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Compactness in Toposes

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Compactness in Toposes

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Introduction

One of the possible interpretations of a topos is as a generalized space. The reason why we can think in these terms is because of the example we are about to give.
Consider a topological space X, we can consider the category $\mathcal{Sh}(X)$ of sheaves over X and morphisms between them. It turns out that such a category is a topos and moreover, if the topological space is regular enough (i.e. sober), then the topos $\mathcal{Sh}(X)$ alone contains enough categorical information in order to reconstruct back X. We might therefore identify (sober) topological spaces with their topos of sheaves and hence see them as a particular case of a broader concept of space represented by toposes.
Broadly speaking, the main objective of this thesis is to translate the classical notion of compactness in topos theoretic terms, that is, to find a suitable definition of compact topos such that in the particular case of $\mathcal{Sh}(X)$ it will coincide with the compactness of X as a topological space.
Actually the situation is more delicate: rather than picking toposes alone as spaces, we should consider toposes $\mathcal{E}$ equipped with a particular morphism (geometric morphism) into another topos $\mathcal{S}$ called *base topos*. In order to understand the reason behind this requirement, we will need to discuss another interpretation of toposes, namely the interpretation as a universe for mathematics. In the classical theory, the base topos is the category $\mathcal{Set}$ of sets and functions, in fact what we usually do in ordinary mathematics is to describe the objects we are working with using sets and functions.
For instance a group G is a set G equipped with two functions $e : 1 \longrightarrow G$ and $\cdot : G \times G \longrightarrow G$, the first selecting the identity and the second describing the multiplication, such that the usual axioms are satisfied. A group though is just a model for a certain theory (Group theory) and it turns out that there is a way to interpret the standard logical formulas in every topos $\mathcal{S}$ so that it becomes possible to talk about models of group theory inside $\mathcal{S}$. For example if X is a topological space, a group in $\mathcal{Sh}(X)$ is a group sheaf, that is, a sheaf F on X where the families $F(U)$ are groups for every open U and the restriction morphisms are group homomorphisms.
Following this principle then, we could just take a topos $\mathcal{S}$, set it as universe for our own mathematics and, working as if it were our category of sets, we could deduce our own interpretation of mathematics. An emblematic example can be seen in Section 2.4 of [LT], where it is shown the construction of a topos $\mathcal{S}$ which is a universe where every real function $f : \mathbb{R} \longrightarrow \mathbb{R}$ must be continuous.
We can introduce a special kind of morphisms called geometric morphisms, which will also provide a way to transfer informations from one topos to another, and if we have a geometric morphism $\mathcal{E} \longrightarrow \mathcal{S}$ we will think of $\mathcal{E}$ as a topos in the universe $\mathcal{S}$ and for this reason we can refer to such a topos $\mathcal{E}$ as a $\mathcal{S}$-topos.

Coming back to the idea of generalized space, we can think that a (sober) topological space X is represented by a geometric morphism of the form $\mathcal{Sh}(X) \longrightarrow \mathcal{Set}$, for in fact a topological space is considered in the universe of sets. We can then think of $\mathcal{Set}$-toposes as generalized spaces in the universe of sets and more in general, for every topos $\mathcal{S}$, we can interpret $\mathcal{S}$-toposes as generalized spaces in the universe $\mathcal{S}$.

We can finally state more precisely the first aim of this thesis, which is to give a good definition of compact $\mathcal{S}$-topos that generalizes the classical compactness (so in particular $\mathcal{Sh}(X)$ will be a compact $\mathcal{Set}$-topos iff X is compact). Once we achieve this goal, we start studying some properties and characterizations of $\mathcal{S}$-compact toposes and in particular the properties of those geometric morphisms $\mathcal{E} \longrightarrow \mathcal{S}$ which render $\mathcal{E}$ a compact $\mathcal{S}$-topos, which will be given the name of proper geometric morphisms.
For these purposes, we follow the ideas already presented in the book [MV], exposing the theoretical preliminaries required to understand it and filling in the gaps left to the reader.

The following is a brief description of the contents of this thesis.

In the first chapter we give the definitions of topos and geometric morphism and we study some fundamental result about them. We then move to the particular case of Grothendieck toposes, which are a generalization of the notion of "category of sheaves", where instead of considering sheaves over opens of a topological space, we consider sheaves over sites, that is, categories equipped with a notion of coverage. Later on we study some special site with nice properties, for they will be needed in the following chapters. We then move to a brief study of 2-categories which will enable us to talk about the 2-category $\mathfrak{Geom}$ of toposes and geometric morphisms, and about Beck-Chevalley conditions.

The second chapter is of a more geometrical nature: here in fact we analyse in a more detailed way the steps that bring us from topological spaces to toposes, passing in particular through locales, which are a generalization of topological spaces where points are no more needed. We study in particular locales and the particularly regular way in which they embed in $\mathfrak{Geom}$.

The third chapter treats the internalization of category theory in toposes and in toposes over a base. More precisely we describe the internal version of categories, diagrams, limits and colimits and then we study some conditions about preservation of filtered internal colimits.

In the fourth chapter we come back to geometry, for we deal with some example of geometric morphism, namely surjections, inclusions and their factorization system; bounded morphisms and their relation with Grothendieck toposes; and the hyperconnected-localic factorization.

In the last chapter we finally deal with compactness, studying first the case of a compact $\mathcal{S}et$-topos and then generalizing it to the general case represented by proper geometric morphisms. We then analyse the relation between proper morphisms and the ones described in the previous chapter, along with some other property and characterization of proper geometric morphisms. We finish it by hinting at some other developments of this theory.

The Appendix contains some technical result about trees and well-founded induction. In particular it also contains some fact about Grothendieck pretopologies (a special kind of coverage) and some useful result used in this thesis.

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Chapter 1

Toposes, Sites and 2-Categories

In this first chapter we are presenting some of the main tools required to understand the topic of this thesis. We will start by exploring some fundamental notions of topos theory in the first two sections. After that we will have to go through some more technical result regarding sites (a sort of wide generalization of a space containing the minimal information needed in order to talk about sheaves). More precisely we will present and study especially structured sites, along with properties about them that will acquire an actual interpretation only later on in this work. The last topics treated in this chapter concern 2-categories, which intuitively are the structure that aims to mimic the one of the category of categories, where besides object and morphisms, we have also arrows between morphisms. The reason why we are dealing with 2-categories is that they will provide us the right environment to work in the category of toposes. In the end, thanks to the previous theory we will also be able to present the Beck-Chevalley conditions which once more will get an interpretation and a use only in later chapters.
In other words, this chapter will treat a good deal of dirty work which is however necessary to develop hopefully more smoothly the main theory of this thesis.

We are not going through all the basic category theory facts and definitions, which can be found in [CWM] or [LC], but we will spend a few words before getting started in order to fix some notation.
First, we will generally write a category using this character $\mathcal{C}$, while we will use $\mathfrak{C}$ to denote a higher category. We denote the collection of objects of a category $\mathcal{C}$ with $\mathcal{C}_0$ and its collection of arrows with $\mathcal{C}_1$. Given two objects A, B of $\mathcal{C}$ we will denote the family of arrows with domain A and codomain B with $\mathcal{C}(A, B)$. If $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, we will denote the composition of f and g with the notation gf.

Given a couple of functors $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ and a natural transformation $\theta : F \Rightarrow G$, for all $C \in \mathcal{C}$, we will denote with θ_C the component at C of θ.
If a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ we will write $F \dashv G$ and sometimes $F \dashv G : \mathcal{D} \longrightarrow \mathcal{C}$ if we need to specify the two categories involved and the direction of G. We will call a full subcategory *reflective* if the inclusion functor has a left adjoint called *reflector*. If instead the inclusion functor has a right adjoint, we call it *coreflective*.
We will call the category of small categories (i.e. such that the collections of objects and arrows are sets) and functors with the name $\mathcal{C}at$, the category of sets and functions with $\mathcal{S}et$, the category of topological spaces and continuous maps with $\mathcal{T}op$ and the category of groups and group homomorphisms with $\mathcal{G}rp$.
We will write $\mathcal{CAT}$ to denote the category[1] of all categories and functors between them.
We will denote with $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ the category of *presheaves* over $\mathcal{C}$ (functors $\mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S}et$) and natural transformations between them. We will call $y_{\mathcal{C}}$ the Yoneda embedding of $\mathcal{C}$ in $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ sometimes we will call it simply y if the category $\mathcal{C}$ is clear from the context.
Let F, G be two functors with common codomain, then we denote with $(F \downarrow G)$ their *comma category* (see [CWM]). Given a category $\mathcal{C}$ and one of its objects c, we will denote with $\mathcal{C}/c$ the usual *slice category* of arrows over c and commutative triangles (i.e. $(\mathrm{id}_{\mathcal{C}} \downarrow c)$ where c is seen as functor $1 \longrightarrow \mathcal{C}$ selecting c).
In a category $\mathcal{C}$ we will denote with $\mathrm{Sub}(C)$ the usual category of subobjects of the object C in $\mathcal{C}$ (not for locales as we shall see later in Section 2.3). If $\mathcal{C}$ is *well-pointed* (the subobjects of any object form a set) and if the category allows it (e.g. if it has pullbacks), we will denote with $\mathrm{Sub} : \mathcal{C} \longrightarrow \mathcal{S}et$ the functor sending each object to the subobject poset and and each arrow $f : X \longrightarrow Y$ to the map sending a subobject U of the codomain to its pullback along f, i.e.

$$\begin{array}{ccc} f^*(U) & \longrightarrow & U \\ {\scriptstyle f^*(i)}\downarrow & \lrcorner & \downarrow{\scriptstyle i} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Recall that a *kernel pair* of an arrow f is the pullback of that arrow along itself. A category is called *regular* if it has the following properties: it has all finite limits, all kernel pairs admit a coequalizer and *regular epimorphisms* (i.e. coequalizers) are pullback stable, that is, if $e : A \longrightarrow B$ is a regular epimorphism and $f : C \longrightarrow B$ is a map, then $f^*(e)$ is again a regular

[1]Some authors present it as a metacategory or a quasicategory, however we don't require the collections of objects and the collection of arrows of a category to be sets or anything of this kind, so no paradox shall arise.

epimorphism where the following is a pullback square

$$\begin{array}{ccc} A \times_B C & \xrightarrow{f^*(e)} & C \\ \big\downarrow & & \big\downarrow f \\ A & \xrightarrow[e]{} & B \end{array}$$

A category is instead called *exact* or *Barr exact* if it is regular and moreover every equivalence relation (see [J] Definition 0.43) is *effective* i.e. the couple of morphisms $a, b : R \longrightarrow X$ defining an equivalence relation is the kernel pair of some morphism f.

1.1 Toposes

We say that a category is *(finitely) complete* if it has all (finite) limits. A category preserving (finite) colimits will be called *(finitely) cocomplete* instead. We call *exponentiable* every object C such that $C \times - : \mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint denoted with $(-)^C$. A category with all finite products and such that every object is exponential will be called *cartesian closed*. Then we have the usual definition of subobject classifier (see [SE] A1.6) which we will denoted as $t : 1 \longrightarrow \Omega$. We will write $\Omega_{\mathcal{C}}$ instead of Ω to avoid confusion about the category which we are using.
Now we can review the definition of the main ingredient of this thesis

Definition 1.1.1. *A* topos $\mathcal{E}$ *is a finitely complete and cartesian closed category with a subobject classifier.*

A first useful property of toposes is the following

Lemma 1.1.2. *Every topos is finitely cocomplete and exact.*

Proof. See [SE] Corollary 2.2.9 for finite cocompleteness and [B3] Corollary 5.9.7 for exactness. □

Example 1.1.3. *As first example of topos we give $\mathcal{S}et_f$, that is the full subcategory of finite sets in $\mathcal{S}et$.*
Note that it is equivalent to a small (countable) category, so toposes can be small.

Now we can define arrows between toposes in different ways depending on the interpretation we give to them. In particular in our case we need to focus on the geometric interpretation of toposes, so we give the following

Definition 1.1.4. *A* geometric morphism *between two toposes, denoted with* $f : \mathcal{E} \longrightarrow \mathcal{F}$*, is an adjunction* $f^* \dashv f_*$ *of two functors* $f^* : \mathcal{F} \longrightarrow \mathcal{E}$ *and* $f_* : \mathcal{E} \longrightarrow \mathcal{F}$ *where the functor* f^* *also preserves finite limits.*
We call f^* *the* inverse image functor *and* f_* *the* direct image functor*.*

Example 1.1.5. *A special example of geometric morphism are equivalences. Let* $f : \mathcal{E} \longrightarrow \mathcal{F}$ *be an equivalence of toposes, then there exist a functor* $g : \mathcal{F} \longrightarrow \mathcal{E}$ *which in particular is left adjoint to* f*. Being also right adjoint,* g *preserves in particular finite limits and thus* $g \dashv f : \mathcal{E} \longrightarrow \mathcal{F}$ *is a geometric morphism.*
In fact we get also a geometric morphism in the opposite direction since the definition of equivalence is self dual.

Example 1.1.6. *Let* $F : \mathcal{C} \longrightarrow \mathcal{D}$ *be a functor of small categories. Consider the functor* $f^* : \mathcal{S}et^{\mathcal{D}^{op}} \longrightarrow \mathcal{S}et^{\mathcal{C}^{op}}$ *defined on objects by* $f^* = - \circ F^{op}$*, i.e.* $G : \mathcal{D}^{op} \longrightarrow \mathcal{S}et$ *is sent to* GF^{op} *and* $\alpha : G \longrightarrow G'$ *is sent to* $\alpha \circ F^{op}$ *i.e. the natural transformation from* GF^{op} *to* $G'F^{op}$ *such that for all* $c \in \mathcal{C}_0$*,* $(\alpha \circ F^{op})_c = \alpha_{F^{op}(c)}$*.*
This functor has both a right adjoint f_* *and a left adjoint* $f_!$*. The functor* f_* *can be computed using the Yoneda Lemma because if it exists it is such that*

$$f_*(G)(d) \cong \mathcal{S}et^{\mathcal{D}^{op}}(y_d, f_*(G)) \cong \mathcal{S}et^{\mathcal{C}^{op}}(f^*(y_d), G)$$

where these isomorphisms are natural in d *and* G*. We can thus define* f_* *in this way obtaining the rightful right adjoint.*
For the left adjoint the situation is a bit more delicate because it is the left Kan extension of the composition of F *with the Yoneda embedding* $y_{\mathcal{D}}F : \mathcal{C} \longrightarrow \mathcal{S}et^{\mathcal{D}^{op}}$*, i.e. given* G *presheaf on* $\mathcal{C}$*,* $f_!(G) = \mathrm{colim} y_{\mathcal{D}}F(c)$ *where the colimit is taken over the diagram of shape given by the representable functors* $\mathcal{C}^{op} \longrightarrow \mathcal{S}et$ *over* G*, i.e. the comma category* $(y \downarrow \{G\})$ *where* $\{G\}$ *is the functor* $1 \longrightarrow \mathcal{S}et^{\mathcal{C}^{op}}$ *with value* G*.*
Notice that the category $(y \downarrow \{G\})$ *is isomorphic to the category of elements thanks to the Yoneda Lemma. The category of elements of* G *is the category having as objects couples* (C, x) *where* C *is an object of* $\mathcal{C}$ *and* $x \in G(C)$*, while arrows from* (C, x) *to* (C', x') *are morphisms* $f : C \longrightarrow C'$ *such that* $G(f)(x') = x$*. We call this category* $el(G)$*. The category of elements is equipped with a forgetful functor* $\pi : el(G) \longrightarrow \mathcal{C}$ *sending* $f : (C, x) \longrightarrow (C', x')$ *to* $f : C \longrightarrow C'$*.*
A more concrete way of seeing $f_!(G)$ *is as the presheaf sending each object* d *of* $\mathcal{D}$ *to the set*

$$\{f : d \longrightarrow F\pi(x) | x \text{ object of } el(G)\} /_{\sim}$$

where $\sim$ *is the smallest equivalence relation such that for all arrows* $h : x \longrightarrow y$ *of* $el(G)$ *and* $f : d \longrightarrow F\pi(x)$ *we have that* $f \sim F\pi(h)f$*.*

Another kind of arrows that we can consider between toposes are logical functors. First note that a topos has power object (see [SE] A2.1 or [LT]) because it has a subobject classifier Ω and it is cartesian closed, so $\mathcal{P}(C) \cong \Omega^C$ for every object C of $\mathcal{C}$. The component in Ω of the counit of the adjunction $- \times A \dashv (-)^A$, is a morphism $\epsilon_\Omega : \mathcal{P}(A) \times A \longrightarrow \Omega$, therefore it corresponds to a subobject $\in_A \longrightarrow \mathcal{P}(A) \times A$ (the element relation). Given a functor $F : \mathcal{E} \longrightarrow \mathcal{F}$ between toposes preserving finite limits, we have in particular that the inclusion of $\in_A$ is preserved and moreover the induced subobject

$$F(\in_A) \longrightarrow F(\mathcal{P}(A) \times A) \cong F(\mathcal{P}(A)) \times F(A)$$

corresponds to a morphism $F(\mathcal{P}(A)) \times F(A) \longrightarrow \Omega$ which induces by the exponential adjunction a morphism $\phi_A : F(\mathcal{P}(A)) \longrightarrow \mathcal{P}(F(A))$ called comparison morphism.

Definition 1.1.7. *A functor F is called* logical functor *if it preserves finite limits and the previously described comparison morphisms are isomorphisms.*

Remark 1.1.8. *A functor F is logical iff it canonically preserves the structure of topos, i.e. preserves finite limits, subobject classifier and exponentials.*
Finite limits are preserved by definition, subobject classifier is preserved because ϕ_1 is an isomorphism and for exponentials see [SE] Proposition A2.3.7(iii). The converse follows from the fact that $\mathcal{P}(A) \cong \Omega^A$.

The following proposition presents some property of logical functors.

Proposition 1.1.9. *Let $F : \mathcal{E} \longrightarrow \mathcal{F}$ be a logical functor, then*

1. *F preserves finite colimits;*
2. *F has a right adjoint iff it has a left adjoint.*

Proof. See [SE] Corollary A2.2.10. □

We can now state the so called fundamental theorem of topos theory.

Theorem 1.1.10 (Fundamental theorem of topos theory)**.** *Let $\mathcal{E}$ be a topos, then for every object A in $\mathcal{E}$, $\mathcal{E}/A$ is still a topos and for all $f : A \longrightarrow B$ in $\mathcal{E}$, the pullback morphism $f^* : \mathcal{E}/B \longrightarrow \mathcal{E}/A$ is logical and has both left and right adjoint.*

Proof. See [SE] Theorem A2.3.2 and Corollary A2.3.3 where a weak topos is a topos by Corollary A2.3.4. □

Remark 1.1.11. *It follows that for every morphism $f : A \longrightarrow B$ in $\mathcal{E}$, f^* is the inverse image of a geometric morphism $f : \mathcal{E}/A \longrightarrow \mathcal{E}/B$ that we will usually denote with the same symbol or as $\mathcal{E}/f$.*

Remark 1.1.12. *Notice also that the left adjoint of f^* is the composition with f, which hence preserves colimits. In particular thus the forgetful functor $\mathcal{E}/A \longrightarrow \mathcal{E}$ preserves colimits.*

Now we will see an important operation that allows us to build new geometric morphisms from given ones. Consider a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ of toposes and an element A of $\mathcal{E}$. We have a new geometric morphism $(f/A) : \mathcal{F}/f^*(A) \longrightarrow \mathcal{E}/A$ called *slicing* or *localization* of f at A. This geometric morphism has as inverse image $(f/A)^* : \mathcal{E}/A \longrightarrow \mathcal{F}/f^*(A)$ the functor f^* applied to morphisms over A and commutative triangles between them, i.e. $(f/A)^*(p : E \longrightarrow A) = f^*p : f^*(E) \longrightarrow f^*(A)$ and in the same way on morphisms. Let now $\eta : \mathrm{id}_{\mathcal{F}} \longrightarrow f_*f^*$ be the unit of the adjunction $f^* \dashv f_*$, then the direct image $(f/A)_* : \mathcal{F}/f^*(A) \longrightarrow \mathcal{E}/A$ is such that for $p : F \longrightarrow f^*(A)$ in $\mathcal{F}/f^*(A)$, $(f/A)_*(p) = (\eta_A)^*(f_*p)$, i.e. we have the following pullback square

$$\begin{array}{ccc}
A \times_{f_*f^*(A)} f_*(F) & \longrightarrow & f_*(F) \\
{\scriptstyle (f/A)_*(p)}\downarrow \;\lrcorner & & \downarrow{\scriptstyle f_*(p)} \\
A & \xrightarrow[\eta_A]{} & f_*f^*(A)
\end{array}$$

On arrows the direct image is defined by universal property of pullbacks. One can verify that $(f/A)^* \dashv (f/A)_*$ and $(f/A)^*$ preserves finite limits because it preserves terminal object and pullbacks: one holds as simple consequence of the definition and the other follows from the preservation of pullbacks by f^* and Remark 1.1.12.

Remark 1.1.13. *Let again $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism of toposes and $g : A \longrightarrow B$ a morphism in $\mathcal{E}$, then we have the following commutative diagram*

$$\begin{array}{ccc}
\mathcal{F}/f^*(A) & \xrightarrow{f^*(g)} & \mathcal{F}/f^*(B) \\
{\scriptstyle f/A}\downarrow & & \downarrow{\scriptstyle f/B} \\
\mathcal{E}/A & \xrightarrow[g]{} & \mathcal{E}/B
\end{array}$$

where the horizontal morphisms are those defined in Theorem 1.1.10.

We say that a property of geometric morphisms is *stable under slicing* or *local* if whenever $f : \mathcal{F} \longrightarrow \mathcal{E}$ has it, then for all A in $\mathcal{E}$, (f/A) has such a property.

1.2 Grothendieck toposes

Now we will give some definitions and notations (which may differ from the standard ones) in order to introduce the idea of sheaves on a site which is a generalization of the notion of sheaf over a topological space.

Definition 1.2.1. *Let $\mathcal{C}$ be a small category, a* coverage *J on $\mathcal{C}$ is a function assigning to any object C in $\mathcal{C}$ a collection $J(C)$ of families $\mathcal{U}$ (called J-*covering families *for C) of morphisms in $\mathcal{C}$ having as codomain C and such that whenever we have a morphism $g : A \longrightarrow B$ in $\mathcal{C}$ and a J-covering family on B, then we can find a J-covering family $\mathcal{W}$ on A such that for all $w \in \mathcal{W}$, the morphism gw factors through some morphism of $\mathcal{U}$. We say that $\{gw|w \in \mathcal{W}\}$* refines *$\mathcal{U}$ if that happens.*
A small category equipped with a coverage will be called a site*.*

This definition of site is quite relaxed if compared to others, but for our purpose this definition is much more easy to handle and we will show in Section 1.3 that there is no harm in this weakening.
Here are some examples that will turn out to be useful through this thesis.

Example 1.2.2. *As first example we may take any small category $\mathcal{C}$ and take the coverage J such that for all $C \in \mathcal{C}_0$, $J(C)$ will be the set of all possible families of morphisms in $\mathcal{C}$ with codomain C. It is immediate to verify that it is a coverage because the empty family is a J-covering for every object providing every needed refinement.*

Example 1.2.3. *Another trivial example consists in taking on any small category $\mathcal{C}$ the coverage J such that $J(C) = \{\{id_C\}\}$ for all $C \in \mathcal{C}_0$. Again the factorization condition is immediately verified.*

A less trivial and far more important example is the following.

Example 1.2.4. *Let X be a topological space, consider the poset of open subsets $\mathcal{O}(X)$ as a category. For every open U, morphisms with codomain U can be identified with open subsets V contained in U, so in this correspondence let $J(U)$ be the collection of all the covering families $\mathcal{U}$ of open subsets of U, that is, such that $\bigcup \mathcal{U} = U$. The coverage property follows from the fact that if we have an inclusion of opens $V \subseteq U$ and an open covering $\mathcal{U}$, then $\{V \cap W | W \in \mathcal{U}\}$ is an open covering for V.*

Intuitively we can think of a site $(\mathcal{C}, J)$ as a generalization of this example, namely the objects of $\mathcal{C}$ can be seen as different sides of the site and a morphism $U \longrightarrow V$ can be viewed as a way of "embedding" U in V, then a J-covering family for U will be a family of these embeddings which together encode all the informations of U.
This definition of site aims to give the minimal instrument to speak about sheaves.

Definition 1.2.5. *Let* $(\mathcal{C}, J)$ *be a site and consider a presheaf* $F : \mathcal{C}^{op} \longrightarrow \mathcal{S}et$. *Let* $\mathcal{U}$ *be a* J*-covering family for an object* $U \in \mathcal{C}$ *and consider a family* $(s_f | f \in \mathcal{U}, s_f \in F(dom(f)))$ *i.e. a sequence of partial sections of the presheaf indexed on* $\mathcal{U}$.
We say that such a family is compatible *if for all* $f, g \in \mathcal{U}$ *with* $dom(f) = U_f$ *and* $dom(g) = U_g$, *whenever there is an object* A *and two morphisms* $h : A \longrightarrow U_f$ *and* $k : A \longrightarrow U_g$ *such that* $fh = gk$, *then* $F(h)(s_f) = F(k)(s_g)$.
We say that such a family has an amalgamation *if there is a section* $s \in F(U)$ *such that* $F(f)(s) = s_f$ *for all* $f \in \mathcal{U}$. *We say that* F *is* separated *if for all* $U \in \mathcal{C}_0$ *and every* J*-coverage* $\mathcal{U}$ *of* U, *every compatible family on* $\mathcal{U}$ *admits at most one amalgamation.*
We say that F *is a* J-sheaf *if for all* $U \in \mathcal{C}_0$ *and every* J*-coverage* $\mathcal{U}$ *of* U, *every compatible family on* $\mathcal{U}$ *admits exactly one amalgamation.*
We call $\mathcal{S}h(\mathcal{C}, J)$ *(or simply* $\mathcal{S}h(\mathcal{C})$ *if the coverage is clear from the context) the full subcategory of* J*-sheaves in* $\mathcal{S}et^{\mathcal{C}^{op}}$.

Example 1.2.6. *In Example 1.2.2, the empty family is a covering for all the objects* U *of the site, so if we take a presheaf* F *over* $\mathcal{C}$, *a family of sections for the empty cover is an empty family and thus it is trivially compatible. On the other hand, every element of* $F(U)$ *is an amalgamation for this empty family, so* $F(U)$ *can contain at most one element. In other words* $F \cong 1$ *is a terminal presheaf.*
This means that $\mathcal{S}h(\mathcal{C}, J)$ *is equivalent to the terminal category.*
From this example we also deduce that whenever the empty family covers an object of $\mathcal{C}$, *for every sheaf* F, $F(C)$ *is bound to be 1.*

Example 1.2.7. *In the site of Example 1.2.3, the only* J*-covering family for an object* U *is* $\{id_U\}$ *so a family of sections is a singleton* (s) *and thus it is trivially compatible and* s *is its only possible amalgamation. Therefore every presheaf is a sheaf and thus* $\mathcal{S}h(\mathcal{C}, J) = \mathcal{S}et^{\mathcal{C}^{op}}$.

Example 1.2.8. *In Example 1.2.4 given an open* U *and a* J*-covering family* $\mathcal{U}$ *on it, a compatible family for a presheaf* $F : \mathcal{O}(X)^{op} \longrightarrow \mathcal{S}et$ *is a sequence* $(s_V \in F(V) | V \in \mathcal{U})$ *such that the restrictions of* s_V *and* s'_V *coincide on* $V \cap V'$ *for all* $V, V' \in \mathcal{U}$. *And the sheaf condition corresponds to the sheaf condition in the classical sense. Thus* $\mathcal{S}h(\mathcal{O}(X)) = \mathcal{S}h(X)$ *and for this reason we will usually call it in this way.*
Sidewise notice that the empty family always covers the bottom element, so for all sheaves $F \in \mathcal{S}h(X)$, $F(\emptyset) = 1$ *for the final observation in Example 1.2.6.*

Proposition 1.2.9. *For every small site* $(\mathcal{C}, J)$, *the category* $\mathcal{S}h(\mathcal{C}, J)$ *is a topos with all small limits and colimits. Moreover it is reflective in* $\mathcal{S}et^{\mathcal{C}^{op}}$, *i.e. the inclusion functor* $\mathcal{S}h(\mathcal{C}, J) \longrightarrow \mathcal{S}et^{\mathcal{C}^{op}}$ *has a left adjoint.*

Proof. See [SE] Proposition C2.2.6. □

Definition 1.2.10. *Any category which is equivalent to a category of sheaves over a small site is called a* Grothendieck topos*.*

Remark 1.2.11. *Examples 1.2.6, 1.2.7 and 1.2.8 show that the terminal category, any pretopos category over a small category and the category of sheaves over a topological space are all Grothendieck toposes.*

We can give a characterization of Grothendieck toposes as follows

Theorem 1.2.12 (Giraud)**.** *A category $\mathcal{E}$ is a Grothendieck topos iff all the following conditions hold*

1. *it is a cocomplete topos;*
2. *it is* locally small *i.e. $\mathcal{E}(A, B)$ is a set for all $A, B \in \mathcal{E}$;*
3. *it has a* separating set of objects *i.e. there is a set S of objects (sometimes called* generators*) such that any couple of parallel arrows $f, g : A \longrightarrow B$ are equal iff $fe = ge$ for every arrow $e : G \longrightarrow A$ such that $G \in S$.*

Proof. See [SE] Theorem C2.2.8. □

Example 1.2.13. *A topos which is not a Grothendieck topos is $\mathcal{S}et_f$ of Example 1.1.3 because it is not complete nor cocomplete.*

Consider now the following definitions

Definition 1.2.14. *Let $\mathcal{C}$ be a category with pullbacks, we say that a coproduct $X = \coprod_{a \in A} X_a$ in $\mathcal{C}$ is* disjoint *if every coproduct inclusion $\iota_a : X_a \longrightarrow X$ is a monomorphism and for all $a, b \in A$ with $a \neq b$, we have that the pullback of the inclusions ι_a and ι_b is initial in $\mathcal{C}$, i.e. $X_a \times_X X_b = 0$.*
We say that a coproduct X as before is universal *or that it is* pullback stable *if for every map $f : Y \longrightarrow X$ we have that Y is the coproduct of the components of X pulled back along f, i.e.*

$$Y = \coprod_{a \in A} (X_a \times Y)$$

More formally and more generally, if we consider a colimiting cocone $\eta : D \Rightarrow \Delta_X$, we say it is a universal colimit *if for every arrow $f : Y \longrightarrow X$, $f^* \circ \eta : f^*D \Rightarrow \Delta_Y$ is still a colimiting cocone in $\mathcal{C}$, where $f^* : \mathcal{C}/X \longrightarrow \mathcal{C}/Y$ is the pullback functor.*

We also have the following important property

Proposition 1.2.15. *In a Grothendieck topos $\mathcal{E}$ coproducts are disjoint and pullback stable.*

Proof. See Theorem 0.45 in [J]. □

1.3 Grothendieck topologies

The definition we gave of coverage is quite elementary and therefore it is quickly applicable to different cases, but because of this there are many different sites that have the same category of sheaves. The aim of this section is to give more rigidity to the notion of coverage in order to reduce this phenomenon.
First, let $\mathcal{C}$ be a small category and let $\{J_i | i \in I\}$ be a family of coverages for $\mathcal{C}$. Call $J = \bigcup_{i \in I} J_i$ the coverage such that $J(C) = \bigcup_{i \in I} J_i(C)$ for every C in $\mathcal{C}$.

Remark 1.3.1. *Note that J is a coverage because each family is a coverage, in fact for every J_i-covering family $\mathcal{U}$ for C and every map $f : D \longrightarrow C$, we can find in J_i a covering family for D providing the required refinement of $\mathcal{U}$ as in Definition 1.2.1 and thus such a family is in J.*

We can thus say that union of coverages over $\mathcal{C}$ is a coverage over $\mathcal{C}$.
If the coverage comes from the union of coverages we can also recover the category of sheaves. In the same notation as before, we have that

$$\mathcal{Sh}(\mathcal{C}, J) = \bigcap_{i \in I} \mathcal{Sh}(\mathcal{C}, J_i)$$

Remark 1.3.2. *It follows immediately from this observation and from Example 1.2.7 that we can always add the covering family $\{id_C\}$ to every covering for all C without changing the corresponding category of sheaves.*

Remark 1.3.3. *Note that coverages can be ordered by pointwise inclusion i.e. say that $J_1 \subseteq J_2$ if for all $C \in \mathcal{C}_0$ one has $J_1(C) \subseteq J_2(C)$. The mapping $J \mapsto \mathcal{Sh}(\mathcal{C}, J)$ from the order of coverages over $\mathcal{C}$ to the one of the full subcategories of $\mathcal{Set}^{\mathcal{C}^{op}}$ is thus order preserving but switching the inclusions. Moreover, thanks to the previous observation, it sends unions to intersections.*

We want now to add more structure to the covering families in order to reduce their variability, so we start by observing that intuitively if we add to a coverage those opens contained in opens of the coverage, we get that what is covered by the first covering is covered also by the second one. We mean thus to take covering families that already contain such smaller objects and this leads to the following definition.

Definition 1.3.4. *Let $\mathcal{C}$ be a small category, a* sieve *in $\mathcal{C}$ is a subset of arrows which is closed by precomposition, i.e. such that if $f : C \longrightarrow D$ is in S and $g : A \longrightarrow C$ is an arrow in $\mathcal{C}$, then fg is in S as well.*
Let c be an object of $\mathcal{C}$, then a sieve over *c is a subset of arrows with codomain c which is closed by precomposition.*

Example 1.3.5. *The first trivial example of sieve is $\mathcal{C}_1$ and as trivial example of sieve over c we have the set of all arrows with codomain c.*
These are usually called maximal sieves *(respectively over c) for they are maximal in the set of sieves (resp. over c) ordered by inclusion.*
A less trivial example of sieve over c is the so called principal sieve*, that is,* $\langle f\rangle = \{fg | g \in \mathcal{C}_1,\ dom(f) = cod(g)\}$ *where f is an arrow in $\mathcal{C}$ with codomain c. The name comes from the analogy with right ideals in a ring, in fact these are additive subgroups which are also closed by right multiplication, and for rings the principal ideal generated by an element a of the ring R is also the set of all* $\{ar | r \in R\}$.

Remark 1.3.6. *Note that if $\mathcal{C}$ has terminal object, then sieves in $\mathcal{C}$ correspond to sieves over it. Conversely a sieve over c is in particular a sieve in $\mathcal{C}$ but can also be thought of as a sieve in $\mathcal{C}/c$ (in fact there is a 1 to 1 correspondence between sieves over c and sieves in $\mathcal{C}/c$).*

Remark 1.3.7. *The set of all sieves in $\mathcal{C}$ (over c) is closed under arbitrary unions and intersection. For instance to prove it in the case of unions and sieves in a category $\mathcal{C}$, consider a family $\{S_i | i \in I\}$ of sieves in $\mathcal{C}$, let $S = \bigcup_{i\in I} S_i$, then if f is in S, there exists $i \in I$ such that S_i contains f, thus for all precomposable g in $\mathcal{C}_1$ the arrow fg is in S_i and hence in S. The proof for the other cases is analogous.*

A consequence of this closure property is that for every subset A of $\mathcal{C}_1$ there exist a minimum sieve containing A, that is, $\langle A\rangle = \bigcap\{S \text{ sieve} | A \subseteq S\}$ called *sieve generated* by A.

Remark 1.3.8. *There is a more explicit way to describe the sieve generated by a set, that is*

$$\langle A\rangle = \{fg | f \in A,\ g \in \mathcal{C}_1 \text{ s.t. } dom(f) = cod(g)\}$$

and this because it is a sieve containing A and every other sieve containing A must contain it for the closure by precomposition.
In particular note that every such sieve is union of the principal sieves generated by the maps in A, namely $\langle A\rangle = \bigcup_{a\in A}\langle a\rangle$.

We will now give an important operation on sieves that allows us to "pull back" a sieve along an arrow.

Definition 1.3.9. *Let $f : d \longrightarrow c$ be an arrow in $\mathcal{C}$ and S a sieve over c, then we define*

$$f^*(S) := \{g \in \mathcal{C}_1 | cod(g) = d,\ fg \in S\}$$

and we call it pullback along f.

This set is a sieve over d because S is a sieve over c. The reason of the name pullback will be justified soon (See Lemma 1.3.11 below).

Now we will present an equivalent way in which we can interpret sieves over an object c, that is, as subpresheaves of the representable functor in c. Let $y_c = \mathcal{C}(-,c) : \mathcal{C}^{\text{op}} \longrightarrow \mathcal{S}et$, i.e. the representable presheaf corresponding to c. Given a sieve S on c, for all $d \in \mathcal{C}$ define $\hat{S}(d) = \{h : d \longrightarrow c | h \in S\}$ and for all $f : d \longrightarrow d'$, let $\hat{S}(f) : \hat{S}(d') \longrightarrow \hat{S}(d)$ be the precomposition by f. Note that the latter correspondence is well defined because of the closure by precomposition of sieves.
In this way we define a functor and in particular a subfunctor of y_c because for all d object of $\mathcal{C}$ we have $\hat{S}(d) \subseteq y_c(d)$ and for all $f : d \longrightarrow d'$, the map $\hat{S}(f)$ is a restriction of $y_c(f)$. In other words the following diagram commutes.

$$\begin{array}{ccc} y_c(d') & \xrightarrow{y_c(f)} & y_c(d) \\ \uparrow & & \uparrow \\ \hat{S}(d') & \xrightarrow{\hat{S}(f)} & \hat{S}(d) \end{array} \tag{1.1}$$

and then a subobject of a presheaf is uniquely determined by the canonical one which in every component is subset of the corresponding component of the bigger one.
Conversely, if T is a subfunctor of y_c with canonical inclusion $i : T \longrightarrow y_c$, consider $S = \bigcup_{d \in \mathcal{C}_0} T(d)$, it is a sieve, in fact for $f : d \longrightarrow c$ in S, $f \in T(d)$ and for every $g : d' \longrightarrow d$, the composition $fg : d' \longrightarrow c$ is s.t. $fg = y_c(g)i(f)$ and hence by naturality of i, this is $iT(g)(f)$ so in particular it is contained in $T(d')$ and hence in S.

Remark 1.3.10. *If we consider the order of inclusion on the set of sieves over c and the usual order on $Sub(y_c)$, then we can see that this correspondence is actually an isomorphism of complete lattices. In particular the maximal presheaf over c corresponds to the maximal subpresheaf of y_c, that is, y_c itself.*
Moreover if $f : d \longrightarrow c$ is an arrow in $\mathcal{C}$, the principal sieve generated by f, i.e. $\langle f \rangle$ corresponds to the subpresheaf image of the morphism $y_f : y_d \longrightarrow y_c$ i.e. the composition with f.

This interpretation of sieves also allows us to justify the name pullback for the sieve $f^*(S)$, in fact we have the following result.

Lemma 1.3.11. *Let $\mathcal{C}$ be a small category and y its Yoneda embedding into its category of presheaves. If $f : d \longrightarrow c$ is an arrow in $\mathcal{C}$ and S a sieve over c, then $f^*(S)$ is the pullback of S along the image of f via the Yoneda embedding, that is, y_f. More explicitly if for now we denote with the hat the* *subpresheaf corresponding to a sieve, we get*

$$\widehat{f^*(S)} = (y_f)^*(\widehat{S})$$

Proof. Pullbacks in presheaves are computed pointwise, so let $a \in \mathcal{C}_0$, then $(y_f)^*(\widehat{S})(a) = y_f(a)^*(\widehat{S}(a))$. Since every $S(a)$ is a subset of $y_c(a)$ for all a, this pullback is actually the preimage along $y_f(a)$ which is the composition with f and hence $y_f(a)^*(\widehat{S}(a)) = \{h : a \longrightarrow d | fh \in \widehat{S}(a)\}$ which is precisely $\widehat{f^*(S)}$ whence we get the equality we are proving. □

We shall from now on denote a sieve and its corresponding subpresheaf in the same way, as soon as this won't cause misunderstandings.

We are now ready to discuss a particular kind of coverage

Definition 1.3.12. *Let $\mathcal{C}$ be a small category, a* Grothendieck coverage *(often called* Grothendieck topology*) is a correspondence assigning to each object c in $\mathcal{C}$ a family $J(c)$ of sieves over c such that*

1. *The maximal sieve y_c is in $J(c)$;*
2. *If S is in $J(c)$ and $f \in \mathcal{C}(d, c)$, then $f^*(S)$ is in $J(d)$;*
3. *If R is in $J(c)$ and S is another sieve on c such that $f^*(S) \in J(dom(f))$ for every $f \in S$, then also $S \in J(c)$.*

In this definition we are not explicitly requiring the axiom for coverages, but it is implicit in the second axiom, in fact, for every sieve S in $J(c)$ and every arrow $f : d \longrightarrow C$, $f^*(S)$ is the required sieve, because for each h in it, $fh \in S$ which thus trivially factors through S (see also Remark 1.3.14).
We call *sifted coverage* a coverage whose coverings are sieves.

Example 1.3.13. *Let $\mathcal{C}$ be a small category, an example of sifted coverage on it is the* canonical *one, which is the maximum sifted coverage such that the representable presheaves are sheaves.*
Note that it is well defined for Remark 1.3.1 and the fact that every coverage generates an equivalent sifted coverage. But we will prove the latter only later on in this section.

Remark 1.3.14. *If J is a sifted family, to check if it is a coverage, i.e. if for every S covering c and $f : d \longrightarrow c$ there is a J-covering sieve T for d such that $fT = \{ft | t \in T\}$ refines S, becomes to check if $fT \subseteq S$. This is true because we are dealing with sieves and if a sieve A refines a sieve B, it means that for all $a \in A$ there is a morphism $b \in B$ such that $a = bh$ for some h but then $a \in B$.*
In other words we have to check that $T \subseteq f^*(S)$ for some covering T.

Actually if we work in the hypotheses of Definition 1.3.12 but for axiom (2), to ask that J is a coverage is equivalent to ask (2). The proof of this fact can be deduced from the following

Lemma 1.3.15. *If J is a sifted coverage on a category $\mathcal{C}$ such that axioms (1) and (3) of Definition 1.3.12 hold, then for every c in $\mathcal{C}$ the family $J(c)$ is upward closed, which means that if R is a J-covering sieve over c and S is a sieve over c containing R, then S is a J-covering as well.*
Moreover $f^*(R)$ is in $J(dom(f))$ for every arrow f with codomain c and every J-covering sieve R on c, thus in particular J is a Grothendieck coverage.

Proof. For all f in R, $f^*(S) = y_{\text{dom}(f)}$ so it is in $J(\text{dom}(f))$ by (1) and hence S is in $J(c)$ by (3). For the second part notice that since R covers and J is a coverage, then there is a coverage T of $\text{dom}(f)$ which is a sieve by hypotheses such that fT is a refinement of R. Now by Remark 1.3.14, $T \subseteq f^*(R)$ and hence $f^*(R) \in J(c)$. By definition we have that (2) holds and thus J is a Grothendieck coverage. □

Another interesting property of $J(c)$ for a Grothendieck coverage J and $c \in \mathcal{C}_0$ is that it is closed under intersections , in fact if $R,\ S \in J(c)$, then for every $f \in R$ we have $f^*(R \cap S) = f^*(S)$ (basically because R is a sieve) and hence, by axiom (3) of Grothendieck coverages, also $R \cap S$ covers c.

Remark 1.3.16. *From these property and Lemma 1.3.15 follows that every $J(c)$ is a filter in the lattice of sieves over c.*
Note that the latter can be seen also as $\Omega(c)$ where Ω is the subobject classifier in the category of presheaves.
In this way we can think of a Grothendieck coverage as a sort of filter in Ω.

Although the definition of Grothendieck coverages looks much more complex than the one of coverages, it has its advantages. For instance the compatibility condition for a family of partial sections translates into a naturality condition and the existence of an amalgamation becomes an extension problem.
Let F be a presheaf over $\mathcal{C}$, c an object in it and R a covering sieve over c, then consider a compatible family $(s_r | r \in R)$ of partial sections in F. We can interpret R as a subpresheaf of y_c and since $s_r \in F(\text{dom}(r))$ we can see this compatible family as $\alpha = (\alpha_d | d \in \mathcal{C})$ where $\alpha_d : R(d) \longrightarrow F(d)$ sends $r \in R$ to s_r. In this setting, compatibility implies that α is natural because given $f : d' \longrightarrow d$, for every $r : d \longrightarrow c$, the composition rf is in R and thus $s_{rf} = F(f)(s_r)$, so $\alpha_{d'} R(f)(r) = \alpha_{d'}(rf) = F(f)\alpha_d(r)$ which proves the naturality of α.
Conversely given a natural transformation $\alpha : R \Rightarrow F$, we get a family $(s_r | r \in R)$ where $s_r = \alpha_{\text{dom}(r)}(r)$. This family is compatible because given $r_1, r_2 \in R$ and an $A \in \mathcal{C}_0$ with morphisms $a_i : A \longrightarrow \text{dom}(r_i)$ for $i = 1, 2$ such that $r_1 a_1 = r_2 a_2$, then since the latter is a map r in $R(A)$, by naturality of α

we have that $F(a_i)(s_{r_i}) = F(a_i)(\alpha_{\mathrm{dom}(r_i)}(r_i)) = \alpha_A R(a_i)(r_i) = \alpha_A(r_i a_i) = \alpha_a(r)$ and thus $F(a_1)(s_{r_1}) = F(a_2)(s_{r_2})$. These correspondences are each other's inverse. We can resume these observations in

Remark 1.3.17. *If R is a J-covering sieve for an object c with J Grothendieck coverage on the category $\mathcal{C}$, we can think of the set of compatible families of partial sections of a presheaf F as*

$$\mathcal{S}et^{\mathcal{C}^{op}}(R, F)$$

Since every such R is subpresheaf of y_c, we can ask when we can extend a natural transformation $\alpha : R \Rightarrow F$ to the whole y_c, i.e. when the following extension problem has solution

$$\begin{array}{ccc} R & \xrightarrow{\alpha} & F \\ \downarrow & \overset{\gamma}{\nearrow} & \\ y_c & & \end{array} \tag{1.2}$$

First notice that thanks to the Yoneda lemma, if such a γ exists, than it must correspond to an element s of $F(c)$ and by this we mean that for every $f \in y_c(d)$, $\gamma_d(f) = F(f)(s)$. This implies that if γ exists, s is an amalgamation for the compatible family corresponding to α as in Remark 1.3.17, because for every $r \in R$, $F(r)(s) = \gamma_{\mathrm{dom}(r)}(r) = \alpha_{\mathrm{dom}(r)}(r) = s_r$, so the extension corresponds to an amalgamation. Conversely, given an amalgamation, defining γ as in the Yoneda Lemma, i.e. $\gamma_d(f) = F(f)(s)$, we get a natural transformation which extends α. Therefore we have

Remark 1.3.18. *In the context of Remark 1.3.17, an extension problem* (1.2) *has solution iff the corresponding compatible family has amalgamation and the extension corresponds to the amalgamation via the natural isomorphism of the Yoneda Lemma.*

Another useful property that we gain from the use of Grothendieck coverages is the fact that they are closed by intersection (again we mean pointwise intersection), more precisely we have

Lemma 1.3.19. *If $\{J_i | i \in I\}$ is a family of Grothendieck topologies on $\mathcal{C}$ indexed by a set I, then their intersection J defined in every component $c \in \mathcal{C}_0$ as $J(c) = \bigcap_{i \in I} J_i(c)$ is again a Grothendieck coverage*

Proof. First of all J is sifted for all the J_i's are and for the same reason it satisfies the axiom (1) of Definition 1.3.12. If R is a covering sieve in $J(c)$, then it is contained in all of the J_i's, so for every $f : d \longrightarrow c$ in $\mathcal{C}$, the sieve $f^*(R)$ is in J_i for all $i \in I$ and thus it is in J which therefore satisfies (2).

We are left to prove (3), so let S be a sieve over c and R a J-covering sieve such that $f^*(S) \in J(\mathrm{dom}(f))$ for all $f \in R$, this means that for every $i \in I$, $f^*(S) \in J_i(\mathrm{dom}(f))$ and since they all satisfy (3), S is in all of the J_i's and thus in J. □

Using Grothendieck coverages we lose the closure by union, but we will solve soon this problem with Remark 1.3.22. Before going on we give the following lemma as reference

Lemma 1.3.20. *Let F be a sheaf on a category $\mathcal{C}$ with respect to a sifted coverage J, then*

1. *If $S \subseteq R$ are two sieves over c and S is in J, then F satisfies the sheaf condition in R.*

2. *If $R \in J(c)$ and S is a sieve on c such that $r^*(S) \in J$ for all $r \in R$, then F satisfies the sheaf condition also in S.*

Proof. 1. Notice that in the interpretation of sieves as subpresheaves of the representable ones given in Remarks 1.3.17 and 1.3.18, we have that an amalgamation for $\alpha : R \Rightarrow F$ is also an amalgamation for its restriction to S. We now want to prove that an amalgamation for the restricton is an amalgamation also for α. If $\alpha = (x_r | r \in R)$ is a compatible family and x the amalgamation of its restriction to S, we just need to prove that $x_r = F(r)(x)$. Since J is a coverage, there exists a sieve $A \subseteq r^*(S)$ (for Remark 1.3.14), then for all $a \in A$, $ra \in S$ so $F(a)F(r)(x) = F(ra)(x) = x_{ra} = F(a)(x_r)$ where the last equality follows from compatibility on R. We can interpret the $(F(a)(x_r)|a \in r^*(S))$ as a compatible family over A which is J-covering and by construction x_r is its amalgamation which is unique since F is a J-sheaf. We said that we can see $F(a)(x_r)$ also as $F(a)F(r)(x)$ and thus it also has $F(r)(x)$ as amalgamation. By uniqueness of the amalgamation we have that $F(r)(x) = x_r$ and this ends the proof.

2. As first step we want to prove that F satisfies the sheaf condition at $S \cap R$, then we would like to prove as in the previous point that this implies the validity of the sheaf condition also in S, but that's not so straightforward as one might think because $S \cap R$ is not a priori a J-cover. However, following the steps of the previous proof we will show that we can still apply the same argument.

Now let's prove that F satisfies the sheaf condition at $S \cap R$. First note that $S \cap R = \bigcup_{r \in R} rr^*(S)$ because, by definition every $rr^*(S)$ is inside S and since R is a sieve, it is also in R, so we have one inclusion. For the other one, let $t \in S \cap R$, then $t = t\mathrm{id} \in tt^*(S)$ and hence also the other inclusion holds, proving the equality above. In particular every element of $R \cap S$ can be written as rh for $r \in R$ and $h \in r^*(S)$.

Consider a compatible family $\alpha = (x_t | t \in R \cap S)$, for all $r \in R$ we have

that the family $(x_{rh}|h \in r^*(S))$ is compatible over $r^*(S)$ because if we interpret α as a natural transformation $S \Rightarrow F$, the latter compatible map is the precomposition of α with the natural transformation $r^*(S) \Rightarrow S \cap R$ pullback of y_r along the inclusion of $S \cap R$ in y_c. Since $r^*(S)$ is a J-covering for $\text{dom}(r)$, we have a unique amalgamation x_r. The family $(x_r|r \in R)$ is also compatible and to prove it we need to show that for every arrow g with $\text{cod}(g) = \text{dom}(r)$, $F(g)(x_r) = x_{rg}$, but for every $h \in (rg)^*(S)$ we have that $F(h)F(g)(x_r) = F(gh)(x_r) = x_{rgh} = F(h)(x_{rg})$, so $F(g)(x_r)$ and x_{rg} are both amalgamation of the same compatible family over the J-covering sieve $(rg)^*(S)$ and hence they coincide. Since $(x_r|r \in R)$ is compatible over $R \in J(c)$, there is a unique amalgamation $x \in F(c)$ for it. Note that for every element $t \in R \cap S$, we get $F(t)(x) = x_t$ because, as proved before, t is of the form rh with $r \in R$ and $h \in r^*(S)$ so $F(rh)(x) = F(h)F(r)(x) = F(h)(x_r) = x_{rh}$. Note also that every amalgamation x' for α must be of this form because $F(h)F(r)(x') = F(rh)(x') = x'_{rh}$, so $F(r)(x')$ is an amalgamation for $(x'_{rh}|h \in r^*(S))$ as was x_r and hence F satisfies the sheaf property at $R \cap S$.
We want now to deduce from this fact that F satisfies the sheaf condition also at S. Consider a compatible family $\alpha = (x_s|s \in S)$, as observed while proving point (1), it induces a compatible family on $R \cap S$ and an amalgamation for α is an amalgamation also for the restricted one. As for point (1) we are done if we prove that that an amalgamation x for the restriction of α is an amalgamation also for α. We have to prove that $F(s)(x) = x_s$ for every $s \in S$ and as before we prove that both terms of this equality are amalgamations for the same compatible family over a J-covering for $\text{dom}(s)$. Now R is in $J(c)$ so, for every $s \in S$, there is $A \in J(\text{dom}(s))$ such that $A \subseteq s^*(R)$. For every $a \in A$, $F(a)F(s)(x) = F(sa)(x) = x_{sa} = F(a)(x_s)$ where the second equality holds because $sa \in R \cap S$ and the last one because α is compatible over S. We got $F(s)(x) = x_s$ for every $s \in S$ and hence x is an amalgamation for S. This implies that F satisfies the sheaf condition at S because it does so in $R \cap S$. □

Now we are ready to fulfil the aim we prefixed at the beginning of this section, that is, to prove that Grothendieck coverages give a more rigid definition of coverage and in particular the most rigid. More precisely we are going to prove that every category of sheaves can be seen as a category of sheaves over a site equipped with a Grothendieck coverage and that two different Grothendieck coverages give different sheaves. Alternatively we can call two coverages on the same category $\mathcal{C}$ *equivalent* if they induce the same category of sheaves (as subcategory of $\mathcal{S}et^{\mathcal{C}^{\text{op}}}$) and we want to prove that each equivalence class of coverages contains exactly one Grothendieck coverage, so that the correspondence of Remark 1.3.3 restricts to an isomorphism between Grothendieck coverages and sheaves subcategories.
Let J be a coverage, we claim that it can be transformed into an equivalent

one $\overline{J}$ which is sifted. We build it defining $\overline{J}(c) = \{\langle \mathcal{U} \rangle | \mathcal{U} \in J(c)\}$ where $\langle \mathcal{U} \rangle$ is the sieve over c generated by $\mathcal{U}$. First of all this new family is a coverage because for every $\langle \mathcal{U} \rangle$ over c and a map $f : d \longrightarrow c$, since J is a covering, we have a $\mathcal{V} \in J(d)$ such that $f\mathcal{V}$ is a refinement of $\mathcal{U}$. From Remark 1.3.14 we just need to find a coverage in $f^*(\langle \mathcal{U} \rangle)$ and since such is $\mathcal{V}$, the same is true for $\langle \mathcal{V} \rangle$ which thus is the sieve we are looking for.
Moreover, note that if F is a presheaf, a compatible family $\alpha = (s_u | u \in \mathcal{U})$ of its partial sections on a J-covering family $\mathcal{U}$ can be uniquely extended to a compatible family $\overline{\alpha}$ on $\langle \mathcal{U} \rangle$, in fact it is enough to define $s_{uh} = F(h)(s_u)$ for every $uh \in \langle \mathcal{U} \rangle$ with $u \in \mathcal{U}$. Compatibility on $\mathcal{U}$ implies that the extended family is well defined and compatible. Then an amalgamation s for α is also an amalgamation for $\overline{\alpha}$ and clearly also the converse holds.
From these facts we have that F is a sheaf with respect to J iff it is a sheaf with respect to $\overline{J}$ so that $\mathcal{Sh}(\mathcal{C}, J) = \mathcal{Sh}(\mathcal{C}, \overline{J})$ and thus J and $\overline{J}$ are equivalent. Therefore without loss of generality we can treat this topic supposing that the coverage that we start with is sifted.
Let now J be a sifted coverage, thanks to Lemma 1.3.19 it makes sense to define $\widetilde{J}$ as the intersection of all the Grothendieck coverages containing J because then $\widetilde{J}$ is a Grothendieck coverage as well. We are just left to prove that it is equivalent to the original coverage.

Lemma 1.3.21. *The coverages J and $\widetilde{J}$ are equivalent.*

Proof. We have to prove that $\mathcal{Sh}(\mathcal{C}, J) = \mathcal{Sh}(\mathcal{C}, \widetilde{J})$ if we see them as subcategories of $\mathcal{Set}^{\mathcal{C}^{\text{op}}}$. In order to do this we will build a new coverage K equivalent to J and such that $J \subseteq \widetilde{J} \subseteq K$, so that thanks to Remark 1.3.3 we will get $\mathcal{Sh}(\mathcal{C}, J) \supseteq \mathcal{Sh}(\mathcal{C}, \widetilde{J}) \supseteq \mathcal{Sh}(\mathcal{C}, K)$, and since K is equivalent to J, we will get that so is $\widetilde{J}$.
Let $c \in \mathcal{C}_0$, we define $K(c)$ as the set of sieves S over c such that F satisfies the sheaf condition for $f^*(S)$ for all f with codomain c. The family K satisfies axiom (1) of the definition of Grothendieck coverage because $f^*(y_c) = y_{\text{dom}(f)}$ and every presheaf satisfies the sheaf condition over it. Now let $R \in K(c)$ and g an arrow with codomain c, in order to prove axiom (2) we have to prove that $g^*(R) \in K(\text{dom}(g))$ i.e. that for every map f with codomain $\text{dom}(g)$, F satisfies the sheaf condition over $f^*(g^*(R))$, but this is $(gf)^*(R)$ and by definition of K, F satisfies the sheaf condition over it. From the latter already follows that K is a coverage, so before we prove the last requirement for K to be a Grothendieck topology, we prove that it contains and is equivalent to J.
First of all $J \subseteq K$ because if S is in $J(c)$, $f^*(S)$ is in J and thus F must satisfy the sheaf condition for it, whence $S \in K(c)$; this implies that $\mathcal{Sh}(\mathcal{C}, K) \subseteq \mathcal{Sh}(\mathcal{C}, J)$. Now let F be a sheaf with respect to J and R a K-covering sieve, then in particular F satisfies the sheaf condition for $R = \text{id}^*(R)$ and hence F is a sheaf with respect to K as well, whence $\mathcal{Sh}(\mathcal{C}, J) = \mathcal{Sh}(\mathcal{C}, K)$.

Finally we prove that the last axiom of Grothendieck topologies holds. If $R \in K(c)$ and S is a sieve over c such that for every $r \in R$ we have $r^*(S) \in K(c)$, we want to prove that S is in $K(c)$ and hence that for every f with codomain c, F satisfies the sheaf condition at $f^*(S)$. Consider the coverage K, we have that $f^*(R)$ is also a K-covering by axiom (2) proved before and moreover for every $a \in f^*(R)$, $a^*(f^*(S)) = (fa)^*(S)$ is in K because $fa \in R$. Applying point (2) of Lemma 1.3.20 in this situation, we get that every K-sheaf F satisfies the sheaf condition at $f^*(S)$, but then since J and K are equivalent, every J-sheaf satisfies the sheaf condition at $f^*(S)$ and hence S is a K-covering.
We have just proved that K is a Grothendieck coverage and hence, by definition of $\widetilde{J}$, $\widetilde{J} \subseteq K$ and thus we have $J \subseteq \widetilde{J} \subseteq K$. As said at the beginning of this proof then, since J and K are equivalent, so are J and $\widetilde{J}$. □

Remark 1.3.22. *Note that that the correspondence $J \mapsto \widetilde{\overline{J}}$ is a closure operator over the poset of coverages over a small category $\mathcal{C}$, in fact both $\overline{(-)}$ and $\widetilde{(-)}$ are idempotent, increasing and order preserving.*
In particular in the poset of Grothendieck coverages over $\mathcal{C}$, the join of a family $\{J_i | i \in I\}$ exists and can be described as

$$\bigvee_{i\in I} J_i = \widetilde{\overline{\bigcup_{i\in I} J_i}}$$

We have proven that every coverage J can be harmlessly transformed in a Grothendieck one (which we will denote with $\widetilde{J}$), but as promised before we have much more, namely that each Grothendieck coverage on a category $\mathcal{C}$ defines a different category of sheaves over $\mathcal{C}$. This fact is proven in the following

Theorem 1.3.23. *Let $\mathcal{C}$ be a small category, consider the correspondence sending a coverage J on $\mathcal{C}$ to the subcategory $\mathcal{Sh}(\mathcal{C}, J) \subseteq \mathcal{Set}^{\mathcal{C}^{op}}$. This correspondence is a bijection between Grothendieck coverages and subcategories of sheaves over $\mathcal{C}$.*

Proof. The surjectivity of this map is given by Lemma 1.3.21. We need to prove that the Grothendieck coverages is uniquely determined from its category of sheaves. In order to prove easily this point we would need to introduce more terminology that won't be of further use in this thesis. For a proof of this fact see [SE] Corollary 2.1.11 or [LT] Theorem 2.5. □

In particular we deduce the following explicit description of $\widetilde{J}$.

Corollary 1.3.24. *If J is a coverage and $\widetilde{J}$ the unique equivalent Grothendieck coverage, then for all $c \in \mathcal{C}_0$, $\widetilde{J}(c)$ is the set of sieves S over c such that every J-sheaf F satisfies the sheaf condition at $f^*(S)$ for every f with codomain c.*

Proof. Consider the sheaf K defined in the proof of Lemma 1.3.21 and note that K is exactly the sheaf in the statement of this corollary. In that lemma we proved that K is a Grothendieck coverage and from Theorem 1.3.23 we have that there is a unique equivalent Grothendieck coverage, so $\widetilde{J} = K$. □

1.4 Coherent sites

In this section we will present a special kind of site that will prove of great use in the last chapter to characterize compactness in toposes.

Definition 1.4.1. *A* coherent ctegory *is a regular category such that* $Sub(c)$ *has finite joins for every object* c *and for every arrow* $f : d \longrightarrow c$*, the map* $f^* : Sub(d) \longrightarrow Sub(c)$ *preserves them.*

Particular coherent categories are *Heyting categories*, which are coherent categories such that for every morphism $f : A \longrightarrow B$, the pullback map $f^* : \mathrm{Sub}(B) \longrightarrow \mathrm{Sub}(A)$ has both a left and a right adjoint, denoted respectively $\exists_f$ and $\forall_f$. In a Heyting category every lattice is a Heyting algebra as proven in Lemma A1.4.13 of [SE].

Example 1.4.2. *Note that toposes are Heyting categories as claimed in Corollary A2.3.5 of [SE] and hence in particular they are coherent.*

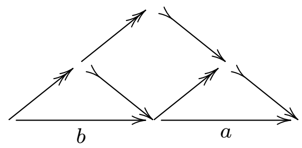
Recall that a family of arrows $\{e_i : A_i \longrightarrow C | i \in I\}$ in $\mathcal{C}$ is called *jointly epimorphic* or simply *epimorphic* if for every couple of parallel morphisms $h, k : C \longrightarrow B$ we have that if $he_i = ke_i$ for all $i \in I$, then $h = k$.
If we are in a coherent category $\mathcal{C}$ we say that a finite family of arrows $\{e_i : c_i \longrightarrow c | i = 1, \dots, n\}$ is *jointly regular-epimorphic* if $\bigvee_{i=1}^{n} \mathrm{im}(e_1) = C$ or equivalently if whenever m is a mono through which every e_i factors, then m is an isomorphism.

Remark 1.4.3. *A finite jointly regular epimorphic family* $\{e_1, \dots, e_n\}$ *is also jointly epimorphic because if* f, g *are such that* $fe_i = ge_i$ *for all* $i = 1, \dots, n$*, then* e_i *factors through the equalizer of* f *and* g*. Since this happens for every* i*, we have that the equalizer of* f *and* g *must contain the join of the images of the* e_i *which is the whole domain of* f *and* g*, so* $f = g$*.*

On a small coherent category $\mathcal{C}$ we have the so called *coherent coverage* P defined for every object $c \in \mathcal{C}_0$ such that $P(c)$ set of finite jointly regular-epimorphic families of arrows with codomain c. We call *coherent topos* a topos which is equivalent to $\mathcal{Sh}(\mathcal{C}, P)$ for some site $(\mathcal{C}, P)$ where $\mathcal{C}$ is coherent and P is the regular coverage.

Before proceding in our study, we make the following observation

Remark 1.4.4. *Consider the composition ab in a regular category, then taking the various factorizations regular epi-mono we get the following diagram*



From it follows by essential uniqueness of the factorization that if m_b is the monic part of b, we have that the monic part of ab coincides with the monic part of am_b.
Moreover we have that if e_a is a regular epi of the factorization of a, then the monic part of ab is obtained composing the monic part of $e_a b$ with m_a. In particular then if a morphism f factors through a as $f = ab$, then the same factorization holds for the monic part of f through the monic part of a.

Lemma 1.4.5. *Let $\mathcal{C}$ be a coherent category, let c in $\mathcal{C}$, $\{e_1, \ldots, e_n\}$ be a jointly regular-epimorphic family of maps over c and for every i, let $\{e_{i,1}, \ldots, e_{i,m_i}\}$ be a jointly regular-epimorphic family over dom(e_i), then*

$$\{e_i e_{i,j} | j = 1, \ldots, m_i; i = 1, \ldots, n\}$$

is again jointly epimorphic.

Proof. Let m be a monomorphism over c through which all of the $e_i e_{i,j}$ factor, then for the universal property of the pullback, all the $e_{i,j}$'s factor through $e_i^*(m)$ which is a mono, being pullback of a mono. Now since $\{e_{i,j} | j = 1, \ldots, m_i\}$ is jointly regular epimorphic over $\mathrm{dom}(e_i)$, $e_i^*(m)$ must be an isomorphism and thus we can choose it to be the identity, getting that every e_i factors as $m(m^*(e_i))$. But the family of the e_i's is jointly regular epimorphic on c, and thus m is forced to be an isomorphism. This implies that the family of compositions is jointly regular epimorphic as well. □

Coming back to the coherent coverage P over $\mathcal{C}$ coherent, we have that it is a coverage, in fact if $\{e_1, \ldots, e_n\} \in P(c)$, then for every arrow $f : d \longrightarrow c$, consider the family $\{f^*(e_1), \ldots, f^*(e_n)\}$ obtained by pulling back every element along f; it is a finite family with common codomain d and we would like to prove that $\bigvee_{i=1}^n \mathrm{im}(f^*(e_i)) = d$. Notice that since $\mathcal{C}$ is regular, the factorization regular epi, mono is pullback stable, so $\mathrm{im}(f^*(e_i)) = f^*(\mathrm{im}(e_i))$ but since we are in a coherent category, f^* preserves finite unions and thus $\bigvee_{i=1}^n f^*(\mathrm{im}(e_i)) = f^* \left(\bigvee_{i=1}^n \mathrm{im}(e_i)\right) = f^*(c)$ which is d (pullback of an identity is an isomorphism). This implies that $\{f^*(e_1), \ldots, f^*(e_n)\} \in P(d)$, proving that P is a coverage.
The coherent coverage has the following property

Proposition 1.4.6. *Representable presheaves over a coherent category are P-sheaves.*

Proof. Let $\mathcal{C}$ be a coherent category, c, d objects in $\mathcal{C}$ and $\{e_1, \ldots, e_n\} \in P(c)$. We want to prove that the representable presheaf y_d is a P-sheaf, so let $x_i \in y_d(\mathrm{dom}(e_i))$ for every i. Every x_i is then an arrow of $\mathcal{C}$ from $\mathrm{dom}(e_i)$ to d and compatibility means that for every $i, j \in \{1, \ldots, n\}$ and a couple of maps a_i, a_j in $\mathcal{C}$ such that $e_i a_i = e_j a_j$ makes sense and holds, we have that also $x_i a_i = x_j a_j$. An amalgamation will than be a map $x : c \longrightarrow d$ such that $x e_i = x_i$ for every i.
If an amalgamation exists, then it is unique because the e_i's form a jointly epimorphic family by Remark 1.4.3, so if we have two amalgamations $x,\ x'$, for every i we get $x e_i = x_i = x' e_i$ and thus $x = x'$.
To prove the existence of an amalgamation we can factor each e_i as $m_i \overline{e_i}$ with m_i mono and $\overline{e_i}$ regular epimorphism. Note that every x_i can be extended along $\overline{e_i}$ in fact if r_1, r_2 are the kernel pair of $\overline{e_i}$, then in particular $e_i r_1 = m_i \overline{e_i} r_1 = m_i \overline{e_i} r_2 = e_i r_2$ so that by compatibility, also $x_i r_1 = x_i r_2$. Then since every regular epimorphism is coequalizer of its kernel pair, x_i factors as $\overline{x_i}\,\overline{e_i}$. The family of $\overline{x_i}$'s is in particular still compatible as one can check using pullback properties and the fact that we are in a regular category (thus pullback and composition of regular epis is a regular epi). So we are basically in the same situation of the beginning but with the e_i's monic.
In this case we use Lemma A1.4.3 of [SE] which says that a binary join is also a pushout in the whole category, so one by one we can glue together all the $\overline{x_i}$'s getting a map x from the pushout of the m_i's to d. Since the e_i's cover, the pushout of the m_i's is c, so $x : c \longrightarrow d$. Then $x e_i = x m_i \overline{e_i} = \overline{x_i}\,\overline{e_i} = x_i$ so that x is an amalgamation. □

A coverage for which representable functors are sheaves is called *subcanonical* and analogously, a *subcanonical site* is a site equipped with a subcanonical coverage. The name is due to the fact that if we consider only sifted coverages, the subcanonical coverages are precisely the subcoverages of the canonical one.

Remark 1.4.7. *It follows from Proposition 1.4.6 that the Yoneda embedding $y : \mathcal{C} \longrightarrow \mathcal{S}et^{\mathcal{C}^{op}}$ factors through the category of P-sheaves for P the coherent coverage. So we get an embedding $y : \mathcal{C} \longrightarrow \mathcal{S}h(\mathcal{C}, P)$.*
In fact for every subcanonical site $(\mathcal{C}, J)$ we have such factorization, and $y : \mathcal{C} \longrightarrow \mathcal{S}h(\mathcal{C}, J)$ is an embedding.

We call the Grothendieck coverage equivalent to P the *coherent Grothendieck coverage*. Note that P is not in general a Grothendieck topology, nor it is its sifted closure, but we can give a nice characterization of the sieves contained in the coherent Grothendieck coverage

Lemma 1.4.8. *Let $\widetilde{P}$ be the Grothendieck coverage generated by P, then $S \in \widetilde{P}$ iff it contains a P-covering family.*

Proof. Let's call for now $J(c)$ the set of sieves on c containing a family in $P(c)$. Let $S \in J(c)$, then there is $\{x_1, \dots, x_n\} \in P(c)$ contained in S, but since $\widetilde{P}$ contains the sifted closure of this family and so does S, we have that $S \in \widetilde{P}(c)$ for Remark 1.3.16; then $J \subseteq \widetilde{P}$.
Now, since J contains the sifted closure of P, if we prove that it is a Grothendieck coverage, then we also get that $\widetilde{P} \subseteq J$ and hence the equality. The first axiom holds because each identity is a P-cover and every maximal sieve contains one. The second follows from the fact that the pullback of a P-covering family is still P-covering (for the category is coherent). For the third, let $R \in J(c)$ and S a sieve over c such that for every $r \in R$, $r^*(S) \in J(\mathrm{dom}(r))$, then in particular there is a P-covering family $\{r_1, \dots, r_n\}$ in R and for every i a covering family $\{r_{i,1}, \dots, r_{i,m_i}\}$ inside $(r_i)^*(S)$. Note that $r_i r_{i,j} \in S$ for any compatible i, j and moreover, thanks to Lemma 1.4.5 we have that $\{r_i r_{i,j} | j = 1, \dots, m_i; i = 1, \dots, n\}$ is a P-cover, so that S is in J as claimed. □

Another coverage that we can put on coherent categories is given in the following definition.

Definition 1.4.9. *Let $\mathcal{C}$ be a coherent category, a sieve S over c is called* dm-sieve *if for every $s \in S$, also the monomorphic part of s (i.e. the inclusion of the image of s in cod(s)) is in S and moreover the monomorphisms of S form a filtered family, i.e. for every finite family of monos $m_1, \dots, m_n \in S$, there is a mono $m \in S$ such that all the m_i's factor through it.*
A dm-coverage *is a sifted coverage such that every covering sieve is a dm-sieve.*

Some properties of this kind of coverages are in the next lemma.

Lemma 1.4.10. *Let $\mathcal{C}$ be a coherent category, then*

1. *maximal sieves are dm-sieves;*

2. *the property of being a dm-sieve is pullback stable.*

Proof. 1. Since every arrow with the right codomain is in the maximal sieve, in particular also their monic part is. The family of monos of the maximal sieve over an object is the family of monos over that objects and all of them factor through the identity.

2. Let S be a dm-sieve over an object c and $f : d \longrightarrow c$, we have to prove that $f^*(S)$ is a dm-sieve. For every arrow h we write its epic part as e_h and his monic one as m_f, so that $h = m_h e_h$. The first property of dm sieves states then that if the sieve contains h, it also contains m_h. Let

now $a \in f^*(S)$, then $fa \in S$ and since S is a dm-sieve, $m_{fa} \in S$. Note that $fa = m_f e_f m_a e_a$ and if we call g the composition $e_f m_a$, we have that $fa = m_f m_g e_g e_a$ so that by uniqueness of factorization we get $m_{fa} \cong m_f m_g$, and thus also $m_f m_g \in S$. Now $fm_a = m_f e_f m_a = m_f g = m_f m_g e_g \in S$ so that $m_a \in f^*(S)$.
For the second property consider a finite family $\{v_1, \dots, v_n\}$ of monos in $f^*(S)$, then $fv_i \in S$ and in particular $m_{fv_i} \in S$, so that we get a finite family of monos. This implies the existence of a mono m in S through which the whole family factors. Let $v = f^*(m)$, it is a mono and it is in $f^*(S)$ for fv factors through m and since for every i, fv_i factors through m, for the universal property of pullbacks we have a factorization of the v_i's through v. □

These two coverages just defined usually need to be considered together, but for this purpose we also require some compatibility condition. We say that a dm-coverage T is *P-compatible* if the following two properties hold

1. Let $R \in T(c)$ and $e : c \longrightarrow d$ a regular epimorphism, then the sieve generated by the monic part of all the er for $r \in R$, is in $T(d)$.

2. Consider two monos $m_i : A_i \longrightarrow B$ for $i = 1, 2$ such that B is the union of the subobjects represented by A_1, A_2. Let $R \in T(A_1)$, then the sieve generated by the arrows corresponding to the subobjects $\mathrm{im}(m_1 r) \vee A_2$ for every $r \in R$ is T-covering.

When instead of P we consider its sifted closure, we still ask these closure axioms for compatibility.
We have that every Grothendieck coverage containing the (sifted closure of the) coherent one can be generated by a combination of the latter and a compatible dm-coverage, as confirmed by the following proposition

Proposition 1.4.11. *Let $\mathcal{C}$ be a small coherent category and P the sifted closure of the coherent coverage over it*

1. *Let J be a Grothendieck coverage on $\mathcal{C}$ containing P and define J_d as the family of those sieves of J which are dm-sieves. Then J_d is a P-compatible coverage and $P \cup J_d$ generate J i.e. the smallest Grothendieck topology containing both P and J_d is J.*

2. *Let T be a P-compatible dm-coverage and $\widetilde{T}$ the Grothendieck coverage it generates. Let J be the minimum Grothendieck coverage containing both P and T. Then $J(c)$ is the set of all sieves R over c such that there exists a sieve $S \in \widetilde{T}(c)$ such that $a^*(R)$ is in $\widetilde{P}(dom(a))$ for every $a \in S$ and in particular T is equivalent to J_d built from J as in the previous point.*

Proof. 1. Thanks to Lemma 1.4.10.(2) we have that J_d defines a coverage. Now we ought to prove the two compatibility conditions with P. For the first, let e be a regular epi and $R \in J_d(\mathrm{dom}(e))$, then let S be the sieve generated by the monic part of all the er for $r \in R$. We want to prove that $S \in J_d(\mathrm{cod}(e))$, so we prove first that $S \in J$. If we consider the sieve A generated by e, then $A \in P(\mathrm{cod}(e)) \subseteq J(\mathrm{cod}(e))$, so if we prove that for every $a \in A$ we have $a^*(S) \in J$, we are done for the third axiom of a Grothendieck topology. Note now that $a = eb$ for some arrow b composable with e, so we can consider $b^*(R)$ and we have that for every $h \in b^*(R)$, $ah = ebh \in S$ because $bh \in R$ and $eR \subseteq S$. Therefore $b^*(R) \subseteq a^*(S)$ and moreover $b^*(R)$ is in J for the second axiom of Grothendieck topologies, so we also get that for Remark 1.3.16, also $a^*(S) \in J$, and thus for what we said before, $S \in J$. In order to prove that $S \in J_d$ we have to prove that if $a \in S$, then its monic part m_a is in S too, but this is true because S is generated by monos. For the other rule instead note that if m_r is the monic part of r, the monic part of er coincides with the monic part of em_r (Remark 1.4.4), so we can see S as generated by the monic part of er using only monos r in R. Consider now a finite family of monos in S, it is of the form $\{m_{er_i}h_i | i = 1, \ldots, n\}$ where m_{er_i} is the monic part of er_i, r_i is a mono in R and h_i a morphism (which is bound to be a mono). Note that already $\{m_{er_i} | i = 1, \ldots, n\}$ is such that the previous family factors through it, so we are left to find a unique factorization for it and since the r_i's form a finite family of monos in R which is a dm-sieve, there exists a mono $r \in R$ such that all the all the r_i's factor through it. The monic part of er is the mono we are looking for, because if r_i factors through r as $r_i = r\overline{r_i}$, then also er_i factors through er and thus there is a factorization also with their monic parts as showed in Remark 1.4.4.

Now we have to prove the second condition of compatibility for J_d, so consider two monos m_1, m_2 satisfying the appropriate hypotheses and $R \in J_d(\mathrm{dom}(m_1))$, then in particular m_1, m_2 are jointly regular-epimorphic, so the sieve B generated by them is in P and hence in J. We want to prove that the sieve S generated by the join of the image of $m_1 r \in R$ and the subobject represented by m_2. Notice that in this way S is generated by the pushout of the pullback of m_2 with monos in $m_1 R$. Again first we prove that $S \in J$ and again we prove it by showing that $b^*(S) \in J$ for every $b \in B$. Now $b = m_1 a$ or $b = m_2 a$ for some arrow a. In the first case we have that $a^*(R) \subseteq b^*(S)$ for if $h \in a^*(R)$, ah factors through a mono r in R and hence $bh = m_1 ah$ factors through the inclusion of $\mathrm{im}(m_1 r) \vee \mathrm{im}(m_2)$. Again from Remark 1.3.16 follows that $b^*(S) \in J$. In the second case instead $b^*(S)$ is the maximal sieve, for b already factors through m_2 and hence through every $\mathrm{im}(m_1 r) \vee \mathrm{im}(m_2)$, so again $b^*(S) \in J$. This implies that $S \in J$, but we still have to prove that S is a dm-sieve. It is generated by monos, so the first condition is satisfied, then if we call ι_r the inclusion of $\mathrm{im}(m_1 r) \vee \mathrm{im}(m_2)$, we have to prove that for a finite family $\{\iota_{r_i} h_i | i = 1, \ldots, n\}$ where the r_i's are in R and h_i are morphisms. Again we can just consider monic r_i's, for $\iota_r = \iota_{m_r}$ where m_r is the monic part of r and again it is enough to prove it for the family $\{\iota_{r_i}\}$, for the previous family factors through it. The r_i's form a finite family of monos in R so there is a mono $r \in R$ such that all the r_i's factor. This implies that all the ι_{r_i}'s factor through ι_r (universal property of pushouts), so even the second property is assured. Note that we are treating joins (i.e. pushouts in the poset of lattices) as pushouts in $\mathcal{C}$ in virtue of Lemma A1.3.4 of [SE].

Now it only remains to prove that the minimum Grothendieck coverage containing P and J_d is actually J. Clearly J contains P and J_d by hypothesis and definition of J_d. Now let $R \in J(c)$ and consider the sieve S generated by all the monomorphisms obtained as unions of the images of finite subsets of R. Note that S is a dm-sieve, in fact the first condition follows from the fact that it is generated by monos and the second holds basically because the family of generating monos is itself filtered. Note also that $R \subseteq S$, so that also S is in $J(c)$ (Remark 1.3.16) and hence in $J_d(c)$. Now for every $s \in S$ let's study $s^*(R)$, we have that s is the inclusion of $\mathrm{im}(r_1) \vee \cdots \vee \mathrm{im}(r_n)$ where $r_i \in R$. Note that all the r_i's factor through s and $\mathrm{dom}(s)$ is covered by the factorizations which we'll call $\overline{r_i}$. Consider now the sieve A generated by the $\overline{r_i}$'s, it is in $P(c)$ and hence in every Grothendieck coverage K containing P, but we also have $A \subseteq s^*(R)$ so that $s^*(R)$ is bound to be K-covering for Remark 1.3.16, but then if $J_d \subseteq K$, then also $S \in K$ and hence by the third axiom of Grotendieck coverages, so is R. Since this operation can be done with every $R \in J$, we have $J \subseteq K$ and thus we get the minimality of J.

2. We call J the family of sieves such that there is $S \in \widetilde{T}$ such that for all $a \in S$, $a^*(R)$ is in $\widetilde{P}$: the Grothendieck coverage generated by P and then we prove that it is the Grothendieck coverage generated by P and T. Note that J is contained in the Grothendieck coverage generated by P and T for axiom (3) of Grothendieck coverages and moreover P and T are contained in J; the first because all of its sieves satisfy the property for every sieve of T and the second because for every sieve $S \in T$, the pullback of S along every element of its is the maximal sieve. If we prove that J is a Grothendieck coverage, then we are done.

Consider a maximal sieve y_c, it is contained in both $\widetilde{P}$ and $T \subseteq \widetilde{T}$ and the pullback of y_c along every map of y_c is still maximal (therefore contained in $\widetilde{P}$). If $R \in J(c)$ and $f : d \longrightarrow c$ we have to prove that $f^*(R) \in J(d)$. Let $S \in \widetilde{T}(c)$ be such that for every $a \in S$, $a^*(S) \in \widetilde{P}$, then since $\widetilde{T}$ is a Grothendieck coverage, $f^*(S) \in \widetilde{T}(\mathrm{dom}(f))$. Note that for every $b \in f^*(S)$, $b^*(f^*(R)) = (fb)^*(R)$ which is in $\widetilde{P}$ for $fb \in S$, thus $f^*(R) \in J$.

For the last one we remind to Corollary A.2.4 in the Appendix, for it requires a bit too many technicalities for a quick treatment.

We have found the form of of the Grothendieck coverage J generated by P and T, now we want to prove that J_d and T are equivalent and hence that $\widetilde{J_d} = \widetilde{T}$. Of course T is in J_d, so we just need to prove that $J_d \subseteq \widetilde{T}$. Let

$R \in J_d$, then it is a dm-sieve and there is $A \in \widetilde{T}$ such that $a^*(R) \in \widetilde{P}$ for all $a \in A$.
Since R is a dm-sieve, also a^*R is a dm-sieve by Lemma 1.4.10 (2), but it is also in $\widetilde{P}$, so in particular it contains a jointly regular epimorphic family $e_1, \ldots, e_m$. Let m_i be the monic part of e_i for all i, $m_i \in a^*(R)$ for $a^*(R)$ is a dm-sieve and for the same reason there is a mono $m \in a^*(R)$ such that all of the m_i's factor through it. The family of e_i's is jointly regular epimorphic, and this means that the only mono through which all of the e_i's (and hence the m_i's) factor is the maximal one, so m is bound to be an isomorphism, and thus $a^*(R) = y_{\mathrm{dom}(a)}$. Now R is a sieve over c such that there is $A \in \widetilde{T}$ with $a^*(R) = y_{\mathrm{dom}(a)} \in \widetilde{T}(\mathrm{dom}(a))$ and thus for the third axiom of Grothendieck topologies, also $R \in \widetilde{T}$. □

Remark 1.4.12. *Despite what claimed by Lemma C3.2.18 in [SE] the coverage J_d is not necessarily a Grothendieck coverage, although it satisfies the first two axioms for it by Lemma 1.4.10.*
In fact in general (3) does not hold. Consider a small topos (hence coherent) without external axiom of choice $\mathcal{C}$, that is, such that there is an epimorphism e which does not split. Let J be the maximal Grothendieck coverage on $\mathcal{C}$, then in particular note that the family $R = \{x \longrightarrow cod(e) | x \cong 0\}$ is a sieve for in a coherent category there are no maps with terminal codomain that are not isomorphisms and moreover it is a dm-sieve for it only contains monos and they are all isomorphic. Consider now the sieve S generated by e, it can't be a dm-sieve because the monic part of e is $id_{cod(e)}$, so if it is contained in S, follows that $id_{cod(e)} = ea$ for some arrow a and hence e splits. So we have that $R \in J_d(cod(e))$ but $S \notin J_d(cod(e))$. Now for every $r \in R$, it is of the form $0 \longrightarrow 1$ so $r^*(S) = \{a | ra \in S\}$ but then every $a \in r^*(S)$ is an isomorphism having codomain 0 and in particular id_0 is in S, so it is maximal. Note that we are in the hypotheses of the third axiom for Grothendieck coverages and yet $S \notin J_d(cod(e))$.
Note also that such a topos exists: for instance consider a small category equivalent to $\mathcal{S}et_f^{A^{op}}$ where A is a finite poset (see Example 1.1.3 for $\mathcal{S}et_f$). It is equivalent to a small topos for every A (inheriting its main objects from $\mathcal{S}et^{A^{op}}$) and in particular if we choose as A the ordinal 2 we have that $\mathcal{S}et_f^{2^{op}}$ is equivalent to the category of arrows in $\mathcal{S}et_f$. Here the morphism from id_2 to $t_2 : 2 \longrightarrow 1$ defined by the couple of morphisms (id_2, t_2) is a regular epimorphism but it is not split because a section would be a couple $(a, b) : t_2 \longrightarrow id_2$ such that a factors through 1 by commutativity, but if this happens, then $id_2 a = a \neq id_2$ and hence in particular it cannot be a section.

We are now going to show that every Grothendieck topos can be seen as a category of sheaves over a site whose underlying category is coherent, but first we need to introduce a particular kind of coherent category.

Definition 1.4.13. *A* pretopos *is a category which is exact and has finite coproducts which are disjoint and pullback stable.*

We can give a more explicit characterization of pretoposes as from the following proposition.

Proposition 1.4.14. *A category is a pretopos iff all the following properties hold*[2]

1. *has all finite limits;*
2. *has finite coproducts and they are disjoint and universal;*
3. *has coequalizers of equivalence relations and they are universal;*
4. *every equivalence relation is effective.*

Proof. A pretopos is exact, so in particular it preserves limits and every equivalence relation is effecive. The second property is common to both definitions, so we are left to prove the third property. Since every equivalence relation is effective, they are all kernel pairs, but since a pretopos is exact, it is in particular regular, which means that they have a coequalizer. Since in regular categories regular epimorphisms are pullback stable, the pullback is still a regular epimorphism, and then using the pullback glueing Lemma and the universal property of pullback one can show that this coequalizer corresponds to the relation pulled back, thus equivalence relations are pullback stable.
To prove the converse we need only to prove that a category satisfying the properties above is exact and, thanks to the last point, we just need to prove that it is regular. It is finitely complete for (1), kernel pairs have coequalizer because of (3), since they are in particular equivalence relations. Finally regular epis are pullback stable because every regular epi is the coequalizer of its kernel pair, so in particular regular epis are coequalizer of equivalence relations and therefore pullback stable by (4). □

Remark 1.4.15. *Every pretopos is coherent, for the existence of finite joins is granted by the existence of finite coproducts and coequalizers, while their universality implies that the pullback map preserves finite joins.*

A first example of pretopos is contained in the following proposition

Proposition 1.4.16. *Every topos is a pretopos.*

Proof. Basically follows from the fact that a topos is finitely complete and cocomplete and an exact category (Lemma 1.1.2). As reference see [SE] Corollary A2.4.5 where the definition of pretopos given there is different but still equivalent to our. □

[2] In some books this is used as definition, see [J] where the category need also be small.

Now we can finally prove the following result

Lemma 1.4.17. *Let $\mathcal{E}$ be a Grothendieck topos, then there is a small full subpretopos $\mathcal{C}$ of $\mathcal{E}$ such that $\mathcal{E} \simeq \mathcal{S}h(\mathcal{C}, J)$ where J is a (Grothendieck) coverage containing the coherent (resp. Grothendieck) coverage.*

Proof. Before beginning the proof, note that for every $A \subset \mathcal{E}$ such that A is (isomorphic to) a small set, we can find a small closuse of A up to limits inside $\mathcal{E}$. Let in fact $A_0 = A$ and for every $n \in \mathbb{N}$, let A_{n+1} be the subset containing a choice of $\lim_I D$ for every $D : I \longrightarrow A_n$, and I a finite category, we just fix the choice of the limit for I terminal to be $D(0)$ where 0 is the unique object of I. Note that here we are making a choice, but the final result will be the same for every choice of this kind and this implies the independence from the axiom of choice. Note that $A_n \subseteq A_{n+1}$ and A_n is small (up to iso) for every $n \in \mathbb{N}$. Since we have a chain of inclusions, it makes sense to define

$$\overline{A} = \bigcup_{n \in \mathbb{N}} A_n$$

Note that $\overline{A}$ is (isomorphic to) a set, being union of small classes.
We also get that the full subcategory of $\mathcal{E}$ with objects in $\overline{A}$ is finitely complete. Consider in fact a diagram $D : I \longrightarrow \overline{A}$, for every $i \in I_0$ there is a $n \in \mathbb{N}$ such that $D(i) \in A_n$ and since I_0 is finite, there is a common such n. In particular then $D : I \longrightarrow A_n$, but this means that A_{n+1} contains the limit of D in $\mathcal{E}$ and hence so does $\overline{A}$. In particular this implies that the full subcategory of $\mathcal{E}$ generated by $\overline{A}$ is finitely complete (and it is the smallest with this property among those containing the objects of A).
Note that the same procedure can be followed with colimits so that from a set A we can get the smallest full cocomplete subcategory of $\mathcal{E}$ which contains A.
Coming back to the lemma, since $\mathcal{E}$ is a Grothendieck topos, from Giraud's Theorem (Theorem 1.2.12) we know that there is a set A of generators for $\mathcal{E}$. We start from this set A and we call it B_0, then for n even we define the set B_{n+1} to be the closure under limits of B_n and for n odd we define B_{n+1} as the closure under colimits of B_n. Again there is an inclusion of B_n in B_{n+1} and all of them are sets. We call $\mathcal{C}$ the full subcategory generated by the union of the B_n's.
Using a similar reasoning to the one used before, we can prove that $\mathcal{C}$ is finitely complete and finitely cocomplete. For finite completeness we take the smallest even n such that the whole diagram is in B_n and then B_{n+1} contains its limit, while for colimits repeat this reasoning with n odd.
Note also that an equivalence relation in $\mathcal{C}$ is already an equivalence relation in $\mathcal{E}$ and here it is effective, then since $\mathcal{C}$ is closed by finite limits and finite colimits, the relation is effective also in $\mathcal{C}$. It follows from Proposition 1.4.14 that $\mathcal{C}$ is a pretopos.

Now we take as coverage J on $\mathcal{C}$, the collection of jointly epimorphic maps and let $\widetilde{J}$ be the generated Grothendieck coverage. Note that J contains P thanks to Remark 1.4.3.
Thanks to Corollary 4.1 in the Appendix of [SGL], we have that every full subcategory of a Grothendieck topos forming a generating set forms a site for the topos as soon as it is equipped with the coverage of families in $\mathcal{C}$ which are jointly epimorphic in $\mathcal{E}$. In our particular case, $\mathcal{C}$ is a full subcategory of $\mathcal{E}$, it generates $\mathcal{E}$ for $A \subseteq \mathcal{E}$ was already a set of generators and finally J is the coverage of jointly epimorphic families, so $\mathcal{E} = \mathcal{Sh}(\mathcal{C}, J)$.
Since $P \subseteq J$, also $\widetilde{P} \subseteq \widetilde{J}$ and this proves the final part of this lemma. □

In this way we are, in principle, able to give informations about a Grothendieck topos studying a site of definition for it which is especially regular. In particular, thanks to Proposition 1.4.11, it is enough to study P-compatible dm-coverages.

Definition 1.4.18. *A* pretopos site *is a site of the form* $(\mathcal{C}, J)$ *with* $\mathcal{C}$ *a pretopos and* J *a coverage such that the Grothendieck coverage it generates contains the coherent one.*

In these terms, what Lemma 1.4.17 states is that there exist a subcanonical pretopos site for every Grothendieck topos.

Before closing this section, we will focus on a particular kind of pretopos sites

Definition 1.4.19. *We say that a pretopos site* $(\mathcal{C}, J)$ *is* compact *if the Grothendieck coverage generated by* $\widetilde{J}_d$ *has as unique covering family of the terminal the maximal sieve.*

We will see in Section 5.3 what exactly means for a pretopos site to be compact. Now we will just make the following observation.

Proposition 1.4.20. *Let* $(\mathcal{C}, J)$ *be pretopos site where* J *is the Grothendieck coverage generated by the Grothendieck coherent one* P *and a coverage* T *of monos such that its sifted closure is a* P*-compatible dm-coverage* $\overline{T}$*, then this site is compact iff every* T*-covering family of* 1 *contains an isomorphism.*

Proof. If the site is compact, then the only covering sieve in $\widetilde{J}_d(1)$ is maximal. Let $A \in T(1)$, then consider the sieve $S = \langle A \rangle$, it is in J and hence in $J_d \subseteq \widetilde{J}_d$, but then S is the maximal sieve, which means that $\mathrm{id}_1 \in S$ and hence there is some $a \in A$ such that $\mathrm{id}_1 = ah$ for some arrow h. Note that a is both a monomorphism and a split epimorphism, and hence it is an isomorphism.
Conversely, suppose that every covering family in $T(1)$ contains an isomorphism, then thanks to Lemma A.2.5, also every sieve of the Grothendieck

topology generated by T contains an isomorphism (consider the (full) subcategory of terminal objects and isomorphisms between them). This implies that the only covering sieve is the maximal one. Note now that by Proposition 1.4.11 (2), the sifted closure of T is equivalent to J_d and thus in particular the Grothendieck topology generated by T is $\widetilde{J_d}$, which means that $\widetilde{J_d}(1) = \{y_1\}$ and thus that the site is compact. □

Thanks to this proposition, we have an easier way to check if a pretopos site is compact. Note that in particular if T is the set of monos in J_d, we are in the hypotheses of this Propositon. It follows that if $(\mathcal{C}, J)$ is a site with J Grothendieck topology, then it is compact iff the only J-covering sieve on 1 which is a dm-sieve is the maximal one.

1.5 2-categories

In order to be more precise later on, we give some basic notion about enriched categories and 2-categories.

Definition 1.5.1. *Let $\mathcal{V}$ be a category with finite products, then a $\mathcal{V}$-*category *(also called $\mathcal{V}$-*enriched category*) $\mathcal{C}$ consists of*

1. *A collection of* objects $\mathcal{C}_0$*;*
2. *For every pair of objects (A, B) an object $\mathcal{C}(A, B)$ of $\mathcal{V}$;*
3. *For every object A a morphism $id_A : 1 \longrightarrow \mathcal{C}(A, A)$ in $\mathcal{V}$ called* identity *of A;*
4. *For every triple of objects (A, B, C) a morphism in $\mathcal{V}$*

$$c_{A,B,C} : \mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$$

called composition.

such that the following properties occur

1. *the composition is associative, meaning that for every quadruple of objects (A, B, C, D), the following diagram commutes*

$$\begin{array}{ccc} \mathcal{C}(C, D) \times \mathcal{C}(B, C) \times \mathcal{C}(A, B) & \xrightarrow{1 \times c_{A,B,C}} & \mathcal{C}(C, D) \times \mathcal{C}(A, C) \\ \downarrow{\scriptstyle c_{B,C,D} \times 1} & & \downarrow{\scriptstyle c_{A,C,D}} \\ \mathcal{C}(B, D) \times \mathcal{C}(A, B) & \xrightarrow[c_{A,B,D}]{} & \mathcal{C}(A, D) \end{array}$$

2. *the identity acts trivially on the composition i.e., if* $t_V : V \longrightarrow 1$ *is the unique morphism in* $\mathcal{V}$ *to the terminal, then*

$$\begin{array}{ccc} \mathcal{C}(A,B) & \xrightarrow{(1, id_A t_{\mathcal{C}(A,B)})} & \mathcal{C}(A,B) \times \mathcal{C}(A,A) \\ {\scriptstyle (id_B t_{\mathcal{C}(A,B)}, 1)}\downarrow & \searrow^{1} & \downarrow{\scriptstyle c_{A,A,B}} \\ \mathcal{C}(B,B) \times \mathcal{C}(A,B) & \xrightarrow[c_{A,B,B}]{} & \mathcal{C}(A,B) \end{array}$$

commutes.

Example 1.5.2. *A locally small category* $\mathcal{C}$ *is a* $\mathcal{S}et$*-enriched category where the so called hom sets are the objects* $\mathcal{C}(A,B)$*, the identity is the map selecting* id_A *in* $\mathcal{C}(A,A)$ *and the composition is the usual composition.*
A less trivial example is the category of groups, in fact we can see $\mathcal{G}rp(G,H)$ *itself as a group where the multiplication is the pointwise multiplication. We also have that the composition preserve this multiplication, so we can interpret* $\mathcal{G}rp$ *as a* $\mathcal{G}rp$*-enriched category.*

As an example we state the following

Lemma 1.5.3. *A finitely complete cartesian closed category* $\mathcal{V}$ *can be seen canonically as an enriched category in itself.*

Proof. We will just sketch the proof. We take as objects the objects of $\mathcal{V}$, then for every pair (A,B) we take $\mathcal{V}(A,B) = B^A$, as identity of A the transpose of $\pi_2 : 1 \times A \longrightarrow A$ and for every triple (A,B,C) as composition we choose the transposed of

$$C^B \times B^A \times A \xrightarrow{1\times ev_c} C^B \times B \xrightarrow{ev_c} C$$

where ev is the evaluation i.e. the unit of the adjunction $- \times A \dashv (-)^A$. One can check that the properties are satisfied using the adjunction. □

After this generalization of categories, we can do the same for functors getting

Definition 1.5.4. *Let again* $\mathcal{V}$ *be a category with finite products and let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* $\mathcal{V}$*-categories. A* $\mathcal{V}$-functor *or* enriched functor *in* $\mathcal{V}$ *consists of a correspondence sending an object* A *to* $\mathcal{C}$ *to an object* $F(A)$ *in* $\mathcal{D}$ *and for every couple* (A,B) *of objects of* $\mathcal{C}$ *a morphism* $F_{A,B} : \mathcal{C}(A,B) \longrightarrow \mathcal{D}(F(A),F(B))$*. These morphisms must be compatible with compositions and identities, which* *means that the following diagrams are commutative in $\mathcal{V}$*

$$\begin{array}{ccc}
\mathcal{C}(B,C) \times \mathcal{C}(A,B) & \xrightarrow{c^{\mathcal{C}}_{A,B,C}} & \mathcal{C}(A,C) \\
\downarrow{\scriptstyle F_{B,C} \times F_{A,B}} & & \downarrow{\scriptstyle F_{A,C}} \\
\mathcal{D}(F(B),F(C)) \times \mathcal{D}(F(A),F(B)) & \xrightarrow[c^{\mathcal{D}}_{F(A),F(B),F(C)}]{} & \mathcal{D}(F(A),F(C))
\end{array}$$

$$1 \xrightarrow{id^{\mathcal{C}}_A} \mathcal{C}(A,A) \xrightarrow{F_{A,A}} \mathcal{D}(F(A),F(A)), \qquad 1 \xrightarrow[id^{\mathcal{D}}_{F(A)}]{} \mathcal{D}(F(A),F(A))$$

For every A, B and C objects in $\mathcal{C}$.
We write $F : \mathcal{C} \longrightarrow \mathcal{D}$ in this case.

The same generalization can be made for natural transformation, obtaining

Definition 1.5.5. *Let $\mathcal{V}$ be a category with finite products, let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{V}$-categories and let F and G two functors $\mathcal{C} \longrightarrow \mathcal{D}$. A $\mathcal{V}$-*natural transformation *from F to G, denoted as $\alpha : F \longrightarrow G$ is a function sending each object A of $\mathcal{C}$ to a morphism $\alpha_A : 1 \longrightarrow \mathcal{D}(F(A), G(A))$ in $\mathcal{V}$ such that the following square in $\mathcal{V}$ is commutative*

$$\begin{array}{ccc}
\mathcal{C}(A,B) & \xrightarrow{\alpha_B \times F_{A,B}} & \mathcal{D}(F(B),G(B)) \times \mathcal{D}(F(A),F(B)) \\
\downarrow{\scriptstyle G_{A,B} \times \alpha_A} & & \downarrow{\scriptstyle c^{\mathcal{D}}_{F(A),F(B),G(B)}} \\
\mathcal{D}(G(A),G(B)) \times \mathcal{D}(F(A),G(A)) & \xrightarrow[c^{\mathcal{D}}_{F(A),G(A),G(B)}]{} & \mathcal{D}(F(A),G(B))
\end{array}$$

We can now define a 2-category as follows

Definition 1.5.6. *A* 2-category $\mathfrak{C}$ *is an enriched category in $\mathcal{CAT}$.*

We can also give a more explicit equivalent definition for 2-categories as follows

Definition 1.5.7. *A* 2-category $\mathfrak{C}$ *is given by a family of objects (0-cells) $\mathfrak{C}_0$, a family of morphisms (1-cells) $\mathfrak{C}_1$ and a family of transformations (2-cells) $\mathfrak{C}_2$ with the following operations (written as arrows) between them.*

$$\mathfrak{C}_2 \underset{t}{\overset{s}{\rightrightarrows}} \mathfrak{C}_1, \quad \mathfrak{C}_2 \xleftarrow{\iota} \mathfrak{C}_1, \qquad \mathfrak{C}_1 \underset{c}{\overset{d}{\rightrightarrows}} \mathfrak{C}_0, \quad \mathfrak{C}_1 \xleftarrow{i} \mathfrak{C}_0$$

such that

1. *i is a simultaneous splitting for d and c*

2. *ι is a simultaneous splitting for s and t*

3. *$ds = dt$ and $cs = ct$*

then there is a partial binary operation on morphisms sending a couple of morphisms (f, g) to gf and defined exactly whenever $c(f) = d(g)$ and such that the following identities hold whenever they make sense for $f, g, h \in \mathfrak{C}_1$ and $A \in \mathfrak{C}_0$

1. *$d(gf) = d(f)$ and $c(gf) = c(g)$*

2. *$h(gf) = (hg)f$*

3. *$i(A)f = f$ and $fi(A) = f$*

Moreover we have two partial binary operations on $\mathfrak{C}_2$ one called vertical composition *and the other called* horizontal composition. *Given a couple of transformations $\alpha, \beta \in \mathfrak{C}_2$, we denote with $\beta\alpha$ the vertical composition and with $\beta \circ \alpha$ the horizontal one. These operations are such that $\beta\alpha$ is defined iff $t(\alpha) = s(\beta)$ while $\beta \circ \alpha$ is defined iff $cs(\alpha) = ds(\beta)$. Then they must satisfy the following identities for all $\alpha, \beta, \gamma, \delta \in \mathfrak{C}_2$, $f \in \mathfrak{C}_1$ and $A \in \mathfrak{C}_0$ for which they make sense.*

1. *$s(\beta\alpha) = s(\alpha)$ and $t(\beta\alpha) = t(\beta)$*

2. *$s(\beta \circ \alpha) = s(\beta)s(\alpha)$ and $t(\beta \circ \alpha) = t(\beta)t(\alpha)$*

3. *$\gamma(\beta\alpha) = (\gamma\beta)\alpha$ and $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$*

4. *$\iota(f)$ act as identity for vertical composition and $\iota(i(A))$ as identity for the horizontal one.*

5. *$(\delta\gamma) \circ (\beta\alpha) = (\delta \circ \beta)(\gamma \circ \alpha)$*

6. *$\iota(g) \circ \iota(f) = \iota(gf)$*

For more details see [SE] Definition B1.1.1.
To pass from Definition 1.5.6 to Definition 1.5.7, let $\mathcal{C}$ be a $\mathcal{CAT}$-enriched category, we define

$$\mathfrak{C}_0 = \mathcal{C}_0$$
$$\mathfrak{C}_1 = \coprod_{(A,B)\in \mathcal{C}_0{}^2} \mathcal{C}(A, B)_0$$
$$\mathfrak{C}_2 = \coprod_{(A,B)\in \mathcal{C}_0{}^2} \mathcal{C}(A, B)_1$$

The composition of 1-cells is obtained from the composition c of $\mathcal{C}$ on objects, the horizontal composition is still c but on arrows and the vertical composition is the internal composition of $\mathcal{C}(A, B)$. The identities morphism i sends an object A to the object selected by $\mathrm{id}_A : 1 \longrightarrow \mathcal{C}(A, A)$ and the map ι sends an object f of $\mathcal{C}(A, B)$ to the identity on f in $\mathcal{C}(A, B)$. One can prove the equivalence between the two definitions.

Let $A \in \mathfrak{C}_0$ and $f \in \mathfrak{C}_1$, we will commonly call $i(A)$ the identity on A and it will be denoted as id_A or 1_A and $\iota(f)$ as ι_f. Moreover, with a little abuse of notation, if $\alpha \in \mathfrak{C}_2$ we will write $\alpha \circ \iota_f$ and $\iota_f \circ \alpha$ as $\alpha \circ f$ and $f \circ \alpha$ respectively.

Remark 1.5.8. *Note that thanks to the last axiom, this abuse of notation will not cause any confusion. In particular we get $g \circ f = gf$ coherently with the usual notation for categories.*

Moreover $d(f)$ and $c(f)$ will be called as usual *domain* and *codomain* of f respectively. Given a transformation α we will call $s(\alpha)$ and $t(\alpha)$ respectively the *source* and the *target* of α, while we will call $ds(\alpha) = dt(\alpha)$ and $cs(\alpha) = ct(\alpha)$ respectively *domain* and *codomain* of α.
Graphically, given two morphisms $f, g : A \longrightarrow B$ we will depict a transformation $\alpha : f \Rightarrow g$ as

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \quad (\Downarrow \alpha)$$

In most cases we will also avoid mentioning the name of the 2-cell, rendering the diagram identical to an ordinary diagram

Example 1.5.9. *The classical example is the category $\mathfrak{Cat}$ of small categories, functors and natural transformations with the usual vertical and horizontal composition.*

Example 1.5.10. *If $\mathcal{V}$ is a category with finite products, taking as objects $\mathcal{V}$-categories, as morphisms $\mathcal{V}$-functors and as transformations $\mathcal{V}$-natural transformations, we get the 2-category of $\mathcal{V}$-categories, denoted with $\mathcal{V}$-$\mathfrak{Cat}$.*

Example 1.5.11. *Every category $\mathcal{C}$ can be seen as a 2-category by taking $\mathfrak{C}_0 := \mathcal{C}_0$ and $\mathfrak{C}_1 := \mathcal{C}_1$ with the same domain and codomain as in the original category and $\mathfrak{C}_2 := \mathcal{C}_1$ with $s = t = \iota = id_{\mathcal{C}_1}$.*

Let $\mathcal{E}$ and $\mathcal{F}$ be two toposes and $f, g : \mathcal{E} \longrightarrow \mathcal{F}$ two geometric morphisms between them.

Definition 1.5.12. *A* geometric transformation $\alpha : f \Rightarrow g$ *is defined to be a natural transformation $\alpha : f^* \Rightarrow g^*$.*

As for the choice of the direction of the geometric morphism, the choice of the direction of the geometric transformations is also arbitrary, for natural transformations $f^* \Rightarrow g^*$ are in bijection with natural transformations between $g_* \Rightarrow f_*$.

Example 1.5.13. *Here we have another example of 2-category i.e. the category of toposes, geometric morphisms and geometric trasformations which throughout this thesis*[3] *will be denoted as* $\mathfrak{Geom}$.

Now that we have a definition of 2-category, we would like to have a notion of arrows between them

Definition 1.5.14. *Let* $\mathfrak{C}$ *and* $\mathfrak{D}$ *be 2-categories, a* pseudofunctor $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ *consists of the following data*

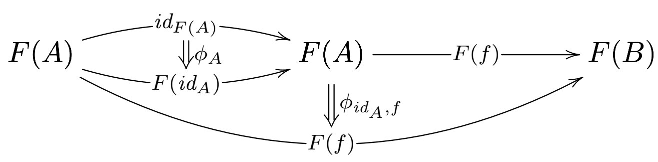
1. *A mapping* $A \mapsto F(A)$ *from* $\mathfrak{C}_0$ *into* $\mathfrak{D}_0$
2. *A mapping* $f \mapsto F(f)$ *from* $\mathfrak{C}_1$ *into* $\mathfrak{D}_1$
3. *A mapping* $\alpha \mapsto F(\alpha)$ *from* $\mathfrak{C}_2$ *into* $\mathfrak{D}_2$
4. *For each object* C *of* $\mathfrak{C}$ *a 2-isomorphism* $\phi_A : id_{F(A)} \Rightarrow F(id_A)$
5. *For each couple of composable morphisms* f *and* g *in* $\mathfrak{C}$ *a 2-isomorphism* $\phi_{f,g} : F(g)F(f) \Rightarrow F(gf)$

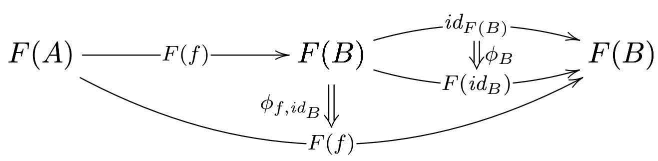
such that for all $f \in \mathfrak{C}_1$ *and* $\alpha,\ \beta \in \mathfrak{C}_2$ *composable transformations, we have*

1. $d(F(f)) = F(d(f))$ *and* $c(F(f)) = F(c(f))$
2. $s(F(\alpha)) = F(s(\alpha))$ *and* $t(F(\alpha)) = F(t(\alpha))$
3. $F(\iota_f) = \iota_{F(f)}$ *and* $(F(\beta\alpha)) = F(\beta)F(\alpha)$

Then we want the ϕ*'s to satisfy the coherence conditions, that is*

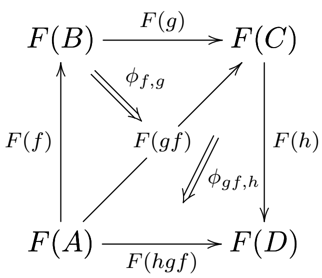
1. *The following 2-cells are equal to* $\iota_{F(f)}$

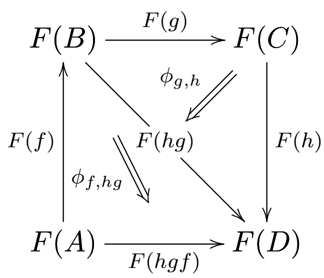




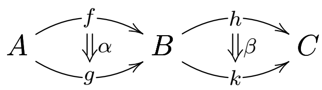
[3] Almost everywhere this category is denoted with $\mathfrak{Top}$ but I prefer to call it in this way to avoid confusion with the category of topological spaces, the 2-category of topological spaces, continuous maps and homotopies or with the category of toposes with a different kind of arrows like logical morphisms

2. The following 2-cell compositions are equal





3. For any couple of horizontally composable transformations



we have

$$\phi_{g,k}(F(\beta)\circ F(\alpha)) = F(\beta\circ\alpha)\phi_{f,h}$$

These conditions are stated here just in order to have a precise definition but we will avoid verifying them throughout this thesis.
Usually the pseudofunctors that we will encounter will be *normalized* that is, for all $A \in \mathfrak{C}$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and ϕ_A is the identity transformation. If a pseudofunctor does not have this property it can always be normalized by defining $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ and then modify accordingly the morphisms $\phi_{f,g}$, so we will tacitly consider the normalized version of a pseudofunctor. In general though, we cannot adopt a similar treatment for the $\phi_{f,g}$'s in order to make them identities. When this happens the pseudofunctor is called *strict* or *2-functor*.
Note that 2-functors are precisely $\mathcal{CAT}$-enriched functors.

Example 1.5.15. *A first example of pseudofunctor comes from Theorem 1.1.10 because given a topos $\mathcal{E}$, we get a pseudofunctor $\mathcal{E} \longrightarrow \mathfrak{Geom}$ where $\mathcal{E}$ is seen as 2-category as in Example 1.5.11. The pseudofunctor sends an object A to $\mathcal{E}/A$ and a morphism $f : A \longrightarrow B$ to the geometric morphism $f : \mathcal{E}/A \longrightarrow \mathcal{E}/B$.*

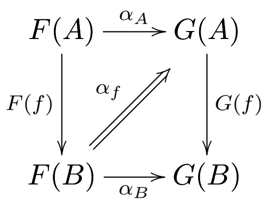
Another example of pseudofunctor will appear in Theorem 2.5.3.
We can also define the equivalent of a natural transformation between pseudofunctors.

Definition 1.5.16. *Let $F, G : \mathfrak{C} \longrightarrow \mathfrak{D}$ be two pseudofunctors, then a* pseudonatural transformation *$\alpha : F \Rightarrow G$ is given by the following data*

1. a mapping $C \mapsto \alpha_C$ from $\mathfrak{C}_0$ into $\mathfrak{D}_1$

2. a mapping $f \mapsto \alpha_f$ *from* $\mathfrak{C}_1$ *into* $\mathfrak{D}_2$

such that for all $f : A \longrightarrow B$ *we have that* α_f *is invertible and*



And moreover some coherence conditions with respect to compositions, unities and naturality. But we will not write them here explicitly.

As we have seen with pseudofunctors, when working with 2-categories it is often too restrictive to require the equality of two morphisms. As an example, if we consider in $\mathcal{C}at$ two parallel functors which are distinct but naturally isomorphic, then they will still share almost all properties, so it is often convenient to consider them as if they were the same.
For this reason, in a 2-category, instead of requiring that two parallel morphisms $f, g : A \longrightarrow B$ are equal, we will often ask them to be *isomorphic*, hence we require the existence of a transformation $\alpha : f \Rightarrow g$ which is invertible with respect to the vertical composition, hence there is another transformation $\beta : g \Rightarrow f$ such that $\alpha\beta = \iota_g$ and $\beta\alpha = \iota_f$. In this case α and β will be called, as expected, *isomorphisms* or *2-isomorphisms*. We will write $f \simeq g$ in this case, or $f \overset{\alpha}{\simeq} g$ if the 2-isomorphism needs to be specified.
The usefulness of this weakening appears already evident if we ask two objects A and B to be isomorphic in this week sense, in fact we expect two morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow A$ that are each other's inverse, but just up to 2-isomorphisms, hence $fg \simeq id_B$ and $fg \simeq id_A$. We will say that two such object are *equivalent* and that f and g are *equivalences*.
In $\mathfrak{Cat}$ the equivalence corresponds precisely to the equivalence of categories and in fact two equivalent categories have basically the same categorical properties.

This weakening leads us to a new interpretation of the commutativity of a diagram in a 2-category. Namely we require, instead of equalities between arrows, just 2-isomorphisms between them, but these isomorphisms need also be compatible.
More precisely let $\mathcal{C}$ be a category and $\mathfrak{C}$ a 2-category, a diagram $D : \mathcal{C} \longrightarrow \mathfrak{C}$ is said to be commutative in a weak sense or *weakly commutative* if for every couple of parallel arrows $f, g : A \longrightarrow B$ in $\mathcal{C}$ we have a 2-isomorphism $\alpha_{f,g} : D(f) \Rightarrow D(g)$ in $\mathfrak{C}$ such that

1. $\alpha_{f,f} = \iota_{D(f)}$ for all f morphism in $\mathcal{C}$

2. $\alpha_{f,h} = \alpha_{g,h}\alpha_{f,g}$ for all parallel morphisms f, g, h in $\mathcal{C}$

3. $\alpha_{kfh,kgh} = k \circ \alpha_{f,g} \circ h$ for all morphisms in $\mathcal{C}$ of the form

$$A \xrightarrow{h} B \underset{g}{\overset{f}{\rightrightarrows}} C \xrightarrow{k} D$$

From now on, when dealing with two categories, unless specifically stated, commutativity of diagram will always be intended in this weak form.
This interpretation of commutativity affects the way we interpret a slice 2-category. Let $\mathfrak{C}$ be a 2-category and $C \in \mathfrak{C}$, then by $\mathfrak{C}/C$ we actually mean the 2-category defined as follows. The objects are morphisms of $\mathfrak{C}$ of the form $f : A \longrightarrow C$ for $A \in \mathfrak{C}$. Given $f : A \longrightarrow C$ and $g : B \longrightarrow C$ two objects of the slice category, morphisms $\mathfrak{C}(f, g)$ are couples (h, α) where h is a $\mathfrak{C}$-morphism from A to B and $\alpha : f \Rightarrow gh$ is a 2-isomorphism. The composition is defined by $(k, \beta)(h, \alpha) = (kh, (\beta \circ h)\alpha)$ and given two parallel morphisms $(h, \alpha), (k, \beta) : f \longrightarrow g$, a transformation between them will be a transformation $\gamma : h \Rightarrow k$ such that $(g \circ \gamma)\alpha = \beta$.
All these details will usually be omitted when working with slice 2-categories to avoid encumbering the notation.

Remark 1.5.17. *The pseudofunctor of Theorem 1.1.10 and Example 1.5.15 can actually be seen as a pseudofunctor with values in a comma 2-category, namely*

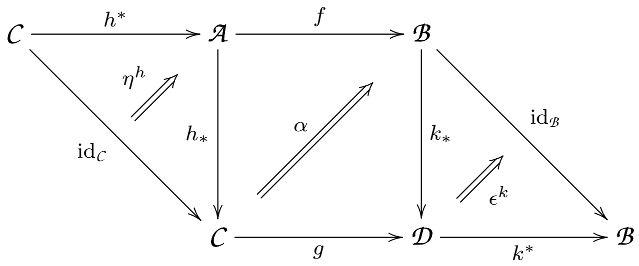
$$\mathcal{E} \longrightarrow \mathfrak{Geom}/\mathcal{E}$$

1.6 Beck-Chevalley conditions

Now we are going to define some conditions on certain weakly commutative squares of functors that will come in handy to describe some relevant properties later on in this thesis.
Consider the following weakly commutative square of functors

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle h_*}\downarrow & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow{\scriptstyle k_*} \\ \mathcal{C} & \xrightarrow[g]{} & \mathcal{D} \end{array} \tag{1.3}$$

where $\alpha : gh_* \Rightarrow k_*f$ is the 2-isomorphism (natural isomorphism) that gives commutativity. Suppose that h_* and k_* have left adjoint, denoted respectively h^* and k^*. Denote the adjunction $h^* \dashv h_*$ with h and the adjunction $k^* \dashv k_*$ with k. For each adjunction $p \in \{h, k\}$ we denote with η^p and ϵ^p respectively the unit and the counit of the adjunction $p^* \dashv p_*$. We have thus the following diagram of functors and natural transformations



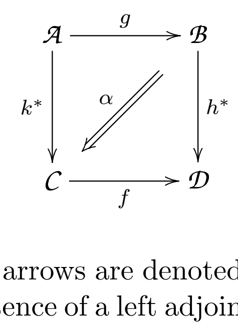
Let $\theta = (k^*g \circ \eta^h)(k^* \circ \alpha \circ h^*)(\epsilon^k \circ fh^*)$, then it is a canonical natural transformation $\theta : k^*g \Rightarrow fh^*$, sometimes called *Beck-Chevalley transformation.*

Definition 1.6.1. *A square of functors which is commutative up to a 2-isomorphism like* (1.3) *with vertical maps which are right adjoints is said to satisfy the* Beck-Chevalley condition *if the natural transformation θ that we have just built is an isomorphism.*

In particular, if the square (1.3) satisfies the Beck-Chevalley condition, we have another commutative square of functors corresponding to $k^*g \cong fh^*$. Sometimes for our purposes we will need a weaker form of this condition, namely

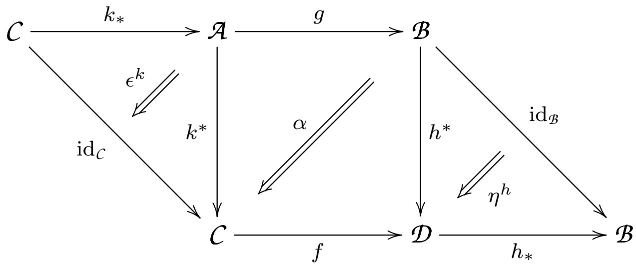
Definition 1.6.2. *A square of functors like the one in Definition 1.6.1 is said to satisfy the* weak Beck-Chevalley condition *if the natural transformation θ is pointwise monic, i.e. for all $c \in \mathcal{C}$ the morphism θ_c is a monomorphism.*

We can also obtain the corresponding dual properties, which will describe different characteristics.
Consider the dual diagram of (1.3) which becomes



(1.4)

where this time the vertical arrows are denoted with the upper star because instead of requiring the existence of a left adjoint, we are asking the existence of a right one, which will be denoted with the lower star. As above we denote the adjunction $p^* \dashv p_*$ with p. This time we have the following diagram of functors and natural transformations



Let $\theta^d = (h_* f \circ \epsilon^k)(h_* \circ \alpha \circ k_*)(\eta^h \circ gk_*)$, so that again we get a canonical natural transformation $\theta^d : gk_* \longrightarrow h_* f$, called *dual Beck-Chevalley transformation.*

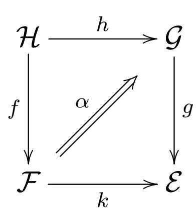
Definition 1.6.3. *A square of geometric morphisms commutative up to a 2-isomorphism like* (1.4) *with vertical maps that are left adjoints is said to satisfy the* dual Beck-Chevalley condition *if the natural transformation* θ^d *that we have just built is an isomorphism.*
It is said to satisfy the weak dual Beck-Chevalley condition *if the natural transformation* θ^d *is* pointwise monic.

Again if (1.4) satisfies the Beck-Chevalley condition, we have another commutative square of functors corresponding to $gk_* \cong h_* f$.

Remark 1.6.4. *Note that in order to build* θ *and* θ^d *it is not needed that* α *is a 2-isomorphism, as long as it respect the direction as showed in diagrams* (1.3) *and* (1.4) *respectively.*

A particular situation where it makes sense to consider all the just presented BC conditions is when the categories of a commutative square like (1.3) and (1.4) are toposes and the vertical functors are direct (inverse) image of a geometric morphism.

Now we will analyse an even more specific situation, that is, when all the functors in the square come from geometric morphism.
Let the following be a weakly commutative square of geometric morphism in $\mathfrak{Geom}$



(1.5)

where $\alpha : kf \Rightarrow gh$ is the 2-isomorphism (invertible geometric transformation) that gives commutativity. By definition α is a natural isomorphism $\alpha : f^* k^* \Rightarrow h^* g^*$ and hence, since right adjoints of isomorphic functors are isomorphic, α corresponds to a natural isomorphisms $\beta : g_* h_* \Rightarrow k_* f_*$. This last isomorphism in particular gives the commutativity of a diagram like the one in (1.3), so one can compute the corresponding BC transformation $\theta : k^*g_* \Rightarrow f_*h^*$. We can now give the following

Definition 1.6.5. *A square in* $\mathfrak{Geom}$ *commutative up to a 2-isomorphism like* (1.5) *is said to satisfy the* Beck-Chevalley condition *if* θ *is an isomorphism. It is said to satisfy the* weak Beck-Chevalley condition *if instead* θ *is a monomorphism in every component.*

Whenever we will deal with a weak commutative square in $\mathfrak{Geom}$ we will by convention consider this construction on the isomorphism in the direction specified by (1.5).

Both the BC and the weak BC conditions for a square in $\mathfrak{Geom}$ are preserved by localization, that is

Lemma 1.6.6. *Suppose the commutative square* (1.5) *satisfies the (weak) Beck-Chevalley condition, then for every* E *object of* $\mathcal{E}$ *also the following diagram does*

$$\begin{array}{ccc} \mathcal{H}/f^*k^*(E) & \xrightarrow{h/g^*(E)} & \mathcal{G}/g^*(E) \\ \Big\downarrow{\scriptstyle f/k^*(E)} & & \Big\downarrow{\scriptstyle g/E} \\ \mathcal{F}/k^*(E) & \xrightarrow[k/E]{} & \mathcal{E}/E \end{array} \tag{1.6}$$

Proof. See [SE] A4.1.16. □

Then we also have a useful characterization of the weak BC condition in terms of the strong version, which uses the following.

Definition 1.6.7. *Let* $\mathcal{C}$ *be a category with terminal object* 1*, a* subterminal *object is an object* U *such that the unique morphism* $U \longrightarrow 1$ *is a monomorphism. In other words it is an object representing a subobject of* 1.

Here are some useful properties of subterminal objects

Proposition 1.6.8. *Let* $\mathcal{C}$ *be a category with terminal object and* U *a subterminal object, then*

1. *for every object* X *there is at most one arrow* $X \longrightarrow U$.
2. *every morphism with domain* U *is a monomorphism.*
3. *there is a map* $1 \longrightarrow U$ *iff* U *is a terminal object*

Proof. 1. Let $a, b : X \longrightarrow U$ and let $t : U \longrightarrow 1$ be the unique morphism into the terminal, then since there is a unique map from any object to the terminal, $ta = tb$ but t is a monomorphism, so $a = b$

2. Let $f : U \longrightarrow Y$ be a morphism in $\mathcal{C}$, if we compose it with the unique arrow $Y \longrightarrow 1$, we get by uniqueness the morphism $U \longrightarrow 1$ which is a monomorphism by definition of subterminal, thus follows that also f is a monomorphism because when a composition is mono, then also the first arrow is a mono.

3. if U is a terminal object, then clearly there is a map $1 \longrightarrow U$. Conversely, if we have a map $s : 1 \longrightarrow U$, if we call $t : U \longrightarrow 1$ the unique map into the terminal, then ts is the identity map of the terminal (again by uniqueness). For the first point of this proposition, st is the identity on U, so s and t are isomorphisms, which implies that U is terminal. □

Proposition 1.6.9. *For a commutative square as* (1.5) *in* $\mathfrak{Geom}$ *the following are equivalent*

1. *The square satisfies the weak Beck-Chevalley condition*

2. *For every monomorphism* $m : A \longrightarrow B$ *in* $\mathcal{F}$, *the naturality square of* θ

$$\begin{array}{ccc} k^*g_*(A) & \xrightarrow{\theta_A} & f_*h^*(A) \\ {\scriptstyle k^*g_*(m)}\downarrow & & \downarrow{\scriptstyle f_*h^*(m)} \\ k^*g_*(B) & \xrightarrow[\theta_B]{} & f_*h^*(B) \end{array}$$

   is a pullback.

3. *For every object* E *in* $\mathcal{E}$, *the square* (1.6) *satisfies the Beck-Chevalley condition at subterminal objects, i.e. the natural transformation* θ/E *associated to this diagram is such that for all* U *subterminal object in* $\mathcal{F}/f^*(A)$, *the morphism* $(\theta/A)_U$ *is an isomorphism.*

Proof. See [SE] Proposition A4.1.17 or [MV] Proposition I.2.7. □

We will find an interpretation of BC condition at the end of Section 3.4.

Chapter 2

Geometry of toposes

In this chapter we are going to review briefly the steps that allow us to view topological spaces as toposes[1]. This chapter is probably the key to understand the purpose of this thesis, since it displays the geometric side of topos theory.
We start by describing the algebraic structure that best mimics a poset of open subsets in a topological space and we will use it to provide a generalization of topological spaces known as locales. We will then study this new idea of space where points are no more the main ingredient, seeing what we gain and what we lose with this generalization. At this point we will provide a way to transform locales into toposes using sheaves, giving also a geometric interpretation of sheaves. With the last section then we will study how locales embed in the 2-category $\mathfrak{Geom}$ presented in Chapter 1. We will conclude by giving a characterization of those toposes that come from a locale and hence from a topological space.

2.1 Frames

The first aim of this chapter is to get rid of the central role that points have in $\mathcal{Top}$. In order to achieve this goal, we need to focus on the other ingredient of a topological space, that is, the poset of subobjects, and in particular we are looking for an algebraic description of this object. For this purpose, we start with the following Definitions.

Definition 2.1.1. *A* lattice *is a partial order which has finite meets and finite joins (therefore in particular top and bottom elements). In pure categorical terms we can define a lattice as a partial order which is finitely complete and finitely cocomplete as a category.*
We say that a lattice is (co)complete *if it is (co)complete as a category.*

About lattices and completeness, we state the following result

[1] This idea is explained more in detail in [SE] Part C.

Lemma 2.1.2. *A lattice (actually an order) P is complete iff it is cocomplete.*

Proof. Since the definition of lattice (order) is self dual (i.e. a category is a lattice iff its dual is a lattice), it is enough to prove that a complete lattice is also cocomplete and in particular that for every subset U of P there is a join $\bigvee U$, for colimits in a poset are joins. Consider the subset $M = \{y \in P | \forall x \in U \; x \leq y\}$ of P, that is, the set of the upper bounds for U. We now claim that the join of U can be constructed as $\bigwedge M$.
For all $y \in M$ and $x \in U$, we have $x \leq y$, so in particular $x \leq \bigwedge M$. If now y is such that $x \leq y$ for all $x \in U$, then $y \in M$ and hence $\bigwedge M \leq y$. We have proven that the universal property of the join is satisfied for $\bigwedge M$, so we have proven the claim. □

We can now state the following

Definition 2.1.3. *A* frame *is a complete distributive lattice, that is, a complete lattice P where the following identity is satisfied:*

$$a \wedge \bigvee_{x \in U} x = \bigvee \{a \wedge x | x \in U\}$$

for all $a \in P$ and $U \subseteq P$.
As before we can state this definition in pure categorical terms by saying that for every object a the functor $a \times -$ commutes with small colimits.
Given two frames A and B, we define a frame homomorphism *from A to B as an order preserving map from A to B which preserves also finite meets and joins. In categorical terms it is a functor preserving finite limits and colimits.*
We can form in this way the category of frames denoted $\mathcal{Frm}$.

These definitions were inspired by the following emblematic example

Example 2.1.4. *Let X be a topological space and consider the poset of its open subsets $\mathcal{O}(X)$ ordered by inclusion. This is a frame because finite meets are intersections, small joins are unions and the distributivity holds because it holds between intersections and unions. For a matter of completeness note that the meet of a small family of open subsets is the interior of their intersection.*
Now let X, Y be topological spaces and $f : X \longrightarrow Y$ a continuous map, then consider the map $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ sending an open of Y to its inverse image via f. Since f is continuous, this map is well defined and from the corresponding properties of the inverse image of functions, we deduce that f^* is a frame homomorphism.
In this way we get a functor

$$\mathcal{O} : \mathcal{Top} \longrightarrow \mathcal{Frm}^{op}.$$

From this example we see that frames are a good algebraic model to describe the poset of subobject and in the next section we will see how.
But first one last remark

Remark 2.1.5. *Thanks to the Adjoint Functor Theorem (Theorem 5.4 in [LC]) we can deduce that a complete lattice is a frame if and only if it is a Heyting algebra.*
Recall that a Heyting algebra *is a cartesian closed lattice.*

2.2 From spaces to locales

Our first aim for this chapter was to obtain a new category of spaces. We have seen how frames can be a good model to represent lattices of opens, so we want to build a new category of "spaces" where we may think that any frame is of the form "lattice of opens" for a certain object of this category. We would also like this category to be similar to $\mathcal{Top}$ and in particular we want it to preserve the order of maps. These ideas lead us to the following definition

Definition 2.2.1. *We define the category of locales to be* $\mathcal{Loc} = \mathcal{Frm}^{op}$, *we will call its objects* locales *and its arrows* continuous maps of locales.
Despite locales are the same as frames, we will denote a locale with a letter X *and the corresponding frame with* $\mathcal{O}(X)$ *while, given a continuous map of locales* $f : X \longrightarrow Y$, *we will denote with* $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ *the corresponding map of frames.*
Moreover, for any locale X *we will also call* X *the top element of* $\mathcal{O}(X)$ *and* $\emptyset$ *the bottom element.*

In this way the functor $\mathcal{O}$ of Example 2.1.4 becomes a functor $\mathcal{Top} \longrightarrow \mathcal{Loc}$ which for now will be called $(-)_l$ but often it will be left nameless. We can thus view $\mathcal{Top}$ inside $\mathcal{Loc}$ but as we will see soon, this step is not lossless.
A really useful locale is the one corresponding to the topological space with one point. We will call such a locale 1. Note that in particular using the notation above $\mathcal{O}(1)$ is $\{\emptyset, 1\}$, so it is isomorphic to the ordinal 2. Moreover this locale is terminal because there exist a unique frame homomorphism from 2 to any frame (2 contains only top and bottom element which are always sent to top and bottom respectively).
As in $\mathcal{Top}$ the points of a space X can be identified with the continuous maps from 1 to X, we give the following definition.

Definition 2.2.2. *Let* X *be a locale, a* point p *of* X *is a continuous map* $p : 1 \longrightarrow X$.

Note that unlike for topological spaces, for locales points are no more a primitive concept, but a derived one.

Interpreted in frames, a point $p : 1 \longrightarrow X$ is a map $p^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(1) \cong 2$, so it corresponds to a subset of $\mathcal{O}(X)$, namely $\{U \in \mathcal{O}(X) | p^*(U) = 1\}$ which we will denote with $\mathcal{N}(p)$. If in particular we take X to be a topological space seen as a locale, $\mathcal{N}(p)$ corresponds to the filter of open neighbourhoods the point p.
Since p^* is not just a map of set, we have some regularity conditions that this subset need to satisfy, namely it needs to be a completely prime filter.

Definition 2.2.3. *Let P be a lattice, a* filter *is a non empty subset $F \subseteq P$ which does not contain the bottom element, is upward closed and closed up to finite intersection, that is, such that respectively*

1. $\perp \notin F$;
2. $\forall x \in F$, *if* $x \leq y$ *then* $y \in F$;
3. *if* $x, y \in F$ *then* $x \wedge y \in F$.

A filter F is said to be completely prime *if for every $S \subseteq P$ we have $\bigvee S \in F$ iff $F \cap S \neq \emptyset$.*

Proposition 2.2.4. *The map $\mathcal{N}$ from points of a locale X to subsets of $\mathcal{O}(X)$ sending p to $\mathcal{N}(p)$ is injective and has as image the subset of completely prime filters on $\mathcal{O}(X)$.*

Proof. Let $p \in \mathcal{L}oc(1, X)$ be a point of X, then $\mathcal{N}(p)$ does not contain $\emptyset$ because p^* preserves initial object, so $p^*(\emptyset) = \emptyset \neq 1$. If $U, V \in \mathcal{O}(X)$ with $U \leq V$ and $U \in \mathcal{N}(p)$, then $p^*(V) \geq p^*(U) = 1$, whence $V \in \mathcal{N}(p)$. Finally, since p^* preserves finite limits, if $U, V \in \mathcal{N}(p)$, then $p^*(U \wedge V) = p^*(U) \wedge p^*(V) = 1 \wedge 1 = 1$, so $U \wedge V \in \mathcal{N}(p)$. This proves that $\mathcal{N}(p)$ is a filter, but with the following argument we also obtain that it is completely prime. Let $S \subseteq \mathcal{N}(p)$, since p^* preserves colimits, $p^*(\bigvee S) = 1$ iff $\bigvee_{U \in S} p^*(U) = 1$ which holds iff there exist a $U \in S$ with $p^*(U) = 1$.
Now let F be a completely prime filter of $\mathcal{O}(X)$, consider the characteristic map of F as hinted before, that is, a map $q^* : \mathcal{O}(X) \longrightarrow 2 = \mathcal{O}(1)$ which is 1 only on F. We will show that it is a frame homomorphism, so that it corresponds to a map of locales $q : 1 \longrightarrow X$ and hence to a point which by construction will force F to be $\mathcal{N}(q)$. Let A be a finite subset of $\mathcal{O}(X)$, then $q^*(\bigwedge A)$ can either be $\emptyset$ or 1. The first case occurs iff $\bigwedge A \notin F$, but F is a filter, so this holds iff at least one of the elements of A is not in F, that is, iff $\bigwedge_{U \in A} q^*(U) = \emptyset$. The second case is exactly the complementary, so q^* preserves finite limits. Let S now be a subset of $\mathcal{O}(X)$, then $q^*(\bigwedge A)$ can either be $\emptyset$ or 1. The second case occurs iff $\bigvee S \in F$ which holds iff at least one of the elements of S is also in F, for this is completely prime. This happens iff $\bigvee_{U \in A} q^*(U) = 1$. Now the first case is exactly the complementary of the second, so reasoning by contradiction we get that q^*

preserves small colimits.
Now we only need to prove that $\mathcal{N}$ is injective, but it is immediate for we defined it using the isomorphism between characteristic maps on $\mathcal{O}(X)$ and its subobjects. □

What we are going to do now is to find a natural way to build a topological space from a locale. Given a locale X we define $X_p := \mathcal{Loc}(1, X)$ i.e. the set of the points of X. We have a map $\phi_X : \mathcal{O}(X) \longrightarrow \mathcal{P}(X_p)$ such that $\phi_X(U) = \{x : 1 \longrightarrow X | x^*(U) = 1\}$. Note that this is a frame homomorphism because if F is a finite subset of $\mathcal{O}(X)$,

$$\phi_X\left(\bigwedge F\right) = \left\{x : 1 \longrightarrow X \middle| x^*\left(\bigwedge F\right) = \bigwedge_{U\in F} x^*(U) = 1\right\} = \bigcap_{U\in F} \phi_X(U)$$

and similarly for all $S \subseteq \mathcal{O}(X)$

$$\phi_X\left(\bigvee S\right) = \left\{x : 1 \longrightarrow X \middle| x^*\left(\bigvee S\right) = \bigvee_{U\in S} x^*(U) = 1\right\} = \bigcup_{U\in S} \phi_X(U)$$

In particular now we have that the image of ϕ_X is a frame, thus it defines a topology on X_p. From now on we will see X_p in $\mathcal{Top}$ with this topology.
Given a continuous map of locales $f : X \longrightarrow Y$, it induces naturally a map $f_p : X_p \longrightarrow Y_p$ by composition, i.e. a point $x : 1 \longrightarrow X$ is sent to $fx : 1 \longrightarrow Y$. Every open of Y_p is of the form $\phi_Y(V)$ with $V \in \mathcal{O}(Y)$, so

$$f_p^{-1}(\phi_Y(V)) = \{x : 1 \longrightarrow X | f_p(x) \in \phi_Y(V)\} =$$
$$= \{x : 1 \longrightarrow X | (fx)^*(V) = 1\} = \{x : 1 \longrightarrow X | (x)^*(f^*V) = 1\} = \phi_X(f^*(V))$$

which says that f_p is continuous, whence $(-)_p$ is a functor from $\mathcal{Loc}$ to $\mathcal{Top}$. The following lemma links this functor to the one that we have called $(-)_l : \mathcal{Top} \longrightarrow \mathcal{Loc}$.

Lemma 2.2.5. *The functor $(-)_p$ is right adjoint to the functor $(-)_l$ and the adjunction is idempotent, i.e. the monad induced by the adjunction is idempotent, or simply ϵ_{X_l} is an isomorphism for all X in $\mathcal{Top}_0$.*

Proof. See [SE] Lemma C1.1.2 for the proof.
For our purpose we just need to know that for every topological space X, the unit of this adjunction is such that $\eta_X : X \longrightarrow X_{lp}$ sends the point x to the point of X_l corresponding to the completely prime filter $\mathcal{N}(x)$. And for every locale X the counit in this component is $\epsilon_X : X_{pl} \longrightarrow X$ corresponding to the frame homomorphism that we called $\phi_X : \mathcal{O}(X) \longrightarrow \mathcal{O}(X_p)$ while defining the topology of X_p. □

Now we are ready to give the following two definitions

Definition 2.2.6. *A topological space X is* sober *if the unit η of the previous adjunction is an isomorphism in the component X or equivalently if every completely prime filter is of the form $\mathcal{N}(x)$ for one and only one $x \in X$. We call $\mathcal{Sob}$ the full subcategory of $\mathcal{Top}$ having as objects the sober spaces.*
A locale X is said to be spatial *if the counit of the previous adjunction is an isomorphism at X, that is, if for all $U, V \in \mathcal{O}(X)$ such that $p^*(U) = p^*(V)$ for every point p of X, then $U = V$. We call $\mathcal{SLoc}$ the full subcategory of $\mathcal{Loc}$ having as objects the spatial locales.*

Remark 2.2.7. *The two definitions of sober space displayed above are equivalent because to ask that for every completely prime filter there exist a unique point $x \in X$ such that the filter is of the form $\mathcal{N}(x)$ is equivalent to ask bijectivity of η_X. Then we claim that η_X is an isomorphism if and only if it is bijective. To prove it we will show that if η_X is bijective, it becomes an open map and thus its inverse is continuous. Let $U \in \mathcal{O}(X)$, then*

$$\eta_X(U) = \{\eta_X(p) | p \in U\} = \{\mathcal{N}(p) | p \in U\} = \{\mathcal{N}(p) | U \in \mathcal{N}(p)\}$$

Now since points of X_l are completely prime filters and thanks to the surjectivity of η_X they are of the form $\mathcal{N}(p)$, this set corresponds to $\{x : 1 \longrightarrow X_l | x^*(U) = 1\}$ and thus to $\phi_X(U)$ which is an open of X_{lp}.

We can also give a more concrete characterization of sober spaces, but first we need to recall the definition of *irreducible* of a topological space X, which is a subspace C such that if C_1 and C_2 are closed subsets of X such that $C \subseteq C_1 \cup C_2$, then $C \subseteq C_1$ or $C \subseteq C_2$. Now we can state the following

Proposition 2.2.8. *A topological space X is sober iff every nonempty closed irreducible subset is the closure of a unique point.*

Proof. Suppose X is sober and let $C \subseteq X$ be nonempty, closed and irreducible. Consider the set $P = \{U \in \mathcal{O}(X) | U \cap C \neq \emptyset\}$. It is nonempty for C is and thus $X \cap C = C \neq \emptyset$. The empty open is not in P by definition and if $V \subseteq U$ with $V \cap C \neq \emptyset$, then also $C \cap U \neq \emptyset$, so if $V \in P$, also $U \in P$. Finally if $U_1,\ U_2 \in P$, we claim that $U_1 \cap U_2 \in P$ and we need to prove this claim by contradiction. Suppose otherwise, then $C \cap U_1 \cap U_2 = \emptyset$ which means $C \subseteq X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2)$, but then C is covered by two closed subsets of X and since C is irreducible, there is $i \in \{1, 2\}$ such that $C \subseteq X \setminus U_i$ and thus $C \cap U_i = \emptyset$, which contradicts the fact that both U_1 and U_2 are in P. We have just proved that P is a filter and moreover if S is a family of opens such that $\bigcup S \in P$, then $\bigcup S \cap C = \bigcup_{U \in S}(U \cap C)$ and since this is nonempty, there exists some U in S such that $U \cap C \neq \emptyset$, which means that $S \cap P$ is nonempty and hence that P is completely prime. Since X is sober, $P = \mathcal{N}(x)$ for a unique point $x \in X$. It follows from this description that an open U is disjoint from C iff it does not contain x which, considering complements, means that a closed K contains C iff it contains

x and hence C and x have the same closure, but since C is closed we get $C = \overline{\{x\}}$.
Finally suppose that C is the closure of y, then an open U is disjoint from C iff its complement contains C and hence y, so iff U does not contain y. This implies that $P = \mathcal{N}(y)$, but then $y = x$ by sobriety.

To prove the converse, let P be a completely prime filter, call W the union of every open in $\mathcal{O}(X) \setminus P$ and consider $C = X \setminus W$. First note that W is open, so C is closed. Now note that W cannot be in P because it is union of opens not contained in P which is completely prime. It follows from the fact that P contains X, that W must be a proper subset and hence that C cannot be empty. Finally, let C_1 and C_2 be closed subsets of X covering C and let $U_i = X \setminus C_i$ for $i = 1, 2$, we have, as we deduced previously, that $C \cap (U_1 \cap U_2) = \emptyset$. Therefore $U_1 \cap U_2$ is contained in W and thus it cannot be in P or also W is, being P upward closed. We also deduce that for some i, U_i is not in P, for this is a filter, so $U_i \subseteq W$ and thus $C \subseteq C_i$, proving that C is irreducible.
C satisfies the right hypotheses to deduce that $C = \overline{\{x\}}$ for a unique $x \in X$. We have that an open U does not contain x iff it does not intersect C and hence iff $U \subseteq W$, which means that $U \notin P$ or also $W \in P$, for P is upward closed. It follows that $\mathcal{N}(x) = P$. Now note that if $\mathcal{N}(y) = P$, by this same reasoning reversed, we get that $C = \overline{\{y\}}$ and hence that $x = y$. □

An example of non sober space is any infinite set X with the cofinite topology because the family of all non empty opens is a completely prime filter but cannot be of the form $\mathcal{N}(p)$ for any point $p \in X$.
An example of sober space is any T_2 space, as showed in [SE] Lemma C1.2.4. Examples of non spatial locales are more delicate. We need in fact a proposition

Proposition 2.2.9. *In a Heyting algebra H we have*

1. $x \leq \neg\neg x$
2. $x \leq y$ *implies* $\neg y \leq \neg x$
3. $\neg x = \neg\neg\neg x$
4. $\neg\neg(x \wedge y) = \neg\neg x \wedge \neg\neg y$

for all $x, t \in H$.

Proof. See [SGL] Proposition I.8.1. □

Example 2.2.10. *Let X be a locale, then in particular $\mathcal{O}(X)$ is a Heyting algebra for Remark 2.1.5, so we can consider the negation operator*

$$\neg : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)^{op}$$

which is also more explicitly defined as the map sending a *to* $a \to 0$ *or stated otherwise, to* $\bigvee\{b \in \mathcal{O}(X) | a \wedge b = 0\}$. *In particular when* X *comes from a topological space,* $\neg(U) = (X \setminus U)^\circ$.
Now consider the map $\neg\neg : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ *(if* X *is a topological space* $\neg\neg(U) = (\overline{U})^\circ$*, i.e. the interior of the closure of* U*). Note that it preserves the order because of the second point of Proposition 2.2.9 applied twice.*
Note also that $\neg\neg$ *preserves the terminal object because* $\neg X = 0$ *and* $\neg 0 = X$ *as one can see using the explicit definition of* $\neg$.
Now the fourth point of Proposition 2.2.9 says that $\neg\neg$ *preserves also binary products, and hence it preserves finite limits.*
Let $\mathcal{O}(X)_{\neg\neg}$ *be the image of* $\neg\neg$ *with the induced order and call the canonical epi-mono factorization of* $\neg\neg$ *as follows*

$$\mathcal{O}(X) \xrightarrow{\ p\ } \mathcal{O}(X)_{\neg\neg} \xrightarrow{\ i\ } \mathcal{O}(X)$$

We have that $p \dashv i$ *in fact for all* $U \in \mathcal{O}(X)$ *and* $V \in \mathcal{O}(X)_{\neg\neg}$*, if* $p(U) \leq V$*, then applying* i *we get* $\neg\neg(U) \leq i(V)$ *and for the first point of Proposition 2.2.9 it implies that* $U \leq i(V)$. *Conversely if* $U \leq i(V)$*, then* $i(V)$ *is of the form* $\neg\neg(W)$ *for some* W*, so applying* $\neg\neg$ *to both terms of the previous inequality one gets* $\neg\neg(U) \leq \neg\neg(V) = \neg\neg\neg\neg(W)$ *but the latter is precisely* $\neg\neg(W)$ *for the first point of Proposition 2.2.9, so we get* $ip(U) = \neg\neg(U) \leq i(V)$. *Since* i *is an inclusion, it is full and faithful, thus we have* $p(U) \leq V$ *proving the adjunction.*
As consequence of this adjunction we have that p *preserves colimits and thus joins and in particular* $\mathcal{O}(X)_{\neg\neg}$ *has all colimits. Because of Proposition 2.2.9,* $\mathcal{O}(X)_{\neg\neg}$ *has also finite limits. Since the join here is defined as the double negation of the join in* $\mathcal{O}(X)$*, one can prove that from distributivity of* $\mathcal{O}(X)$ *follows the distributivity of* $\mathcal{O}(X)_{\neg\neg}$ *which thus is a frame.*
We deduce that p *is a frame homomorphism. We can call* $X_{\neg\neg}$ *the locale corresponding to the frame* $\mathcal{O}(X)_{\neg\neg}$ *and let* $j : X_{\neg\neg} \longrightarrow X$ *be the continuous map of locales corresponding to* p.
Note that if $x : 1 \longrightarrow X_{\neg\neg}$ *is a point, then* jx *is a point in* X*, and thus we have that* $((jx)^*)^{-1}(1)$ *is of the form* $\mathcal{N}(t)$ *for some point* t *in* X. *But* $((jx)^*)^{-1}(1) = (j^*)^{-1}(x^*)^{-1}(1) = (p)^{-1}(x^*)^{-1}(1)$*, now if we call* $P = (x^*)^{-1}(1)$*, this must be a totally prime filter, and moreover this equality says that* $p^{-1}(P) = \mathcal{N}(t)$*, so* $P = p(\mathcal{N}(t))$.

Now instead of a generic locale X *we choose* $\mathbb{R}_l$. *We have that* P *must be of the form* $\{p(U) | U \in \mathcal{O}(\mathbb{R}), t \in U\}$*, thus it is the set of all open neighbourhoods of* t *such that they coincide with the internal of their closure. The filter* P *contains for example all of* $\mathbb{R}$*, but in* $\mathcal{O}(\mathbb{R}_l)_{\neg\neg}$ *we also have* $(-\infty, t)$ *and* (t, ∞) *which are not in* P *and yet their join is* $\neg\neg((-\infty, t) \cup (t, \infty)) = (\overline{(-\infty, t) \cup (t, \infty)})^\circ = \mathbb{R}^\circ = \mathbb{R}$*, which is in* P*, so the latter can't be a totally prime filter.*
From this contradiction follows that $\mathbb{R}_{\neg\neg}$ *does not have any point, and yet it*

contains at least two different opens, e.g. $(-\infty, t)$ *and* (t, ∞)*, thus it can't be spatial.*

Remark 2.2.11. *A topological space* X *is* T_0 *iff* η_X *is injective as follows from the definition, so in particular sober implies* T_0*. Joining this fact with previous observations we get the following inclusions of categories*

$$\mathcal{T}_2 \subset \mathcal{Sob} \subset \mathcal{T}_0$$

There is no relation of inclusion between being sober and being T_1*.*

From Lemma 2.2.5 we get the following

Corollary 2.2.12.

1. *The category* $\mathcal{Sob}$ *is reflective in* $\mathcal{Top}$ *with* $(-)_{lp}$ *as reflector called* soberification.
2. *The category* $\mathcal{SLoc}$ *is coreflective in* $\mathcal{Loc}$ *and its coreflector is* $(-)_{pl}$*.*
3. *The adjunction of Lemma 2.2.5 restricts to an equivalence of categories between* $\mathcal{Sob}$ *and* $\mathcal{SLoc}$*.*

From Lemma 2.2.5 and this corollary we can better see what we lose exactly when passing from $\mathcal{Top}$ to $\mathcal{Loc}$, namely the distinction between a space and its soberification. In other words, seen as locales, a topological space and its soberification are isomorphic.
Despite this loss of information, a lot of topological properties like compactness and connectedness depend only on the lattice of opens and in fact a space has such properties iff its soberification does. From now on we will focus only on properties of this kind.

Example 2.2.13. *For every set we can build as usual the corresponding topological space with the discrete topology and we get a functor* $D : \mathcal{Set} \longrightarrow \mathcal{Top}$ *where* $D(f) = f$ *for it becomes continuous. This functor is injective on objects, full and faithful and moreover it is left adjoint to the forgetful functor* $U : \mathcal{Top} \longrightarrow \mathcal{Set}$*.*
Now note that if we compose this adjunction with the one proved in Lemma 2.2.5 we get a new adjunction $d \dashv p : \mathcal{Loc} \longrightarrow \mathcal{Set}$*. In particular given a set* A*, the locale* $d(A) = D(A)_l$ *is such that the corresponding frame is the lattice* $\mathcal{P}(A)$ *ordered by inclusion and given a function* $f : A \longrightarrow B$*, the map* $d(f)$ *is such that the corresponding frame homomorphism is* $d(f)^* = f^{-1}$ *sending every subset of* B *to its preimage in* A *via* f*.*
On the other side, for every locale X*,* $p(X) = U(X_p) = \mathcal{Loc}(1 \longrightarrow X)$ *is the set of points of* X*.*
We call discrete *every locale isomorphic to one of the form* $d(A)$ *for some set* A*.*
Note that the equivalence of Corollary 2.2.12 restricts to an equivalence between the full subcategories of discrete topological spaces and discrete locales. Hence the full subcategory of discrete locales is equivalent to $\mathcal{Set}$*.*

2.3 Maps of locales

Before going further we will display some meaningful objects and maps related to locales.

Definition 2.3.1. *A* sublocale *is a regular subobject. We will denote with $Sub(X)$ the poset of sublocales of X.*

Note that this is in contrast with the notation given in Chapter 1, but we will never consider the lattice of subobjects. This fact is not so unexpected because also for topological spaces we never consider all the subobjects but just regular ones i.e. subspaces.

Remark 2.3.2. *Note that regular subobjects of locales correspond to regular quotients of frames and regular quotients are precisely frame homomorphisms which are surjective.*

Example 2.3.3. *In Example 2.2.10, the map $j : X_{\neg\neg} \longrightarrow X$ is an example of sublocale.*

Remark 2.3.4. *From Example 2.2.10 we deduce also that the property of being a spatial locale is not inherited by sublocales, for $\mathbb{R}$ is spatial and $\mathbb{R}_{\neg\neg}$ isn't.*

In topological spaces every open set can be seen as a subspace of X, so we want to mimic the idea for locales.
Let X be a locale and $U \in \mathcal{O}(X)$, consider the poset $\downarrow(U)$, that is the set $\{V \in \mathcal{O}(X) | V \leq U\}$ with the induced order. We can easily show that it is a frame because joins and meets are the same as in $\mathcal{O}(X)$. Now consider the map

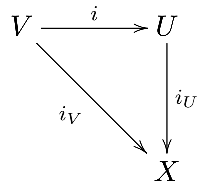
$$- \wedge U : \mathcal{O}(X) \longrightarrow \downarrow(U)$$

From distributivity and commutativity of $\wedge U$ with meets (and the fact that the bottom is sent to the bottom) follows that this is a frame homomorphism. Now let's call U the locale corresponding to the frame $\downarrow(U)$ and $i_U : U \longrightarrow X$ the map corresponding to $- \wedge U : \mathcal{O}(X) \longrightarrow \downarrow(U)$ which will represent the inclusion of U in X. Now this is more than an inclusion, in fact for Remark 2.3.2 it represents a regular sublocale because $- \wedge U$ is surjective.

Remark 2.3.5. *Note also that whenever $V \leq U$ in $\mathcal{O}(X)$, there is a commutative diagram of frame homomorphisms*

$$\mathcal{O}(X) \xrightarrow{-\wedge U} \downarrow(U) \xrightarrow{-\wedge V} \downarrow(V), \qquad \mathcal{O}(X) \xrightarrow{-\wedge V} \downarrow(V)$$

which leads to a diagram of continuous maps of locales



We can see that we have built a functor

$$\mathcal{O}(X) \longrightarrow \mathcal{L}oc/X$$

Moreover as we noticed before, this functor factors through the subcategory of $\mathcal{L}oc/X$ *in the regular monomorphisms and thus leads to a functor*

$$\mathcal{O}(X) \longrightarrow Sub(X)$$

This functor is injective on objects and faithful, so we can see $\mathcal{O}(X)$ *inside the lattice of sublocales.*

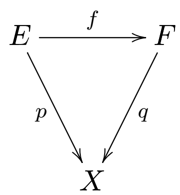
We call *open sublocales* the sublocales obtained as image of this functor and we still denote this subset as $\mathcal{O}(X)$.
Proceding in our analogies with topological spaces we have

Definition 2.3.6. *A continuous map of locales* $f : X \longrightarrow Y$ *is a* local homeomorphism *if there is a covering* $\mathcal{U}$ *of* X *such that for all* $U \in \mathcal{U}$, *if* $i_U : U \longrightarrow X$ *represents the corresponding open sublocale, we have that* $fi_U : U \longrightarrow Y$ *represents an open sublocale of* Y.
We call LH *the subcategory of* $\mathcal{L}oc$ *selecting only the local homeomorphisms.*

First of all LH is actually a category for we have the following

Lemma 2.3.7. *The following facts are true*

1. *Composition of local homeomorphisms is a local homeomorphism*
2. *A pullback of a local homeomorphism is a local homeomorphism*
3. *Given a commutative triangle of continuous maps of locales*



if p *and* q *are local homeomorphisms then so is* f

4. *the inclusion functor* $LH \longrightarrow \mathcal{L}oc$ *creates pullbacks.*
5. *if* $f : X \longrightarrow Y$ *is a local homeomorphism and* Y *is a spatial locale, then so is* X*.*

Proof. See [SE] Lemma C1.3.2 □

Note also that a local homeomorphism between topological spaces becomes a local homeomorphism of locales.

In general in $\mathcal{L}oc$, epimorphisms are not pullback stable (see example C1.2.12 in [SE]), but if we pull back along local homeomorphisms, then this stability holds.

Lemma 2.3.8. *Let the following be a pullback diagram in* $\mathcal{L}oc$

$$\begin{array}{ccc} F & \xrightarrow{p^*(f)} & E \\ {\scriptstyle f^*(p)}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y \end{array}$$

such that f *is a local homeomorphism and* p *an epimorphism, then* $f^*(p)$ *is an epimorphism as well.*

Proof. Suppose first that f is the inclusion of an open sublocale in Y, then one can prove directly that the pullback of f along p is the inclusion of the open $p^*(X)$ in E and then $p^*(f)$ on frames is the restriction of f^* to the opens of Y contained in X. Then if p is an epimorphism, p^* is injective because of Lemma C1.2.11 of [SE] and hence also $p^*(f)$ is. It follows that the pullback of an epi along inclusions of open sublocales is still epi.
Now consider the case of a local homeomorphism and consider the open cover $\{U_i | i \in I\}$ such that $f|_{U_i}$ is an open inclusion $V_i \longrightarrow Y$. If we pull back $f^*(p)$ along the inclusion $\iota_i : U_i \longrightarrow X$ we get a map h_i which is also the pullback of p along the inclusion $V_i \longrightarrow Y$ and hence it is an epi.

$$\begin{array}{ccccc} F_i & \longrightarrow & F & \xrightarrow{p^*(f)} & E \\ {\scriptstyle h_i}\downarrow & \lrcorner & {\scriptstyle f^*(p)}\downarrow{\scriptstyle f^*(p)} & \lrcorner & \downarrow{\scriptstyle p} \\ U_i & \xrightarrow[\iota_i]{} & X & \xrightarrow[f]{} & Y \end{array}$$

Now let $a, b : X \longrightarrow A$ two parallel arrows such that $af^*(p) = bf^*(p)$, then by commutativity we also have that $a\iota_i h_i = b\iota_i h_i$ and since h_i is an epi, we have $a\iota_i = b\iota_i$ for all i. This is a cocone with vertex A and hence, since the U_i's cover X, by universal property of the colimit there is a unique compatible map $X \longrightarrow A$ and follows that $a = b$ and thus that $f^*(p)$ is epi.

□

2.4 Sheaves of locales

The main purpose of this section is to adapt the construction of the sheaf category seen in Example 1.2.8 to locales.
Let X be a locale and consider the corresponding frame $\mathcal{O}(X)$. As for topological spaces, for all $U \in \mathcal{O}(X)$, arrows with codomain U in $\mathcal{O}(X)$ can be identified with their domain and thus with opens $V \leq U$. Now we can say that a family $\mathcal{U}$ of arrows over U is a covering if it is a covering for U in the intuitive sense, that is, if $\bigvee_{f \in \mathcal{U}} \mathrm{dom}(f) = U$.
Let's take any $V \leq U$ and any family $\mathcal{U}$ covering U, if we consider the family $\mathcal{V} = \{V \wedge W | W \in \mathcal{U}\}$, all of its elements are smaller than V and thus it corresponds to a family of arrows over V. From distributivity we deduce that

$$\bigvee \mathcal{V} = \bigvee_{W \in \mathcal{U}} (W \wedge V) = \left(\bigvee_{W \in \mathcal{U}} W \right) \wedge V = U \wedge V = V$$

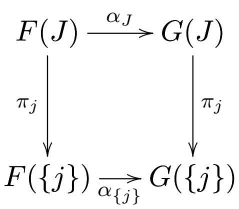
Thus $\mathcal{V}$ is a covering family for V and this family is such that for all $W \wedge V \longrightarrow V$ in $\mathcal{V}$, the map $W \wedge V \longrightarrow V \longrightarrow U$ factors through $W \longrightarrow U$ in $\mathcal{U}$, so we have just defined a coverage for the category $\mathcal{O}(X)$. We will refer to such a coverage as the *open coverage* and to its covering families as *open coverings* in agreement with the topological case. From now on, unless explicitly stated, this will be the standard way to view a frame as a site.
Finally, given a locale X we can consider the topos of sheaves over its frame with this topology and thus we call $\mathcal{Sh}(X)$ the category $\mathcal{Sh}(\mathcal{O}(X))$ of sheaves over the site $\mathcal{O}(X)$.

Example 2.4.1. *Let X be a discrete locale and suppose it is of the form $X = d(I)$ for some set I as in Example 2.2.13, then $\mathcal{Sh}(X) \simeq \mathcal{Set}^I$.*
First note that for every sheaf F on X, its partial sections at every open are uniquely determined by its partial sections at the singletons. In fact for every $i \in I$, $\{i\}$ is an open, so for all $J \subseteq I$, the family $\{\{j\} | j \in J\}$ is a coverage for J. Every family $\{s_j \in F(\{j\}) | j \in J\}$ is compatible since pairwise intersections of different covering opens is empty, and thus it admits a unique amalgamation in $F(J)$. It follows from this observation that $F(J) = \prod_{j \in J} F(\{j\})$ and for all $K \subseteq J$ the restriction map is the projection to the components corresponding to the elements of J contained also in K.
Now consider the correspondence

$$\psi : \mathcal{Sh}(X) \longrightarrow \mathcal{Set}^I$$

sending a sheaf F to the sequence of sets $(F(\{i\}))_I$ and an arrow $\alpha : F \longrightarrow G$ to the sequence of arrows $\left(\alpha_{\{i\}}\right)_{i \in I}$. This mapping preserves identities and composition, so it is a functor.
This functor is faithful because for every $J \subseteq I$ and every $j \in J$ we have a

commutative diagram of the form

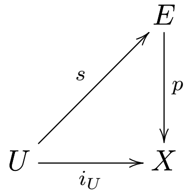


and there is exactly one such arrow with this property for the universal property of the product, in fact we have that $\alpha_J = \prod_{j\in J} \alpha_{\{j\}}$*. Therefore the natural transformation* $\alpha : F \Rightarrow G$ *is uniquely determined by* $\psi(\alpha)$*, that is by its components in the singletons.*
Consider now two sheaves F *and* G *and a family* $(\alpha_i)_{i\in I} : \psi(F) \longrightarrow \psi(G)$*. For all* $J \subseteq I$*, since* $F(J)$ *is the product of the* $F(\{j\})$*'s for* $j \in J$*, we can define a function* $\alpha_J = \prod_{j\in J} \alpha_j$*. The restriction maps of the sheaves are projection maps, so the family of the* α_J*'s for* $J \subseteq I$ *is natural and hence they form a natural transformation* $\alpha : F \longrightarrow G$*. Then note that* $\alpha_{\{i\}} = \alpha_i$ *for all* $i \in I$*, so* $\psi(\alpha) = (\alpha_i)_{i\in I}$ *and thus the functor* ψ *is also full.*
Finally note that ψ *is also essentially surjective. For every* I*-indexed family of sets* $(A_i)_{i\in I}$ *we can build a presheaf on* $\mathcal{O}(X) = \mathcal{P}(I)$ *sending a subset* J *to* $\prod_{j\in J} A_j$ *and having as restriction maps the projections as it should be in order to mimic the structure of a sheaf in* $\mathcal{Sh}(X)$ *seen at the beginning of this example. Again using the same observations, we notice that* F *is a sheaf and in particular* $F(\{i\}) = A_i$ *for all* $i \in I$*.*
We have just proven that ψ *is an equivalence and thus that* $\mathcal{Sh}(X) \simeq \mathcal{Set}^I$*.*

We can now give a more geometric description of sheaves over a locale. Let X be a locale, consider the slice category $\mathcal{Loc}/X$, then we can build a functor

$$\Gamma : \mathcal{Loc}/X \longrightarrow \mathcal{Set}^{\mathcal{O}(X)^{\mathrm{op}}} \tag{2.1}$$

Sending a continuous map $p : E \longrightarrow X$ to the presheaf $\Gamma(p)$ of partial sections of p, that is, for every $U \in \mathcal{O}(X)$, we can view U as a locale as showed in Section 2.3, so let $i_U : U \longrightarrow X$ be its inclusion, then a section of p on U is a continuous map $s : U \longrightarrow E$ such that the following diagram commutes



Thanks to Remark 2.3.5 we can give a structure of presheaf to $\Gamma(p)$ because for all $V \leq U$ in $\mathcal{O}(X)$ we can define $\Gamma(p)(V \leq U) : \Gamma(p)(U) \longrightarrow \Gamma(p)(V)$ to be $- \circ i$ where $i : V \longrightarrow U$ is the corresponding inclusion of locales.

Let now $f : p \longrightarrow q$ be an arrow in $\mathcal{L}oc/X$, we define $\Gamma(f)$ to be the composition with f or more precisely if $U \in \mathcal{O}(X)$, then $\Gamma(f)(U) : \Gamma(p)(U) \longrightarrow \Gamma(q)(U)$ is the composition of a section s on U with f. The naturality of this map follows from the fact that composition commutes with precomposition, so this is a map of presheaves.
In this way we have defined as promised the functor Γ which we will call *partial sections functor*.

Remark 2.4.2. *The functor Γ factors through $\mathcal{S}h(X)$. For a proof of this see [SE] Lemma C1.3.4.*

This functor is important because we have the following

Theorem 2.4.3. *For a locale X the partial sections functor restricts to an equivalence of categories $\Gamma : LH/X \longrightarrow \mathcal{S}h(X)$.*

Proof. We will just see a sketch of the proof, namely the construction of a pseudoinverse $\mathcal{R}$. For a more complete and different proof see [SE] Theorem C1.3.11.
Let F be a sheaf over X and consider $el(F)$ the category of elements. Notice that as a category it is a preorder because by definition an arrow between two elements (s, U) and (t, V) is an arrow $f : U \subseteq V$ such that $F(f)(t) = s$ but there is already at most a unique arrow from U to V. Actually the category of elements of F is an ordered set, in fact if $(s, U) \leq (t, V) \leq (s, U)$, then in particular $U \leq V \leq U$ and thus $U = V$, but then $F(U \leq V)$ is the identity over $F(U)$ because F is a functor, and thus $s = F(U \leq V)(t) = t$.
Analysing better this order and interpreting the elements as generalised functions or sections, we notice that $(s, U) \leq (t, V)$ can be interpreted as "t extends s". To simplify the notation we will write $t|_U$ to denote $F(U \subseteq V)(t)$ and call it the *restriction* to U of t.
Now we can define the set $\mathcal{O}(\mathcal{R}(F))$ of subcategories closed by precomposition and colimits, i.e. subcategories $\mathcal{A}$ such that for all morphisms $f : (s, U) \longrightarrow (t, V)$ in $el(F)$ with $(t, V) \in \mathcal{A}$, then also $(s, U) \in \mathcal{A}$ and such that the inclusion functor creates colimits. Note that such subcategories are full.
Since we are dealing with an order, we can make this definition more explicit. A subcategory closed by composition is a *downward closed* suborder, i.e. a suborder such that if it contains an element U then it contains all the elements $V \subseteq U$. A full subcategory such that the inclusion creates colimits corresponds to a subset $\mathcal{A}$ such that for every $\mathcal{U} \subseteq \mathcal{A}$, if $\bigvee \mathcal{U}$ exists in $el(F)$, then it is already in $\mathcal{A}$.
We can give to $\mathcal{O}(\mathcal{R}(F))$ the order relation induced by the inclusion in $\mathcal{P}(el(F))$. Note that the intersection of elements in $\mathcal{O}(\mathcal{R}(F))$ is again in $\mathcal{O}(\mathcal{R}(F))$ and this implies that $\mathcal{O}(\mathcal{R}(F))$ is complete. It is thus also cocomplete because of Lemma 2.1.2. Note that the colimit of a family $\mathcal{A}$ in

$\mathcal{O}(\mathcal{R}(X))$ is the full cocompletion of $\bigcup \mathcal{A}$ and in particular it is the set of all colimits made using elements contained in elements of $\mathcal{A}$.
The join also distributes along the finite meet, in fact, let $A \in \mathcal{O}(\mathcal{R}(F))$ and $\mathcal{B} \subseteq \mathcal{O}(\mathcal{R}(F))$, we have as usual $\bigvee_{B\in\mathcal{B}}(B \cap A) \leq A \cap \bigvee \mathcal{B}$ so we just need to prove the converse inclusion. To do this, let $(t, V) \in A \cap \bigvee \mathcal{B}$, then $(t, V) \in A$ and it is the colimit of a family $\{(t_i, V_i)|i \in I\}$ of elements in elements of $\mathcal{B}$, but since all of the (t_i, V_i) are restrictions of (t, V), they already are in A. Thus (t, V) is colimit of elements (t_i, V_i) where for all $i \in I$ there is a $B \in \mathcal{B}$ such that $(t_i, V_i) \in A \cap B$ and hence $(t, V) \in \bigvee_{B\in\mathcal{B}}(B \cap A)$ proving the distributivity. In particular thus it is a frame. We call of course the corresponding locale $\mathcal{R}(F)$.
We actually have more structure, in fact, consider the map

$$p^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(\mathcal{R}(F))$$

sending an open U of X to the set $\bigcup_{V\leq U} F(V)$, or more formally into $\pi^{-1}(\downarrow U)$ where $\pi : el(F) \longrightarrow \mathcal{O}(X)$ is the forgetful functor that presents the category of elements as a category over the domain of F (see Example 1.1.6). First let's check that p^* is well defined, hence that $\pi^{-1}(\downarrow U)$ is closed with respect to precomposition and colimits. Let $(s_1, U_1) \leq (s_2, U_2)$, then $U_1 \leq U_2$ and $s_2|_{U_1} = s_1$, now suppose $(s_2, U_2) \in p^*(U) = \pi^{-1}(\downarrow U)$, so in particular $U_2 = \pi(s_2, U_2) \leq U$ and hence $\pi(s_1, U_1) = U_1 \leq U_2 \leq U$, so also (s_1, U_1) is inside $p^*(U)$, which is thus closed under precomposition.
Let's consider now a family $\{(t_i, V_i)|i \in I\}$ in $p^*(U)$ that admits a colimit i.e. a join (t, V) in $el(F)$. Then (t, V) extends all of the (t_i, V_i), so $t_i = t|_{V_i}$ and in particular for all $i, j \in I$,

$$t_i|_{V_i\wedge V_j} = t|_{V_i}|_{V_i\wedge V_j} = t|_{V_i\wedge V_j} = t|_{V_j}|_{V_i\wedge V_j} = t_j|_{V_i\wedge V_j}$$

so $((t_i, V_i)|i \in I)$ is a compatible family and thus it admits a unique amalgamation (and here we use the fact that F is a sheaf). Therefore this amalgamation is $t|_{\bigvee_{i\in I} V_i}$, but $V = \bigvee_{i\in I} V_i$ otherwise $\left(t|_{\bigvee_{i\in I} V_i}, \bigvee_{i\in I} V_i\right)$ is the join of $\{(t_I, V_i)|i \in I\}$. Since $V_i \leq U$ for all $i \in I$, also $V \leq U$, so in particular $(t, V) \in p^*(U)$.
Now clearly if $V \leq U$, then $p^*(V) \subseteq p^*(U)$, so p^* is also order preserving. Further, p^* preserves finite limits since the preimage preserves intersections and since for a finite subset $\mathcal{F}$ of $\mathcal{O}(X)$ we have $\bigcap_{U\in\mathcal{F}} \downarrow U = \downarrow(\bigwedge \mathcal{F})$ by universal property of the meet. Let now $\mathcal{U} \subseteq \mathcal{O}(X)$, we have that $p^*(U) \subseteq p^*(\bigvee \mathcal{U})$ for all $U \in \mathcal{U}$, so $\bigcup_{U\in\mathcal{U}} p^*(U) \subseteq p^*(\bigvee \mathcal{U})$ and hence $\bigvee_{U\in\mathcal{U}} p^*(U) \leq p^*(\bigvee \mathcal{U})$. Using the explicit form of the colimits in $\mathcal{O}(\mathcal{R}(F))$, we want to prove that $p^*(\bigvee \mathcal{U}) \leq \bigvee p^*(\mathcal{U})$, so in particular that every element (t, V) such that $\pi(t, V) \leq \bigvee \mathcal{U}$ is join (colimit) of elements in $\bigcup_{U\in\mathcal{U}} p^*(U)$. This is true because t is the amalgamation of $\{t|_{V\wedge U}|U \in \mathcal{U}\}$ where for all $U \in \mathcal{U}$, $t|_{V\wedge U} \in p^*(U)$ and hence $(t, U) = \bigvee_{U\in\mathcal{U}}(t|_{V\wedge U}, V \wedge U)$ proving the

needed inequality. We have just proved that p^* preserves also joins and thus the latter is a frame homomorphism. Call $p : \mathcal{R}(F) \longrightarrow X$ the corresponding continuous map of locales.
This p is a local homeomorphism, but before proving it, for every (s, U) in $el(F)$ let's call $s(U) = {\downarrow}(s, U) = \{(t, V) \in el(F) | (t, V) \leq (s, U)\}$, it is an open of $\mathcal{R}(F)$ because it is downward closed and has all colimits. Note also that as suborder of $el(F)$ is it isomorphic to ${\downarrow}(U)$ for we can send $V \leq U$ to $(s|_V, V)$. The latter is also isomorphic to ${\downarrow}s(U) = \mathcal{O}(s(U))$ inside $\mathcal{O}(\mathcal{R}(F))$. To prove it consider an open of $\mathcal{R}(F)$, it is less or equal than $s(U)$ iff its elements are restrictions of s and thus every subfamily of $s(U)$ is compatible, which means that there is a unique amalgamation and it is still a restriction of s, so such open is of the form $t(V)$ for some restriction t of s. Call the composition of these isomorphisms $e : \mathcal{O}(U) \longrightarrow \mathcal{O}(s(U))$. Now if we take the inclusion of $s(U)$ in X, it corresponds to the frame homomorphism $- \cap s(U) : \mathcal{O}(\mathcal{R}(F)) \longrightarrow \mathcal{O}(s(U))$ we have the following commutative square

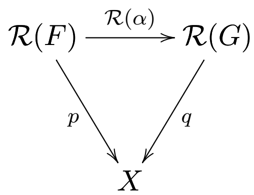
$$\begin{array}{ccc} \mathcal{O}(X) & \xrightarrow{p^*} & \mathcal{O}(\mathcal{R}(F)) \\ {\scriptstyle -\wedge U}\downarrow & & \downarrow{\scriptstyle -\cap s(U)} \\ \mathcal{O}(U) & \xrightarrow[e]{} & \mathcal{O}(s(U)) \end{array}$$

To prove it, for every $V \in \mathcal{O}(X)$ we have thus to prove that $p^*(V) \cap s(U) = e(V \wedge U)$ or more explicitly that $\pi^{-1}({\downarrow}V) \cap s(U) = s|_{V \wedge U}(V \wedge U)$. We know that $s|_{V \wedge U}(V \wedge U) \subseteq s(U)$ and $\pi(s|_{V \wedge U}(V \wedge U)) = V \wedge U$ so $s|_{V \wedge U}(V \wedge U) \subseteq \pi^{-1}({\downarrow}V) \cap s(U)$. To prove the other inclusion consider an element of $\pi^{-1}({\downarrow}V) \cap s(U)$, since it is in $s(U)$ it is of the form $(s|_W, W)$ for some $W \leq U$, then $W = \pi(s|_W, W) \leq V$, so $W \leq U \wedge V$ which means that $(s|_W, W)$ is in $s|_{V \wedge U}(V \wedge U)$ too, proving the equality and thus the commutativity of the square above.
Notice now that $\{s(U) | (s, U) \in el(X)\}$ is an open covering for $\mathcal{R}(F)$ and hence p is a local homeomorphism.
Let now $\alpha : F \longrightarrow G$ be an arrow in $\mathcal{Sh}(X)$, it induces a map on the respective categories of elements $el(\alpha) : el(F) \longrightarrow el(G)$ mapping a morphism $f : (s, U) \longrightarrow (t, V)$ in $el(F)$ to $f : (\alpha_U(s), U) \longrightarrow (\alpha_V(t), V)$. This function $el(\alpha)$ is order preserving.
Consider now the preimage $el(\alpha)^{-1} : \mathcal{P}(el(G)) \longrightarrow \mathcal{P}(el(F))$. If we restrict this map to $\mathcal{O}(\mathcal{R}(G))$ we claim that we land inside $\mathcal{O}(\mathcal{R}(F))$. To prove it let $\mathcal{U}$ be an open of $\mathcal{R}(G)$, we have to prove that $el(\alpha)^{-1}(\mathcal{U})$ is closed by precomposition and colimits. Let $(s, U) \in el(\alpha)^{-1}(\mathcal{U})$, and let $(t, V) \leq (s, U)$ in $el(F)$, then in partcular $el(\alpha)(t, V) \leq el(\alpha)(s, U) \in \mathcal{U}$, so $el(\alpha)(t, V) \in \mathcal{U}$ and then $(t, V) \in el(\alpha)^{-1}(\mathcal{U})$. To prove the closure by colimits, take a family $\{(s_i, U_i) | i \in I\}$ in $el(\alpha)^{-1}(\mathcal{U})$ that has a colimit (s, U) in $el(F)$, then as showed before it is a compatible family and the colimit is the amalgamation.

All we have to prove now is that $el(\alpha)(s, U) = (\alpha_U(s), U) \in \mathcal{U}$. Note that $U = \bigvee_{i \in I} U_i$ ans $s|_{U_i} = s_i$ by definition of amalgamation, and since α is a natural transformation, $\alpha_{U_i}(s|_{U_i}) = \alpha_U(s)|_{U_i}$, so it follows that $\alpha_U(s)$ is an amalgamation of the $\alpha_{U_i}(s_i)$'s and thus it is in $\mathcal{U}$, so $(s, U) \in el(\alpha)^{-1}(\mathcal{U})$ proving the closure by colimits.
Call the restriction of $el(\alpha)^{-1}$

$$\mathcal{R}(\alpha)^* : \mathcal{O}(\mathcal{R}(G)) \longrightarrow \mathcal{O}(\mathcal{R}(F))$$

it is a map of frames because it is defined using a preimage which preserves intersections, unions and, as we have just seen, it also preserves colimits. It induces therefore a continuous map of locales that we will call $\mathcal{R}(\alpha)$.
If $p : \mathcal{R}(F) \longrightarrow X$ and $q : \mathcal{R}(G) \longrightarrow X$ are the projections on X as defined above, we have that the following diagram commutes



to prove it let $U \in \mathcal{O}(X)$, we have to show that $el(\alpha)^{-1}\pi^{-1}(\downarrow U) = \pi^{-1}(\downarrow U)$, but we are done if we prove that $\pi el = \pi$ and this is true by definition of $el(\alpha)$, in fact $\pi(el(\alpha)(s, U)) = \pi(\alpha_U(s), U) = U = \pi(s, U)$ for every element (s, U) of F.
Using Lemma 2.3.7 (3) we get that $\mathcal{R}(\alpha)$ is a map in LH/X and this construction is functorial, so we have just defined a functor

$$\mathcal{R} : \mathcal{Sh}(X) \longrightarrow LH/X$$

One can prove that this functor is the pseudoinverse of Γ. □

Thanks to this theorem, the category of sheaves over a locale X is equivalent to the category of local homeomorphisms over X and here we have a geometrical interpretation of sheaves.

2.5 From locales to toposes

Now that we have generalized the idea of topological space to locales we want to generalize locales to toposes. We have already found a way to view a locale as the topos of sheaves over it, so the aim of this section will be to create a functor

$$\mathcal{Sh} : \mathcal{Loc} \longrightarrow \mathfrak{Geom}$$

and study its properties.
Let $f : X \longrightarrow Y$ be a continuous map of locales, then the functor $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ induces by precomposition a functor

$$- \circ (f^*)^{\mathrm{op}} : \mathcal{Set}^{\mathcal{O}(X)^{\mathrm{op}}} \longrightarrow \mathcal{Set}^{\mathcal{O}(Y)^{\mathrm{op}}}$$

Moreover, since f^* preserves intersections and covers, we can see that if F is a sheaf, then $F \circ (f^*)^{\mathrm{op}}$ is a sheaf, so this map factors through a map $f_* : \mathcal{Sh}(X) \longrightarrow \mathcal{Sh}(Y)$ which we call *direct image functor* associated with f.
Now with the same $f : X \longrightarrow Y$ we can produce a pullback functor

$$f^* : LH/Y \longrightarrow LH/X$$

sending an object $p : E \longrightarrow Y$ to a pullback $f^*(p) : f^*(E) = E \times_Y X \longrightarrow X$ and a map to the corresponding universal map from the two pullback. Now using Theorem 2.4.3 we get a functor

$$f^* : \mathcal{Sh}(Y) \longrightarrow \mathcal{Sh}(X)$$

which we call *inverse image functor* associated with f.

Proposition 2.5.1. *The functor f^* preserves finite limits.*

Proof. See [SE] Lemma C1.4.1. □

Theorem 2.5.2. *For any $f : X \longrightarrow Y$ continuous map of locales, we have an adjunction $f^* \dashv f_* : \mathcal{Sh}(X) \longrightarrow \mathcal{Sh}(Y)$.*

Proof. See [SE] Theorem C1.4.3. □

Therefore combining the last two results we get that every continuous map of locales $f : X \longrightarrow Y$ induces a geometric morphism

$$\mathcal{Sh}(f) = (f^* \dashv f_*) : \mathcal{Sh}(X) \longrightarrow \mathcal{Sh}(Y)$$

Moreover we have the following

Theorem 2.5.3. *The mapping $\mathcal{Sh} : \mathcal{Loc} \longrightarrow \mathfrak{Geom}$ sending a locale X to the topos of sheaves over it and a continuous map of locales f to $\mathcal{Sh}(f)$ is a pseudofunctor.*

Proof. Let X be a locale, then $(\mathrm{id}_X)_*$ is the identity functor on $\mathcal{Sh}(X)$ and the pullback along an identity is (isomorphic to) the identity, so $(\mathrm{id}_X)^*$ is isomorphic to the same identity functor, so $\mathcal{Sh}(\mathrm{id}_X)$ is (isomorphic to) the identity geometric morphism. Now given two composable arrows f and g we have by definition that $(gf)_* = g_* f_*$, now since composition of adjunctions is an adjunction we have $(gf)^* \dashv (gf)_* = g_* f_* \vdash f^* g^*$ and since given a functor there exists only one left adjoint up to unique natural isomorphism, we also have $(gf)^* \cong f^* g^*$ (not necessarily equal). □

By including in this way $\mathcal{Loc}$ in $\mathfrak{Geom}$ we are not losing any information because we can always recover back the original locale from its topos of sheaves as showed in the following result.

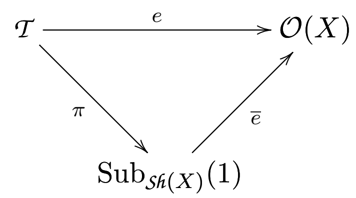
Proposition 2.5.4. *Let X be a locale, then the full subcategory of subterminal objects in $\mathcal{Sh}(X)$ is equivalent to $\mathcal{O}(X)$.*

Proof. The category $\mathcal{Sh}(X)$ is reflective in $\mathcal{Set}^{\mathcal{O}(X)^{\mathrm{op}}}$, so in particular the inclusion functor preserves limits and thus both monos and terminal. A subterminal object $C \longrightarrow 1$ then is a presheaf such that each component is a subset of 1 and thus either 0 or 1. But C is also a sheaf, so if we consider the set $\mathcal{U}$ of all opens W such that $C(W)$ is 1, then this is a covering for $U = \bigvee \mathcal{U}$ and thus, since C is a sheaf, there must be an amalgamation at U, so in particular U is in $\mathcal{U}$ which thus has a maximum element. Note also that for all $V \leq U$ we must have a map $C(U) \longrightarrow C(V)$, so also $C(V)$ is non empty and thus $\mathcal{U}$ is downward closed and coincides with the set $\downarrow U$.
Denote with $\mathcal{T}$ the full subcategory of subterminal objects in $\mathcal{Sh}(X)$. Consider on the correspondence $e : \mathcal{T}_0 \longrightarrow \mathcal{O}(X)$ such that $e(C) = U$ the maximal open for which C has a section. Let now $f : C \longrightarrow D$ be a morphism in $\mathcal{T}$, we have that $C(e(C)) = 1$, so $f_{e(C)}$ is a map from 1 into $D(e(C))$ which therefore must be 1. This implies that $e(D)$ is greater than or equal to $e(C)$ and thus we can send the arrow $f : C \longrightarrow D$ to the arrow $e(f)$ representing $e(C) \leq e(D)$, getting a correspondence $e : \mathcal{T}_1 \longrightarrow \mathcal{O}(X)_1$. These maps e define the components on objects and arrows of a functor $e : \mathcal{T} \longrightarrow \mathcal{O}(X)$, in fact identities and composition are preserved thanks to the fact that $\mathcal{O}(X)$ is an order.
This functor is surjective on objects because for every open U, the presheaf sending an open V to 1 if $V \subseteq U$ and to $\emptyset$ otherwise is actually a sheaf. It is faithful because for every couple of subterminal objects C, D there is at most one map $f : C \longrightarrow D$ for Proposition 1.6.8 (1). Now if $e(C) \leq e(D)$, we can define for $U \in \mathcal{O}(X)$ a map $f_U : C(U) \longrightarrow D(U)$ because, if $U \leq e(C)$, then $C(U)$ and $D(U)$ are both terminal, so we define $f(U)$ as the unique map between them, otherwise $C(U) = 0$ so f_U must be the empty map. The family f is a natural transformation $f : C \longrightarrow D$, so e is also full.
It follows that $e : \mathcal{T} \longrightarrow \mathcal{O}(X)$ is an equivalence of categories. □

Corollary 2.5.5. *Let X be a locale, then the lattice $Sub_{\mathcal{Sh}(X)}(1)$ is isomorphic to $\mathcal{O}(X)$.*

Proof. Let's use the same notation used in the proof of Proposition 2.5.4, so we have an equivalence $e : \mathcal{T} \longrightarrow \mathcal{O}(X)$.
Note that $\mathcal{T}$ actually contains only monomorphisms of sheaves because of Proposition 1.6.8 (2), thus we can build the quotient category under the relation of isomorphism which coincides with the lattice of subobjects of 1 in $\mathcal{Sh}(X)$ and the corresponding projection functor π from $\mathcal{T}$ into $\mathrm{Sub}_{\mathcal{Sh}(X)}(1)$

sending thus a subterminal object U to the corresponding subobject of 1. We want to prove that e factors through π into an isomorphism



This factorization takes place if and only if e is constant on equivalence classes of objects and arrows. The equivalence relation is isomorphism and e preserves them, but the only isomorphisms in $\mathcal{O}(X)$ are identities, thus this property is true and we have a functor $\overline{e}$ as above.
Note that π is also an equivalence because it is full and faithful (it is a functor of preorders) and surjective. Let $s : \mathrm{Sub}_{\mathcal{Sh}(X)}(1)$ be a quasi inverse for π, then $es \cong \overline{e}$ and thus it is an equivalence of categories as well. In particular it preserves limits and colimits, so it is a frame homomorphism and moreover, since the isomorphisms of its domain and codomain are the identities, $\overline{e}$ is bound to be an isomorphism, concluding the proof. □

Remark 2.5.6. *If we think of $\mathcal{Sh}(X)$ as LH/X, since monos $A \longrightarrow X$ are the same in LH and LH/X, we see that the local homeomorphisms which are monos are precisely the inclusions of open sublocales. We got a characterization of the open sublocales.*

Note also that if we have a continuous map $f : X \longrightarrow Y$, of locales, the inverse image functor $f^* : \mathcal{Sh}(Y) \longrightarrow \mathcal{Sh}(X)$ preserves finite limits for Proposition 2.5.1, so in particular it preserves terminals and monos. We thus have that it restricts to a map $f^* : \mathrm{Sub}_{\mathcal{Sh}(Y)}(1) \longrightarrow \mathrm{Sub}_{\mathcal{Sh}(X)}(1)$ and thus to a map $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$. This apparent ambiguity of notation does not lead to any confusion because the inverse image frame homomorphism and the one obtained by the inverse map functor are actually the same map. This is evident in the interpretation of $\mathcal{Sh}(X)$ as LH/X and f^* as pullback using Remark 2.5.6.

We can even say more. For every couple of continuous maps of locales $f, g : X \longrightarrow Y$, we say that $f \leq g$ if $f^*(V) \leq g^*(V)$ for all $V \in \mathcal{O}(Y)$, viewing each of these inequalities as a transformation from f to g.

Remark 2.5.7. *In this way $\mathcal{Loc}(X, Y)$ becomes a poset, so $\mathcal{Loc}$ can be seen as a category enriched in posets.*
Moreover, if we think it in terms of frames, $f \leq g$ corresponds exactly a natural transformation $\alpha : f^* \Rightarrow g^*$ where $f^*, g^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ are seen as functors and for all $V \in \mathcal{O}(X)$ the component $\alpha_V : f^*(V) \longrightarrow g^*(V)$ is the inclusion.

In particular then if X and Y are locales, this order corresponds to the order induced on $\mathcal{Top}(X_p, Y_p)$ by the order of specialization *on the sober space corresponding to X.*
This is worked out in detail in [SE] before Lemma C1.2.17.

This confers to the category of locales a structure of a 2-category which we will call $\mathfrak{Loc}$. Note that $\mathfrak{Loc}$ contains the 2-category obtained from $\mathcal{Loc}$ as described in Example 1.5.11 and what actually happens is that the pseudofunctor $\mathcal{Sh} : \mathcal{Loc} \longrightarrow \mathfrak{Geom}$ can be extended to $\mathfrak{Loc}$ as follows.
Let $f, g : X \longrightarrow Y$ such that $f \leq g$, then as showed in the previous remark there is a corresponding natural transformation $\alpha : f^* \Rightarrow g^*$. If now we consider the opposite functors $(f^*)^{\mathrm{op}}$, $(g^*)^{\mathrm{op}} : \mathcal{O}(Y)^{\mathrm{op}} \longrightarrow \mathcal{O}(X)^{\mathrm{op}}$, there is a natural transformation $\alpha^{\mathrm{op}} : (g^*)^{\mathrm{op}} \Rightarrow (f^*)^{\mathrm{op}}$. Now for any sheaf $F \in \mathcal{Sh}(X)$ we can get a transformation $F \circ \alpha^{\mathrm{op}} : g_*(F) \Rightarrow f_*(F)$ which is also natural in F and thus it forms a natural transformation $g_* \Rightarrow f_*$. This transformation corresponds by adjunction to a natural transformation $f^* \Rightarrow g^*$ and thus by definition to a geometric transformation from $\mathcal{Sh}(f) \Rightarrow \mathcal{Sh}(g)$ which will be set as image of $f \leq g$ via $\mathcal{Sh}$.
We have that $\mathcal{Sh} : \mathfrak{Loc}(X, Y) \longrightarrow \mathfrak{Geom}(\mathcal{Sh}(X), \mathcal{Sh}(Y))$ is a functor for all $X, Y \in \mathfrak{Loc}_0$, so we get a pseudofunctor

$$\mathcal{Sh} : \mathfrak{Loc} \longrightarrow \mathfrak{Geom}$$

Proposition 2.5.8. *For each couple of locales X and Y, the functor $\mathcal{Sh}$: $\mathfrak{Loc}(X, Y) \longrightarrow \mathfrak{Geom}(\mathcal{Sh}(X), \mathcal{Sh}(Y))$ is one half of an equivalence of categories.*

Proof. See [SE] Proposition C1.4.5. □

This in particular means that if we have a geometric morphism of the form $\phi : \mathcal{Sh}(X) \longrightarrow \mathcal{Sh}(Y)$ where X, Y are locales, then there is a continuous map of locales $f : X \longrightarrow Y$ such that $\phi \cong \mathcal{Sh}(f)$, so there is a geometric transformation between them which is invertible. Because of this, from now on we shall often identify the category of locales with the full sub-2-category of $\mathfrak{Geom}$ having as objects those toposes equivalent to a topos of sheaves over a locale.

Our final aim for this section is to characterize this subcategory.
First note that $\mathcal{Loc}$ has a terminal object 1 as noticed in Section 2.2. The terminal locale has as category of sheaves $\mathcal{Sh}(1) = \mathcal{Set}$ because presheaves are $\mathcal{Set}^{2^{\mathrm{op}}} = \mathcal{Set}^2$ that is the category of functions and commutative squares between them. From the final observation made in Example 1.2.8, every presheaf A (i.e. function $A_1 \longrightarrow A_0$) is such that in $\emptyset$ it is terminal (i.e. it is of the form $A_1 \longrightarrow 1$) and thus for the universal property of the terminal object, it is isomorphic to $\mathcal{Set}$.
In particular, for every locale X there exist a geometric morphism γ :

$\mathcal{Sh}(X) \longrightarrow \mathcal{Set}$ which is unique up to invertible 2-cells i.e. geometric transformations. Thus the functor $\mathcal{Sh}$ factors through the inclusion $\mathfrak{Geom}/\mathcal{Set} \longrightarrow \mathfrak{Geom}$.

Remark 2.5.9. *In particular the existence and essential uniqueness of a geometric morphism $\gamma : \mathcal{E} \longrightarrow \mathcal{Set}$ hold in more general toposes.*
More precisely, since every set can be written as coproduct of its points and the inverse image preserves finite limits and is a left adjoint, for all sets A, if we suppose the existence of coproducts in $\mathcal{E}$ we have

$$\gamma^*(A) = \gamma^* \left(\coprod_{p \in A} 1_{\mathcal{Set}} \right) = \coprod_{p \in A} \gamma^*(1_{\mathcal{Set}}) = \coprod_{p \in A} 1_{\mathcal{E}}$$

where all these isomorphisms are natural in A. On the other side $\gamma_* : \mathcal{E} \longrightarrow \mathcal{Set}$ has image in $\mathcal{Set}$, so $\gamma_*(E) \cong \mathcal{Set}(1, \gamma_*(E))$ where the isomorphism is natural in E and moreover, since it must be right adjoint to γ^*, we get

$$\gamma_* \cong \mathcal{Set}(1, \gamma_*(-)) \cong \mathcal{E}(\gamma^*(1), -) \cong \mathcal{E}(1_{\mathcal{E}}, -)$$

The only hypotheses we made for the construction were the existence in $\mathcal{E}$ of small coproducts and that it is locally small. In these two hypotheses one can prove that defining $\gamma^*(A) = \coprod_A 1_{\mathcal{E}}$ and $\gamma_*(E) = \mathcal{E}(1, E)$ one gets an adjunction and, since in toposes coproducts are pullback stable, we also get that γ^* preserves finite limits. In particular in a Grothendieck topos this geometric morphism exists.
Note also that the essential uniqueness holds for every topos that admits such a morphism because, the choice of the directed image is essentially unique, and thus by essential uniqueness of the adjoint functor, also the inverse image is such.

If we have a topos of sheaves over a locale X, interpreting it as LH/X we have that the terminal sheaf corresponds to X. We thus have that arrows $1 \longrightarrow E$ are in particular global sections of $E \longrightarrow X$ which are also local homeomorphisms, but thanks to Lemma 2.3.7(3) we have that these are precisely all the global sections.
For this reason in a topos $\mathcal{E}$ with $E \in \mathcal{E}_0$ we call morphisms of the form $1 \longrightarrow E$ *global sections* of E. In particular when we have a geometric morphism into $\mathcal{Set}$, its direct image coincides with the global sections functor and this explains why we denote this unique geometric morphism with γ.
Now we want to see which of these toposes over $\mathcal{Set}$ is equivalent to a topos of sheaves over a locale. For this reason we need the following definition

Definition 2.5.10. *A geometric morphism of toposes $f : \mathcal{E} \longrightarrow \mathcal{F}$ is called* localic *if every E in $\mathcal{E}_0$ is expressible as quotient of a subobject of one of the* *form* $f^*(A)$ *with* A *in* $\mathcal{F}_0$*, that is, we have*

$$\begin{array}{ccc} D & \overset{e}{\twoheadrightarrow} & E \\ {\scriptstyle m}\big\downarrow & & \\ f^*(A) & & \end{array}$$

where e *is a regular epimorphism and* m *a monomorphism. When this happens we say that* E *is a* subquotient *of* $f^*(A)$*.*

We can finally state the following

Theorem 2.5.11. *Let* $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$ *be a topos over* $\mathcal{S}et$*. The following are equivalent:*

1. $\mathcal{E}$ *is equivalent to one of the form* $\mathcal{S}h(X)$ *for some locale* X*.*
2. *The subterminal objects of* $\mathcal{E}$ *form a generating set.*
3. *The geometric morphism* $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$ *is localic.*

Proof. See [SE] Theorem C1.4.7. □

This theorem provides the characterization of toposes in $\mathfrak{Loc}$ that we were looking for.

Definition 2.5.12. *A topos* $\mathcal{E}$ *will be called* localic *if one of the equivalent conditions of Theorem 2.5.11 holds.*

Chapter 3

Internal structures

In this chapter we are going to describe a way to interpret some objects and tools of category theory inside another category. In the first part we will define an internal notion of categories, functors and diagrams, then we will give a way to describe limits and colimits in this environment. We will move to study particular cases of these internal structures, like in the case of filtered internal categories or in the case where we are working inside a topos instead of just a finitely complete category. Finally in the last section we will treat the case of the interpretation in a topos over a base, studying in particular the notion of preservation of certain colimits and giving an interpretation of the Beck-Chevalley condition in a special case.
Before we start this chapter, we make some consideration about intuitionistic higher order logic. First of all a topos has enough structure to allow us to interpret higher order logic in it. More explicitly, given a topos $\mathcal{S}$ and a certain higher order language $\mathcal{L}$, we can create what is generally called an $\mathcal{L}$-structure in $\mathcal{S}$, which consists of certain data in $\mathcal{S}$ for which we can give an interpretation of a higher order formula in $\mathcal{L}$ and in particular we can determine whether a statement is true or not for a certain structure. Given a higher order theory, we can define a model for it in $\mathcal{S}$ to be an $\mathcal{L}$-structure where every formula of this theory is satisfied.
We have thus vaguely displayed the reason why toposes may be interpreted as a universes for mathematics as we hinted at in the Introduction. Another important fact is that if a statement is proved constructively (which roughly speaking means that it is proved without the use of the law of excluded middle or the axiom of choice), such a proof is still valid in every topos. It follows from this fact that we can immediately prove in any topos every constructive property that holds for a model in the classical case, that is, within the topos $\mathcal{S}et$. The details concerning the interpretation of higher order logic in a topos can be found in [LS].
Most of the structures defined in this chapter are just models of certain higher order theories when seen in a topos, however we provide for each of

them an explicit description, possibly valid in categories with less structure, rather than seeing them in this fashion.

3.1 Internal category theory

In ordinary category theory, a small category is made of a set of objects and a set of arrows, while the information of domain, codomain and identities can be represented as functions between them. What we want to do in this section is to replace the role of $\mathcal{S}et$ with another sufficiently regular category. Let $\mathcal{S}$ be a finitely complete category throughout the section.

Definition 3.1.1. *An* internal category $\mathcal{C}$ *in* $\mathcal{S}$ *consists of the following data:*

1. *two objects* C_0 *and* C_1 *of* $\mathcal{S}$ *called respectively* object of objects *and* object of morphisms*;*
2. *three morphisms* $d, c : C_1 \longrightarrow C_0$ *and* $i : C_0 \longrightarrow C_1$ *of* $\mathcal{S}$ *called respectively* domain*,* codomain *and* identities*;*
3. *a morphism* $m : C_1 \times_{C_0} C_1 \longrightarrow C_1$ *called* composition*, where the pullback is the following*

$$\begin{array}{ccc} C_1 \times_{C_0} C_1 & \xrightarrow{p_2} & C_1 \\ {\scriptstyle p_1}\downarrow & \lrcorner & \downarrow{\scriptstyle d} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array}$$

and is called object of composable morphisms*.*

such that the following conditions are satisfied

1. $di = 1_{C_0} = ci$
2. $dp_1 = dm$ *and* $cp_2 = cm$
3. *(identity axiom)* $m(1_{C_0}, i\ c) = 1_{C_0} = m(i\ d, 1_{C_0})$
4. *(associativity of m)* $m(1_{C_1} \times_{C_0} m) = m(m \times_{C_0} 1_{C_1})$ *as morphisms*

$$C_1 \times_{C_0} C_1 \times_{C_0} C_1 \longrightarrow C_1$$

where the ternary pullback can be seen as the limit of the diagram

$$\begin{array}{ccccccc} C_1 & & & C_1 & & & C_1 \\ & {\scriptstyle c}\searrow & \swarrow{\scriptstyle d} & & {\scriptstyle c}\searrow & \swarrow{\scriptstyle d} & \\ & & C_0 & & C_0 & & \end{array}$$

Example 3.1.2. *Let X be an object of $\mathcal{S}$, we can always form an internal category called discrete internal category corresponding to X. Let $C_0 = C_1 = X$ and $d, c, i = 1_X$ (so that in particular also the object of composable morphisms is X) and then we choose as composition again the identity on X. We denote this category X as the object that it represents.*
Note that if $\mathcal{S} = \mathcal{S}et$, then internal discrete categories are the discrete categories, that is, sets.
Since $\mathcal{S}$ has terminal object, a particular case of this example is when we choose X terminal.

In general we say that an internal category $\mathcal{C}$ in $\mathcal{S}$ is *discrete* iff the identities morphism i is an isomorphism in $\mathcal{S}$.
Now that we have a definition of internal category we can give the definition of internal functor

Definition 3.1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two internal categories in $\mathcal{S}$, an* internal functor *$F : \mathcal{C} \longrightarrow \mathcal{D}$ is given by a couple of morphisms in $\mathcal{S}$*

$$F_0 : C_0 \longrightarrow D_0$$

$$F_1 : C_1 \longrightarrow D_1$$

such that the following conditions are satisfied

1. $dF_1 = F_0 d$ *and* $cF_1 = F_0 c$
2. $F_1 i = iF_0$
3. $F_1 m = m(F_1 \times_{F_0} F_1)$ *where the last pullback has to be seen in the category of arrows of $\mathcal{S}$ (which makes sense thanks to the first condition)*

If we consider as objects internal categories and as morphisms the functors with as identity functor on $\mathcal{C}$ the functor $\mathrm{id}_{\mathcal{C}} = (\mathrm{id}_{C_0}, \mathrm{id}_{C_1})$ and as composition the pairwise composition, we get a category called the *category of internal categories* in $\mathcal{S}$ and denoted with $\mathcal{C}at(\mathcal{S})$.

Example 3.1.4. *As for Example 3.1.2, for all morphisms $f : X \longrightarrow Y$ in $\mathcal{S}$ we can build a functor $f : X \longrightarrow Y$ between the two discrete categories corresponding to X and Y by taking $f_0 = f_1 = f$.*

Remark 3.1.5. *Combining Examples 3.1.2 and 3.1.4 we created a functor*

$$\mathcal{S} \longrightarrow \mathcal{C}at(\mathcal{S})$$

Moreover note that this functor is full and faithful and a category is isomorphic to one of this form if and only if it is discrete. This functor becomes then one half of an equivalence of categories if the codomain is restricted to discrete internal categories. For this reason we can always interpret a finitely complete category as the full subcategory of discrete categories inside the category of its internal categories.

Remark 3.1.6. *In the end of Example 3.1.2 we formed an internal category* 1 *in* $\mathcal{S}$*, now note that this name is justified because this category is terminal in* $\mathcal{C}at(\mathcal{S})$*. Given any internal category* $\mathcal{C}$ *in* $\mathcal{S}$*, we have in fact a unique couple of morphisms* $f_0 : C_0 \longrightarrow 1$ *and* $f_1 : C_1 \longrightarrow 1$ *and since 1 is terminal in* $\mathcal{S}$*, these two arrows satisfy the axioms of functor* $\mathcal{C} \longrightarrow 1$*, thus* 1 *is the terminal category in* $\mathcal{C}at(\mathcal{S})$*.*
We call object *of an internal category* $\mathcal{C}$ *an internal functor* $1 \longrightarrow \mathcal{C}$*. We actually have that functors* $1 \longrightarrow \mathcal{C}$ *correspond bijectively to morphisms* $1 \longrightarrow C_0$*. To prove that this correspondence is injective we can use the second axiom of functors, while to prove that it is surjective we show that every* $1 \longrightarrow C_0$ *is the object component of an internal functor* $1 \longrightarrow \mathcal{C}$*, defining the other component* $1 \longrightarrow C_1$ *by composition of* $1 \longrightarrow \mathcal{C}_0$ *with* $i : C_0 \longrightarrow C_1$*.*

Now we can proceed a step further

Definition 3.1.7. *Let* $F, G : \mathcal{C} \longrightarrow \mathcal{D}$ *be two internal functors in* $\mathcal{S}$*, an* internal natural transformation $\alpha : F \Rightarrow G$ *is a morphism* $\alpha : C_0 \longrightarrow D_1$ *satisfying the following conditions*

1. $d\alpha = F_0$ *and* $c\alpha = G_0$
2. *(naturality)* $m(F_1, \alpha c) = m(\alpha d, G_1)$

As expected, adding the internal natural transformations as 2-cells for $\mathcal{C}at(\mathcal{S})$ we get a 2-category called *2-category of internal categories* and denoted with $\mathfrak{Cat}(\mathcal{S})$. For more details see [B1] Proposition 8.1.4.

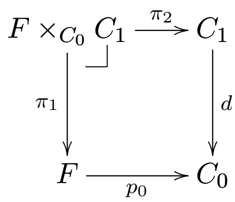
Example 3.1.8. *If we choose* $\mathcal{S} = \mathcal{S}et$*, then* $\mathfrak{Cat}(\mathcal{S}et)$ *is equivalent*[1] *to* $\mathfrak{Cat}$*.*

For small categories it also makes sense to define functors landing in the category $\mathcal{S}et$ even though it is not small. The same can be done for internal categories as well, but with some caution. A functor F from a small category $\mathcal{C}$ into $\mathcal{S}et$ can be seen as the set of all its partial sections $E = \coprod_{c \in C_0} F(c)$ with a projection $p : E \longrightarrow C_0$. The action of F on arrows instead can be interpreted as an action of the arrows C_1 on E, namely given $f \in \mathcal{C}(a, b)$ and $s \in F(a)$ we define the action as $(s, f) \mapsto F(f)(s)$.
Following this idea we are now able to describe this kind of functors which are usually called internal diagrams (or internal base-valued functors[2]).

[1] It is not equal nor isomorphic because if we change the pullback representing composable morphisms with an isomorphic one, we get a different structure which in $\mathcal{S}et$ theoretic terms corresponds to the same category.

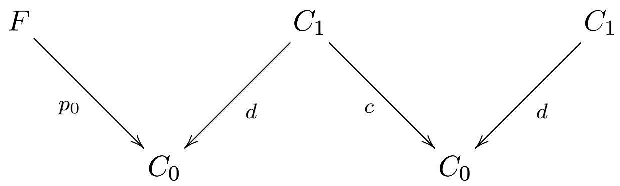
[2] As in [B1].

Definition 3.1.9. *In a finitely complete category* $\mathcal{S}$ *an* internal diagram F *over an internal category* $\mathcal{C}$ *consists of an object* F *and a couple of morphisms:* $p_0 : F \longrightarrow C_0$ *and* $p_1 : F \times_{C_0} C_1 \longrightarrow F$*, where the pullback is*



Such data must then have the following properties

1. $p_0 p_1 = c\pi_2$
2. $p_1(1_F, ip_0) = 1_F$
3. $p_1(p_1 \times_{C_0} 1_{C_1}) = p_1(1_{F_0} \times m) : F \times_{C_0} C_1 \times_{C_0} C_1 \longrightarrow F$ *where the ternary pullback is the limit of the following diagram*



We can write it as $F : \mathcal{C} \longrightarrow \mathcal{S}$ *to emphasise its interpretation as functor.*

We will usually use the term internal diagram to avoid confusing it with internal functors and because it will be mostly used in this thesis precisely as diagram. We can in fact give a precise definition of limit and colimit for an internal diagram, as we shall see below.

Definition 3.1.10. *Let* $F, G : \mathcal{C} \longrightarrow \mathcal{S}$ *be two internal diagrams. We will call the defining morphisms of both of them as in Definition 3.1.9, for the diagram they belong to is clear from the context. An* internal natural transformation $\alpha : F \Rightarrow G$ *is given by a morphism* $\alpha : F \longrightarrow G$ *in* $\mathcal{S}$ *such that*

1. $p_0 \alpha = p_0$
2. $\alpha p_1 = p_1(\alpha \times_{C_0} 1_{C_1})$

Internal diagrams and internal natural transformations between them form a category, for composition of natural transformations is the composition in $\mathcal{S}$. We shall call this category $\mathcal{S}^{\mathcal{C}}$.

Example 3.1.11. *Let $\mathcal{S}$ be a finitely complete category and $\mathcal{C}$ be an internal category in $\mathcal{S}$. Let X be an object of $\mathcal{S}$, then we always create an internal diagram*

$$\Delta_X : \mathcal{C} \longrightarrow \mathcal{S}$$

as follows

$$\Delta_X = X \times C_0$$

$$p_0 = \pi_2 : X \times C_0 \longrightarrow C_0$$

where π_2 is the projection on the second component and finally as p_1 we choose the following

$$p_1 = (\pi_X, c\pi_{C_1}) : \Delta_X \times_{C_0} C_1 \longrightarrow \Delta_X$$

A more explicit description of p_1 can be given in this case: since $\Delta_X \times_{C_0} C_1 = X \times C_0 \times_{C_0} C_1 \cong X \times C_1$, if we choose $\Delta_X \times_{C_0} C_1 = X \times C_1$, we see that $p_1 : \Delta_X \times_{C_0} C_1 \longrightarrow \Delta_X$ is

$$p_1 = (1_X \times c) : X \times C_1 \longrightarrow X \times C_0$$

We call the diagram of this example the *constant internal diagram* for X.
If $f : X \longrightarrow Y$ is a morphism in $\mathcal{S}$, then we get an internal natural transformation $\Delta_f : \Delta_X \longrightarrow \Delta_Y$

$$\Delta_f = (f \times 1_{C_0}) : X \times C_0 \longrightarrow Y \times C_0$$

In this way we get a functor

$$\Delta : \mathcal{S} \longrightarrow \mathcal{S}^{\mathcal{C}}$$

We shall see below more properties of this functor in particular settings.

Remark 3.1.12. *Let $\mathcal{C}$ be a discrete category on $\mathcal{S}$, then $\mathcal{S}^{\mathcal{C}} \simeq \mathcal{S}/C_0$ and in fact an internal diagram $F : \mathcal{C} \longrightarrow \mathcal{S}$ is uniquely defined by $p_0 : F \longrightarrow C_0$. This holds because $F \times_{C_0} C_1 \cong F$ and by axiom 2 of internal diagrams, since $(1_F, ip_0)$ is bound to be an isomorphism, than also p_1 is such and in particular it is the inverse of $(1_F, ip_0)$ which is uniquely determined by p_0. On the other side a morphism $f : A \longrightarrow C_0$ determines a diagram if we choose $p_0 = f$ and $p_1 = (1_A, ip_0)^{-1}$. Both these maps are extendable to functors by sending transformations $\alpha : F \Rightarrow G$ to the underlying morphism $\alpha : F \longrightarrow G$ and vice versa. So we get the desired equivalence.*
Note in particular that for the terminal internal category, $\mathcal{S}^{1} \simeq \mathcal{S}$.

Let $F : \mathcal{C} \longrightarrow \mathcal{S}$ be an internal diagram in a finitely complete category $\mathcal{S}$, then by definition we have $p_0 : F \longrightarrow C_0$ and $p_1 : F \times_{C_0} C_1 \longrightarrow F$. Now we will show how we can see F as a category, more precisely we will build what

in set theoretic category theory corresponds to the category of elements of a functor seen in Example 1.1.6.
Let $F_0 = F$ and $F_1 = F \times_{C_0} C_1$, then take as identities morphism $(1_F, p_0) : F \longrightarrow F \times_{C_0} C_1$, as domain the projection on the first component $\pi_1 : F \times_{C_0} C_1 \longrightarrow F$ and as codomain the morphism $p_1 : F \times_{C_0} C_1 \longrightarrow F$. Then as composition we need a morphism $(F \times_{C_0} C_1) \times_F (F \times_{C_0} C_1) \longrightarrow (F \times_{C_0} C_1)$, so if we denote the projections $(F \times_{C_0} C_1) \times_F (F \times_{C_0} C_1) \longrightarrow (F \times_{C_0} C_1)$ with π_1' and π_2' respectively for the left and the right, we can take as composition the morphism $(\pi_1\pi_1', m(\pi_2\pi_1', \pi_2\pi_2'))$. The properties of internal diagram of F assure that this is an internal category which we will call *internal category of elements* and we will denote with $el(F)$.
Actually we have more structure, in fact the remaining properties of F as a diagram tell us that we have a functor $el(F) \longrightarrow \mathcal{C}$ having as components p_0 on objects and π_2 on morphisms. But there is more: if we have a natural transformation of diagrams $\alpha : F \Rightarrow G$, then $el(\alpha)_0 = \alpha : F \longrightarrow G$ and $el(\alpha)_1 = \alpha \times 1_{C_1}$ produce an internal functor $el(\alpha) : el(F) \longrightarrow el(G)$ such that

$$\begin{array}{ccc} el(F) & \xrightarrow{el(\alpha)} & el(G) \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

This because of the properties of α as an internal natural transformation of diagrams.
As we can see from the definition of el, identities and composition of natural transformations of internal diagrams are preserved, so we have found a functor

$$el : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{C}at(\mathcal{S})/\mathcal{C}$$

This functor will be fundamental in the next section.
Before going on though note that this functor is injective on objects because for every F in $\mathcal{S}^{\mathcal{C}}$, $el(F)$ provides all the informations of F. Note then that an object $f : \mathcal{D} \longrightarrow \mathcal{C}$ is isomorphic to one in the image of el if and only if we have the following pullback

$$\begin{array}{ccc} D_1 & \xrightarrow{d} & D_0 \\ {\scriptstyle f_1}\downarrow & \lrcorner & \downarrow{\scriptstyle f_0} \\ C_1 & \xrightarrow[d]{} & C_0 \end{array}$$

Any such functor is called *discrete opfibration*. This fact holds because for all $el(F)$ we have the pullback by definition of internal diagram and conversely if a functor $f : \mathcal{D} \longrightarrow \mathcal{C}$ happens to have such a pullback, then $(D_0, f_0, c^{\mathcal{D}})$ is an internal diagram. Moreover el is faithful by definition and it is full because given a functor $f : el(F) \longrightarrow el(G)$ then in particular $f_0 : F \longrightarrow G$ turns out to be a natural transformation. From these observations we have that el into the full subcategory of discrete opfibrations is one half of an equivalence of categories.

For further reference we note that

Lemma 3.1.13. *Let $\mathcal{S}$ be a finitely complete category and $f : \mathcal{C} \longrightarrow \mathcal{D}$ an internal functor of internal categories. Then we have a functor*

$$f^* : \mathcal{S}^{\mathcal{D}} \longrightarrow \mathcal{S}^{\mathcal{C}}$$

Proof. On objects we send a diagram $F : \mathcal{D} \longrightarrow \mathcal{S}$ defined by the data (F, p_0, p_1) to the diagram $f^*(F) : \mathcal{C} \longrightarrow \mathcal{S}$ defined by the data $(f^*(F), q_0, q_1)$ such that q_0 is the pullback of p_0 along f_0 (and thus $f^*(F) = F \times_{D_0} C_0$) and q_1 is the morphism $p_1 \times c : F \times_{D_0} D_1 \times_{D_1} C_1 \longrightarrow F \times_{D_0} C_0$. On arrows we send an internal natural transformation $\alpha : F \Rightarrow G$ to $\alpha \times 1_{C_0}$ and thus f^* is by definition a functor.
In other terms we can find this functor using the characterization given before, in fact it can be found as in [J] from the restriction to discrete opfibrations of the pullback functor

$$f^* : \mathcal{C}at(\mathcal{S})/\mathcal{D} \longrightarrow \mathcal{C}at(\mathcal{S})/\mathcal{C}$$

This functor exists for $\mathcal{C}at(S)$ is finitely complete (see [J] Lemma 2.16) and sends discrete opfibrations to discrete opfibrations because of the pullback pasting Lemma. □

We won't enter in further details here, but in the next sections we will give more properties for this functor.

Let $F : \mathcal{E} \longrightarrow \mathcal{F}$ be a functor preserving finite limits between two finitely complete categories, let $\mathcal{C}$ be an internal category in $\mathcal{E}$, then applying F to every data defining $\mathcal{C}$, since F preserves pullbacks and is a functor, we still get a category $F(\mathcal{C})$ but this time inside $\mathcal{F}$. In the same way applying F to both the components of an internal functor $f : \mathcal{C} \longrightarrow \mathcal{D}$ we get an internal functor $F(f) : F(\mathcal{C}) \longrightarrow F(\mathcal{D})$ in $\mathcal{F}$. Note that also internal natural transformations are sent to natural transformations in this fashion. We have just built a pseudofunctor

$$\mathfrak{Cat}(\mathcal{E}) \longrightarrow \mathfrak{Cat}(\mathcal{F})$$

and in particular this pseudofunctor is strict. Moreover, if $\mathcal{C}$ is an internal category in $\mathcal{E}$, this 2-functor leads to a 2-functor

$$\mathfrak{Cat}(\mathcal{E})/\mathcal{C} \longrightarrow \mathfrak{Cat}(\mathcal{F})/F(\mathcal{C})$$

Remark 3.1.14. *Since F preserves pullbacks, then in particular the functor that we have just found sends discrete opfibrations to discrete opfibrations and hence from the previous characterization we get a functor*

$$\mathcal{E}^{\mathcal{C}} \longrightarrow \mathcal{F}^{F(\mathcal{C})}$$

3.2 Limits and colimits of internal diagrams

Let $\mathcal{S}$ be a finitely complete category and $\mathcal{C}$ an internal category in $\mathcal{S}$.

Definition 3.2.1. *An* internal limit *for an internal diagram $F : \mathcal{C} \longrightarrow \mathcal{S}$ is given by an object $\lim_{\mathcal{C}} F$ of $\mathcal{S}$ and a morphism $\epsilon_F : \Delta(\lim_{\mathcal{C}} F) \longrightarrow F$ such that for every object A of $\mathcal{S}$ and every morphism $b : \Delta(A) \longrightarrow F$ in $\mathcal{S}^{\mathcal{C}}$ there exist a unique morphism $a : A \longrightarrow \lim_{\mathcal{C}} F$ such that $\epsilon_F \Delta(a) = b$.*
An internal colimit *for an internal diagram $F : \mathcal{C} \longrightarrow \mathcal{S}$ is given by an object $\mathrm{colim}_{\mathcal{C}} F$ of $\mathcal{S}$ and a morphism $\eta_F : F \longrightarrow \Delta(\mathrm{colim}_{\mathcal{C}} F)$ such that for every object A of $\mathcal{S}$ and every morphism $b : F \longrightarrow \Delta(A)$ in $\mathcal{S}^{\mathcal{C}}$ there exist a unique morphism $a : \mathrm{colim}_{\mathcal{C}} F \longrightarrow A$ in $\mathcal{S}$ such that $\Delta(a)\eta_F = b$.*
We say then that $\mathcal{S}$ has all internal (co)limits of shape $\mathcal{C}$ if every internal diagram $F : \mathcal{C} \longrightarrow \mathcal{S}$ has internal (co)limit.
A category is said to be internally (co)complete *if it has all internal (co)limits of shape $\mathcal{C}$ for all internal categories $\mathcal{C}$.*

Example 3.2.2. *If the internal category $\mathcal{C}$ is discrete, then the (co)limit of an internal diagram of shape $\mathcal{C}$ is called* internal (co)product*.*
A category is said to have all (co)products if for every discrete category the (co)limit exists.

Remark 3.2.3. *With the same notation as in Definition 3.2.1, $\mathcal{S}$ has all internal limits of shape $\mathcal{C}$ iff the functor Δ has a right adjoint*

$$\lim_{\mathcal{C}} : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}$$

dually, $\mathcal{S}$ has all internal colimits of shape $\mathcal{C}$ iff the functor Δ has a left adjoint

$$\mathrm{colim}_{\mathcal{C}} : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}$$

Definition 3.2.4. *Let $\mathcal{S}$ be a finitely complete category with reflexive coequalizers i.e. coequalizers of morphisms $p, q : A \longrightarrow B$ with a common section $s : B \longrightarrow A$, let $\mathcal{C}$ be an internal category in $\mathcal{S}$.*
We define $\pi_0(\mathcal{C}) := \mathrm{Coeq}(d, c)$ the coequalizer of domain and codomain (which have as common section the identities morphism i).

Remark 3.2.5. *In $\mathcal{S}et$, let $\mathcal{C}$ be a small category, we can define a connected component of $\mathcal{C}$ as an equivalence class of objects with respect to the minimal equivalence relation which identifies those objects such that there is an arrow having one as domain and the other as codomain regardless of the order. In other words it is a connected component of the (unoriented) graph obtained from $\mathcal{C}$ once we forget the direction of its arrows.*
In this way $\pi_0(\mathcal{C})$ corresponds to the set of connected components of $\mathcal{C}$, whence the name.

The object just defined is the object component of a functor $\pi_0 : \mathcal{C}at(\mathcal{S}) \longrightarrow \mathcal{S}$ whose arrows component is uniquely determined by the universal property of coequalizers.
Such a functor is left adjoint to the functor described in Remark 3.1.5.

Proposition 3.2.6. *Let $\mathcal{S}$ be a finitely complete category with reflexive coequalizers, then it is internally cocomplete where for every internal category $\mathcal{C}$ the left adjoint* $\mathrm{colim}_{\mathcal{C}} : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}$ *to Δ is the composition*

$$\mathcal{S}^{\mathcal{C}} \xrightarrow{el} \mathcal{C}at(\mathcal{S}) \xrightarrow{\pi_0} \mathcal{S}$$

More explicitly the colimit of an internal diagram $F : \mathcal{C} \longrightarrow \mathcal{S}$ is the coequalizer of the following parallel morphisms

$$F \times_{C_0} C_1 \underset{p_1}{\overset{\pi_1}{\rightrightarrows}} F$$

(Notation as in definition 3.1.9)

Proof. See [B1] Proposition 8.3.2, it is proved if $\mathcal{S}$ has all coequalizers, but the proof only uses reflective ones. □

For the case of internal limits the situation is a bit more delicate, in fact we have the following

Proposition 3.2.7. *Let $\mathcal{S}$ be a finitely complete category, the following are equivalent:*

1. *$\mathcal{S}$ is cartesian closed;*
2. *$\mathcal{S}$ admits all internal products;*
3. *$\mathcal{S}$ is internally complete.*

Proof. See [B1] Proposition 8.3.5. □

3.3 Internal categories in a topos

Now we will apply this internal category theory to particular cases in order to see which additional structures we can gain.
A (pre)topos is finitely complete, so we can make sense to the previously studied internal objects. A first important result is the following

Proposition 3.3.1. *If $\mathcal{S}$ is a topos, then so is $\mathcal{S}^{\mathcal{C}}$ for every internal category in $\mathcal{S}$.*

Proof. See [SE] Corollary B2.3.18. □

Let now $\mathcal{S}$ be a topos and $\mathcal{C}$ be an internal category in $\mathcal{S}$, then let $\Delta : \mathcal{S} \longrightarrow \mathcal{S}^{\mathcal{C}}$ be the functor defined in Section 3.1. Since a topos is cartesian closed, by Proposition 3.2.7 we have all limits, so in particular the functor Δ has a right adjoint $\lim_{\mathcal{C}} : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}$. Then for Lemma 1.1.2 we have all finite colimits and thus in particular coequalizers, so by Proposition 3.2.6, Δ also has a left adjoint and therefore it preserves finite limits.

Remark 3.3.2. *For every topos $\mathcal{S}$ and every internal category $\mathcal{C}$ in $\mathcal{S}$ we have a geometric morphism*

$$\pi : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}$$

such that $\pi^* = \Delta$ and $\pi_* = \lim_{\mathcal{C}}$

Now we will consider the case of special internal categories which will be particularly useful throughout this thesis, namely internal filtered categories.

Definition 3.3.3. *Let $\mathcal{S}$ be a regular category and let $\mathcal{C} = (C_0, C_1, d, c, i, m)$ be an internal category in $\mathcal{S}$, we say that $\mathcal{C}$ is filtered if the following conditions hold*

1. *the unique morphism $C_0 \longrightarrow 1$ is a regular epimorphism;*
2. *The morphism $(ds_0, ds_1) : S \longrightarrow C_0 \times C_0$ is a regular epimorphism where (S, s_0, s_1) is a kernel pair of c;*
3. *Let (T, t_0, t_1) and (R, r_0, r_1) be the kernel pairs of $(\pi_1, m) : C_1 \times_{C_0} C_1 \longrightarrow C_1 \times C_1$ and $(d, c) : C_1 \longrightarrow C_0 \times C_0$ respectively. Let $t : T \longrightarrow R$ be the unique morphism such that $r_0 t = \pi_0 t_0$ and $r_1 t = \pi_0 t_1$. We ask t to be a regular epimorphism.*

Remark 3.3.4. *These axioms correspond in $\mathcal{S}et$ to the requirements*

1. $\exists C \in C_0$
2. $\forall A, B \in C_0 \exists f : A \longrightarrow C, g : B \longrightarrow C$
3. $\forall f, g : A \longrightarrow B \exists h : B \longrightarrow C$ *s.t.* $hf = hg$

which are precisely the axioms of a filtered small category.

Remark 3.3.5. *Suppose we have a functor $f : \mathcal{F} \longrightarrow \mathcal{E}$ between regular categories, suppose than that it preserves finite limits and finite colimits (e.g. the inverse image of a geometric morphism). Then let $\mathcal{C}$ be a filtered category in $\mathcal{F}$. Since f preserves finite limits, $f(\mathcal{C})$ is a category as noticed towards the end of Section 3.1 and since f also preserves finite colimits, then in particular it sends regular epimorphisms to regular epimorphisms, so $f(\mathcal{C})$ satisfies the axioms of a filtered category.*

Theorem 3.3.6. *Let $\mathcal{S}$ be an exact category with reflexive coequalizers, then a category $\mathcal{C}$ is filtered iff the functor* $\mathrm{colim}_{\mathcal{C}} : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}$ *preserves finite limits.*

Proof. See [SE] Theorem 2.6.8 where an exact category here is called effective regular. □

Remark 3.3.7. *Note that if $\mathcal{S}$ is a topos we are in the hypotheses of Theorem 3.3.6, so in particular if $\mathcal{C}$ is a filtered category, then in virtue of this theorem we have a geometric morphism*

$$\infty : \mathcal{S} \longrightarrow \mathcal{S}^{\mathcal{C}}$$

defined by $\infty^* = \mathrm{colim}_{\mathcal{C}}$ and $\infty_* = \Delta$. Sometimes we may call it $\infty_{\mathcal{C}}$ to specify the internal category we are using.

Note that ∞ is a section of $\pi : \mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}$ (see Remark 3.3.2) in $\mathfrak{Geom}$ (in the usual weak sense), in fact we have

$$(\pi\infty)^* = \infty^*\pi^* = \mathrm{colim}_{\mathcal{C}}\Delta$$

which is the identity functor on $\mathcal{S}$ (or at least naturally isomorphic to it) as we will prove below, but by essential uniqueness of the adjoint functor, also $(\pi\infty)_*$ must be isomorphic to the identity on $\mathcal{S}$. To prove this identity we can make explicitly the computation of the colimit given in Proposition 3.2.6. The colimit of an internal constant diagram Δ_X of shape $\mathcal{C}$ is $\pi_0 el(\Delta_X)$ and hence, the coequalizer of $(1_X \times d), (1_X \times c) : X \times C_1 \longrightarrow X \times C_0$ but since $\mathcal{S}$ is cartesian closed, the functor $X \times -$ preserves colimits and thus this colimit is $X \times \mathrm{Coeq}(d, c) = X \times \pi_0(\mathcal{C})$ and as proved in [SE] Lemma B2.6.4, $\pi_0(\mathcal{C}) = 1$ if $\mathcal{C}$ is filtered, so $\mathrm{colim}_{\mathcal{C}}\Delta_X \cong X$.
To better understand the notation of this geometric morphism, we need the following results

Theorem 3.3.8. *Let $\mathcal{S}$ be a topos, let $f : \mathcal{C} \longrightarrow \mathcal{D}$ be an internal functor in $\mathcal{S}$ and let $f^* : \mathcal{S}^{\mathcal{D}} \longrightarrow \mathcal{S}^{\mathcal{C}}$ as in Lemma 3.1.13, then f^* has both left and right adjoint called respectively* colim_f *and* $\lim_f$.
In particular we have a geometric morphism $\mathcal{S}^{\mathcal{C}} \longrightarrow \mathcal{S}^{\mathcal{D}}$ having f^* as inverse image and $\lim_f$ *as direct image.*

Proof. See [J] Theorem 2.34. □

Sometimes the left and right adjoint of f^* are called respectively left and right *internal Kan extensions* along f.
Moreover note that if $\mathcal{D} = 1$ the terminal internal category, then $f : \mathcal{C} \longrightarrow 1$ and f^* is up to equivalence the functor $\Delta : \mathcal{S} \longrightarrow \mathcal{S}^{\mathcal{C}}$ thanks to Remark 3.1.12 and the definition of Δ.

Corollary 3.3.9. *Let $\mathcal{S}$ be a topos, then the assignment sending an internal category $\mathcal{C}$ to $\mathcal{S}^{\mathcal{C}}$ and an internal functor f to the geometric morphism of Theorem 3.3.8 gives a pseudofunctor*

$$\mathcal{C}at(\mathcal{S}) \longrightarrow \mathfrak{Geom}/\mathcal{S}$$

Moreover if we precompose this pseudofunctor with the functor defined in Remark 3.1.5 we get up to isomorphisms the pseudofunctor

$$\mathcal{S} \longrightarrow \mathfrak{Geom}/\mathcal{S}$$

obtained from Theorem 1.1.10 (see Remark 1.5.17).

Proof. Corollary 2.35 of [J]. □

Actually this pseudofunctor can be extended as showed in [SE] after Corollary B2.3.22 to a pseudofunctor

$$\mathfrak{Cat}(\mathcal{S}) \longrightarrow \mathfrak{Geom}/\mathcal{S}$$

Remark 3.3.10. *Let $\mathcal{S}$ be a topos and $\mathcal{C}$ an internal category, consider its objects i.e. internal functors $x : 1 \longrightarrow \mathcal{C}$. Note that objects are the sections of the unique internal functor $\mathcal{C} \longrightarrow 1$ in $\mathcal{C}at(\mathcal{S})$.*
If we apply the pseudofunctor of Corollary 3.3.9, again thanks to Remark 3.1.12 we get that an object of $\mathcal{C}$ corresponds up to isomorphisms to a section $\mathcal{S} \longrightarrow \mathcal{S}^{\mathcal{C}}$ of the geometric morphism π of Remark 3.3.2.
So both these morphisms and the morphism ∞ of Remark 3.3.7 are sections of π. Now analysing the inverse image of these sections we get that a section s_x of π coming from an object x of $\mathcal{C}$ has as inverse image x^* which is a sort of evaluation of a diagram in x (in $\mathcal{S}et$ this is precisely what happens). On the other hand the inverse image of ∞ is $\mathrm{colim}_{\mathcal{C}}$ *and thus we may think of it as a sort of evaluation at infinity of a diagram, whence the name ∞.*

In Remark 3.1.14 we found that a functor preserving finite limits induces a functor between categories of internal diagrams, but if we have the regularity of toposes we have even more.

Lemma 3.3.11. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism of toposes and let $\mathcal{C}$ be an internal category in $\mathcal{E}$, then we have a geometric morphism*

$$f^{\mathcal{C}} : \mathcal{F}^{f^*(\mathcal{C})} \longrightarrow \mathcal{E}^{\mathcal{C}}$$

Proof. First of all, notice that both f^* and f_* preserve finite limits, so from the constructions made towards the end of Section 3.1 we have two functors

$$f^* : \mathcal{C}at(\mathcal{E}) \longrightarrow \mathcal{C}at(\mathcal{F})$$

$$f_* : \mathcal{C}at(\mathcal{F}) \longrightarrow \mathcal{C}at(\mathcal{E})$$

Moreover, since $f^* \dashv f_*$ by definition of geometric morphisms, we get that also these two functors are one adjoint to each other in the same way. The isomorphism for this adjunction is obtained directly by applying the isomorphism of the geometric morphism to each component of the functor and it is well defined by naturality. Now an adjunction between two categories induces an adjunction between slice categories as follows.

$$f^* \dashv f_* : \mathcal{C}at(\mathcal{F})/f^*(\mathcal{C}) \longrightarrow \mathcal{C}at(\mathcal{E})/\mathcal{C}$$

And moreover both these functors send discrete opfibrations to discrete opfibrations (still because f^* and f_* preserve pullbacks), so restricting to opfibrations we get the desired adjunction

$$(f^{\mathcal{C}})^* \dashv (f^{\mathcal{C}})_* : \mathcal{F}^{f^*(\mathcal{C})} \longrightarrow \mathcal{E}^{\mathcal{C}}$$

We can also give a more explicit construction of these adjoints. We have that $(f^{\mathcal{C}})^* : \mathcal{E}^{\mathcal{C}} \longrightarrow \mathcal{F}^{f^*(\mathcal{C})}$ is the functor obtained from f^* as showed in Remark 3.1.14 while $(f^{\mathcal{C}})_* : \mathcal{F}^{f^*(\mathcal{C})} \longrightarrow \mathcal{E}^{\mathcal{C}}$ is the composition of functors

$$\mathcal{F}^{f^*(\mathcal{C})} \longrightarrow \mathcal{E}^{f_*f^*(\mathcal{C})} \longrightarrow \mathcal{E}^{\mathcal{C}}$$

where the first is the one obtained from f_* again as in Remark 3.1.14 on the category $f^*(\mathcal{C})$ and the second is the functor obtained as in Lemma 3.1.13 from the functor $\eta_{\mathcal{C}} : \mathcal{C} \longrightarrow f_*f^*(\mathcal{C})$ which is the component in $\mathcal{C}$ of the unit of the adjunction defined before between categories of internal categories. More explicitly it is the functor with components η_{C_0} and η_{C_1} where $\eta : \mathrm{id}_{\mathcal{E}} \longrightarrow f_*f^*$ is the unit of the geometric morphism f.
Finally $(f^{\mathcal{C}})^*$ preserves finite limits because from Lemma 2.16 in [J] we have that finite limits in $\mathcal{E}^{\mathcal{C}}$ are computed as in $\mathcal{C}at(\mathcal{E})/\mathcal{C}$ and that here limits are computed componentwise. Therefore, since $(f^{\mathcal{C}})^*$ corresponds to the componentwise application of f^* to both components and it preserves finite limits, than also $(f^{\mathcal{C}})^*$ does. So we have just proved that $f^{\mathcal{C}}$ is a geometric morphism. □

The following diagram in $\mathfrak{Geom}$ commutes

$$\begin{array}{ccc} \mathcal{F}^{f^*(\mathcal{C})} & \xrightarrow{\pi_{f^*(\mathcal{C})}} & \mathcal{F} \\ {\scriptstyle f^{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{E}^{\mathcal{C}} & \xrightarrow[\pi_{\mathcal{C}}]{} & \mathcal{E} \end{array} \tag{3.1}$$

where $\pi_{\mathcal{C}}$ and $\pi_{f^*(\mathcal{C})}$ are the geometric morphisms defined in Remark 3.3.2. In fact explicitly $(\pi_{f^*(\mathcal{C})})^* f^* = \Delta f^*$ and $(f^{\mathcal{C}})^*(\pi_{\mathcal{C}})^* = f^*\Delta$, but since f^* preserves finite limits, we have $\Delta f^* = f^*\Delta$. Thus by essential uniqueness of the adjoint, the diagram of geometric morphisms commutes up to isomorphism. But actually we have more

Proposition 3.3.12. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism of toposes and $\mathcal{C}$ be an internal category in $\mathcal{E}$, then the diagram* (3.1) *is a pullback. In other words it is a product in $\mathfrak{Geom}/\mathcal{E}$.*

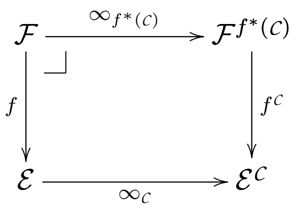
Proof. See [SE] Corollary B3.2.12. □

From this result we get the following two corollaries

Corollary 3.3.13. *The pseudofunctor $\mathfrak{Cat}(\mathcal{S}) \longrightarrow \mathfrak{Geom}/\mathcal{S}$ of Remark 3.3.9 preserves finite products.*

Proof. See [SE] Corollary B3.2.13. □

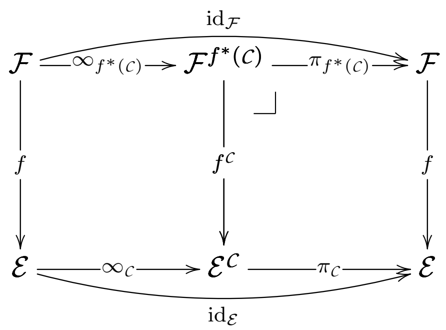
Corollary 3.3.14. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism of toposes and let $\mathcal{C}$ be a filtered internal category in $\mathcal{E}$, then we have the following pullback.*



Proof. First of all the morphism $\infty_{f^*(\mathcal{C})}$ is well defined because of Remarks 3.3.5 and 3.3.7. Then this diagram commutes up to isomorphism because the corresponding diagram of inverse images does so, for f^* preserves colimits. In fact given a diagram F in $\mathcal{E}^{\mathcal{C}}$ we have

$$\infty^* \left(f^{\mathcal{C}}\right)^* (F) = \mathrm{colim}_{f^*(\mathcal{C})} \left(f^{\mathcal{C}}\right)^* (F) = \pi_0 el \left(\left(f^{\mathcal{C}}\right)^* (F)\right) =$$
$$= \pi_0(f^*(el(F))) = f^*(\pi_0(el(F))) = f^*\mathrm{colim}_{\mathcal{C}} F = \infty^* f^*(F)$$

where the first three equalities and the last two are basically definitions and the fourth holds because f^* preserves colimits.
Now we have the following commutative diagram



Where the internal right and the external squares are pullbacks, so from the pullback pasting lemma we deduce that also the internal left diagram is a pullback, proving the thesis. □

3.4 Internal categories in toposes over a base

In the previous section we worked in $\mathfrak{Geom}$, now we will work inside $\mathfrak{Geom}/\mathcal{S}$ where $\mathcal{S}$ is a topos.
If $\mathcal{C}$ is an internal category in $\mathcal{S}$, then we can interpret it in every topos over $\mathcal{S}$. Namely if $p : \mathcal{E} \longrightarrow \mathcal{S}$ is a topos over $\mathcal{S}$, we can interpret $\mathcal{C}$ in $\mathcal{E}$ by taking $p^*(\mathcal{C})$. In the same way we can talk about diagrams in $\mathcal{E}$ of shape $\mathcal{C}$ by taking internal diagrams of shape $p^*(\mathcal{C})$. With a little abuse of notation we will call simply $\mathcal{E}^{\mathcal{C}}$ what we have so far called $\mathcal{E}^{p^*(\mathcal{C})}$. Note that this will not cause many troubles because with the previously used notation, the object $\mathcal{S}^{\mathcal{C}}$ coincides with the same object in the new notation, for clearly $\mathrm{id}^*_{\mathcal{S}}(\mathcal{C}) = \mathcal{C}$. Now consider a geometric morphism over $\mathcal{S}$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad f \quad} & \mathcal{E} \\ & {\scriptstyle p}\searrow \qquad \nearrow{\scriptstyle q} & \\ & \mathcal{S} & \end{array}$$

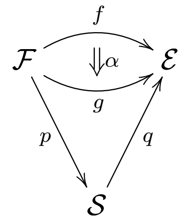
we will call $f^{\mathcal{C}}$ the geometric morphism over $\mathcal{S}^{\mathcal{C}}$ corresponding in the previous notation to

$$\begin{array}{ccc} \mathcal{F}^{p^*(\mathcal{C})} & \xrightarrow{\quad f^{q^*(\mathcal{C})} \quad} & \mathcal{E}^{q^*(\mathcal{C})} \\ & {\scriptstyle p^{\mathcal{C}}}\searrow \qquad \nearrow{\scriptstyle q^{\mathcal{C}}} & \\ & \mathcal{S}^{\mathcal{C}} & \end{array}$$

Note again that there is no serious danger of confusion because the only common case with the previous notation is when the codomain of f is $\mathcal{S}$ (here considered as terminal object in $\mathfrak{Geom}/\mathcal{S}$), but then the two notations represent the same arrow, for $f^{\mathcal{C}} : \mathcal{E}^{f^*(\mathcal{C})} \longrightarrow \mathcal{S}^{\mathcal{C}}$ coincides in meaning with $f^{\mathcal{C}} : \mathcal{E}^{\mathcal{C}} \longrightarrow \mathcal{S}^{\mathcal{C}}$.
With the new notation we have then

$$\begin{array}{ccc} \mathcal{F}^{\mathcal{C}} & \xrightarrow{\quad f^{\mathcal{C}} \quad} & \mathcal{E}^{\mathcal{C}} \\ & {\scriptstyle p^{\mathcal{C}}}\searrow \qquad \nearrow{\scriptstyle q^{\mathcal{C}}} & \\ & \mathcal{S}^{\mathcal{C}} & \end{array}$$

Let now $f, g : \mathcal{F} \longrightarrow \mathcal{E}$ be two geometric morphisms over $\mathcal{S}$, let $\mathcal{C}$ be an internal category in $\mathcal{S}$ and let $\alpha : f \Rightarrow g$ be a geometric transformation.



We can create a geometric transformation $\alpha^{\mathcal{C}} : f^{\mathcal{C}} \Rightarrow g^{\mathcal{C}}$ taking for every internal diagram F in $\mathcal{E}^{\mathcal{C}}$ as component $(\alpha^{\mathcal{C}})_F : (f^{\mathcal{C}})^*(F) \longrightarrow (g^{\mathcal{C}})^*(F)$ the arrow α_F. We have that $\alpha^{\mathcal{C}}$ is an internal natural transformation of diagrams and this follows from naturality of α, its compatibility with the isomorphisms $qf \cong p$ and $qg \cong p$ and the fact that both f^* and g^* preserve finite limits, so that $\alpha_{A \times B} = \alpha_A \times \alpha_B$.

Remark 3.4.1. *Note that the correspondences just defined give rise to a pseudofunctor*

$$(-)^{\mathcal{C}} : \mathfrak{Geom}/\mathcal{S} \longrightarrow \mathfrak{Geom}/\mathcal{S}^{\mathcal{C}}$$

We can also consider the pseudofunctor

$$\pi \circ - : \mathfrak{Geom}/\mathcal{S}^{\mathcal{C}} \longrightarrow \mathfrak{Geom}/\mathcal{S}$$

given by composition with π.

Remark 3.4.2. *From Proposition 3.3.12 and Corollary 3.3.14 we get the following two pseudonatural transformations.*

$$\pi : (\pi \circ -)(-)^{\mathcal{C}} \longrightarrow id_{\mathfrak{Geom}/\mathcal{S}}$$

$$\infty : id_{\mathfrak{Geom}/\mathcal{S}} \longrightarrow (\pi \circ -)(-)^{\mathcal{C}}$$

by taking for every $p : \mathcal{E} \longrightarrow \mathcal{S}$ topos over $\mathcal{S}$ the corresponding morphisms $\pi_{\mathcal{C}} : \mathcal{E}^{\mathcal{C}} \longrightarrow \mathcal{E}$ and $\infty_{\mathcal{C}} : \mathcal{E} \longrightarrow \mathcal{E}^{\mathcal{C}}$ and for every geometric morphism over $\mathcal{S}$ the corresponding invertible geometric transformation that gives commutativity in the two previously cited results.
Note also that they are such that $\pi\infty = \iota_{id_{\mathfrak{Geom}/\mathcal{S}}}$.

Thinking in these terms we have an interpretation for an internal category in a topos $\mathcal{S}$ inside every topos over $\mathcal{S}$. We can thus give the following definition

Definition 3.4.3. *Let $\mathcal{C}$ be an internal category in $\mathcal{S}$ and $p : \mathcal{E} \longrightarrow \mathcal{S}$ a topos over $\mathcal{S}$. We say that $\mathcal{E}$ has all (co)limits of shape $\mathcal{C}$ if it has all (co)limits of shape $p^*(\mathcal{C})$. We say that $\mathcal{E}$ is* $\mathcal{S}$-(co)complete *if it has all (co)limits of diagrams over every internal category in $\mathcal{S}$.*

An interesting case is when $\mathcal{S} = \mathcal{S}et$ because here we will see that $\mathcal{S}et$-(co)completeness corresponds to the standard small (co)completeness.
Let $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$ be a topos over $\mathcal{S}et$ and let $\mathcal{C}$ be a small category. We have two possible interpretations for the notation $\mathcal{E}^{\mathcal{C}}$: one is the usual category of diagrams over $\mathcal{C}$ i.e. of functors from $\mathcal{C}$ to $\mathcal{E}$ and natural transformations between them and the other is as the category of internal diagrams of shape $\gamma^*(\mathcal{C})$, so for the moment to distinguish them we will call the latter $\mathcal{E}^{\gamma^*(\mathcal{C})}$.
Consider a diagram $F : \mathcal{C} \longrightarrow \mathcal{E}$, we send it to an internal diagram $\widetilde{F}$ of $\mathcal{E}^{\gamma^*(\mathcal{C})}$ built as follows. Let $\widetilde{F} = \coprod_{x \in C_0} F(x)$, then note that from Remark 2.5.9 $\gamma^*(C_0) = \coprod_{x \in C_0} 1$ and the canonical inclusions are of the form $\gamma^*(x)$ where $x : 1 \longrightarrow C_0$ corresponds to the object x. Now for all $x : 1 \longrightarrow C_0$ object of C_0, there is a unique morphism from $F(x)$ to $\gamma^*(C_0)$ which factors through $\gamma^*(x) : 1 \longrightarrow \gamma^*(C_0)$, call such morphism $(p_0)_x$. Now we will define

$$p_0 = \coprod_{x \in C_0} (p_0)_x : \widetilde{F} \longrightarrow \gamma^*(C_0)$$

Let's now work in the topos $\mathcal{E}/\gamma^*(C_0)$, here we have the objects

$$p_0 : \widetilde{F} \longrightarrow \gamma^*(C_0) \qquad \gamma^*(d) : \gamma^*(C_1) \longrightarrow \gamma^*(C_0)$$

Consider their product $\widetilde{F} \times_{\gamma^*(C_0)} \gamma^*(C_1)$. Note that by definition of $\widetilde{F}$ and again from Remark 2.5.9, this object is

$$\left(\coprod_{x \in C_0} F(x)\right) \times_{\gamma^*(C_0)} \left(\coprod_{f \in \gamma^*(C_1)} 1\right)$$

Thus since $\mathcal{S}$ is cartesian closed and from Remark 1.1.12 coproducts in $\mathcal{E}/\gamma^*(C_0)$ remain the same in $\mathcal{E}$, we get that this product is isomorphic to

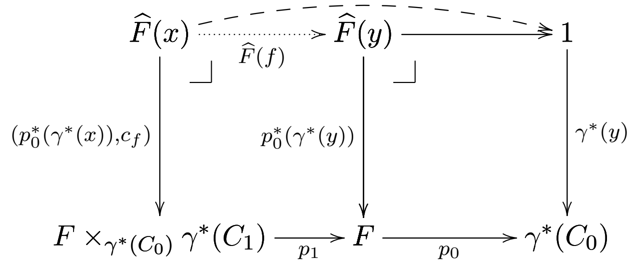
$$\coprod_{x \in C_0, f \in C_1, df = x} F(x)$$

So in order to define a morphism $p_1 : \widetilde{F} \times_{\gamma^*(C_0)} \gamma^*(C_1) \longrightarrow \widetilde{F}$ one can simply define one of the form $F(x) \longrightarrow \widetilde{F}$ for all $x \in C_0$ and $f \in C_1$ such that f has domain x. Let y be the codomain of f, then we choose $F(f) : F(x) \longrightarrow F(y)$ composed with the canonical inclusion of $F(y)$ in $\widetilde{F}$ to be this morphism. In this way we get the morphism p_1 that we were looking for. One can prove that $(\widetilde{F}, p_0, p_1)$ just defined is an internal diagram.
Given two functors $F, G : \mathcal{C} \longrightarrow \mathcal{E}$ and a natural transformation $\alpha : F \Rightarrow G$, one can also define an internal natural transformation $\widetilde{\alpha} : \widetilde{F} \Rightarrow \widetilde{G}$ by taking $\widetilde{\alpha} = \coprod_{x \in C_0} \alpha_x$. One can prove that $\widetilde{(-)}$ defines a functor.
On the other hand, given an internal diagram $F : \gamma^*(\mathcal{C}) \longrightarrow \mathcal{E}$ we can build a functor $\widehat{F} : \mathcal{C} \longrightarrow \mathcal{E}$ as follows. For all $x \in C_0$, $\widehat{F}(x)$ is obtained as pullback of p_0 along $\gamma^*(x) : 1 \longrightarrow \gamma^*(C_0)$. For all $f : 1 \longrightarrow C_1$, say f :

$x \longrightarrow y$, there exist a unique arrow $\widehat{F}(x) \longrightarrow \gamma^*(C_1)$ which factors through $\gamma^*(f) : 1 \longrightarrow \gamma^*(C_1)$ which we will call c_f. Now consider the morphism $(p_0^*(\gamma^*(x)), c_f) : \widehat{F}(x) \longrightarrow F \times_{\gamma^*(C_0)} \gamma^*(C_1)$, we have that

$$p_0 p_1(p_0^*(\gamma^*(x)), c_f) = c\pi_2(p_0^*(\gamma^*(x)), c_f) = cc_f$$

Now since the codomain of f is y, $\gamma^*(c)c_f$ factors through $\gamma^*(y) : 1 \longrightarrow \gamma^*(C_0)$ and thus the arrow $p_1(p_0^*(\gamma^*(y)), c_f)$ factors uniquely through $\widehat{F}(y)$ which is by definition the pullback of $\gamma^*(y)$ and p_0.



One can check that this defines a functor. Now given an internal natural transformation $\alpha : F \Rightarrow G$ between internal functors, we can build $\widehat{\alpha} : \widehat{F} \longrightarrow \widehat{G}$ by taking as $\widehat{\alpha}_x$ the pullback along $\gamma^*(x) : 1 \longrightarrow \gamma^*(C_0)$ of the morphism $\alpha : F \longrightarrow G$ over $\gamma^*(C_0)$. Again one can verify that this transformation is natural and that $\widehat{(-)}$ is a functor.

Remark 3.4.4. *We have that $\widetilde{(-)}$ and $\widehat{(-)}$ form an equivalence of categories and thus the category of internal diagrams in $\mathcal{E}$ over a small category $\mathcal{C}$ is equivalent to the category of functors from $\mathcal{C}$ to $\mathcal{E}$.*
Moreover one can prove that for every element $X \in \mathcal{E}$, the constant internal diagram Δ_X is such that $\widehat{\Delta_X}$ is the constant functor Δ_X, so the functor Δ defined in Section 3.1 is isomorphic to the usual functor $\Delta : \mathcal{E} \longrightarrow \mathcal{E}^{\mathcal{C}}$. We also deduce that, via this equivalence, internal (co)limits of diagrams over the internal category $\gamma^*(\mathcal{C})$ correspond to (co)limits over the corresponding diagrams of shape $\mathcal{C}$.
Thus a topos over $\mathcal{S}et$ is $\mathcal{S}et$-(co)complete iff it is (co)complete in the standard sense.

Another important question that we can ask is whether the direct image of a geometric morphism over a certain base preserves filtered colimits of a certain shape.
Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism over $\mathcal{S}$ and let $\mathcal{C}$ be an internal filtered category in $\mathcal{S}$. Suppose that both $\mathcal{F}$ and $\mathcal{E}$ have all colimits of diagrams of shape $\mathcal{I}$, then thanks to Corollary 3.3.14, the following diagram

commutes in $\mathfrak{Geom}$

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\infty} & \mathcal{F}^{\mathcal{C}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{\mathcal{C}}} \\ \mathcal{E} & \xrightarrow[\infty]{} & \mathcal{E}^{\mathcal{C}} \end{array} \tag{3.2}$$

we have a situation like (1.5) in Section 1.6, hence we have a 2-isomorphism between the direct images in the opposite direction $\alpha : (f^{\mathcal{C}})_* \infty_* \Rightarrow \infty_* f_*$. Using the definition of ∞ given in Remark 3.3.7 we get that this isomorphism is

$$\alpha : (f^{\mathcal{C}})_* \Delta \Rightarrow \Delta f_*$$

where Δ is the constant diagram functor.
Now we would like to internalize the idea of preservation of a certain colimit. What we are going to do is to follow the reasoning in set theoretical terms and apply it to the internal case. If $\mathcal{S} = \mathcal{S}et$, to see if f_* preserves limits of a certain diagram D, we consider the canonical map that we get from the universal property of colimits and than we see if it is an isomorphism. Explicitly we do as follows. We take the universal cocone i.e. $\eta_D : D \longrightarrow \Delta\text{colim}_{\mathcal{C}}$, then we apply the functor we are studying on this map of diagrams, i.e. f_*. Note that this operation corresponds to applying the functor $\left(f^{\mathcal{C}}\right)_*$ in the component D because we are working with diagrams. After this operation we get $\left(f^{\mathcal{C}}\right)_* (\eta_D) : \left(f^{\mathcal{C}}\right)_* (D) \longrightarrow \left(f^{\mathcal{C}}\right)_* (\Delta(\text{colim}_{\mathcal{C}}(D)))$ and here we can compose with the isomorphism $\alpha_{\text{colim}_{\mathcal{C}}(D)}$ to change the codomain in $\Delta f_* \text{colim}_{\mathcal{C}}(D)$. Now this is a cocone over $f_*\text{colim}_{\mathcal{C}}(D)$, so from the universal property of colimits (i.e. by adjunction) we get a map $\text{colim}_{\mathcal{C}} \left(f^{\mathcal{C}}\right)_* (D) \longrightarrow f_*\text{colim}_{\mathcal{C}}(D)$. Note that the adjoint is computed explicitly by applying $\text{colim}_{\mathcal{C}}$ and then composing with the counit at $f_*\text{colim}_{\mathcal{C}}(D)$. As recapitulation, the final map is

$$\epsilon_{f_*\text{colim}_{\mathcal{C}}(D)} \text{colim}_{\mathcal{C}}(\alpha_{\text{colim}_{\mathcal{C}}(D)} \left(f^{\mathcal{C}}\right)_* (\eta_D))$$

Note first that all the operations we did were natural in D, in fact we could have just omitted it and work with the natural transformations, obtaining that this map is the component at D of

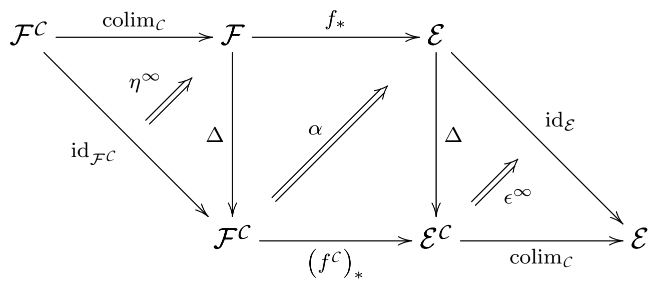
$$(\epsilon \circ f_* \circ \text{colim}_{\mathcal{C}}) \left(\text{colim}_{\mathcal{C}} \circ \left((\alpha \circ \text{colim}_{\mathcal{C}})(\left(f^{\mathcal{C}}\right)_* \circ \eta)\right)\right)$$

and hence

$$(\epsilon \circ f_* \circ \text{colim}_{\mathcal{C}})(\text{colim}_{\mathcal{C}} \circ \alpha \circ \text{colim}_{\mathcal{C}})(\text{colim}_{\mathcal{C}} \circ \left(f^{\mathcal{C}}\right)_* \circ \eta) \tag{3.3}$$

Secondly note that we never really used $\mathcal{S}et$, in fact we can formally repeat this construction for every $\mathcal{S}$ getting the same expression as (3.3).

Graphically the situation is the following



The natural transformation (3.3) is the canonical transformation that we were looking for and note that it is by construction the Beck-Chevalley transformation of the square (3.2), so we will call it θ.

Definition 3.4.5. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism in $\mathfrak{Geom}/\mathcal{S}$ and $\mathcal{C}$ an internal filtered category in $\mathcal{S}$. Let D be an object in $\mathcal{F}^{\mathcal{C}}$, we say that the direct image of f* preserves internal colimits *of D if the morphism θ_D is an isomorphism.*
We say that f_* preserves internal filtered colimits of shape $\mathcal{C}$ if it preserves the internal filtered colimits of the family $\mathcal{F}^{\mathcal{C}}$.
We say that f_* preserves internal filtered colimits if for every internal filtered category $\mathcal{C}$, it preserves all internal colimits of shape $\mathcal{C}$.

Remark 3.4.6. *Note that if $\mathcal{S}$ is $\mathcal{S}et$, then following the previous reasoning we have that f_* preserves the internal colimit of an internal filtered diagram D iff f_* preserves the colimit of the corresponding diagram $\widehat{D}$ in the usual sense.*

Remark 3.4.7. *As noticed before θ is the Back-Chevalley transformation of the diagram* (3.2)*, so it follows that f_* preserves colimits of a family $\mathcal{D}$ iff it satisfies the BC condition on $\mathcal{D}$.*
In particular then f_* preserves internal filtered colimits of shape $\mathcal{C}$ iff the square (3.2) *satisfies the BC condition.*

Chapter 4

Special geometric morphisms

In this section we start to deal with particular kinds of geometric morphism that often correspond to the topos theoretic generalization of geometrical properties. In particular in the first and in the last sections we will explore two factorizations of geometric morphisms: one with surjections and inclusions and the other with hyperconnected and localic maps. In both cases we will analyse the geometrical meaning and some properties as pullback stability. In the Second section instead we will internalize the concept of Grothendieck toposes in the case of toposes over a base different from $\mathcal{S}et$ by using the so called bounded morphisms. We will also introduce the idea of internal sites, internal locales and internal sheaves and for the first time in this thesis it will appear a hint of the relation between toposes and intuitionistic higher order logic.

4.1 Surjections and inclusions

We start this chapter by studying surjections and inclusion that, as the name suggests, aim to describe the properties of surjectivity and injectivity.

Definition 4.1.1. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism of toposes, we call it* surjection *if f^* is faithful and we call it* inclusion *if f_* is full and faithful. In some texts inclusions are called* embeddings.

Before going further we give the following example which will be used later on.

Example 4.1.2. *Let $\mathcal{E}$ be a topos and $f : A \longrightarrow B$ an epimorphism, then the geometric morphism $\phi : \mathcal{E}/A \longrightarrow \mathcal{E}/B$ (also denoted as $\mathcal{E}/f$) obtained by the fundamental Theorem of topos theory (Theorem 1.1.10) is a surjection. To prove it consider $x_1, x_2 : X \longrightarrow Y$ in $\mathcal{E}/B$, since ϕ^* is the pullback along*

f, we have for $i = 1, 2$ the following diagram

$$\begin{array}{ccc} A \times_B X & \xrightarrow{A \times_B x_i} & A \times_B Y \\ \downarrow{\scriptstyle \pi_X} & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[x_i]{} & Y \end{array}$$

Suppose that $\phi^*(x_1) = \phi^*(x_2)$, then $A \times_B x_1 = A \times_B x_2$ and thus in particular if we compose with π_Y we get

$$x_1 \pi_X = \pi_Y (A \times_B x_1) = \pi_Y (A \times_B x_2) = x_2 \pi_X$$

Now note that π_X is the pullback of f along the map $X \longrightarrow B$, hence it is an epimorphism, for in a topos epis are pullback stable. We deduce that $x_1 = x_2$ and thus ϕ^* is faithful, which means by definition that ϕ is a surjection.

We can give equivalent definitions thanks to the following

Lemma 4.1.3. *Let $L \dashv R : \mathcal{C} \longrightarrow \mathcal{D}$ be an adjunction, then*

1. *L is faithful iff the unit η of the adjunction is a monomorphism in every component*
2. *R is full and faithful iff the counit ϵ is an isomorphism*

Proof. 1. Let $h : A \longrightarrow B$ be an arrow in $\mathcal{D}$, then the mate of $\eta_B h$ is $\epsilon_{L(B)} L(\eta_B h) = ((\epsilon \circ L)(L \circ \eta))_B \, L(h) = L(h)$ for the triangular identities. Let now $h, k : A \longrightarrow B$ two arrows in $\mathcal{D}$, then $\eta_B h = \eta_B k$ iff $L(h) = L(k)$; this implies that if η_B is mono, then L is faithful and vice versa, so L is faithful iff every component of η is mono.

2. If we consider the dual of the previous case, we have that R is faithful iff every component of ϵ is an epimorphism. If R is also full, then in particular for all C in $\mathcal{C}$, there is a map $s : C \longrightarrow LR(C)$ such that $R(s) = \eta_{R(C)}$, so in particular if we take $s\epsilon_C$ and we transpose it, we get

$$R(s\epsilon_C)\eta_{R(C)} = R(s)(R \circ \epsilon)_C(\eta \circ R)_C = R(s) = \eta_{R(C)}$$

So transposing it back we get the identity and hence $s\epsilon_C = \mathrm{id}_{LR(C)}$, which means that s is a split monomorphism too and thus an isomorphism, making ϵ itself an isomorphism.
Conversely if ϵ is an isomorphism, then for all $k : R(A) \longrightarrow R(B)$, transposing k and precomposing it with ϵ_A^{-1} we get a map $h : A \longrightarrow B$ which is such that $R(h) = k$, for the transpose of $R(h)$ is $h\epsilon_A$ (follows from the dual of the computation made in the proof of the first point of this lemma) which coincides with the transpose of k. □

Therefore if η and ϵ are respectively unit and counit of the adjunction $f^* \dashv f_*$ given by a geometric morphism f, then f is a surjection iff η is a mono (in a topos a natural transformation is mono iff it is pointwise mono) while it is an inclusion iff ϵ is an iso.

Remark 4.1.4. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism and consider the functor $f^* : \mathcal{C}at(\mathcal{E}) \longrightarrow \mathcal{C}at(\mathcal{F})$ defined in Section 3.1. Thanks to Lemma 4.1.3(2) we have that the counit of f is an isomorphism so, by naturality, we have that an internal category $\mathcal{C}$ is isomorphic to one of the form $f^* f_*(\mathcal{C})$. This proves that $f^* : \mathcal{C}at(\mathcal{E}) \longrightarrow \mathcal{C}at(\mathcal{F})$ is essentially surjective.*

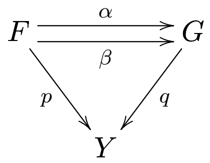
The choice of the names inclusion and surjection becomes visible if we restrict to the localic case, in fact surjections correspond to epimorphisms and inclusions correspond to regular monomorphisms, i.e. maps representing inclusions of sublocales.

Proposition 4.1.5. *Let $f : X \longrightarrow Y$ be a continuous map of locales, consider the geometric morphism $\mathcal{S}h(f) : \mathcal{S}h(X) \longrightarrow \mathcal{S}h(Y)$, we have that*

1. *f is a regular monomorphism iff $\mathcal{S}h(f)$ is an inclusion*
2. *f is an epimorphism iff $\mathcal{S}h(f)$ is a surjection*

Proof. 1. If f is a regular mono, then $f^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(Y)$ is surjective. Let now F, G in $\mathcal{S}h(X)$ and consider a natural transformation $\alpha : f_*(F) \Rightarrow f_*(G)$. By definition of f_*, we have that α is a natural transformation $F \circ f^* \Rightarrow G \circ f^*$. If we have a transformation $\beta : F \Rightarrow G$ such that $f_*(\beta) = \alpha$, then for every $U \in \mathcal{O}(X)$, it must be $\beta_U = \alpha_V$ for a $V \in \mathcal{O}(Y)$ such that $f^*(V) = U$ in fact then $\alpha_V = f_*(\beta)_V = \beta_{f^*(V)} = \beta_U$. We use this equality as a definition for β and once we prove that this is a well defined natural transformation, then we are done because it is the unique such that its image via f_* is α and hence f_* is full and faithful. To see that it is well defined, let V be such that $f^*(V) = U$, such a V exists for f^* is surjective. Now consider $V_m = \bigvee (f^*)^{-1}(U)$, since f is a map of locales, f^* preserves joins, so we also have $f^*(V_m) = U$, but we have $V \leq V_m$ and f_* sends this map on a map from U to U and hence to the identity of U. From naturality one gets that $\alpha_V = \alpha_{V_m}$ and hence β_U is well defined for all $U \in \mathcal{O}(X)$. Now to prove naturality, given $U_1 \leq U_2$ in $\mathcal{O}(X)$ let V_1, V_2 in the preimage via f^* of U_1 and U_2 respectively. We have that $f^*(V_1 \wedge V_2) = f^*(V_1) \wedge f^*(V_2) = U_1 \wedge U_2 = U_1$, so without loss of generality we can take a $V_1 \leq V_2$ (otherwise just take $V_1 \wedge V_2$ instead of V_1). Then the naturality of β in the map $U_1 \leq U_2$ follows from naturality of α in $V_1 \leq V_2$ and thus β is a natural transformation. We proved that f_* is full and faithful, so $\mathcal{S}h(f)$ is an inclusion.
Conversely if $\mathcal{S}h(f)$ is an inclusion, then the counit ϵ of $f^* \dashv f_*$ is an isomorphism, so for every U subterminal in $\mathcal{S}h(X)$ we have that $U \cong f^* f_*(U)$ and hence the open V in $\mathcal{O}(Y)$ corresponging to $f_*(U)$ is such that $f^*(V) = U$ proving that f^* is surjective, which implies that f is a regular inclusion.

2. Suppose f is an epimorphism, then consider F, G in $\mathcal{Sh}(Y)$, thanks to the interpretation of $\mathcal{Sh}(Y)$ as LH/Y (Theorem 2.4.3), they correspond to local homeomorphisms p and q over Y. Now consider $\alpha, \beta : F \longrightarrow G$, they correspond to morphisms in LH/Y as follows



Suppose now $f^*(\alpha) = f^*(\beta)$, then in this interpretation as local homeomorphisms, it means that the pullback of α and β along $f : X \longrightarrow Y$ is the same, and thus in particular α and β are equalized by $f^*(F) \longrightarrow F$, i.e. the pullback of f along p. Now thanks to Lemma 2.3.8, $f^*(F) \longrightarrow F$ is an epimorphism and hence $\alpha = \beta$ which means that f^* is faithful and hence that $\mathcal{Sh}(f)$ is a surjection.
Conversely if $\mathcal{Sh}(f)$ is a surjection, then f^* is faithful and thus it reflects isomorphisms because a topos is a *balanced category*, i.e. monomorphisms that are also epimorphisms are isomorphisms. If we restrict f^* to subterminal objects, thanks to the observation made after Remark 2.5.6 we get the map $f^* : \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$ which thus reflects isomorphisms. We have that this f^* is injective because if U and V are opens in Y such that $f^*(U) = f^*(V)$, then $f^*(U \wedge V) = f^*(U) \wedge f^*(V) = f^*(U)$ so, since $U \wedge V \leq U$ and $U \wedge V \leq V$ and these arrows are sent to isos by f^* which reflects isos, we have that $U = U \wedge V = V$. So f is a surjection as claimed. □

We state now some properties that will come in handy in the following chapters.

Lemma 4.1.6. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a surjection and $\mathcal{C}$ an internal category in $\mathcal{E}$, then $f^{\mathcal{C}}$ is a surjection as well.*

Proof. A geometric morphism is a surjection iff its inverse image is faithful by definition, but the inverse image of $f^{\mathcal{C}}$ is obtained by applying f^* to both components of the discrete opfibration defining an internal diagram over $\mathcal{C}$. Therefore since f^* is faithful, also $\left(f^{\mathcal{C}}\right)^*$ is such. □

Moreover, the properties of being a surjection and an inclusion are local, in fact

Proposition 4.1.7. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism, let E in $\mathcal{E}$, then*

1. *if f is an surjection then f/E is a surjection;*

2. *if f is an inclusion then f/E is an inclusion.*

Proof. 1. For the first case, if f^* is faithful, then since $(f/E)^*$ is simply the application of f^* to elements over E, it is faithful as well.

2. The direct image of f/E is the composition of f_* applied to objects over f^*E, which is faithful, and the unit of $f^* \dashv f_*$ which is an isomorphism because f is an inclusion. We thus have that $(f/E)_*$ is faithful as well, implying that f/E is an inclusion. □

Another interesting fact about inclusions and surjections is that as in topological spaces and locales we can factor every continuous map as a surjective one followed by a monomorphism (representing the inclusion of the image), we have a similar property also in toposes.

Theorem 4.1.8. *Every geometric morphism can be factored uniquely up to a canonical isomorphism as a surjection followed by an inclusion.*

Proof. See [SE] Theorem A4.2.10. □

This implies the following

Corollary 4.1.9. *If a geometric morphism is both an inclusion and a surjection, then it is an equivalence.*

Proof. Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be such a morphism, then we can see it as $f1_{\mathcal{F}}$ and $1_{\mathcal{E}}f$, both are surjection-inclusion factorization, so for the previous theorem they are canonically isomorphic. This in particular implies that there exists a map $e : \mathcal{E} \longrightarrow \mathcal{F}$ such that $fe \cong 1_{\mathcal{E}}$ and $ef \cong 1_{\mathcal{F}}$, so f is an equivalence. □

4.2 Bounded geometric morphisms

We often treat geometric morphisms $\mathcal{E} \longrightarrow \mathcal{S}$ as if $\mathcal{S}$ were a generalized universe of sets and $\mathcal{E}$ a $\mathcal{S}$-topos. In this interpretation we would like to describe what is the condition on such a geometric morphism which makes it resemble a sheaf topos over this $\mathcal{S}$, and thus such that this morphism plays the role of the partial section morphism. The aim of this section is therefore to adapt the notion of a Grothendieck topos to the general case of an $\mathcal{S}$-topos and then study some useful property that will come in handy in the remaining sections.
The property we need is boundedness

Definition 4.2.1. *A geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{S}$ is called* bounded *if there exists an object B in $\mathcal{E}$ called* bound *such that every object of $\mathcal{E}$ is the quotient of a subobject of $f^*(I) \times B$ for some I in $\mathcal{S}$.*

Example 4.2.2. *An example of bounded geometric morphisms are localic ones, for we can take as bound the terminal object.*

In the particular case of a geometric morphism $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$, as we have seen in Remark 2.5.9, $\gamma^*(I)$ is $\coprod_{i \in I} 1$ and $- \times B$ is a left adjoint, so in particular preserves colimits, and thus $f^*I \times B \cong B^I$. A subobject of B^I represented by the monomorphism $m : U \longrightarrow B^I$ is such that

$$U \cong \coprod_{i \in I} m^*(B_i)$$

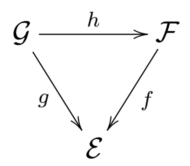
where B_i is the i-th component of the coproduct for $i \in I$.
If $\mathcal{E}$ is a Grothendieck topos and G a set of generators, then if we take $B = \coprod_{c \in G} c$, it is a bound for γ because given E, it is colimit of these generators and hence a quotient of the coproduct of some of them and as we have just seen, this coproduct is subobject of B.
Conversely, subobjects of B form a generating set and thus one can prove that $\mathcal{E}$ is a Grothendieck topos using Giraud's Theorem. We have thus sketched the proof of

Proposition 4.2.3. *A geometric morphism $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$ is bounded iff $\mathcal{E}$ is a Grothedieck topos.*

Proof. For a more complete proof see [SE] Theorem C2.2.8 and thus Example B3.1.8(b), or just wait until Giraud-Diaconescu Theorem is proved. □

Now that we have proved that boundedness is a good generalization of Grothendieck toposes, we will give some new property that wasn't visible in the latter case

Lemma 4.2.4. *Consider the following commutative diagram in $\mathfrak{Geom}$*

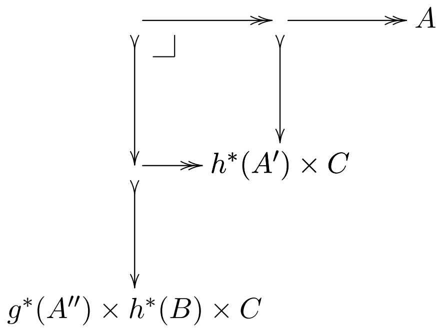


1. *If f and h are bounded, then also g is bounded.*

2. *If g is bounded, then h is bounded.*

Proof. 1. Let B be a bound for f and C a bound for h, then we claim that $h^*(B) \times C$ is a bound for g. For every $A \in G$, since h is bounded, there is a A' in $\mathcal{F}$ such that A is a subquotient of $h^*(A') \times C$, but also f is bounded, so there is an object A'' in $\mathcal{E}$ such that A' is a subquotient of $f^*(A'') \times B$. Note that both h^* and $- \times C$ are left adjoints and thus preserve epimorphisms. Moreover they both preserve pullbacks, so they also preserve monomorphisms and hence the functor $h^*(-) \times C$ preserves monos and epis. This implies that $h^*(A') \times C$ is a subquotient of $h^*(f^*(A'') \times B) \times C \cong$

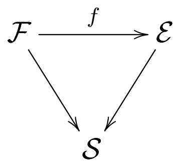
$g^*(A'') \times h^*(B) \times C$. Now combine the two subquotient diagrams that we have just studied and take the pullback as below



Since in a topos monos and epis are pullback stable, we have that A is a subquotient of $g^*(A'') \times h^*(B) \times C$, which proves our claim.

2. For this point instead, if B is a bound for g, then it is a bound also for h, in fact every object A in $\mathcal{G}$ can be written as $g^*(A') \times B \simeq h^* f^*(A') \times B$ for some A' in $\mathcal{E}$ and thus in particular $X = f^*(A')$ is such that A is a subquotient of $h^*(X) \times B$, proving that there is such an object in $\mathcal{F}$ and hence that h is bounded. □

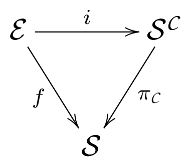
Remark 4.2.5. *It follows from Lemma 4.2.4(2) that if we fix a topos $\mathcal{S}$ and consider a morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ of toposes over $\mathcal{S}$, i.e.*



If the morphism $\mathcal{E} \longrightarrow \mathcal{S}$ is bounded, then also f must be a bounded geometric morhism.
In particular every geometric morphism between Grothendieck toposes is bounded.

We will state now a theorem which will point out another hint of the fact that bounded geometric morphisms are the right generalization of Grothendieck toposes to toposes over a base.

Theorem 4.2.6 (Giraud-Diaconescu)**.** *Let $\mathcal{S}$ be a topos and consider a geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{S}$, it is bounded iff there is an internal category $\mathcal{C}$ of $\mathcal{S}$ such that f factors in $\mathfrak{Geom}$ as*



where $\pi_{\mathcal{C}}$ is the geometric morphism of Remark 3.3.2 and i is an inclusion. Equivalently if we consider f and $\pi_{\mathcal{C}}$ as objects of $\mathfrak{Geom}/\mathcal{S}$ for every internal category $\mathcal{C}$ in $\mathcal{S}$, we have that f is bounded iff there is a morphism in $\mathfrak{Geom}/\mathcal{S}$ from f to $\pi_{\mathcal{C}}$ for some internal category $\mathcal{C}$.

Proof. See Theorem B3.3.4 of [SE]. □

Before commenting on this result, we note that Proposition 4.2.3 follows immediately from this theorem because in the case $\mathcal{S} = \mathcal{S}et$, an internal category $\mathcal{C}$ is a small category and $\mathcal{S}^{\mathcal{C}}$ is the usual functor category $\mathcal{S}et^{\mathcal{C}}$. Giraud-Diaconescu's Theorem states that f is bounded iff $\mathcal{E}$ has an inclusion in $\mathcal{S}et^{\mathcal{C}} = \mathcal{S}et^{(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}}$ for some small category $\mathcal{C}$ and hence iff it has an inclusion in some category of presheaves over a small category, which is equivalent to be a Grothendieck topos.
The name Giraud-Diaconescu comes from the fact that it was proved by Diaconescu, but its statement resembles the one of Giraud's Theorem. As we observed before Proposition 4.2.3 in fact, the condition of boundedness resembles the condition of having a set of generating objects (here substituted by the bound). The other two properties of Giraud's Theorem are implicit in that we are dealing with toposes and geometric morphisms rather than just categories.
Before closing this section, we give a corollary of Theorem 4.2.6 which will assure the presence of pullbacks of bounded maps and a sort of base change stability for boundedness

Corollary 4.2.7. *Let $f : \mathcal{E} \longrightarrow \mathcal{S}$ and $g : \mathcal{F} \longrightarrow \mathcal{S}$ be geometric morphisms such that f is bounded (respectively localic), then there exists in $\mathfrak{Geom}$ a pullback square*

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\;k\;} & \mathcal{F} \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ \mathcal{E} & \xrightarrow[\;f\;]{} & \mathcal{S} \end{array}$$

such that k is bounded (resp. localic).

Proof. For a complete proof see Proposition B3.3.6 of [SE]. The idea behind the bounded case is that we factor f as inclusion followed by $\pi_{\mathcal{C}}$ for some $\mathcal{C}$ in $\mathcal{C}at(\mathcal{S})$. We have proved in Proposition 3.3.12 that we can pull back $\pi_{\mathcal{C}}$ along g as $\pi_{g^*\mathcal{C}}$. One can then prove that inclusions are pullback stable, so that we conclude by pullback glueing Lemma and Giraud-Diaconescu's Theorem. □

A site is a model for an intuitionistic higher order theory and as such it can be internalized in every topos $\mathcal{S}$ getting the so called *internal sites* (See

[LS] for the interpretation of higher order logic in a topos). An explicit way to do it is to consider an internal category $\mathcal{C}$ and an indexing for covering families represented by an object J of $\mathcal{S}$ with two maps

$$\mathcal{C}_0 \xleftarrow{\ b\ } J \xrightarrow{\ c\ } \mathcal{P}(\mathcal{C}_1)$$

such that some properties are satisfied. We won't enter further in these details, but a more complete treatment can be found in [SE] at Section C2.4. Here we will just give an interpretation of these two maps b and c: the latter has a rule to associate to a cover the family of arrows that it represents, while the first sends a cover to the object it covers. In particular then we would like the object covered by a covering family to be the codomain of every arrow in the family that it represents. This can be translated in a higher order formula and we will require an internal site to satisfy it.
Once we have an idea of internal site in $\mathcal{S}$, we can develop the theory of sheaves in a similar way. We can in fact consider an internal site $(\mathcal{C}, J)$ in $\mathcal{S}$ and the category of presheaves, which in this environment is $\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$. Let F be in $\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$, we interpret in higher order logic the idea of a J-compatible family and the one of amalgamation, so that we can select those F which satisfy the sheaf condition and we call them *internal sheaves*. We denote the full subcategory of internal sheaves as $\mathcal{Sh}_{\mathcal{S}}(\mathcal{C}, J)$.
In the classical theory one gets that a geometric morphism $i : \mathcal{G} \longrightarrow \mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ is an inclusion iff there is a coverage J on $\mathcal{C}$ such that $\mathcal{G} \cong \mathcal{Sh}(\mathcal{C}, J)$ and hence iff it is a Grothendieck topos and i is isomorphic to its inclusion in the category of presheaves. Since these properties can be proved constructively, one can translate this result from $\mathcal{S}et$ to $\mathcal{S}$, so that inclusions in $\mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$ are precisely those inclusions of the form $\mathcal{Sh}_{\mathcal{S}}(\mathcal{C}, J) \longrightarrow \mathcal{S}^{\mathcal{C}^{\mathrm{op}}}$ for some internal site $(\mathcal{C}, J)$ in $\mathcal{S}$.
It follows from Giraud-Diaconescu's theorem that every bounded geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{S}$ is isomorphic to one of the form $\mathcal{Sh}_{\mathcal{S}}(\mathcal{C}, J) \longrightarrow \mathcal{S}$.
If we take a closer look at Corollary 4.2.7, using the ideas that we have just introduced, we can say moreover that we have

Theorem 4.2.8. *Let $f : \mathcal{E} \longrightarrow \mathcal{S}$ be a geometric morphism and $(\mathcal{C}, J)$ an internal sifted site in $\mathcal{S}$, then there is an internal site $(f^*(\mathcal{C}), f^{\#}J)$ in $\mathcal{E}$ such that the following diagram*

$$\begin{array}{ccc} \mathcal{Sh}_{\mathcal{E}}(f^*(\mathcal{C}), f^{\#}J) & \longrightarrow & \mathcal{Sh}_{\mathcal{S}}(\mathcal{C}, J) \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{S} \end{array}$$

is a pullback in $\mathfrak{Geom}$.

Proof. For a complete proof see Theorem C2.4.6 in [SE].
Here we will just describe explicitly the site $(f^*(\mathcal{C}), f^{\#}J)$. The underlying category $f^*(\mathcal{C})$ is obtained applying f^* to both components of $\mathcal{C}$ as showed at the end of Section 3.1. On this category $f^{\#}J$ is obtained as suggested by the following diagram

$$\mathcal{C}_0 \xleftarrow{\ b\ } f^*J \xrightarrow{\ \phi_{\mathcal{C}_1} f^*(c)\ } \mathcal{P}(f^*(\mathcal{C}_1))$$

where $\phi_{\mathcal{C}_1} : f^*(\mathcal{P}(\mathcal{C}_1)) \longrightarrow \mathcal{P}(f^*(\mathcal{C}_1))$ is the comparison morphism defined right before Definition 1.1.7. From this definition it turns out that such a cover is sifted and moreover every axiom of Grothendieck topology that J satisfies is satisfied also by $f^{\#}J$. $\square$

Thanks to this theorem we have a canonical way to pull back sheaves and sites in another base topos.

4.3 Hyperconnected and localic morphisms

In this section we are going to see another useful factorization of geometric morphisms, namely the hyperconnected-localic factorization. We have already presented localic geometric morphisms in Section 2.5, so now we will start with the other half of the factorization.

Definition 4.3.1. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism, we call it* hyperconnected *if f^* is full and faithful and the image of f^* is closed under subquotients, i.e. if in $\mathcal{F}$ an object A is quotient of a subobject of $f^*(B)$ for some B in $\mathcal{E}$, then $A = f^*(E)$ for some E in $\mathcal{E}$.*

The name hyperconnected is due to the fact that a geometric morphism such that its direct image is full and faithful is called *connected*. In turn, the name connected depends as expected on its geometric interpretation. A topological space X is connected if whenever X is the disjoint union of two of its open sets A and B, then either A or B is X. Such a definition can readily be translated in localic terms saying that a locale X is connected if the only complemented open in $\mathcal{O}(X)$ are the top and the bottom elements. Recall that an open U is complemented if there is an open V which is disjoint and such that $U \vee V = X$ and in the particular case of a Heyting algebra, if such a V exists it is $\neg U$. Sometimes[1] one asks that there are precisely two complemented objects, but in this way the initial locale is excluded. On the other hand, requiring that there are precisely two complemented objects gives the following characterization which explains the name connected geometric morphisms.

[1] For instance [SE].

Proposition 4.3.2. *Let X be a locale, then $\gamma : \mathcal{Sh}(X) \longrightarrow \mathcal{Set}$ is connected iff $\mathcal{O}(X)$ has precisely two complemented objects.*

Proof. See in [SE] Lemma C1.5.7. □

Consider now a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ and select in $\mathcal{F}$ the full subcategory $\mathcal{G}$ having as objects subquotients of objects in the image of f^*. This subcategory has two important properties contained in the following result.

Lemma 4.3.3. *In the situation defined right above*

1. *$\mathcal{G}$ is a coreflective subcategory of $\mathcal{F}$;*
2. *$\mathcal{G}$ is a topos and its inclusion in $\mathcal{F}$ preserves finite limits.*

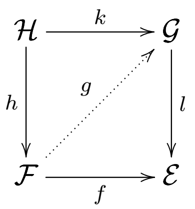
Proof. See [SE] Lemma A4.6.3. □

Note that f^* always factors through $\mathcal{G}$ because the latter is full and contains the image of f^*. In particular we have that f is hyperconnected iff f^* factors as an equivalence $\mathcal{E} \longrightarrow \mathcal{G}$. On one hand if f is hyperconnected, then f^* is full and faithful; further on, since every object A in $\mathcal{G}$ is a subquotient of an object in the image of f^* and since this image is closed under subquotients, an isomorphic copy of A is in the image of f^*, which thus is essentially surjective, implying that it is one half of an equivalence. Conversely if f^* factors as an equivalence $\mathcal{E} \longrightarrow \mathcal{G}$, then in particular f^* is also full and faithful for $\mathcal{G}$ is a full subcategory and since the image of f^* is up to isomorphisms $\mathcal{G}$ which contains subquotients of the image of f^*, the latter also contains an isomorphic copy of them.
We can now give the factorization theorem, even though we won't enter in the details of the proof.

Theorem 4.3.4. *Every geometric morphism of toposes factors as a hyperconnected map followed by a localic one and this factorization is unique up to equivalence.*

Proof. Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism and let $\mathcal{G}$ be as before the full subcategory of $\mathcal{F}$ of subquotients of elements in the image of f^*. We have seen in Lemma 4.3.3 that $\mathcal{G}$ is a topos and that it is coreflective in $\mathcal{F}$, which means that the inclusion functor has a right adjoint. Moreover, again from Lemma 4.3.3, the inclusion of $\mathcal{G}$ preserves finite limits, so we can define a geometric morphism $h : \mathcal{F} \longrightarrow \mathcal{G}$ such that h^* is the inclusion of $\mathcal{G} \longrightarrow \mathcal{F}$. Further, we noticed that f^* factors through h^*, so let's call l^* this factorization, which preserves finite limits (for f^* does so) and has a right adjoint $l_* = f_* h^*$ (for h^* is full and faithful). We thus have that $l = (l^* \dashv l_*) : \mathcal{G} \longrightarrow \mathcal{F}$ is a geometric morphism.
Note now that h is hyperconnected, in fact h^* is full and faithful and the

image of h^* is closed under subquotients because the subquotient of a subquotient is itself a subquotient as we can see in a diagram similar to the one in the proof of Lemma 4.2.4. On the other hand, l is localic, in fact every object of $\mathcal{G}$ is by definition subquotient of one in the image of f^* which coincides with the image of l^* by definition. Making computations on inverse images, we get thus that $f \simeq lh$, so every geometric morphism can be factored as a hyperconnected morphism followed by a localic one.
The essential uniqueness of this factorization follows from the fact that hyperconnected and localic morphisms satisfy an orthogonality condition as showed in [SE] at Lemma A4.6.4, which means that for every commutative square in $\mathfrak{Geom}$ as the one in the diagram below



(4.1)

where h is hyperconnected and l localic, there is an essentially unique map g as above making the whole diagram (4.1) commutative, usually called *lifting*[2].
Let in fact $lh \cong l'h'$ be two hyperconnected-localic factorizations of the same arrow, then we can build a diagram as (4.1) in four ways: taking as morphisms k, h, l, f respectively h, h', l, l' in the first, h', h, l', l in the second, h, h, l, l in the third and h', h', l', l' in the fourth. Note that in the last two the identity makes for a good lift, while in the first two we get two morphisms respectively g and g'. Such morphisms are each other's quasi-inverse, in fact gg' is also a lift for the third diagram while $g'g$ is a lift for the fourth, making them isomorphic to identities by uniqueness of the lift. □

For completeness and further reference we just cite the following result about the hyperconnected-localic factorization which corresponds to Corollary C2.4.12 in [SE].

Corollary 4.3.5. *The hyperconnected-localic factorization of an arbitrary (resp. bounded) geometric morphism is stable under pullback along bounded (resp. arbitrary) geometric morphism in* $\mathfrak{Geom}$.

Note that geometrically hyperconnected morphisms are not that interesting, in fact consider a geometric morphism built from a continuous map of locales; it is already localic as one can see proving the equivalent of Lemma 4.2.4(2) in the localic case (just repeat the same proof with bound 1). Thanks to Theorem 4.3.4 such a map must be an equivalence and therefore it must come from an isomorphism of locales thanks to the results in

[2]The problem to find a lifting for a diagram of this form is called *lifting problem*.

Section 2.5.
In fact this shows that hyperconnected morphisms are the farthest morphisms from localic ones, hence those defined from a continuous map of locales.

Note that as we did for bounded maps in the end of Section 4.2, we can describe localic maps with internal sites. With localic maps though we have a luckier situation, in fact instead of internal sites we can just use internal locales.
Locales are the same as frames and these are models for a higher order theory. If instead of taking models of this theory in $\mathcal{S}et$ we take them in a topos $\mathcal{S}$ we get the so called *internal frames* (which can also be seen as internal categories) and hence *internal locales*. The theory developed for locales in $\mathcal{S}et$ can be translated in terms of $\mathcal{S}$, as soon as it is built in constructive terms. Therefore, as for internal sheaves cited at the end of Section 4.2, we have the concept of sheaf over an internal locale X in $\mathcal{S}$ and the full subcategory of $\mathcal{S}^{\mathcal{O}(X)^{\mathrm{op}}}$ having as objects these sheaves is denoted with $\mathcal{S}h_{\mathcal{S}}(X)$.
Since the proof of Theorem 2.5.11 is constructive, it is valid even if instead of $\mathcal{S}et$ we have another topos $\mathcal{S}$, so we obtain the following result.

Corollary 4.3.6. *Every localic geometric morphism $\mathcal{E} \longrightarrow \mathcal{S}$ is equivalent to one of the form $\mathcal{S}h_{\mathcal{S}}(X) \longrightarrow \mathcal{S}$ for some internal locale X in $\mathcal{S}$.*

Chapter 5

Compact toposes and proper morphisms

Finally we arrived at the core of this thesis. In this chapter we will explore the concept of compactness following essentially the first chapter of [MV]. After reviewing the classical definitions, we will extend the concept of compactness to toposes over $\mathcal{Set}$, giving some explicit example along with some characterization in familiar situations. With the second section we will then take this generalization a step further treating the case of toposes over a base, obtaining in this way the definition of proper geometric morphisms. We will proceed by studying first some intrinsic property of proper morphisms such as stability by composition and localization, and then we will develop the interaction between proper maps and those geometric morphisms studied in Chapter 4. We will then give a characterization of compact toposes using pretopos sites introduced in Section 1.4. The generalization of this characterization to the case of proper maps will enable us to prove the pullback stability of propriety, which in turn will permit another characterization of proper morphisms involving the weak Beck-Chevalley condition of Section 1.6. We will end this chapter with a display of possible developments of the theory of proper maps that we didn't treat in detail in this work.

5.1 Compactness

In this first section we aim to extend the definition of compactness first to locales and then to toposes over $\mathcal{Set}$.

Classically the definition of compact topological space is the following.

Definition 5.1.1. *A topological space X is* compact *if for every open covering there is a subcovering which is finite. Explicitly for every family $\mathcal{U} \subseteq \mathcal{O}(X)$ such that $\bigcup \mathcal{U} = X$, there exist a finite subset $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}$ such that $U_1 \cup \cdots \cup U_n = X$.*

The first thing that we notice about this definition is that it only depends on the lattice of open subsets, therefore it is easily generalizable to locales as follows

Definition 5.1.2. *A locale X is* compact *if for every open covering there is a subcovering which is finite, which in this case it means that for every family $\mathcal{U} \subseteq \mathcal{O}(X)$ such that $\bigvee \mathcal{U} = X$, there exist a finite subset $\{U_1, \ldots, U_n\} \subseteq \mathcal{U}$ such that $U_1 \vee \cdots \vee U_n = X$.*

Remark 5.1.3. *Note that since the Definitions 5.1.1 and 5.1.2 are formally the same and only depend on the lattice of opens, we have that a topological space is compact if and only if it is compact as a locale.*

In order to speak of compactness in toposes, we have to translate this definition on a more comfortable way. First of all we will show that we can restrict our study of coverings to filtered ones, where the existence of a finite subcover corresponds to the existence of the maximal open in it. Then we will see how to translate this in terms of preservation of filtered colimits.
In fact we have the following

Theorem 5.1.4. *Let X be a locale, then the following are equivalent*

1. *X is compact;*
2. *every filtered open cover of X contains X;*
3. *the direct image of the (unique) geometric morphism $\gamma : \mathcal{Sh}(X) \longrightarrow \mathcal{Set}$ preserves filtered colimits of subterminal objects.*

But before proving it, we need the following two results that will allow us to see filtered colimits of $\mathcal{O}(X)$ as filtered colimits in $\mathcal{Sh}(X)$.

Lemma 5.1.5. *Let $\gamma : \mathcal{E} \longrightarrow \mathcal{Set}$ be a topos over sets and let $\mathcal{T}$ be the full subcategory of $\mathcal{E}$ of its subterminal objects. Then the inclusion functor $\mathcal{T} \longrightarrow \mathcal{E}$ creates filtered colimits.*

Proof. The inclusion functor of a full subcategory creates a colimit if and only if the colimit taken in the bigger category is already contained in the smaller one. For this reason we will have to prove that the colimit of a filtered subterminal diagram is again subterminal.
Let I be a small filtered category and let $D : I \longrightarrow \mathcal{T}$ be a diagram of subterminal objects. We can also see it as a functor $I \longrightarrow \mathcal{E}$ which we will call with the same name. For all $i \in I$ there is a unique mono $D(i) \longrightarrow 1$, so we have a natural transformation $\alpha : D \Rightarrow \Delta_1$ which is a mono in $\mathcal{E}^I$.
We are done if we manage to prove that the functor $\mathrm{colim}_I : \mathcal{E}^I \longrightarrow \mathcal{E}$ preserves finite limits, in fact this functor sends $\alpha : D \longrightarrow \Delta_1$ to the unique arrow $\mathrm{colim}_I D \longrightarrow 1$ and if colim_I preserves finite limits, it also preserves

monos, proving that $\mathrm{colim}_I D$ is subterminal.
We might prove this fact directly, but we already have a result for the internal case, so we will use that. We can see $\mathcal{I}$ as internal category in $\mathcal{S}et$ for Remark 3.3.4 and thus thanks to Remark 3.3.5, $\gamma^*(\mathcal{I})$ is an internal filtered category in $\mathcal{E}$. Now thanks to Theorem 3.3.6 the functor $\mathrm{colim}_{\gamma^*(\mathcal{I})} : \mathcal{E}^{\gamma^*(\mathcal{I})} \longrightarrow \mathcal{E}$ of internal colimits preserves finite limits and finally, thanks to Remark 3.4.4, also the functor $\mathrm{colim}_I : \mathcal{E}^I \longrightarrow \mathcal{E}$ of colimits preserves finite limits. □

Thanks to Proposition 2.5.4, we have an equivalence $e : \mathcal{T} \longrightarrow \mathcal{O}(X)$ and thanks to Lemma 5.1.5 we have that the colimit in $\mathcal{S}h(X)$ of a filtered diagram $D : \mathcal{I} \longrightarrow \mathcal{S}h(X)$ of subterminal objects is also subterminal, so it makes sense to consider $e(\mathrm{colim}_I)$. The following corollary clarifies what this object is.

Corollary 5.1.6. *Let X be a locale and $D : \mathcal{I} \longrightarrow \mathcal{S}h(X)$ be a filtered diagram of subterminal objects, then*

$$e(\mathrm{colim}_I D) = \bigvee_{i \in I_0} e(D(i))$$

in $\mathcal{O}(X)$.

Proof. Thanks to Lemma 5.1.5, the colimit of D in $\mathcal{S}h(X)$ is also the colimit of D in $\mathcal{T}$. Then e is an equivalence so in particular it preserves colimits, therefore $e(\mathrm{colim}_I D) = \mathrm{colim}_I eD$. The last colimit is taken in $\mathcal{O}(X)$ where every colimit is the join of the components of the diagram, so $\mathrm{colim}_I D = \bigvee_{i \in I_0} e(D(i))$ and thus we have our thesis. □

Now we are finally ready to prove Theorem 5.1.4.

Proof (of Theorem 5.1.4). $1 \Rightarrow 2$: If X is compact, then in particular if we take a filtered open cover $\mathcal{U}$, we can find a finite subcover $\{U_1, \ldots, U_n\}$. Since $\mathcal{U}$ is filtered, there exist an open $U \in \mathcal{U}$ containing all of the U_i for $i = 1, \ldots n$, but the join of these U_i's is X so we have that U must be X.

$2 \Rightarrow 1$: Let $\mathcal{U}$ be a covering for X, then consider the family $\mathcal{V} = \{\bigvee \mathcal{F} | \mathcal{F} \subseteq \mathcal{U} \text{ finite}\}$. First note that since the singletons of $\mathcal{U}$ are finite subsets, $\mathcal{U} \subseteq \mathcal{V}$ and therefore in particular, also $\mathcal{V}$ covers X. Moreover $\mathcal{V}$ is filtered because we can index it with the lattice $\mathcal{P}_f(\mathcal{U})$ of finite subsets of $\mathcal{U}$ using the join as indexing map. We notice first that if $\mathcal{F}_1 \subseteq \mathcal{F}_2$ in $\mathcal{P}_f(\mathcal{U})$, then $\bigvee \mathcal{F}_1 \leq \bigvee \mathcal{F}_2$ and secondly for all $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{P}_f(\mathcal{U})$, we have that $\mathcal{F}_1 \cup \mathcal{F}_2$ is greater than both $\mathcal{F}_1$ and $\mathcal{F}_2$ and thus $\bigvee(\mathcal{F}_1 \cup \mathcal{F}_2)$ is greater than $\bigvee \mathcal{F}_1$ and $\bigvee \mathcal{F}_2$, so that $\mathcal{V}$ is a filtered cover for X. Condition (2) implies that X is contained in $\mathcal{V}$ and hence that there is a finite subset $\mathcal{F}$ of $\mathcal{U}$ such that $\bigvee \mathcal{F} = X$, but this is precisely the condition required in (1). Since this holds for every covering $\mathcal{U}$, then also (1) holds.

$2 \Leftrightarrow 3$ If we interpret $\mathcal{S}et$ as $\mathcal{S}h(1)$, from Proposition 2.5.4 we have two equivalences

$$e : \mathcal{T} \longrightarrow \mathcal{O}(X)$$
$$e' : \mathcal{T}' \longrightarrow \mathcal{O}(1)$$

where as before $\mathcal{T}$ and $\mathcal{T}'$ are the full subcategories of subterminal objects in $\mathcal{S}h(X)$ and $\mathcal{S}h(1)$ respectively. Since γ_* is a right adjoint, it preserves limits, so in particular it preserves subterminal objects.
Let now $D : I \longrightarrow \mathcal{S}h(X)$ be a filtered diagram of subterminal objects, thanks to the previous deduction we have that $\gamma_* D$ lands inside $\mathcal{T}$. Now in Lemma 5.1.5 we got that the colimit of a diagram of subterminal objects is also subterminal, so $\mathrm{colim}_I D$ and $\mathrm{colim}_I \gamma_* D$ are subterminals in $\mathcal{S}h(X)$ and $\mathcal{S}h(1)$ respectively and hence also $\gamma_*(\mathrm{colim}_I D)$ is subterminal in $\mathcal{S}h(1)$.
Translated explicitly, the condition (3) of this theorem says that for every D as before we have

$$\gamma_*(\mathrm{colim}_I D) \cong \mathrm{colim}_I \gamma_* D \tag{5.1}$$

but for what we have just deduced, this is an isomorphism of subterminal objects of $\mathcal{S}h(1)$, so it holds if and only if it holds in $\mathcal{T}'$ for it is a full subcategory. Now e' is an equivalence and thus preserves and reflects isomorphisms, so (5.1) holds if and only if $e'\gamma_*(\mathrm{colim}_I D) = e'\mathrm{colim}_I \gamma_* D$ holds (note that the isomorphism becomes an equality because in $\mathcal{O}(1)$ the identities are the only isomorphisms). Using Corollary 5.1.6 we have that $e'\mathrm{colim}_I \gamma_* D = \bigvee_{i \in I_0} e'\gamma_* D(i)$. Note that now we are working inside $\mathcal{O}(1)$ which is isomorphic to the ordinal 2 and has only two possible objects: 0 and 1. Therefore to prove an equality is equivalent to prove that the first member of that equality is 1 if and only if the second member of the equality is 1, so in particular (5.1) is equivalent to prove that $e'\gamma_*\mathrm{colim}_I D$ is 1 if and only if $\bigvee_{i \in I_0} e'\gamma_* D(i)$ is 1. Further, in $\mathcal{O}(1)$, a join is 1 iff it is 1 in at least one component, so (5.1) is equivalent to the following condition: $e'\gamma_*(\mathrm{colim}_I D) = 1$ iff $\exists i \in I_0$ such that $e'\gamma_* D(i)$ is 1. Since e' reflects terminals, this condition holds precisely when $\gamma_*\mathrm{colim}_I D = 1$ iff $\exists i \in I_0$ such that $\gamma_* D(i)$ is 1.
Now we want to prove that from the latter condition we can delete γ_* so we claim that (5.1) becomes equivalent to ask that $\mathrm{colim}_I D$ is terminal iff $\exists i \in I_0$ such that $D(i)$ is terminal. Note first that thanks to Remark 2.5.9, we know that γ_* is the covariant representable functor $\mathcal{S}h(X)(1, -)$. Thanks to Proposition 1.6.8 (3) we have that for every subterminal U, it is terminal iff there is a map $1 \longrightarrow U$, so iff $\gamma_*(U) \simeq \mathcal{S}h(X)(1, U) \neq 0$ and hence iff $\gamma_*(U) = 1$, for $\gamma_*(U)$ must be subterminal in $\mathcal{S}h(X) = \mathcal{S}et$ where the only non initial subterminal objects are terminal. Hence we have proven the previous claim.
Now we can apply the equivalence e, for it preserves and reflects colimits

and limits and we get that (5.1) holds precisely when $e\mathrm{colim}_{I}D$ is terminal iff $\exists i \in I_0$ such that $eD(i)$ is terminal. Note first that now we are working in $\mathcal{O}(X)$ where the terminal is X and secondly we can apply again Corollary 5.1.6 so that (5.1) becomes $\bigvee_{i\in I_0} eD(i) = X$ iff $\exists i \in I_0$ such that $eD(i) = X$. In other words this means that the family $\{eD(i)|i \in I_0\}$ covers X iff it contains X but since one of these implications always happens, at the end of the day, to prove (5.1) is equivalent to prove that if the family $\{eD(i)|i \in I_0\}$ covers X, then it must contain X.
Now there is a striking similarity to condition (2) of this theorem, in fact suppose that (2) holds. To prove condition (3) we have to prove that for every filtered diagram $D : I \longrightarrow \mathcal{Sh}(X)$ of subterminal objects, the isomorphism (5.1) holds, but thanks to the previous deductions, this corresponds to prove that for every such D, if the family $\{eD(i)|i \in I_0\}$ covers X, then it must contain X. But now if D is filtered, then also eD is filtered and thus in particular $\{eD(i)|i \in I_0\}$ is filtered, so thanks to (2) it contains X proving thus (5.1) for every D and hence (3).
Conversely suppose that (3) holds, if we have a filtered cover $\mathcal{U}$ for X, then we can see it as a filtered full subcategory of $\mathcal{O}(X)$ and thus the inclusion $\iota : \mathcal{U} \longrightarrow \mathcal{O}(X)$ is a filtered diagram in $\mathcal{O}(X)$. Let $q : \mathcal{O}(X) \longrightarrow \mathcal{T}$ be a quasi inverse of e, then $q\iota$ is a filtered diagram in $\mathcal{T}$ and hence it can be seen as a filtered diagram $D : \mathcal{U} \longrightarrow \mathcal{Sh}(X)$ of subterminal objects. Now, since (3) holds, then (5.1) holds for D and this also means that if the family $\{eD(U)|U \in \mathcal{U}\}$ covers X, then it must contain X. Note that $eD = eq\iota = \iota$ again because in $\mathcal{O}(X)$ isomorphisms are identities, therefore the previous family becomes $\{\iota(U)|U \in \mathcal{U}\}$ and thus it is $\mathcal{U}$. But $\mathcal{U}$ covers X and thus it must contain X. Since this holds for all filtered covers, then we proved (2). Hence (2) and (3) are equivalent. □

Remark 5.1.7. *The previous theorem also holds if instead of X locale we write X topological space, because of Remark 5.1.3 and the fact that $\mathcal{Sh}(X) = \mathcal{Sh}(X_l)$ for every topological space X as observed in Example 1.2.8.*

Using the previous theorem we can give the following

Definition 5.1.8. *Let $\gamma : \mathcal{E} \longrightarrow \mathcal{Set}$ be a topos over $\mathcal{Set}$, we say that it is* compact *if the direct image of γ preserves filtered colimits of subterminal objects.*

Remark 5.1.9. *A direct consequence of Theorem 5.1.4 is that a locale (or a topological space) X is compact if and only if $\mathcal{Sh}(X)$ is compact as topos over $\mathcal{Set}$.*

Example 5.1.10. *Let I be a set and consider the topos $\mathcal{Set}/I$, we can see it as the category of I indexed families, that is the category having as objects sequences $(A_i)_{i\in I}$ of sets and as maps sequences of index preserving functions $(f_i)_{i\in I}$. This can be deduced directly from Remarks 3.1.12 and*

3.4.4. Thanks to the observations made in Example 2.4.1, we have that $\mathcal{S}et^I$ is equivalent to the category of sheaves over a discrete locale, i.e. over a discrete topological space over I. This implies that $\mathcal{S}h(I)$ is compact iff I is compact as a topological space and this happens iff I is finite.

For a category of presheaves we can find the following characterization.

Proposition 5.1.11. *Let $\mathcal{C}$ be a small category and consider the category of presheaves over it. Then $\mathcal{S}et^{\mathcal{C}^{op}}$ is compact iff $\mathcal{C}$ has a finite* final set of objects*, that is a set F of objects in $\mathcal{C}_0$ such that for all $d \in \mathcal{C}_0$ there exists $c \in F$ and an arrow $f \in \mathcal{C}(d, c)$.*

Proof. Note first that since monomorphisms are natural transformations with injective components, a subterminal presheaf is a presheaf of subterminal sets, so their components are either empty or contain exactly one element. Let $U \longrightarrow 1$ be a subterminal object, then we can consider the set of objects $S(U) \subseteq \mathcal{C}_0$ containing the objects c such that $U(c) = 1$. In other words the objects of the subcategory equalizer of U and Δ_1. Note that this is not just a subcategory, but it also has the following property: for all $f : d \longrightarrow c$, if c is in $S(U)$, then also d is. We will denote such a property with (c). Conversely, every subset S of $\mathcal{C}$ with (c) defines a subterminal functor, $\chi(S)$ which sends an object $d \in S$ to 1 if $d \in S$ and to 0 otherwise. For the value of $\chi(S)$ on arrows instead we have a unique possible choice for we are dealing with subterminal sets. We only have to check that such a map exists, hence that for a map $f : c \longrightarrow d$, we don't get $\chi(S)(d) = 1$ and $\chi(S)(c) = 0$. This case never happens though because S satisfies (c), in fact if $\chi(S)(d) = 1$, then $d \in S$ and thus for (c), because of f, also c is in S and hence $\chi(S)(c) = 1$.
If we consider the full subcategory $\mathcal{T}$ of subterminal presheaves in $\mathcal{S}et^{\mathcal{C}^{op}}$ and the ordered set $\mathcal{P}$ of subsets of $\mathcal{C}$ with (c) ordered by inclusion, we can extend S and χ to functors

$$S : \mathcal{T} \longrightarrow \mathcal{P}$$
$$\chi : \mathcal{P} \longrightarrow \mathcal{T}$$

Namely if we have a natural transformation $\alpha : U \Rightarrow V$ in $\mathcal{T}$, then if $c \in S(U)$, it means that $U(c) = 1$ and thus, since we have $\alpha_c : U(c) \longrightarrow V(c)$, also $V(c)$ must be 1, so $c \in S(V)$ and therefore $S(U) \subseteq S(V)$. For χ instead, if $A \subseteq B$ in $\mathcal{P}$, then for all $c \in \mathcal{C}$, if $\chi(A)(c) = 0$ we have a unique map $\alpha_c : \chi(A)(c) \longrightarrow \chi(B)(c)$ and if $\chi(A)(c) = 1$, then $c \in A \subseteq B$ and hence $\chi(B)(c) = 1$, so we can define $\alpha_c = \mathrm{id}_1$. The family $(\alpha_c)_{c \in \mathcal{C}_0}$ is a natural transformation because the only objects appearing are 0 and 1 which admit at most one map between them. Call this natural transformation $\chi(A \subseteq B)$. Observe that χ and S form an equivalence of categories because $\chi S = \mathrm{id}_{\mathcal{P}}$ while $S\chi(U) \cong U$ for all subterminal object by definition and these isomorphisms are natural because of Proposition 1.6.8 (1).

Following the reasoning used in the proof of Theorem 5.1.4 we have that γ_* preserves filtered colimits of subterminals iff for every filtered diagram D of subterminals, whenever $\gamma_*(\mathrm{colim}_I D) = 1$, there exist $i \in I_0$ such that $\gamma_*(D(i)) = 1$. But again thanks to Lemma 5.1.5, $\mathrm{colim}_I D$ is subterminal and for a subterminal object U, to have $\gamma_*(U) = 1$ means precisey that $U = 1$ because of Proposition 1.6.8 (3). So γ_* preserves filtered colimits iff for all D as above, $\mathrm{colim}_I D = 1_{\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}$ iff there is $i \in I_0$ s.t. $D(i) = 1_{\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}}$.
The functor S is an equivalence, so this condition holds iff it holds when we apply S. Namely $\mathrm{colim}_I SD = \mathcal{C}$ iff $\exists i \in I_0$ s.t. $S(D(i)) = \mathcal{C}$ because S creates limits and colimits.
Note that union of subsets with (c) still satisfies (c), so in particular the colimit of a family $\{A_i\}_{i\in I}$ is $\bigcup_{i\in I} A_i$. This implies that $\mathcal{S}et^{\mathcal{C}}$ is compact iff any filtered cover of $\mathcal{C}_0$ by subcategories with (c), already contains $\mathcal{C}_0$.
If $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ is compact, consider for every finite subset $\mathcal{F} \subset \mathcal{C}_0$, the subset $D(\mathcal{F}) \subset \mathcal{C}_0$ which closes $\mathcal{F}$ under (c), hence containing all the $d \in \mathcal{C}_0$ such that there is a morphism $f \in \mathcal{C}(d, c)$ for some $c \in \mathcal{F}$.
Note that this family defines a filtered diagram $D : \mathcal{P}_f(\mathcal{C}_0) \longrightarrow \mathcal{P}$ where $\mathcal{P}_f(\mathcal{C}_0)$ is the lattice of finite subsets of $\mathcal{C}_0$. (The reasoning behind this is similar to the one used in the beginning of the proof of Theorem 5.1.4) Moreover note that for every object $d \in \mathcal{C}_0$, it is inside $D(\{d\})$, so the family $\{D(\mathcal{F})|\mathcal{F} \in \mathcal{P}_f(\mathcal{C}_0)\}$ covers $\mathcal{C}_0$ and hence, since $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ is compact, there is a finite subset $\mathcal{F}$ of $\mathcal{C}_0$ such that $D(\mathcal{F}) = \mathcal{C}_0$. So there is a finite set of objects $\mathcal{F}$ such that for all $d \in \mathcal{C}$, there is a morphism $f : d \longrightarrow c$ and thus $\mathcal{F}$ is a final set of objects.
Conversely if $\mathcal{C}$ has a finite final set of objects $\mathcal{F}$, then for every filtered cover $\{D(i)|i \in I\}$ for $\mathcal{C}_0$, there is a finite subset $\{i_c|c \in \mathcal{F}\}$ of I such that for all $c \in \mathcal{F}$, we have $c \in D(i_c)$. Since it was a filtered cover, there is an element i such that $D(i) \supseteq D(i_c)$ for all $c \in \mathcal{F}$ and thus such that $\mathcal{F} \subseteq D(i)$, but then for all d there is an element $c \in \mathcal{F}$ and an arrow $f : d \longrightarrow c$, so $d \in D(i)$. This means that $D(i)$ covers $\mathcal{C}_0$ and since this holds for every filtered cover of $\mathcal{C}_0$, we have that $\mathcal{S}et^{\mathcal{C}^{\mathrm{op}}}$ is compact. □

In particular then we have the following interesting examples.

Example 5.1.12. *1. If $\mathcal{C}$ has a terminal object, then the corresponding topos of presheaves over it is compact. Hence for example the category of simplicial sets is compact for* 1 *is terminal in the simplicial category. Recall that the category of* simplicial sets *is the category of presheaves over the* simplicial categoty Δ *where objects are finite ordinals and a map $f : n \longrightarrow m$ is any order preserving map.*

2. If $\mathcal{C}$ is a group or a monoid G, then the category of presheaves over it is a compact topos. Moreover we know also that $\mathcal{S}et^{G^{op}}$ is isomorphic to the category G-$\mathcal{S}et$ of G-sets, i.e. the category of right actions and equivariant *maps between them (functions $f : X \longrightarrow Y$ between G-sets such that $f(xg) = f(x)g$ for all $x \in X$ and $g \in G$).*

3. Let ω be the first infinite ordinal (isomorphic to $\mathbb{N}$ with the natural order), then $\mathcal{S}et^{\omega}$ is not compact because ω does not have a finite final set of objects, in fact for every finite set F, there is always an element $n \in \omega$ such that $c < n$ for all $c \in F$ and there are no arrows from n to any of the elements of F by antisymmetry.

5.2 Proper maps

We often interpret a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ as an object in $\mathfrak{Geom}/\mathcal{E}$ and thus as a topos over $\mathcal{E}$. The main idea that will lead us in the creation of a suitable definition of proper geometric morphism is to generalize the concept of compactness to toposes over a base different from $\mathcal{S}et$. In other words, a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ will be a proper morphism of toposes if it renders $\mathcal{F}$ compact as a topos over $\mathcal{E}$ in a suitable sense.
First let's analyse the condition of compactness from the internal point of view. The objects appearing in the definition of compact topos are (small) filtered categories, colimits and subterminal objects. Subterminal objects remain subterminal objects also internally and using ideas from Section 3.4 we can translate also colimits and small filtered categories as internal colimits and internal filtered categories.

Remark 5.2.1. *Diagrams of subterminal objects are precisely subterminal diagrams. Let D be a diagram of subterminals over a category $\mathbb{I}$, then we have a monomorphism to the terminal Δ_1. Conversely, if D is a subterminal diagram, then for every object $x : 1 \longrightarrow \mathbb{I}$, the component at x of the monomorphism inside the terminal diagram is computed as the pullback of such mono along x and therefore it is a monomorphism.*

Consider now an internal filtered category $\mathbb{I}$ in $\mathcal{S}et$, thanks to Corollary 3.3.14 we have the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\infty_{\gamma^*(\mathbb{I})}} & \mathcal{E}^{\gamma^*(\mathbb{I})} \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma^{\mathbb{I}}} \\ \mathcal{S}et & \xrightarrow[\infty_{\mathbb{I}}]{} & \mathcal{S}et^{\mathbb{I}} \end{array} \tag{5.2}$$

whose commutativity represents the fact that γ^* preserves colimits.
Let now D be an internal subterminal diagram D in $\mathcal{E}^{\gamma^*(\mathbb{I})}$ (corresponding by the previous observations to a filtered diagram of subterminals). The condition that γ_* preserves the colimit of D can also be translated internally as showed in Section 3.4. Explicitly we have the same case of Definition 3.4.5

where now the f in that definition is $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$ in $\mathfrak{Geom}/\mathcal{S}et$.
So, as showed in Remark 3.4.7, we have that γ_* preserves filtered colimits of subterminal diagrams iff it satisfies the Beck-Chevalley condition at the subterminal diagrams.
It would seem that we are done internalizing the condition of compactness, but the problem is that as it is this definition is not necessarily a local property, which is what we need for our purpose. To solve this problem we ask that the commutativity of internal colimits holds for all the localizations of f. So finally we can give the following

Definition 5.2.2. *A geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ is said to be* proper *if for any object E in $\mathcal{E}$ and for every filtered internal category I, the square*

$$\begin{array}{ccc} \mathcal{F}/f^*(E) & \xrightarrow{\infty_{(f/E)^*(I)}} & (\mathcal{F}/f^*(E))^{(f/E)^*(I)} \\ {\scriptstyle f/E}\downarrow & & \downarrow{\scriptstyle (f/E)^I} \\ \mathcal{E}/E & \xrightarrow[\infty_I]{} & (\mathcal{E}/E)^I \end{array} \tag{5.3}$$

satisfies the BC condition on all subterminal objects, i.e. if θ is the Beck-Chevalley transformation for the previous diagram, then for every subterminal diagram D in $(\mathcal{F}/f^*(E))^{f^*I}$, we have that

$$\theta_D : (\infty_I)^*(f/E)^I_*(D) \longrightarrow (f/E)_* \left(\infty_{(f/E)^*(I)}\right)^* (D)$$

is an isomorphism.
If $f : \mathcal{F} \longrightarrow \mathcal{E}$ is proper we also say that $\mathcal{F}$ is compact *as a $\mathcal{E}$-topos.*

We begin by checking that this definition is coherent with the purpose of generalizing compactness.

Proposition 5.2.3. *Let $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$ be a topos over $\mathcal{S}et$, then it is compact iff γ is proper.*

Proof. If γ is proper, then choosing $E = 1$ in $\mathcal{S}et$ we have that the diagram (5.3) becomes

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\infty_{\gamma^* I}} & \mathcal{E}^{\gamma^* I} \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \gamma^I} \\ \mathcal{S}et & \xrightarrow[\infty_I]{} & \mathcal{S}et^I \end{array} \tag{5.4}$$

because slicing under the terminal object is a neutral operation. Thus in particular, by definition of proper map, we have that the BC transformation θ is an isomorphism at every D subterminal diagram in $\mathcal{E}^{\gamma^* I}$.
Now translating everything in set theoretical facts using Remarks 3.4.4,

5.2.1 and 3.4.7 we get that for every filtered (small) category $\mathbb{I}$ and every diagram of subterminal objects D, the functor γ_* preserves the colimit of D. Therefore $\mathcal{E}$ is compact.
Conversely, suppose $\mathcal{E}$ is compact, then reversing what we have just said we have that for every subterminal diagram D in $\mathcal{F}^{\gamma^*\mathbb{I}}$, the square (5.4) satisfies the BC condition at D. We are left to prove that for every set E, this property holds also on the slice diagram.
First note that $\mathcal{S}et/E \cong \mathcal{S}et^E$ and $\mathcal{E}/E \cong \mathcal{E}^{\gamma^*(E)}$ for Remark 3.1.12. Now note that from the definition of $(\gamma^E)^* : \mathcal{S}et^E \longrightarrow \mathcal{E}^{\gamma^*(E)}$ given in Remark 3.1.14, we can observe that it is isomorphic to $(\gamma/E)^* : \mathcal{S}et/E \longrightarrow \mathcal{E}/\gamma^*(E)$, for in both cases the map corresponds to the application of γ^* at every object and arrow. Note also that thanks to Remark 2.5.9 this map corresponds to the map $(\gamma^*)^E : \mathcal{S}et^E \longrightarrow \mathcal{E}^E$, i.e. the map of product categories which is γ^* in every component.
We can also see that an internal (filtered) category in $\mathcal{S}et^E$ is a family of small (filtered) categories indexed by E and the corresponding functor $((\gamma^*)^E)^{\mathbb{I}}$ is isomorphic to $\prod_{e\in E}(\gamma^{\mathbb{I}})^*$. All the right adjoints, colimits, limits, units and counits of the adjunctions are computed componentwise, so the diagram of the form (5.3) with $f = \gamma$ and $\mathbb{I}$ filtered in $\mathcal{S}et/E$ is isomorphic to the product for $e \in E$ of the diagrams of the form (5.4) where here $\mathbb{I}$ is actually $\mathbb{I}_e$, so the e-th component of the internal category $\mathbb{I}$.
Note that again because all the objects are computed pointwise, we have that the product diagram satisfies BC condition for all subterminal diagrams and filtered categories iff every component does. Since every component of $\mathbb{I}$ is again filtered and every component of a subterminal diagram in $\mathcal{E}/\gamma^*(E) \cong \mathcal{E}^E$ is again subterminal, then we have that all the latter squares satisfies the BC condition in this case and hence so does the product diagram, proving that γ is proper □

As said before we would like propriety to be a local property, and in fact

Proposition 5.2.4. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism. If f is proper, then for every object E of $\mathcal{E}$, the map f/E is proper.*

Proof. Let f be proper, then we want to prove that f/E is proper. Using the definition, we have to prove that for every object $p : A \longrightarrow E$ in $\mathcal{E}/E$, we have that the morphism $(f/E)/p$ is equivalent to $f/A : \mathcal{F}/f^*(A) \longrightarrow \mathcal{E}/A$, but then the square

$$\begin{array}{ccc}
(\mathcal{F}/f^*(E))/(f/E)^*(p) & \xrightarrow{\infty_{((f/E)/p)^*\mathbb{I}}} & ((\mathcal{F}/f^*(E))/(f/E)^*(p))^{f^*\mathbb{I}} \\
\Big\downarrow{\scriptstyle (f/E)/p} & & \Big\downarrow{\scriptstyle ((f/E)/p)^{\mathbb{I}}} \\
(\mathcal{E}/E)/p & \xrightarrow[\infty_{\mathbb{I}}]{} & ((\mathcal{E}/E)/p)^{\mathbb{I}}
\end{array}$$

is equivalent the one corresponding to the localization of f at A, so explicitly it is of the form (5.3) with A instead of E and hence it satisfies the BC condition for subterminal objects because f is proper. It follows by definition that f/E is proper too. □

Before going further with the other results about proper morphisms, we make a remark that will simplify considerably the following proofs.

Remark 5.2.5. *1. Consider a weakly commutative square in* $\mathfrak{Geom}$

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{h} & \mathcal{G} \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle g} \\ \mathcal{F} & \xrightarrow[k]{} & \mathcal{E} \end{array}$$

let θ be the corresponding Beck-Chevalley transformation, then to prove that θ_D is an isomorphism for a certain subterminal D in $\mathcal{G}$, it is enough to show that

$$k^*g_*(D) \cong f_*h^*(D) \tag{5.5}$$

The reason behind this fact lays in Proposition 1.6.8, in fact direct and inverse images of geometric morphisms preserve finite limits, so in particular $f_*h^*(D)$ is a subterminal object. Thanks to the first point of said proposition, we have a unique map $k^*g_*(D) \longrightarrow f_*h^*(D)$ and since we have (5.5) *and θ_D, it follows that the isomorphism* (5.5) *is given precisely by the component of the Beck-Chevalley transformation in D.*

In particular we have that a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ is proper iff for every internal filtered category $\mathcal{C}$ in $\mathcal{E}$, for every object E in $\mathcal{E}$ and for every subterminal diagram D in $(\mathcal{F}/f^*(E))^{(f/E)^*(\mathcal{C})}$, we have

$$\infty_{\mathcal{C}}^*(((f/E)^{\mathcal{C}})_*(D) \cong (f/E)_*\infty_{(f/E)^*(\mathcal{C})}^*(D)$$

2. We can even do more, in fact if we manage to find an arrow $e : f_*h^*(D) \longrightarrow k^*g_*(D)$, since $k^*g_*(D)$ is as well subterminal, then $\theta_D e$ and $e\theta$ are forced to be identities.

For the particular case above, the morphism f is proper iff for all $\mathcal{C}$, E and D as above we can find a morphism

$$(f/E)_*\infty_{(f/E)^*(\mathcal{C})}^*(D) \longrightarrow \infty_{\mathcal{C}}^*(((f/E)^{\mathcal{C}})_*(D)$$

Thanks to this remark we can prove the following proposition which gives us two elementary properties of proper maps

Proposition 5.2.6. *1. every equivalence of toposes is proper.*

2. *if f, g are composable proper geometric morphisms, then fg is proper too.*

Proof. 1. Let $e : \mathcal{F} \longrightarrow \mathcal{E}$ be an equivalence of toposes. Note that the slicing of an equivalence is still an equivalence, so we are done if we prove that every equivalence preserves internal filtered colimits of subterminal diagrams. In particular, thanks to Remark 5.2.5 we just need to prove that for all filtered internal categories $\mathcal{C}$ in $\mathcal{E}$ and for all D subterminal in $\mathcal{F}^{e^*(\mathcal{C})}$ we have $(\infty_{\mathcal{C}})^*(e^{\mathcal{C}})_*(D) \cong e_*(\infty_{e^*(\mathcal{C})})^*(D)$. More explicitly we have to prove $\text{colim}_{\mathcal{C}}(e^{\mathcal{C}})_*(D) \cong e_*\text{colim}_{e^*(\mathcal{C})}(D)$. Thanks to Proposition 3.2.6 we know that the functors colim are computed as coequalizers of maps which correspond one another via e_*. This implies that the previous isomorphism holds if e_* preserves coequalizers, but this is true because e_* is one half of an equivalence.

2. Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ and $g : \mathcal{G} \longrightarrow \mathcal{F}$, let E be an object in $\mathcal{E}$, let $\mathcal{C}$ be an internal filtered category in $\mathcal{E}/E$ and let D be a subterminal in $(\mathcal{G}/(fg)^*(\mathcal{C}))^{(fg)^*(\mathcal{C})}$. Note first that $(fg)/E = (f/E)(g/f^*(E))$ as one can see combining Proposition 3.3.12 and Remark 3.1.12, in fact one can see the localization as the pullback along π_E for the discrete internal category E and use the pullback pasting Lemma. Moreover we have $((fg)/E)^{\mathcal{C}} = (f/E)^{\mathcal{C}}(g/f^*(E))^{(f/E)^*(\mathcal{C})}$, so

$$\begin{aligned} &\text{colim}_{\mathcal{C}}\left(((fg)/E)^{\mathcal{C}}\right)_*(D) \cong \\ &\cong \text{colim}_{\mathcal{C}}\left((f/E)^{\mathcal{C}}\right)_*\left((g/f^*(E))^{(f/E)^*(\mathcal{C})}\right)_*(D) \cong \\ &\cong (f/E)_*\text{colim}_{(f/E)^*(\mathcal{C})}\left((g/f^*(E))^{(f/E)^*(\mathcal{C})}\right)_*(D) \end{aligned}$$

because f is proper, and $\left((g/f^*(E))^{(f/E)^*(\mathcal{C})}\right)_*$ preserves subterminals. Since g is proper and $(f/E)^*$ preserves filtered internal categories we have

$$\begin{aligned} &(f/E)_*\text{colim}_{(f/E)^*\mathcal{C}}\left((g/f^*(E))^{(f/E)^*(\mathcal{C})}\right)_*(D) \cong \\ &\cong (f/E)_*(g/f^*(E))_*\text{colim}_{(f/E)^*\mathcal{C}}(D) \cong \\ &\cong ((fg)/E)_*\text{colim}_{(f/E)^*\mathcal{C}}(D) \end{aligned}$$

So pasting these isomorphisms together we get that fg is proper. □

Thanks to Remark 5.2.5 we can also prove a sort of inversion of Proposition 5.2.4 given by the following result.

Proposition 5.2.7. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism and let $\{e_j : E_j \longrightarrow 1 | j \in J\}$ be a jointly epimorphic family of arrows in $\mathcal{E}$ such that f/E_j is proper for all $j \in J$. If either $\mathcal{E}$ is a Grothendieck topos or J is finite, then also f is proper.*

Proof. Note that whether $\mathcal{E}$ is a Grothendieck topos or J is finite, then the coproduct $\coprod_{j\in J} E_j$ exists and this is what we need for our proof.
Thanks to the universal property of coproduct, from a family of the form $\mathcal{U} = \{e_j : E_j \longrightarrow 1 | j \in J\}$ we get a unique arrow $e : E = \coprod_{j\in J} E_j \longrightarrow 1$ that precomposed with the j-th inclusion gives e_j for all $j \in J$. Note that by definition and universal property of coproduct, the family $\mathcal{U}$ is jointly epimorphic iff e is an epimorphism.
Now we divide the proof in two steps: first we prove that if f/E_j is proper for all $j \in J$, then $f/\coprod_{j\in J} E_j$ is proper and secondly that if f/E is proper and $E \longrightarrow 1$ is an epimorphism, then f is proper.

Let's start from the first step, so we have that f/E_j is proper for all $j \in J$. We can define the following functor

$$\mathcal{E}/\coprod_{j\in J} E_J \xrightarrow{p} \prod_{j\in J} \mathcal{E}/E_j$$

where the product is taken in categories. On objects p is defined by sending $A \longrightarrow \coprod_{j\in J}$ to the list $(A \times_{\coprod_{j\in J} E_j} E_j | j \in J)$, hence in the component j it is the pullback along the inclusion of E_j in the coproduct. On arrows this functor is defined by universal property of pullback.
We can also build a functor q in the opposite direction by sending a list $(A_j | j \in J)$, where A_j is an object over E_j for all $j \in J$, to $\coprod_{j\in J} A_j$ which is an object over $\coprod_{j\in J} E_j$ (the universal property of coproduct defines it uniquely on maps).
Thanks to the pullback stability of coproducts in toposes, we have that p and q are each other's quasi-inverse, in fact $\coprod_{j\in J}(A \times E_j) \cong A$. Instead of studying $\mathcal{E}/\coprod_{j\in J} E_j$ we can thus study $\prod_{j\in J} \mathcal{E}/E_j$.
Now note that an internal filtered category $\mathcal{C}$ in $\prod_{j\in J} \mathcal{E}/E_j$ is a list $(\mathcal{C}_j | j \in J)$ where $\mathcal{C}_j$ is an internal filtered category in $\mathcal{E}/E_j$ for all j. Note also that f^* preserves coproducts, so $\coprod_{j\in J} f^* E_j$ makes sense in $\mathcal{F}$ and it is isomorphic to $f^*(\coprod_{j\in J} E_j)$. Checking the definitions of diagram categories, we can see that the diagram

$$\begin{array}{ccc}
\mathcal{F}/f^*(\coprod_{j\in J} E_j) & \xrightarrow{\infty_{f^*\mathcal{C}}} & (\mathcal{F}/f^*(\coprod_{j\in J} E_j))^{f^*\mathcal{C}} \\
\downarrow{\scriptstyle f/\coprod_{j\in J} E_j} & & \downarrow{\scriptstyle (f/\coprod E_j)^{\mathcal{C}}} \\
\mathcal{E}/\coprod_{j\in J} E_j & \xrightarrow[\infty_{\mathcal{C}}]{} & (\mathcal{E}/\coprod_{j\in J} E_j)^{\mathcal{C}}
\end{array} \tag{5.6}$$

is isomorphic to the product of diagrams of the form

$$\begin{array}{ccc}
\mathcal{F}/f^*(E_j) & \xrightarrow{\infty_{f^*\mathcal{C}_j}} & (\mathcal{F}/f^*(E_j))^{f^*\mathcal{C}_j} \\
\downarrow{\scriptstyle f/E_j} & & \downarrow{\scriptstyle (f/E_j)^{\mathcal{C}_j}} \\
\mathcal{E}/E_j & \xrightarrow[\infty_{\mathcal{C}_j}]{} & (\mathcal{E}/E_j)^{\mathcal{C}_j}
\end{array} \tag{5.7}$$

for all $j \in J$. Let now U be a subterminal in $(\mathcal{F}/f^*(\coprod_{j\in J} E_j))^{f^*C} \cong \prod_{j\in J}\left((\mathcal{F}/f^*(E_j))^{f^*C_j}\right)$, then it is of the form $(U_j|j \in J)$ where U_j is subterminal in $\mathcal{E}/E_j$. Since f/E_j is proper for all j, we have that

$$\infty_{C_j}^*((f/E_j)^{C_j})^*(U_j) \cong (f/E_j)_*\infty_{f^*C_j}^*(U_j)$$

and hence thanks to the previous description of the square (5.6) as products of the (5.7)'s for all $j \in J$, we also have

$$(\infty_C)^*\left(\left(f/\coprod E_j\right)^C\right)_*(U) \cong \left(f/\coprod_{j\in J} E_j\right)_*(\infty_{f^*C})^*(U)$$

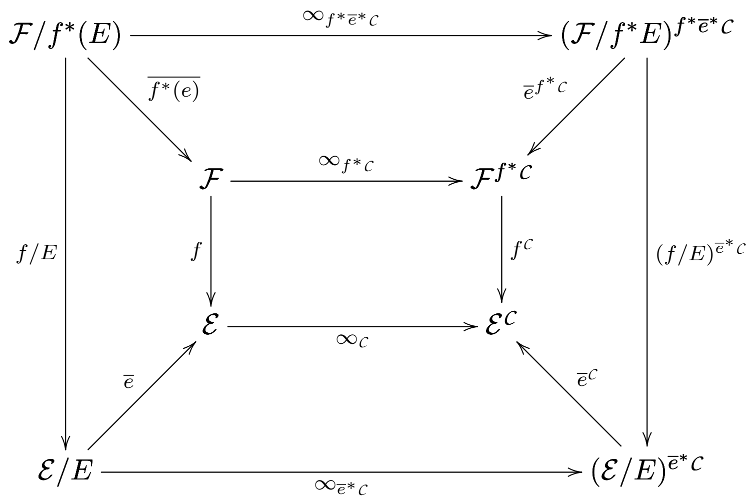
Now note that the slicing behaves well with the equivalence $\mathcal{E}/\coprod E_j \simeq \prod \mathcal{E}/E_j$, i.e. if A is an object over $\coprod E_j$, then

$$\left(\mathcal{E}/\coprod E_j\right)/A \simeq \prod\left((\mathcal{E}/E_j)/(A \times_{\coprod E_j} E_j)\right)$$

This implies that since f/E_j is proper for all $j \in J$, also $f/\coprod E_j$ is proper, proving the first step.

For the second one, let $e : E \longrightarrow 1$ be an epimorphism, then also $f^*(e) : f^*(E) \longrightarrow 1$ is an epimorphism, for f^* preserves epis. Now as showed in Example 4.1.2 we have that $\overline{e} = \mathcal{E}/e : \mathcal{E}/E \longrightarrow \mathcal{E}$ and $\overline{f^*(e)} = \mathcal{F}/f^*(e) : \mathcal{F}/f^*E \longrightarrow \mathcal{F}$ are surjections. Thanks to Lemma 4.1.6, also $(\overline{e})^C$ and $\overline{f^*(e)}^{f^*(C)}$ are surjections.

Consider now the following commmutative diagram in 𝔊𝔢𝔬𝔪



(5.8)

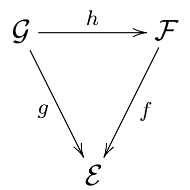
we must show that $\infty_{f^*C}^*(f^C)_*(U) \cong f_*\infty_C^*(U)$ for all U subterminal in $\mathcal{F}^{f^*C}$. If we apply $\overline{e}^*$ to both terms we get the two terms of the analogous condition for the exterior square in (5.8) (applied at the subterminal $(\overline{e}^{f^*C})^*(U)$),

but this isomorphism holds for the exterior square because f/E is proper. Moreover since $\overline{e}$ is a surjection, $\overline{e}^*$ is faithful an in particular it reflects isomorphisms. Thus we also get the isomorphism that we needed to prove. Let now A be an object in $\mathcal{E}$, then if we take $\mathcal{E}/A$ instead of $\mathcal{E}$, $e \times A$ instead of e, $\mathcal{F}/f^*(A)$ instead of $\mathcal{F}$ and f/A instead of f, then we are in the same conditions as before for $e \times A$ is still an epimorphism and $f/(E \times A) = (f/E)/A$ is proper by Proposition 5.2.4 (actually we just need that the BC transformation is an isomorphism at subterminals which is already true for f/E is proper). We can thus deduce the isomorphism $\infty^*_{\mathcal{C}}(((f/A)^{\mathcal{C}})_*(U) \cong (f/A)_* \infty^*_{(f/A)^*\mathcal{C}}(U)$ for every filtered internal category $\mathcal{C}$ in $\mathcal{E}/A$.
We have just proved that f is proper, and hence this proposition. □

This allows us to check if a map is proper by checking at its localizations at objects which form a jointly epimorphic family. For instance if the map comes from a continuous map of locales $f : X \longrightarrow Y$, given an open cover $\{V_i | i \in I\}$ of Y, once we see that $\mathcal{Sh}(f)/V_i$ (which corresponds to the map $f|_{f^*(V_i)} : f^*(V_i) \longrightarrow V_i$) is proper, we can deduce that $\mathcal{Sh}(f)$ is proper. This might be useful in cases where the codomain has an especially regular cover, such as for manifolds, which allow us to restrict the study to the case $X \longrightarrow U$ where U is isomorphic to $\mathbb{R}^n$ for some $n \in \mathbb{N}$.

In some case we can also deduce the propriety of a map from the propriety of a composition in which it appears. Namely we have the following case.

Proposition 5.2.8. *Consider the following diagram in* $\mathfrak{Geom}$



1. *If h is a surjection and g is proper, then f is proper as well.*

2. *If f is an inclusion and g is proper, then h is proper.*

Proof. 1. let $\mathcal{C}$ be an internal category in $\mathcal{E}$ and D a subterminal diagram in $\mathcal{F}^{\mathcal{C}}$. We will use the notation of Section 3.4. Consider $(h^{\mathcal{C}})^*(D)$, it is still a subterminal diagram and since g is proper, we have that.

$$\infty^*(g^{\mathcal{C}})_*(h^{\mathcal{C}})^*(D) \cong g_* \infty^*(h^{\mathcal{C}})^*(D)$$

Now by hypothesis h is a surjection and thus so is $h^{\mathcal{C}}$ by Lemma 4.1.6. Thanks to counit and unit of the adjunction $(h^{\mathcal{C}})^* \dashv (h^{\mathcal{C}})_*$ we have two arrows: $(h^{\mathcal{C}})^*(h^{\mathcal{C}})_*(h^{\mathcal{C}})^*(D) \longrightarrow (h^{\mathcal{C}})^*(D)$ and $(h^{\mathcal{C}})^*(D) \longrightarrow (h^{\mathcal{C}})^*(h^{\mathcal{C}})_*(h^{\mathcal{C}})^*(D)$. Since we are dealing with subterminals, from Proposition 1.6.8 (1) we have that these maps are each other's inverse and moreover, since $h^{\mathcal{C}}$ is a surjection, its inverse image reflects isomorphisms, so we have

$$(h^{\mathcal{C}})_*(h^{\mathcal{C}})^*(D) \cong D$$

Therefore $(g^{\mathcal{C}})_*(h^{\mathcal{C}})^*(D) \cong (f^{\mathcal{C}})_*(h^{\mathcal{C}})_*(h^{\mathcal{C}})^*(D) \cong (f^{\mathcal{C}})_*(D)$.
From all of these observations we get $g_*\infty^*(h^{\mathcal{C}})^*(D) \cong \infty_*(f^{\mathcal{C}})^*(D)$ but we also have

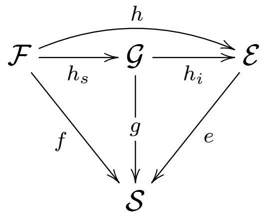
$$g_*\infty^*(h^{\mathcal{C}})^*(D) \cong g_*h^*\infty^*(D) \cong f_*h_*h^*\infty^*(D) \cong f_*\infty^* D$$

The last isomorphism is due to the fact that $\infty^*(D)$ is again subterminal and the others follow from commutativity of diagrams.
We have proved that $\infty^*(f^{\mathcal{C}})_*(D) \cong f_*\infty^*(D)$ and this holds for every D and $\mathcal{C}$. Since localizing we are in the same hypotheses for Proposition 4.1.7, we get that f is proper by Remark 5.2.5.

2. A proof of this point can be found in [SE] as Lemma 3.2.16(iii).
Note that the proof given in [MV] corresponding to Proposition I.2.3 actually requires that, the functor $f^* : \mathcal{C}at(\mathcal{E}) \longrightarrow \mathcal{C}at(\mathcal{F})$ as in Remark 4.1.4 is such that for every filtered category $\mathcal{C}$ in $\mathcal{F}$ there is an internal filtered category $\mathcal{D}$ in $\mathcal{E}$ such that $f^*(\mathcal{D}) \cong \mathcal{C}$. Even though Remark 4.1.4 assures that f^* is essentially surjective, a priori we don't know if filtered categories are images of filtered ones. In this hypothesis however the proof should be as follows.
Let $\mathcal{C}$ be a filtered internal category in $\mathcal{F}$ and let D be subterminal in $\mathcal{G}^{\mathcal{C}}$. We want to show that $\infty^*(h^{\mathcal{C}})_*(D) \cong h_*\infty^*(D)$ and using Remark 5.2.5 (2), we just have to find an arrow $h_*\infty^*(D) \longrightarrow \infty^*(h^{\mathcal{C}})_*(D)$ but since f is an inclusion, f_* is full and faithful, so it is enough to find an arrow $f_*h_*\infty^*(D) \longrightarrow f_*\infty^*(h^{\mathcal{C}})_*(D)$.
Thanks to the hypothesys just introduced, $\mathcal{C}$ can be seen as an internal filtered category over the base topos $\mathcal{E}$. Then we can prove the previous isomorphisms considering an internal filtered category $\mathcal{C}$ in $\mathcal{E}$, using the notation of Section 3.4.
We have that $f_*h_*\infty^*(D) \cong g_*\infty^*(D)$ and since g is proper (and $\mathcal{C}$ is internal in $\mathcal{E}$), the right hand side is isomorphic to $\infty^*(g^{\mathcal{C}})_*(D) \cong \infty^*(f^{\mathcal{C}})_*(h^{\mathcal{C}})_*(D)$. Now we just need to prove that there is a map from $\infty^*(f^{\mathcal{C}})_*(h^{\mathcal{C}})_*(D)$ into $f_*\infty^*(h^{\mathcal{C}})_*(D)$. But that is the component $(h^{\mathcal{C}})_*(D)$ of the BC transformation corresponding to f. As before, thanks to Proposition 4.1.7, we can repeat the same reasoning for the localized case, so h is proper. □

Note that in particular the first point of Proposition 5.2.8 has an interesting geometric interpretation. Consider in fact a geometric morphism $h : \mathcal{F} \longrightarrow \mathcal{E}$ of toposes over a base topos $\mathcal{S}$ and let $f : \mathcal{F} \longrightarrow \mathcal{S}$ and $e : \mathcal{E} \longrightarrow \mathcal{S}$ the unique such geometric morphisms in $\mathfrak{Geom}/\mathcal{S}$. We can factor h in $\mathfrak{Geom}/\mathcal{S}$

as follows



where h_s is a surjection, h_i an inclusion and $g \simeq eh_i$. We can think of $\mathcal{G}$ as the image of the geometric morphism h, as suggests the geometric interpretation of this factorization (Proposition 4.1.5).
What Proposition 5.2.8(1) says is that if f is proper, than also g is, or in other words we have that if $\mathcal{F}$ is compact as an $\mathcal{S}$-topos, then so is its image $\mathcal{G}$ via a geometric morphism. Which agrees with the classical theory in topological spaces where the image of a compact space along a continuous function is compact.

Another consequence of Proposition 5.2.8 is that a geometric morphism is proper iff its surjective-injective factorization is, that is

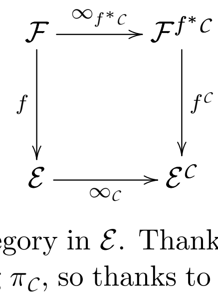
Corollary 5.2.9. *Let f be a geometric morphism, suppose $f = f_i f_s$ with f_i inclusion and f_s surjection, then f is proper iff both f_s and f_i are proper.*

Proof. If f_i and f_s are both proper, Proposition 5.2.6 says that f is proper. Conversely, if f is proper, apply Proposition 5.2.8 which implies that f_i is proper (for f_s is a surjection) and f_e is proper (for f_i is an inclusion). □

In Section 4.3 we have seen another factorization, which in the case of proper maps might even be more useful. We have in fact

Proposition 5.2.10. *Every hyperconnected geometric morphism is proper.*

Proof. Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a hyperconnected geometric morphism and consider the following diagram



for $\mathcal{C}$ an internal filtered category in $\mathcal{E}$. Thanks to Lemma 3.3.12 we can see $f^{\mathcal{C}}$ as the pullback of f along $\pi_{\mathcal{C}}$, so thanks to Corollary 4.3.5, we know that $f^{\mathcal{C}}$ is hyperconnected as well.
For every subterminal object U in $\mathcal{F}^{f^*\mathcal{C}}$, we have that U is a particular subquotient of 1 which is necessarily of the form $(f^{\mathcal{C}})^*(1)$. Since $f^{\mathcal{C}}$ is hyperconnected, the image of $(f^{\mathcal{C}})^*$ is closed under subquotients and in particular $U = (f^{\mathcal{C}})^*(V)$ for some object V in $\mathcal{E}^{\mathcal{C}}$. There is a unique morphism $V \longrightarrow 1$

and we can factor it as a regular epi followed by a mono; factorization which is preserved by $(f^{\mathcal{C}})^*$, being an inverse image. The unique map $U \longrightarrow 1$ is a mono, so the epimorphic component of $V \longrightarrow 1$ is sent to an isomorphism, but $(f^{\mathcal{C}})^*$ is also full and faithful, so it reflects isomorphisms and thus the epimorphic component of $V \longrightarrow 1$ is an isomorphism, proving that V must be a subterminal.
Consider $(\infty_{\mathcal{C}})^*(V)$, it is isomorphic to $f_*f^*(\infty_{\mathcal{C}})^*(V)$ and the isomorphism is the unit of the adjunction $f^* \dashv f_*$ because f is hyperconnected, so f^* is full and faithful and thus from the dual of Lemma 4.1.3, such a unit is an isomorphism.
Now $f_*f^*(\infty_{\mathcal{C}})^*(V)$ is isomorphic to $f_*(\infty_{f^*\mathcal{C}})^*(f^{\mathcal{C}}) * (V)$ by commutativity of the square above and the latter is by definition of V isomorphic to $f_*(\infty_{f^*\mathcal{C}})^*(U)$. For the same reason stated above, since $f^{\mathcal{C}}$ is hyperconnected, we have that $V \cong (f^{\mathcal{C}})_*(f^{\mathcal{C}})^*(V) = (f^{\mathcal{C}})_*(U)$. Putting together all these isomorphisms we get

$$(\infty_{\mathcal{C}})^*(f^{\mathcal{C}})_*(U) \cong f_*(\infty_{f^*\mathcal{C}})^*(U)$$

Now note that we have the following pullback square

$$\begin{array}{ccc} \mathcal{F}/f^*(E) & \xrightarrow{\pi_{f^*(E)}} & \mathcal{F} \\ {\scriptstyle f/E}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathcal{E}/E & \xrightarrow[\pi_E]{} & \mathcal{E} \end{array}$$

thus in particular if f is hyperconnected, also f/E is, because of Corollary 4.3.5. In particular we can apply the reasoning of the first part of this proof, for we are in the same situation. We can thus prove that for every internal filtered category $\mathcal{C}$ in $\mathcal{E}/E$, the square

$$\begin{array}{ccc} \mathcal{F}/f^*(E) & \xrightarrow{\infty_{(f/E)^*\mathcal{C}}} & (\mathcal{F}/f^*(E))^{(f/E)^*\mathcal{C}} \\ {\scriptstyle f/E}\downarrow & & \downarrow{\scriptstyle (f/E)^{\mathcal{C}}} \\ \mathcal{E}/E & \xrightarrow[\infty_{\mathcal{C}}]{} & (\mathcal{E}/E)^{\mathcal{C}} \end{array}$$

is such that for every subterminal U in $(\mathcal{F}/f^*(E))^{(f/E)^*\mathcal{C}}$, we have

$$\infty_{\mathcal{C}}^*(f^{\mathcal{C}})_*(U) \cong f_*(\infty_{f^*\mathcal{C}})^*(U)$$

and thus f is proper as claimed. □

From this proposition it follows the following result

Corollary 5.2.11. *A geometric morphism is proper iff its localic component is proper.*

Proof. One implication follows from Propositions 5.2.10 and 5.2.6(2). The other follows from Proposition 5.2.8(1) since hyperconnected maps are surjections by definition. □

We have then that the propriety of a geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{S}$ only depends on the propriety of its localic component, but the latter, thanks to Corollary 4.3.6, is of the form $\mathcal{Sh}_{\mathcal{S}}(X) \longrightarrow \mathcal{S}$ for some internal locale X in $\mathcal{S}$. This suggests another characterization of proper maps which follows from the generalization of Theorem 5.1.4 to proper maps.

Corollary 5.2.12. *Let $f : \mathcal{E} \longrightarrow \mathcal{S}$ be a geometric morphism and X an internal locale in $\mathcal{S}$ such that the localic component of f is $\mathcal{Sh}_{\mathcal{S}}(X) \longrightarrow \mathcal{S}$. Then f is proper iff X is compact as internal site, that is, it satisfies the compactness condition formulated in the internal logic of $\mathcal{S}$.*

Proof. Since Theorem 5.1.4 can be proved constructively, it is valid also in this general case. □

5.3 Pretopos sites and weak BC condition

In this section we will use the theory we have developed in Section 1.4 to characterize compact toposes and therefore proper morphisms in terms of sites (at least when dealing with Grothendieck toposes and Bounded geometric morphisms respectively). As we will see, this characterization will enable us to prove stronger statements, like pullback stability of proper maps and a characterization of proper maps using the weak Beck-Chevalley condition.

Let's start with the main theorem

Theorem 5.3.1. *Let $\mathcal{E}$ be a Grothendieck topos, the following are equivalent*

1. *$\mathcal{E}$ is compact*

2. *There exists a small site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$ with $\mathcal{C}$ coherent and J the union of a dm-coverage T (see Definition 1.4.9) compatible with the coherent one P such that the only T-covering family of the terminal of $\mathcal{C}$ is the maximal sieve*

3. *There exists a small site of definition $(\mathcal{C}, J)$ for $\mathcal{E}$ with $\mathcal{C}$ coherent and J the Grothendieck topology obtained as join of the coherent Grothendieck coverage P and $\widetilde{T}$ generated by a P-compatible dm-coverage T. Moreover the only T-covering family of the terminal of $\mathcal{C}$ is the maximal sieve.*

4. *Any subcanonical pretopos site for $\mathcal{E}$ is compact.*

Proof. $1 \Rightarrow 4$: Let $(\mathcal{C}, J)$ be a subcanonical pretopos site for $\mathcal{E}$ (which exists by Lemma 1.4.17. Being the site subcanonical, the Yoneda embedding factors through $\mathcal{E}$ in $y : \mathcal{C} \longrightarrow \mathcal{E}$ and then we can see $\mathcal{C}$ as a full subcategory of $\mathcal{E}$ (up to equivalence). Representable presheaves are a set of generators, so $\mathcal{C}$ is a generating set and hence if we equip it with the coverage H made of families which are jointly epimorphic in $\mathcal{E}$, we get a subcanonical pretopos site for Corollary 4.1 of the Appendix of [SGL]. Since now $\mathcal{E}$ is obtained both as $\mathcal{Sh}(\mathcal{C}, J)$ and $\mathcal{Sh}(\mathcal{C}, H)$, we get that J and H are equivalent.
We will now prove that the sifted closure of H is also a Grothendieck coverage, and hence the one generated by H.
The maximal sieve is covering for it contains the identity.
A sieve S on c is jointly epimorphic iff the map

$$e : \coprod_{s \in S} \mathrm{dom}(s) \longrightarrow c$$

is an epimorphism (as one can see with the universal property of coproduct). Epimorphisms in a topos are regular and thus pullback stable. Since also coproducts are pullback stable, follows that for every $f : d \longrightarrow c$, the family of pullbacks of arrows in S along f is jointly epimorphic. This family is contained in $f^*(S)$, so we get the second axiom of Grothendieck coverages. For the last axiom, let R and S be sieves over c with R jointly epimorphic and S such that $r^*(S)$ jointly covering. Consider the set $\{ra | r \in R, a \in r^*(S)\}$, it is jointly epimorphic since given x, y arrows such that $xra = yra$ for all such ar, in particular $xr = yr$ for $r^*(S)$ is jointly epimorphic and then $x = y$ for R is. Moreover this set of arrows is contained in S and thus also the latter is jointly epimorphic.
It follows that $\widetilde{J}$ is the coverage of jointly epimorphic sieves. In particular the Grothendieck coverage generated by covering dm-sieves in $\widetilde{J}$ is also generated by the jointly epimorphic families which are dm-sieves, i.e. H_d.
Let T be the coverage whose covering families are the filtered families of monos which are also jointly epimorphic. The sifted closure of T is precisely H_d as follows from Lemma A.2.1, and hence by previous deduction it is $\widetilde{J}_d$. Let now $A \in T(1)$ and consider the full subcategory of $\mathcal{E}$ with objects $\mathrm{dom}(a)$ for $a \in A$, call D the inclusion of this subcategory. We have that D is a diagram of subterminal objects and the fact that A is filtered implies that D is filtered, moreover note that A can be seen as a cocone for D. As observed before, a jointly epimorphic family in a topos is jointly regular epimorphic and this fact, joined with Lemma 5.1.5 assures that A is actually a universal cocone. In particular $\mathrm{colim}(D) = 1$ and since $\mathcal{E}$ is compact, $\gamma : \mathcal{E} \longrightarrow \mathcal{S}et$ preserves filtered colimits, which means that the colimit of γD is 1 in $\mathcal{S}et$, so that the union of $\mathrm{dom}(a)$'s for $a \in A$ is 1 and thus that A contains an isomorphism.
We can now apply Proposition 1.4.20 with T, which implies directly that the site $(\mathcal{C}, J)$ is compact.

$4 \Rightarrow 3$: It follows from the fact that a pretopos is a coherent category and from the fact that a subcanonical pretopos site actually exists by Lemma 1.4.17.

$3 \Rightarrow 2$: It is true if we take J to be the union of P and T instead of the join.

$2 \Rightarrow 3$: It is true if we take J to be the join of the Grothendieck coherent coverage with $\widetilde{T}$. This Grothendieck coverage is the one generated by $\widetilde{P}$ and $\widetilde{T}$ and hence by $P \cup T$.

$3 \Rightarrow 1$: Let $h : \mathcal{C} \longrightarrow \mathcal{Sh}(\mathcal{C}, J) \simeq \mathcal{E}$ be the composition of the Yoneda embedding and the sheafification functor. h in particular preserves monos for both functors preserve them. Any object E of $\mathcal{E}$ can naturally be seen as colimit of sheaves image of h because every presheaf is colimit of representable ones and sheafification preserves colimits. In particular if E is subterminal we see this colimit as a union (for in every component there is at most one partial section) and in particular we can use only $h(u)$ for u subterminal in $\mathcal{C}$ because for every $c \in \mathcal{C}$ such that the map $c \longrightarrow 1$ factors through the subterminal u we have that $E(c) = E(u)$. The latter equality comes from the fact that $E(u) \longrightarrow E(c)$ is a mono and $c \longrightarrow u$ is a P-cover being a regular epimorphism and thus for every section in $E(c)$ there is a unique section in $E(u)$, creating a section $E(c) \longrightarrow E(u)$ for the previous mono which thus becomes an iso.
Viewing subterminal objects of $\mathcal{E}$ as union of $h(u)$'s for u subterminal in $\mathcal{C}$, we can refine a subterminal cover of 1 in $\mathcal{E}$ by a cover of images via h of subterminals in $\mathcal{C}$ (still subterminals in $\mathcal{E}$) call the set of such subterminals U. Since $\mathcal{C}$ is coherent, it makes sense to consider finite joins of subterminals in U, getting a new family V of subterminals in $\mathcal{C}$ whose union in $\mathcal{E}$ gives 1. The sieve S generated by maps $v \longrightarrow 1$ for $v \in V$ is a dm-sieve thanks to Lemma A.2.1 and it is sent by h in a family of jointly epimorphic maps in $\mathcal{E}$. One can prove that such a sieve is bound to be in T, but then, since the only sieve of this form is the maximal one, $\mathrm{id}_1 \in S$. Hence we can see 1 as a finite union of subterminals in U and thus as finite union of subterminals in $\mathcal{E}$, but then, being a filtered cover of 1 by subterminals of $\mathcal{E}$, such a cover must already contain 1. We thus get that γ preserves filtered colimits of subterminal objects. □

This result can be formulated also in terms of proper maps, where instead of $\gamma : \mathcal{E} \longrightarrow \mathcal{Set}$ we have a generic geometric morphism. The problem here is to adapt the notion of Grothendieck topos to this general case, but as we have seen in Section 4.2, the generalization we need is represented by bounded geometric morphisms.
Interpreting the statement of Theorem 5.3.1 in the general case of a topos over a certain base, we can give a characterization of proper maps. Namely we get that a geometric morphism $\phi : \mathcal{F} \longrightarrow \mathcal{E}$ where $\mathcal{E}$ has a *natural numbers object* (see [LT] Section 3.2.3), then there is an internal site in $\mathcal{E}$ which

satisfies also the internal version of the property of compactness. Note that the existence of a natural numbers object is necessary for, in the proof of Theorem 5.3.1, we are using induction to prove $(1 \Rightarrow 4)$ and more precisely in Lemma 1.4.17 when we are building the closure under limits and colimits of a set of generators.
We can now state one of the most important consequences of this formulation, i.e. the following

Theorem 5.3.2. *Let the following be a pullback square in* $\mathfrak{Geom}$

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{k} & \mathcal{G} \\ {\scriptstyle h}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ \mathcal{F} & \xrightarrow[f]{} & \mathcal{E} \end{array}$$

where f is proper and either g is bounded or f is bounded and $\mathcal{E}$ has a natural numbers object, then also k is proper.
Moreover if f is a surjection, so is k.

Proof. More complete proofs of this theorem can be found in [SE] Theorem C3.2.21 or in [MV] as Theorem I.5.8 (the latter for the case of Grothendieck toposes).
Here we will just sketch it briefly, for it would require a more specific analysis which is unnecessary for the purpose of this thesis.
In the second case, when f is bounded, we can find an internal sieve $(\mathcal{C}, J)$ satisfying the properties of Theorem 5.3.1(2). Note that for this step and more precisely for the creation of an internal subcanonical pretopos site, we need $\mathcal{E}$ to have a natural numbers object, for we use induction in order to build the closure under limits and colimits of a set of generators.
We can pull back the site $(\mathcal{C}, J)$ along g as showed in Theorem 4.2.8 getting $(g^*\mathcal{C}, g^\# J)$. We can see by construction that $g^\#(J)$ is union of $g^\# P$ and $g^\# T$. Now $g^\# P$ turns out to be the coherent coverage of $g^*(\mathcal{C})$ and from the P-compatibility of the internal dm-coverage T we get that $g^\# T$ is a $g^\# P$-compatible internal dm-sieve, so that also $(f^*\mathcal{C}, f^\# J)$ is a pretopos site. Moreover, this site inherits the property of compactness from T. We thus get that $(g^*\mathcal{C}, g^\# J)$ is an internal sieve in $\mathcal{G}$ satisfying the internal version of Theorem 5.3.1(2) and thus by this same theorem internalized, we get that k is proper.

For the pullback stability under bounded maps instead the proof is built on a different principle: we factor the bounded morphism as in Giraud-Diaconescu Theorem and then we make one pullback at the time using the pullback glueing Lemma. The two cases that need to be studied separately are the case of the pullback along the morphism $\pi_{\mathcal{C}} : \mathcal{E}^{\mathcal{C}} \longrightarrow \mathcal{E}$ and the case of the inclusion which is a particular case of localic morphism. For the first case the proof is direct with some effort, while for the second we need again a factorization, this time of the proper map which is factored in hyperconnected and localic. Then we study again the two pullback squares separately: the first is the pullback of a hyperconnected along a localic which can be proved to be hyperconnected (See Theorem C2.4.11(ii) in [SE] or apply Corollary 4.3.5) and hence proper by Proposition 5.2.10; the second pullback square instead is the pullback of a localic proper map along a localic map which is proper in virtue of the internal generalization of a result true for locales (Proposition C3.2.6 in [SE]). We are done because composition of proper maps is proper. □

This theorem is in particular useful to show the link between Propriety and Beck-Chevalley conditions. Before we do so, we give the following definitions

Definition 5.3.3. *Let $f : \mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism*

1. *we say that f satisfies the* (weak) Beck-Chevalley condition *or that f is a* (weak) Beck-Chevalley morphism *if for every bounded morphism $g : \mathcal{G} \longrightarrow \mathcal{E}$ the pullback square*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{h} & \mathcal{F} \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathcal{G} & \xrightarrow[g]{} & \mathcal{E} \end{array}$$

   satisfies the (weak) Beck-Chevalley condition, so the Beck-Chevalley transformation $\theta : g^* f_* \Rightarrow k_* h^*$ is a natural isomorphism (monomorphism).

2. *We say that f satisfies the* stable (weak) Beck-Chevalley condition *or that it is a* stable (weak) Beck-Chevalley morphism *if the pullback of f along every bounded geometric morphism is again a (weak) Beck-Chevalley morphism.*

Remark 5.3.4. *We can readily see that stable (weak) BC morphisms are stable under pullback along bounded maps. Let f satisfy the stable (weak) BC condition and g be bounded, we want to prove that the pullback k of f along g is a stable (weak) BC morphism, hence that the pullback of k along any bounded morphism b is again a BC morphism. Consider the following* *diagram in* $\mathfrak{Geom}$

$$\begin{array}{ccccc}
\mathcal{A} & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{F} \\
{\scriptstyle a}\downarrow & \lrcorner & {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\
\mathcal{B} & \xrightarrow[b]{} & \mathcal{G} & \xrightarrow[g]{} & \mathcal{E}
\end{array}$$

For the pullback pasting Lemma, the exterior square is a pullback and therefore a, which is already the pullback of k along b, is also the pullback of f along gb. Now thanks to Lemma 4.2.4(1), gb is bounded and since f is a stable (weak) BC morphism, a is a (weak) BC morphism.

In Section 1.6 we saw with Proposition 1.6.9 that the weak BC condition can be expressed in a form which is very similar to the condition defining a proper map. In fact one has the following result.

Theorem 5.3.5. *Let* $f : \mathcal{F} \longrightarrow \mathcal{E}$ *be a geometric morphism, it is a proper map iff it satisfies the stable weak Beck-Chevalley condition.*

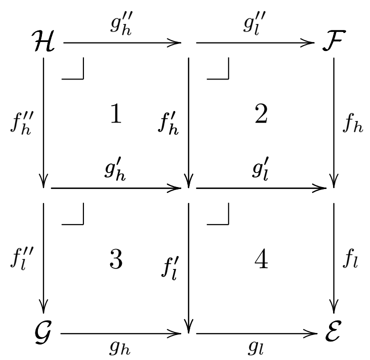
Proof. For every $E \in \mathcal{E}$ and $\mathcal{C}$ internal filtered category in $\mathcal{E}/E$, consider the diagram

$$\begin{array}{ccccccc}
\mathcal{F}/f^*(E) & \xrightarrow{\infty_{(f/E)^*\mathcal{C}}} & (\mathcal{F}/f^*(E))^{(f/E)^*\mathcal{C}} & \xrightarrow{\pi_{(f/E)^*(\mathcal{C})}} & \mathcal{F}/f^*(E) & \xrightarrow{\pi_{f^*(E)}} & \mathcal{F} \\
{\scriptstyle f/E}\downarrow & & \downarrow{\scriptstyle (f/E)^{\mathcal{C}}} & & \downarrow{\scriptstyle f/E} & & \downarrow{\scriptstyle f} \\
\mathcal{E}/E & \xrightarrow[\infty_{\mathcal{C}}]{} & (\mathcal{E}/E)^{\mathcal{C}} & \xrightarrow[\pi_{\mathcal{C}}]{} & \mathcal{E}/E & \xrightarrow[\pi_E]{} & \mathcal{E}
\end{array} \quad (5.9)$$

The morphism π_E can be seen as the projection $\pi_E : \mathcal{E}^E \longrightarrow \mathcal{E}$ where we see E as a discrete internal category (Remark 3.1.12) and the same happens with $\pi_{f^*(E)}$, so that the rightmost square of (5.9) can be treated in the same guise as the central one.
These two squares are actually pullbacks as we proved with Proposition 3.3.12 and moreover, a map of the form π_I for some internal category I is a bounded one as follows from Giraud-Diaconescu's Theorem. Note also that the composition of bounded maps is bounded thanks to Lemma 4.2.4(1) and that for the pullback pasting Lemma we can glue the central and the right square of (5.9) getting a pullback where the horizontal lower map is bounded.
If f satisfies the stable weak BC condition, then $(f/E)^{\mathcal{C}}$ is by definition a weak Beck-Chevalley morphism, for it is the pullback of f along $\pi_E\pi_{\mathcal{C}}$ which is bounded. By definition this means that if we pull back f along a bounded morphism, the pullback square satisfies the weak BC condition. Note that $\pi_{\mathcal{C}}\infty_{\mathcal{C}} \simeq \mathrm{id}_{\mathcal{E}/E}$ which is bounded, being localic (every object A in $\mathcal{E}/E$ is trivially a subquotient of $\mathrm{id}^*_{\mathcal{E}/E}(A)$) and thus by Lemma 4.2.4(2),

also ∞_C is bounded. Moreover, if we regard at the leftmost square in (5.9), thanks to Corollary 3.3.14, it is a pullback and thus it satisfies the weak BC condition. Now we can apply Proposition 1.6.9 and by ($1 \Rightarrow 3$) we get that the BC transformation of this pullback square is an isomorphism at subterminal diagrams, but this is precisely the definition of proper map.

For the converse statement, if we prove that proper maps satisfy the weak Beck-Chevalley condition, then we are done because, thanks to Theorem 5.3.2, the pullback of a proper map is again proper and hence a weak Beck-Chevalley morphism. Suppose that $f : \mathcal{F} \longrightarrow \mathcal{E}$ is proper, we have to prove that the pullback square in $\mathfrak{Geom}$ along any bounded morphism $g : \mathcal{G} \longrightarrow \mathcal{E}$ satisfies the weak BC condition. We factor both f and g in hyperconnected and localic geometric morphisms getting $f = f_l f_h$ and $g = g_l g_h$. The situation is the following



Thanks to the pullback pasting Lemma this is a pullback square of f and g. Moreover using Corollary 4.3.5 we have that the squares (1), (2) and (3) have at least a hyperconnected morphism and thus satisfy the weak BC condition (see for instance Lemma A4.6.8 in [SE]). What is left to study is the square (4), where we have a pullback of a localic proper map along a localic map. This case can be deduced from the corresponding $\mathcal{S}et$-theoretic result as soon as the latter can be proved constructively, but this is precisely what happens (see for instance Proposition C3.2.6 in [SE]). Now we have to prove that the exterior square satisfies the weak BC condition and thus that for every subterminal U in $\mathcal{F}$ we get the isomorphism $g^* f_*(U) = (f'')_*(g'')^*(U)$ if we call $f'' := f''_l f''_h$ and $g'' := g''_l g''_h$. Since both direct and inverse images preserves limits, they also preserve subterminal objects so that, since the weak BC condition is valid for each of the previous subsquares, we have the following chain of isomorphisms

$$\begin{aligned} g^* f_*(U) = (g_h)^*(g_l)^*(f_l)_*(f_h)_*(U) = (g_h)^*(f'_l)_*(g'_l)^*(f_h)_*(U) = \\ = (g_h)^*(f'_l)_*(f'_h)_*(g''_l)^*(U) = (f''_l)_*(g'_h)^*(f'_h)_*(g''_l)^*(U) = \\ = (f''_l)_*(f''_h)_*(g''_h)^*(g''_l)^*(U) = (f'')_*(g'')^*(U) \end{aligned}$$

The isomorphism that we have just obtained holds between subterminal objects, thus it is unique and in particular it coincides with the component in

U of the Beck-Chevalley transformation. The BC condition is thus satisfied ending the proof. □

5.4 Further developments

In this section we will briefly describe some other important results and definitions related to compactness or more in general to propriety, which we are not treating in detail in this thesis.

Another property of geometric morphisms that we could have defined is closedness (see in [SE] C3.2). Intuitively a closed geometric morphism is such that sends closed subtoposes ([SE] A4.5) to closed ones. This idea corresponds to the localic case where a map is closed ([SE] C3.2.1) when it sends closed sublocales ([SE] C1.2.6(b)) to closed sublocales, which in turn corresponds to the usual idea of closed continuous function of spaces (sending closed subsets to closed ones).
The problem with closedness is that it is not necessarily stable by pullback and hence it is not well behaved with respect to the change of base. We can then prove the following theorem which is not only another characterization of proper maps, but it describes the link between proper and closed morphisms.

Theorem 5.4.1. *Let $\mathcal{F} \longrightarrow \mathcal{E}$ be a geometric morphism, then it is proper iff it is* stably closed*, i.e. the pullback of f along every bounded geometric morphism $g : \mathcal{G} \longrightarrow \mathcal{E}$ is closed.*

Proof. See [SE] Lemma C3.2.26. □

In particular note that every proper map is closed. This theorem basically means that propriety is a sort of pullback stable version of closedness. Moreover note that Theorem 5.4.1 resembles the geometrical definition of proper function as given in [BT], which is

Definition 5.4.2. *A continuous map $f : X \longrightarrow Y$ of topological spaces is* proper *if $f \times Z : X \times Z \longrightarrow Y \times Z$ is closed for every topological space Z.*

Coming back to the affinity between proper morphisms and closed ones, it is worth to mention that an inclusion is closed iff it is proper and this leads to the definition of separated morphisms.
A *separated morphism* is a (bounded) geometric morphism $f : \mathcal{E} \longrightarrow \mathcal{S}$ which have a proper diagonal morphism, where by diagonal map we mean

$$\Delta_f = (\mathrm{id}_{\mathcal{E}}, \mathrm{id}_{\mathcal{E}}) : \mathcal{E} \longrightarrow \mathcal{E} \times_{\mathcal{S}} \mathcal{E}$$

This is reminiscent of the definition of separated (Hausdorff) topological spaces where we can characterize separable spaces as those spaces X such that the diagonal subspace $\Delta_X = \{(x, x)|x \in X\}$ is closed in $X \times X$. We can in this fashion define a *separated locale* to be a locale X where the diagonal map $\Delta_X : X \longrightarrow X \times X$ is closed. We have then that a locale X is separated iff the morphism $\gamma : \mathcal{Sh}(X) \longrightarrow \mathcal{Set}$ is a separated geometric morphism. However we are not as lucky with topological spaces, in fact in general a separated topological space X need not be separated as a locale, for in general the latter is bigger. We deduce that, even if X is Hausdorff, $\gamma : \mathcal{Sh}(X) \longrightarrow \mathcal{Set}$ is not necessarily separated.

Another direction that we could have explored concerns a strengthening of the definition of proper map and Theorem 5.3.5. For a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$, to be proper means that for every object E in $\mathcal{E}$ and every filtered category $\mathcal{C}$ in $\mathcal{E}/E$, if $\theta : (\mathcal{F}/f^*E)^{f^*\mathcal{C}} \longrightarrow \mathcal{E}/E$ is the BC transformation of the pullback square of f/E along $\infty_{\mathcal{C}}$, then θ is an isomorphism at subterminals. If instead it is the whole θ to be an isomorphism we get what is called a *tidy geometric morphism*. Every tidy morphism is in particular proper but this property is too strong to represent compactness and in fact it is the relative version of the $\mathcal{Set}$-theoretic strong compactness, where a topos $\mathcal{E}$ is *strongly compact* if the direct image of the unique geometric morphism $\gamma : \mathcal{E} \longrightarrow \mathcal{Set}$ preserves all filtered colimits. A particular case where these properties coincide is when we are dealing with separated geometric morphisms (Proposition III.2.8 in [MV]).
However, it turns out that most of the results which were true for proper maps have their own tidy version. Such adaptations can be found in Chapter III of [MV]. In particular note that, as one would expect by the description that we have made of BC conditions, the tidy version of Theorem 5.3.5 is the following

Theorem 5.4.3. *A geometric morphism is tidy iff it satisfies the stable Beck-Chevalley condition.*

Proof. Corollary III.4.9 in [MV]. □

In passing by, note that one could also define the tidy version of separated morphism, where the diagonal instead of being proper is tidy. This case is known with the name of *strongly separated geometric morphism*.

One last generalization one could do is to interpret internally what does it mean to be a proper or a tidy map. Namely we say that a geometric morphism $f : \mathcal{F} \longrightarrow \mathcal{E}$ in $\mathfrak{Geom}/\mathcal{S}$ for a topos $\mathcal{S}$ is *relatively proper* (resp *tidy*) if for every object S in $\mathcal{S}$ and every filtered internal category $\mathcal{C}$ in $\mathcal{S}$, we get the corresponding conditions where we interpret both $\mathcal{C}$ and S in $\mathcal{E}$ and $\mathcal{F}$ using the inverse image of the respective unique geometric morphism in $\mathfrak{Geom}/\mathcal{S}$.

Appendix A

Trees and induction

The aim of this appendix is to complete proofs left behind during the previous discussions. The main reason why these proofs were not given before is that they required the introduction of some technicalities that would have interrupted the flow of the explanation while not being so relevant for the overall theory.

The main ingredient of this Appendix will be well founded posets and the induction rules that we obtain from them.

Definition A.0.1. *Let X be a set and $\prec$ a binary relation, we say that a subset A is inductive if for every $x \in X$ the following holds*

$$\{y \in X | y \prec x\} \subseteq A \to x \in A$$

The relation $\prec$ is called well-founded *if the only inductive subset of X is X itself.*

Example A.0.2. *For every set E, the poset E_* obtained by adding a bottom element to E with the discrete order is well founded because if $A \subset E_*$ is inductive, then for the bottom $\perp$ we have that $\{y|y < \perp\}$ is empty and thus $\perp$ is contained in A. Then for every $e \in E$ the set $\{y|y < e\} = \{\perp\} \subseteq A$, so $e \in A$ and hence $A = E_*$.*
Analogously one proves that it is well founded also with the order reversed, in fact if A is inductive here, then since $e \in E$ have $\{y \in E_*|y > e\} = \emptyset$, $E \subseteq A$, but $\{y \in E_*|y > \perp\} = E$, so $A = E_*$.

Example A.0.3. *Let α be an ordinal, then using transfinite induction one proves that it is well-founded. If we consider the reversed relation, classically we have that only finite ordinals are well founded.*

Note that well-founded relations enable us to define a new induction principle. If in fact we want to prove a propriety ϕ valid for elements of X well-founded, we consider the subset A of those elements in X satisfying ϕ, for every x we prove that when every y s.t. $y < x$ has this property,

then also x does. By doing so, we are proving that A is inductive and thus it is bound to be X, so this makes ϕ valid on the whole of X. Note that this principle includes transfinite induction using ordinals and in particular, standard induction using the first infinite ordinal ω.
As example of use of this induction we prove the following proposition, where writing $x \leq y$ we mean that the formula $(x = y) \vee (x \leq y)$ holds. From now on we will use this convention.

Proposition A.0.4. *Let $(X, <)$ be a well-founded poset, then for every $x \in X$, the subset $\downarrow(x) = \{y \in X | y \leq x\}$ is well-founded.*

Proof. Let A be the subset of X containing those $x \in A$ such that $\downarrow(x)$ is well founded. Suppose now $x \in X$ such that $\{y \in X | y < x\} \subseteq A$, then we want to prove that $x \in A$, and hence that $\downarrow(x)$ is inductive in X. Let B be an inductive subset of $\downarrow(x)$ and consider for every $y \leq x$ the set $B \cap \downarrow(y)$. Since $\downarrow(y)$ is downward closed in $\downarrow(x)$, $B \cap \downarrow(y)$ is inductive in $\downarrow(y)$, but since by hypothesis $y \in A$, $\downarrow(y)$ is well founded, we have $\downarrow(y) \subseteq B$. This can be done for every $y < x$, so in particular B contains every $y < x$, but B is inductive, so it must also contain x and thus $B = \downarrow(x)$ implying that $\downarrow(x)$ is well-founded and thus that $x \in A$ which in turn implies that A is inductive. Now X is well founded, so $A = X$ and in particular $\downarrow(x)$ must be well founded for every $x \in X$. □

We will not use the induction principle with general binary relations, but only on posets, and more precisely on trees, that is, the second main ingredient of this Appendix.

Definition A.0.5. *A* tree *is a strict poset $(T, <)$ such that for every $x \in T$ the subset $\downarrow(x)$ is a chain (totally ordered set).*
A tree is said to be rooted *if it has a bottom element $\perp$ which we will call* root.
We then call leaves *the maximal elements l of T, that is, such that for all $x \in T$, $l \leq x$ implies $x = l$. Then we say that the tree is* leafy *if for every $x \in T$ there exist a leaf l such that $x \leq l$. We will call* internal *an element $x \in T$ such that there is an element $y > x$ in T.*

Remark A.0.6. *In a leafy tree we have constructively that an element is either internal or a leaf, in fact for every x there is a leaf $l \geq x$ and this means $(l > x) \vee (l = x)$, in the first case x is internal and in the second a leaf.*

In particular we will need to work with nice trees in order to achieve our purpose

Definition A.0.7. *We will call a tree* nice *if it is rooted, leafy and the the relation $<$ reversed is well-founded.*
We will keep the notation inductive and well founded even if we should add co- before them.

Remark A.0.8. *Let T be a nice tree, then for every $x \in T$, the set $\uparrow(x) = \{y \in T | y \geq x\}$ is as well a nice tree with the induced order.*
The element x becomes the root and if $y \geq x$, then every leaf over y is also a leaf over x and hence a leaf in $\uparrow(x)$, which is thus leafy. Finally from the dual of Proposition A.0.4 we get that $\uparrow(x)$ is also well-founded (still with the reversed relation.

The reason why we use trees for our issues depends on the concept of composition or concatenation of trees. Let T_0 be a rooted and leafy tree and for every $l \in T_0$ leaf let T_l be a leafy rooted tree with root l. Define their concatenation as the tree T obtained by glueing T_l to T_0 such that the root l of T_l coincides with the leaf l of T_0. As order relation on T we put the one generated by the relations on T_0 and the T_l's. Note that the relation we obtain is compatible with the glueing, in fact we have that $x < y$ iff either $x \in T_0$ and $y \in T_l$ or both $x,\ y$ are contained in the same subtree and here $x < y$.

Lemma A.0.9. *The concatenation T as above is a rooted leafy tree whose root is the root of T_0 and whose set of leaves is the union of the sets of leaves in T_l for every l leaf of T_0.*
If then T_0 and T_l for all l leaf in T_0 are nice, then so is T.

Proof. The relation $\leq$ on is an order relation for it is glueing of order relations. The poset $(T, <)$ is a tree because for every $x \in T$, $\downarrow(x)$ in T is the same as $\downarrow(x)$ in T_0 if $x \in T_0$, while if it is in T_l it is the glueing of $\downarrow(x)$ with $\downarrow(l)$ and the glueing of chains is still a chain.
The root $\perp$ of T_0 is by definition the bottom of T as well and thus T is rooted and in the same way the leaves of T_l are leaves of T. Conversely, if e is maximal in T, then if it is in T_0, it must be a leaf, but since every element of T_e is thus greater than e, e must be the unique element of T_e and hence a leaf of T_e. If instead it is in T_l, then it must again be a leaf, whence the previous claim.
For every $x \in T$, if $x \in T_l$, since T_l is leafy, there is a leaf e of T_l such that $x \leq e$ and it is also a leaf in T. If instead $x \in T_0$, then there is l leaf of T_0 such that $x \leq l$, but this implies that every leaf of T_l is a leaf of T over x and in particular, since T_l is leafy, there exist a leaf over the root l, and hence over x in T.
Suppose now that T_0 and T_l are nice for every l, then what is left to prove is that T is well-founded. Let A be inductive in T, since T_l is upward closed in T, also $A \cap T_l$ is inductive in T_l and since T_l is well-founded, $T_l \subseteq A$. Now let $x \in T_0$, the set $C = \{y \in T | y > x\}$ is the union of $C_0 = \{y \in T_0 | y > x\}$ with T_l for every leaf l over x. Since A always contains all of the T_l, it contains the set C iff it contains the set C_0. Therefore A is inductive in T iff $A \cap T_0$ is inductive in T_0 and hence again, since T_0 is well founded, T_0 is

in A. We have that A is bound to be T and thus the latter is well founded as claimed. □

This lemma will be the key to solve the problem of closing a coverage under composition, fact that will allow us to describe far more easily a Grothendieck coverage generated by a coverage, that is, in the fashion of Lemma 1.4.8 with the coherent coverage. We shall see it in the following section.

A.1 Pretopologies

In this section we are going to study a particular case of coverage which will turn out to be very useful in the study of Grothendieck topologies.

Definition A.1.1. *A* pretopology *or* base coverage *over a small category $\mathcal{C}$ is a coverage J satisfying the following properties*

1. *(identities) $\{id_C\} \in J(C)$ for all $C \in \mathcal{C}_0$*
2. *(transitivity) If $\{f_i | i \in I\}$ covers C and for all $i \in I$ the family $\{g_{i,k} | k \in K_i\}$ covers $dom(f_i)$, then the family $\{f_i g_{i,k} | i \in I,\ k \in K_i\}$ covers C as well.*

Example A.1.2. *The coverage of Examples 1.2.2, 1.2.3 and 1.2.4 are examples of pretopologies. The first two trivially and the third because for a topological space (locale) X, $\{X\}$ covers and for every open cover $\mathcal{U}$, if every open U has an open cover $\mathcal{V}_U$, then $\bigcup_{U \in \mathcal{U}} \mathcal{V}_U$ is an open cover for X.*

Example A.1.3. *If $\mathcal{C}$ is a coherent category, then the coherent coverage is a pretopology, in fact every identity is a regular epi and composition of jointly regular-epimorphic families is jointly regular epimorphic as showed in the proof of Lemma 1.4.8.*

The reason why such a coverage is so useful is that we can characterize the Grothendieck coverage generated by a pretopology as follows

Lemma A.1.4. *Let $\mathcal{C}$ be a small category and J a pretopology on it, then the Grothendieck coverage generated by J contains for every object $c \in \mathcal{C}_0$ precisely all the sieves on c which contain a J-covering family on c.*

Proof. In Section 1.3 we saw that the Grothendieck topology generated by J can be obtained first by taking the sifted closure $\overline{J}$ and then the intersection of all Grothendieck coverages containing the latter. Let K be the sifted coverage such that $K(c)$ is the set of sieves over c containing a family of $J(c)$, we want to prove that K is $\widetilde{J}$. Note that a sieve S contains a family $A \in J$ iff it contains $\langle A \rangle$ and in particular then $\overline{J} \subseteq K$. Moreover, if H is a Grothendieck coverage containing $\overline{J}$ then, since $H(c)$ is upward closed in

the lattice of sieves for every c (Remark 1.3.16), for every sieve S in K that contains a sieve R in $\overline{J}$ (and thus in H), also S is in H, whence $K \subseteq H$ and then $K \subseteq \widetilde{J}$.
We only need to prove that $\widetilde{J} \subseteq K$, but this follows if we prove that K is a Grothendieck topology, and that is what we are going to do. The first axiom is satisfied because the maximal sieve over an object c contains in particular $\{\mathrm{id}_c\}$ which is in J by the axiom of identities. The second axiom follows from the fact that J is a coverage, in fact, let $S \in K(c)$ and $f : d \longrightarrow c$, then there is $A \in J$ with $A \subseteq S$, but J is a coverage, so there is B such that $B \subseteq f^*(A) \subseteq f^*(S)$ and thus $f^*(S) \in K(d)$. For the final axiom, let $R \in K(c)$ and S a sieve over c such that for every $r \in R$, $r^*(S) \in K(\mathrm{dom}(r))$. We have then $A \in J(c)$ with $A \subseteq R$ and for every $r \in R$, a $B_r \in K(\mathrm{dom}(r))$ such that $B_r \subseteq r^*(S)$. Consider the set $C = \{ab | a \in A,\ b \in B_a\}$, it is a J-covering family by transitivity of J and for every such ab, since $a \in A \subseteq R$ and $b \in B_a \subseteq a^*(S)$, $ab \in S$. We deduce that $C \subseteq S$ and thus $S \in K(c)$, proving the third and final axiom. □

Remark A.1.5. *Lemma 1.4.8 is basically a corollary of Lemma A.1.4 once we prove that P is a pretopology.*

Not every coverage is a pretopology (for instance Grothendieck coverages aren't, for the identities axiom does not necessarily hold), but sometimes it is really useful to be able to characterize the coverings of a Grothendieck coverage with the coverings of a poorer generating coverage. Our aim now is to find a way to obtain a pretopology from a generic coverage J equivalent to it.
As showed in Remark 1.3.2, adding the identities is a harmless operation, so in this chapter from now on we shall suppose that a coverage J also satisfies the axiom of identities. Therefore we are now left to close a coverage J under composition of coverings and thus to prove transitivity. This is precisely the point when trees become essential.

Let J be a coverage (with identities), let $\mathcal{D}(J)$ be the directed graph having as objects all the J covering families, then arrows are triples (b, A, B) where $b \in B$ for some J-covering B and $A \in J(\mathrm{dom}(b))$ and domain and codomain of such a map are respectively A and B. In particular an arrow $A \longrightarrow B$ is a triple (b, A, B) where $b \in B$. For simplicity we will call such an arrow simply b if A and B are clear from the context or just write $b : A \longrightarrow B$ otherwise.
Let now $\mathcal{F}(J)$ be the free category generated by the graph $\mathcal{D}(J)$. A more explicit description of $\mathcal{F}(J)$ is the following: objects are still J-covering families and arrows from B_0 to B_n are (possibly empty) finite sequences $(b_n, \dots, b_1)$ of arrows $b_i : B_{i-1} \longrightarrow B_i$ for $i = 1, \dots, n$.

Remark A.1.6. *Being $\mathcal{F}(J)$ free over $\mathcal{D}(J)$, there is a universal inclusion of directed graphs $i : \mathcal{D}(J) \longrightarrow \mathcal{F}(J)$ sending an arrow $b : A \longrightarrow B$ to $(b) : A \longrightarrow B$.*

Universality means that for every category $\mathcal{A}$ with a morphism of directed graphs $\phi : \mathcal{D}(J) \longrightarrow \mathcal{A}$, there is a unique functor $\overline{\phi} : \mathcal{F}(J) \longrightarrow \mathcal{A}$ such that as graph morphisms we get $\phi = \overline{\phi} i$.

Note also that if J is a coverage on $\mathcal{C}$, we have a correspondence $\pi : \mathcal{D}(J) \longrightarrow \mathcal{C}$ sending a morphism $b : A \longrightarrow B$ to b and hence a coverage $S \in J(c)$ is sent to c, so that this is a directed graphs morphism. It follows from Remark A.1.6 that there is a unique functor $k : \mathcal{F}(J) \longrightarrow \mathcal{C}$ which we will call composition. Explicitly on objects it is as π while on morphisms it sends a finite sequence $(b_n, \dots, b_1)$ to the composition $b_n \dots b_1$, whence the name.
Let now T be a nice tree, consider a functor

$$X : T^{\mathrm{op}} \longrightarrow \mathcal{F}(J)$$

with the following two properties

1. If $x \in T$ is internal, for all $f \in X(x)$ there is $y \in T$, $y \geq x$ such that $X(x \leq y) = f : X(y) \longrightarrow X(x)$
2. If $l \in T$ is a leaf, $X(l) = \{\mathrm{id}_c\}$ for some $c \in \mathcal{C}$.

We will call $\chi(T)$ the set of all such functors.
Let now $\hat{J}$ be the set of families $F_X = \{kX(\perp_T \leq l) | l \text{ leaf in } T\}$ for every nice tree T and $X \in \chi(T)$, where k is the composition defined before. We can see $\hat{J}$ as indexed over the objects of $\mathcal{C}$, where $\hat{J}(c) = \{F_X | X \in \chi(T),\ X(\perp) \in J(c),\ T \text{ nice tree}\}$. In the following lemma we prove that this is the family we need

Lemma A.1.7. *Let J be a coverage over $\mathcal{C}$, then $\hat{J}$ built above is a pretopology containing J and equivalent to it.*

Proof. Note first that for every tree T and $X \in \chi(T)$, $kX(\perp) = c$ implies that $X(\perp) \in J(c)$ and thus that every map in F_X has codomain c as required.
Without loss of generality we can assume that J already contains identities, so take as tree the singleton 1. It is a nice tree in fact it is well-founded and its unique element 0 is both root and leaf. In this case $\chi(1)$ contains exactly those X such that $X(0) = \{\mathrm{id}_c\}$ for $c \in \mathcal{C}_0$, for there are only leaves and they are bound to be of this form. In particular for every such X, $F_X = \{\mathrm{id}_c\}$ and thus $\hat{J}$ contains all the identities.
We need now to prove that $\hat{J}$ is a coverage, so let $S \in \hat{J}(c)$, it contains then F_X for some $X \in \chi(T)$ and a nice tree T. We will achieve our goal once we show that for every f there exist a nice tree T' and a functor $Y \in \chi(T')$ such that fF_Y refines F_X. Let A be the subset of those $x \in T$ such that for $X|_{\uparrow(x)}$ and for every map f with codomain $kX(x)$, there exist a nice tree T' and a map $Y \in \chi(T')$ such that fF_Y refines $F_{X|_{\uparrow(x)}}$. The aim is to prove that

it is inductive, so that since T is well-founded, follows that in particular $\perp$ has this property and hence that there exists a tree T' and $Y \in \chi(T')$ such that fF_Y refines F_X proving that $\hat{J}$ is a coverage. First note that $\uparrow(x)$ is a nice tree for Remark A.0.8, so it makes sense to consider the class $\chi(\uparrow(x))$; then $X|_{\uparrow(x)}$ is in $\chi(\uparrow(x))$ because the two properties defining the sets $\chi(-)$ are stable under restriction to upward closed subsets of the domain.
Now, to prove that A is inductive, suppose $\{y \in T | y > x\} \subseteq A$. If there are no elements $y > x$, then x is a leaf, so for the first axiom of functors in $\chi(T)$, $X(x)$ must be $\{\mathrm{id}_c\}$ for some object c of $\mathcal{C}$ and, for every $f : d \longrightarrow c$, $\{\mathrm{id}_d\} \in \hat{J}$ for previous deductions, so $x \in A$ for every leaf x.
If instead x is internal, suppose $X(x) \in \hat{J}(c)$, then for every $f : d \longrightarrow c$ there is a family $E \in J$ such that fE refines $X(x)$, hence for every $e \in E$ there is a $u_e \in X(x)$ such that $fe = u_e\phi_e$ for some arrow $\phi_e : \mathrm{dom}(e) \longrightarrow d$ in $\mathcal{C}$. Since x is internal, for the first property of functors in $\chi(T)$, for every u_e there is a $y_e > x$ such that $X(x \leq y_e) = u_e$. Since for every $e \in E$ all the $y_e > x$, $y_e \in A$ and thus $X|_{\uparrow(y_e)}$ has the desired property and in particular for every ϕ_e, we have a nice tree T_e and $Y_e \in \chi(T_e)$ such that $\phi_e F_{Y_e}$ refines $F_{X|_{\uparrow(y_e)}}$.
More explicitly this means that for every leaf l in T_e, there is an arrow ψ_e and a leaf $l_e \in \uparrow(y_e)$ such that $\phi_e kY_e(\perp \leq l) = kX|_{\uparrow(y_e)}(y_e \leq l_e)\psi_e$.
Consider the poset T_0 obtained by adding a bottom element $\perp$ to E with the discrete order, that is, the only relations are $\perp < e$ for all $e \in E$ (for praticity call $\perp_e$ the element $e \in T_0$. We have that T_0 is rooted and leafy and from Example A.0.2 it is also well founded.
Now we compose this tree after the T_e, that is, we glue each T_e with T_0 by identifying the root of T_e with $\perp_e \in T_0$ (whence the choice of the name), obtaining thus a new poset T'.
Thanks to Lemma A.0.9 we have that T' is a nice tree. Consider now $Y : T \longrightarrow \mathcal{F}(J)$ uniquely defined by taking $Y(\perp) = E \in J(d)$, then for every $e \in E$, $Y(\perp \leq \perp_e) = e$ and for every element $y \in T_e$, $Y(y) = Y_e(y)$.
Note first that it makes sense for $\mathrm{dom}(e) = \mathrm{dom}(\phi_e)$ and $Y_e(\perp_e) \in J(\mathrm{dom}(\phi_e))$. Then if l is a leaf in T', again thanks to Lemma A.0.9, l is a leaf of Y_e for some e and thus $Y(l) = Y_e(l)$ which is an identity singleton. Therefore Y has the second property for functors in $\chi(T')$.
If x is internal, then either it is the root $\perp$, in which case we have that for every $e \in E = Y(\perp)$ there is a $\perp_e > \perp$ such that $Y(\perp \leq \perp_e) = e$, or it is in Y_e for some e and hence again it satisfies the first axiom of functors in $\chi(T')$.
We have that $Y \in \chi(T')$ and thus it makes sense to consider F_Y.
Note now that every arrow in F_y is of the form $kY(l)$ for some leaf l, but then for every leaf $l \in T'$, l is also a leaf in T_e for some e (Lemma A.0.9) and thus there is ψ_e such that

$$\phi_e kY_e(\perp_e \leq l) = kX|_{\uparrow(y_e)}(y_e \leq l_e)\psi_e$$

but then we have

$$\begin{aligned} fkY(\bot \leq l) = f\,(kY(\bot \leq \bot_e))\,(kY(\bot_e \leq l)) = \\ = fe\,(kY_e(\bot_e \leq l)) = u_e\phi_e\,(kY_e(\bot_e \leq l_e)) = \\ = u_e(kX|_{\uparrow(y_e)}(y_e \leq l_e))\psi_e = kX(\bot \leq y_e)kX(y_e \leq l_e)\psi_e = \\ = kX(\bot \leq l_e)\psi_e \end{aligned}$$

Therefore by definition fF_Y refines F_X and this proves that $\hat{J}$ is a coverage. We need to prove transitivity, but it basically follows from Lemma A.0.9 because if we have a family $F_{X_0} \in \hat{J}(c)$ for $X_0 \in \chi(T_0)$ and for every $h \in F_X$ a coverage $F_{X_h} \in \hat{J}(\mathrm{dom}(h))$ where $X_h \in \chi(T_h)$ for a nice tree T_h, then the family $\{hq|h \in F_{X_0}, q \in F_{X_h}\}$ is actually the family F_X where X is obtained by glueing X_0 to X_h. To be more precise, each $h \in F_{X_0}$ is of the form $kX_0(\bot \leq l_0)$ for some leaf $l_0 \in T_0$, so for every such a leaf, glue the root of T_h with the corresponding leaf l_0 in T_0. Call their glueing $\bot_{l_0}$. For the mentioned lemma, T is nice and if we define X to be equal to X_0 on the internal elements of T_0 and on all arrows different from the identity on leaves, and X_h on the whole T_h, we get a well defined functor (one should just check that $X_0(l_0) = X_h(\bot_{l_0})$ but that's a consequence of our choices). Internal points of T are internal points of T_0 or of T_h and leaves of T are leaves of some T_h, so by definition of X, we have $X \in \chi(T)$. It makes sense then to consider F_X and by functoriality of X and k one gets that for a leaf $l \in T$, if it is in T_h,

$$kX(\bot \leq l) = (kX_0(\bot \leq l_0))(kX_h(\bot_{l_0} \leq l)) = h\ (kX_h(\bot_h \leq l))$$

Since now l is any leaf of T_h and we can do this for every h, we have what claimed, hence that $F_X = \{hq|h \in F_{X_0}, q \in F_{X_h}\} \in \hat{J}(c)$. This implies transitivity of $\hat{J}$.
We are left to prove $J \subseteq \hat{J}$ and their equivalence. For the first, note that every $R \in J$ is of the form F_X where $X \in \chi(T)$ for a nice tree T. This because we can obtain T by adding a bottom element on R with the discrete order and then X sends $\bot$ to R and every r to the corresponding $\{\mathrm{id}_{\mathrm{dom}(r)}\}$, while $X(\bot \leq r) = r$. The functor X satisfies by construction the two properties required for being in $\chi(T)$ and in particular $F_X = R$.
For the second, since $J \subseteq \hat{J}$, we only have to prove that $\widetilde{\hat{J}} \subseteq \widetilde{J}$ where with $\widetilde{(-)}$ we mean the Grothendieck topology generated by the respective coverage. Thanks to Lemma A.1.4, elements of $\widetilde{\hat{J}}$ are sieves containing a family F_X for $X \in \chi(T)$ and T nice tree and thus containing $\langle F_X \rangle$ which we will denote with S_X. Thanks to Remark 1.3.16, if we prove that all of the S_X are in $\widetilde{J}$ for $X \in \chi(T)$ and T nice tree, we are done, for then every sieve in $\widetilde{\hat{J}}$ must contain some S_X and hence it is bound to be in $\widetilde{J}$.
Again we proceed by well founded induction to prove that $S_X \in \widetilde{J}$ for

$X \in \chi(T)$ and T nice tree. Let $A = \{x \in T | S_{X|_{\uparrow(x)}} \in \widetilde{J}\}$ and let $x \in T$ such that $U(x) = \{y \in T | y > x\} \subseteq A$. If $U(x)$ is empty, x is a leaf and thus $X|_{\uparrow(x)}$ sends x to $X(x) = \{\mathrm{id}_c\}$ for $c \in \mathcal{C}_0$, so that $S_{X|_{\uparrow(x)}} = y_c$ which is in $\widetilde{J}$. Otherwise, x is internal and thus for every $a \in X(x)$ there is $y_a > x$ such that $X(x \leq y_a) = a$. Let $c \in \mathcal{C}_0$ be such that $X(x) \in J(c)$, we have that $\langle X(x) \rangle$ is in $\widetilde{J}(c)$ and then we have the sieve $S_{X|_{\uparrow(x)}}$ also over the same c. We want to prove that the latter is in $\widetilde{J}(c)$ and thus we use the third axiom of a Grothendieck topology. Every element of $\langle X(x) \rangle$ is of the form ah for $a \in X(x)$ and a morphism h. Then $(ah)^* \left(S_{X|_{\uparrow(x)}} \right) = h^* a^* \left(S_{X|_{\uparrow(x)}} \right)$ in particular contains $h^* \left(S_{X|_{\uparrow(y_a)}} \right)$ which is in $\widetilde{J}$ by inductive hypothesis and the second axiom of Grothendieck topologies. Again we can conclude that $(ah)^* \left(S_{X|_{\uparrow(x)}} \right)$ is in $\widetilde{J}$ and thus so is $S_{X|_{\uparrow(x)}}$ which implies that $x \in A$, implying in turn that $A = X$ and thus in particular $S_{X|_{\perp}} = S_X$ is in $\widetilde{J}$. It follows from previous deductions that J and $\hat{J}$ are equivalent. □

Corollary A.1.8. *The Grothendieck coverage generated by a coverage J for $\mathcal{C}$ is such that a sieve S over c is a covering iff there exist some nice tree T and some $X \in \chi(T)$ with $X(\perp) \in J(c)$ such that $F_X \subseteq S$.*

Proof. It follows from Lemma A.1.7 and Lemma A.1.4. □

We will now use this characterisation to complete the proofs left unsolved about coherent sites.

A.2 Applications to Section 1.4

In Section 1.4 we worked with the notion of dm-sieves. Before passing to the proofs we will give the following characterization for dm-sieves

Lemma A.2.1. *A sieve S is a dm-sieve iff it is generated by a filtered family of monos.*

Proof. If S is a dm-sieve, then it is generated by the family $M = \{m \in S | m \text{ mono}\}$, in fact if $a \in S$, then its monic part m_a is in S, and hence in M, but then a is generated by m_a and thus S is generated by M. The set M is filtered for the second property of dm-sieves.
Conversely if we have a filtered family M of monomorphisms, then consider $\langle M \rangle$. If $a \in \langle M \rangle$, it is of the form mh where $m \in M$ and h is a morphism, but as follows from Remark 1.4.4, the monic part of a factors then through m and thus it is in $\langle M \rangle$. Then given a finite family of monos in $\langle M \rangle$, by definition they all factor through monos in M which in turn factor through a unique mono $m \in M \subseteq \langle M \rangle$. □

A dm-cover is thus the sifted closure of a coverage whose covering families are filtered and made of monos. Unfortunately, trees of such families are not necessarily filtered, and in fact as we saw in Remark 1.4.12, the set of dm-sieves in a Grothendieck coverage is not necessarily a Grothendieck coverage. For this reason, a good way to describe that coverage is to consider trees of such families.
Let now $\mathcal{C}$ be a coherent category and D be a dm-coverage which is compatible with P: the coherent coverage. Call S the coverage where the covering families are obtained by taking all the monomorphisms of a D-covering family for every covering family. Before proving the crucial lemma we make the following

Remark A.2.2. *The family S is a coverage because if $A \in S(c)$ and $f : d \longrightarrow c$, if we take $B = \{f^*(a) | a \in A\}$, then $f^*A \in S$ for every $A \in S$. From pullback stability of monos and the universal property of pullbacks one gets that the pullback of a filtered family of monos is again a filtered family of monos. Then again thanks to the universal property of pullbacks we see that the family B generates $f^*(\langle A \rangle)$. Therefore follows that $B \in S$.
One can prove then using well-founded induction and pullback pasting lemma, that the same happens in $\hat{S}$. Moreover one has that if $A = F_X$ for some $X \in \chi(T)$, then $B = F_Y$ for some $Y \in \chi(T)$, i.e. they can be built using the same underlying tree.*

This remark will turn out to be really useful in the next result. As matter of notation, in the next lemma we will consider functors of $\chi(T)$ into different coverages, so we will distinguish the different cases by writing the corresponding coverage as subscript. For instance $\chi_J(T)$ will be the set of functors $T^{\text{op}} \longrightarrow \mathcal{F}(J)$ with the suitable properties.
And now the promised result which can be interpreted as a sort of commutativity between the coherent coverage and a compatible dm-sieve.

Lemma A.2.3. *Let $B \in P(c)$ and for every $b \in B$ let $E_b \in \hat{S}(dom(b))$, then there is $E \in \hat{S}(c)$ and for every $e \in E$ there is a $B_e \in P(dom(e))$ such that*

$$\{uv | u \in E, v \in B_u\} \quad \textit{refines} \quad \{vu | v \in B, u \in E_v\}$$

Proof. First let's prove this result if instead of $\hat{S}$ we have S, then we will proceed by induction for the general result. With the notation above, B contains a finite jointly regular epimorphic family $\{b_1, \dots, b_n\}$, now for every $i = 1, \dots, n$, thanks to P-compatibility of S, the sieve generated by the inclusions of $\text{im}(b_1 u_1) \vee \cdots \vee \text{im}(b_n u_n)$ is S-covering for any combination of $u_i \in E_{b_i}$. To prove this claim we proceed by steps: the sieve generated by the images of $\overline{b_i} u$ where $\overline{b_i}$ is the epic part of b_i and $u \in E_{b_i}$, is an S-covering sieve because of P-compatibility. If then we consider for every $i = 1, \dots, n$ the monomorphisms of inclusion of $\text{im}(b_i)$ and $\bigvee_{j \neq i} \text{im}(b_j)$, we

have again by P-compatibility that the family is generated by $\mathrm{im}(b_i u) \vee \bigvee_{j \neq i} \mathrm{im}(b_j)$ for every $u \in E_{b_i}$ (this because $m_{b_i u} = m_{b_i} m_{\overline{b_i} u}$ by Remark 1.4.4). Note also that intersection of the sieves generated by the $\mathrm{im}(b_i u) \vee \bigvee_{j \neq i} \mathrm{im}(b_j)$ for $u \in E_{b_i}$ must be in the Grothendieck coverage generated by D (the coverage generated by S) and hence it contains a family E of $\hat{S}$. In particular note also that the intersection described above is generated by the $\mathrm{im}(b_1 u_1) \vee \cdots \vee \mathrm{im}(b_n u_n)$ for u_i's as above, in fact all of these inclusions are in the intersection above and every element which factors through all of the $\mathrm{im}(b_i u_i) \vee \bigvee_{j \neq i} \mathrm{im}(b_j)$ for $i = 1, \ldots, n$ also factors through $\bigvee_{i=1}^{n} \mathrm{im}(b_i u_i)$.
Note also that if we call $w = \bigvee_{i=1}^{n} \mathrm{im}(b_i u_i)$, then the family B_w of $\overline{b_i} u_i$ for $i = 1, \ldots n$ is a P-coverage such that the family of all the wb with w as above and B in B_w refines the family of $\{uv | u \in B, v \in E_u\}$. In particular every $e \in E$ factors through some $w = \bigvee \mathrm{im}(b_i u_i)$ and hence there is a $B_e := B_w$ such that $\{vu | v \in E, u \in B_v\}$ refines $\{uv | u \in B, v \in E_u\}$.
Now we need to prove the general case: with $B \in P$ and $E_b \in \hat{S}$ for every $b \in B$. Each E_b is of the form F_{X_b} for $X_b \in \chi_S(T_b)$. Let T be the disjoint union of the T_b's to which we add a root $\perp$, note then that in particular the family $\{bu | b \in B, u \in E_u\}$ is of the form F_X for $X \in \chi_{S \cup P}(T)$ defined to be X_b on each T_b and B in $\perp$ (where arrows from the root are sent to the corresponding arrows of B). We will proceed by induction on T, but this time the inductive hypothesis needs to be more complex.
Let $x \in T$, for every P-covering V, for every family of $y_v > x$ (one for each $v \in V$ and hence in finite number) and for every $X_v \in \chi_S(\uparrow(y_v))$ with $X_v(y_v) \in S(\mathrm{dom}(v))$, if we glue the root of every tree $\uparrow(y_v)$ to corresponding leaf of the tree obtained by adding a root to the discrete poset V (getting a tree T'), then as for X above, we can build $Z \in \chi_{S \cup P}(T')$ such that Z is X_v on $\uparrow(y_v)$ and $Z(\perp) = V$. Let A be the set of $x \in T$ such that for every possible situation as above, we can build a tree T'', $Y \in \chi_S(T'')$ and for every leaf $l \in T''$, a P-covering family B_l such that, the family $\{uv | u = kY(\perp \leq l), l \text{ leaf}, v \in B_l\}$ refines F_Z.
If we prove that this holds for all $x \in T$, then it also holds for $\perp \in T$ and hence, if we choose for all $b \in B$ the element y_b whose existence is assured from X being in $\chi_{S \cup P}(T)$, then we are in the right hypotheses, so there is a tree T'', Y and B_l's as above. If we call $E = F_Y$ and for $e \in E$, $B_e = B_l$ for some leaf $l \in T''$ such that $kY(\perp \leq l) = e$, we get the claim of this lemma.
Now let's prove that A is inductive. Let $x \in T$ and $\{y \in T | y > x\} \subseteq A$, as usual we want to prove that $x \in A$. If x is a leaf, the result is trivial because there are no elements over x, so we can suppose that x is internal. In the notation of the inductive hypothesis, we have $V \in P$, $y_v > x$ and $X_v \in \chi_S(\uparrow(y_v))$. In particular then, $X_v(y_v) \in S$, so applying the specific case showed at the beginning of this proof, with $B = V$ and $E_v = X_v(y_v)$ we can get an $E' \in \hat{S}$ and for every $e \in E'$, $V_e \in P$ such that $\{uv | u \in E', v \in V_u\}$ refines $\{vu | v \in V, u \in X_v(y_v)\}$. Now for all $q \in X_v(y_v)$ there is a $y_q > y_v$,

hence we can apply the inductive hypothesis, being $y_v \in A$. Note that every $W_q \in \chi_S(\uparrow(y_q))$, can be pulled back along the factorizations given by the refinement, so that for every $v \in V_e$ we can find by Remark A.2.2 a $W'_v \in \chi_S(\uparrow(y_q))$ with $W'_v(y_q) \in S(\mathrm{dom}(w))$. In particular for every e we can glue V_e with all these W'_v for $v \in V_e$. We are precisely in the right hypotheses to apply the induction, so we get $E_e \in \hat{S}$ and for every $f \in E_e$ a $B_f \in P$ with the needed property of refinement. Composing the trees generating E_e and E', we still get a tree in S and hence a family in $\hat{S}$, then the B_f's give the P-covering families that prove the induction. Up to the remaining details, we have proven that A is inductive and thus the lemma. □

Using this theorem we obtain the following result, which completes Proposition 1.4.11.

Corollary A.2.4. *Let $\mathcal{C}$, P, D and S as right before Remark A.2.2 and again we will use $\widetilde{(-)}$ to denote the Grothendieck coverage generated by a coverage. For every $c \in \mathcal{C}_0$, let $J(c)$ be the set of sieves R over c such that there is $A \in \widetilde{D}(c)$ such that $a^*(R) \in \widetilde{P}(dom(a))$ for every $a \in A$. In these hypotheses we have that J satisfies the third axiom of Grothendieck coverages.*

Proof. Let $R \in J(c)$ and H a sieve over c such that $r^*(H) \in J(c)$, we want to prove that $H \in J(c)$. Since $R \in J(c)$, there is $A \in \widetilde{D}$ such that $a^*(R) \in \widetilde{P}$ for all $a \in A$; moreover, for every $r \in R$, there is $B_r \in \widetilde{D}(\mathrm{dom}(r))$ such that for all $b \in B_r$, $b^*(r^*(H)) \in \widetilde{P}(\mathrm{dom}(b))$. Since $A \in \widetilde{D}$, it contains a family A' of $\hat{S}$. Choose $a \in A'$, then $a^*(R) \in \widetilde{P}$, so there is a finite jointly epimorphic family $e_1, \dots, e_n$ in it, thanks to Lemma 1.4.8. We have that $ae_i \in R$, so it makes sense to consider B_{ae_i} and in particular we have that it contains as well a covering family in $\hat{S}$, call it B'_{ae_i} and finally for every $b \in B'_{ae_i}$, $b^*(ae_i)^*H \in \widetilde{P}$, so there is a $V_{ae_ib} \in P$ contained in it.
Now if we apply Lemma A.2.3, using the families $\{e_1, \dots, e_n\}$ and B'_{ae_i}, we get a family $E_a \in \hat{S}(\mathrm{dom}(a))$ and for every $x \in E_a$ a family $Q_x \in P(\mathrm{dom}(x))$ such that $\{xq | x \in E_a, q \in Q_x\}$ refines $\{e_ib | b \in B'_{ae_i}, i = 1, \dots, n\}$. Consider now the composition of E_a with A', i.e. $E' = \{ax | a \in A', x \in E_a\}$, it is in $\hat{S}$ because of Lemma A.1.7, so in particular the sieve E generated by E' is in $\widetilde{D}$. Moreover, for every $ax \in E'$ and $q \in Q_x$, there is $i = 1, \dots, n$, $b \in B'_{ae_i}$ and a map $\phi : \mathrm{dom}(x) \longrightarrow \mathrm{dom}(b)$ such that $xq = e_ib\phi$, so consider V_{ae_ib} and pull it back along ϕ, so that we get $U_q \in P(\mathrm{dom}(q))$. Consider the composition of the U_q's with Q_x, i.e. $W_x = \{qu | q \in Q_x, u \in U_q\}$, it is in $P(\mathrm{dom}(x))$ for Lemma 1.4.8. Now, take the previously constructed E, it is in $\widetilde{D}$ and all of its elements are of the form axk for some arrow k, $a \in A'$ and $x \in E_a$, then consider $(axk)^*(H) = k^*(ax)^*(H)$. We have that $k^*(ax)^*(H)$ contains $k^*(W_x)$ because $(ax)^*(H)$ contains W_x and the latter is true because every element $w \in W_x$ is of the form qu with $q \in Q_x$ and $u \in U_x$, so $axqu = ae_ib\phi u$. Then by construction ϕu factors through

some $v \in V_{ae_ib}$ as $\phi u = vp$ and thus $axqu = ae_ibvp$, but $v \in (ae_ib)^*(H)$, so $axqu \in H$ as claimed.
We have finally found $E \in \widetilde{D}$ such that for all $ax \in E$, $(ax)^*(H) \in \widetilde{P}$ (for it contains $k^*(W_x) \in P$) and thus $H \in J$ as needed. □

We prove also the following technical fact

Lemma A.2.5. *Let $\mathcal{C}$ be a category, J a cover on it and $\mathcal{D}$ a subcategory of $\mathcal{C}$. If for every $d \in \mathcal{D}_0$, every family in $J(d)$ contains an arrow of $\mathcal{D}$, so does $\hat{J}$ and hence $\widetilde{J}$.*

Proof. If we prove it for $\hat{J}$, then for $\widetilde{J}$ is immediate because every covering in $\widetilde{J}$ contains one in $\hat{J}$ which contains an arrow of $\mathcal{D}$.
Let's prove it for $\hat{J}$. First notice that without loss of generality we can consider J with identities, for $\{\mathrm{id}_c\}$ is in $\mathcal{D}$ iff $c \in \mathcal{D}$. Now let $d \in \mathcal{D}_0$ and consider a $\hat{J}$-covering family, from Theorem A.1.7 it is of the form F_X for some $X \in \chi(T)$ for some nice tree T. We work again by induction on T. Let A be the set of $x \in T$ such that if $X(x) \in J(d)$ for some $d \in \mathcal{D}_0$, then $F_{X|_{\uparrow(x)}}$ contains an arrow of $\mathcal{D}$. The set A satisfies the inductive condition on leaves l because then $F_{X|_{\uparrow(l)}} = \{\mathrm{id}_c\}$ and $c \in \mathcal{D}_0$ iff $\mathrm{id}_c \in \mathcal{D}_1$. If instead $x \in T$ is internal and such that $\{y \in T | y > x\} \subseteq A$, then if $X(x) \in J(d)$, it contains an arrow $f : d \longrightarrow c$ in $\mathcal{D}$ by hypothesis on J. Since $X \in \chi(T)$, there is a $y > x$ such that $X(x \leq y) = f$, but then $X(y) \in J(d)$ and since $y \in A$, there is a leaf $l \geq y$ such that $X(y \leq l) \in \mathcal{D}_1$. Now l is a leaf in $\uparrow(x)$ and $X(x \leq l) = X(x \leq y)X(y \leq l) = fX(y \leq l)$ which is in $\mathcal{D}$ being composition of arrows of $\mathcal{D}$. From the definition of A, it follows that $x \in A$ and hence that A is inductive. We have then $A = T$ and thus the family $F_X \in J(d)$ for $d \in \mathcal{D}_0$, being such that $X(\perp) \in J(d)$, contains an arrow of $\mathcal{D}$ ending the proof. □

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