak equivalence

$$(P_i \to X) \overset{\sim}{\to} (Z_i \to X) \times^h_{(Z \to X)} (Y \to X)$$

in $\mathscr{M}/X$ that we have by hypothesis also witnesses that we get functorial homotopy pullback squares

in $\mathscr{M}$. (P1) for $\mathscr{M}$ now gives a homotopy pullback square in $\mathscr{M}$

---

[7] Given a cospan $(A \to X) \to (B \to X) \leftarrow (C \to X)$ in $\mathscr{M}/X$, we can form the pullback $A \times_B C$ in $\mathscr{M}$. The pullback square in $\mathscr{M}$ having $A \times_B C$ as its northwest corner is actually a commutative square over X and witnesses that the canonical map $A \times_B C \to X$ given as the composite $A \times_B C \to C \to X$ (or $A \times_B C \to A \to X$) is indeed the pullback object of the starting cospan in $\mathscr{M}/X$.

and this square is actually a homotopy pullback square

in $\mathcal{M}/X$. Since $\operatorname{colim}_{i\in\mathscr{I}} P_i \to X$ is the colimit object in $\mathcal{M}/X$ of the diagram in $\mathcal{M}/X$ given by

$$\mathscr{I} \ni i \mapsto (P_i \to X)$$

and analogously for $\operatorname{colim}_{i\in\mathscr{I}} Z_i \to X$ (see Remark 2.1.4), we also get that $\operatorname{hocolim}_{i\in\mathscr{I}} P_i \to X$ and $\operatorname{hocolim}_{i\in\mathscr{I}} Z_i \to X$ are homotopy colimits in $\mathcal{M}/X$ of the diagrams $i \mapsto (P_i \to X)$ and $i \mapsto (Z_i \to X)$ respectively. Indeed, if $\operatorname{hocolim}_{\mathscr{I}} = \operatorname{colim}_{\mathscr{I}} \circ Q^{\mathscr{I}}$ for a left $(\operatorname{colim}_{\mathscr{I}})$−deformation $Q\colon \mathcal{M}^{\mathscr{I}} \longrightarrow \mathcal{M}^{\mathscr{I}}$, then, since Q comes equipped with a natural weak equivalence $q\colon Q \stackrel{\sim}{\Longrightarrow} \operatorname{Id}_{\mathcal{M}^{\mathscr{I}}}$, there is an induced functor $Q/X\colon (\mathcal{M}/X)^{\mathscr{I}} \longrightarrow (\mathcal{M}/X)^{\mathscr{I}}$ sending a functor $T_\bullet\colon i \mapsto (T_i \to X)$ to

$$(Q/X)(T)\colon i \mapsto ((QT)_i \to T_i \to X)$$

(here T is the functor $\mathscr{I} \longrightarrow \mathcal{M}$ sending $i \in \mathscr{I}$ to $T_i \in \mathcal{M}$). It is immediate to see that Q/X is homotopical and the obvious natural transformation $q/X\colon Q/X \stackrel{\sim}{\Longrightarrow} \operatorname{Id}_{(\mathcal{M}/X)^{\mathscr{I}}}$ induced by q makes $(Q/X,\ q/X)$ a left deformation retract for the colimit functor $(\mathcal{M}/X)^{\mathscr{I}} \longrightarrow \mathcal{M}/X$. We can then conclude that $\mathcal{M}/X$ has (P1). $\qquad\square$

6.3 Local model structures and homotopy sheaves.

As announced in the introduction to this chapter, given a Grothendieck site $(\mathscr{C}, \tau)$ (see Definition 3.1.9), we outline in this last section how to obtain a model category structure on $\mathrm{sPsh}(\mathscr{C})$ using the given Grothendieck topology τ. Looking for the fibrant objects with respect to such a model category $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$, we will discover a meaningful notion of *homotopy sheaves* on such a site. Actually, we will see that there are different model categories which, being Quillen equivalent to $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$, present the same homotopy theory build upon $(\mathscr{C}, \tau)$: their interplay will allow us to show that $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$ is a model topos. Our exposition is heavily based on [Jar1], [Jar2], [DHI] and [Rzk2] to which we will refer for the proof of the most important results.

Let then $(\mathscr{C}, \tau)$ be a Grothendieck site, which we fix once and for all throughout the whole section. We also denote by

$$\mathrm{PSh}(\mathscr{C}) \overset{a}{\underset{i}{\rightleftarrows}} \perp\ \mathrm{Sh}(\mathscr{C}, \tau)$$

the associated adjunction, so that i is the inclusion and a is the sheafification functor (see Section 3.2). We would like to construct a homotopy theory out of $(\mathscr{C}, \tau)$.

As anticipated above, the first approach we will take is to define a model category structure on $\mathrm{sPsh}(\mathscr{C})$ based on τ. Since this model structure will have as cofibrations all and only the monomorphism, we actually need to construct a suitable class of weak equivalences. To do so, we start with an observation about weak equivalences of simplicial sets.

Given a simplicial set K, one of the equivalent ways to define its homotopy groups is via its geometric realization (see Example 4.1.39). Namely, one can set, for all $n \in \mathbb{N} \setminus \{0\}$ and all $p \in K_0$,

$$\pi_n(K, p) := \pi_n(|K|, |p|) \quad \text{and} \quad \pi_0(K) := \pi_0(|K|). \tag{6.16}$$

Then, by the description of the Kan-Quillen model structure on $\mathbf{sSet}$ (see Example 4.1.19), a map $f\colon K \longrightarrow L$ of simplicial sets is a weak equivalence if and only if the obvious induced maps

$$\pi_0(f)\colon \pi_0(K) \to \pi_0(L) \quad \text{and} \quad \pi_n(f, p)\colon \pi_n(K, p) \to \pi_n(L, f_0(p))$$

are isomorphisms, for all $n \geq 1$ and all $p \in K_0$. Now, there is a way to characterise weak equivalences eliminating the base points. Indeed, given a simplicial set K, define, for each $n \geq 1$,

$$\pi_n(K) := \coprod_{p \in K_0} \pi_n(K, p).$$

This is naturally an object in the overcategory $\mathbf{Set}/K_0$ by

$$\pi_n(K) \to K_0, \quad \pi_n(K, p) \ni \alpha \mapsto p \in K_0$$

and the operations on each $\pi_n(K, p)$ gives a group-object structure to $\pi_n(K) \to K_0$ in $\mathbf{Set}/K_0$[8]. Each map $f\colon K \to L$ of simplicial sets gives rise to maps $\pi_n(f)\colon \pi_n(K) \to \pi_n(L)$ of sets fitting into the following commutative square

$$\begin{array}{ccc} \pi_n(K) & \xrightarrow{\ \pi_n(f)\ } & \pi_n(L) \\ \downarrow & & \downarrow \\ K_0 & \xrightarrow{\ f_0\ } & L_0 \end{array} \tag{6.17}$$

for all $n \geq 1$. In other words, for every positive integer n we get functors

$$\pi_n\colon \mathbf{sSet} \longrightarrow \mathrm{Arr}(\mathbf{Set}),$$

where $\mathrm{Arr}(\mathbf{Set})$ is the arrow category of $\mathbf{Set}$. From the definitions of all the objects involved and the description of pullbacks in $\mathbf{Set}$, we get the following

---

[8] This is easily seen when one notices that, given a category $\mathscr{D}$ and an object $A \in \mathscr{D}$, the product of two objects in $\mathscr{D}/A$ is given by the pullback of the corresponding cospan in $\mathscr{D}$, together with the canonical arrow into A.

Lemma 6.3.1. *A map $f\colon K \longrightarrow L$ of simplicial sets is a weak equivalence if and only if both the following properties are satisfied:*

(i) $\pi_0(f)$ *is an isomorphism;*

(ii) *for all $n \geq 1$, the square* (6.17) *is a pullback square.*

Functoriality of π_0 and of each π_n allow us to extend the above constructions to simplicial presheaves on $\mathscr{C}$. Namely, for each $F \in \mathrm{sPsh}(\mathscr{C})$ and all $m \in \mathbb{N}$ we get presheaves

$$\pi_m(F)\colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}, \quad A \mapsto \pi_m(F(A))$$

and, for all $n \geq 1$, we also get maps of presheaves

$$\pi_n(F) \longrightarrow F_0,$$

where F_0 is the presheaf of sets on $\mathscr{C}$ sending $A \in \mathscr{C}$ to the set of 0-simplexes of the simplicial set $F(A)$. We can now give the following

Definition 6.3.2. A map $\tau\colon F \Longrightarrow G$ of simplicial presheaves is a *local weak equivalence* if the following properties are satisfied:

(i) the induced map $a\pi_0(F) \to a\pi_0(G)$ is an isomorphisms of sheaves on $(\mathscr{C},\ \tau)$;

(ii) the induced commutative square

$$
\begin{array}{ccc}
a\pi_n(F) & \longrightarrow & a\pi_n(G) \\
\downarrow & & \downarrow \\
aF_0 & \longrightarrow & aG_0
\end{array}
$$

is a pullback square of sheaves on $(\mathscr{C},\ \tau)$, for every $n \geq 1$.

Remark 6.3.3. From Lemma 6.3.1, since the sheafification functor is left exact, we get that every pointwise weak equivalence of simplicial presheaves is a local weak equivalence.

Remark 6.3.4. Suppose that $(\mathscr{C},\ \tau)$ is the site $(\mathrm{Op}(X),\ \tau)$ where X is a topological space, $\mathrm{Op}(X)$ is the poset of open subsets in X and τ is the open cover topology on $\mathrm{Op}(X)$ (see Example 3.1.4). For each $x \in X$ and any $F \in \mathrm{sPsh}(X) := \mathrm{sPsh}(\mathrm{Op}(X))$, we can consider the simplicial set F_x which sends $[n] \in \Delta^{\mathrm{op}}$ to $(F_n)_x$, the stalk at x of the presheaf F_n. We will call such an $F_x \in \mathbf{sSet}$ the *stalk of F at x*. We then get that a map $\tau\colon F \Longrightarrow G$ in $\mathrm{sPsh}(X)$ is a local weak equivalence if and only if, for all $x \in X$, the induced map of stalks $\tau_x\colon F_x \longrightarrow G_x$ are weak equivalences of simplicial sets (see [Lur], Remark 6.5.2.2). Essentially, this follows using Lemma 6.3.1 and the fact that homotopy groups commute with filtered colimits of simplicial sets, together with the observations that isomorphisms of (pre)sheaves can be checked stalkwise and stalks do not change upon sheafification. This explains the choice of the attribute *local* in Definition 6.3.2.

The following Theorem is due to J. F. Jardine and says that we can make local weak equivalences part of a very nice model category structure on $\mathrm{sPsh}(\mathscr{C})$.

Theorem 6.3.5. *There is a (unique) model category structure on $\mathrm{sPsh}(\mathscr{C})$ with weak equivalences given by local weak equivalences and cofibrations consisting of all monomorphisms. The associated model category is denoted by $\mathrm{sPsh}(\mathscr{C})_{Jar}$ and is a proper, simplicial and cofibrantly generated model category in which every object is cofibrant. (Here the simplicial, tensored and cotensored structures are the pointwise ones given in Remark 4.2.31).*

Proof. See [Jar1], Theorem 2 and Corollary 4 as well as [Jar2], Theorem 18. □

Remark 6.3.6. If $\mathrm{Sh}(\mathscr{C},\ \tau) = \mathrm{PSh}(\mathscr{C})$ (this happens, for example, when τ is the trivial topology on the small category $\mathscr{C}$, see Example 3.1.4), the model category structure given by Theorem 6.3.5 is the injective model structure on $\mathrm{sPsh}(\mathscr{C})$ (see Theorem 4.3.5).

The homotopy theory defined by $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$ can be presented also via a model structure on the category of simplicial *sheaves*. Namely, we have the following result.

Theorem 6.3.7. *Let* $\mathrm{sSh}(\mathscr{C},\,\tau)$ *be the category* $\mathrm{Sh}(\mathscr{C},\,\tau)^{\Delta^{\mathrm{op}}}$ *of simplicial objects in* $\mathrm{Sh}(\mathscr{C},\,\tau)$. *Declare a map of simplicial sheaves to be a local weak equivalence if it is so when seen as a map of simplicial presheaves. Then the following hold.*

1. *There is a (unique) model category structure on* $\mathrm{sSh}(\mathscr{C},\,\tau)$ *with weak equivalences given by local weak equivalences and cofibrations consisting of all monomorphisms. The associated model category is denoted by* $\mathrm{sSh}(\mathscr{C},\,\tau)_{Joy}$ *or simply by* $\mathrm{sSh}(\mathscr{C})_{Joy}$ *and is a proper, simplicial and combinatorial model category in which every object is cofibrant.*

2. *The adjoint pair*

$$\mathrm{sPsh}(\mathscr{C})_{Jar} \quad \overset{a\circ-}{\underset{i\circ-}{\rightleftarrows}} \perp \quad \mathrm{sSh}(\mathscr{C},\,\tau)_{Joy}$$

   induced by $a \dashv i$ *is a Quillen equivalence.*

Proof. See Theorem 5 of [Jar1]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 6.3.8. The model category structure of Theorem 6.3.7 is denoted by $\mathrm{sSh}(\mathscr{C})_{\mathrm{Joy}}$ since it is due to A. Joyal who communicated it to A. Grothendieck in a private letter ([Joy]).

Note that, since both $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$ and $\mathrm{sSh}(\mathscr{C})_{\mathrm{Joy}}$ are simplicial combinatorial model categories, Theorem 5.3.8 applies and gives small simplicial presentations for them. The point is that they also have descent. To show this, observe first that, in view of Theorem 6.3.7 above, it is enough to prove that $\mathrm{sSh}(\mathscr{C})_{\mathrm{Joy}}$ satisfies descent (see Proposition 6.1.14). Now, Theorem 6.1.17 is proven to be true in [Rzk2] not only for simplicial sets but also for $\mathrm{sSh}(\mathscr{C})_{\mathrm{Joy}}$. In Section 6.1 we exploited Theorem 6.1.17 to obtain descent for **sSet** and our proof relied on the fact that **sSet** is a Grothendieck topos and a proper model category. Since $\mathrm{sSh}(\mathscr{C})_{\mathrm{Joy}}$ shares the same properties (recall that, if $\mathscr{E}$ is a Grothendieck topos, then so is $\mathscr{E}^{\mathscr{D}}$ for every small category $\mathscr{D}$, see Corollary 2.2.5), that proof for simplicial sets applies verbatim to show that $\mathrm{sSh}(\mathscr{C})_{\mathrm{Joy}}$ has descent as well. Giraud's Theorem for model topoi then gives

Theorem 6.3.9. $\mathrm{sPsh}(\mathscr{C})_{Jar}$ *and* $\mathrm{sSh}(\mathscr{C})_{Joy}$ *are model topoi.*

As announced, it is possible to give a description of the fibrant objects in $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$ which leads to a meaningful notion of *homotopy sheaves*. The idea, of course, is that of mimicking the sheaf condition (3.9) and adapting it to our homotopical context, where we deal with simplicial presheaves instead of presheaves of sets.

As a motivation, we start by looking at (3.9) from another perspective. From now on, we suppose that $\mathscr{C}$ has pullbacks and that β is a basis for our Grothendieck topology τ on $\mathscr{C}$ (see Definition 3.1.6, Proposition 3.1.7 and Proposition 3.1.8). Fix $X \in \mathscr{C}$, a covering sink $\{f_i \colon U_i \to X\}_{i \in I}$ in $\beta(X)$ and a presheaf of sets $P \in \mathrm{PSh}(\mathscr{C})$. In (3.9), we considered the diagram

$$\prod_{i\in I} P(U_i) \overset{p}{\underset{q}{\rightrightarrows}} \prod_{(i,j)\in I^2} P(U_i \times_X U_j) \tag{6.18}$$

where the maps p and q are induced by the pullback projections. Setting, for each $n \in \mathbb{N} \setminus \{0\}$ and every $n-$uple $(i_1,\dots,i_n)$ of elements in I, $U_{i_1 i_2 \cdots i_n} := U_{i_1} \times_X U_{i_2} \times_X \cdots \times_X U_{i_n}$, we can actually prolong that diagram further to

$$P(U_\bullet) := \left(\prod_{i\in I} P(U_i) \rightrightarrows \prod_{(i,j)\in I^2} P(U_{ij}) \rightrightarrows\!\!\!\!\rightarrow \prod_{(i,j,k)} P(U_{ijk}) \overset{\longrightarrow}{\rightrightarrows\!\!\!\!\rightarrow} \prod_{(i,j,k,l)} P(U_{ijkl}) \overset{\longrightarrow}{\overset{\longrightarrow}{\rightrightarrows}} \cdots \right) \tag{6.19}$$

where again all the maps are induced by the various projections from the pullbacks. Now, it turns out that the equalizer of (6.18) and the limit of $P(U_\bullet)$ coincide. So we can reinterpret the sheaf condition (3.9) for P (with respect to the given covering $\{f_i \colon U_i \to X\}_{i \in I}$) as asking the natural map

$$P(X) \to \lim P(U_\bullet)$$

to be an isomorphism. We can rewrite this conclusion in a fancier way. The category **Set** has a trivial model category structure where weak equivalences are given by isomorphisms, whereas the class of cofibrations and the class of fibrations coincide and consist of all maps of sets (see Example 4.1.2). Clearly,

homotopy limits and colimits with respect to such a model structure coincide with ordinary limits and colimits, so that we can actually restate the sheaf condition (3.9) by saying that the natural map

$$P(X) \rightarrow \mathrm{holim} \left(\prod_{i \in I} P(U_i) \rightrightarrows \prod_{(i,j) \in I^2} P(U_{ij}) \underset{\Rrightarrow}{\rightrightarrows} \prod_{(i,j,k)} P(U_{ijk}) \underset{\Rrightarrow}{\overset{\Rrightarrow}{\rightrightarrows}} \cdots \right) \qquad (6.20)$$

is a weak equivalence with respect to the trivial model category structure on **Set** (and for all covering sink in $\beta(X)$ for any $X \in \mathscr{C}$).

Remark 6.3.10. The request that the map (6.20) is a weak equivalence actually makes sense when **Set** is just given the structure of a trivial *homotopical category* where weak equivalences are all and only the isomorphisms. This is exactly the structure of homotopical category that **Set** inherits from the homotopical category **sSet** when it is considered as the full subcategory of **sSet** consisting of discrete simplicial sets (the so-called *(homotopy) 0-types*).

The above Remark says that one should think of (6.20) as asking for a certain map of *simplicial sets* to be a weak equivalence. Indeed, the condition in (6.20) is meaningful not only when P is a presheaf of sets (i.e. a discrete simplicial presheaf) but also for *any* simplicial presheaf. We give the following

Definition 6.3.11. A simplicial presheaf $P \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathbf{sSet}$ which is objectwise fibrant is said to *satisfies Čech descent* for the covering sink $\{f_i \colon U_i \to X\}_{i \in I}$ in $\beta(X)$ ($X \in \mathscr{C}$) if the canonical map (6.20) is a weak equivalence of simplicial sets. If $P \in \mathrm{sPsh}(\mathscr{C})$ is not objectwise fibrant, we say that P satisfies Čech descent for the covering sink $\{f_i \colon U_i \to X\}_{i \in I}$ if some objectwise fibrant approximation of P does so.

Note that Definition 6.3.11 is given so that if $P \Longrightarrow F$ is a pointwise weak equivalence of simplicial presheaves, then P satisfies Čech descent for a covering sink if and only if F does so.

One may now guess that we will call homotopy sheaves those simplicial presheaves satisfying Čech descent for all covering sinks. However, this would not allow us to completely characterise the fibrant objects in $\mathrm{sPsh}(\mathscr{C})_{\mathrm{Jar}}$. The problem with Čech descent (as we will see at the end of the section), is that it gives a strong enough homotopical condition on an objectwise fibrant simplicial presheaf P only when there is a positive integer $n \in \mathbb{N}$ such that $P(X)$ has no non-trivial homotopy groups in dimension n or higher, uniformly in $X \in \mathscr{C}$ (that is, when there is an n such that each $P(X)$ is a *(homotopy) $(n-1)-$type*).

The issue is solved by substituting Čech descent with *descent with respect to a hypercover*. To introduce this concept properly, we first need some auxiliary definitions. Recall that, for integers $n \in \mathbb{N}$ and $0 \leq k \leq n$, $\Lambda^k[n] \in \mathbf{sSet}$ denotes the $(n,k)-$th horn (see Example 4.1.19). As in section 5.2, we denote by

$$\mathbf{y} \colon \mathscr{C}^{\mathrm{op}} \longrightarrow \mathrm{sPsh}(\mathscr{C})$$

the simplicial Yoneda embedding which sees each representable functor associated to $X \in \mathscr{C}$ as a discrete simplicial set.

Definition 6.3.12. Let $\tau \colon F \Longrightarrow G$ be a map in $\mathrm{sPsh}(\mathscr{C})$ and let $X \in \mathscr{C}$.

1. We say that a commutative diagram such as

$$\begin{array}{ccc} \Lambda^k[n] \otimes \mathbf{y}X & \longrightarrow & F \\ \downarrow & & \downarrow {\scriptstyle \tau} \\ \Delta[n] \otimes \mathbf{y}X & \longrightarrow & G \end{array} \qquad (6.21)$$

for $n \in \mathbb{N}$ and $0 \leq k \leq n$, *has local liftings* if there is a covering sieve S of X such that, for every arrow $V \to X$ in S, the induced diagram

$$\begin{array}{ccc} \Lambda^k[n] \otimes \mathbf{y}V & \longrightarrow & F \\ \downarrow & & \downarrow {\scriptstyle \tau} \\ \Delta[n] \otimes \mathbf{y}V & \longrightarrow & G \end{array}$$

has a diagonal filler $\Delta[n] \otimes \mathbf{y}V \to F$.

2. We say that τ is a *local fibration* if every square (6.21) has local liftings.

3. We say that τ is a *local acyclic fibration* if it is both a local weak equivalence and a local fibration.

We can now give the following

Definition 6.3.13. Let $X \in \mathscr{C}$. A *hypercover of* X is a pair

$$(U, \ U \Longrightarrow \mathbf{y}X),$$

where $U \in \mathrm{sPsh}(\mathscr{C})$ and $U \Longrightarrow \mathbf{y}X$ is a map of simplicial presheaves such that:

(i) for every $n \in \mathbb{N}$, U_n is a coproduct of representable presheaves. Namely, for every $n \in \mathbb{N}$, there is a set A_n and a family $\{U_{\alpha_n}\}_{\alpha_n \in A_n}$ of objects of $\mathscr{C}$ such that

$$U_n = \coprod_{\alpha_n \in A_n} \mathbf{y}(U_{\alpha_n}) \quad ;$$

(ii) $U \Longrightarrow \mathbf{y}X$ is a local acyclic fibration (see Definition 6.3.12).

Given a covering sink $\{U_i \to X\}_{i \in I}$, setting, for each $n \in \mathbb{N}$,

$$U_n := \coprod_{(i_0, i_1, \cdots, i_n) \in I^{n+1}} \mathbf{y}(U_{i_0 i_i \cdots i_n}) \quad ,$$

where $U_{i_0 i_i \cdots i_n}$ is the iterated pullback over X as above, the natural map $U \Longrightarrow \mathbf{y}X$ turns U into an hypercover of X which is sometimes called the *Čech complex* associated to the given covering sink. We then generalize Definition 6.3.11 as follows

Definition 6.3.14. Let $X \in \mathscr{C}$ and let $(U, \ U \Longrightarrow \mathbf{y}X)$ be a hypercover of X so that each U_n is a coproduct of representables $\mathbf{y}(U_{\alpha_n})$, for $\{U_{\alpha_n}\}_{\alpha_n \in A_n} \subseteq \mathrm{Ob}(\mathscr{C})$.

1. An objectwise fibrant simplicial presheaf P is said to *satisfy descent* for the hypercover $U \Longrightarrow \mathbf{y}X$ if the canonical map

$$P(X) \to \mathrm{holim} \left(\prod_{\alpha_0 \in A_0} P(U_{\alpha_0}) \rightrightarrows \prod_{\alpha_1 \in A_1} P(U_{\alpha_1}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \prod_{\alpha_2 \in A_2} P(U_{\alpha_2}) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right)$$

is a weak equivalence. Here, the displayed map within brackets are induced by the simplicial structure of U. If P is an arbitrary simplicial presheaf, we say that it *satisfies descent* for the hypercover $U \Longrightarrow \mathbf{y}X$ if some objectwise fibrant approximation of P does so.

2. An objectwise fibrant simplicial presheaf P is called a *homotopy sheaf* if it satisfies descent for all hypercovers $U \Longrightarrow \mathbf{y}X$, for any $X \in \mathscr{C}$.

Remark 6.3.15. There are different ways to characterise hypercovers and the descent property with respect to them for a simplicial presheaf. We just mention the following elegant paraphrasing of descent for a simplicial presheaf which follows from Lemma 4.4. in [DHI]. A simplicial presheaf P satisfies descent for a hypercover $U \Longrightarrow \mathbf{y}X$ if and only if the induced map of homotopy function complexes

$$\mathrm{map}^{\mathrm{h}}{}_{\mathrm{sPsh}(\mathscr{C})}(\mathbf{y}X, P) \to \mathrm{map}^{\mathrm{h}}{}_{\mathrm{sPsh}(\mathscr{C})}(U, P)$$

is a weak equivalence of simplicial sets. Here $\mathrm{map}^{\mathrm{h}}{}_{\mathrm{sPsh}(\mathscr{C})}(-, \bullet)$ denotes a homotopy function complex on either $\mathrm{sPsh}(\mathscr{C})_{\mathrm{inj}}$ or on $\mathrm{sPsh}(\mathscr{C})_{\mathrm{proj}}$ (see Definition 4.4.12).

We finally get the promised

Theorem 6.3.16. *Let $(\mathscr{C}, \ \tau)$ be a Grothendieck site. Let $\mathrm{sPsh}(\mathscr{C})_{inj}$ and $\mathrm{sPsh}(\mathscr{C})_{proj}$ denote the category of simplicial presheaves on $\mathscr{C}$ endowed with the injective and the projective model structure respectively (see Definition 4.3.2). Then the following hold.*

1. *The Bousfield localization of $\mathrm{sPsh}(\mathscr{C})_{inj}$ with respect to the class H of all hypercovers exists and coincides with $\mathrm{sPsh}(\mathscr{C})_{Jar}$.*

2. *The fibrant objects in* $\mathrm{sPsh}(\mathscr{C})_{Jar}$ *are given by the fibrant objects in* $\mathrm{sPsh}(\mathscr{C})_{inj}$ *which are homotopy sheaves.*

3. *The Bousfield localization* $\mathrm{sPsh}(\mathscr{C})_{proj,H}$ *of* $\mathrm{sPsh}(\mathscr{C})_{proj}$ *with respect to the class* H *of all hypercovers exists. The fibrant objects of such a localization are given by the homotopy sheaves.*

4. $\mathrm{sPsh}(\mathscr{C})_{proj,H}$ *and* $\mathrm{sPsh}(\mathscr{C})_{Jar}$ *are Quillen equivalent.*

A proof of the above Theorem can be found in Section 6 of [DHI]. Observe that 4. follows from 1. and 3. because $\mathrm{sPsh}(\mathscr{C})_{\mathrm{proj}}$ and $\mathrm{sPsh}(\mathscr{C})_{\mathrm{inj}}$ are Quillen equivalent via the identity functors and this Quillen equivalence passes to the Bousfield localizations (see [Hir1], Theorem 3.3.20). Such a Quillen equivalence essentially tells us that, in order to present the homotopy theory of homotopy sheaves, we can get rid of the injective-fibrancy condition in 2. of Theorem 6.3.16, by considering the projective model structure on $\mathrm{sPsh}(\mathscr{C})$ to start with. This is desirable, as in $\mathrm{sPsh}(\mathscr{C})_{\mathrm{proj}}$ the fibrant objects are exactly the objectwise fibrant simplicial presheaves, whereas the description of fibrant objects in $\mathrm{sPsh}(\mathscr{C})_{\mathrm{inj}}$ is more inexplicit. (Note however that passing from the injective to the projective model structure causes the loss of the cofibrancy property for all objects in $\mathrm{sPsh}(\mathscr{C})$ and in its Bousfield localizations).

We conclude our work by reporting the following result (see [DHI], Corollary A.9), which explains our previous comment on sufficiency of Čech descent for $n-$types.

Proposition 6.3.17. *Let* P *be an objectwise fibrant simplicial presheaf. Assume that there is an* $n \in \mathbb{N}$ *such that, for all* $X \in \mathscr{C}$, $\pi_k(P(X), p) \cong 0$ *for every* $0-simplex$ p *of* $P(X)$ *and all* $k \geq n$. *Then* P *is a homotopy sheaf if and only if it satisfies Čech descent for all covering sinks.*

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